# Effective field theory approach to the gravitational two-body dynamics, at fourth post-Newtonian order and quintic in the Newton constant 

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#### Abstract

Working within the post-Newtonian (PN) approximation to General Relativity, we use the effective field theory (EFT) framework to study the conservative dynamics of the two-body motion at fourth PN order, at fifth order in the Newton constant. This is one of the missing pieces preventing the computation of the full Lagrangian at fourth PN order using EFT methods. We exploit the analogy between diagrams in the EFT gravitational theory and 2 -point functions in massless gauge theory, to address the calculation of 4loop amplitudes by means of standard multi-loop diagrammatic techniques. For those terms which can be directly compared, our result confirms the findings of previous studies, performed using different methods.


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## 1 Introduction

The post-Newtonian (PN) approximation to the 2-body problem in General Relativity has been subject of intense investigation in the last decades as it describes the dynamics of gravitationally bound binary systems in the weak curvature, slow velocity regime, reviewed in [1, 2] and [3].

From the phenomenological point of view its results have been of paramount importance in constructing the waveforms which have been eventually used as templates [4, 5] for the LIGO/Virgo data analysis pipeline leading to the detection [6], along with numerical simulation allowing to solve for the space time in the strong curvature regime [7] and earlier in the analysis of the Hulse-Taylor pulsar arrival times [8, 9].

Interferometric detectors of gravitational waves are particularly sensitive to the time varying phase of the signal of coalescing binaries, which thus must be computed with better than $\mathcal{O}(1)$ precision [10]. Such a phase can be determined from short-circuiting the information of the energy and luminosity function of binary inspirals with at least 3PN order accuracy.

Focusing on the conservative sector of the two body problem without spins (see [3] for results involving spins), we recall that within the EFT formalism, initially proposed in [11] and reviewed in [3, 12-14], the 1PN, 2PN [15] and 3PN [16] dynamics have been
computed, reproducing results obtained with more traditional methods; moreover the 4PN Lagrangian, quadratic in the Newton constant $G_{N}$, was first derived in the EFT framework [17].

The complete 4PN dynamics has been obtained recently by two groups within the Arnowitt-Deser-Misner Hamiltonian formalism [18, 19] and by iterating the PN equation in the harmonic gauge in [20, 21]; in both approaches an arbitrary coefficient has been fixed by using results for the gravitational wave tail effect from self-force computations [22-24]. It is worth mentioning that the two results did not initially agree at orders $G_{N}^{4}$ and $G_{N}^{5}$ and, as it is argued in [25], the discrepancy has been overcome by a suitable regularization of the infrared and ultraviolet divergencies in the approach based on the equations of motion, although the new regularization could not fix yet the value of the second ambiguity parameter in [21].

This work goes in the direction of providing a third-party computation with an independent methodology by filling one of the missing pieces to obtain the full 4PN result within EFT methods. Using the virial relation $v^{2} \sim G_{N} M / r$, being $r$ and $v$ respectively the relative distance and velocity of the binary constituents with $M$ the total mass, the terms contributing to the 4PN order dynamics can be parametrized as $G_{N}^{5-n} v^{2 n}$ with $0 \leq n \leq 5$, the leading term being the Newtonian potential, scaling simply as $G_{N}$. By following on the way paved by [17], we present in this work some results concerning the $G_{N}^{5}$ order.

The Lagrangian contains in general terms with high derivative of the dynamical variables: it is however possible to keep the equations of motion of second order without altering the dynamics by adding to the Lagrangian terms quadratic at least in the equations of motions tuned to cancel the high derivative terms at the price of introducing additional terms with higher $G_{N}$ powers, according to the standard procedure first proposed in [26] and dubbed double zero technique. The $G_{N}^{5}$ sector of the Lagrangian receives contributions from $G_{N}, G_{N}^{2}$ and $G_{N}^{3}$ Lagrangian terms which are at least quadratic in accelerations (computed in [17] up to $G_{N}^{2}$ ) via the double zero trick, as well as from genuine $G_{N}^{5}$ terms: in the present article, we focus on the genuine $G_{N}^{5}$ contribution, that is terms that do not contain $a b$ initio any power of velocity $v$ or acceleration $\dot{v}$, and leave the very last contribution, coming from $\mathcal{O}\left(G_{N}^{3} \dot{v}^{2}\right)$ terms, to a forthcoming paper dedicated to the whole $G_{N}^{3}$ sector.

In this work, we evaluate the 50 diagrams contributing to the classical effective Lagrangian in the gravitational theory at order $G_{N}^{5}$. They are non-trivial integrals over 3momenta which can be computed by means of multi-loop diagrammatic techniques. We exploit the analogy between diagrams in the EFT gravitational theory and diagrams corresponding to 2-point functions in massless gauge theory, to address the calculation of the $\mathcal{O}\left(G_{N}^{5}\right)$ diagrams as 2-point 4-loop dimensionally regulated integrals in $d$ dimensions. In particular, we use integration-by-parts identities (IBPs) [27-29] in two ways: according to the topology of the graph, IBPs allow to carry out the multiloop integration recursively loop-by-loop; alternatively, they can be used to express the result of the amplitudes as linear combination of irreducible integrals, known as master integrals (MIs). The latter are evaluated independently. The contribution to the three-dimensional Lagrangian coming from each graph is then determined by taking the $d \rightarrow 3$ limit of the Fourier transform to position-space.

The paper is organized as follows. In sec. 2 we review the EFT formalism applied to the two-body dynamics in the PN approximation to General Relativity and in sec. 3 we present the details of the 4PN computation at $G_{N}^{5}$ order. We summarize in sec. 4 and conclude in sec. 5. Appendix A contains the expressions of the master integrals needed for the computation, in Appendix B we give the contribution to the Lagrangian coming from the individual diagrams and in Appendix C details of the computation of selected amplitudes are reported.

## 2 The method

The application of the EFT framework to post-Newtonian calculations in binary dynamics has now been extensively investigated. It was first formulated in this context in [11] and subsequently applied to various aspects of the binary problem (see reviews [3, 13] and references therein).

We summarize here the basic features of this approach, along the lines and notations of $[16,17]$, while referring the reader to the literature for a more complete account. The starting point is the action

$$
\begin{equation*}
S=S_{b u l k}+S_{p p}, \tag{2.1}
\end{equation*}
$$

with the world-line point particle action representing the binary components (we only consider here spinless point masses and neglect tidal effects)

$$
\begin{equation*}
S_{p p}=-\sum_{i=1,2} m_{i} \int \mathrm{~d} \tau_{i}=-\sum_{i=1,2} m_{i} \int \sqrt{-g_{\mu \nu}\left(x_{i}\right) \mathrm{d} x_{i}^{\mu} \mathrm{d} x_{i}^{\nu}}, \tag{2.2}
\end{equation*}
$$

as well as the usual Einstein-Hilbert action ${ }^{1}$ plus a gauge fixing term

$$
\begin{equation*}
S_{\text {bulk }}=2 \Lambda^{2} \int \mathrm{~d}^{d+1} x \sqrt{-g}\left[R(g)-\frac{1}{2} \Gamma_{\mu} \Gamma^{\mu}\right], \tag{2.3}
\end{equation*}
$$

which corresponds to the same harmonic gauge condition adopted in refs. [1, 20], where $\Gamma^{\mu} \equiv g^{\rho \sigma} \Gamma_{\rho \sigma}^{\mu}$. Here $\Lambda^{-2} \equiv 32 \pi G_{N} L^{d-3}$, with $G_{N}$ the 3-dimensional Newton constant and $L$ an arbitrary length scale which keeps the correct dimensions of $\Lambda$ in dimensional regularization, and always cancels out in the expression of physical observables.
In this framework, a Kaluza-Klein (KK) parametrization of the metric [30, 31] is usually adopted (a somehow similar parametrization was first applied within the framework of a PN calculation in [32]):

$$
g_{\mu \nu}=e^{2 \phi / \Lambda}\left(\begin{array}{cc}
-1 & A_{j} / \Lambda  \tag{2.4}\\
A_{i} / \Lambda & e^{-c_{d} \phi / \Lambda} \gamma_{i j}-A_{i} A_{j} / \Lambda^{2}
\end{array}\right),
$$

with, $\gamma_{i j} \equiv \delta_{i j}+\sigma_{i j} / \Lambda, c_{d} \equiv 2 \frac{(d-1)}{(d-2)}$ and $i, j$ running over the $d$ spatial dimensions. The field $A_{i}$ is not actually needed in the present computation, so it will henceforth be set to zero; we refer to [16] for the general treatment and formulae including $A_{i}$.

