OVERCONVERGENT EICHLER-SHIMURA ISOMORPHISMS FOR QUATERNIONIC MODULAR FORMS OVER $\mathbb{Q}$

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Abstract. In this work we construct overconvergent Eichler-Shimura isomorphisms over Shimura curves over $\mathbb{Q}$. More precisely, for a prime $p > 3$ and a wide open disk U in the weight space, we construct a Hecke-Galois-equivariant morphism from the space of families of overconvergent modular symbols over U to the space of families of overconvergent modular forms over U. In addition, for all but finitely many weights $\lambda \in U$, this morphism provides a description of the finite slope part of the space of overconvergent modular symbols of weight λ in terms of the finite slope part of the space of overconvergent modular forms of weight $\lambda + 2$. Moreover, for classical weights these overconvergent isomorphisms are compatible with the classical Eichler-Shimura isomorphism.

Introduction

The classical Eichler-Shimura isomorphism describes the space of weight $k \in \mathbb{N}$ modular symbols in terms of elliptic modular forms of weight $k+2$. Faltings in [12] gave an arithmetic version of this isomorphism.

In [9] and [10] the authors show that modular eigenforms of finite slope can be p-adically interpolated, in fact there exists a geometric object parametrizing such modular eigenforms called the *eigencurve*. On the other hand, modular symbols have interesting p-adic properties. In fact, Stevens in [19] was able to define overconvergent modular symbols and showed that classical modular symbols can be interpolated in p-adic families.

A natural question one could raise is if Faltings' Eichler-Shimura isomorphorphism could be p-adically interpolated in the weight variable. In [2] the authors answer affirmatively to this question. More precisely, they give a description of the finite slope part of p-adic families of overconvergent modular symbols in terms of the finite slope part of p-adic families of overconvergent modular forms. We can think about this result as a comparison between two different approaches to construct eigenvarieties: one using the theory of p-adic and overconvergent modular eigenforms, and the other using cohomology of arithmetic groups (overconvergent modular eigensymbols).

In the present work we carry out the same study as in [2] in the context of Shimura curves over $\mathbb{Q}$. Let $p > 3$ be a rational prime. Let B be an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $\delta > 1$ and suppose that $p \nmid \delta$. Let $N \geq 5$ be such that $(N, p\delta) = 1$. Let L be a finite extension of $\mathbb{Q}_p$ and denote by G_L the absolute Galois group of L. Let $M(N,p)$ be the Shimura curve of level $H(N,p)$ (see 1.1) defined over L. If $w > 0$ is a rational number such that $w < \frac{p}{p+1}$ and there exists an element $p^w \in L$ of valuation w, we

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denote by $M(w)$ the strict neighbourhood of width p^w of the connected component of the ordinary locus of $M(N,p)^{\mathrm{an}}$ where the canonical subgroup coincides with the level subgroup of order p.

The theory of p-adic modular forms attached to B was developed by Kassaei (see [15]) for integral weights. In [5] Brasca extended this theory to all weight in the weight space and constructed the relevant eigencurve. More precisely for every weight λ Brasca constructed an invertible sheaf ω_w^λ on $M(w)$ (for w and L depending on λ) such that if $\lambda = k \in \mathbb{Z}$ then ω_w^k is the restriction to $M(w)$ of the appropriate weight k classical modular sheaf. Moreover, if U is a wide open disk in the weight space and λ_U its universal weight, then Brasca constructed a sheaf $\omega_w^{\lambda_U}$ (with w and L depending on U) interpolating the sheaves ω_w^λ for every $\lambda \in U(L)$. The elements of $\mathrm{H}^0(M(w), \omega_w^\lambda)$ are called *overconvergent modular forms* of weight λ and degree of overconvergence w and those of $\mathrm{H}^0(M(w), \omega_w^{\lambda_U})$ are p-adic families of overconvergent modular forms over U.

Let Γ be a fundamental group of $M(N,p)(\mathbb{C})$ (here we need to fix an embedding of $\overline{L}$ in $\mathbb{C}$). Let $r \in \mathbb{N}$ be such that if $\lambda \in U(L)$ then λ is r-admissible (see notations below). For each $\lambda \in U(L)$ let D_λ be the L-Banach space of the r-locally analytic distributions on $\mathbb{Z}_p$ endowed with an action of Γ depending on λ (see section 3). Moreover, let D_U be the $\Lambda_U \otimes_{\mathcal{O}_L} L$-Banach module of the r-locally analytic distributions on $\mathbb{Z}_p$ with values in $\Lambda_U \otimes_{\mathcal{O}_L} L$, here Λ_U is the algebra of bounded by 1 rigid analytic functions on U. There is a natural action of Γ on D_U. In the context of Shimura curves over $\mathbb{Q}$, $\mathrm{H}^1(\Gamma, D_\lambda)$ is the space of overconvergent modular symbols of weight λ and $\mathrm{H}^1(\Gamma, D_U)$ that of p-adic families of overconvergent modular symbols over U.

Let $h > 0$ and suppose that U satisfies:

1) there exists an integer $k_0 \in U(L)$ such that $k_0 > h-1$;

2) both $\mathrm{H}^1(\Gamma, D_U)$ and $\mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})$ have slope $\le h$ decompositions (see remark 5.1).

The main result of this paper is:

Theorem 0.1. *There exists a finite subset of weights $Z \subset U(\mathbb{C}_p)$ such that:*

(a) For each $\lambda \in U(L) - Z$ there is a finite dimensional $\mathbb{C}_p$-vector space $S_\lambda^{\le h}$ endowed with an action of the Hecke operators and trivial semilinear G_L-action, such that we have a natural Hecke and G_L-equivariant isomorphism

$$\mathrm{H}^1(\Gamma, D_\lambda)^{\le h} \otimes_L \mathbb{C}_p(1) \cong \Big(\mathrm{H}^0(M(w), \omega_w^{\lambda+2})^{\le h} \otimes_L \mathbb{C}_p\Big) \oplus \Big(S_\lambda^{\le h}(\lambda+1)\Big).$$

(b) For every wide open disk $V \subset U$ defined over L such that $V(\mathbb{C}_p) \cap Z = \emptyset$ there is a finite free Hecke $\Lambda_V[1/p]\hat{\otimes}_L \mathbb{C}_p$-module, $S_V^{\le h}$, with trivial semilinear action of G_L and a Hecke and G_L-equivariant exact sequence:

$$0 \longrightarrow S_V^{\le h}(\chi \cdot \chi_V^{univ}) \longrightarrow \mathrm{H}^1(\Gamma, D_V)^{\le h} \hat{\otimes}_L \mathbb{C}_p(1) \xrightarrow{\Psi_V^{\le h}} \mathrm{H}^0(M(w), \omega_w^{\lambda_V+2})^{\le h} \hat{\otimes}_L \mathbb{C}_p \longrightarrow 0.$$

Moreover there exists a non-zero $b \in \Lambda_V \otimes_{\mathcal{O}_L} L$ such that the last exact sequence localized at b splits canonically and uniquely as a sequence of G_L-modules.

This article follows the same general line of arguments as in [2] with the following differences:

a) We work with all the weights in $\mathcal{W}$ not only the accessible ones.

b) Working on Shimura curves instead of modular curves, simplifies some problems and complicates others. Namely, the non-existence of cusps simplifies the log structures on Faltings' sites. On the other hand, the universal abelian scheme over the Shimura scheme has relative dimension 2 and one has to use the quaternionic multiplication in order to obtain objects (Tate modules, sheaves of differentials, canonical subgroups, etc.) of the right size.

In the recent article [8] the authors have obtained results analogue to ours but their approach is different. They used the perfectoid Shimura curve associated to the tower of Shimura curves of variable p-power level, the Hodge-Tate period morphism and the pro-étale site. Instead, we work with Faltings' sites attached to formal Shimura curves of finite level in order to compare p-adic families of overconvergent modular symbols and p-adic families of modular eigenforms

The structure of this paper is as follows. In section 1 we introduce the Faltings' sites attached to the Shimura curves. Section 2 is devoted to the classical Eichler-Shimura isomorphism in the context of Shimura curves. In section 3 we introduce the spaces of overconvergent modular symbols. Section 4 is the technical part of this work, we define modular sheaves on Faltings' sites and we construct the map from overconvergent modular symbols to overconvergent modular forms. Finally in section 5 we prove our main theorem.

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Notations. Let $p > 3$ be a prime integer and L a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{L}$. We fix $\pi \in \mathcal{O}_L$ a uniformizer. We denote by $\overline{L}$ an algebraic closure of L and write G_L for the Galois group of $\overline{L}/L$. We also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{L}$, where $\overline{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. We denote by $\mathbb{C}_p$ the completion of $\overline{L}$, by v the valuation of $\mathbb{C}_p$ normalized such that $v(p) = 1$ and let $|\cdot|$ denote the absolute value of $\mathbb{C}_p$ defined by $|x| = p^{-v(x)}$ for any $x \in \mathbb{C}_p$.

We denote by $\mathcal{W}$ the weight space, i.e., the rigid analytic space associated to the complete noetherian semilocal algebra $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$. For any affinoid algebra A, we have $\mathcal{W}(A) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, A^\times)$. We embed $\mathbb{Z}$ in $\mathcal{W}$ attaching to any $k \in \mathbb{Z}$ the weight $t \to t^k$. We say that $\lambda \in \mathcal{W}(L)$ is *r-admissible* if the map $\mathbb{Z}_p \to L$ given by $z \to \lambda(1 + p^{r+1}z)$ is analytic. We can check that λ is r-admissible if

$$|\lambda(1+p) - 1| < p^{-\frac{1}{p^{r-1}(p+1)}}.$$

We fix an admissible covering, $\{\mathcal{W}_r\}_{r\geq 1}$, of $\mathcal{W}$ as defined in [5, §5]. Remark that any $\lambda \in \mathcal{W}_r(L)$ is r-admissible.

