Optimization of an n-Person Game Under Linear Side Conditions

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Summary. This paper is concerned with n-person games which typically occur in mathematical conflict models [cf. [4], [7],[8]]. These games are so called cost-games, in which every actor tries to minimize his own costs and the costs are interlinked by a system of linear inequalities. It is shown that, if the players cooperate, i.e., minimize the sum of all the costs, they achieve a Nash Equilibrium. In order to determine Nash Equilibria, the simplex method can be applied with respect to the dual problem. An important special case is discussed and numerical examples are presented.

1. Introduction

The conferences of Rio de Janeiro 1992 and Kyoto 1997 demand for new economic instruments which have a focus on environmental protection in the macro and micro economy. An important economic tool being part of the treaty of Kyoto in that area is *Joint-Implementation*. It is an international program being part of the treaty of Kyoto which intends to strenghten international cooperations between enterprises in order to reduce CO_2-reductions. A sustainable development can only be guaranteed if the instrument is embedded in an optimal energy management. Optimal energy management according to *Joint-Implementation* means in this context that it must work on a micro level with minimal costs and it should be protected against misuse on a macro level.
For that reason, the TEM model (Technology-Emissions-Means model) was developed, giving the possibility to simulate such an extraordinary market situation.

2. The TEM model

Technology-Emissions-Means model

The realization of Joint-Implementation (JI) is determined by technical and financial constraints. In a JI Program the reduced emissions resulting from technical cooperations are registrated at the *Clearing House*. The TEM model integrates both the simulation of the technical and financial parameters. For that reason we want to give a short introduction into the TEM model at the beginning. In Pickl (1999) the TEM model is treated as a time-discrete control problem. Furthermore, the analysis of the feasible set is examined in Pickl (2000). In the following, after having introduced the TEM model we want to present a new
GAME

The presented TEM-model describes the economical interaction between several actors (players) which intend to maximize their emissions reduced (E_i) caused by technologies (T_i) by means of expenditures of money (M_i) or financial means, respectively. The index stands for the i-th player, $i \in \{1, \ldots, n\}$. The players are linked by technical cooperations and the market, which expresses itself in the nonlinear time-discrete dynamics of the Technology-Emission-Means model, in short: TEM model.

$$E_i(t+1) = E_i(t) + \sum_{j=1}^{n} em_{ij}(t) M_j(t) \quad (1)$$

$$M_i(t+1) = M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \Delta E_i(t)\} \quad (2)$$

We want to explain in the following the TEM model. Let us begin with a description of the following parameters. (For a deeper insight see Pickl (1999)):

E_i	emissions reduced of actor i in percent
M_i	financial means of actor i
em_{ij}	*effectivity measure parameter*
	describes the effect on the emissions of the i-th player
	if the j-th actor invests money for his technologies
φ_i	*memory parameter*
λ_i	*growth parameter*

Here, em_{ij} describes the effect on the emissions of the i-th actor, if the j-th actor invests money. We can say that it expresses **how effective technology cooperations are (like an innovation factor)**, which is the central element of a JI Program.

Furthermore, we are able to determine the em_{ij}-parameter **empirically**. In the first equation the level of the reduced emissions at the t-th time-step depends upon the last value plus a *market effect*. This effect expresses itself in the additive terms which might be negative or positive. In general, $E_i > 0$ implies that the actors have reached yet the demanded value $E_i = 0$ (normalized *Kyoto-level*). A value $E_i < 0$ expresses that the emissions are less than the requirements of the treaty. In the second equation we see that for such a situation the financial means will increase whereas $E_i > 0$ leads to a reduction of $M_i(t+1)$:

$$M_i(t+1) = M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \Delta \mathbf{E_i(t)}\}$$

The second equation contains the logistic functional dependence and the memory parameter φ_i which describes the effect of the preceeding investment of financial means. The dynamics does not guarantee, that the parameter $M_i(t)$ lies in the interval, which can be regarded as a budget for the i-th actor. For that reason we have to add restrictions to the dynamical representation.

$$0 \leq M_i(t) \leq M_i^*, \quad i = 1, \ldots, n \quad \text{and} \quad t = 0, \ldots, N.$$

Then it is easy to show that

$$-\lambda_i M_i(t)[M_i^* - M_i(t)] \leq 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad t = 0, \ldots, N.$$

We have guaranteed that $M_i(t+1)$ increases if $E_i(t) + \varphi_i \Delta E_i(t) \leq 0$ and decreases if $E_i(t) + \varphi_i \Delta E_i(t) \geq 0$. Applying the memory parameter φ_i we have developed a reasonable model for the *money expenditure - emission* - interaction, where the influence of the technologies is integrated in the *em*-matrix of the system.

