Nonlinear Dyn
https://doi.org/10.1007/s11071-017-3958-3

ORIGINAL PAPER

A Normal Form analysis in a finite neighborhood of a Hopf bifurcation: on the Center Manifold dimension

M. Eugeni · D. Dessi · F. Mastroddi

Received: 24 February 2017 / Accepted: 18 November 2017

Abstract The problem of determining the bounds of applicability of perturbation expansions in terms both of the system parameters and the state-space variable amplitude is a key point in the perturbation analysis of nonlinear systems. In the present paper an analysis in a finite neighborhood of a Hopf bifurcation is presented in order to analyze the conditions under which a Normal Form zero-divisors-based approach fails to describe the local dynamics and, therefore, a small divisor approach is required. The condition of "smallness" referred to the divisors is analyzed from both a qualitative and a quantitative point of view. Finally, a simple but effective analytical and numerical example is introduced to illustrate the theoretical issues along with an interpretation within a codimension-two framework.

Keywords Normal Form · Small divisors · Hopf bifurcation · Nonlinear systems

1 Introduction

In the present paper, a finite neighborhood analysis of a Hopf bifurcation experienced by an autonomous system is presented. The scope beyond such analysis is to evaluate the conditions under which the Center Manifold approach, based on a zero-divisor Normal Form, is applicable.

The problem of finding upper bounds to the applicability of perturbation expansions in terms of both the system parameter or the state-space variable amplitude (Ref. [1–3]) is a key point in the perturbation analysis of nonlinear systems. Indeed, if the increasing loss of accuracy with respect to the perturbation parameter is part of these techniques, less acceptable is missing more basic features of the solution like the bifurcation scenario or the mechanism of the onset of chaotic behavior (Ref. [4–7]).

The local Hopf bifurcation analysis is based upon the hypothesis that the Center Manifold consists of two critical, complex conjugate modes (see, e.g., Ref. [8]). Indeed, it is always possible to find a sufficiently small neighborhood of the bifurcation point in the control parameter space (in the following indicated as μ), where the other modes are damped. In other words, there exists a neighborhood of the bifurcation point where the state space can be decomposed in the Center Manifold, where the persistent dynamics lies, and a Stable Manifold whose tangent space is determined by the stable modes where the orbits are diffeomorphic to an exponential decay (see the Hartman–Grobman linearization Theorem in Ref. [8]). This assumption on the Center Manifold dimension will be referred in the following to as the Hopf Bifurcation Center Manifold assumption (HBCM assumption, for short). How-

M. Eugeni (✉) · F. Mastroddi
Department of Mechanical and Aerospace Engineering, University of Rome La Sapienza, Via Eudossiana 18, 00184 Rome, Italy
e-mail: marco.eugeni@uniroma1.it

D. Dessi
CNR INSEAN, Via di Vallerano 139, 00128 Rome, Italy
e-mail: daniele.dessi@cnr.it

Published online: 02 December 2017

ever, there are cases when the HBCM assumption is clearly violated as long as the parameter μ is increased above a certain threshold value (see, for instance, Refs. [9,10]). This occurs when the amplitude of the modes associated with the negative real-part eigenvalues is no longer damped. In this case, the singular perturbation analysis has to include variations of the linear terms as well as nonlinear terms satisfying the so-called "resonance condition," implying that the dynamics of some non-critical modes has to be taken into account as well (Refs. [4,5]).

This analysis in the present paper is performed by utilizing the Normal Form method (see Refs. [8,11,12]) which is based on the idea that a nonlinear system can be simplified by a suitable transformation in the state space, so as to eliminate in the reduced equations the nonlinear terms that do not contribute to the essence of the solution. The generality of the followed approach makes it applicable to systems depending on a control parameter without any restriction on nature and dimension of the system itself. A reduced order for the dynamical system is obtained by identifying which state-space variables are damped, thereby removing them from the analysis (as permitted by the Hartman–Grobman Theorem). However, it is possible that a state-space variable that is damped when the system undergoes the Hopf bifurcation, becomes effective when the parameter μ is above a certain threshold. Thus, even if it is always possible to determine a neighborhood of the bifurcation point where the HBCM is true, no information is typically given about how far from the critical value μ_{cr} of the control parameter the HBCM is still fairly accurate. Therefore, in this paper an analytical expression for μ_{cr} is given along with a related interpretation of the overall system behavior as a codimension-two bifurcation.

In Sect. 2 a suitable state-space variable transformation is introduced by using the eigenvector matrix evaluated at the bifurcation point, and then the nonlinear autonomous system is recast in perturbation form, which allows focusing on the ordering of the involved terms. In Sect. 3 the Normal Form approach within a standard zero-divisor condition is formulated. Next, in Sect. 4 the Normal Form equations are specialized for the Hopf bifurcation case so discussing the stability characteristics of the limit cycles. In Sect. 5, the Normal Form analysis is extended to the second bifurcation relaxing the zero-divisor condition and introducing a small (not zero) divisor concept so as to identify the essential terms for the second bifurcation description. Finally, in Sect. 6 a simple but effective analytical and numerical example is introduced. Specifically a system of two coupled Van der Pol-like oscillators is studied to illustrate the theoretical issues along with an interpretation within a codimension-two framework.

2 Formulation of the problem, modal coordinates and perturbation ordering

In the following, it will be considered a system of the general type

$$\frac{\mathrm{d}\check{\mathbf{v}}}{\mathrm{d}t} = \mathbf{h}_\mu(\check{\mathbf{v}}) \tag{1}$$

where we assume the vector field $\mathbf{h}_\mu(\check{\mathbf{v}}) = \mathbf{h}(\check{\mathbf{v}};\mu)$ to be analytic with respect to both $\check{\mathbf{v}}$ and μ and that $\mathbf{h}_\mu(\mu_H) = 0$ (otherwise $\check{\mathbf{v}}$ represents the difference with respect to a solution $\mathbf{h}_\mu(\check{\mathbf{v}}) = 0$). Taking a Taylor expansion around $\check{\mathbf{v}} = 0$, one has

$$\frac{\mathrm{d}\check{\mathbf{v}}}{\mathrm{d}t} = \check{\mathbf{A}}_\mu\check{\mathbf{v}} + \check{\mathbf{b}}_\mu(\check{\mathbf{v}},\check{\mathbf{v}}) + \check{\mathbf{c}}_\mu(\check{\mathbf{v}},\check{\mathbf{v}},\check{\mathbf{v}}) + \mathcal{O}(\|\check{\mathbf{v}}\|^4) \tag{2}$$

where $\check{\mathbf{A}}_\mu$ is an $N \times N$ matrix, whereas $\check{\mathbf{b}}_\mu$ and $\check{\mathbf{c}}_\mu$ are, respectively, a bilinear and a trilinear forms.

In the following, it is assumed that the system undergoes a Hopf bifurcation, i.e., that for a given value of $\mu = \mu_H$ a couple of complex conjugate modes becomes unstable. Specifically, setting $\lambda_i = \check{\lambda}_i(0)$, where $\check{\lambda}_i(\mu)$ are the eigenvalues of $\check{\mathbf{A}}_\mu$, we assume that $\lambda_1 = \bar{\lambda}_2 = i\,\omega_1$ and that $\lambda_i^R = Real(\lambda_i) < 0$ $(i = 3,\ldots,N)$.