[^0]In terms of the metric parametrization (2.4), with $A_{i}=0$, each world-line coupling to the gravitational degrees of freedom $\phi, \sigma_{i j}$ reads

$$
\begin{equation*}
S_{p p}=-m \int \mathrm{~d} \tau=-m \int \mathrm{~d} t e^{\phi / \Lambda} \sqrt{1-e^{-c_{d} \phi / \Lambda}\left(v^{2}+\frac{\sigma_{i j}}{\Lambda} v^{i} v^{j}\right)} \tag{2.5}
\end{equation*}
$$

and its Taylor expansion provides the various particle-gravity vertices of the EFT.
Also the pure gravity sector $S_{\text {bulk }}=S_{E H}+S_{G F}$ can be explicitly written in terms of the KK variables; we report here only those terms which are needed for the present calculation ${ }^{2}$ :

$$
\begin{align*}
& S_{\text {bulk }} \supset \int \mathrm{d}^{d+1} x \sqrt{\gamma}\left\{\frac{1}{4}\left[(\vec{\nabla} \sigma)^{2}-2\left(\vec{\nabla} \sigma_{i j}\right)^{2}\right]-c_{d}(\vec{\nabla} \phi)^{2}\right. \\
&-\frac{1}{\Lambda}\left(\frac{\sigma}{2} \delta^{i j}-\sigma^{i j}\right)\left(\sigma_{i k},{ }^{l} \sigma_{j l},{ }^{, k}-\sigma_{i k}, k\right.  \tag{2.6}\\
&\left.\left.\sigma_{j l} l^{l}+\sigma_{, i} \sigma_{j k}, k-\sigma_{i k, j} \sigma^{, k}\right)\right\} .
\end{align*}
$$



Figure 1. The diagrams contributing at order $G_{N}^{5}$. As in the EFT approach the massive objects are non-dynamical, the horizontal black lines have to be seen as classical sources, and not as propagators. Green solid lines stand for $\sigma$ field propagators, blue dashed lines for $\phi$ fields.

[^1]The 2-body effective action can be found by integrating out the gravity fields from the above-derived actions

$$
\begin{equation*}
\exp \left[\mathrm{i} S_{e f f}\right]=\int D \phi D \sigma_{i j} \exp \left[\mathrm{i}\left(S_{b u l k}+S_{p p}\right)\right] \tag{2.7}
\end{equation*}
$$

As usual in field theory, the functional integration can be perturbatively expanded in terms of Feynman diagrams involving the gravitational degrees of freedom as internal lines only ${ }^{3}$, regarded as dynamical fields emitted and absorbed by the point particles which are taken as non-dynamical sources.

In order to make manifest the $v$ scaling necessary to classify the results according to the PN hierarchy, it is convenient to work with the space-Fourier transformed fields

$$
\begin{equation*}
W_{p}^{a}(t) \equiv \int \mathrm{d}^{d} x W^{a}(t, x) e^{-\mathrm{i} p \cdot x} \quad \text { with } W^{a}=\left\{\phi, \sigma_{i j}\right\} . \tag{2.8}
\end{equation*}
$$

The fields defined above are the fundamental variables in terms of which we are going to construct the Feynman graphs; the action governing their dynamics can be found from eqs. $(2.5,2.6)$.

The next step is to lay down all the diagrams which contribute at this $\mathcal{O}\left(G_{N}^{5}\right)$ in the static limit, following the rule that each vertex involving $n$ gravitational fields carries a factor $G_{N}^{n / 2-1}$ if it is a bulk one, and a factor $G_{N}^{n / 2}$ if it is attached to an external particle.

The diagrams in fig. 1 schematically represent the exchange of gravitational potential modes through the field $\phi$ (blue dotted lines) and $\sigma_{i j}$ (green solid line) which mediate the gravitational interaction. Massive objects represented by the thick horizontal black solid line are non-dynamical sources or sinks of gravitational modes. Their dynamics is described by the world line $S_{p p}$ hence no massive particle propagator is present in between two different insertions of gravitational modes on the same particle.

The amplitudes corresponding to each diagram can be built from the Feynman rules in momentum-space derived from $\mathcal{S}_{p p}, \mathcal{S}_{\text {bulk }}$. By looking in particular at the quadratic parts, one can explicitly write the propagators:

$$
\begin{equation*}
P\left[W_{p}^{a}\left(t_{a}\right) W_{p^{\prime}}^{b}\left(t_{b}\right)\right]=\frac{1}{2} P^{a a} \delta_{a b}(2 \pi)^{d} \delta^{d}\left(p+p^{\prime}\right) \mathcal{P}\left(p^{2}, t_{a}, t_{b}\right) \delta\left(t_{a}-t_{b}\right), \tag{2.9}
\end{equation*}
$$

where $P^{\phi \phi}=-\frac{1}{c_{d}}, P^{\sigma_{i j} \sigma_{k l}}=-\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}+\left(2-c_{d}\right) \delta_{i j} \delta_{k l}\right)$ and

$$
\begin{equation*}
\mathcal{P}\left(p^{2}, t_{a}, t_{b}\right)=\frac{\mathrm{i}}{p^{2}-\partial_{t_{a}} \partial_{t_{b}}} \simeq \frac{\mathrm{i}}{p^{2}} \tag{2.10}
\end{equation*}
$$

has been truncated to its instantaneous non-relativistic part. The terms involving time derivatives (which acting on the $e^{i p \cdot x}$, generate extra factors of $v$ ) can be indeed neglected. In fact, in the present work, we are interested in the pure 4PN $G_{N}^{5}$ contribution, which, by power counting, can be accessed in the limit of zero velocity and instantaneous interactions. In other words, gravitational mode momenta have scaling of the types $(v / r, 1 / r)$, therefore

[^2]the temporal component of their momenta can be neglected, since we are computing the $G_{N}^{5} v^{0}$ sector.

From the previous discussion, one can derive the following Feynman rules, respectively for the $\phi$-propagator,

$$
\begin{equation*}
--------\quad \rightarrow-\frac{i}{2 c_{d} p^{2}} \tag{2.11}
\end{equation*}
$$

and for the $\sigma$-propagator,

$$
\begin{equation*}
\frac{\mathrm{rj}}{\mathrm{P}} \quad \rightarrow \frac{\mathrm{i} P^{\sigma_{r j} \sigma_{k l}}}{2 p^{2}} \tag{2.12}
\end{equation*}
$$

The Feynman rules for the interaction vertices can be derived in a similar fashion and are reported below:

$$
\begin{align*}
\ddots & \rightarrow \mathrm{i} \frac{2 c_{d}}{\Lambda}\left[\frac{1}{2}(p-k) \cdot k \delta^{r j}-k^{r}(p-k)^{j}+(r \leftrightarrow j)\right], \\
& \rightarrow \mathrm{i} \frac{4 c_{d}}{\Lambda^{2}}\left[k^{r} p^{l} \delta^{j m}-\frac{1}{2} k^{l} p^{m} \delta^{r j}-\frac{1}{8} p \cdot k \mathcal{Q}^{r j l m}+(r \leftrightarrow j, l \leftrightarrow m)\right] \\
& \rightarrow \mathrm{i} \frac{1}{8 \Lambda}\left\{(p-k) \cdot k\left(\frac{1}{2} \delta^{t r} \mathcal{I}^{l m j q}-\frac{1}{4} \delta^{q r} \mathcal{I}^{t j l m}-\frac{1}{8} \delta^{t j} \mathcal{Q}^{q r l m}\right)+\right.  \tag{2.13}\\
& +\frac{1}{4}(p-k)^{t} k^{j} \mathcal{Q}^{q r l m}+\left[\left(\frac{1}{2} \delta^{t j} \delta^{m r}-\delta^{t r} \delta^{j m}\right)(p-k)^{q} k^{l}-(l \leftrightarrow q)\right]+ \\
& \left.+\delta^{l m} \delta^{t r}(p-k)^{q} k^{j}-\delta^{t m} \delta^{q r}(p-k)^{l} k^{j}+(t \leftrightarrow j, l \leftrightarrow m, q \leftrightarrow r)\right\}
\end{align*}
$$

$$
\therefore \because \because, \quad \rightarrow-\frac{\mathrm{i}}{n!\Lambda^{n}}
$$

with $\mathcal{I}^{i j l m} \equiv \delta^{i l} \delta^{j m}+\delta^{i m} \delta^{j l}$ and $\mathcal{Q}^{i j l m} \equiv \mathcal{I}^{i j l m}-\delta^{i j} \delta^{l m}$.
Finally, the contribution of each amplitude to the two body Lagrangian $\mathcal{L}$ can be derived from its Fourier transform,

$$
\begin{equation*}
\mathcal{L}_{a}=-\mathrm{i} \lim _{d \rightarrow 3} \int \frac{\mathrm{~d}^{d} p}{(2 \pi)^{d}} e^{\mathrm{i} p \cdot r} \square \square a \tag{2.14}
\end{equation*}
$$

where the box diagram stands for the generic diagram $a=1, \ldots, 50$ of fig. 1 , and $p$ is the momentum transfer of the source.