We can use the TEM model as a time-discrete model where we start with a special parameter set and observe the resulting trajectories. Normally, the actors start with a negative value, i.e., they lie under the baseline mentioned in Kyoto Protocol, see Kyoto (1997). They try to reach a positive value of E_i . If we add control parameters, we enforce this development by an additive financial term. For that reason the control parameter are added only to the second equation.

$$\begin{aligned} M_i(t+1) &= M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \Delta E_i(t)\} + u_i(t) \\ u_i(t) & \quad \text{control parameter} \end{aligned}$$

In the sense of environmental protection, the aim is to reach a state which is mentioned in the treaty of *Kyoto* by choosing the control parameters such that the emissions of each player become minimized. The focal point is the realization of the necessary optimal control parameters via a played cost game, which is determined by the way of cooperation of the actors.

3. The Cost-Game in the TEM Model

If we regard the nonlinear time-discrete dynamics of the TEM-model

$$\begin{aligned} E_i(t+1) &= E_i(t) + \sum_{j=1}^{n} em_{ij}(t) M_j(t) \qquad (1) \\ M_i(t+1) &= M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \Delta E_i(t)\} \end{aligned}$$

we can also formulate

$$\begin{aligned} E_i(t+1) &= E_i(t) + \sum_{j=1}^{n} em_{ij}(t) M_j(t) \qquad (2) \\ M_i(t+1) &= M_i(t) - \lambda_i M_i(t)[M_i^* - M_i(t)]\{E_i(t) + \varphi_i \sum_{j=1}^{n} em_{ij}(t) M_j(t)\} \end{aligned}$$

considering that $\Delta E_i(t) = E_i(t+1) - E_i(t)$

In order to reach steady states, which are determined in Pickl (1999), an independent institution may influence the *trade relations* between the actors. In practice, the imposing of *taxes* or the giving of *incentives* means that in the TEM-model the *em*-parameter will change. Now, the principle of JI implies that technical cooperation will be benefitted:

$$\begin{pmatrix} em_{11} & em_{12}+\epsilon & em_{13} \\ em_{21}+\epsilon & em_{22} & em_{23} \\ em_{31} & em_{32} & em_{33} \end{pmatrix}$$

Actor 1 and Actor 2 do cooperate

$$\begin{pmatrix} em_{11}+\omega & em_{12}+\omega & em_{13}+\omega \\ em_{21}+\omega & em_{22}+\omega & em_{23}+\omega \\ em_{31}+\omega & em_{32}+\omega & em_{33}+\omega \end{pmatrix}$$

All players do cooperate

According to (1) and (2) let us begin with the construction of the cost-game in the TEM-model

$$\begin{aligned} v_t(K) &:= \underbrace{\sum_{j \in K} M_j(t)}_{Without\ Cooperation} - \underbrace{M(K)}_{Cooperation} \qquad (3) \\ &= \begin{pmatrix} K_1^*(t) & K_2^*(t) & K_3^*(t) \end{pmatrix} \begin{pmatrix} 0 & \epsilon & \delta \\ \epsilon & 0 & \gamma \\ \delta & \gamma & 0 \end{pmatrix}_{Ind(K)} \begin{pmatrix} M_1(t) \\ M_2(t) \\ M_3(t) \end{pmatrix} \end{aligned}$$

with $K_i^*(t) = \varphi_i \tilde{M}_i(t), \tilde{M}_i(t) := [M_i^* - M_i(t)], (i = 1, \ldots, n)$ and $K \in Pot(\mathcal{N})$.
In the sequel, we have

$$(B)_{Ind(K)} := A, \text{with} \left\{ \begin{array}{lcl} a_{ij} & = & b_{ij} \quad , \text{if} \quad i \in K \quad \text{and} \quad j \in K \\ a_{ij} & = & 0 \end{array} \right.$$

