HIGHER ORDER FRACTIONAL LEINBIZ RULE

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Abstract. The fractional Leibniz rule is generalized by the Coifman-Meyer estimate. It is shown that the arbitrary redistribution of fractional derivatives for higher order with the corresponding correction terms.

1. Introduction

One of the most important tools to obtain local well-posedness of nonlinear equations of mathematical physics is based on the bilinear estimate of the form

$$\|D^s(fg)\|_{L^p} \le C\|D^s f\|_{L^{p_1}}\|g\|_{L^{p_2}} + C\|f\|_{L^{p_3}}\|D^s g\|_{L^{p_4}}, \tag{1.1}$$

where $D^s = (-\Delta)^{s/2}$ is the standard Riesz potential of order $s \in \mathbb{R}$, $L^p = L^p(\mathbb{R}^n)$ and $f, g \in \mathcal{S}(\mathbb{R}^n)$. A typical domain for parameters $s, p, p_j, j = 1, \cdots, 4$, where (1.1) is valid is

$$s > 0, \quad 1 < p, p_1, p_2, p_3, p_4 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

Classical proof can be found in [20]. The estimate can be considered as natural homogeneous version of the non-homogeneous inequality of type (1.1) involving Bessel potentials $(1-\Delta)^{s/2}$ in the place of D^s, obtained by Kato and Ponce in [23] (for this the estimates of type (1.1) are called Kato-Ponce estimates, too). More general domain for parameters can be found in [17].

Another estimate showing the flexibility in the redistribution of fractional derivatives can be deduced when $0 < s < 1$. More precisely, Kenig, Ponce, and Vega [24] obtained the estimate

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \le C\|D^{s_1} f\|_{L^{p_1}}\|D^{s_2} g\|_{L^{p_2}}, \tag{1.2}$$

provided

$$0 < s = s_1 + s_2 < 1, \quad s_1, s_2 \ge 0,$$

and

$$1 < p, p_1, p_2 < \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}. \tag{1.3}$$

One can interpret the bilinear form

$$\mathrm{Cor}_s(f, g) = fD^s g + gD^s f$$

as a correction term such that for any redistribution of the order s of the derivatives, i.e. for any $s_1, s_2 \ge 0$, such that $s_1 + s_2 = s$, we have

$$\|D^s(fg) - \mathrm{Cor}_s(f, g)\|_{L^p} \le C\|D^{s_1} f\|_{L^{p_1}}\|D^{s_2} g\|_{L^{p_2}}, \tag{1.4}$$

i.e., we have flexible redistribution of the derivatives of the remainder $D^s(fg) - \mathrm{Cor}_s(f, g)$.

Estimates of the form (1.2) are of interest on their own in harmonic analysis [1, 2, 3, 5, 6, 8, 16, 18, 19, 20, 21, 23, 28, 31] as well as in applications to nonlinear partial differential equations [4, 7, 11, 22, 24, 26, 27, 29, 30]. Our goal is to generalize (1.2) in the case where $s \ge 1$. It is

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shown in [13] that (1.2) holds even for $s = 1$ in one space dimension. This means that we could expect (1.4) with appropriate correction terms in a general setting. In fact, for $s = 2$, we have $D^2 = -\Delta$ and

$$D^2(fg) - fD^2g - gD^2f + 2\nabla f \cdot \nabla g = 0.$$

Therefore, some additional correction terms might be necessary for $s > 1$.

Typically, one can use paraproduct decompositions and reduce the proof of (1.4) separating different frequency domains for the supports of $\widehat{f}$ and $\widehat{g}$. In the case, when $\widehat{f}$ is localized in low-frequency domain and $\widehat{g}$ is localized in high-frequency domain, the estimate (1.4) can be derived from the commutator estimate

$$\|[D^s, f]g\|_{L^p} \le C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}},$$

where the assumption $s \le 1$ plays a crucial role. More precisely, if we assume

$$\text{supp}\, \widehat{f} \subset \{\xi \in \mathbb{R}^n;\, |\xi| \le 2^{k-2}\}, \quad \text{supp}\, \widehat{g} \subset \{\xi \in \mathbb{R}^n;\, 2^{k-1} \le |\xi| \le 2^{k+1}\}, \tag{1.5}$$

then we can use the relation

$$[D^s, f]g(x) = A_s(Df, D^{s-1}g)(x),$$

where

$$A_s(F, G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} a_s(\xi, \eta)\widehat{F}(\xi)\widehat{G}(\eta) d\xi d\eta$$

is a Coifman-Meyer type bilinear operator with a symbol $a_s(\xi, \eta)$ of Coifman-Meyer class supported in the cone

$$\Gamma = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n; 0 < |\xi| \le |\eta|/2\}, \tag{1.6}$$