$T_{1}$


28


34


40


46

$T_{2}$


29


35


41


47

$T_{3}$


30


36


42


48

$T_{4}$


31


37


43


49


26


32


38


44


50


33


39


45

Figure 2. Four-loop 2-point topologies corresponding to the diagrams in fig.1.

## 3 Amplitudes and Feynman Integrals

In general, within the EFT approach, since the sources (black lines) are static and do not propagate, any gravity-amplitude of order $G_{N}^{\ell}$ can be mapped into an ( $\ell-1$ )-loop 2-point function with massless internal lines and external momentum $p$, where $p^{2} \equiv s \neq 0$,


Accordingly, the 50 diagrams in fig. 1 can be mapped onto the 29 topologies of fig. 2 , where the sets $T_{1}=\{1,2,3,4,5,6\}, T_{2}=\{7,8,10,11,14,16,17,20,21,25\}, T_{3}=\{9,12,13,22\}$, $T_{4}=\{15,18,19,23,24\}$, collect the diagrams that share the same topology. For instance, the diagrams 1 to 6 of fig. 1 correspond to integrals which have the same five denominators of the graph indicated by $T_{1}$ in fig. 2 , but different numerators, due to the different terms associated to $1,2,3$ or $4 \phi$ emission or absorption from the massive particle.

The representation of the gravity-amplitudes as 4-loop 2-point integrals yields the possibility of evaluating the latter by means of by-now standard multi-loop techniques based on integration-by-parts identities (IBPs) [27, 28].

Accordingly, we collect the 50 amplitudes of fig. 1 in two sets, $\mathcal{A}_{I}=\{1: 28,31,32,35$ : $37,39,41,45: 47\}$ and $\mathcal{A}_{I I}=\{29,30,33,34,38,40,42,43,44,48,49,50\}$, and address their computation separately.


Figure 3. The master integrals which appear in the calculation of the amplitudes in the set $\mathcal{A}_{I I}$. The names of the diagrams follow refs. [36-38].

The set $\mathcal{A}_{I}$ contains diagrams with a simpler internal structure, and they have been computed by using the kite rule [27, 28]

where the dots stand for squared denominators, and by using the standard identity holding for 2-point 1-loop graphs,

$$
\begin{equation*}
\int \frac{\mathrm{d}^{d} k}{(2 \pi)^{d}} \frac{1}{k^{2 a}(p-k)^{2 b}}=-\overbrace{b}^{a}-\frac{\left(p^{2}\right)^{d / 2-a-b}}{(4 \pi)^{d / 2}} \frac{\Gamma(d / 2-a) \Gamma(d / 2-b) \Gamma(a+b-d / 2)}{\Gamma(a) \Gamma(b) \Gamma(d-a-b)}, \tag{3.3}
\end{equation*}
$$

where $a$ and $b$ are generic denominators' powers. Alternatively we also performed an IBPreduction using the program Reduze [34, 35], identifying 5 master integrals (MIs), namely $\mathcal{M}_{0,1}, \mathcal{M}_{1,1}, \mathcal{M}_{1,2}, \mathcal{M}_{1,3}, \mathcal{M}_{1,4}$ of fig. 3. Both strategies gave the same results.

The amplitudes $\mathcal{A}_{I I}$, instead, have a less trivial internal structure. By means of IBPs, they have been systematically reduced to linear combinations of 7 MIs , all shown in fig. 3 . In this case, the reduction to MIs has been performed in two ways, by an in-house implementation of Laporta's algorithm which is based on Form [39-41], as well as by means of Reduze.

The 4-loop MIs in fig. 3 can be considered as a complete set of independent integrals, such that any amplitude of the sets $\mathcal{A}_{I}$ and $\mathcal{A}_{I I}$ can be written as a linear combination of them. The results of the 4-loop MIs are well-known in $d=4+\varepsilon$ euclidean space-time dimensions since long [36, 37], while the values around $d=3+\varepsilon$ of $\mathcal{M}_{2,2}, \mathcal{M}_{3,6}$ became available more recently [38]. In particular, $\mathcal{M}_{0,1}, \mathcal{M}_{1,1}, \mathcal{M}_{1,2}, \mathcal{M}_{1,3}, \mathcal{M}_{1,4}$ can be computed in a straightforward way by means of eq. (3.3), and admit closed analytic expressions, exact in $d$, which can be expanded in Laurent series in $\varepsilon$ around $d=3$. The series expansions of $\mathcal{M}_{2,2}$ and $\mathcal{M}_{3,6}$ were first obtained numerically in ref. [38] by using the difference equations method, exploiting the fact that dimensionally regulated Feynman integrals obey dimensional recurrence relations [29, 42-45]. For instance, owing to IBPs, $\mathcal{M}_{3,6}$ is solution of the
following recursive formula,

with

$$
\begin{align*}
& a_{1}= \frac{5(d-3)(d-4)^{2}(5-d)(5 d-26)(5 d-24)(5 d-22)(5 d-18)}{3(d-6)^{2}(3 d-16)(3 d-14) s^{4}},  \tag{3.5}\\
& a_{2}= 80(d-3)^{3}(2 d-7)(5 d-26)(5 d-24)(5 d-22)(5 d-18)(5 d-16) \times \\
& \frac{(14-5 d)\left(63872-40162 d+8403 d^{2}-585 d^{3}\right)}{9(d-6)^{2}(d-4)^{2}(3 d-16)^{2}(3 d-14)^{2}(3 d-10) s^{6}},  \tag{3.6}\\
& a_{3}= 40(d-3)^{2}(8-3 d)(5 d-26)(5 d-24)(5 d-22)(5 d-18) \times \\
& \frac{(5 d-16)(5 d-14)(7 d-32)}{3(d-6)^{2}(d-4)^{2}(3 d-16)(3 d-14) s^{6}},  \tag{3.7}\\
& a_{4}=(d-3)^{2}(3 d-10)^{2}(3 d-8)^{2} \times \\
& \frac{2897664-2445164 d+772948 d^{2}-108475 d^{3}+5702 d^{4}}{3(d-6)^{2}(d-4)^{2}(3 d-16)(3 d-14) s^{6}},  \tag{3.8}\\
& a_{5}=20(d-3)(2 d-7)(2 d-5)(5 d-26)(5 d-24) \times
\end{aligned} \quad \begin{aligned}
& \quad(5 d-22)(5 d-18)(5 d-16)(5 d-14)(5 d-12) \times \\
& \\
& \quad \frac{\left(1972736-1666418 d+527297 d^{2}-74070 d^{3}+3897 d^{4}\right)}{9(d-6)^{2}(d-5)(d-4)^{3}(3 d-16)^{2}(3 d-14)^{2} s^{7}}, \tag{3.9}
\end{align*}
$$

which links $M_{3,6}$ in $d-2$ dimensions (on the l.h.s.) to $M_{3,6}$ in $d$ dimension, and to other MIs belonging to subtopologies, also defined in $d$ dimensions (on the r.h.s). The MIs belonging to subtopologies have to be considered as the non-homogeneous term of the dimensional recurrence relation: they are known terms in a bottom-up approach (where simpler integrals, with less denominators, are computed first) ${ }^{4}$.

The solving strategy of dimensional recurrence equations for Feynman integrals has been discussed in [45] and implemented in the code SummerTime [38], which provides numerical values for the coefficients of the Laurent series in the $\varepsilon \rightarrow 0$ limit, at very high-accuracy (hundreds of digits).

Let us observe that $\mathcal{M}_{2,2}$ is finite in three dimensions, and, within the amplitudes' evaluation, it always appears multiplied by positive powers of $\varepsilon$, therefore it drops out of the final result.

[^3]In Appendix A, we provide the list of the results for the MIs of fig.3.