1. Faltings' sites and topoi

1.1. **The geometric set-up.** Let $N \geq 5$ be a positive integer not divisible by p. Let B denote an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant δ satisfying $(pN, \delta) = 1$ and let $\mathcal{O}_B \subseteq B$ denote a maximal order. We fix isomorphisms $B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{R})$ and $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_l \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Z}_l)$ for all primes $l \nmid \delta$. Note that we obtain an isomorphism $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{Z}/N\mathbb{Z})$. Let $\cdot' : B \to B$ denote the canonical involution of B. Since $\mathbb{Q}(\sqrt{-\delta})$ splits B there is an element $t \in B$ such that $t^2 + \delta = 0$ and let $\cdot^* : B \to B$ denote the map $x \mapsto t^{-1}x't$ which defines another involution of B.

Let m be a positive integer such that $(m, \delta) = 1$ then we have a group homomorphism:

$$\beta_m : (\mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times \longrightarrow GL_2(\mathbb{Z}/m\mathbb{Z}).$$

We denote by V_m the kernel of β_m and let $V_1(m) := \beta_m^{-1}\left(\left\{\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)\right\}\right)$. Let H be a compact open subgroup of $(\mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$ satisfying the following conditions:

- $(N_H, \delta) = 1$, where N_H is the minimal positive integer m such that $V_m \subseteq H$;
- $H \subseteq V_1(m)$ for some integer $m \geq 4$ such that $(m, \delta) = 1$;
- $det(H) = \hat{\mathbb{Z}}^\times$.

In particular, we may take $H = V_1(N)$ for $N > 4$ we fixed before.

Let S be any scheme over $\mathbb{Z}_p$. A false elliptic curve over S is a pair (A, i), where A is an abelian surface over S and $i : \mathcal{O}_B \hookrightarrow \mathrm{End}_S(A)$ is an injection of rings with identity. It can be proved that there is a unique principal polarization on A such that for any geometric point $x \in S$, the corresponding Rosati involution on $\mathrm{End}(A_x)$ restricts to $\cdot^*$ on $\mathcal{O}_B$. This means that any false elliptic curve is canonically principally polarized for fixed t. A full level N structure on a false elliptic curve (A, i) over S is an isomorphism of S-group schemes $\alpha : A[N] \xrightarrow{\sim} (\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z})_S$ compatible with left actions of $\mathcal{O}_B$. If H is as above, an H-level structure on (A, i) is a full level N_H structure defined up to right H-equivalence. Now we fix a compact open subgroup $H \subseteq (\mathcal{O}_B \otimes_{\mathbb{Z}} \hat{\mathbb{Z}})^\times$ of level N satisfying the above conditions, and denote it by $H(N)$. The functor $\mathbb{Z}_p$-schemes $\to$ **Sets** which sends S to the set of isomorphism classes of false elliptic curves with $H(N)$-level structure is represented by a geometrically connected scheme $\mathcal{M}(N)$ defined over $\mathbb{Z}_p$. Moreover, the map $\mathcal{M}(N) \to \mathrm{Spec}(\mathbb{Z}_p)$ is smooth, proper, and of relative dimension 1. The universal object will be denoted $\mathcal{A}(N) \to \mathcal{M}(N)$.

Let $\mathcal{C}$ be a pseudo-abelian category (see [7, §1.3]) and let X be an object of $\mathcal{C}$ with an action of $\mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Fixing a non trivial idempotent $e \in \mathcal{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong \mathrm{M}_2(\mathbb{Z}_p)$ such that $e^* = e$, we obtain a decomposition: $X = X^1 \oplus X^2$, where each X^i has an action of $\mathbb{Z}_p$, $i = 1, 2$.

Now, for any integer $r \geq 0$, we define $H(N, p^r) := H(N) \cap \beta_{p^r}^{-1}\left(\left\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\right\}\right)$ and $H(Np^r) := H(N) \cap V_{p^r}$. It is easy to see that they satisfy the above conditions. There is a regular scheme $\mathcal{M}(N, p^r)$ (resp. $\mathcal{M}(Np^r)$) that represents the functor $\mathbb{Z}_p-$schemes $\to$ **Sets** that sends S to the set of isomorphism classes of quadruples (A, i, α, C) (resp. (A, i, α, P)), where (A, i) is a false elliptic curve over S, α is an $H(N)$-level structure on (A, i) and C is a finite flat subgroup of order p^r of $A[p^r]^1$ (resp. P is a point of exact order p^r of

$A[p^r]^1$ in the sense of Drinfeld). We have that $\mathcal{M}(N,p^r) \to \mathrm{Spec}(\mathbb{Z}_p)$ is proper, and of relative dimension 1, generically smooth but not smooth. If $r=1$ the special fiber is a normal crossing divisor with two components that intersect at the supersingular points (to the definition of supersingulr look at [5, §2]). Moreover, $\mathcal{M}(Np^r) \to \mathrm{Spec}(\mathbb{Z}_p)$ is proper, generically smooth, and of relative dimension 1. The universal object will be denoted $\mathcal{A}(N,p^r) \to \mathcal{M}(N,p^r)$ (resp. $\mathcal{A}(Np^r) \to \mathcal{M}(Np^r)$).

Now let H be one of $H(N)$, $H(N,p^r)$ or $H(Np^r)$ and let $\mathcal{M}$ be the corresponding Shimura curve with universal false elliptic curve $\mathcal{A}$. If $e : \mathcal{M} \to \mathcal{A}$ is the zero section, we define $\underline{\omega} := \underline{\omega}_H := \left(e^*\Omega^1_{\mathcal{A}/\mathcal{M}}\right)^1$, which is an invertible sheaf on $\mathcal{M}$.

We fix an integer $r \geq 1$ and let w be a rational number such that $0 < w < \frac{1}{p^{r-2}(p+1)}$. Let L be a finite extension of $\mathbb{Q}_p$ such that there is an element (which will be denoted by p^w) in $\mathcal{O}_L$ whose valuation is w. Define

$$\mathcal{M}(N)(w) := \mathrm{Spec}_{\mathcal{M}(N)}(\mathrm{Sym}(\underline{\omega}^{\otimes(p-1)})/ < E_{p-1} - p^w >),$$

where E_{p-1} is a lift of the Hasse invariant (see [15, §5] for details). Note that $\mathcal{M}(N)(w)$ has a natural moduli interpretation. It classifies couples $\big((\mathcal{A},i,\alpha),Y\big)$, where $(\mathcal{A},i,\alpha)$ is a false elliptic curve with level structure $H(N)$ and Y is a global section of $\underline{\omega}^{\otimes(1-p)}$ such that $Y \cdot E_{p-1} = p^w$.

From now on to simplify the notation we continue to use $\mathcal{M}(N)$, $\mathcal{M}(N)(w)$, $\mathcal{M}(N,p^r)$ and $\mathcal{M}(Np^r)$ to denote the corresponding formal schemes over $\mathrm{Spf}(\mathcal{O}_L)$ which are the formal completions of the respective schemes along their special fibers. We denote by $M(N)$, $M(N)(w)$, $M(N,p^r)$, $M(Np^r)$ the rigid analytic generic fibers of the respective formal schemes. The rigid analytic generic fiber of the natural morphism $\mathcal{M}(N)(w) \to \mathcal{M}(N)$ is an open immersion $M(N)(w) \hookrightarrow M(N)$.

Remark 1.1. The above definition of $\mathcal{M}(N)(w)$ works well for any rational number $0 \leq w < 1$. The condition $0 < w < \frac{1}{p^{r-2}(p+1)}$ implies that $\mathcal{A}[p^r]^1$ has a canonical finite and flat subgroup scheme of rank p^r, where $\mathcal{A}$ is an object of the moduli problem of $\mathcal{M}(N)(w)$. Then we have a morphism $\mathcal{M}(N)(w) \to \mathcal{M}(N,p^r)$ defined by the canonical subgroup. Moreover its rigid analytic generic fiber is a section of the morphism $M(N,p^r) \to M(N)$ over $M(N)(w)$. We also denote by $M(N)(w)$ the image of this section.

Now let $M^r(w)$ be the preimage of $M(N)(w)$ under the map $M(Np^r) \to M(N,p^r)$, i.e., $M^r(w) := M(Np^r) \times_{M(N,p^r)} M(N)(w)$. The natural morphism $M^r(w) \to M(N)(w)$ is finite and étale. We consider $M(w)$ (resp. $M^r(w)$) as connected affinoid subdomains of $M(N)$ and $M(N,p^r)$ (resp. $M(Np^r)$). Then we define $\mathcal{M}^r(w)$ to be the normalization of $\mathcal{M}(N)(w)$ in $M^r(w)$, which is a flat and normal formal scheme over $\mathrm{Spf}(\mathcal{O}_L)$.

Remark 1.2. For any normal, flat, and p-adically complete $\mathcal{O}_L$-algebra, there is a natural bijection between $\mathcal{M}^r(w)(R)$ and the set of isomorphism classes of couples $\big((\mathcal{A}/R,i,\alpha,P),Y\big)$, where $(\mathcal{A}/R,i,\alpha,P)$ is an object of the moduli problem of $\mathcal{M}(Np^r)$ where P generates the canonical subgroup of $\mathcal{A}[p^r]^1$ and Y is a section of $\underline{\omega}^{\otimes(1-p)}_{\mathcal{A}/R}$ satisfying $Y \cdot E_{p-1} = p^w$ (see [5, Prop. 2.2]).

We have the following natural commutative diagram of formal schemes and rigid analytic spaces:

$$\begin{array}{ccccc}
M(Np^r) & \longrightarrow & M(N,p^r) & \longrightarrow & M(N) \\
\uparrow & & \uparrow & & \uparrow \\
M^r(w) & \longrightarrow & M(w) & = & M(w) \\
{\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow & & {\scriptstyle u}\downarrow \\
\mathcal{M}^r(w) & \longrightarrow & \mathcal{M}(w) & = & \mathcal{M}(w) \\
 & & \downarrow & & \downarrow \\
\mathcal{M}(Np^r) & \longrightarrow & \mathcal{M}(N,p^r) & \longrightarrow & \mathcal{M}(N).
\end{array}$$

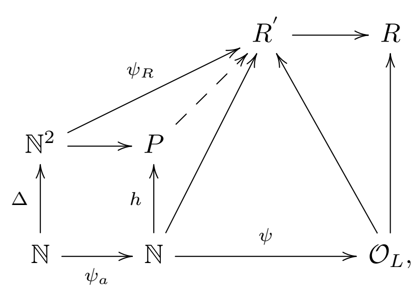
1.2. **Log structures.** We fix N, r and w as before and denote by $\underline{\mathcal{M}}(w)$, $\underline{\mathcal{M}}(N,p)$ the formal schemes denoted by $\mathcal{M}(w)$, $\mathcal{M}(N,p)$ in the previous section. We know that $\underline{\mathcal{M}}(w)$ has an open covering by affines of the type $\mathcal{U} = \mathrm{Spf}(R)$ such that:

- R is a p-adically complete $\mathcal{O}_L$-algebra;
- $\mathrm{Spec}(R)$ is connected, i.e., R has no nontrivial idempotents;
- there is a formally étale morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R')$, where $R' := \mathcal{O}_L\{X,Y\}/(XY - \pi^a)$ and $a \in \mathbb{N}$.