Recall the definition of Coifman-Meyer class:

Definition 1. *We say that a symbol*

$$\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\})$$

belongs to the Hörmander class S^0, *if for all multi-indices* $\alpha \in \mathbb{N}_0^n$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, *we have*

$$|\partial_\xi^\alpha \sigma(\xi)| \le C_\alpha |\xi|^{-|\alpha|}, \quad \forall \xi \ne 0.$$

We say that a bilinear symbol

$$a \in C^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(0, 0)\})$$

belongs to the Coifman-Meyer (CM) class, if

$$|\partial_\xi^\alpha \partial_\eta^\beta a(\xi, \eta)| \le C_{\alpha,\beta}(|\xi| + |\eta|)^{-|\alpha|-|\beta|}.$$

for all multi-indices $\alpha, \beta : |\alpha| + |\beta| < m_n$, *where* m_n *depends on the dimension only.*

It is well-known that operators with symbols in S^0 give rise to bounded operators on L^p : $1 < p < \infty$ spaces. The result of Coifman and Meyer (see [9, 10, 14, 25]) generalizes this result to bilinear symbols. Namely, it states that bilinear operators

$$A(F, G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} a(\xi, \eta)\widehat{F}(\xi)\widehat{G}(\eta) d\xi d\eta$$

with symbols in the CM class satisfy

$$\|A(F, G)\|_{L^p} \le C_{p,p_1,p_2}\|F\|_{L^{p_1}}\|G\|_{L^{p_2}} \tag{1.7}$$

for all $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$.

Applying Coifman-Meyer bilinear estimate for A_s we can deduce the following estimate

Lemma 1.1. *Suppose* f, g *satisfy the assumptions* (1.5) *and* p, p_1, p_2 *satisfy* $1 < p, p_1, p_2 < \infty$ *and* $1/p = 1/p_1 + 1/p_2$. *Then for any* $s \ge 0$ *we have*

$$\|[D^s, f]g\|_{L^p} \le C\|D^1 f\|_{L^{p_1}}\|D^{s-1}g\|_{L^{p_2}}. \tag{1.8}$$

This estimate and the assumptions (1.5) explains the possibility to redistribute the fractional derivatives. Namely, if f and g satisfy (1.5), we have the possibility to replace the right hand side of (1.8) by $C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$ for any couple (s_1, s_2) of non-negative real numbers with $0 < s_1 + s_2 = s < 1$.

Our main goal is to study a similar effect of arbitrary redistribution of fractional derivatives for $s \geq 2$ in the scale of Lebesgue and Triebel-Lizorkin spaces in $\mathbb{R}^n$.

First, we shall try to explain the correction term in (1.4), such that estimate of type

$$\|[D^s, f]g - \mathrm{Cor}_s(f, g)\|_{L^p} \leq C\|D^2 f\|_{L^{p_1}}\|D^{s-2}g\|_{L^{p_2}}$$

will be fulfilled.

Let $a_s(\xi, \eta, \theta) = |\eta + \theta\xi|^s$. We also define

$$\begin{aligned} A_s^m(\theta)(f, g) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)}\frac{1}{m!}\partial_\theta^m a_s(\xi, \eta, \theta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta, \\ \widetilde{A}_s^\alpha(\theta)(f, g) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)}\frac{\alpha!}{|\alpha|!}\partial_\eta^\alpha a_s(\xi, \eta, \theta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta. \end{aligned} \tag{1.9}$$

Then $A_s^0(1)(f, g) = D^s(fg)$, $A_s^0(0)(f, g) = fD^s g$, and $A_s^1(0)(f, g) = s\nabla f \cdot D^{s-2}\nabla g$. Moreover, we have the following estimate:

Lemma 1.2. *For any multi - indices α, β one can find a constant $C > 0$ so that for*

$$(\xi, \eta) \in \Gamma = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n;\ 0 < |\xi| \leq |\eta|/2\},$$

one has the estimate

$$\sup_{0\leq\theta\leq1} |\partial_\xi^\alpha\partial_\eta^\beta a_s(\xi, \eta, \theta)| \leq C|\eta|^{s-|\alpha|-|\beta|}.$$

Lemma 1.2 and the Coifman-Meyer estimate show that for any $f, g \in \mathcal{S}$ which satisfy (1.5),

$$\|\widetilde{A}_s^\alpha(f, g)\|_{L^p} \leq C\|f\|_{L^{p_1}}\|D^{s-|\alpha|}g\|_{L^{p_2}}.$$