Example. As an illustrative example, we apply our algorithm to diagram 49 of fig. 1. The corresponding amplitude reads

$$
\begin{equation*}
\mathcal{A}_{49}=\square=-2 \mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(\frac{(d-2)}{(d-1)} m_{1} m_{2}\right)^{3} \rightarrow\left[N_{49}\right] \tag{3.10}
\end{equation*}
$$

with

and

$$
\begin{align*}
& N_{49} \equiv\left(k_{1} \cdot k_{3} k_{12} \cdot k_{23}-k_{1} \cdot k_{12} k_{3} \cdot k_{23}-k_{1} \cdot k_{23} k_{3} \cdot k_{12}\right) \times \\
& \left(p_{2} \cdot k_{23} p_{4} \cdot k_{34}+p_{4} \cdot k_{23} p_{2} \cdot k_{34}-p_{2} \cdot p_{4} k_{23} \cdot k_{34}\right) \tag{3.12}
\end{align*}
$$

where we define $\int_{k} \equiv \int \frac{\mathrm{~d}^{d} k}{(2 \pi)^{d}}$ and $p_{a} \equiv p-k_{a}, k_{a b} \equiv k_{a}-k_{b}$. By means of IBPs, we express the 2-point amplitude in terms of MIs,

with

$$
\begin{align*}
& c_{1}=\frac{(d-3)^{2}(d-2)^{2} s^{2}}{(d-4)^{2}(5 d-14)(12-5 d)}, \quad c_{2}=\frac{(d-2)^{2}\left(432-512 d+203 d^{2}-27 d^{3}\right) s}{8(d-4)^{3}(5-2 d)(5 d-12)},  \tag{3.14}\\
& c_{3}=\frac{(d-2)^{2}\left(76-58 d+11 d^{2}\right) s}{4(d-4)^{2}(14-5 d)(5 d-12)}, \quad c_{4}=\frac{(d-2)^{2} s}{2(d-4)^{2}}  \tag{3.15}\\
& c_{5}=\frac{(d-2)^{2}\left(1096-1598 d+870 d^{2}-210 d^{3}+19 d^{4}\right)}{(d-4)^{4}(3-d)(3 d-8)} \tag{3.16}
\end{align*}
$$

This result can be expanded around $d=3+\varepsilon$, using the expressions of the MIs given in Appendix A,

$$
\begin{align*}
& \mathcal{A}_{49}=-\mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(m_{1} m_{2}\right)^{3} 2^{-4}(4 \pi)^{-(4+2 \varepsilon)} e^{2 \varepsilon \gamma_{E}} s^{(1+2 \varepsilon)} \times \\
& {\left[\frac{1}{\varepsilon}\left(\frac{\pi^{2}}{16}-\frac{2}{3}\right)+\frac{29}{18}-\frac{13}{144} \pi^{2}-\frac{\pi^{2}}{8} \log 2+\mathcal{O}(\varepsilon)\right] } \tag{3.17}
\end{align*}
$$

where $\gamma_{E}=0.57721 \ldots$ is the Euler-Mascheroni constant. Finally, by means of the Fourier transform formula

$$
\begin{equation*}
\int_{p} \mathrm{e}^{\mathrm{i} p \cdot r} p^{-2 a}=\frac{\Gamma(d / 2-a)}{(4 \pi)^{d / 2} \Gamma(a)}\left(\frac{r}{2}\right)^{(2 a-d)}, \tag{3.18}
\end{equation*}
$$

one obtains the following Lagrangian term,

$$
\begin{equation*}
\mathcal{L}_{49}=-\mathrm{i} \lim _{d \rightarrow 3} \int_{p} \mathrm{e}^{\mathrm{i} p \cdot r} \mathcal{A}_{49}=\left(32-3 \pi^{2}\right) \frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}} \tag{3.19}
\end{equation*}
$$

## 4 Results and discussion

The complete $4 \mathrm{PN}, \mathcal{O}\left(G_{N}^{5}\right)$ Lagrangian was already presented in [20],

$$
\begin{align*}
\mathcal{L}_{4 P N}^{G_{N}^{5}} & =\frac{3}{8} \frac{G_{N}^{5} m_{1}^{5} m_{2}}{r^{5}}+\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[\frac{1690841}{25200}+\frac{105}{32} \pi^{2}-\frac{242}{3} \log \frac{r}{r_{1}^{\prime}}-16 \log \frac{r}{r_{2}^{\prime}}\right] \\
& +\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[\frac{587963}{5600}-\frac{71}{32} \pi^{2}-\frac{110}{3} \log \frac{r}{r_{1}^{\prime}}\right]+\left(m_{1} \leftrightarrow m_{2}\right), \tag{4.1}
\end{align*}
$$

where $r_{1}^{\prime}, r_{2}^{\prime}$ are two UV scales which do not contribute to physical observables. Such a Lagrangian gets contributions from the 50 genuine $\mathcal{O}\left(G_{N}^{5}\right)$ diagrams depicted in fig.1, and from diagrams at lower orders in $G_{N}$ which are at least quadratic in the accelerations:

$$
\begin{equation*}
\mathcal{L}_{4 P N}^{G_{N}^{5}}=\sum_{a=1}^{50} \mathcal{L}_{a}+\sum_{j=1}^{3} \mathcal{L}_{4 P N}^{G_{N}^{j} \rightarrow G_{N}^{5}}+\left(m_{1} \leftrightarrow m_{2}\right) \tag{4.2}
\end{equation*}
$$

The evaluation of $\sum_{a=1}^{50} \mathcal{L}_{a}$ represents the main result of this work, and it amounts to

$$
\begin{equation*}
\sum_{a=1}^{50} \mathcal{L}_{a}=\frac{3}{8} \frac{G_{N}^{5} m_{1}^{5} m_{2}}{r^{5}}+\frac{31}{3} \frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}+\frac{141}{8} \frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}} \tag{4.3}
\end{equation*}
$$

The individual contributions $\mathcal{L}_{a}$ are presented in Appendix B. We observe that, although there appear contributions which are divergent in the $d \rightarrow 3$ limit, the sum of all contributions is finite, hence $L$ does not show up in physical observables.

To obtain the whole expression for the $4 \mathrm{PN} \mathcal{O}\left(G_{N}^{5}\right)$ corrections, one would need to add contributions generated from lower $G_{N}$ terms when using the equations of motion, in order to eliminate terms quadratic at least in the accelerations. All such contributions have been computed also in the EFT framework [17], except for $\mathcal{L}_{4 P N}^{G_{N}^{3} \rightarrow G_{N}^{5}}$. We can nevertheless perform partial checks between eq.(4.3) and eq.(4.1).

The $m_{1}^{5} m_{2}$-term. It can be proven that this term does not receive any contribution from lower $G_{N}$ terms ${ }^{5}$, and the corresponding coefficient for the two-body Lagrangian of eq.(4.3)

[^4]agrees with the Lagrangian term reported in eq.(4.1).
The $\pi^{2}$-term. The contributions coming from the lower $G_{N}$ orders come entirely from the still unpublished $\mathcal{L}_{4 P N}^{G_{N}^{3} \rightarrow G_{N}^{5}}$ : for dimensional reasons terms at least quadratic in the accelerations can appear only in $G_{N}^{m \leq n-1}$ sectors at $n$-th PN order, and all the terms up to $\mathcal{O}\left(G_{N}^{2}\right)$ do not contain $\pi^{2}$. Although the computational details will be given elsewhere, such contributions have been computed in the EFT framework and found to be
\[

$$
\begin{equation*}
\frac{105}{32} \pi^{2} \frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}-\frac{71}{32} \pi^{2} \frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}} \tag{4.4}
\end{equation*}
$$

\]

This result, alone, already accounts for the Lagrangian $\pi^{2}$-term of eq. (4.1), presented in [20] and previously computed also in [19]. Athough some of the $\mathcal{L}_{a}$ 's listed in Appendix B (namely, $a=33,49,50$ ) contain terms proportional to $\pi^{2}$, these terms cancel in the sum of all the diagrams (as shown in ref. [46]), thus providing agreement with the literature.
Other terms. The other terms are not directly comparable without full knowledge of the $\mathcal{L}_{4 P N}^{G_{N}^{3} \rightarrow G_{N}^{5}}$ contribution, and without taking into account the different regularization schemes used here and in [20].

## 5 Conclusion

We studied the conservative dynamics of the two-body motion at fourth post-Newtonian order (4PN), at fifth order in the Newton constant $G_{N}$, within the effective field theory (EFT) framework to General Relativity. We determined an essential contribution of the complete 4PN Lagrangian at $\mathcal{O}\left(G_{N}^{5}\right)$, coming from 50 Feynman diagrams. By exploiting the analogy between such diagrams in the EFT gravitational theory and 2-point 4-loop functions in massless gauge theory, we addressed their calculation by means of multi-loop diagrammatic techniques, based on integration-by-parts identities and difference equations. We performed the calculation within the dimensional regularization scheme, and the contribution to the Lagrangian of each graph was given as Laurent series in $d=3+\varepsilon$, being $d$ the number of dimensions. Although some individual amplitudes are divergent in the $\varepsilon \rightarrow 0$ limit and others contain the irrational factor $\pi^{2}$, the sum of the fifty terms is found to be finite at $d=3$ and rational, in agreement with previous calculations performed with other techniques.