Such affine open is called a *small affine*.

Let $\underline{S} := \mathrm{Spf}(\mathcal{O}_L)$ and $S := (\underline{S}, M)$ be the associated log formal scheme where M is the log structure on $\underline{S}$ defined by its closed point. It has a local chart given by $\psi : \mathbb{N} \to \mathcal{O}_L$ sending 1 to π.

Now take a small open affine $\mathcal{U} = \mathrm{Spf}(R) \hookrightarrow \mathcal{M}(w)$. Let $P := \mathbb{N}^2 \oplus_{\mathbb{N}} \mathbb{N}$ the amalgamated sum of the morphisms $\Delta : \mathbb{N} \to \mathbb{N}^2$, $n \to (n,n)$ and $\psi_a : \mathbb{N} \to \mathbb{N}$, $n \to an$. Then we have the following commutative diagram of monoids:



where $\psi_R(m,n) = X^m Y^n$ and $\mathcal{O}_L$, R, R' are the multiplicative monoids associated to the respective rings. Let $N_{\mathcal{U}}$ be the log structure on $\mathcal{U}$ associated to the prelog structure given by the composition $P \dashrightarrow R' \to R$. Moreover the morphism $h : \mathbb{N} \to P$ is a local chart for the morphism $f : \mathcal{M}(w) := (\underline{\mathcal{M}}(w), N) \to S$. Actually $\mathcal{M}(w)$ is a fine saturated (we use the short hand notation fs) log scheme and $f : \mathcal{M}(w) \to S$ is log smooth (see [2, §2.2] for details).

Now let N_r be the inverse image log structure on $\underline{\mathcal{M}}^r(w)$ via the morphism $\underline{\mathcal{M}}^r(w) \to \underline{\mathcal{M}}(w)$ and denote by $\mathcal{M}^r(w) := (\underline{\mathcal{M}}^r(w), N_r)$ the log formal scheme. Let $M(w)$, $M^r(w)$ be the log rigid generic fibres of the log formal schemes $\mathcal{M}(w)$ and $\mathcal{M}^r(w)$ respectively. Both these rigid analytic generic fibers have trivial log structures.

Moreover let $\mathcal{M}(N,p)$ be the log formal scheme whose underlying formal scheme is $\underline{\mathcal{M}}(N,p)$ and the log structure is defined by its special fibre which is a divisor with normal crossings. Give $\underline{M}(N,p) := \underline{\mathcal{M}}(N,p)^{\mathtt{rig}}$ the trivial log structure and denote by $M(N,p)$ the corresponding log rigid space. We have the following commutative diagram of log formal schemes and log rigid spaces:

$$\begin{array}{ccc} \mathcal{M}(w) & \xrightarrow{\nu} & \mathcal{M}(N,p) \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle u} \\ M(w) & \hookrightarrow & M(N,p). \end{array}$$

1.3. **The site $\mathfrak{M}$ and $\mathfrak{M}^r$.** Let $(\mathcal{M}, M)$ be one of the pairs $(\mathcal{M}(w), M(w))$ or $(\mathcal{M}(N,p), M(N,p))$. We will define Faltings' site associated to this pair and denote it by $\mathfrak{M}$. First let $\mathcal{M}^{\mathtt{ket}}$ be the Kummer étale site of $\mathcal{M}$. Then for any object $\mathcal{U}$ in $\mathcal{M}^{\mathtt{ket}}$ we denote by $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$ the finite étale site attached to $\mathcal{U}$ over $\overline{L}$ as defined in [1, §2.2]. More explicitly an object in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$ is a pair (W,K) where K is a finite extension of L contained in $\overline{L}$ and W is an object of the finite étale site of $\mathcal{U}_K$, which is denoted by $\mathcal{U}_K^{\mathtt{fet}}$. Given two objects (W,K) and (W',K') in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$, the morphisms between them are defined to be:

$$\mathrm{Hom}_{\mathcal{U}_{\overline{L}}^{\mathtt{fet}}}\Big((W,K),(W',K')\Big) := \varinjlim \mathrm{Hom}_{\mathcal{U}_{K''}}\Big(W\otimes_K K'', W'\otimes_{K'} K''\Big),$$

where the direct limit is over all finite extensions K'' of L contained in $\overline{L}$ and containing both K and K'.

Now we define the category $E_{\mathcal{M}_{\overline{L}}}$ as follows:

(i) the objects consist of pairs $\big(\mathcal{U},(W,K)\big)$ such that $\mathcal{U}\in\mathcal{M}^{\mathtt{ket}}$ and $(W,K)\in\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$;

(ii) a morphism $\big(\mathcal{U},(W,K)\big)\to\big(\mathcal{U}',(W',K')\big)$ in $E_{\mathcal{M}_{\overline{L}}}$ is a pair of morphisms (α,β), where $\alpha:\mathcal{U}\to\mathcal{U}'$ is a morphism in $\mathcal{M}^{\mathtt{ket}}$ and $\beta:(W,K)\to(W'\times_{\mathcal{U}'_{K'}}\mathcal{U}_{K'},K')$ is a morphism in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$.

Remark 1.3. A good reference for the Kummer étale site $\mathcal{M}^{\mathtt{ket}}$ is [14]. In particular the definition of a Kummer étale morphism is given in subsection 1.6 of *op. cit.*.

Remark 1.4. The pair $\big(\mathcal{M},(M,K)\big)$ is a final object in the category $E_{\mathcal{M}_{\overline{L}}}$ and fiber products exist in this category (see [11, Prop. 2.6] for an explicit description of the fiber product). For convenience for the rest of this paper we write $(\mathcal{U},W)$ instead of $\big(\mathcal{U},(W,K)\big)$ for an object in $E_{\mathcal{M}_{\overline{L}}}$, whenever this does not create confusion.

A family of morphisms $\Big\{(\mathcal{U}_i,W_i)\to(\mathcal{U},W)\Big\}_{i\in I}$ is a covering family in $E_{\mathcal{M}_{\overline{L}}}$ if it is in one of the following two cases:

(a) $\{\mathcal{U}_i\to\mathcal{U}\}_{i\in I}$ is a covering in $\mathcal{M}^{\mathtt{ket}}$ and $W_i\cong W\times_{\mathcal{U}_L}\mathcal{U}_{i,L}$ for each $i\in I$;

or

(b) $\mathcal{U}_i\to\mathcal{U}$ is an isomorphism for each $i\in I$ and $\{W_i\to W\}_{i\in I}$ is a covering family in $\mathcal{U}_{\overline{L}}^{\mathtt{fet}}$.

We denote by $\mathfrak{M}$ the site associated to the category $E_{\mathcal{M}_{\overline{L}}}$ endowed with the topology generated by the covering families defined above. We will denote by $\mathfrak{M}(w)$, resp. $\mathfrak{M}(N,p)$ the Faltings' sites and topoi associated to the pairs $(\mathcal{M}(w), M(w))$, resp. $(\mathcal{M}(N,p), M(N,p))$.

Now let $\mathfrak{M}$ be either the site $\mathfrak{M}(w)$ or $\mathfrak{M}(N,p)$. We will describe two important sheaves on this site. We denote by $\mathcal{O}_{\mathfrak{M}}$ the presheaf of $\mathcal{O}_{\overline{L}}$-algebras on $\mathfrak{M}$ defined as:

$$\mathcal{O}_{\mathfrak{M}}(\mathcal{U}, W) := \texttt{the normalization of } \Gamma(\underline{\mathcal{U}}, \mathcal{O}_{\underline{\mathcal{U}}}) \texttt{ in } \Gamma(\underline{W}, \mathcal{O}_{\underline{W}}).$$

We also define the sub-presheaf of $\mathbb{W}(\mathbb{L})$-algebras $\mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}$ of $\mathcal{O}_{\mathfrak{M}}$ whose sections over $(\mathcal{U}, W)$ consist of elements $x \in \mathcal{O}_{\mathfrak{M}}(\mathcal{U}, W)$ for which there exist a finite unramified extension M of L contained in $\overline{L}$, a Kummer log étale morphism $\mathcal{V} \to \mathcal{U} \times_{\mathcal{O}_L} \mathcal{O}_M$ and a morphism $W \to \mathcal{V}_L$ over $\mathcal{U}_L$ such that x, viewed as an element of $\Gamma(\underline{W}, \mathcal{O}_{\underline{W}})$, lies in the image of $\Gamma(\underline{\mathcal{V}}, \mathcal{O}_{\underline{\mathcal{V}}})$.

Actually these presheaves are sheaves (see [1, §2.2] for details). We then denote by $\widehat{\mathcal{O}_{\mathfrak{M}}}$ and $\widehat{\mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}}$ the continuous sheaves on $\mathfrak{M}$ defined by the projective systems $\{\mathcal{O}_{\mathfrak{M}}/p^n\mathcal{O}_{\mathfrak{M}}\}_{n\geq 0}$ and $\{\mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}/p^n\mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}\}_{n\geq 0}$ respectively.

Finally, we define the site $\mathfrak{M}^r(w)$ and $\mathfrak{M}(Np^r)$ to be the induced sites

$$\mathfrak{M}(w)_{/(\mathcal{M}(w), M^r(w))} \text{ and } \mathfrak{M}(N,p)_{/(\mathcal{M}(N,p), M(Np^r))}$$

respectively (see [2, §2.4] about induced sites).

