Published for SISSA by Springer

Received: *July 18, 2017*
Accepted: *October 12, 2017*
Published: *October 18, 2017*

Heterotic sigma models on T^8 and the Borcherds automorphic form Φ_{12}

Sarah M. Harrison,[a] Shamit Kachru,[b] Natalie M. Paquette,[b] Roberto Volpato[b,c,d] and Max Zimet[b]

[a] *Department of Physics, Harvard University, 17 Oxford St., Cambridge, MA 02138, U.S.A.*
[b] *Department of Physics, Stanford University, 382 Via Puebla Mall, Stanford, CA 94305, U.S.A.*
[c] *Department of Physics and Astronomy, University of Padua, Via Marzolo 8, 35131 Padova, Italy*
[d] *Department of Particle Physics and Astrophysics, SLAC, 2575 Sand Hill Road, Menlo Park, CA 94309, U.S.A.*

E-mail: sarharr@gmail.com, skachru@stanford.edu, npaquette09@gmail.com, roberto.volpato@pd.infn.it, maxzimet@stanford.edu

Abstract: We consider the spectrum of BPS states of the heterotic sigma model with $(0,8)$ supersymmetry and T^8 target, as well as its second-quantized counterpart. We show that the counting function for such states is intimately related to Borcherds' automorphic form Φ_{12}, a modular form which exhibits automorphy for $O(2,26;\mathbb{Z})$. We comment on possible implications for Umbral moonshine and theories of AdS_3 gravity.

Keywords: Superstring Vacua, Superstrings and Heterotic Strings

ArXiv ePrint: 1610.00707

https://doi.org/10.1007/JHEP10(2017)121

Contents

1 Introduction

Studies of the entropy of supersymmetric black holes in string theory have led to the discovery of beautiful and unexpected relations between basic objects in the theory of automorphic forms, algebraic geometry, and indices of supersymmetric sigma models. A basic example is the formula of Dijkgraaf, Verlinde and Verlinde [1], capturing the degeneracy of 1/4-BPS dyons in the $\mathcal{N} = 4, d = 4$ string theory obtained from compactifying type II string theory on $K3 \times T^2$ or, equivalently, the heterotic string on T^6. This theory has duality group

$$\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SO}(6,22;\mathbb{Z}),$$

where the electric and magnetic charges are given by 28-dimensional vectors $Q, P \in \Gamma^{6,22}$, and the $\mathrm{SL}(2,\mathbb{Z})$ factor is the electric-magnetic S-duality group of the theory.

The degeneracy of 1/4-BPS dyons, $D(Q,P)$, with charges (Q,P) is then

$$(-1)^{Q\cdot P+1} D(Q,P) = \oint d\Omega \frac{e^{\pi i \mathrm{Tr}\,(\Omega\cdot\Lambda)}}{\Phi_{10}(\Omega)}$$

where

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$$

parametrizes the Siegel upper half-space of degree 2, and

$$\Lambda = \begin{pmatrix} Q\cdot Q & Q\cdot P \\ Q\cdot P & P\cdot P \end{pmatrix}$$

encapsulates the three T-duality invariants of the black hole charges. The beautiful function in the denominator of the integrand is the Igusa cusp form $\Phi_{10}(\Omega)$, a weight 10 Siegel

JHEP10(2017)121

automorphic form for the modular group $SP(2,\mathbb{Z})$. For large charges this formula for the degeneracies has asymptotic growth

$$D(Q,P) \sim e^{\pi\sqrt{Q^2P^2-(Q\cdot P)^2}},$$

reproducing the expected result for the Bekenstein-Hawking entropy of these black holes,

$$S = \pi\sqrt{Q^2P^2 - (Q\cdot P)^2}.$$

The automorphic function Φ_{10} also has a connection to elliptic genera of symmetric powers of the K3 surface, derived in [2] by considering the D1-D5 system on $K3\times S^1$, which engineers 5D black holes. The end result is an elegant formula for the generating function,

$$\frac{\phi_{10,1}(q,y)}{\Phi_{10}(p,q,y)} = \sum_{n=0}^{\infty} p^{n-1} Z_{EG}(q,y;K3^{[n]}) \ .$$

(Here, $\phi_{10,1}$ is a named weak Jacobi form of weight 10 and index 1; explicit expressions can be found in [3]). Note the correction factor $\phi_{10,1}$ spoils the automorphy of this function. In [4], a connection between 4D and 5D black holes [5] was used to explain the appearance of $1/\phi_{10,1}$ upon compactification and re-derive the result of [1].