Since

$$\partial_\theta^m a_s(\xi, \eta, \theta) = \sum_{|\alpha|=m} \alpha!\,\partial_\eta^\alpha a_s(\xi, \eta, \theta)\xi^\alpha,$$

we have for any $f, g \in \mathcal{S}$ which satisfy (1.5),

$$\begin{aligned} \|[D^s, f]g\|_{L^p} &= \|A_s^0(1)(f, g) - A_s^0(0)(f, g)\|_{L^p} \\ &\leq \int_0^1 \|A_s^1(\theta)(f, g)\|_{L^p}d\theta \\ &\leq \sum_{|\alpha|=1}\int_0^1 \|\widetilde{A}_s^\alpha(\theta)(\partial^\alpha f, g)\|_{L^p}d\theta \\ &\leq C\|Df\|_{L^{p_1}}\|D^{s-1}g\|_{L^{p_2}}, \end{aligned} \tag{1.10}$$

$$\begin{aligned} \|[D^s, f]g - s\nabla f \cdot D^{s-2}\nabla g\|_{L^p} &= \|A_s^0(1)(f, g) - A_s^0(0)(f, g) - A_s^1(0)(f, g)\|_{L^p} \\ &\leq \int_0^1 \|A_s^2(\theta)(f, g)\|_{L^p}d\theta \\ &\leq \sum_{|\alpha|=2}\int_0^1 \|\widetilde{A}_s^\alpha(\theta)(\partial^\alpha f, g)\|_{L^p}d\theta \\ &\leq C\|D^2 f\|_{L^{p_1}}\|D^{s-2}g\|_{L^{p_2}}. \end{aligned} \tag{1.11}$$

These estimates and the assumptions (1.5) explain the redistribution the fractional derivatives, since we have the possibility to replace the right hand sides of the last inequalities of (1.10) and (1.11) by $C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$ and $C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$, respectively, for any couple (s_1, s_2) of non-negative real numbers with $s_1 + s_2 = s$. For details, see Lemma 2.2.

To state the main results in this article, we introduce the following notation. Let $\Phi \in \mathcal{S}$ be radial function and satisfy $\hat{\Phi} \geq 0$,

$$\operatorname{supp} \hat{\Phi} \subset \{\xi \in \mathbb{R}^n;\ 2^{-1} < |\xi| < 2\}, \qquad \sum_{j\in\mathbb{Z}} \hat{\Phi}(2^{-j}\xi) = 1$$

for all $\xi \in \mathbb{R}^n\backslash\{0\}$, where $\hat{\Phi} = \mathfrak{F}\Phi$ is the Fourier transform of Φ. We define $\Phi_j = \mathfrak{F}^{-1}(\hat{\Phi}(2^{-j}\cdot)) = 2^{jn}\Phi(2^j\cdot)$, $\widetilde{\Phi}_j = \sum_{k=-2}^{2}\Phi_{j+k}$, and $\Psi_j = 1-\sum_{k>j}\Phi_k$ for $j \in \mathbb{Z}$. For simplicity, we denote $\widetilde{\Phi} = \widetilde{\Phi}_0$ and $\Psi = \Psi_0$. For $f \in \mathcal{S}'$, we define $P_j f$, $P_{\leq j} f$, and $P_{>j} f$ as

$$P_j f = \Phi_j * f, \qquad P_{\leq j} f = \Psi_j * f, \qquad P_{>j} f = \left(\sum_{k>j}\Phi_k\right) * f,$$

respectively, where $*$ denotes the convolution.

We are ready now to state our main results.

Theorem 1.1. *Let $\ell \in \mathbb{N}$. Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \leq s_1, s_2$ and $\ell - 1 \leq s = s_1 + s_2 \leq \ell$. Then the following bilinear estimate*

$$\begin{aligned}&\left\|D^s(fg) - \sum_{k\in\mathbb{Z}}\sum_{m=0}^{\ell-1} A_s^m(0)(P_{\leq k-3}f, P_k g) - \sum_{j\in\mathbb{Z}}\sum_{m=0}^{\ell-1} A_s^m(0)(P_{\leq j-3}g, P_j f)\}\right\|_{L^p}\\ &\leq C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}\end{aligned}$$

holds for all $f, g \in \mathcal{S}$, where C is a constant depending only on n, p, p_1, p_2.

Moreover, we have the generalization of (1.2) and simple correction term for $s \geq 2$ as a corollary of Theorem 1.1.