1.4. **Continuous functors.** Now we have defined several sites, namely $\mathcal{M}^{\mathtt{ket}}(w)$, $\mathcal{M}^{\mathtt{ket}}(N,p^r)$, $M_{\overline{L}}^{\mathtt{ket}}$, $\mathfrak{M}(w)$, $\mathfrak{M}(N,p^r)$ and $\mathfrak{M}^r(w)$. We have the following natural functors which send covering families to covering families, commute with fibre products and send final objects to final objects. In particular they induce morphisms of topoi.

(1) $\mu : \mathcal{M}^{\mathtt{ket}}(N,p^r) \longrightarrow \mathcal{M}^{\mathtt{ket}}(w)$ induced by the natural morphism of formal log schemes $\mathcal{M}(w) \hookrightarrow \mathcal{M}(N,p^r)$.
(2) $\nu : \mathfrak{M}(N,p) \longrightarrow \mathfrak{M}(w)$ with $\nu\big((\mathcal{U}, W)\big) := \big(\mathcal{U} \times_{\mathcal{M}(N,p)} \mathcal{M}(w), W \times_{M(N,p)} M(w)\big)$.
(3) $v_{\mathfrak{M}} : \mathcal{M}^{\mathtt{ket}} \longrightarrow \mathfrak{M}$ with $v_{\mathfrak{M}}(\mathcal{U}) := (\mathcal{U}, \mathcal{U}_K)$, where $\mathcal{M}$ be either $\mathcal{M}(w)$ or $\mathcal{M}(N,p)$ and $\mathfrak{M}$ be either $\mathfrak{M}(w)$ or $\mathfrak{M}(N,p)$, respectively. Moreover, we have $v_{\mathfrak{M}}^*(\mathcal{O}_{\mathcal{M}^{\mathtt{ket}}}) \cong \mathcal{O}_{\mathfrak{M}}^{\mathtt{un}}$. We also have the following commutative diagram of sites:

$$\begin{array}{ccc} \mathcal{M}^{\mathtt{ket}}(N,p) & \xrightarrow{v_{\mathfrak{M}(N,p)}} & \mathfrak{M}(N,p) \\ \mu \downarrow & & \downarrow \nu \\ \mathcal{M}^{\mathtt{ket}}(w) & \xrightarrow[v_{\mathfrak{M}(w)}]{} & \mathfrak{M}(w). \end{array}$$

(4) $u : \mathfrak{M} \longrightarrow M_{\overline{L}}^{\mathtt{ket}}$ with $(\mathcal{U}, W) \longmapsto W$.
(5) $j_r : \mathfrak{M}(w) \longrightarrow \mathfrak{M}^r(w)$ sending $(\mathcal{U}, W) \mapsto \big(\mathcal{U}, W \times_{M(w)} M^r(w), \mathtt{pr}_2\big)$. This morphism of topoi has the following properties (see [2, §2.5] for proofs):
   (i) The functor $j_{r,*} : \mathrm{Sh}(\mathfrak{M}^r(w)) \to \mathrm{Sh}(\mathfrak{M}(w))$ is an exact functor.

(ii) $R^i j_{r,*} = 0$ for all $i \geq 1$.

(6) $v_r : \mathcal{M}^{\mathtt{ket}}(w) \longrightarrow \mathfrak{M}^r(w)$, which is defined to be the composite $v_r := j_r \circ v_{\mathfrak{M}(w)}$. Actually $v_r(\mathcal{U}) = \big(\mathcal{U}, \mathcal{U}_K \times_{M(w)} M^r(w), \mathtt{pr}_2\big)$ and $v_r(\mathcal{M}(w)) = (\mathcal{M}(w), M^r(w), \mathtt{id})$. Moreover, we have $R^i v_{r,*} = R^i v_* \circ j_{r,*}$.

We denote by $\mathcal{O}_{\mathfrak{M}^r(w)} := j_r^*(\mathcal{O}_{\mathfrak{M}(w)})$ and by $\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)} := j_r^*(\widehat{\mathcal{O}}_{\mathfrak{M}(w)})$. By the construction of $\mathcal{M}^r(w)$, we have natural isomorphisms of sheaves on $\mathcal{M}^{\mathtt{ket}}(w)$:

$$\Big(v_{r,*}(\mathcal{O}_{\mathfrak{M}^r(w)})\Big)^{G_r} \cong \mathcal{O}_{\underline{\mathcal{M}(w)}} \texttt{ and } \Big(v_{r,*}(\widehat{\mathcal{O}}_{\mathfrak{M}^r(w)})\Big)^{G_r} \cong \widehat{\mathcal{O}}_{\underline{\mathcal{M}(w)}},$$

where $G_r \cong (\mathbb{Z}/p^r\mathbb{Z})^\times$ is the Galois group of $M^r(w)/M(w)$.

(7) $\beta_{\mathcal{U}} : \mathcal{U}_{\overline{K}}^{\mathtt{fet}} \longrightarrow \mathfrak{M}$, for any object $\mathcal{U}$ in $\mathcal{M}^{\mathtt{ket}}$, sending $W \longmapsto (\mathcal{U}, W)$.

1.5. **The localization functor.** This section is a brief recall of [2, §2.7]. Now let $\mathfrak{M}$ be one of the sites $\mathfrak{M}(w)$ or $\mathfrak{M}(N,p)$ and $\mathcal{U} = \big(\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})\big)$ a connected small affine object of $\mathcal{M}^{\mathtt{ket}}$. We denote by $U := \mathcal{U}_L$ the log rigid generic fibre of $\mathcal{U}$. Write $R_{\mathcal{U}} \otimes \overline{L} = \prod_{i=1}^n R_{\mathcal{U},i}$ with $\mathrm{Spf}(R_{\mathcal{U},i})$ connected, let $N_{\mathcal{U},i}$ be the monoids giving the respective log structures and U_i the respective log rigid generic fibres. Let $\mathbb{C}_{\mathcal{U},i} := \overline{\mathrm{Frac}(R_{\mathcal{U},i})}$ and $\eta_{\mathcal{U},i}$ denote the log geometric point of $\mathcal{U}_i := \big(\mathrm{Spf}(R_{\mathcal{U},i}), N_{\mathcal{U},i}\big)$ over $\mathbb{C}_{\mathcal{U},i}$. Let $\mathcal{G}_{\mathcal{U},i}$ be the étale fundamental group of U_i. Then the category $U_i^{\mathtt{fet}}$ is equivalent to the category of finite sets with continuous actions of $\mathcal{G}_{\mathcal{U},i}$. Write $(\overline{R}_{\mathcal{U},i}, \overline{N}_{\mathcal{U},i})$ for the direct limit of all the normal extensions S of $R_{\mathcal{U},i}$ in $\mathbb{C}_{\mathcal{U},i}$ such that $\mathrm{Spm}(S_L) \to U_i$ is finite étale,. Finally, we let $\overline{R}_{\mathcal{U}} := \prod_{i=1}^n \overline{R}_{\mathcal{U},i}$, $\overline{N}_{\mathcal{U}} := \prod_{i=1}^n \overline{N}_{\mathcal{U},i}$ and $\mathcal{G}_{U_{\overline{L}}} := \prod_{i=1}^n \mathcal{G}_{\mathcal{U},i}$. Then we have an equivalence of categories,

$$\mathrm{Sh}(U_{\overline{L}}^{\mathtt{fet}}) \xrightarrow{\sim} \mathrm{Rep}(\mathcal{G}_{U_{\overline{K}}}), \qquad \mathcal{F} \longmapsto \varinjlim \mathcal{F}(\mathrm{Spm}(S_L)),$$

where $\mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$ is the category of discrete abelian groups with continuous $\mathcal{G}_{U_{\overline{L}}}$-action. Composing with $\beta_{\mathcal{U},*}$, we obtain a localization functor $\mathrm{Sh}(\mathfrak{M}) \longrightarrow \mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$ and we donote by $\mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}})$ the image of $\mathcal{F}$ in $\mathrm{Rep}(\mathcal{G}_{U_{\overline{L}}})$.

Let $\mathcal{F} \in \mathrm{Sh}(\mathfrak{M}^r(w))$ and fix $\mathcal{U} = \big(\mathrm{Spf}(R_{\mathcal{U}}, N_{\mathcal{U}})\big)$ a connected small affine object of $\mathcal{M}(w)^{\mathtt{ket}}$ as before. Let

$$\Upsilon_{\mathcal{U}} := \Big\{\text{homomorphisms of } R_{\mathcal{U}} \otimes \overline{L}\text{-algebras } \Gamma_{\mathcal{U}} := \Gamma\big(U^r(w), \mathcal{O}_{U^r(w)}\big) \longrightarrow \overline{R}_{\mathcal{U}}[p^{-1}]\Big\},$$

where $U^r(w) := \mathcal{U}_L \times_{M(w)} M^r(w)$. For any $g \in \Upsilon_{\mathcal{U}}$, write $\mathcal{F}(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g) := \varinjlim \mathcal{F}(\mathcal{U}, \mathrm{Spm}(S_L))$, where the limit is taken over all $\Gamma_{\mathcal{U}}$-subalgebra S of $\overline{R}_{\mathcal{U}}$ (via g) such that $\mathrm{Spm}(S_L) \to U^r(w)$ is finite and étale. Let $\mathcal{G}_{U_{\overline{K}},r,g} \subseteq \mathcal{G}_{U_{\overline{L}}}$ be the subgroup fixing $\Gamma_{\mathcal{U}}$. Similarly as before we get a localization functor $\mathrm{Sh}(\mathfrak{M}^r(w)) \longrightarrow \mathrm{Rep}(\mathcal{G}_{U_{\overline{L}},r,g})$ and donote by $\mathcal{F}\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\big)$ the image of $\mathcal{F} \in \mathrm{Sh}(\mathfrak{M}^r(w))$.