On the basis of the above analytic-function assumption, it is possible to write (without loss of generality we assume $\mu_H = 0$)

$$\check{\mathbf{A}}_\mu = \check{\mathbf{A}}_0 + \mu\check{\mathbf{A}}_1 + \mathcal{O}(\mu^2) \tag{3}$$

For simplicity, in this paper the eigenvalues are assumed to be distinct so that $\check{\mathbf{A}}_0$ will have linearly independent eigenvectors $\mathbf{z}_i$, with $\mathbf{Z} = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N]$, such that

$$\mathbf{Z}^{-1}\check{\mathbf{A}}_0\mathbf{Z} = \mathbf{\Lambda} \tag{4}$$

where $\mathbf{\Lambda}$ is a diagonal matrix with elements equal to λ_i. Then, setting $\check{\mathbf{v}} = \mathbf{Z}\mathbf{v}$, and taking a Taylor series of $\check{\mathbf{b}}$ and $\check{\mathbf{c}}$ with respect to μ, Eq. 2 may be written as

$$\begin{aligned}\dot{\mathbf{v}} = {} & (\mathbf{\Lambda} + \mu\mathbf{A}_1)\,\mathbf{v} + \mathbf{b}(\mathbf{v},\mathbf{v}) + \mathbf{c}(\mathbf{v},\mathbf{v},\mathbf{v}) \\ & + \mathcal{O}(\|\mathbf{v}\|^4) + \mathcal{O}(\mu^2) + \mathcal{O}(\mu\|\mathbf{v}\|^2)\end{aligned} \tag{5}$$

where $\mathbf{A}_1 = \mathbf{Z}^{-1}\check{\mathbf{A}}_1\mathbf{Z}$, $\mathbf{b}(\mathbf{v}, \mathbf{v}) = \mathbf{Z}^{-1}\check{\mathbf{b}}(\mathbf{Zv}, \mathbf{Zv}; 0)$, and $\mathbf{c}(\mathbf{v}, \mathbf{v}, \mathbf{v}) = \mathbf{Z}^{-1}\check{\mathbf{c}}(\mathbf{Zv}, \mathbf{Zv}, \mathbf{Zv}; 0)$. Note that the above expansion, see Eq. 5, implies that $\mu \simeq \|\mathbf{v}\|^2$, that is a typical assumption of perturbation methods. This implies that the amplitude of the perturbed solution is of the same order of the perturbation of control parameter with respect to its critical value (see also Eqs. 8, 9).

Next, we introduce a parameter ε such that $\mathbf{v} = \varepsilon\mathbf{x}$. Hence,

$$\dot{\mathbf{x}} = \mathbf{\Lambda x} + \mu\mathbf{A}_1\mathbf{x} + \varepsilon\mathbf{b}(\mathbf{x}, \mathbf{x}) + \varepsilon^2\mathbf{c}(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(\varepsilon^3) + \mathcal{O}(\mu^2) + \mathcal{O}(\mu\varepsilon) \tag{6}$$

In the following we assume that the even-order nonlinear terms in $\mathbf{x}$ vanish (this is true, for instance, when the system is symmetric, so that the equations remain unchanged if $\mathbf{x}$ is replaced by $-\mathbf{x}$). The reason for this assumption is that in general the mechanism of the bifurcation is not altered by the presence of even-order terms (see, e.g., Ref. [8,13]). Indeed, it is always possible (if there are not zero eigenvalues) to use a suitable coordinate transformation which eliminates the even terms. Thus, one has

$$\dot{\mathbf{x}} = \mathbf{\Lambda x} + \mu\mathbf{A}_1\mathbf{x} + \varepsilon^2\mathbf{c}(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \mathcal{O}(\varepsilon^4) + \mathcal{O}(\mu^2) + \mathcal{O}(\mu\varepsilon^2) \tag{7}$$

since the even-order terms in $\mathbf{x}$ (including the fourth-order terms) have been assumed to vanish. It is worth noting that Eq. 7 is essentially the same of the original Eq. 2 (except for the presence of even terms), but it has been rewritten in a parameter perturbation form after diagonalizing its linear part.

Within the present system representation, the case of interest occurs when the variation of the linear terms and the nonlinear terms is of the same order. Thus, in order to balance the nonlinear terms with the variation of the linear terms, $\mu\mathbf{A}_1\mathbf{x}$ (this issue will be addressed in depth later), we set

$$\mu = \pm\varepsilon^2 \tag{8}$$

Substituting the previous relation into Eq. 7, it yields

$$\dot{\mathbf{x}} = \mathbf{\Lambda x} + \varepsilon^2\left[\mathbf{Ax} + \mathbf{c}(\mathbf{x}, \mathbf{x}, \mathbf{x})\right] + \mathcal{O}(\varepsilon^4) \tag{9}$$

with

$$\mathbf{A} = \pm\mathbf{A}_1 \tag{10}$$

3 Standard normal form

The above problem may be written as

$$\dot{\mathbf{x}} = \mathbf{\Lambda x} + \varepsilon^2\mathbf{f}(\mathbf{x}) + \mathcal{O}(\varepsilon^4) \tag{11}$$

with

$$\mathbf{f}(\mathbf{x}) = \mathbf{Ax} + \mathbf{c}(\mathbf{x}, \mathbf{x}, \mathbf{x}). \tag{12}$$

If one limits the analysis to $\mathcal{O}(\varepsilon^2)$ terms, the standard Normal Form method consists of seeking a solution of the type

$$\mathbf{x} = \mathbf{y} + \varepsilon^2\mathbf{w}(\mathbf{y}) + \mathcal{O}(\varepsilon^4) \tag{13}$$

where $\mathbf{w}(\mathbf{y})$ is to be chosen so as to simplify the formulation. One wishes to use Eq. 13 to transform Eq. 11 into an equation of the type

$$\dot{\mathbf{y}} = \mathbf{\Lambda y} + \varepsilon^2\mathbf{g}(\mathbf{y}) + \mathcal{O}(\varepsilon^4) \tag{14}$$

Combining Eqs. 11 and 13 one obtains

$$\dot{\mathbf{x}} = \dot{\mathbf{y}} + \varepsilon^2\mathbf{W}\dot{\mathbf{y}} = \mathbf{\Lambda}\left(\mathbf{y} + \varepsilon^2\mathbf{w}(\mathbf{y})\right) + \varepsilon^2\mathbf{f}(\mathbf{y}) + \mathcal{O}(\varepsilon^4) \tag{15}$$

where $\mathbf{W} = \partial\mathbf{w}/\partial\mathbf{y} := [\partial w_n/\partial y_k]$. Then, combining Eqs. 14 and 15, one obtains

$$\left(\mathbf{I} + \varepsilon^2\mathbf{W}\right)\left(\mathbf{\Lambda y} + \varepsilon^2\mathbf{g}(\mathbf{y})\right) = \mathbf{\Lambda}\left(\mathbf{y} + \varepsilon^2\mathbf{w}(\mathbf{y})\right) + \varepsilon^2\mathbf{f}(\mathbf{y}) + \mathcal{O}(\varepsilon^4) \tag{16}$$

which yields

$$\mathbf{g}(\mathbf{y}) = -\mathbf{W\Lambda y} + \mathbf{\Lambda w}(\mathbf{y}) + \mathbf{f}(\mathbf{y}) + \mathcal{O}(\varepsilon^2). \tag{17}$$

Hence, combining with Eq. 14, one finally obtains

$$\dot{\mathbf{y}} = \mathbf{\Lambda y} + \varepsilon^2\left(-\mathbf{W\Lambda y} + \mathbf{\Lambda w}(\mathbf{y}) + \mathbf{f}(\mathbf{y})\right) + \mathcal{O}(\varepsilon^4) \tag{18}$$

Equation 18 is formally equal to Eq. 11. However, it contains the arbitrary vector $\mathbf{w}$, which may be chosen so as to remove all the terms that cause unnecessary complexity in Eq. 11. The objective stated above (of choosing $\mathbf{w}(\mathbf{y})$ in Eq. 13 so as to render Eq. 14 simpler to solve than Eq. 11) may be obtained by choosing for $\mathbf{w}(\mathbf{y})$ the same functional dependence as for $\mathbf{f}(\mathbf{y})$. Shifting to indicial notations, Eq. 12 yields $f_n(\mathbf{y}) = \sum a_{np}y_p + \sum c_{npqr}y_py_qy_r$. Thus, one chooses

$$w_n(\mathbf{y}) = \sum_p \alpha_{np}y_p + \sum_{pqr}\gamma_{npqr}y_py_qy_r. \tag{19}$$