## Notes

In a first version of this manuscript, $\mathcal{L}_{50}$ appeared to have a different value, yielding to a disagreement with the literature. Subsequently, the authors of ref. [46] pointed us to a missing overall factor of " -3 " in $\mathcal{L}_{50}$, which we have been able to find and correct: the value of $\mathcal{L}_{50}$ reported in this version is the amended one. Let us also notice, that the analytic result for the master integral $\mathcal{M}_{3,6}$ obtained in [46] agrees with the semi-analytic expression given in our current work.

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## A Master integrals

In this appendix, we provide the expressions of the master integrals. They are defined by

$$
\begin{array}{ll}
\mathcal{M}_{0,1}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{14}}, & \mathcal{M}_{1,1}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{9} D_{12}}, \\
\mathcal{M}_{1,2}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{10} D_{11}}, & \mathcal{M}_{1,3}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{8} D_{10}}, \\
\mathcal{M}_{1,4}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{7} D_{13}}, & \mathcal{M}_{2,2}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{10} D_{15} 1} \\
\mathcal{M}_{3,6}=\int_{k_{1 \ldots 4}} \frac{1}{D_{1 \ldots 4} D_{5} D_{6} D_{10} D_{14}}, &
\end{array}
$$

where $k_{i}(i=1,2,3,4)$ are the loop momenta and $p$ is the external momentum of the diagrams depicted in fig. 3. The integral measure is the same as used in sec. 3 and given by $\int_{k_{1} \ldots 4}=\int_{k_{1}} \int_{k_{2}} \int_{k_{3}} \int_{k_{4}}$ with $\int_{k_{i}} \equiv \int \frac{\mathrm{~d}^{d} k_{i}}{(2 \pi)^{d}}(i=1,2,3,4)$. The denominators read

$$
\begin{aligned}
& D_{1 \ldots 4}=k_{1}^{2} k_{2}^{2} k_{3}^{2} k_{4}^{2}, \quad D_{5}=\left(k_{2}-k_{3}\right)^{2}, \quad D_{6}=\left(k_{1}-k_{4}\right)^{2} \text {, } \\
& D_{7}=\left(k_{2}+k_{3}-k_{4}\right)^{2}, \quad D_{8}=\left(k_{1}+k_{2}+k_{3}-k_{4}\right)^{2}, \quad D_{9}=\left(k_{1}-p\right)^{2}, \\
& D_{10}=\left(k_{1}+k_{2}-p\right)^{2}, \quad D_{11}=\left(k_{3}+k_{4}+p\right)^{2}, \quad D_{12}=\left(k_{2}-k_{3}-k_{4}+p\right)^{2}, \\
& D_{13}=\left(k_{1}-k_{2}-k_{3}+p\right)^{2}, \quad D_{14}=\left(k_{1}+k_{2}-k_{3}-k_{4}-p\right)^{2}, \\
& D_{15}=\left(k_{1}+k_{4}-p\right)^{2}, \quad D_{16}=\left(k_{2}+k_{3}-p\right)^{2} .
\end{aligned}
$$

## A. 1 Master integrals known in $d$ dimensions

The following master integrals are known in closed analytical form, exact in $d$ :

$$
\begin{align*}
& \mathcal{M}_{0,1}=(4 \pi)^{-2 d} s^{2 d-5} \frac{\Gamma(5-2 d) \Gamma\left(\frac{d}{2}-1\right)^{5}}{\Gamma\left(\frac{5}{2} d-5\right)}  \tag{A.1}\\
& \stackrel{d=3+\varepsilon}{=} c(\varepsilon) s\left[\frac{1}{24 \varepsilon}-\frac{13}{36}+\varepsilon\left(\frac{481}{216}-\frac{11}{288} \pi^{2}\right)\right. \\
&\left.-\varepsilon^{2}\left(\frac{3943}{324}-\frac{143}{432} \pi^{2}-\frac{113}{72} \zeta_{3}\right)+\mathcal{O}\left(\varepsilon^{3}\right)\right],  \tag{A.2}\\
& \mathcal{M}_{1,1}=(4 \pi)^{-2 d} s^{2 d-6} \frac{\Gamma\left(4-\frac{3}{2} d\right) \Gamma\left(2-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}-1\right)^{6}}{\Gamma(d-2) \Gamma(2 d-4)} \tag{A.3}
\end{align*}
$$

$$
\begin{align*}
\begin{aligned}
& d=3+\varepsilon \\
&= \\
&-c(\varepsilon) \pi^{2}\left[\frac{1}{8}+\mathcal{O}\left(\varepsilon^{1}\right)\right], \\
& \mathcal{M}_{1,2}=(4 \pi)^{-2 d} s^{2 d-6} \frac{\Gamma(3-d)^{2} \Gamma\left(\frac{d}{2}-1\right)^{6}}{\Gamma\left(\frac{3}{2} d-3\right)^{2}} \\
& \stackrel{d=3+\varepsilon}{=} c(\varepsilon)\left[\frac{1}{4 \varepsilon^{2}}-\frac{3}{2 \varepsilon}+\left(\frac{27}{4}-\frac{7}{48} \pi^{2}\right)\right. \\
&\left.-\varepsilon\left(27-\frac{7}{8} \pi^{2}-\frac{11}{3} \zeta_{3}\right)+\mathcal{O}\left(\varepsilon^{2}\right)\right], \\
& \mathcal{M}_{1,3}=(4 \pi)^{-2 d} s^{2 d-6} \frac{\Gamma(6-2 d) \Gamma(3-d) \Gamma\left(2-\frac{d}{2}\right) \Gamma\left(\frac{d}{2}-1\right)^{6} \Gamma(2 d-5)}{\Gamma\left(5-\frac{3}{2} d\right) \Gamma(d-2) \Gamma\left(\frac{3}{2} d-3\right) \Gamma\left(\frac{5}{2} d-6\right)} \\
& d=3+\varepsilon c(\varepsilon)\left[\frac{1}{8 \varepsilon^{2}}-\frac{1}{\varepsilon}+\frac{49}{8}-\frac{19}{96} \pi^{2}\right. \\
&\left.-\varepsilon\left(34-\frac{19}{12} \pi^{2}-\frac{107}{24} \zeta_{3}\right)+\mathcal{O}\left(\varepsilon^{2}\right)\right], \\
& \mathcal{M}_{1,4}=(4 \pi)^{-2 d} s^{2 d-6} \frac{\Gamma(6-2 d) \Gamma\left(2-\frac{d}{2}\right)^{2} \Gamma\left(\frac{d}{2}-1\right)^{6} \Gamma\left(\frac{3}{2} d-4\right)}{\Gamma(4-d) \Gamma(d-2)^{2} \Gamma\left(\frac{5}{2} d-6\right)} \\
& d=3+\varepsilon-c(\varepsilon) \pi^{2}\left[\frac{1}{16 \varepsilon}-\left(\frac{5}{16}+\frac{1}{8} \log 2\right)+\mathcal{O}\left(\varepsilon^{1}\right)\right],
\end{aligned} \tag{A.4}
\end{align*}
$$

with the Euler $\Gamma$ function $\Gamma(z)=\int_{0}^{\infty} t^{z-1} e^{-t} \mathrm{~d} t$, the Riemann zeta function $\zeta_{n}=\sum_{k=1}^{\infty} \frac{1}{k^{n}}$, and $s=p^{2}$. The coefficient function $c(\varepsilon)$ is given by

$$
\begin{equation*}
c(\varepsilon)=e^{2 \varepsilon \gamma_{E}} s^{2 \varepsilon} /(4 \pi)^{4+2 \varepsilon} \tag{A.11}
\end{equation*}
$$