Corollary 1.1. *Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \leq s_1, s_2 \leq 1$, and $s = s_1 + s_2$. Then the following bilinear estimate*

$$\|D^s(fg) - fD^s g - gD^s f\|_{L^p} \leq C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$$

holds for all $f, g \in \mathcal{S}$.

Corollary 1.2. *Let p, p_1, p_2 satisfy $1 < p, p_1, p_2 < \infty$ and $1/p = 1/p_1 + 1/p_2$. Let s, s_1, s_2 satisfy $0 \leq s_1, s_2 \leq 2$ and $s = s_1 + s_2 \geq 2$. Then the following bilinear estimate*

$$\|D^s(fg) - fD^s g - gD^s f + sD^{s-2}(\nabla f \cdot \nabla g)\|_{L^p} \leq C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}$$

holds for all $f, g \in \mathcal{S}$.

This article is organized as follows. In Section 2, we collect some basic estimates and key estimates for the commutators. In Section 3, we prove Lemma 1.2, Theorem 1.1 and Corollaries 1.1, and 1.2.

2. Preliminaries

We collect some preliminary estimates needed in the proofs of the main results. For the purpose, we introduce some notations. Let $\mu(p) = \max\{p, (p-1)^{-1}\}$. For $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, and $s \in \mathbb{R}$, let $\dot{F}^s_{p,q} = \dot{F}^s_{p,q}(\mathbb{R}^n)$ be the usual homogeneous Triebel-Lizorkin space with

$$\|f\|_{\dot{F}^s_{p,q}} = \|(2^{sj}P_j f)\|_{L^p(l^q_j)} = \|\|(2^{sj}P_j f)\|_{l^q_j}\|_{L^p}.$$

It is well known that for $s \in \mathbb{R}$ and $1 < p < \infty$, $\dot{F}^s_{p,2}$ may be identified with $\dot{H}^s_p$, where $\dot{H}^s_p = D^{-s}L^p(\mathbb{R}^n)$ is the usual homogeneous Sobolev space and $\dot{F}^s_{p,q}$ is continuously embedded into $\dot{F}^s_{p,\infty}$. We also define the Hardy-Littlewood maximal operator by

$$(Mf)(x) = \sup_{r>0}\frac{1}{|B(r)|}\int_{B(r)}|f(x+y)|dy,$$

where $B(r) = \{\xi \in \mathbb{R}^n; |\xi| \le r\}$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, we put $\langle x \rangle = (1+|x|^2)^{1/2}$, where $|x|^2 = x_1^2 + \cdots + x_n^2$. We adopt the standard multi-index notation such as $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, where $\partial_m = \partial/\partial x_m$, $m = 1, \cdots, n$.

Lemma 2.1 ([15, Theorem 5.1.2])**.** *The estimates*

$$\mu(p)^{-1}\|f\|_{L^p} \le \|f\|_{\dot F^0_{p,2}} \le \mu(p)\|f\|_{L^p}$$

hold for $1 < p < \infty$ and $f \in L^p$.

Lemma 2.2 ([15, Theorem 2.1.10])**.** *Let $s \ge 0$. Then $x \cdot \nabla D^s \Psi \in L^1$. The estimate*

$$|D^s P_{\le k} f(x)| \le 2^{sk} \|x \cdot \nabla D^s \Psi\|_{L^1} Mf(x)$$

holds for any $f \in L^1_{\mathrm{loc}}$, $k \in \mathbb{Z}$, and $x \in \mathbb{R}^n$, where C depends only on n.

Proof. For completeness, we give its proof here: Recall that (Ψ_k) and Ψ are radial Schwartz functions satisfying

$$\widehat{P_{\le k} f}(\xi) = \widehat{\Psi}_k(\xi)\widehat{f}(\xi), \quad \widehat{P_{\le 0} f}(\xi) = \widehat{\Psi}(\xi)\widehat{f}(\xi).$$

Using a rescaling argument, combined with the relation

$$D^s P_{\le k} = D^s S^*_{2^k} P_{\le 0} S_{2^k} = 2^{sk} S^*_{2^k} D^s P_{\le 0} S_{2^k}, \quad S^*_{2^k} M S_{2^k} = M,$$

one can reduce the proof of Lemma 2.2 to the case when $k = 0$, where $S_{2^k} f = f(2^{-k}x)$ and $S^*_{2^k} f = f(2^k x)$. Let $\rho \in C^\infty([0,\infty); [0,1])$ satisfy

$$\rho = \begin{cases} 1 & \text{if} \quad 0 \le x \le 1/2, \\ \searrow & \text{if} \quad 1/2 < x < 1, \\ 0 & \text{if} \quad x \ge 1, \end{cases}$$

and $\rho_R(\cdot) = \rho(\cdot/R)$ for any $R > 0$. Let

$$F_x(r) = \int_{S^{n-1}} f(x + r\omega) d\omega, \qquad G_x(r) = \int_0^r F_x(r') r'^{n-1} dr'.$$