This implies, combining the above equations with Eq. 17,

$$g_n(\mathbf{y}) = -\sum_p W_{nk}\lambda_py_p + \lambda_nw_n + f_n(\mathbf{y}) = -\sum_p \alpha_{np}\lambda_py_p - \sum_{pqr}\gamma_{npqr}\left(\lambda_p + \lambda_q + \lambda_r\right)y_py_qy_r$$

$$+\lambda_n \sum_p \alpha_{np} y_p + \lambda_n \sum_{pqr} \gamma_{npqr} y_p y_q y_r$$
$$+\sum_p a_{np} y_p + \sum_{pqr} c_{npqr} y_p y_q y_r \tag{20}$$
$$= \sum_p \left[a_{np} + (\lambda_n - \lambda_p)\, \alpha_{np}\right] y_p$$
$$+\sum_{pqr} \left[c_{npqr} + (\lambda_n - \lambda_p - \lambda_q - \lambda_r)\, \gamma_{npqr}\right] y_p y_q y_r \tag{21}$$

Next, one chooses α_{np} and γ_{npqr} so as to eliminate from Eq. 20 as many terms as possible. Specifically, one sets

$$\alpha_{np} = \frac{a_{np}}{\lambda_p - \lambda_n}$$
$$= 0 \quad \text{if } \lambda_p \neq \lambda_n \quad \text{otherwise} \tag{22}$$
$$\gamma_{npqr} = \frac{c_{npqr}}{\lambda_p + \lambda_q + \lambda_r - \lambda_n}$$
$$= 0 \quad \text{if } \lambda_p + \lambda_q + \lambda_r \neq \lambda_n \quad \text{otherwise} \tag{23}$$

The terms on the denominator are known as divisors. These equations, combined with Eq. 19, define completely the vector $\mathbf{w}(\mathbf{y})$ and, therefore, the transformation in Eq. 13. On the other hand, the contribution to $\mathbf{g}(\mathbf{y})$ arises only from the zero-divisors terms (for which $\alpha_{np} = 0$ and $\gamma_{npqr} = 0$, see Eqs. 22 and 23). Thus,

$$g_n(\mathbf{y}) = \sum_{p \in I_p^n} a_{np} y_p + \sum_{pqr \in I_{pqr}^n} c_{npqr} y_p y_q y_r \tag{24}$$

where, according to Eqs. 22 and 23, $I_p^n = \{(n;\, p) | \, \lambda_p = \lambda_n\}$ and $I_{pqr}^n = \left\{(n;\, p, q, r) | \lambda_p + \lambda_q + \lambda_r = \lambda_n\right\}$.

Equation 14, with $\mathbf{g}(\mathbf{y})$ given by Eqs. 24, is indeed simpler to solve then Eq. 11 (for which the sums range over all the possible combinations). This will be apparent in the following section.

4 First Hopf bifurcation

In Sect. 2, we have assumed the eigenvalues λ_i to be distinct (see below Eq. 3). Then, $\lambda_p - \lambda_n = 0$ implies $p = n$. Hence, $I_p^n = \{(n; n)\}$. Next consider Eq. 23. The condition $\lambda_p + \lambda_q + \lambda_r - \lambda_n = 0$ is obviously satisfied for instance if $\lambda_p + \lambda_q = 0$ and $\lambda_r - \lambda_n = 0$. Although this is not the only way to satisfy the above condition, here we assume this to be the case. Recalling that, $\lambda_1^R = \lambda_2^R = 0$ and $\lambda_i^R < 0$ $(i = 3, 4, \ldots)$, we have that $\lambda_n + \lambda_p = 0$ only if $n = 1$ and $p = 2$, and viceversa. Therefore, I_{pqr}^n contains at most six elements: $(n; n, 1, 2)$, $(n; n, 2, 1)$, $(n; 1, n, 2)$, $(n; 2, n, 1)$, $(n; 1, 2, n)$, $(n; 2, 1, n)$. Thus, Eq. 14 is given by

$$\dot{y}_n = \lambda_n y_n + \varepsilon^2 \left(\beta_n y_n + \gamma_n |y_1|^2 y_n\right) \tag{25}$$

where

$$\beta_n = a_{nn}$$
$$\gamma_1 = c_{1112} + c_{1121} + c_{1211}$$
$$\gamma_2 = c_{2221} + c_{2212} + c_{2122}$$
$$\gamma_n = c_{nn12} + c_{nn21} + c_{n1n2} + c_{n2n1} + c_{n12n} + c_{n21n}$$
$$\text{if} \quad n \neq 1, 2. \tag{26}$$

By setting

$$y_n = Y_n(t) e^{i\,\varphi_n(t)} \tag{27}$$

and separating real and imaginary parts one obtains

$$\dot{Y}_n = \left[\lambda_n^R + \varepsilon^2 \left(\beta_n^R + \gamma_n^R Y_1^2\right)\right] Y_n \tag{28}$$
$$\dot{\varphi}_n = \lambda_n^I + \varepsilon^2 \left(\beta_n^I + \gamma_n^I Y_1^2\right) \tag{29}$$

Next, following the HBCM assumption, we assume, naively, that the Center Manifold is composed by Y_1 and Y_2, in that the terms corresponding to $n \geq 3$ are assumed to be damped and hence disregardable for sufficiently large time. For $n = 1$, Eqs. 28 and 29 yield

$$\dot{Y}_1 = \varepsilon^2 \left(\beta_1^R Y_1 + \gamma_1^R Y_1^3\right) \tag{30}$$
$$\dot{\varphi}_1 = \omega_1 + \varepsilon^2 \left(\beta_1^I + \gamma_1^I Y_1^2\right) \tag{31}$$

The solution of Eq. 30 (easily obtained by setting $Y_1 = 1/\sqrt{u}$) is

$$Y_1(t) = \left(\frac{-\beta_1^R/\gamma_1^R}{1 + \kappa e^{-2\varepsilon^2 \beta_1^R t}}\right)^{1/2} \tag{32}$$

where κ is determined by the initial conditions, whereas noting that

$$\dot{Y}_1/Y_1 = \varepsilon^2 (\beta_1^R + \gamma_1^R Y_1^2), \tag{33}$$

Equation 31 yields

$$\varphi_1(t) = \varepsilon^2 \left(\beta_1^I - \gamma_1^I \beta_1^R/\gamma_1^R\right) t +$$
$$+ \varepsilon^2 \frac{\gamma_1^I}{\gamma_1^R} \ln |Y_1/Y_{10}| + \varphi_{10} \tag{34}$$

with φ_0 determined by the initial conditions as well along with Y_{1_0}.

Note that $\lambda_i(\mu) = \lambda_i + \varepsilon^2 \beta_i + \mathcal{O}(\varepsilon^4)$. Hence, recalling that $\mathbf{A} = \pm \mathbf{A}_1$, we have $\beta_1^R \gtrless 0$ for $\mu \gtrless 0$. Therefore (see Fig. 1), for $\mu > 0$ $(\beta_1^R > 0)$, we have that if

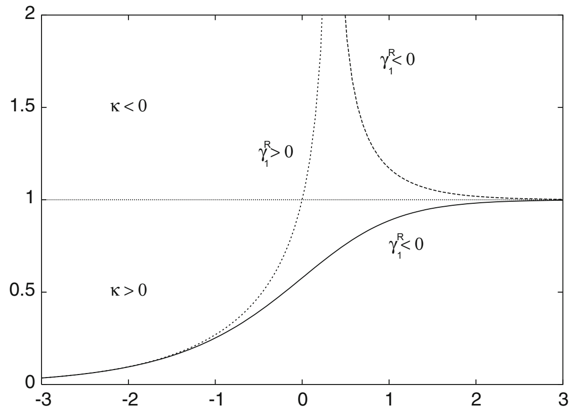

Fig. 1 Stability behavior as given by the time response envelope ($Y_1(t)/Y_1^{LC}$) in the right neighborhood ($\mu > 0$, and $\beta_R > 0$) of a Hopf bifurcation with respect to the initial conditions (k) and nonlinear terms nature (γ_1^R)