The purpose of this note is to report analogous formulae governing BPS state counts in the heterotic sigma model with T^8 target. This is the model which would naturally arise on the worldsheet of heterotic string compactifications preserving half-maximal supersymmetry in two space-time dimensions. We will focus in this note on the physics of the 2d field theory and its supersymmetry-preserving excitations, and mostly limit any discussion of possible space-time interpretations to the concluding section. This work was largely motivated by trying to develop an understanding of BPS counts at Niemeier points in the moduli space of compactifications to 3 and 2 dimensions, and their possible interpretation in light of Mathieu and Umbral moonshine [6–8], in the picture advocated in [9].

2 The Borcherds modular form Φ_{12}

The hero of our story will be the Borcherds modular form Φ_{12} [10]. A nice description of the relevant aspects of this form can be found in the work of Gritsenko [11], from which we borrow heavily.

Let $\Pi_{2,26}$ denote the (unique) even unimodular lattice of signature $(2,26)$. The Borcherds modular form Φ_{12} is of weight 12 with respect to $O^+(\Pi_{2,26})$. It is the unique cusp form with this property.

Of great interest for us will be the following explicit multiplicative lift formulae for Φ_{12}. Consider the split

$$\Gamma^{2,26} = \Gamma^{1,1} \oplus \Gamma^{1,1} \oplus N(-1)$$

with $N \equiv N(R)$ chosen from among the 24 even unimodular positive-definite lattices in dimension 24, i.e., the 23 Niemeier lattices and the Leech lattice.[1] These lattices are

[1]We use the notation $N(-1)$ to denote taking the signature of N to be $(0,24)$ instead of the usual $(24,0)$.

uniquely classified by their root systems, R, which are unions of simply-lace ADE root systems of the same Coxeter number, and the Leech lattice is the unique 24-dimensional even positive-definite unimodular lattice with no roots. We will denote by $h(R)$ the Coxeter number of the root system R associated with N, where we set $h(R) = 1$ when N is the Leech lattice.

To each possible choice of N, we can associate the refined lattice theta series

$$\theta_N(\tau,\xi) = \sum_{\lambda\in N} e^{\pi i\tau(\lambda,\lambda)+2\pi i(\xi,\lambda)} .$$

Here, (v, w) denotes the lattice inner product of $v, w \in N$, and $\xi \in N \otimes \mathbb{C}$ can be thought of as a complex 24-vector of 'flavor' chemical potentials refining the lattice theta function.

Then the automorphic form $\Phi_{12}(\tau, \xi, \sigma)$, defined on

$$\{(\tau,\xi,\sigma) \in \mathbb{C} \times (N\otimes\mathbb{C}) \times \mathbb{C} \mid 2\Im(\tau)\Im(\sigma) - (\Im(\xi),\Im(\xi)) > 0,\ \Im(\tau) > 0\}\,,$$

can be constructed as a multiplicative lift of a conventional Jacobi form. Define the Fourier coefficients $f(n, \lambda)$ via

$$\frac{\theta_N(\tau,\xi)}{\eta^{24}(\tau)} = \sum_{n\in\mathbb{Z},\lambda\in N} f(n,\lambda)\; q^n\; e^{2\pi i(\xi,\lambda)} .$$

This function is a weakly holomorphic Jacobi form of weight zero and index one for the lattice N. Then, Φ_{12} is given by the formula

$$\Phi_{12}(\tau,\xi,\sigma) = q^A r^{\vec{B}} p^C \prod_{\substack{n,m\in\mathbb{Z}\\ \lambda\in N\\ (n,\lambda,m)>0}} (1-q^n r^\lambda p^m)^{f(mn,\lambda)}$$

$$p \equiv e^{2\pi i\sigma}, \qquad r^\lambda = e^{2\pi i(\xi,\lambda)}.$$

In the above, we have defined $A \equiv \frac{1}{24}\sum_{\lambda\in N} f(0,\lambda), \vec{B} \equiv \frac{1}{2}\sum_{\lambda>0} f(0,\lambda)\lambda \in \frac{1}{2}N, C \equiv \frac{1}{48}\sum_{\lambda\in N} f(0,\lambda)(\lambda,\lambda)$,[2] and used the notation $(n, \lambda, m) > 0$ to mean $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$ and $\lambda < 0$. Furthermore, $\lambda > 0$ (or < 0) means that $\lambda \in N$ has positive (respectively, negative) scalar product with a reference vector $x \in N \otimes \mathbb{R}$. The vector x must be chosen so that $(x, \lambda) \neq 0$ for all $\lambda \in N$, and different choices of x are related to one each other by automorphisms in $O(\Gamma^{2,26})$.