Since Ψ and $D^s\Psi$ are radial functions, it is useful to introduce the notation $\psi_s(|\cdot|) = D^s\Psi(\cdot)$. By integration by parts,

$$\begin{aligned}
|D^s P_{\le 0} f| &= \lim_{R\to\infty} \left| \int f(x+y)\rho_R(|y|) D^s\Psi(y) dy \right| \\
&= \lim_{R\to\infty} \left| \int_0^R F_x(r) r^{n-1} \rho_R(r)\psi_s(r) dr \right| \\
&= \lim_{R\to\infty} \left| \underbrace{G_x(R)\rho_R(R)\psi_s(R)}_{=0} - \underbrace{G_x(0)\rho_R(0)\psi_s(0)}_{=0} - \int_0^R G_x(r) \frac{d}{dr}(\rho_R \psi_s)(r) dr \right| \\
&\le |S^{n-1}| \int_0^\infty r^{n-1} \left| r \frac{d}{dr} \psi_s(r) \right| dr Mf(x) \\
&= \int_{\mathbb{R}^n} |x \cdot \nabla D^s \Psi(x)| dx \; Mf(x).
\end{aligned}$$

□

Remark 2.1. *One can show that $\|x \cdot \nabla D^s \Psi\|_{L^1}$ is bounded as follows:*

$$\begin{aligned}
\int_{\mathbb{R}^n} |x \cdot \nabla D^s \Psi(x)| dx &= \int_{\mathbb{R}^n} |(n+s) D^s \Psi(x) + D^s \nabla(x\Psi)(x)| dx \\
&\le (n+s)\|D^s\Psi\|_{L^1} + \|D^s \nabla(x\Psi)\|_{L^1}.
\end{aligned}$$

For any $s \ge 0$,

$$\|D^s\Psi\|_{L^1} \le C\|\Psi\|_{\dot B^s_{1,1}} \le C(\|\Psi\|_{\dot B^0_{1,\infty}} + \|\Psi\|_{\dot B^{2\lceil s/2\rceil}_{1,\infty}}) \le C\|\Psi\|_{H_1^{2\lceil s/2\rceil}},$$

where $\lceil s \rceil = \min\{a \in \mathbb{Z};\, a \geq s\}$. *Moreover, since* $\operatorname{supp} \nabla \hat{\Psi} \subset \mathbb{R}^n \backslash B(1)$, $D^s \nabla (x\Psi) \in \mathcal{S}$ *and* $\|D^s \nabla (x\Psi)\|_{L^p} < \infty$.

Lemma 2.3 ([15, Theorem 2.1.6])**.** *Let* $1 < p \leq \infty$ *and* $f \in L^p(\mathbb{R}^n)$. *Then the estimate*

$$\|Mf\|_{L^p} \leq 3^{n/p} p' \|f\|_{L^p}$$

holds.

Lemma 2.4 (Fefferman-Stein[12][15, Theorem 1.2])**.** *Let* $(f_j)_{j\in\mathbb{Z}}$ *be a sequence of mesurable functions on* $\mathbb{R}^n$. *Let* $1 < p < \infty$ *and* $1 < q \leq \infty$. *Then the estimate*

$$\|(Mf_j)\|_{L^p(l^q_j)} \leq C_n \mu(p)\mu(q) \|(f_j)\|_{L^p(l^q_j)}$$

holds.

Lemma 2.5. *Let* s_1, s_2, s_3, s_4, s_5 *be non-negative real numbers satisfying* $s_1 + s_2 + s_3 = s_4 + s_5$ *and let* $1 < p, p_1, p_2 < \infty$ *satisfy* $1/p = 1/p_1 + 1/p_2$. *Then*

$$\left\| D^{s_1} \sum_{j\in\mathbb{Z}} P_j D^{s_2} f P_j D^{s_3} g \right\|_{L^p} \leq Cp\mu(p_1)\mu(p_2) \|f\|_{\dot{F}^{s_4}_{p_1,2}} \|g\|_{\dot{F}^{s_5}_{p_2,2}}.$$