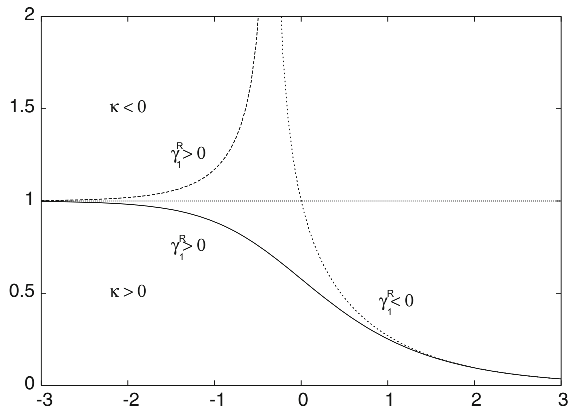

Fig. 2 Stability behavior as given by the time response envelope ($Y_1(t)/Y_1^{LC}$) in the left neighborhood ($\mu > 0$, and $\beta_R < 0$) of a Hopf bifurcation with respect to the initial conditions (k) and nonlinear terms nature (γ_1^R)

$\gamma_1^R < 0$ (stabilizing cubic terms) there exists a stable limit cycle with amplitude $Y_1^{LC} = \sqrt{-\beta_1^R/\gamma_1^R}$, as it can be seen directly from the differential equation. On the other hand (see Fig. 2), still for $\mu > 0$ ($\beta_1^R > 0$), if $\gamma_1^R > 0$ (destabilizing cubic terms), the solution would go to infinity in finite time. On the other hand, if $\mu < 0$ ($\beta_1^R < 0$) and if $\gamma_1^R < 0$ (stabilizing cubic terms) the solution is unconditionally stable, whereas for $\gamma_1^R > 0$ (destabilizing nonlinear terms) there exists an unstable limit cycle.

The above results coincide with the multiple time-scale results of Ref. [13] applied to a dynamical system with cubic nonlinearities experiencing a Hopf bifurcation at a certain value of the system parameter, undergoing the same HBCM hypothesis.

5 Second bifurcation: a small divisor approach

In the present section, the solution for the system motion including all the non-critical modes y_n ($n \geq 3$) is presented (refer to Eq. 27). Specifically, from their analytical perturbative expression, the implicit coupling with the critical modes and the mechanism by which they become no longer damped are highlighted. Thus, following Mastroddi, Ref. [14], combining Eqs. 27, 28 and 29 with Eq. 32, one has

$$Y_n = Y_n(t) = Y_{n0}\, e^{\sigma_n(t)}, \tag{35}$$

with[1]
Y_{n0} given by initial conditions and

$$\sigma_n(t) = \left(\lambda_n^R + \varepsilon^2\beta_n^R - \varepsilon^2\gamma_n^R\frac{\beta_1^R}{\gamma_1^R}\right)t - \frac{\gamma_n^R}{\gamma_1^R}\ln\left|\frac{Y_1}{Y_{10}}\right| \tag{36}$$

In the presence of a stable limit cycle (that is, for $\gamma_1^R < 0$ and $\beta_1^R > 0$), for t sufficiently large, Eq. 36 tends to:

$$\sigma_n(t) \simeq \left(\lambda_n^R + \varepsilon^2\beta_n^R - \varepsilon^2\gamma_n^R\frac{\beta_1^R}{\gamma_1^R}\right)t - \frac{\gamma_n^R}{\gamma_1^R}\ln\left|\frac{\beta_1^R}{\gamma_1^R}\right| \tag{37}$$

From this equation it is apparent that, for any n, the assumption $\lim_{t\to\infty} Y_n = 0$ is no longer valid if

$$\lambda_n^R + \varepsilon^2\left(\beta_n^R - \gamma_n^R\beta_1^R/\gamma_1^R\right) > 0, \tag{38}$$

implying alternatively $\|Y_n\| \to +\infty$ for $t \to +\infty$ if $n \neq 1, 2$. Therefore, recalling that $\lambda_n^R < 0$ and $\beta_1^R > 0$,

[1] From Eq. 27 the motion along the $n-th$ mode can be written as:

$$y_n = Y_n(t)e^{i\varphi_n(t)}$$

where

$$\varphi_n(t) = \left(\lambda_n^I + \varepsilon^2\beta_n^I - \varepsilon^2\gamma_n^I\frac{\beta_1^R}{\gamma_1^R}\right)t - \frac{\gamma_n^I}{\gamma_1^R}\ln\left|\frac{Y_1}{Y_{10}}\right| + \varphi_{n0},$$

By observing Eq. 28 it appears that

$$Y_n(t) = Y_{n0}e^{\sigma_n(t)}$$

with $\sigma_n(t)$ given by Eq. 36

if $\beta_n^R/\beta_1^R > \gamma_n^R/\gamma_1^R$ is verified, there exists a critical value of ε and consequently of μ (see Eq. 8)

$$\mu_{cr} = \varepsilon_{cr}^2 = |\lambda_n^R| / \left(\beta_n^R - \gamma_n^R \beta_1^R/\gamma_1^R\right) \tag{39}$$

which implies that the extension of the HBCM assumption to a finite interval is limited by the threshold given above. Thus, for $\mu > \mu_{cr}$ or $\varepsilon > \varepsilon_{cr}$ the system behavior is not predicted by the standard Normal Form and this evidence claims for reviewing the HBCM analysis.

The condition given by Eq. 39 implies that the assumption, that the Center Manifold is determined from the zero divisors, is violated. Therefore, its determination must be based upon a small divisor approach (Ref. [8]). As Eqs. 35 and 36 show, the motion along this mode remains an exponential decay if $\sigma_n(t) < 0$. If $\sigma_n(t) > 0$, the motion increases exponentially in time. Thus, the considered mode is no longer stable. Equation 39 gives the limit condition implying that $\sigma_n(t) > 0$. By solving the condition with respect to ε the value of the perturbation parameter activating the n-th modes is obtained and given in Eq. 39. This implies that if a zero-divisors approach is used in order to evaluate the Normal Form of the system there exists a value of the control parameter which gives the actual boundary of validity of this method. Indeed, for larger perturbations, other modes could be activated. Since the zero-divisors approach does not work, it is necessary to introduce a small divisors approach based on the condition given by Eq. 39.

To facilitate the interpretation of this result, it is convenient to set $\check{\lambda}_n^R(\mu) = \lambda_n^R + \varepsilon^2 \beta_n^R$, where $\check{\lambda}_n^R(\mu)$ represents the eigenvalue associated with the n-th mode and the right-hand side is its truncated perturbation expression in ε (it turns out to be linear in μ), whereas $\varepsilon^2 \beta_1^R/\gamma_1^R$ is the squared limit-cycle amplitude of the first mode

$$\|x_1\|_{LC}^2 := \varepsilon^2 \beta_1^R/\gamma_1^R. \tag{40}$$

Thus, the condition given in Eq. 38 can be recast as

$$\check{\lambda}_n^R(\mu) + \gamma_n^R \|x_1(\mu)\|_{LC}^2 > 0 \tag{41}$$

Highlighting the dependence on μ for sake of clarity, Eq. 28 is written as:

$$\dot{Y}_n = \left[\check{\lambda}_n^R(\mu) + \gamma_n^R \|x_1(\mu)\|_{LC}^2\right] Y_n \tag{42}$$

Equations 41 and 42 show that the nonlinear coupling between the modes influences those assumed to be damped and, furthermore, this damping is dependent on the control system parameter μ. Therefore, if for a certain value of μ (or ε) the condition given by Eq. 41 is verified, the corresponding n-th mode cannot be considered as damped and the HBCM assumption must be reconsidered. This result represents the main contribution of the present paper. Indeed, as it will be shown, it permits to analyze how the activation of slave modes influences the Center Manifold dimension and thus the resonance conditions. The condition of activation of the generic slave mode, see Eq. 41, is an equality at the critical point (second bifurcation). Thus, moving back to the perturbation parameter ε, one has:

$$\check{\lambda}_n^R(\mu) - \gamma_n^R \|x_1(\varepsilon)\|_{LC}^2 = 0 \tag{43}$$