## A. 2 Master integrals known in $d=3+\varepsilon$ dimensions

The master integrals $\mathcal{M}_{2,2}$ and $\mathcal{M}_{3,6}$ are known numerically [38]. In three dimensions $\mathcal{M}_{2,2}$ is finite, i.e. $\mathcal{M}_{2,2}=\mathcal{O}\left(\varepsilon^{0}\right)$, and does not contribute to our amplitudes, since it always appears multiplied by a positive power of $\varepsilon$. The Laurent expansion in $\varepsilon$ around $d=3$ for $\mathcal{M}_{3,6}$ reads,

$$
\begin{align*}
& \mathcal{M}_{3,6}^{d=3+\varepsilon}= \frac{c(\varepsilon)}{s^{2}}[ \\
& 0.50000000000000000000000000000000000000000000000000000000000 / \varepsilon^{2} \\
&-0.50000000000000000000000000000000000000000000000000000000000 / \varepsilon \\
&-3.58876648328794339088189620833849370269526252469830039056611 \\
&+15.6234156117945512067218751269082577384023065736147735689317 \varepsilon \\
&\left.+\mathcal{O}\left(\varepsilon^{2}\right)\right]  \tag{A.12}\\
& \stackrel{\text { PSLQ }}{=} \frac{c(\varepsilon)}{s^{2}}\left[\frac{1}{2 \varepsilon^{2}}-\frac{1}{2 \varepsilon}-4+\frac{\pi^{2}}{24}-\varepsilon\left(9-\pi^{2}\left(\frac{13}{8}-\log 2\right)-\frac{77}{6} \zeta_{3}\right)+\mathcal{O}\left(\varepsilon^{2}\right)\right] . \tag{A.13}
\end{align*}
$$

The analytical coefficients in the $\varepsilon$ expansion have been obtained from the high precision numerical result with the PSLQ algorithm [47]. We observe that, according to the arguments in footnote 4 , the value of the coefficient of the double pole can be obtained analytically from
the recurrence relation: its numerical reconstruction agrees with the analytic determined value.

Moreover, in order to perform a consistency check of the other analytical coefficients of eq. (A.13), we determined $\mathcal{M}_{3,6}$ also in 1- and 5 -dimensions with SummerTime [38] numerically and used the PSLQ algorithm to obtain again the analytical coefficients of the $\varepsilon$ expansion, respectively reading,

$$
\begin{align*}
& \mathcal{M}_{3,6}^{d=1+\varepsilon}=(4 \pi)^{4} \frac{c(\varepsilon)}{s^{6}}[ \\
& 11.0000000000000000000000000000000000000000000000000000000000 / \varepsilon \\
&+750.157936507936507936507936507936507936507936507936507936508 \\
&-5333.19383013044510985261411265298578814107960018433010670281 \varepsilon \\
&-3509.80936167055655677303026105319710926833682220819489993426 \varepsilon^{2} \\
&\left.+\mathcal{O}\left(\varepsilon^{3}\right)\right]  \tag{A.14}\\
& \stackrel{\text { PSLQ }}{=}(4 \pi)^{4} \frac{c(\varepsilon)}{s^{6}}\left[\frac{11}{\varepsilon}+\frac{945199}{1260}-\varepsilon\left(\frac{35338924}{6615}-\frac{11}{12} \pi^{2}\right)+\varepsilon^{2}\left(\frac{160485605363}{27783000}\right.\right. \\
&\left.\left.-\frac{14515601}{15120} \pi^{2}-22 \pi^{2} \log 2+\frac{847}{3} \zeta_{3}\right)+\mathcal{O}\left(\varepsilon^{3}\right)\right],  \tag{A.15}\\
& \mathcal{M}_{3,6}^{d=5+\varepsilon}= \frac{1}{(4 \pi)^{4}} \frac{c(\varepsilon) s^{2}}{2520}[ \\
& 1.00000000000000000000000000000000000000000000000000000000000 / \varepsilon^{2} \\
&-7.49665930774956257270733971502880747383208927084097052723419 / \varepsilon \\
&+33.1813244635562837450781924787207309198665172698916969562612 \\
&+\mathcal{O}(\varepsilon)]  \tag{A.16}\\
& \text { PS. } 15 \\
&= \frac{1}{(4 \pi)^{4}} \frac{c(\varepsilon) s^{2}}{2520}\left[\frac{1}{\varepsilon^{2}}-\frac{1}{\varepsilon}\left(\frac{467}{7}-6 \pi^{2}\right)\right.  \tag{A.17}\\
&\left.+\frac{123478}{147}-\frac{1651}{21} \pi^{2}+54 \pi^{2} \log 2-333 \zeta_{3}+\mathcal{O}(\varepsilon)\right] .
\end{align*}
$$

We verified that the analytical ansätze for $\mathcal{M}_{3,6}^{d=1+\varepsilon}, \mathcal{M}_{3,6}^{d=3+\varepsilon}, \mathcal{M}_{3,6}^{d=5+\varepsilon}$ fulfill the dimensional recurrence relation (3.4) analytically, order-by-order in $\varepsilon$, therefore we have high confidence in their correctness.

## B Results for all the amplitudes

In this appendix we collect the contributions to the Lagrangian in eq. (4.2), coming from all the amplitudes of fig. 1:

$$
\begin{gathered}
0=\mathcal{L}_{9}=\mathcal{L}_{12}=\mathcal{L}_{13}=\mathcal{L}_{22}=\mathcal{L}_{26}=\mathcal{L}_{27}=\mathcal{L}_{31}=\mathcal{L}_{36}=\mathcal{L}_{46}=\mathcal{L}_{47}, \\
\frac{1}{2} \frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}=\mathcal{L}_{1}=\mathcal{L}_{3}=4 \mathcal{L}_{5}=3 \mathcal{L}_{14}=\frac{\mathcal{L}_{19}}{8}=\frac{3 \mathcal{L}_{20}}{2}=\frac{3 \mathcal{L}_{21}}{4}=\frac{\mathcal{L}_{23}}{4}=\frac{\mathcal{L}_{24}}{4}=\frac{3 \mathcal{L}_{25}}{2},
\end{gathered}
$$

$$
\begin{array}{clr}
\frac{1}{2} \frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}=\mathcal{L}_{2}=3 \mathcal{L}_{4}=\frac{3 \mathcal{L}_{8}}{2}=\frac{3 \mathcal{L}_{10}}{2}=\frac{3 \mathcal{L}_{11}}{2}=\frac{\mathcal{L}_{15}}{4}=\frac{3 \mathcal{L}_{16}}{4}=\frac{3 \mathcal{L}_{17}}{4}=\frac{\mathcal{L}_{18}}{4}, \\
\frac{1}{120} \frac{G_{N}^{5} m_{1}^{5} m_{2}}{r^{5}}=\mathcal{L}_{6}=\frac{\mathcal{L}_{7}}{20}=\frac{3 \mathcal{L}_{30}}{20}=-\frac{3 \mathcal{L}_{35}}{56}=\frac{\mathcal{L}_{39}}{24}=\frac{\mathcal{L}_{45}}{12}, \\
\mathcal{L}_{28}=\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[\frac{428}{75}+\frac{4}{15} \mathcal{P}\right], & \mathcal{L}_{29}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[-\frac{409}{450}+\frac{1}{5} \mathcal{P}\right], \\
\mathcal{L}_{32}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[-\frac{91}{450}+\frac{1}{15} \mathcal{P}\right], & \mathcal{L}_{33}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left(16-\pi^{2}\right), \\
\mathcal{L}_{34}=\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[\frac{13}{5}-\frac{2}{3} \mathcal{P}\right], & \mathcal{L}_{37}=-\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}[17+2 \mathcal{P}], \\
\mathcal{L}_{38}=\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[\frac{147}{25}+\frac{8}{15} \mathcal{P}\right], & \mathcal{L}_{40}=\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[-\frac{39}{25}+\frac{4}{15} \mathcal{P}\right], \\
\mathcal{L}_{41}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[\frac{49}{18}+\frac{1}{3} \mathcal{P}\right], & \mathcal{L}_{42}=-\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[\frac{97}{225}+\frac{1}{15} \mathcal{P}\right], \\
\mathcal{L}_{43}=-\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[\frac{53}{150}+\frac{2}{15} \mathcal{P}\right], & \mathcal{L}_{49}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left[37-3 \pi^{2}\right), \\
\mathcal{L}_{48}=\frac{G_{N}^{5} m_{1}^{4} m_{2}^{2}}{r^{5}}\left[\frac{578}{75}+\frac{8}{5} \mathcal{P}\right], & \\
\mathcal{L}_{50}=\frac{G_{N}^{5} m_{1}^{3} m_{2}^{3}}{r^{5}}\left(4 \pi^{2}-\frac{124}{3}\right), & \tag{B.1}
\end{array}
$$

where the pole part $\mathcal{P} \equiv \frac{1}{\varepsilon}-5 \log \frac{r}{L_{0}}$ (with $L_{0}$ defined by $L=\sqrt{4 \pi \mathrm{e}^{\gamma_{E}}} L_{0}$ ) cancels exactly in the sum of all the terms.