Proof. By the Hölder and Fefferman-Stein inequalities, for any $h \in L^{p'}$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} D^{s_1} \sum_{j\in\mathbb{Z}} P_j D^{s_2} f(x) P_j D^{s_3} g(x) h(x) dx \right| \\
&= \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |(D^{s_1}\Psi_{j+2}(x-y) D^{s_2} P_j f(y) D^{s_3} P_j g(y) h(x)| dy dx \\
&\leq \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}} |D^{s_1}\Psi_{j+2}| * |h|(y) |D^{s_2} P_j f(y) D^{s_3} P_j g(y)| dy \\
&\leq Cp \left\| \|2^{s_4 j} M P_j f(y)\|_{l^2_j} \|2^{s_5 j} M P_j g\|_{l^2_j} \right\|_{L^p} \|h\|_{L^{p'}} \\
&\leq Cp\mu(p_1)\mu(p_2) \|h\|_{L^{p'}} \|f\|_{\dot{F}^{s_1}_{p_1,2}} \|g\|_{\dot{F}^{s_2}_{p_2,2}}.
\end{aligned}$$

□

Recall the definition of the Hörmander class S^s.

Definition 2. *Let* $s \in \mathbb{R}$. *We say that a symbol*

$$\sigma \in C^\infty(\mathbb{R}^n \setminus \{0\})$$

belongs to the Hörmander class S^s, *if for all multi-indices* α, *we have*

$$|\partial^\alpha_\xi \sigma(\xi)| \leq C_\alpha |\xi|^{s-|\alpha|}, \quad \forall \xi \neq 0.$$

Lemma 2.6. *Let* $s \geq 0$. *If* $a \in S^s$, *then for all multi-indices* α, β *and* (ξ, η) *in the cone* Γ, *defined in* (1.6), *we have*

$$\sup_{0\leq\theta\leq 1} |\partial^\alpha_\xi \partial^\beta_\eta a(\eta + \theta\xi)| \leq C_{\alpha+\beta} |\eta|^{s-|\alpha|-|\beta|}.$$

Proof. For $\alpha, \beta \in \mathbb{N}^n_0$,

$$\partial^\alpha_\xi \partial^\beta_\eta a(\eta + \theta\xi) = (\partial^{\alpha+\beta}_\eta a)(\eta + \theta\xi)\theta^{|\alpha|}.$$

and for $(\xi, \eta) \in \Gamma$,

$$\frac{1}{2}|\eta| \leq |\eta + \theta\xi| \leq \frac{3}{2}|\eta|.$$

The required estimate is established and the proof is complete. □

3. Proofs of Lemma 1.2, Theorem 1.1, and Corollaries 1.1 and 1.2.

Proof of Lemma 1.2. Since $|\cdot|^s \in S^s$ and Lemma 2.6, we are done. □

To prove Theorem 1.1, Corollaries 1.1 and 1.2, we introduce the following notation. For bilinear operator B, defined by

$$B(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} b(\xi,\eta)\widehat{F}(\xi)\widehat{G}(\eta)d\xi d\eta,$$

we can define

$$B_{\ll}(f,g) = \sum_{k\in\mathbb{Z}} B(P_{\leq k-3}f, P_k g), \quad B_{\sim}(f,g) = \sum_{j\in\mathbb{Z}}\sum_{k=j-2}^{j+2} B(P_j f, P_k g).$$

Obviously, we have the decomposition

$$B(f,g) = B_{\ll}(f,g) + B_{\sim}(f,g) + B_{\ll}(g,f) \tag{3.1}$$

and the symbol $b_{\ll}(\xi,\eta)$ of $B_{\ll}$ is defined by

$$b_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi}_{k-3}(\xi)\widehat{\Phi}_k(\eta)b(\xi,\eta). \tag{3.2}$$

We have the following useful property.

Lemma 3.1. *Let $s \geq k \geq 0$ and s_1, s_2 are non-negative real numbers satisfying*

$$s_1 \leq k, \ \ s_1 + s_2 = s$$

and let $1 < p, p_1, p_2 < \infty$ satisfy $1/p = 1/p_1 + 1/p_2$. Then the bilinear form $B_{\ll}(f,g)$ with symbol of type (3.2) with b in the Coifman - Meyer class satisfies

$$\sup_{|\alpha|=k} \|B_{\ll}(\partial^\alpha f, D^{s-k}g)\|_{L^p} \leq C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}. \tag{3.3}$$

The proof follows from the Coifman - Meyer estimate (1.7) and we skip it.
Lemma 2.6 implies:

Lemma 3.2. *Let $s \geq 0$. If $a \in S^s(\mathbb{R}^n)$, then with $a^s(\xi,\eta,\theta) = |\eta+\theta\xi|^s$ we have*

$$\sup_{0\leq\theta\leq 1} |\partial_\xi^\alpha \partial_\eta^\beta a^s_{\ll}(\xi,\eta,\theta)| \leq C|\eta|^{s-|\alpha|-|\beta|}.$$

Another useful application of the Coifman - Meyer estimate (1.7) concerns the bilinear form

$$B(f,g) = D^{s_1}(\partial^\alpha f D^{s_2} g). \tag{3.4}$$

Lemma 3.3. *Let α be a multi-index and s_1, s_2, s_3, s_4 be non-negative numbers satisfying*

$$s_1 + |\alpha| + s_2 = s_3 + s_4, \ \ s_3 \leq |\alpha|$$

and let $1 < p, p_1, p_2 < \infty$ satisfy $1/p = 1/p_1 + 1/p_2$. Then the bilinear form (3.4) satisfies

$$\|B_{\ll}(f,g)\|_{L^p} \leq C\|D^{s_3}f\|_{L^{p_1}}\|D^{s_4}g\|_{L^{p_2}}. \tag{3.5}$$

Proof. By Lemma 3.2 and Coifman - Meyer estimate (1.7)

$$\|B_{\ll}(f,g)\|_{L^p} \leq C\|\partial^\alpha f\|_{L^{p_1}}\|D^{s_1+s_2}g\|_{L^{p_2}} \leq C\|D^{|\alpha|}f\|_{L^{p_1}}\|D^{s_1+s_2}g\|_{L^{p_2}}$$

and

$$\|B_{\ll}(f,g)\|_{L^p} \leq C\|f\|_{L^{p_1}}\|D^{s_1+s_2+|\alpha|}g\|_{L^{p_2}}.$$

By interpolating these two estimate, we obtain (3.5). □

Proof of Theorem 1.1. Consider the bilinear form $B(f,g) = D^s(fg)$. We have the decomposition (3.1). For the term $B_\sim(f,g)$ we can apply the estimate of Lemma 2.5. Therefore, it is sufficient to show that

$$B_\ll(f,g) = \sum_{m=0}^{\ell-1} A^m_{s,\ll}(0)(f,g) + \sum_{|\alpha|=\ell} T^\alpha_\ll(\partial^\alpha f, D^{s-\ell}g), \tag{3.6}$$

where $T^\alpha_\ll$ is a Coifman-Meyer bilinear form

$$T^\alpha_\ll(F,G)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{ix(\xi+\eta)} t^\alpha_\ll(\xi,\eta)\widehat{F}(\xi)\widehat{G}(\eta)d\xi d\eta$$

with symbol $t^\alpha_\ll(\xi,\eta)$ in the CM class supported in $\{|\xi| \le |\eta|/2\}$, so it satisfies the estimate

$$\|T^\alpha_\ll(F,G)\|_{L^p} \le C_{p,p_1,p_2}\|F\|_{L^{p_1}}\|G\|_{L^{p_2}} \tag{3.7}$$

for all $1 < p, p_1, p_2 < \infty$ with $1/p = 1/p_1 + 1/p_2$.

We can use the Taylor expansion with respect to θ:

$$a_s(\xi,\eta,1) = \sum_{m=0}^{\ell-1} \frac{1}{m!}\partial^m_\theta a_s(\xi,\eta,0) + \frac{1}{(\ell-1)!}\int_0^1 (1-\theta)^{\ell-1}\partial^\ell_\theta a_s(\xi,\eta,\theta)d\theta$$

and note that (1.9) implies

$$B_\ll(f,g) = A^0_{s,\ll}(1)(f,g)$$

so the Taylor expansion for $a_s(\xi,\eta,1)$ implies (3.6) with symbol

$$t^\alpha_\ll(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi}_{k-3}(\xi)\widehat{\Phi}_k(\eta)\int_0^1 (1-\theta)^{|\alpha|-1}\theta^{|\alpha|}\partial^\alpha_\eta a_s(\xi,\eta,\theta)\frac{d\theta}{(|\alpha|-1)!}|\eta|^{-s+|\alpha|}.$$

An application of Lemma 3.2 shows that $t^\alpha_\ll(\xi,\eta)$ belongs to the CM class so the Coifman-Meyer estimate proves (3.7) and completes the proof of the theorem. □