More precisely, the Niemeier and Leech points define cusps in the domain of definition of Φ_{12}. These formulae should be thought of as expansions of the modular form around the cusps.

To make contact with the earlier work [9], it is useful to specialize the chemical potentials as follows. Choose a fixed lattice vector $\delta \in N$. Then we can define

$$\Theta_{N,\delta}(\tau,z) = \sum_{\lambda\in N} q^{\frac{(\lambda,\lambda)}{2}} y^{(\delta,\lambda)}$$

[2]In the interpretation of Φ_{12} as a denominator for the fake Monster Lie algebra, one views $(A, \vec{B}, C)$ as a Weyl vector; see [10, 11] for details and section 4 for further comments on potential applications of the algebraic structure to physics.

with $y = e^{2\pi i z}$. One now obtains a Jacobi form of weight 0 and index $(\delta,\delta)/2$:

$$F^{N,\delta}(\tau,z) = \frac{\Theta_{N,\delta}(\tau,z)}{\eta^{24}(\tau)} = \sum_{n,l\in\mathbb{Z}} f_\delta(n,l) q^n y^l$$

where

$$f_\delta(n,l) = \sum_{\substack{\lambda\in N,\\ (\lambda,\delta)=l}} f(n,\lambda) \ .$$

These Jacobi forms, for suitable choices of δ, are the BPS counting functions discussed in [9]. That is, they control the coefficients in the expansion of a certain "F^4" term in the low-energy effective action of heterotic string compactification to three dimensions, when the moduli are deformed a slight distance away from a point with Niemeier symmetry (the enhanced symmetry point itself having singular couplings).

A specialized form of Φ_{12} can be obtained as a lift of these BPS counting functions as well:

$$\Phi_{12}^{N,\delta}(\tau,z,\sigma) \ = \ q^A y^B p^C \prod_{\substack{n,m,l\in\mathbb{Z}\\ (n,l,m)>0}} (1-q^n y^l p^m)^{f_\delta(mn,l)}$$

with the prefactors A and C as above and $B \equiv \frac{1}{2}\sum_{\lambda>0} f(0,\lambda)(\lambda,\delta)$. This object is an automorphic form on the Siegel upper half-space

$$\{(\tau,z,\sigma)\in\mathbb{C}^3 \mid \Im(\tau)\Im(\sigma)-\Im(z)^2>0,\ \Im(\tau)>0\}$$

for the group $SO^+(L_\delta)$, where L_δ is the lattice of signature (2,3) and with quadratic form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -(\delta,\delta) & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 T^8 sigma models

3.1 Basic connection

The Narain moduli space [12] of compactifications of heterotic strings on T^8 is the double coset

$$\mathcal{M} = O(8,24;\mathbb{Z})\backslash O(8,24)/O(8)\times O(24) \ .$$

This structure also arises in the non-perturbative description of heterotic strings on T^7 [13].

One can think of $\mathcal{M}$ as parametrizing even unimodular lattices of signature $(8,24)$, $\Gamma_{8,24}(\mathcal{M})$. The worldsheet field theory at a given point in moduli space consists of 2d bosons propagating on the relevant lattice.

Let us consider this theory on a toroidal worldsheet. At any point in moduli space, there are 24 abelian currents of conformal dimension $(1,0)$. One can consider coupling these to background chemical potentials ("Wilson lines"). In this setup, the parameter τ of section 2 can be considered as the modular parameter of the torus, while the 24 complex chemical potentials ξ should be thought of as these Wilson line degrees of freedom. The parameter p will emerge upon second-quantization.

Now, let us make a precise connection to the discussion of section 2. Consider a point in $\mathcal{M}$ where

$$\Gamma_{8,24} = \Gamma_{8,0} \oplus N(-1)$$

where N is again one of the 24 even unimodular positive-definite lattices, and $\Gamma_{8,0}$ the E_8 lattice. We will call this CFT V_N.