For the time-dependent grand coalition we get:

$$\begin{aligned} v_t(\mathcal{N}) &:= \underbrace{\sum_{j \in \mathcal{N}} M_j(t)}_{Without\,Cooperation} - \underbrace{M(\mathcal{N})}_{Cooperation} \\ &= \left(K_1^*(t) \quad K_2^*(t) \quad K_3^*(t) \right) \begin{pmatrix} 0 & \omega & \omega \\ \omega & 0 & \omega \\ \omega & \omega & 0 \end{pmatrix} \begin{pmatrix} M_1(t) \\ M_2(t) \\ M_3(t) \end{pmatrix} \end{aligned}$$

For $\tilde{M}_i(t)\varphi_i M_i(t) \geq 0$ $(i, j \in \{1, \ldots, n\})$ the difference between the cooperative and the non-cooperative case is always positive. So we have constructed a reasonable cost-game. Now we want to steer the system in a cost-minimal way in order to reach the states $M_i(\hat{t}) = 0$ and $E_i(\hat{t}) = 0\,(i = 1, \ldots, n)$, for some $\hat{t} \geq 0$.

The method is that at each time step, the amount of our cost-game is put into a central fund, which can also be used as feasible set for our control process. In the following we want to analyse a special allocation principle.

This minimization problem leads directly to the following allocation problem which we will solve in the next section. Using linear programming techniques and the simplex method Nash Equilibria are determined. Together with the basic theory [5] then we are able to simulate and analyse an economical Joint-Implementation Program [7] with the TEM model in the sense of Gustav Feichtinger [2], [3].

4. The Allocation Problem

In connection with the TEM-Model [7] which is based on a general conflict model [6] the following allocation problem is in the center of interest. In order to develop a Joint-Implementation Program we begin with the following formulation:

Given n players who pursue n goals which are given by an n-vector

$$b = (b_1, \ldots, b_n)^T \quad \text{with} \quad b_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n.$$

In order to achieve these goals every player has to put in a certain amount of money, say $x_i \geq 0$ for the i-th player. The share of the player j at the goal b_i (where b_j is his own goal) when he spends one unit is assumed to be c_{ij} where (for good reasons)

$$c_{ii} > 0 \quad \text{for} \quad i = 1, \ldots, n\ . \tag{4}$$

If $i \neq j$, however, $c_{ij} \leq 0$ is also allowed for. In such a case player j can be considered as an opponent of player i. The requirement to achieve all the goals is expressed by the following system of linear inequalities

$$\sum_{j=1}^{n} c_{ij}\ x_j \geq b_i \quad \text{for} \quad i = 1, \ldots, n\ . \tag{5}$$

In the sequel we assume that there is a vector $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$ for $i = 1, \ldots, n$ for which the inequalities (2) are satisfied.

Then, for every i, the i-th player, of course, is interested in minimizing his own contribution x_i. In general, this will not be possible simultaneously. So the players will have to cooperate. Let us assume that they choose $\hat{x} \in I\!R^n$ with (2) and $x_j \geq 0$, $j = 1, \ldots, n$, for a $x = \hat{x}$ such that

$$s(x) = \sum_{j=1}^{n} x_j \tag{6}$$

for $x = \hat{x}$, is as small as possible. Now let, for $i \in \{1, \ldots, n\}$, $x_i \geq 0$ be chosen such that

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} c_{kj}\ \hat{x}_j + c_{ki}\ x_i \geq b_k \quad \text{for all} \quad k = 1, \ldots, n\ , \tag{7}$$

then

$$\sum_{j=1}^{n} \hat{x}_j \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} \hat{x}_j + x_i \quad \Longrightarrow \quad \hat{x}_i \leq x_i \tag{8}$$

which implies that $\hat{x}$ is a Nash equilibrium.

5. On the Determination of Nash-Equilibria

The problem of minimizing (3) subject to (2) and

$$x_j \geq 0 \quad \text{for} \quad j = 1, \ldots, n \tag{9}$$

is a typical problem of linear programming whose dual problem consists of maximizing

$$t(y) = \sum_{i=1}^{n} b_i \; y_i \tag{10}$$

subject to

$$\sum_{i=1}^{n} c_{ij} \; y_i \leq 1 \quad \text{for} \quad j = 1, \ldots, n \tag{11}$$

and

$$y_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n \; . \tag{12}$$

If we put $y_i = 0$ for $i = 1, \ldots, n$, then we obtain a solution of (11) and (12). Under the above assumption that there exists a solution of (5) and (7) we can apply a well known duality theorem and conclude that there exists a solution $\hat{x} \in I\!R^n$ of (5) and (9) which minimizes (6) and a solution $\hat{y} \in I\!R^n$ of (11) and (12) which maximizes (10) and that $s(\hat{x}) = t(\hat{y})$ which is equivalent to

$$\hat{x}_j > 0 \quad \Longrightarrow \quad \sum_{i=1}^{n} c_{ij} \; \hat{y}_i = 1$$

and

$$\hat{y}_i > 0 \Longrightarrow \sum_{j=1}^{n} c_{ij} \; \hat{x}_j = b_i \; . \tag{13}$$