3.1. **Proof of Corollary 1.1.** Let $B(f,g) = D^s(fg) - fD^sg - gD^sf$. Then the term $B_\sim(f,g)$ can be estimated by Lemma 2.5. So it is sufficient to check the estimate

$$\|B_\ll(f,g)\|_{L^p} \le C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}. \tag{3.8}$$

The term $B_\ll(f,g)$ can be represented as

$$B_\ll(f,g) = B^I_\ll(f,g) + B^{II}_\ll(f,g),$$

where

$$B^I(f,g) = D^s(fg) - fD^sg$$

and

$$B^{II}(f,g) = -gD^sf.$$

The symbol of

$$B^I_\ll(f,g) = A^0_s(1)(P_{\le k-3}f, P_kg) - A^0_s(0)(P_{\le k-3}f, P_kg),$$

can be represented as by the aid of the Taylor expansion

$$a_s(\xi,\eta,1) - a_s(\xi,\eta,0) = \int_0^1 \partial_\theta a_s(\xi,\eta,\theta)d\theta$$

so as in (3.6) we have

$$B^I_\ll(f,g) = \sum_{|\alpha|=1} T^\alpha_\ll(\partial^\alpha f, D^{s-1}g)$$

with symbol

$$t^\alpha_\ll(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi}_{k-3}(\xi)\widehat{\Phi}_k(\eta)\int_0^1 \theta\partial^\alpha_\eta a(\xi,\eta,\theta)d\theta|\eta|^{-s+1}$$

in the CM class. Applying Lemma 3.1, we get

$$\|B^I_\ll(f,g)\|_{L^p} \le C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}.$$

The term $B^{II}_{\ll}(f,g)$ can be estimated by the aid of Lemma 3.1 again, so we get (3.8) and the proof is complete.

3.2. **Proof of Corollary 1.2.** Let $B(f,g) = D^s(fg) - fD^sg - gD^sf + sD^{s-2}(\nabla f \cdot \nabla g)$. The term $B_{\sim}(f,g)$ can be estimated by using Lemma 2.5. As in the proof of Corollary 1.1, it is sufficient to show

$$\|B_{\ll}(f,g)\|_{L^p} \le C\|D^{s_1}f\|_{L^{p_1}}\|D^{s_2}g\|_{L^{p_2}}. \tag{3.9}$$

The term $B_{\ll}(f,g)$ can be represented as follows

$$B_{\ll}(f,g) = B^I_{\ll}(f,g) + B^{II}_{\ll}(f,g) + B^{III}_{\ll}(f,g),$$

where

$$\begin{aligned} B^I(f,g) &= D^s(fg) - fD^sg + s\nabla f \cdot D^{s-2}\nabla g, \\ B^{II}(f,g) &= sD^{s-2}(\nabla f \cdot \nabla g) - s\nabla f \cdot D^{s-2}\nabla g, \\ B^{III}(f,g) &= -gD^sf. \end{aligned}$$

Then

$$B^I_{\ll}(f,g) = A^0_{s,\ll}(1)(f,g) - A^0_{s,\ll}(0)(f,g) - A^1_{s,\ll}(0)(f,g) = \sum_{|\alpha|=2} T^{\alpha}_{\ll}(\partial^\alpha f, D^{s-2}g),$$

$$\begin{aligned} B^{II}_{\ll}(f,g) &= \sum_{m=1}^{n} s\{A^0_{s-2,\ll}(1)(\partial_m f, \partial_m g) - A^0_{s-2,\ll}(0)(\partial_m f, \partial_m g)\} \\ &= \sum_{m=1}^{n}\sum_{|\alpha|=1} s\widetilde{T}^{\alpha}_{\ll}(\partial^\alpha\partial_m f, D^{s-3}\partial_m g) \end{aligned}$$

with symbol

$$t^{\alpha}_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi}_{k-3}(\xi)\widehat{\Phi}_k(\eta)\int_0^1 (1-\theta)\theta^2\partial^\alpha_\eta a_s(\xi,\eta,\theta)d\theta|\eta|^{-s+2},$$

$$\widetilde{t}^{\alpha}_{\ll}(\xi,\eta) = \sum_{k\in\mathbb{Z}} \widehat{\Psi}_{k-3}(\xi)\widehat{\Phi}_k(\eta)\int_0^1 \theta\partial^\alpha_\eta a_{s-2}(\xi,\eta,\theta)d\theta|\eta|^{-s+3}$$

in the CM class. Applying Lemma 3.1, we can estimate $B^I_{\ll}(f,g)$, $B^{II}_{\ll}(f,g)$ and $B^{III}_{\ll}(f,g)$ and deduce (3.9).

This completes the proof.

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