One obtains BPS states in the heterotic sigma model by putting right-movers in their ground state and performing a trace counting left-moving excitations. We restrict the counting to BPS states carrying no right-moving momentum, i.e. carrying zero charge with respect to the right-moving E_8 lattice:

$$\mathrm{Tr}_{V_N}\left((-1)^F q^{L_0}\bar{q}^{\bar{L}_0} y^{\xi^a \cdot J_a}\right)$$

(with J_a the 24 left-moving abelian currents). A subtlety arises because of the extra right-moving zero modes of this string background; without inserting extra factors of the right-moving fermion number operator F_R in the trace, the computation will formally vanish due to the existence of fermion zero modes. One can circumvent this vanishing by inserting powers of F in the trace to soak up zero modes, as in [14]; however, this will alter the modularity properties and spoil the connection we intend to make to Φ_{12}. Instead, we consider, following [15], a 'twisted' computation where we trace over left-moving excitations but consider the ground states of the $\mathbb{Z}_2$ orbifold of the right-moving sector — that is, we consider the right-movers to live in the Ramond sector of the Conway module constructed by Duncan [16].[3] The orbifold action kills the extra zero modes, but leaves the left-moving sector untouched. The ground-state degeneracy of the right-movers gives an overall factor of 24, as in [15]. Including this degeneracy, the result (for a given N) is

$$\text{BPS counting function}(\tilde{V}_N) \equiv \mathrm{Tr}_{\tilde{V}_N}\left((-1)^F q^{L_0}\bar{q}^{\bar{L}_0} y^{\xi^a \cdot J_a}\right) = 24\frac{\Theta_N(\tau,\xi)}{\eta^{24}(\tau)}$$

(where $\tilde{V}_N$ is the twisted Hilbert space described above). Specializing chemical potentials as before by choosing a $\delta \in N$, we see that the BPS counting function is $F^{N,\delta}(\tau, z)$.

We note here that the need to consider the twisted index computation to find a non-zero answer plays well with the structure of Φ_{12}. While the heterotic compactification without the twist has a moduli space obtained by choosing left and right moving sub-lattices of $\Gamma^{8,24}$, after twisting the E_8 lattice is rigidified. Therefore, compactifying on an additional T^2 to compute the index, one expects a moduli space based on $\Gamma^{2,26}$. This makes it natural to expect a connection to an automorphic form for the Narain modular group $O(2, 26; \mathbb{Z})$, such as Φ_{12}.

[3]This is related by modular invariance to computations with a $\mathbb{Z}_2$ insertion in the trace, which give character valued indices of the original model — hence, we view it as a trick to extract certain BPS degeneracies of the original theory.

JHEP10(2017)121

3.2 Second quantization

Now, let us consider the second-quantized version of this counting function. To do this, consider the heterotic sigma model based on the conformal field theory $\mathrm{Sym}^n(\tilde{V}_N)$. ($\tilde{V}_N$ appears because again, one can count BPS states without additional insertions to absorb fermion zero modes by quotienting by the natural $\mathbb{Z}_2$ action on the E_8 lattice, before taking the symmetric product and putting right movers into their ground state). We still restrict ourselves to BPS states carrying zero charge with respect to the right-moving U(1) currents. Following the logic of Dijkgraaf, Moore, Verlinde and Verlinde [2], we see that

$$\log\left(\sum_{n=0}^{\infty} p^n F^{\mathrm{Sym^n}(\tilde{V}_N)}(\tau,z)\right) \;=\; 24\,\log\left(\frac{\psi_{N,\delta}(\tau,z,\sigma)}{\Phi_{12}^{N,\delta}(\tau,z,\sigma)}\right)\,.$$

Here, $F^{\mathrm{Sym^n}(\tilde{V}_N)}$ is the specialized BPS counting function for the CFT based on the nth symmetric product of the (twisted) heterotic sigma model, and

$$\psi_{N,\delta} = \pm p^{h(R)}\eta^{24}(\tau)\prod_{\alpha\in R^+}\frac{\theta_1(\tau,z(\delta,\alpha))}{\eta(\tau)}$$

where R^+ is the set of positive roots of the lattice N, and the sign depends on the choice of the set of positive roots. The factor of 24 again arises from the right-moving ground state degeneracy, as in [15].