On introducing slack variables

$$z_j \geq 0 \quad \text{for} \quad j = 1, \ldots, n \tag{14}$$

the inequalities (11) can be rewritten as equations in the form

$$z_j + \sum_{i=1}^{n} c_{ij}\ y_i = 1 \quad \text{for} \quad j = 1, \ldots, n\ . \tag{15}$$

The dual problem is then equivalent to the minimization of

$$\sum_{j=1}^{n} 0 \cdot z_j + \sum_{i=1}^{n} b_i\ y_i \tag{16}$$

subject to (12), (14), and (15). This problem can be solved immediately by the simplex method starting with the basic solution

$$z_j = 1 \quad \text{for} \quad j = 1, \ldots, n \quad \text{and} \quad y_i = 0 \quad \text{for} \quad i = 1, \ldots, n\ . \tag{17}$$

5.1 A Special Case

Now let us assume that for some $j \in \{1, \ldots, n\}$ it is true that

$$c_{ij} \leq 0 \quad \text{for all} \quad i = 1, \ldots, n\ , \quad i \neq j\ , \tag{18}$$

i.e., the player j can be considered as an opponent of all the other players. If $\hat{x} \in I\!R^n$ is a solution of (5) and (9) that minimizes (6), it follows that

$$\sum_{k=1}^{n} c_{jk}\ \hat{x}_k = b_j\ . \tag{19}$$

For otherwise $(\hat{x}_1, \ldots, \hat{x}_{j-1}, x_j^*, \hat{x}_{j+1}, \ldots, \hat{x}_n)$ with

$$x_j^* = \frac{1}{c_{jj}}\ (b_j - \sum_{\substack{k=1 \\ k \neq j}}^{n} c_{jk}\ \hat{x}_k) < \hat{x}_j$$

also solves (5) and (9) and it follows that

$$x_j^* + \sum_{\substack{k=1 \\ k \neq j}}^{n} \hat{x}_k < \sum_{k=1}^{n} \hat{x}_k$$

contradicting the assumption that $\sum_{k=1}^{n} \hat{x}_k$ is minimal.

Now let us assume that

$$c_{ij} \leq 0 \quad \text{for all} \quad i \neq j\ , \tag{20}$$

i.e. all players can be considered as opponents to each other. Then, for every solution $\hat{x} \in I\!R^n$ of (5) and (9) that minimizes (6), it follows that

$$\sum_{j=1}^{n} c_{ij}\ \hat{x}_j = b_i \quad \text{for all} \quad i = 1, \ldots, n \ . \tag{21}$$

If we assume (20) and

$$\sum_{j=1}^{n} c_{ij} > 0 \quad \text{for all} \quad i = 1, \ldots, n \ , \tag{22}$$

then (1) is satisfied and the matrix C is inverse monotone, i.e., the inverse C^{-1} exists and is positive (see [1]). In this case the solution of (21) is given by

$$\hat{x} = C^{-1}\, b\, (\geq \Theta_n\,)\,, \quad \Theta_n \quad n\text{-dimensional zero-vector}\ .$$

If $x = (x_1, \ldots, x_n)^T$ is any solution of (5) and (9), then it even follows that

$$x \geq C^{-1}\, b = \hat{x}, \quad \text{i.e.} \quad x_i \geq \hat{x}_i \quad \text{for all} \quad i = 1, \ldots, n \ .$$

This means that, if all players oppose each other but every player's contribution to achieving his own goal is larger than the negative sum of his opponents, then everybody can reach an absolutely minimal amount of money.