The ordering process in the perturbation parameter (see Sect. 5) implies that $\gamma_n^R \beta_1^R/\gamma_1^R = \mathcal{O}(1)$. Indeed, if the condition $\gamma_n^R \beta_1^R/\gamma_1^R = \mathcal{O}(1)$ not be valid, the considered term should not be present in Eq. 11 at the considered order, and thus the present analysis would not apply (see the calculation steps in Sect. 2). Thus, if Eq. 43 is verified, it yields:

$$\check{\lambda}_n^R(\mu) = \mathcal{O}(\varepsilon^2) \tag{44}$$

and, consequently,

$$\check{\lambda}_n(\mu) + \bar{\check{\lambda}}_n(\mu) = 2\check{\lambda}_n^R(\mu) = \mathcal{O}(\varepsilon^2) \tag{45}$$

The above Eq. 45 implies that, for a fixed n and by assuming that the $(n+1)$-th eigenvalue is the complex conjugated of the n-th eigenvalue, the following condition holds:

$$\check{\lambda}_p(\mu) - (\check{\lambda}_p(\mu) + \check{\lambda}_n(\mu) + \check{\lambda}_{n+1}(\mu)) = \mathcal{O}(\varepsilon^2) \quad \forall p = 1, 2 \tag{46}$$

Considering Eq. 23 clearly points out that Eq. 46 corresponds to a small divisor condition associated with nonlinear terms of type $y_p|y_n|^2$, which are activated by the condition given by Eq. 38.

Thus, I_{pqr}^i will contain 12 nonzero elements obtained from all the permutations of the last three terms of the quadruples $(i; i, 1, 2)$ and $(i; i, n, n+1)$, where n indicates the generic activated mode and $n+1$ its complex conjugated. It is worth noting that the quadruples $(i; i, 1, 2)$ are those associated with the Hopf bifurcation (first bifurcation) related to the HBCM assumption. On the other hand the quadruples $(i; i, n, n+1)$ correspond to a new bifurcation associated with the activation of the n-th mode through the mechanism illustrated by Eqs. 38–39.

Being the activation of the generic n-th mode dependent on the value of the perturbation parameter (or equivalently the system control parameter), the above analysis permits achieving easily a measure (accurate up to second order in ε) of the neighborhood in which the HBCM assumption is capable of generating a Normal Form fully representing the system dynamics. For this reason, it is relevant to point out that this condition does not conflict with the Center Manifold Theorem, but it totally agrees with it. From a more theoretical point of view, the previous results point out that it is possible to extend the HBCM analysis to a finite neighborhood with a measure determined by Eq. 41. Finally, it can be observed that, the activation of the generic damped mode may be caused by the linear eigenvalue crossing the imaginary axis (if $\gamma_n^R = 0$) or by the nonlinear contribution making the n-th mode unstable (when $\beta_n^R = 0$), or, in general, by a combination of the two effects. In the next section a numerical example is used in order to illustrate the analyzed phenomenon.

6 Numerical results

In this section the above theoretical findings are numerically explored through a simple but effective example. Moreover, it is shown that the obtained results can be interpreted in codimension-two bifurcation framework.

6.1 Nonlinear coupled oscillators exhibiting analytical LCs

In order to illustrate the above results, the Normal Form method developed in the previous sections is here applied to the following set of equations:

$$\ddot{x}+x+\dot{x}\left[-\mu+c_1\left(x^2+\dot{x}^2\right)+c_2\left(y^2+\alpha^2\dot{y}^2\right)\right]=0$$
$$\ddot{y}+\omega_y^2 y+\dot{y}\left[\delta-c_3\left(x^2+\dot{x}^2\right)\right]=0 \tag{47}$$

where $\omega_x = 1$ and ω_y are the uncoupled natural frequencies, μ is the system parameter, δ, α, c_1, c_2 and c_3 are real positive coefficients. Setting $\hat{\mathbf{v}} = \{x, \dot{x}, y, \dot{y}\}^T$, Eq. 47 is recast in first-order form as Eq. 2.

This equation provides an interesting test case for the Normal Form theory presented in the previous sections. Specifically, following the standard procedure also known as *harmonic balance*, Ref. [8], we concentrate on the case $\alpha = 1/\omega_y$ for which an exact harmonic limit-cycle solution is available in the form:

$$x(t) = X\cos(t+\theta_1) \qquad y(t) = Y\cos(\omega_y t+\theta_2) \tag{48}$$

Before moving to calculate analytically the LC amplitudes X and Y, it is useful to recast the system in first-order form:

$$\frac{\mathrm{d}\check{\mathbf{v}}}{\mathrm{d}t} = \check{\mathbf{A}}_\mu + \check{\mathbf{c}}_\mu(\check{\mathbf{v}}, \check{\mathbf{v}}, \check{\mathbf{v}}) \tag{49}$$

with $\check{\mathbf{v}}^T = \{x, \dot{x}, y, \dot{y}\}$ and

$$\check{\mathbf{A}}_\mu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & \mu & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_y^2 & -\delta \end{bmatrix} \tag{50}$$

$$\check{\mathbf{c}}_\mu(\check{\mathbf{v}}) = \begin{bmatrix} 0 \\ -c_1\check{v}_2(\check{v}_1^2+\check{v}_2^2) - c_2\check{v}_2(\check{v}_3^2+\frac{\check{v}_4^2}{\omega_y^2}) \\ 0 \\ c_3\check{v}_4(\check{v}_1^2+\check{v}_2^2) \end{bmatrix} \tag{51}$$

which shows clearly that exists a couple of complex conjugate eigenvalues (and associated eigenvectors) at the critical value of the control parameter $\mu = \mu_H = 0$. Thus, the system undergoes a Hopf bifurcation. By substituting the above relationships into Eq. 47, one obtains

$$X\left(-\mu+c_1X^2+c_2Y^2\right)=0$$
$$Y\left(\delta-c_3X^2\right)=0 \tag{52}$$

Assuming the damped mode amplitude to vanish (accordingly to the HBCM assumption), one obtains that $Y = 0$ implies that the second equation is not required, whereas the first one yields $X = \sqrt{\mu/c_1}$. This solution is stable as long as $\delta - c_3X^2 = \delta - c_3\mu/c_1 < 0$. Of course we have assumed $\delta > 0$ (otherwise the solution is unstable for small values of μ). Thus, the above condition is no longer satisfied if $\mu \geq \mu_{cr}$, with $\mu_{cr} = \delta\, c_1/c_3$. In this case the (stable) solution is $X = \sqrt{\delta/c_3}$ and $Y^2 = (\mu - c_1\delta/c_3)/c_2$, which gives a limit-cycle solution as long as $\mu \geq \delta\, c_1/c_3 =: \mu_{cr}$. This (positive) quantity μ_{cr} is the limit value for the validity of the Hopf bifurcation theorem (and the Center Manifold hypothesis).