Diagrams which are symmetric under $(1 \leftrightarrow 2)$ exchange, i.e. $3,5,22,23,24,32,33$, $41,42,43,49,50$ have been multiplied by $1 / 2$.

## C Evaluation of $\mathcal{A}_{33}$ and $\mathcal{A}_{50}$

We describe the evaluation of amplitudes 33 and 50 which, along with amplitude 49 already discussed in detail in section 3, are the only ones containing $\pi^{2}$ terms.

## C. 1 Amplitude 33

$$
\begin{equation*}
\mathcal{A}_{33}=\square=-\mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(\frac{(d-2)}{(d-1)} m_{1} m_{2}\right)^{3} \backsim\left[N_{33}\right] \tag{C.1}
\end{equation*}
$$

with

$$
\begin{equation*}
\bigcirc\left[N_{33}\right] \equiv \int_{k_{1}, k_{2}, k_{3}, k_{4}} \frac{N_{33}}{k_{1}^{2} k_{2}^{2} k_{3}^{2} k_{4}^{2} k_{14}^{2} p_{12}^{2} p_{34}^{2} p_{123}^{2}}, \tag{C.2}
\end{equation*}
$$

and

$$
N_{33} \equiv k_{3} \cdot k_{4}\left(k_{2} \cdot p_{12} k_{1} \cdot p_{34}+k_{1} \cdot k_{2} p_{12} \cdot p_{34}-k_{1} \cdot p_{12} k_{2} \cdot p_{34}\right)
$$

$$
\begin{align*}
& +k_{2} \cdot k_{4}\left(k_{1} \cdot p_{12} k_{3} \cdot p_{34}+k_{1} \cdot k_{3} p_{12} \cdot p_{34}-k_{3} \cdot p_{12} k_{1} \cdot p_{34}\right) \\
& +k_{1} \cdot k_{4}\left(k_{3} \cdot p_{12} k_{2} \cdot p_{34}-k_{2} \cdot p_{12} k_{3} \cdot p_{34}-k_{2} \cdot k_{3} p_{12} \cdot p_{34}\right) \\
& +k_{2} \cdot k_{3}\left(k_{4} \cdot p_{12} k_{1} \cdot p_{34}+k_{1} \cdot p_{12} k_{4} \cdot p_{34}\right) \\
& +k_{1} \cdot k_{3}\left(k_{2} \cdot p_{12} k_{4} \cdot p_{34}-k_{4} \cdot p_{12} k_{2} \cdot p_{34}\right) \\
& +k_{1} \cdot k_{2}\left(k_{4} \cdot p_{12} k_{3} \cdot p_{34}-k_{3} \cdot p_{12} k_{4} \cdot p_{34}\right), \tag{C.3}
\end{align*}
$$

where $p_{123} \equiv p-k_{1}-k_{2}-k_{3}, p_{a b} \equiv p-k_{a}-k_{b}, k_{14} \equiv k_{1}-k_{4}$. By means of IBPs, we express the 2 -point amplitude in terms of MIs,

and

$$
\begin{align*}
& c_{1}=\frac{(d-2)(3 d-10)\left(d^{2}-12 d+24\right) s^{3}}{4(d-3)(5 d-16)(5 d-14)(5 d-12)},  \tag{C.5}\\
& c_{2}=\frac{(d-2)\left(19 d^{4}+225 d^{3}-2708 d^{2}+8140 d-7680\right) s}{4(d-4)^{2}(2 d-5)(3 d-10)(5 d-12)},  \tag{C.6}\\
& c_{3}=\frac{(d-2)\left(33 d^{5}-44 d^{4}-1936 d^{3}+11024 d^{2}-22512 d+16128\right) s}{4(d-4)^{2}(d-3)(5 d-16)(5 d-14)(5 d-12)},  \tag{C.7}\\
& c_{4}=-\frac{2(d-2)\left(d^{3}+7 d^{2}-55 d+78\right) s}{(d-4)^{2}(d-3)(5 d-12)},  \tag{C.8}\\
& c_{5}=\frac{(d-2)(2 d-5)\left(3 d^{4}+204 d^{3}-1856 d^{2}+5296 d-4944\right)}{2(d-4)^{2}(d-3)^{2}(3 d-10)(3 d-8)} . \tag{C.9}
\end{align*}
$$

This result can be expanded around $d=3+\varepsilon$, using the expressions of the MIs given in Appendix A,

$$
\begin{align*}
& \mathcal{A}_{33}=-\mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(m_{1} m_{2}\right)^{3} 2^{-4}(4 \pi)^{-(4+2 \varepsilon)} e^{2 \varepsilon \gamma_{E}} S^{(1+2 \varepsilon)} \times \\
& {\left[\frac{1}{\varepsilon}\left(\frac{\pi^{2}}{48}-\frac{1}{3}\right)+\frac{49}{18}-\frac{5 \pi^{2}}{16}+\frac{7 \pi^{2}}{8} \log 2-\frac{37 \zeta_{3}}{8}+\mathcal{O}(\varepsilon)\right] . } \tag{C.10}
\end{align*}
$$

Finally, by applying the Fourier transform formula (3.18) to $-\mathrm{i} \mathcal{A}_{33}$, one gets the result for $\mathcal{L}_{33}$ reported in appendix B.

## C. 2 Amplitude 50

Coming to amplitude 50, we have

$$
\begin{equation*}
\mathcal{A}_{50}=\square=-\mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(\frac{(d-2)}{(d-1)} m_{1} m_{2}\right)^{3} \backsim\left[N_{50}\right], \tag{C.11}
\end{equation*}
$$

with

$$
\begin{equation*}
\oiiint\left[N_{50}\right] \equiv \int_{k_{1}, k_{2}, k_{3}, k_{4}} \frac{N_{50}}{k_{1}^{2} k_{2}^{2} k_{3}^{2} k_{4}^{2} k_{12}^{2} k_{34}^{2} \hat{k}_{24}^{2} p_{13}^{2} \hat{p}_{14}^{2}}, \tag{C.12}
\end{equation*}
$$

and

$$
\begin{align*}
N_{50} \equiv & \left(k_{3} \cdot p_{13} k_{12} \cdot \hat{p}_{14}-k_{12} \cdot p_{13} k_{3} \cdot \hat{p}_{14}-k_{3} \cdot k_{12} p_{13} \cdot \hat{p}_{14}\right) \\
& \times\left(k_{2} \cdot k_{34} k_{1} \cdot k_{4}+k_{1} \cdot k_{34} k_{2} \cdot k_{4}-k_{4} \cdot k_{34} k_{1} \cdot k_{2}\right) \\
& +\left(k_{12} \cdot k_{34} p_{13} \cdot \hat{p}_{14}-k_{34} \cdot p_{13} k_{12} \cdot \hat{p}_{14}-k_{12} \cdot p_{13} k_{34} \cdot \hat{p}_{14}\right) \\
& \times\left(k_{1} \cdot k_{2} k_{3} \cdot k_{4}-k_{1} \cdot k_{3} k_{2} \cdot k_{4}-k_{1} \cdot k_{4} k_{2} \cdot k_{3}\right) \\
& +\left(k_{34} \cdot p_{13} k_{1} \cdot \hat{p}_{14}+k_{1} \cdot p_{13} k_{34} \cdot \hat{p}_{14}-k_{1} \cdot k_{34} p_{13} \cdot \hat{p}_{14}\right)\left(k_{4} \cdot k_{12} k_{2} \cdot k_{3}-k_{2} \cdot k_{12} k_{3} \cdot k_{4}\right) \\
& +\left(k_{1} \cdot k_{34} k_{3} \cdot k_{12}+k_{1} \cdot k_{3} k_{12} \cdot k_{34}-k_{1} \cdot k_{12} k_{3} \cdot k_{34}\right)\left(k_{2} \cdot \hat{p}_{14} k_{4} \cdot p_{13}+k_{2} \cdot p_{13} k_{4} \cdot \hat{p}_{14}\right) \\
& +\left(k_{2} \cdot k_{12} k_{4} \cdot k_{34}-k_{4} \cdot k_{12} k_{2} \cdot k_{34}\right)\left(k_{1} \cdot k_{3} p_{13} \cdot \hat{p}_{14}-k_{1} \cdot p_{13} k_{3} \cdot \hat{p}_{14}\right) \\
& -2 k_{1} \cdot k_{4} k_{3} \cdot k_{34} k_{2} \cdot p_{13} k_{12} \cdot \hat{p}_{14} \\
& -2 k_{1} \cdot p_{13} k_{3} \cdot k_{34}\left(k_{2} \cdot k_{4} k_{12} \cdot \hat{p}_{14}+k_{4} \cdot k_{12} k_{2} \cdot \hat{p}_{14}\right) \\
& +k_{1} \cdot \hat{p}_{14} k_{4} \cdot k_{12}\left(k_{2} \cdot k_{34} k_{3} \cdot p_{13}-2 k_{2} \cdot p_{13} k_{3} \cdot k_{34}\right) \\
& +k_{2} \cdot k_{4} k_{12} \cdot k_{34}\left(k_{3} \cdot p_{13} k_{1} \cdot \hat{p}_{14}+k_{1} \cdot p_{13} k_{3} \cdot \hat{p}_{14}\right) \\
& +2 k_{1} \cdot k_{4} k_{12} \cdot p_{13}\left(k_{2} \cdot k_{34} k_{3} \cdot \hat{p}_{14}-k_{3} \cdot k_{34} k_{2} \cdot \hat{p}_{14}\right) \\
& +2 k_{1} \cdot k_{12} k_{4} \cdot k_{34}\left(k_{3} \cdot p_{13} k_{2} \cdot \hat{p}_{14}+k_{2} \cdot p_{13} k_{3} \cdot \hat{p}_{14}\right) \\
& +2 k_{3} \cdot \hat{p}_{14} k_{12} \cdot p_{13}\left(k_{1} \cdot k_{34} k_{2} \cdot k_{4}-k_{4} \cdot k_{34} k_{1} \cdot k_{2}\right) \\
& +k_{1} \cdot \hat{p}_{14} k_{2} \cdot k_{4}\left(k_{3} \cdot k_{12} k_{34} \cdot p_{13}-2 k_{3} \cdot k_{34} k_{12} \cdot p_{13}\right) \\
& +2 k_{1} \cdot k_{12} k_{3} \cdot k_{4}\left(k_{34} \cdot p_{13} k_{2} \cdot \hat{p}_{14}+k_{2} \cdot p_{13} k_{34} \cdot \hat{p}_{14}\right) \\
& +k_{2} \cdot k_{4}\left(k_{34} \cdot \hat{p}_{14} k_{3} \cdot k_{12} k_{1} \cdot p_{13}+p_{13} \cdot \hat{p}_{14} k_{1} \cdot k_{12} k_{3} \cdot k_{34}\right) \\
& -k_{1} \cdot \hat{p}_{14} k_{2} \cdot k_{12} k_{4} \cdot k_{34} k_{3} \cdot p_{13}, \tag{C.13}
\end{align*}
$$