Moreover, given a covering of $\mathcal{M}(w)^{\mathtt{ket}}$ by open small affines $\{\mathcal{U}_i\}_{i\in I}$, choosing $g_i \in \Upsilon_{\mathcal{U}_i}$ for every $i \in I$, the map $\mathrm{Sh}(\mathfrak{M}^r(w)) \longrightarrow \prod_{i\in I} \mathrm{Rep}(\mathcal{G}_{U_{i,\overline{K}},r,g_i})$ is faithful. We also have

$$j_{r,*}(\mathcal{F})(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}) \cong \bigoplus_{g\in\Upsilon_{\mathcal{U}}} \mathcal{F}\big(\overline{R}_{\mathcal{U}}, \overline{N}_{\mathcal{U}}, g\big).$$

2. The classical Eichler-Shimura isomorphism

First we fix some notations for this section. Let $\mathcal{M} := \mathcal{M}(N,p)$, $M = M(n,p)$ and $\mathfrak{M} := \mathfrak{M}(N,p)$, the Faltings' site associated to the pair $(\mathcal{M}, M)$. Let $\mathcal{A} \to \mathcal{M}$ be the universal false elliptic curve and $e : \mathcal{M} \to \mathcal{A}$ the zero section. Denote $\mathcal{T} := T_p\big((\mathcal{A}[p^\infty]^1)^\vee\big)$, and $\underline{\omega} := \underline{\omega}_L = \big(e^*\Omega^1_{\mathcal{A}/\mathcal{M}}\big)^1$. For any integer $k \geq 0$, let $\mathcal{V}_k := \mathrm{Sym}^k(\mathcal{T}) \otimes L$ and consider it as an étale sheaf on $M^{et}_{\overline{L}}$.

Proposition 2.1. *With the above notations we have a canonical isomorphism compatible with the actions of G_L and all Hecke operators*

$$\mathrm{H}^1_{\acute{e}t}\left(M_{\overline{L}}, \mathcal{V}_k\right) \otimes \mathbb{C}_p(1) \cong \Big(\mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p\Big) \oplus \Big(\mathrm{H}^1(M, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1)\Big).$$

Proof. Recall that $\mathrm{H}^1_{\acute{e}t}\left(M_{\overline{L}}, \mathcal{V}_k\right) \otimes \mathbb{C}_p(1) \cong \mathrm{H}^1\left(\mathfrak{M}, \mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}}\right) \otimes \mathbb{C}_p(1)$. Take a small affine $\mathcal{U} = (\mathrm{Spf}(R), N)$ of $\mathcal{M}^{\mathtt{ket}}$ such that $\underline{\omega}_R$ restricted to U is a free R-module of rank 1. Let A be the corresponding false elliptic curve defined over R, $T := T_p\big((A[p^\infty]^1)^\vee\big)$, $V := \mathrm{Sym}^k(T) \otimes \hat{\overline{R}}[p^{-1}]$ for a non negative integer k and $\underline{\omega}_R$ the pullback of $\underline{\omega}$ to $\mathcal{U}$. We have a continuous functor $v : \mathcal{M}^{\mathtt{ket}} \to \mathfrak{M}$ sending $\mathcal{U}$ to $(\mathcal{U}, \mathcal{U}_L)$. The Leray spectral sequence

$$\mathrm{H}^i\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^j\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \Longrightarrow \mathrm{H}^{i+j}\left(\mathfrak{M}, \mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}}\right)$$

for $i + j = 1$ degenerates to the exact sequence

$$0 \to \mathrm{H}^1\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \to \mathrm{H}^1\left(\mathfrak{M}, \mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}}\right) \to \mathrm{H}^0\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^1\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \to \mathrm{H}^2\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right)$$

By [2, Lemma 4.10], the sheaf $\mathrm{R}^j\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})$ is just the sheaf associated to the presheaf on $\mathcal{M}^{\mathtt{ket}}$:

$$\mathcal{U} \longmapsto \mathrm{H}^j\left(\Delta, (\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})(\overline{R}, \overline{N})\right),$$

where $\Delta := \mathrm{Gal}\left(\overline{R}[p^{-1}]/R_{\overline{L}}\right)$ is a subgroup of $\mathcal{G} := \mathrm{Gal}\left(\overline{R}[p^{-1}]/R[p^{-1}]\right)$ and the localization

$$(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})(\overline{R}, \overline{N}) = \mathrm{Sym}^k(T) \otimes \hat{\overline{R}}[p^{-1}] = V.$$

First we claim that:

$$\mathrm{H}^0(\Delta, V) \cong \underline{\omega}_R^{-k} \otimes \mathbb{C}_p(k),$$
$$\mathrm{H}^1(\Delta, V) \cong \underline{\omega}_R^{k+2} \otimes \mathbb{C}_p(-1).$$

Granted this two claims we deduce:

$$\mathrm{H}^0\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^1\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \otimes \mathbb{C}_p \cong \mathrm{H}^0\left(M, \underline{\omega}^{k+2}\right) \otimes \mathbb{C}_p(-1),$$
$$\mathrm{H}^1\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \otimes \mathbb{C}_p \cong \mathrm{H}^1\left(M, \underline{\omega}^{-k}\right) \otimes \mathbb{C}_p(k).$$

Moreover we have $\mathrm{H}^2\left(\mathcal{M}^{\mathtt{ket}}, \mathrm{R}^0\, v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \otimes \mathbb{C}_p \cong \mathrm{H}^2\left(M, \underline{\omega}^{-k}\right) \otimes \mathbb{C}_p(k) = 0$ since M has dimension 1. Therefore we have an exact sequence of $\mathbb{C}_p$-modules compatible with the actions of G_L and the Hecke operators

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{H}^1\left(\mathcal{M}^{\mathrm{ket}}, \mathrm{R}^0 v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \otimes \mathbb{C}_p(1) & \longrightarrow & \mathrm{H}^1\left(\mathfrak{M}, \mathcal{V}_k \otimes \widehat{\mathcal{O}}_{\mathfrak{M}}\right) \otimes \mathbb{C}_p(1) & \longrightarrow & \mathrm{H}^0\left(\mathcal{M}^{\mathrm{ket}}, \mathrm{R}^1 v_*(\mathcal{V}_k \otimes \hat{\mathcal{O}}_{\mathfrak{M}})\right) \otimes \mathbb{C}_p(1) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \mathrm{H}^1(M, \underline{\omega}^{-k}) \otimes \mathbb{C}_p(k+1) & & \mathrm{H}^1_{\mathrm{\acute{e}t}}\left(M_{\overline{L}}, \mathcal{V}_k\right) \otimes \mathbb{C}_p(1) & & \mathrm{H}^0(M, \underline{\omega}^{k+2}) \otimes \mathbb{C}_p. & \end{array}$$

By the main result of [21], the above sequence splits canonically and we deduce the proposition.

Now we prove our claims. We start with the following Hodge-Tate sequence of $\hat{\overline{R}}$-modules with semilinear Δ-actions, associated to A:

$$0 \longrightarrow \underline{\omega}_R^{-1} \otimes_R \hat{\overline{R}}(1) \xrightarrow{a} T \otimes_{\mathbb{Z}_p} \hat{\overline{R}} \xrightarrow{\mathtt{dlog}_{\mathcal{A}}} \underline{\omega}_R \otimes_R \hat{\overline{R}} \longrightarrow 0.$$

Here a is defined by the $\mathtt{dlog}$ maps of $T_p(\mathcal{A}^1)$ and (1) is the Tate twist (see [5]). This sequence is $\mathcal{G}$-equivariant but not exact in general, it is always exact after inverting p, i.e. the sequence of $\mathcal{G}$-modules

$$0 \longrightarrow \underline{\omega}_R^{-1} \otimes_R \hat{\overline{R}}[p^{-1}](1) \xrightarrow{a} T \otimes_{\mathbb{Z}_p} \hat{\overline{R}}[p^{-1}] \xrightarrow{\mathtt{dlog}_{\mathcal{A}}} \underline{\omega}_R \otimes_R \hat{\overline{R}}[p^{-1}] \longrightarrow 0$$

is exact. Let e_0, e_1 be an $\hat{\overline{R}}[p^{-1}]$-basis of $T \otimes_{\mathbb{Z}_p} \hat{\overline{R}}[p^{-1}]$ such that

- e_1 is a R-basis of $\underline{\omega}_R^{-1}$

  and
- $\mathtt{dlog}(e_0)$ is a basis of $\underline{\omega}_R$, i.e., $\sigma\mathtt{dlog}(e_0) = \mathtt{dlog}(e_0)$ for any $\sigma \in \mathcal{G}$.

This gives us the following filtration of V:

$$0 =: \mathrm{Fil}^{-1}(V) \subseteq \mathrm{Fil}^0(V) \subseteq \mathrm{Fil}^1(V) \subseteq \cdots \subseteq \mathrm{Fil}^{k-1}(V) \subseteq \mathrm{Fil}^k(V) := V,$$

where $\mathrm{Fil}^i(V) := \sum_{n=0}^{i} \hat{\overline{R}}[p^{-1}] e_1^{k-n} e_0^n$, for $i = 0, 1, \ldots, n$. For example,

$$\mathrm{Fil}^0(V) = \hat{\overline{R}}[p^{-1}] e_1^k \text{ and } \mathrm{Fil}^1(V) = \hat{\overline{R}}[p^{-1}] e_1^k + \hat{\overline{R}}[p^{-1}] e_1^{k-1} e_0.$$

In [12, Thm.3], Falting proved the following two important results:

(i) $\mathrm{H}^0(\Delta, \hat{\overline{R}}[p^{-1}]) = R_{\mathbb{C}_p}$,

(ii) $\mathrm{H}^1(\Delta, \hat{\overline{R}}[p^{-1}]) = \underline{\omega}_R^2 \hat{\otimes} \mathbb{C}_p(-1)$,

where $R_{\mathbb{C}_p}$ denote the completed tensor product $R \hat{\otimes} \mathbb{C}_p$. Remark that in (ii) we used the Kodaira-Spencer isomorphism (see for example [15, Cor. 3.2.]). Using these we have:

$$\mathrm{H}^0(\Delta, \mathrm{Fil}^0(V)) \cong \mathrm{H}^0(\Delta, \underline{\omega}_R^{-k} \otimes \hat{\overline{R}}[p^{-1}](k)) \cong \underline{\omega}_R^{-k} \otimes \mathrm{H}^0(\Delta, \hat{\overline{R}}[p^{-1}](k)) \cong \underline{\omega}_R^{-k} \hat{\otimes} \mathbb{C}_p(k),$$
$$\mathrm{H}^1(\Delta, \mathrm{Fil}^0(V)) \cong \mathrm{H}^1(\Delta, \underline{\omega}_R^{-k} \otimes \hat{\overline{R}}[p^{-1}](k)) \cong \underline{\omega}_R^{-k} \otimes \mathrm{H}^1(\Delta, \hat{\overline{R}}[p^{-1}](k)) \cong \underline{\omega}_R^{-k+2} \hat{\otimes} \mathbb{C}_p(k-1).$$