6.2 Time marching solution, analytical envelopes and Normal Form solution

Next, different combinations of the coefficients c_i will be discussed for the case $\alpha = 1/\omega_y$, with $\omega_y = 2.234$. First, let us assume

$$c_1 = 1 \quad c_2 = 0 \quad c_3 = 1 \quad \delta = 0.04468 \tag{53}$$

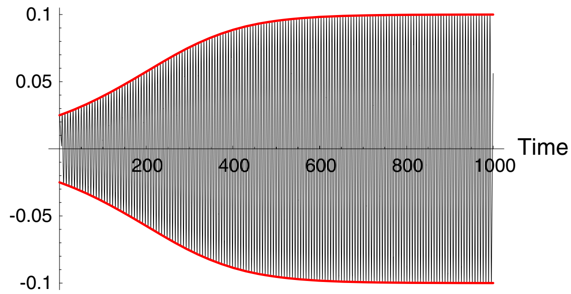

Fig. 3 Normal Form envelope of the LCO for $x(t)$ with $\mu = 0.01 < \mu_{cr}$, $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $\delta = 0.04468$

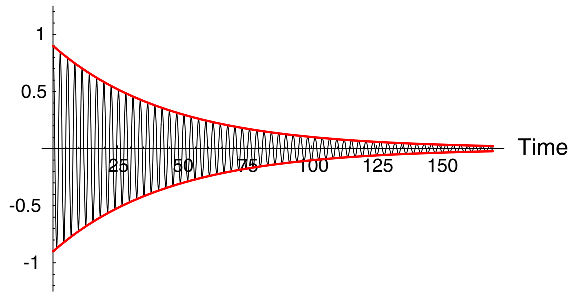

Fig. 4 Normal Form envelope of the LCO for $y(t)$ with $\mu = 0.01 < \mu_{cr}$, $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $\delta = 0.04468$

The first equation in the variable x shows a limit-cycle amplitude, and it drives the slave variable y in the second equation. For small values of the control parameter μ the standard Hopf bifurcation theory gives exactly the critical-mode amplitude $X(\mu)$ obtained above. In Figs. 3 and 4 the numerical solutions $x(t)$, $y(t)$ of Eq. 47 for $\mu = 0.01$ are plotted along with the analytical envelopes given by Eqs. 32 and 36. By using Eq. 39 together with Eq. 26 (see Sect. 4) the critical value for the activation of the slave modes is obtained so giving that for $\mu > \mu_{cr} = 0.04468$ the slave-mode amplitudes are no longer damped. This is numerically confirmed by considering $\mu = 0.05 > \mu_{cr}$. This choice of μ does not affect the stability of the first mode (it modifies only the limit-cycle amplitude, see Fig. 5) because the first of Eq. 47 is independent from the second one. Otherwise, as already predicted, the second mode is no longer damped, and its envelope amplitude is still well approximated as shown in Fig. 6.

Finally, consider

$$c_1 = 1 \quad c_2 = 0.01 \quad c_3 = 1 \quad \delta = 0.04 \tag{54}$$

The Center Manifold hypothesis is violated for $\mu \geq \mu_{cr} = 0.04$ (see Eq. 39), as it can be seen in Fig. 7

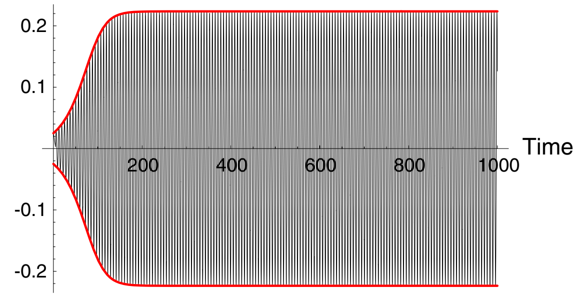

Fig. 5 Normal Form envelope of the LCO for $x(t)$ with $\mu = 0.05 > \mu_{cr}$, $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $\delta = 0.04468$

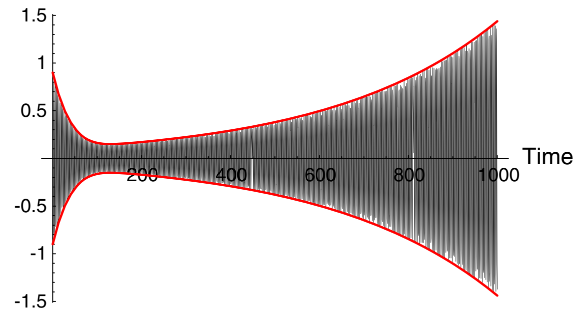

Fig. 6 Normal Form envelope of the LCO for $y(t)$ with $\mu = 0.05 > \mu_{cr}$, $c_1 = 1$, $c_2 = 0$, $c_3 = 1$, $\delta = 0.04468$

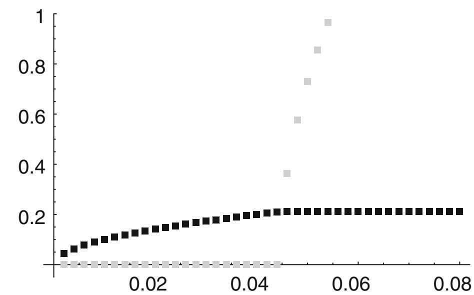

Fig. 7 Bifurcation diagram (LC amplitudes vs control parameter μ) for Eq. 47 with $c_1 = 1$ $c_2 = 0.01$ $c_3 = 1$ $\delta = 0.04$

where, by plotting the amplitudes of the numerically computed LCO amplitudes, the bifurcation diagram for the considered system is shown. The transient behavior obtained with time-step integration is compared again with the Normal Form approximation. First, a value of $\mu = 0.001$ is considered. As shown in Figs. 8 and 9, both the critical (undamped) and slave (damped) mode are well approximated for all the time-range $[0, \infty)$. If it is assumed $\mu = 0.1 > \mu_{cr}$, the approximation provided by the theory is good only for a limited time-range, as shown in Figs. 10 and 11.

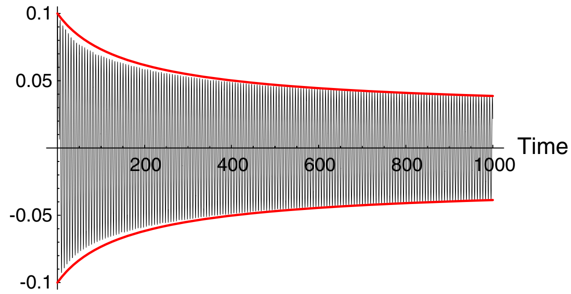

Fig. 8 Normal Form envelope of the LCO for $x(t)$ with $\mu = 0.001 < \mu_{cr}$ $c_1 = 1$ $c_2 = 0.01$ $c_3 = 1$ $\delta = 0.04$

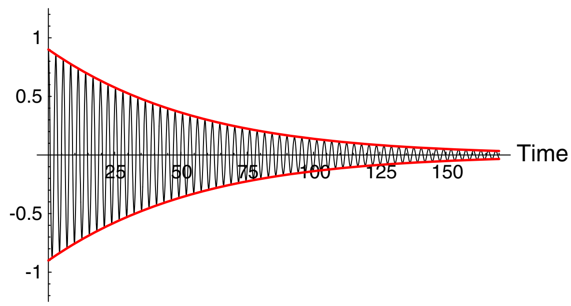

Fig. 9 Normal Form envelope of the LCO for $y(t)$ with $\mu = 0.001 < \mu_{cr}$ $c_1 = 1$ $c_2 = 0.01$ $c_3 = 1$ $\delta = 0.04$

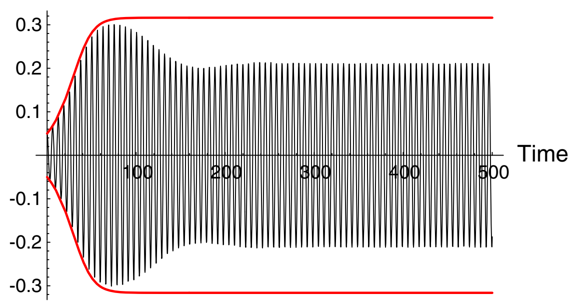

Fig. 10 Normal Form envelope of the LCO for $x(t)$ with $\mu = 0.1 > \mu_{cr}$ $c_1 = 1$ $c_2 = 0.01$ $c_3 = 1$ $\delta = 0.04$