where $k_{a b} \equiv k_{a}-k_{b}, \hat{k}_{24} \equiv k_{2}+k_{4}, p_{13} \equiv p-k_{1}-k_{3}$ and $\hat{p}_{14} \equiv p-k_{1}+k_{2}-k_{3}+k_{4}$. By means of IBPs, we express the 2-point amplitude in terms of MIs,

and

$$
\begin{align*}
& c_{1}=-\frac{(d-2)(3 d-10)\left(3 d^{3}-41 d^{2}+165 d-204\right) s^{3}}{4(d-3)(2 d-7)(5 d-16)(5 d-14)(5 d-12)},  \tag{C.15}\\
& c_{2}=\frac{(d-2)\left(51 d^{4}-769 d^{3}+4018 d^{2}-8868 d+7080\right) s}{2(d-4)^{2}(2 d-5)(3 d-10)(5 d-12)},  \tag{C.16}\\
& c_{3}=\frac{(d-2)\left(164 d^{5}-3543 d^{4}+26298 d^{3}-90056 d^{2}+146592 d-92160\right) s}{12(d-4)^{2}(d-3)(5 d-16)(5 d-14)(5 d-12)}, \tag{C.17}
\end{align*}
$$

$$
\begin{align*}
& c_{4}=-\frac{(d-2)(9 d-23)\left(d^{2}-12 d+24\right) s}{2(d-4)^{2}(d-3)(5 d-12)},  \tag{C.18}\\
& c_{5}=-\frac{(d-2)\left(609 d^{5}-8946 d^{4}+52176 d^{3}-151096 d^{2}+217360 d-124320\right)}{2(d-4)^{3}(d-3)^{2}(3 d-10)(3 d-8)} . \tag{C.19}
\end{align*}
$$

This result can be expanded around $d=3+\varepsilon$, using the expressions of the MIs given in Appendix A,

$$
\begin{align*}
& \mathcal{A}_{50}=-\mathrm{i}\left(8 \pi G_{N}\right)^{5}\left(m_{1} m_{2}\right)^{3} 2^{-4}(4 \pi)^{-(4+2 \varepsilon)} e^{2 \varepsilon \gamma_{E}} s^{(1+2 \varepsilon)} \times \\
& {\left[\frac{1}{\varepsilon}\left(\frac{31}{36}-\frac{\pi^{2}}{12}\right)-\frac{985}{216}+\frac{61 \pi^{2}}{144}-\frac{3 \pi^{2}}{4} \log 2+\frac{37 \zeta_{3}}{8}+\mathcal{O}(\varepsilon)\right] . } \tag{C.20}
\end{align*}
$$

Finally, by applying the Fourier transform formula (3.18) to $-\mathrm{i} \mathcal{A}_{50}$, one gets the result for $\mathcal{L}_{50}$ reported in appendix B.

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[^0]:    ${ }^{1}$ We adopt the "mostly plus" convention $\eta_{\mu \nu} \equiv \operatorname{diag}(-,+,+,+)$, and the Riemann and Ricci tensors are defined as $R_{\nu \rho \sigma}^{\mu}=\partial_{\rho} \Gamma_{\nu \sigma}^{\mu}+\Gamma_{\alpha \rho}^{\mu} \Gamma_{\nu \sigma}^{\alpha}-\rho \leftrightarrow \sigma, R_{\mu \nu} \equiv R_{\mu \alpha \nu}^{\alpha}$.

[^1]:    ${ }^{2}$ It is understood that spatial indices in this expression, including those implicit in terms carrying a $(\vec{\nabla})^{2}$, are contracted by means of the spatial metric $\gamma_{i j}$, which implies the appearance of extra $\sigma$ fields, e.g. $(\vec{\nabla} \sigma)^{2} \equiv \gamma^{a b} \gamma^{c d} \gamma^{i j} \sigma_{a b, i} \sigma_{c d, j}$ and $\gamma^{i j}=\left(\gamma^{-1}\right)_{i j}$ (and on the second line $\sigma^{i j}=\sigma_{i j}, \sigma=\delta^{i j} \sigma_{i j}$ ).

[^2]:    ${ }^{3}$ As we focus on the conservative part of the dynamics, we are not interested in diagrams where gravitational radiation is released to infinity, even though tail effects [33] involving emitted and absorbed radiation are relevant at $G_{N}^{2}$ order also in the conservative sector.

[^3]:    ${ }^{4}$ The dimensional recurrence (3.4) implies that $\mathcal{M}_{3,6}(d=3+\varepsilon) \equiv \sum_{k=-2}^{\infty} \mathcal{M}_{3,6}(3, k) \varepsilon^{k}$ can be obtained from the knowledge of the MIs on the r.h.s., $\mathcal{M}_{i, j}(d=5+\varepsilon) \equiv \sum_{k=-2}^{\infty} \mathcal{M}_{i, j}(5, k) \varepsilon^{k}$. It is interesting to notice that in eq. (3.4) the coefficient $a_{1}$ is proportional to $(d-5)$. Therefore, by expanding both sides of the equation in a Laurent series, the Laurent coefficient $\mathcal{M}_{3,6}(3, k)$ gets a contribution from $\mathcal{M}_{3,6}(5, k-1)$ and from the Laurent coefficients of the other MIs at $d=5$. In particular, the coefficient of the double pole $\mathcal{M}_{3,6}(3,-2)$ is completely determined by the series expansions of the MIs of the subtopologies only, because when $k=-2, \mathcal{M}_{3,6}(d=5+\varepsilon)$ does not give any contribution.

[^4]:    ${ }^{5}$ Contributions to this term from lower $G_{N}$ orders would come from terms of the type $G_{N}^{5-n} m_{1}^{5-n} m_{2} a_{2}^{n}$ with $2 \leq n \leq 4$. However, diagrams giving rise to such terms would have exactly one propagator attached to particle 2 , hence $a_{2}^{2}$ or higher power of $a_{2}$ can be taken out by integration by parts instead of by using the doube zero trick. It can be checked explicitly in [17] that $G_{N}^{5-n} m_{1}^{5-n} m_{2} a_{2}^{n}$ terms do not appear in the Lagrangian for $n=3,4$.