Moreover, for any $0 \le i \le k-1$, we have

$$\mathrm{H}^0(\Delta, \mathrm{Fil}^{i+1} / \mathrm{Fil}^i) \cong \mathrm{H}^0(\Delta, \underline{\omega}_R^{2i+2-k} \otimes \hat{\overline{R}}[p^{-1}](k-i-1)) \cong \underline{\omega}_R^{2i+2-k} \hat{\otimes} \mathbb{C}_p(k-i-1),$$
$$\mathrm{H}^1(\Delta, \mathrm{Fil}^{i+1} / \mathrm{Fil}^i) \cong \mathrm{H}^1(\Delta, \underline{\omega}_R^{2i+2-k} \otimes \hat{\overline{R}}[p^{-1}](k-i-1)) \cong \underline{\omega}_R^{2i+4-k} \hat{\otimes} \mathbb{C}_p(k-i-2).$$

The class of extension

$$0 \to \mathrm{Fil}^i / \mathrm{Fil}^{i-1} \to \mathrm{Fil}^{i+1} / \mathrm{Fil}^{i-1} \to \mathrm{Fil}^{i+1} / \mathrm{Fil}^i \to 0$$

in $\mathrm{H}^1\left(\Delta, \underline{\omega}_R^{-2} \otimes_R \hat{\bar{R}}[p^{-1}](1)\right) \cong \underline{\omega}_R^{-2} \otimes_R \mathrm{H}^1(\Delta, \hat{\bar{R}}[p^{-1}](1)) \cong R_{\mathbb{C}_p}$ can be computed from the Kodaira-Spencer class and turns out to be a unit. Then by induction, for any $i = 1, 2, \ldots, k$, we have:

$$\mathrm{H}^0(\Delta, \mathrm{Fil}^i) = \underline{\omega}_R^{-k} \hat{\otimes} \mathbb{C}_p(k), \qquad \mathrm{H}^1(\Delta, \mathrm{Fil}^i) = \underline{\omega}_R^{-k+2+2i} \hat{\otimes} \mathbb{C}_p(k-1-i).$$

In particular,

$$\mathrm{H}^0(\Delta, V) = \mathrm{H}^0(\Delta, \mathrm{Fil}^k) = \underline{\omega}_R^{-k} \hat{\otimes} \mathbb{C}_p(k), \qquad \mathrm{H}^1(\Delta, V) = \mathrm{H}^1(\Delta, \mathrm{Fil}^k) = \underline{\omega}_R^{k+2} \hat{\otimes} \mathbb{C}_p(-1).$$

This proves the claims and the proposition. □

Remark 2.2. The analogue result for modular curves was proved by Faltings in [12]. The above proof follows the main lines of the arguments in Faltings' paper.

3. Distributions

3.1. **Definitions.** Let $r \in \mathbb{N}$ and $U \subset \mathcal{W}_r$ be a wide open disk defined over L. We denote by $\mathcal{O}(U)$ the L-algebra of rigid functions on U and $\Lambda_U \subset \mathcal{O}(U)$ the $\mathcal{O}_L$-algebra of bounded by 1 rigid functions i.e. the set of $f \in \mathcal{O}(U)$ such that $\mid f(\lambda) \mid \leq 1$ for each $\lambda \in U$. The algebra Λ_U is a local algebra and let $\mathfrak{m}_U$ be its maximal ideal. Let $\mathrm{ord} : \Lambda_U \to \mathbb{Z} \cup \{\infty\}$ be defined by $\mathrm{ord}(x) = \sup\{n \in \mathbb{N} \,|\, x \in \pi^n \Lambda_U\}$, recall that $\pi \in \mathcal{O}_L$ is a uniformizer. We denote by $\lambda_U : \mathbb{Z}_p^\times \to \Lambda_U^\times$ the character defined by $\lambda_U(s)(\lambda) = \lambda(s)$ for each $s \in \mathbb{Z}_p^\times$ and $\lambda \in U(\overline{\mathbb{Q}}_p)$. Then there exists a sequence in Λ_U, $\{b_n\}_{n \in \mathbb{N}}$, such that $\mathrm{ord}(b_n) \to +\infty$ if $n \to +\infty$ and $\lambda_U(1 + p^{r+1} z) = \sum_{n \geq 0} b_n z^n$ for any $z \in \mathbb{Z}_p$.

In the rest of this section we will denote by (B, λ) one the following pairs: (Λ_U, λ_U) or $(\mathcal{O}_L, \lambda)$, where $\lambda \in U(L)$. We denote by $\mathfrak{m}$ the maximal ideal of B. Remark that there exits a $b_\lambda \in B_L := B \otimes_{\mathcal{O}_L} L$ such that $\lambda(1 + p^{r+1} y) = \exp(b_\lambda \log(y))$ for all $z \in \mathbb{Z}_p$ (see [5, §5]).

Definition 3.1. Let A_λ° be the set of functions $f : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to B$ such that:

i) $f(aw, az) = \lambda(a) f(w, z)$ for each $a \in \mathbb{Z}_p^\times$ and $(w, z) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$;
ii) for each $i \in \{0, ..., p^r - 1\}$ there exist $\{c_{i,m}\}_{m \geq 0} \in B^{\mathbb{N}}$ such that $\mathrm{ord}(c_{i,m}) \to \infty$ if $m \to \infty$, and $f(1, z) = \sum_{m \geq 0} c_{i,m} \left(\frac{z-i}{p^r}\right)^m$ for each $z \in i + p^r \mathbb{Z}_p$.

We denote $D_\lambda^\circ := \mathrm{Hom}_{\mathcal{C}^0, B}(A_\lambda^\circ, B)$ the B-module of continuous homomorphisms where we consider A_λ° (respectively B) endowed with the $\mathfrak{m}$-adic topology i.e. the topology given by the family $\{\mathfrak{m}^h A_\lambda^\circ\}_{h \in \mathbb{N}}$ (respectively the family $\{\mathfrak{m}^h\}_{h \in \mathbb{N}}$). In general, if M is a B-module then the topology given by the family $\{\mathfrak{m}^h M\}_{h \in \mathbb{N}}$ will be called the $\mathfrak{m}$-*topology*.

We consider the following B_L-modules $D_\lambda := D_\lambda^\circ \otimes_{\mathcal{O}_L} L$ and $A_\lambda := A_\lambda^\circ \otimes_{\mathcal{O}_L} L$. We know that B_L is an L-Banach algebra. Moreover A_λ is an orthonormalizable Banach B_L-module. Explicitly an orthonormal basis of A_λ, $\{f_{i,m} \mid i = 0, ..., p^r - 1, m \in \mathbb{N}\}$ which is defined by:

$$f_{i,m}(1, z) = \left(\frac{z - i}{p^r}\right)^m 1_{i+p^r \mathbb{Z}_p}(z), \tag{1}$$

where $1_{i+p^r \mathbb{Z}_p}$ is the characteristic function of the set $i + p^r \mathbb{Z}_p$.

We denote $\Delta_p = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}_p) \mid ad - bc \neq 0, a \in \mathbb{Z}_p^\times, c \in p\,\mathbb{Z}_p\}$. We define an action of Δ_p on A_λ° and D_λ° as follows. Let $f \in A_\lambda^\circ$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_p$ then the function $\gamma f : \mathbb{Z}_p^\times \times \mathbb{Z}_p \to B$ is defined by $(\gamma f)(w, z) = f(aw + cz, bw + dz)$.

Lemma 3.2. *For any $f \in A_\lambda^\circ$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_p$ we have $\gamma f \in A_\lambda^\circ$.*

Proof. Let $e \in \mathbb{Z}_p^\times$, then it is straightforward to verify that $(\gamma f)(ew, ez) = \lambda(e)(\gamma f)(w, z)$ for any $(w, z) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$.

Let $i \in \{0, ..., p^r - 1\}$, then there is an unique $j \in \{0, ..., p^r - 1\}$ such that $\frac{b+dz}{a+cz} \in j + p^r \mathbb{Z}_p$ for any $z \in i + p^r \mathbb{Z}_p$. By definition there exist $\{c_m\}_{m\geq 0} \in B^{\mathbb{N}}$ such that $\mathrm{ord}(c_m) \to \infty$ if $m \to \infty$, and $f(1, z) = \sum_{m\geq 0} c_m \left(\frac{z-j}{p^r}\right)^m$ for any $z \in j + p^r \mathbb{Z}_p$. Then for any $z \in i + p^r \mathbb{Z}_p$ we have:

$$(\gamma f)(1, z) = \lambda(a + cz) \sum_{m\geq 0} c_m p^{-rm} \left(\frac{b + dz}{a + cz} - j\right)^m.$$

The function $\lambda(a + cz)$ is analytic for $z \in i + p^r \mathbb{Z}_p$.

Moreover if $z \in i + p^r \mathbb{Z}_p$, $z = i + p^r y$ with $y \in \mathbb{Z}_p$ then:

$$\sum_{m\geq 0} c_m p^{-rm} \left(\frac{b + dz}{a + cz} - j\right)^m = \sum_{m\geq 0} c_m \left(\frac{b' + d'y}{a' + c'y}\right)^m,$$

where $a' = a+ic, b' = p^{-r}(b+id-j(a+ci)), c' = cp^r, d' = d-jc$. Remark that $b' \in \mathbb{Z}_p$ and then $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Delta_p$. Moreover, by hypothesis on λ we can write $\lambda(1 + a^{-1}cz) = \lambda(1 + a^{-1}ci)\lambda(1 + p^{r+1}\beta y) = \sum_{m\geq 0} b'_m y^m$ where $\beta = \frac{c}{p(a+ci)}$ and $\mathrm{ord}(b'_m) \to \infty$. Then we can write $(\gamma f)(1, z) = \sum_{m\geq 0} d_m y^m$ for any $z = i + p^r y \in i + p^r \mathbb{Z}_p$ where $d_m \in B$ and $\mathrm{ord}(d_m) \to +\infty$. So we deduce condition ii) for $(\gamma f)(1, z)$. □

Remark 3.3. **(1)** We obtain a left action of Δ_p on A_λ°. For each $\gamma \in \Delta_p$ the morphism induced on A_λ° is clearly continuous, then we get a right action of Δ_p on D_λ°: for $\gamma \in \Delta_p$ and $\mu \in D_\lambda^\circ$ we have $(\mu \mid \gamma)(f) = \mu(\gamma f)$ for any $f \in A_\lambda^\circ$. We also obtain actions of Δ_p on A_λ and D_λ.