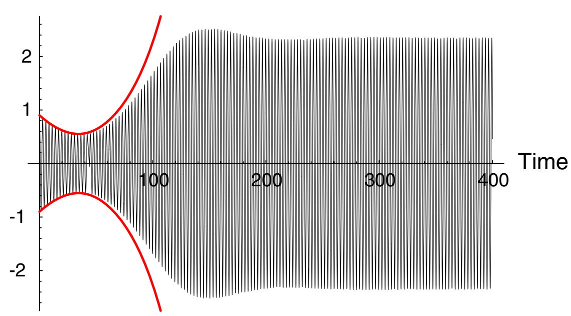

Fig. 11 Normal Form envelope of the LCO for $y(t)$ with $\mu = 0.1 > \mu_{cr}$ $c_1 = 1$ $c_2 = 0.01$ $c_3 = 1$ $\delta = 0.04$

6.3 An interpretation as codimension-two dynamical system

The calculations performed in the previous subsections can be interpreted by considering a codimension-two bifurcation (see Refs. [8,15,16]) with the adjoint of a linear constraint between the two unfolding parameters governing the bifurcation. By repeating the approach presented in Sect. 2, Eq. 49 can be transformed by using the eigenbasis associated with the state-space matrix evaluated at the critical vale of the control parameter $\mu = \mu_H = 0$ and ordered according to the perturbation parameter ε. Indeed, by using the transformations $\check{\mathbf{v}} = \varepsilon \mathbf{Z}\mathbf{x}$ and $\mu = \varepsilon^2$ one obtains:

$$\dot{\mathbf{x}} = \mathbf{\Lambda}\mathbf{x} + \varepsilon^2 \left[\mathbf{A}_1\mathbf{x} + \mathbf{c}(\mathbf{x}, \mathbf{x}, \mathbf{x})\right] \tag{55}$$

with

$$\mathbf{Z} = \begin{bmatrix} -i & i & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\delta-\sqrt{\delta^2-4\omega_y^2}}{2\omega_y^2} & -\frac{\delta+\sqrt{\delta^2-4\omega_y^2}}{2\omega_y^2} \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{56}$$

$$\mathbf{Z}^{-1}\check{\mathbf{A}}_0\mathbf{Z} = \mathbf{\Lambda} = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \tag{57}$$

$$\mathbf{Z}^{-1}\check{\mathbf{A}}_1\mathbf{Z} = \mathbf{A}_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{58}$$

with $\lambda_1 = \bar{\lambda}_2 = i$ and $\lambda_3 = \bar{\lambda}_4 = \frac{1}{2}\left(-\delta + i\sqrt{4\omega_y^2 - \delta^2}\right) = -\frac{\delta}{2} + i\hat{\omega}_y$. Since the system is experiencing a Hopf bifurcation, the analysis can be performed as in Sect. 4 with the same associated zero divisors. In particular, the Normal Form associated with the occurring bifurcation is:

$$\dot{Y}_1 = \left[\frac{\varepsilon^2}{2}Y_1 - 2c_1\varepsilon^2 Y_1^3\right] \tag{59}$$

$$\dot{Y}_3 = \left[-\frac{\delta}{2} + 2\varepsilon^2 c_3 Y_1^2\right] Y_3 \tag{60}$$

$$\dot{\varphi}_1 = 1$$

$$\dot{\varphi}_3 = \hat{\omega}_y \tag{61}$$

If the new variables (r_1, r_2) are defined as:

$$r_1 = \sqrt{2}\varepsilon Y_1 \tag{62}$$

$$r_2 = \frac{\sqrt{2}\varepsilon}{\hat{\omega}_y} Y_3 \tag{63}$$

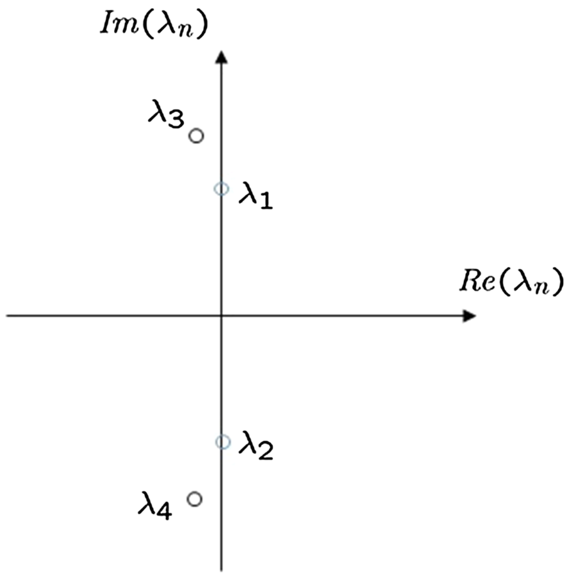

Fig. 12 Qualitative sketch for codimension-one Hopf bifurcation

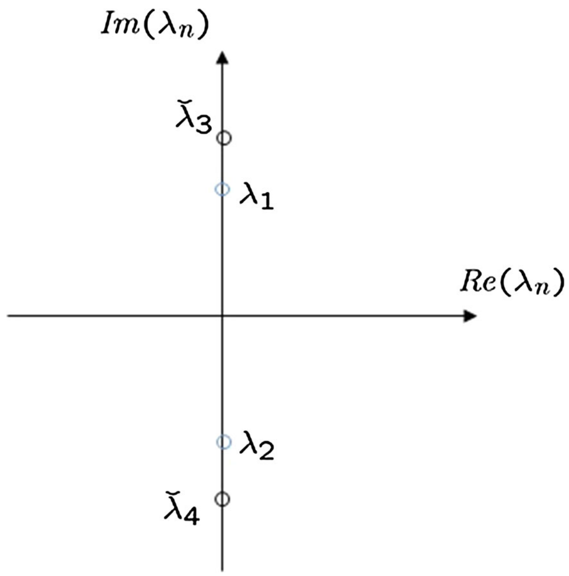

Fig. 13 Qualitative sketch for codimension-two Hopf bifurcation

the Normal Form equations assume the general form (for the sake of simplicity only the equations for the LC amplitudes are presented here):

$$\dot{r}_1 = \left[\frac{\varepsilon^2}{2} r_1 - c_1 r_1^3\right] = \left[\frac{\mu}{2} r_1 - c_1 r_1^3\right] \tag{64}$$

$$\dot{r}_2 = \left[-\frac{\delta}{2} + c_3 r_1^2\right] r_2 = \left[-\frac{\delta}{2} + c_3 r_1^2\right] r_2 \tag{65}$$

$$\mu = \varepsilon^2 \tag{66}$$

It is relevant to point out that Eqs. 65–66 represent the original problem rewritten in the form predicted by the Hartman–Grobman Theorem (codimension-one bifurcation, see Fig. 12 for eigenvalues qualitative sketch). Equation 65 is associated with the LC dynamics embedded in the Center Manifold, whereas Eq. 66 represents the dynamics on the Stable Manifold whose orbits are diffeomorphic to an exponential decay. By applying the same analysis performed in Sect. 5, one can observe that, if $\mu \geq \mu_{cr}$, the solution of Eq. 61 diverges with time. This behavior can be brought back to the bifurcation analysis framework by increasing the bifurcation codimension: thus, new resonance conditions must be considered (see the end of Sect. 5). In order to clarify this point let us observe that Eq. 65 has the fix-point solution:

$$r_1 = \sqrt{\frac{\mu}{2c_1}} \tag{67}$$

which implies that the term in the square bracket in Eq. 66 is zero if $\mu = \mu_{cr} = \delta \frac{c_1}{c_3}$. Thus, if the control parameter exceeds this identified critical value the bifurcation analysis, as performed until this point, is no longer valid and it is necessary to consider terms of the form $r_i r_j^2$ in the Normal Form Equations:

$$\dot{r}_1 = \left[\frac{\mu}{2} r_1 - c_1 r_1^3 - c_2 r_1 r_2^2\right] \tag{68}$$

$$\dot{r}_2 = \left[-\frac{\delta}{2} + c_3 r_1^2\right] r_2. \tag{69}$$

These are the same Normal Form equations shown in Ref. [8] (see Sect. 7.5.3) for a codimension-two Hopf bifurcation (see Fig. 13) in the case that a constraint on the second parameter is assumed and only the coefficient of $r_1 r_2^2$ is different from zero. Indeed, in the parameters plane (μ, δ) the considered problem lies on the line $\delta = const$. With respect to the general case presented in Ref. [8] it is relevant to point out that, since terms of type r_2^3 in Eq. 69 are not present, the LC amplitude associated with the first bifurcating mode will remain constant after the activation of the second mode. It is worth to note that, as already explained in Sect. 5, the activation of the so-called slave mode corresponds to a condition of zero real part of a "nonlinear eigenvalue." Thus, the activation of the damped mode can be interpreted as an event related to a codimension-two Hopf bifurcation where two couples of complex conjugate eigenvalues have a real part which becomes positive as system parameters are increased.