We see that there is a precise analogy to the findings in [1, 2]: just as an (automorphic correction of) $\frac{1}{\Phi_{10}}$ governs the BPS states of the sigma models on the Hilbert scheme of K3 surfaces, (an automorphic correction of) $\frac{1}{\Phi_{12}}$ governs the BPS states of the heterotic sigma models $\mathrm{Sym^n}(\tilde{V}_N)$. The role of the factor $\psi_{N,\delta}$ is superficially similar to that of the automorphic correction $\phi_{10,1}$ in the former story. As mentioned earlier, this factor achieves an interpretation in the 4d/5d lift [4, 5], and it would be interesting to give a precise similar interpretation to $\psi_{N,\delta}$.

4 Discussion

In this note, we've described how the (inverse of the) Borcherds modular form Φ_{12} serves as a generating function for the BPS state degeneracies of heterotic sigma models on T^8 (after suitably twisting to kill right-moving zero modes), and their symmetric products. We conclude with several comments and possible avenues for further development.

- In light of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, it is natural to conjecture that these heterotic conformal field theories, $\mathrm{Sym}^n(V_N)$, (at least at large central charge, i.e. large n) are dual to AdS_3 gravity theories with discrete symmetry groups corresponding to those of the associated Niemeier lattice — i.e. the Umbral groups and Conway's largest sporadic group. A criterion was developed in [17], using BPS degeneracies of 2d SCFTs to test for a possible large radius gravity dual. By modifying and checking this criterion for the case at hand, it has been found that these theories will have (at best) 'stringy' gravity duals — they will not achieve a (parametric) separation between the inverse AdS radius and M_{string} as the central charge $c\to\infty$ [18].

- The 2d heterotic compactifications governed by the CFTs V_N are the dimensional reductions of the Niemeier points studied in [9] in relation to moonshine. It is tempting to try and connect Borcherds' modular form Φ_{12} to Umbral moonshine. Making these thoughts precise is difficult because the notion of counting BPS states in gravity theories in $d \leq 3$ flat dimensions is fraught with subtlety; charged or gravitating particles have strong infra-red effects in low dimensions. Possibly, finding an interpretation of the present formulae in the setting of the supersymmetry-protected amplitudes studied in [19] is a route forward.

- While the models V_N depend on the choice of a Niemeier lattice N, the function counting the second quantized BPS states is the same in all cases. Indeed, these counting functions are just Fourier expansions at different cusps of the same automorphic function Φ_{12}. This is non-trivial: the points where the lattice $\Gamma^{8,24}$ splits into an E_8 lattice plus a Niemeier lattice are isolated points in the Narain moduli space of perturbative heterotic strings on T^8. The basic reason behind this phenomenon is that these isolated points are the different decompactification limits in the Narain moduli space of heterotic strings on T^9, with the condition that the Narain lattice $\Gamma^{9,25}$ splits as an orthogonal sum of E_8 and $\Gamma^{1,25}$. This suggests that there might be an interpretation of Φ_{12} in terms of compactification of the heterotic string to one dimension, possibly along the lines suggested in [15].

- It is known that $1/\Phi_{10}$ is related to the square of the denominator of a generalized Kac-Moody (GKM) algebra. There is a beautiful story relating this algebra to wall-crossing of 1/4-BPS dyons in $\mathcal{N} = 4, d = 4$ string theory [20]. As we mentioned earlier, $1/\Phi_{12}$ is also the denominator of a GKM algebra-the fake monster Lie algebra. It would be interesting to explore a similar story relating this algebra to the BPS states in the theories we discuss.

- The Gromov-Witten theory of $K3 \times T^2$ has recently been conjectured to be governed by the Igusa cusp form [21]. This follows from the role of Φ_{10} in black hole entropy counts, and string duality. Given that the heterotic string on T^8 is dual to type IIA on $K3 \times T^4$, it is tempting to think that the appearance of Φ_{12} in BPS counts in the present setting presages a similar role for Φ_{12} in (flavored) enumerative geometry, perhaps of $K3 \times T^4$ [22].