Using the family (1), we obtain the following description of D_λ°:

Lemma 3.4. *There exists an isomorphism of B-modules $\psi : D_\lambda^\circ \longrightarrow \prod_{i=0}^{p^r-1} \prod_{m\in\mathbb{N}} B$, defined by $\mu \to (\mu(f_{i,m}))_{i,m}$. This morphism is a homeomorphism, where we consider the $\mathfrak{m}$-adic topology on the left side and the product topology of the $\mathfrak{m}$-adic topologies on each term of the right side.*

3.2. **Filtrations.** In this subsection we will define a filtration of D_λ° which is stable under the action of Δ_p.

Definition 3.5. Let $h \in \mathbb{N}$. We define the following B-module:

$$\mathrm{Fil}^h(D_\lambda^\circ) := \{\mu \in D_\lambda^\circ \mid \mu(f_{i,m}) \in \mathfrak{m}^{h-m} \quad \forall m = 0,..,h-1 \text{ and } i = 0,..,p^r-1\}$$

Remark 3.6. Consider another system of representatives $\Omega \subset \mathbb{Z}_p$ of $\mathbb{Z}_p/p^r\mathbb{Z}_p$ other than $\{0,1,2,...,p^r-1\}$. We can define a basis of A_λ as before: $f_{j,m}(1,z) = \left(\frac{z-j}{p^r}\right)^m 1_{j+p^r\mathbb{Z}_p}(z)$ for $j \in \Omega$. The same definition of the filtration of D_λ° also works and it is not difficult to verify that the filtration does not depend on the choice of Ω.

Proposition 3.7. *The module* $\mathrm{Fil}^h(D_\lambda^\circ)$ *is stable under the action of* Δ_p *and* $D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ *is an artinian* $\mathcal{O}_L$*-module with an action of* Δ_p*. Moreover, we have a natural isomorphism of* B*-modules:*

$$D_\lambda^\circ \longrightarrow \varprojlim_h D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$$

Proof. The image of $\mathrm{Fil}^h(D_{n,\lambda}^\circ)$ under the map ψ defined in 3.4, is given by $\prod_{i=0}^{p^r-1}\left(\prod_{m=0}^{h-1}\mathfrak{m}^{h-m} \times \prod_{m\geq h} B\right)$. Then $D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ is isomorphic to $\prod_{i=0}^{p^r-1}\prod_{m=0}^{h-1} B/\mathfrak{m}^{h-m}$ as a B-module. By hypothesis each term $B/\mathfrak{m}^{h-m}$ is an artinian $\mathcal{O}_L$-module, then $D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ is an artinian $\mathcal{O}_L$-module. Finally the map $D_\lambda^\circ \longrightarrow \varprojlim_h D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ is an isomorphism because ψ is an isomorphism.

Now we verify that $\mathrm{Fil}^h(D_\lambda^\circ)$ is invariant under the action of Δ_p. Let $\mu \in \mathrm{Fil}^h(D_\lambda^\circ)$ and $\gamma \in \Delta_p$. Fix any i and $m < h$ we need to verify that $\mu(\gamma f_{i,m}) \in \mathfrak{m}^{h-m}$. To do that remark that we have the following decomposition $\Delta_p = N^-T^+N$ where:

$$N^- = \{\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right) \mid c \in p\mathbb{Z}_p\},\ N = \{\left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right) \mid b \in \mathbb{Z}_p\},\ T^+ = \{\left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right) \mid a \in \mathbb{Z}_p^\times,\ d \in \mathbb{Z}_p - \{0\}\}.$$

If $\gamma = \left(\begin{smallmatrix}1&b\\0&1\end{smallmatrix}\right) \in N$ there is a unique $j \in \{0,..,p^r-1\}$ such that $i - b = j - b'$ for some $b' \in p^r\mathbb{Z}_p$, then it is straightforward to obtain $\gamma f_{i,m} = \sum_{k=0}^m \binom{m}{k}(b')^k f_{j,m-k}$. So by definition we get that $\mu(\gamma f_{i,m}) \in \sum_{k=0}^m \mathfrak{m}^{h-m+k} \subset \mathfrak{m}^{h-m}$.

If $\gamma = \left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right) \in N^-$ using the notations of remark 3.6 we have $\gamma f_{i,m}(w,z) = G_m(\frac{z}{w})f_{j,m}(w,z)$ where $G_m(x) = (1-ip)^m\lambda(1+cx)/(1+cx)^m$ and $j = \frac{i}{1-ic}$. By the hypothesis on λ if $x \in j + p^r\mathbb{Z}_p$ we can write $G_m(x) = \sum_{k\geq 0} d_k(\frac{x-j}{p^r})^k$ with $d_r \in \mathfrak{m}^r$ and $\mathrm{ord}(d_r) \to \infty$. To prove that $\mu(\gamma f_{i,m}) \in \mathfrak{m}^{h-m}$ it is enough to prove that: if $s \in \mathbb{N}$ and $d \in \mathfrak{m}^s$ then $\mu(df_{i,m+s}) \in \mathfrak{m}^{h-m}$. This last claim is trivial if $s \geq h-m$, finally if $s < h-m$ we have $m+s < h$ then $\mu(df_{i,m+s}) = d\mu(f_{i,m+s}) \in \mathfrak{m}^{h-m-s+s} = \mathfrak{m}^{h-m}$.

If $\gamma = \left(\begin{smallmatrix}a&0\\0&d\end{smallmatrix}\right) \in T^+$ by definition we have $\gamma f_{i,m}(1,z) = \lambda(a)(\frac{d}{a}z - i)^m 1_{i+p^r\mathbb{Z}_p}(\frac{d}{a}z)$. We deduce that there exists $J \subset \{0,...,p^r-1\}$ such that $\gamma f_{i,m} = \sum_{j\in J}\sum_{k=0}^m a_{j,k}f_{j,m-k}$ for some $a_{j,k} \in B$, so we obtain that $\mu(\gamma f_{i,m}) \in \sum_{k=0}^m \mathfrak{m}^{h-m+k} \subset \mathfrak{m}^{h-m}$. □

Let Γ be the fundamental group associated to $\mathrm{M}(N,p)(\mathbb{C})$ for some base point. As Γ is torsion free it can be embedded as a subgroup of the étale fundamental group of $\mathrm{M}(N,p)(\mathbb{C})$ for the same base point. The argument of the beginning of section 3.5 shows that there is a group homomorphism $\Gamma \to I$, where $I \subset GL_2(\mathbb{Z}_p)$ is the Iwahori subgroup, which induces left actions of Γ on D_λ° and $D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ for any $h \in \mathbb{N}$.

Corollary 3.8. *We have canonical isomorphisms:*

$$\mathrm{H}^1(\Gamma, D_\lambda^\circ) \cong \varprojlim_h \mathrm{H}^1(\Gamma, D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)) \cong \mathrm{H}^1_{\mathrm{cont}}(\Gamma, (D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ))_{h\in\mathbb{N}}),$$

where on the right side we consider the continuous Γ*-cohomology of the projective system* $(D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ))_{h\in\mathbb{N}}$.

Proof. The proof is exactly the same as that of theorem 3.15 of [2]. For each $h \in \mathbb{N}$ the quotient $D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)$ is an artinian $\mathcal{O}_L$-module, then the projective system $(D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ))_{h\in\mathbb{N}}$ satisfies the Mittag-Leffler condition and so we have a natural isomorphism:

$$\mathrm{H}^1_{\mathrm{cont}}(\Gamma, (D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ))_{h\in\mathbb{N}}) \cong \varprojlim_h \mathrm{H}^1(\Gamma, D_\lambda^\circ/\mathrm{Fil}^h(D_\lambda^\circ)).$$

The first isomorphism in the statement of the corollary follows in the same way as lemma 3.13 in [2]. □

3.3. **Specialization.** We will introduce some notations:

- if $(B, \lambda) = (\Lambda_U, \lambda_U)$, then we write $A_U^\circ := A_{\lambda_U}^\circ$ and $D_U^\circ := D_{\lambda_U}^\circ$. Recall that on A_U° we consider the $\mathfrak{m}_U$-topology. Moreover, we denote $A_U := A_{\lambda_U}$ and $D_U := D_{\lambda_U}$.
- now let $(B, \lambda) = (\mathcal{O}_L, \lambda)$, where $\lambda \in U(L)$. In this case we use the notations of subsection 3.1 i.e. A_λ°, D_λ°, A_λ and D_λ.

These two cases are related as follows. We fix a function $\pi_\lambda \in \Lambda_U$ which vanishes of order 1 at λ and nowhere else; we call this function a *uniformizer at* λ. We define $\rho_\lambda : A_U^\circ \to A_\lambda^\circ$ by $\rho_\lambda(f)(w,z) = f(w,z)(\lambda)$, here $f \in A_U^\circ$ and $(w,z) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$. Moreover we define $\eta_\lambda : D_U^\circ \to D_\lambda^\circ$ as follows. For each $f \in A_\lambda^\circ$ we denote $f_U \in A_U^\circ$ the function defined by $f_U(w,z) = \lambda_U(w) f(1, z/w)$. Then for each $\mu \in D_U^\circ$ we define $\eta_\lambda(\mu)$ by the formula $\eta_\lambda(\mu)(f) = \mu(f_U)$, for any $f \in A_\lambda^\circ$.