6. Inverse Monotony

Let C be inverse monotone and let $C^* \geq C$. If $x \in I\!R^n$ is a solution of (5) and (7), then x also solves

$$C^*\, x \geq b \tag{1}$$

and, if $x^* \in I\!R^n$ solves (7) and (1) and minimizes (6), then

$$s(x^*) \leq s(\hat{x}) \ ,$$

if $\hat{x} \in I\!R^n$ solves (5) and (7) and minimizes (6).

Moreover, if

$$C^* \, x^* = b \ , \tag{2}$$

then

$$C \, x^* \leq C^* \, x^* = b \leq C \, \hat{x}$$

which implies $x^* \leq \hat{x}$. The last considerations can be interpreted as follows: If C is inverse monotone, then the players can achieve the best possible individual results by solving the linear system

$$C \, \hat{x} = b \ .$$

If they replace C by a matrix C^* with $C^* \geq C$ such that there exists $x^* \in I\!R^n$ with (9) and (2), then $x^* \leq \hat{x}$, i.e., every player gets a better result x^* which is a Nash equilibrium.

6.1 The General Case

We assume that there is a solution of (4) and (5) which implies that the dual problem has a solution. If this is obtained by $r \leq n$ steps of the simplex method, the result can be assumed to be of the following form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_r \\ z_{r+1} \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ d_r \\ d_{r+1} \\ \vdots \\ d_n \end{pmatrix} + \tilde{D} \begin{pmatrix} z_1 \\ \vdots \\ -z_r \\ -y_{r+1} \\ \vdots \\ -y_n \end{pmatrix}$$

$$\tilde{D} = \begin{pmatrix} d_{11} & \dots & d_{1r} & d_{1r+1} & \dots & d_{1n} \\ \vdots & & & & & \\ d_{r1} & \dots & d_{rr} & d_{r+r1} & \dots & d_{rn} \\ d_{r+11} & \dots & d_{r+1r} & d_{r+1\,r+1} & \dots & d_{r+1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{n1} & \dots & d_{n1} & d_{nr+1} & \dots & d_{nn} \end{pmatrix}$$

$$\sum_{j=1}^{r} b_j \, y_j = \sum_{j=1}^{r} b_j \, d_j + \sum_{k=1}^{r} (\sum_{j=1}^{r} d_{jk} \, b_j) \, (-z_k) + \sum_{k=r+1}^{n} (\sum_{j=1}^{r} d_{jk} \, b_j) \, (-y_k)$$

where $d_j \geq 0 \quad \text{for} \quad j = 1, \dots, n$ and $\sum_{j=1}^{r} d_{jk} \, b_j \geq 0 \quad \text{for} \quad k = 1, \dots, n$.

The solution of the dual problem reads $\hat{y}_j = d_j$ for $j = 1, \ldots, r$ and $\hat{y}_j = 0$ for $j = r+1, \ldots, n$.
Let us assume that

$$C_r^T = \begin{pmatrix} c_{11} & \ldots & c_{r1} \\ \vdots & & \vdots \\ c_{1r} & & c_{rr} \end{pmatrix}$$

is invertible. Then

$$D_r = \begin{pmatrix} d_{11} & \ldots & d_{1r} \\ \vdots & & \vdots \\ d_{r1} & \ldots & d_{rr} \end{pmatrix} = (C_r^T)^{-1} = (C_r^{-1})^T .$$

If we put

$$x^r = \begin{pmatrix} \sum\limits_{j=1}^{r} d_{j1} \quad b_j \\ \vdots \\ \sum\limits_{j=1}^{r} d_{jr} \quad b_j \end{pmatrix} = D_r^T \ b^r \quad \text{with} \quad b^r = \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} ,$$

then $x^r \geq 0$ and $C_r \ x^r = b^r$. Further we obtain

$$d^r = \begin{pmatrix} d_1 \\ \vdots \\ d_r \end{pmatrix} = (C_r^T)^{-1} \ e^r \quad \text{with} \quad e^r = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} ,$$

hence,

$$C_r^T \ d^r = e^r \quad \text{and} \quad (d^r)^T \ b^r = (e^r)^T \ C_r^{-1} \ b^r = (e^r)^T \ x^r .$$

All this implies that

$$\begin{aligned} \hat{x}_j &= x_j^r \quad \text{for} \quad j = 1, \ldots, r , \\ \hat{x}_j &= 0 \quad \text{for} \quad j = r+1, \ldots, n \end{aligned}$$

minimizes $s(x)$ subject to (4) and (5).

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