Acknowledgments

We are grateful to N. Benjamin and A. Tripathy for discussions. S.M.H. is supported by a Harvard University Golub Fellowship in the Physical Sciences and DOE grant DE-SC0007870. The research of S.K. was supported in part by the National Science Foundation under grant NSF-PHY-1316699. N.M.P. is supported by a National Science Foundation Graduate Research Fellowship under grant DGE-114747. R.V. is supported by a grant from 'Programma per giovani ricercatori Rita Levi Montalcini' 2013. M.Z. is supported by the Mellam Family Fellowship at the Stanford Institute for Theoretical Physics.

JHEP10(2017)121

References

[1] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [INSPIRE].

[2] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [hep-th/9608096] [INSPIRE].

[3] A. Dabholkar, S. Murthy and D. Zagier, *Quantum Black Holes, Wall Crossing and Mock Modular Forms*, arXiv:1208.4074 [INSPIRE].

[4] D. Shih, A. Strominger and X. Yin, *Recounting Dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [INSPIRE].

[5] D. Gaiotto, A. Strominger and X. Yin, *New connections between $4 - D$ and $5 - D$ black holes*, *JHEP* **02** (2006) 024 [hep-th/0503217] [INSPIRE].

[6] T. Eguchi, H. Ooguri and Y. Tachikawa, *Notes on the K3 Surface and the Mathieu group M_{24}*, *Exper. Math.* **20** (2011) 91 [arXiv:1004.0956].

[7] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral Moonshine*, *Commun. Num. Theor. Phys.* **08** (2014) 101 [arXiv:1204.2779] [INSPIRE].

[8] M.C.N. Cheng, J.F.R. Duncan and J.A. Harvey, *Umbral Moonshine and the Niemeier Lattices*, *Res. Math. Sci.* **1** (2014) 3 [arXiv:1307.5793] [INSPIRE].

[9] S. Kachru, N.M. Paquette and R. Volpato, *3D String Theory and Umbral Moonshine*, *J. Phys.* **A 50** (2017) 404003 [arXiv:1603.07330] [INSPIRE].

[10] R.E. Borcherds, *Automorphic forms on $O_{s+2,2}(R)$ and infinite products*, *Invent. Math.* **120** (1995) 161.

[11] V. Gritsenko, *24 faces of the Borcherds modular form Φ_{12}*, arXiv:1203.6503 [INSPIRE].

[12] K.S. Narain, M.H. Sarmadi and E. Witten, *A Note on Toroidal Compactification of Heterotic String Theory*, *Nucl. Phys.* **B 279** (1987) 369 [INSPIRE].

[13] A. Sen, *Strong - weak coupling duality in three-dimensional string theory*, *Nucl. Phys.* **B 434** (1995) 179 [hep-th/9408083] [INSPIRE].

[14] J.M. Maldacena, G.W. Moore and A. Strominger, *Counting BPS black holes in toroidal Type II string theory*, hep-th/9903163 [INSPIRE].

[15] N.M. Paquette, D. Persson and R. Volpato, *Monstrous BPS-Algebras and the Superstring Origin of Moonshine*, *Commun. Num. Theor. Phys.* **10** (2016) 433 [arXiv:1601.05412] [INSPIRE].

[16] J. Duncan, *Super-moonshine for Conway's largest sporadic group*, *Duke Math. J.* **139** (2007) 255.

[17] N. Benjamin, M.C.N. Cheng, S. Kachru, G.W. Moore and N.M. Paquette, *Elliptic Genera and 3d Gravity*, *Annales Henri Poincaré* **17** (2016) 2623 [arXiv:1503.04800] [INSPIRE].

[18] N. Benjamin, unpublished.

JHEP10(2017)121

[19] G. Bossard, C. Cosnier-Horeau and B. Pioline, *Protected couplings and BPS dyons in half-maximal supersymmetric string vacua*, Phys. Lett. **B 765** (2017) 377 [arXiv:1608.01660] [INSPIRE].

[20] M.C.N. Cheng and E.P. Verlinde, *Wall Crossing, Discrete Attractor Flow and Borcherds Algebra*, *SIGMA* **4** (2008) 068 [arXiv:0806.2337] [INSPIRE].

[21] G. Oberdieck and R. Pandharipande, *Curve counting on K3 $\times$ E, the Igusa cusp form χ_{10} and descendent integration*, arXiv:1411.1514 [INSPIRE].

[22] S. Kachru and A. Tripathy, in progress.

JHEP10(2017)121