In order to complete the analysis it is interesting to compare the codimension-one Eqs. 65–66 and the codimension-two Eqs. 68–69 with the analytical envelope of the solution obtained in Sect. 6.1 (see Eqs. 52). By using the fixed-point solution of Eq. 65 (see Eq. 67), going through the calculation steps shown in Sect. 2 and remembering that $\hat{\mathbf{v}} = \{x, \dot{x}, y, \dot{y}\}^T$, one obtains for sufficiently large times:

$$x(t) = \sqrt{\frac{\mu}{c_1}} \cos(t + \frac{\pi}{2}) \tag{70}$$

$$y(t) = 0 \tag{71}$$

Equations 70 and 71 coincide exactly with Eq. 48 representing the long-term analytical solution of Eq. 47 for $\mu < \mu_{cr}$. Thus, the analytical envelope shown in Fig. 5 is captured also by the Normal Form for t sufficiently large. In the case $\mu \geq \mu_{cr}$, the dynamical system is described by Eqs. 67–69. In this case, the system presents the fixed-point solution:

$$r_1 = \sqrt{\frac{\delta}{2c_3}} \tag{72}$$

$$r_2 = \sqrt{\frac{\mu - c_1\delta/c_3}{2c_2}} \tag{73}$$

By repeating the same calculations as before one obtains:

$$x(t) = \sqrt{\frac{\delta}{c_3}} \cos\left(t + \frac{\pi}{2}\right) \tag{74}$$

$$y(t) = \sqrt{\frac{\mu - c_1\delta/c_3}{c_2}} \cos(\hat{\omega}_y t + \frac{\pi}{2} + \angle\lambda_3) \tag{75}$$

where $\angle(\cdot)$ indicates the phase of the argument. Equations 74 and 75 coincide exactly with Eq. 48 representing the long-term analytical solution of Eq. 47 for $\mu \geq \mu_{cr}$, and also in this case the Normal Form analysis is equivalent to the one presented in Sect. 6.1. Therefore, for this simple but representative example, the Normal Form theory succeeds in obtaining the same solutions and the same value of the parameter μ also given by the analytical approach based on the harmonic balance, Ref. [8], (Eqs. 74–75).

7 Concluding remarks

In the present paper the problem of determining the applicability bounds of perturbation expansions in terms of both the system parameter and the state-space variable amplitude has been dealt with by using as key parameter the smallness of the divisors defining the near-identity transformation in the Normal Form solution procedure. Specifically, a finite neighborhood analysis of the involved Hopf bifurcation has been carried out to set the conditions under which a Normal Form zero-divisors-based approach fails to consistently approximate the system dynamics. To recover the Normal Form solution in this case, a small divisor approach is required. More in deep, it has been pointed out that whenever the condition given by Eq. 38 is satisfied for the considered problem, there exists a value of the perturbation parameter such that the dynamics outside the Center Manifold is relevant for a satisfactory description of the system behavior. In this case, the criterion for including more terms in the near-identity transformation is provided by a small divisors approach. These terms are exactly defined by the perturbation order. Conversely, the small divisor condition introduces a specific bound for the Hopf bifurcation parameter (Eq. 39, $\varepsilon < \varepsilon_{cr}$) in which the standard Normal Form is still valid. Indeed, the concept of small divisor is related to the eigenvalues with a (negative) real part close to the imaginary axis. The question we are asking ourselves is: "How small should the real part be?". To us, the answer, in the past, has always been quite a vague concept, especially for non-conservative system and in practical application. Specifically, Eq. 39 gives a precise definition of the smallness of the real part of λ_i. Thus, "how small a small divisor" is not an absolute concept since it depends upon a balancing between the perturbed contribution of the linear term versus the nonlinear counterpart (i.e., $\beta_n^R/\beta_1^R > \gamma_n^R/\gamma_1^R$) and also upon the perturbation level, ε (i.e., μ). It is also important to underline that without considering the small divisors, the analysis, even if performed at higher orders introducing terms of type $y_n|y_1|^4$, would not lead to any improvement of the perturbed solution. Finally, the specific example introduced in this paper has also shown that the activations of the damped modes can be interpreted as a bifurcation occurring in a system with two couples of marginal eigenvalues (purely complex conjugated) and then can be classified as a codimension-two bifurcation.

Acknowledgements The authors want to thank Prof. Luigi Morino for the helpful contribution to the results presented in the paper.

References

1. Lamarque, C.H., Touzé, C., Thomas, O.: An upper bound validity limits of asymptotic analytical approaches based on normal form theory. Nonlinear Dyn. **70**, 1931–1949 (2012)
2. Poincarré, H.: Sur le problme des trois corps et les equations de la dynamique. Acta Math. **13**, 1–270 (1890)
3. Arnold, V.I.: Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, New York (1982)
4. Eugeni, M., Dowell, E.H., Mastroddi, F.: Post-buckling longterm dynamics of a forced nonlinear beam: a perturbation approach. J. Sound Vib. (2014). https://doi.org/10.1016/j.jsv.2013.12.026
5. Eugeni, M., Mastroddi, F., Dowell, E.H.: Normal form analysis of a forced aeroelastic plate. J. Sound Vib. **390**, 141–163 (2016)
6. Morino, L., Mastroddi, E., Cutroni, M.: Lie transformation method for dynamical system having chaotic behavior. Nonlinear Dyn. **7**(4), 403–428 (1995)
7. Abdechiri, M., Faez, K., Amindavar, H., Bilotta, E.: The chaotic dynamics of high-dimensional systems. Nonlinear Dyn. (2017). https://doi.org/10.1007/s11071-016-3213-3
8. Guckenheimer, J., Holmes, P.: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Applied Mathematical Sciences. Springer, New-York, p. 459 (1983)
9. Dessi, D., Mastroddi, F., Morino, L.: Limit-cycle stability reversal near a Hopf bifurcation with aeroelastic applications. J. Sound Vib. **256**, 347–365 (2002)
10. Dessi, D., Mastroddi, F.: Limit-cycle stability reversal via singular perturbation and wing-flap flutter. J. Fluids Struct. **19**, 765783 (2004)
11. Chow, S., Hale, J.K.: Methods of Bifurcation Theory, A Series of Comprehensive Studies in Mathematics. Springer, New-York, p. 525 (1982)
12. Huang, K., Feng, Q., Qu, B.: Bending aeroelastic instability of the structure of suspended cable-stayed beam. Nonlinear Dyn. (2017). https://doi.org/10.1007/s11071-016-3226-y
13. Smith, L.L., Morino, L.: Stability analysis of nonlinear differential autonomous system with applications to flutter. AIAA J. **14**, 333–341 (1976)
14. Mastroddi, F.: Aeroservoelasticità: Problematiche Nonlineari, Università di Roma "La Sapienza", Rome, Italy, p. 203 (in Italian) (1994)
15. Takens, F.: Singularities of vector fields. Publications Mathmatiques de l'Institut des Hautes tudes Scientifiques **43**(1), 47–100 (1974)
16. Takens, F.: Forced oscillations of vector fields. Communications of the Mathematical Institute, Rijksuniversiteit Utrecht vol. 3, pp. 1–59 (1974)