It is not difficult to prove that for each $h \in \mathbb{N}$ we have $\eta_\lambda(\mathrm{Fil}^h(D_U^\circ)) \subset \mathrm{Fil}^h(D_\lambda^\circ)$. Moreover following the arguments in [13, §3] we can prove that the following two sequences of Δ_p-modules are exact:

$$0 \longrightarrow A_U^\circ \xrightarrow{\cdot\pi_\lambda} A_U^\circ \xrightarrow{\rho_\lambda} A_\lambda^\circ \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow D_U^\circ \xrightarrow{\cdot\pi_\lambda} D_U^\circ \xrightarrow{\eta_\lambda} D_\lambda^\circ \longrightarrow 0$$

Consider the case when $\lambda \in U(L)$ is an algebraic weight i.e. there exists $k \in \mathbb{N}$ such that $\lambda(t) = t^k$ for each $t \in \mathbb{Z}_p^\times$. Let $V_\lambda^\circ \subset A_\lambda^\circ$ be the $\mathcal{O}_L$-module of homogeneous polynomials of degree k. We can verify that V_λ° is stable under the action of Δ_p. We denote $V_\lambda := V_\lambda^\circ \otimes_{\mathcal{O}_L} L$, we have a natural inclusion $V_\lambda \hookrightarrow A_\lambda$, then we obtain a natural map of Δ_p-modules: $D_\lambda \to V_\lambda^\vee$.

3.4. **Slope decompositions.** Let l be a prime integer such that $(l, \delta) = 1$. Using the action of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)$ we define, in the usual way, a Hecke operator on $\mathrm{H}^1(\Gamma, D_U)$, $\mathrm{H}^1(\Gamma, D_\lambda)$ and on $\mathrm{H}^1(\Gamma, V_\lambda^\vee)$ if λ is an algebraic weight. As usual we denote this operator by T_l if $l \nmid Np$ and U_l if $l \mid Np$.

The L-vector space $\mathrm{H}^1(\Gamma, D_\lambda)$ admits a slope decomposition with respect to U_p. To prove this statement we can proceed as follows. Firstly, we remark that the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & l \end{smallmatrix}\right)$ on D_λ induces a compact operator, secondly, we use [20, Prop. 2.3.13] and the arguments given in [20, §4.2] to deduce the statement. Moreover, suppose that λ is an algebraic weight attached to $k \in \mathbb{N}$, then the space $\mathrm{H}^1(\Gamma, V_\lambda^\vee)$ is finite-dimensional, so it is clear that it admits a slope decomposition with respect to U_p. The map of Δ_p-modules $D_\lambda \to V_\lambda^\vee$, induces a morphism of cohomology groups: $\mathrm{H}^1(\Gamma, D_\lambda) \to \mathrm{H}^1(\Gamma, V_\lambda^\vee)$. This map is Hecke equivariant and if $h < k+1$, then we have a canonical isomorphism:

$$\mathrm{H}^1(\Gamma, D_\lambda)^{\leq h} \xrightarrow{\sim} \mathrm{H}^1(\Gamma, V_\lambda^\vee)^{\leq h}$$

To proof this sentence we follow the same steps that in the proof of [18, Thm. 5.4]. Alternatively, it is a consequence of [20, Prop. 4.3.10].

Now we deal with the problem of the existence of slope decomposition of $\mathrm{H}^1(\Gamma, D_U)$ with respect to U_p. Following the proofs of [2, Lemma 3.5] and [2, Cor. 3.6] we obtain:

Lemma 3.9. *Let $\{\mu_j\}_{j\in J}$ be a family of elements in D_U° such that its image in the $\mathbb{L}$-vector space $D_U^\circ/\mathfrak{m}D_U^\circ$ is a basis. Then for each $m > 0$ the natural map $\oplus_{j\in J}(\Lambda_U/\mathfrak{m}^m)\mu_j \to D_U^\circ/\mathfrak{m}^m D_U^\circ$ is an isomorphism of Λ_U-modules. Moreover, for each $\mu \in D_U^\circ$ there exists a unique family $\{a_j\}_{j\in J}$ in Λ_U such that **i)** for the weak topology $a_j \to 0$ in the cofinite filter on J, and **ii)** $\mu = \sum_{j\in J} a_j\mu_j$.*

Proposition 3.10. *Let $\lambda \in \mathcal{W}(L)$ then there exists a wide open disk $U \subset \mathcal{W}$ defined over L such that $\lambda \in U(L)$ and $\mathrm{H}^1(\Gamma, D_U)$ admits a slope $\leq h$ decomposition with respect to U_p.*

Proof. The proof follows the same steps as those of [2, Thm. 3.17]. If $U \subset \mathcal{W}$ is a wide open disk, using [20, §4.2.1] we obtain a complex $C^\bullet(\Gamma, D_U)$ which calculates $\mathrm{H}^\bullet(\Gamma, D_U)$ such that each term $C^i(\Gamma, D_U)$ is isomorphic to finitely many copies of D_U and we can lift the Hecke operators on it. It is enough to prove the theorem for this complex.

Fix a wide open disk $U' \subset \mathcal{W}$ defined over L such that $\lambda \in U'(L)$. Using lemma 3.9 and [4, Thm. 4.5.1], there exists a closed disk $V \subset U'$ centered at λ such that the Fredholm determinant $F_V^\bullet$ of U_p acting on $C^\bullet(\Gamma, D_V)$ admits a slope $\leq h$-decomposition. Let $U \subset V$ be a wide open disk centered at λ. Using the family of lemma 3.9, we can define the Fredholm series, $F_U^\bullet$, of U_p acting on $C^\bullet(\Gamma, D_U)$. In fact, we have $F_V^\bullet = F_U^\bullet$ because both series are computed using the same basis. Using the results of [4, §4.6] we deduce the theorem. □

3.5. **Distributions on the Faltings' Site.** Let (B, λ) be as in 3.1 and consider the modules defined there: A_λ°, A_λ, D_λ°, $\mathcal{D}_\lambda$ and $\mathrm{Fil}^h(D_\lambda^\circ)$ for $h \in \mathbb{N}$.

We fix $\eta = \mathrm{Spec}(\mathbb{K})$ a geometric generic point of $M(N,p)$. Let $\mathcal{G}$ be the *geometric étale fundamental group* attached to $M(N,p)$ and η. Let $\mathcal{C}_1 \to \mathcal{M}(N,p)$ be the level p-subgroup of the universal object $\mathcal{A} \to \mathcal{M}(N,p)$. Let $T := \varprojlim_n \mathcal{A}^\vee[p^n]^1_{L,\eta}$ and let π be the composition of the natural maps $T \to \mathcal{A}^\vee[p]^1_{L,\eta} \to (\mathcal{C}_1^\vee)_{L,\eta}$. Remark that T is a free $\mathbb{Z}_p$-module of rank 2 and moreover we have an action of $\mathcal{G}$ on it. We fix a basis $\{\epsilon_0, \epsilon_1\}$ of T such that $\pi(\epsilon_1) = 0$. The action of $\mathcal{G}$ on this basis gives us a group homomorphism $\mathcal{G} \to GL_2(\mathbb{Z}_p)$ and the condition $\pi(\epsilon_1) = 0$ implies that the image of this homomorphism is contained in the Iwahori subgroup $I \subset GL_2(\mathbb{Z}_p)$. Then for each $h \in \mathbb{N}$ the finite $\mathcal{O}_L$-modules $A_\lambda^\circ/\mathfrak{m}^h A_\lambda^\circ$ and $D_\lambda^\circ/\mathrm{Fil}^h(D^\circ)$ can be considered as $\mathcal{G}$-modules and we denote $\mathcal{A}_{\lambda,h}^\circ, \mathcal{D}_{\lambda,h}^\circ \in \mathbf{Sh}(M(N,p)_{\bar{L}}^{\mathrm{\acute{e}t}})$ the étale sheaves attached to this $\mathcal{G}$-modules. We obtain continuous sheaves $\mathcal{A}^\circ_\lambda := (\mathcal{A}^\circ_{\lambda,h})_{h\in\mathbb{N}}, \mathcal{D}^\circ_\lambda := (\mathcal{D}^\circ_{\lambda,h})_{h\in\mathbb{N}} \in \mathbf{Sh}(M(N,p)^{\text{ét}}_{\overline{L}})^{\mathbb{N}}$ and ind-continuous sheaves $\mathcal{A}_\lambda, \mathcal{D}_\lambda \in \mathrm{Ind}-\mathbf{Sh}(M(N,p)^{\text{ét}}_{\overline{L}})^{\mathbb{N}}$.

Using the natural functor $u_* : \mathrm{Sh}(M(N,p)^{\text{ét}}_{\overline{L}}) \to \mathrm{Sh}(\mathfrak{M}(N,p))$, from the sheaves $\mathcal{A}^\circ_{\lambda,h}$ and $\mathcal{D}^\circ_{\lambda,h}$ on $M(N,p)^{\text{ét}}_{\overline{L}}$ we obtain sheaves on $\mathfrak{M}(N,p)$. We will use the same symbols to denote these sheaves. In the same way as before we get $\mathcal{A}^\circ_\lambda, \mathcal{D}^\circ_\lambda \in \mathbf{Sh}(\mathfrak{M}(N,p))^{\mathbb{N}}$ and $\mathcal{A}_\lambda, \mathcal{D}_\lambda \in \mathrm{Ind}-\mathbf{Sh}(\mathfrak{M}(N,p))^{\mathbb{N}}$.

On the modules $\mathrm{H}^1(M(N,p)^{\text{ét}}_{\overline{L}}, \mathcal{D}_\lambda)$ and $\mathrm{H}^1(\mathfrak{M}(N,p), \mathcal{D}_\lambda)$ we can define Hecke operators in the same way as in [2, §5].

Proposition 3.11. *We have the following isomorphisms of Hecke modules:*

$$\mathrm{H}^1(\Gamma, D_\lambda) = \mathrm{H}^1(M(N,p)^{\text{ét}}_{\overline{L}}, \mathcal{D}_\lambda) = \mathrm{H}^1(\mathfrak{M}(N,p), \mathcal{D}_\lambda)$$

Proof. Let F be a finite representation of $\mathcal{G}$ and let $\mathcal{F}$ the sheaf on $M(N,p)^{\text{ét}}_{\overline{L}}$ attached to F. Using the fact that $M(N,p)_{\mathbb{C}}$ is $K(\pi,1)$ and using the same argument as in [2, Prop. 3.18] (in fact, our case is easier because there are no cusps) we obtain that $\mathrm{H}^1(\Gamma, F) \cong \mathrm{H}^1(M(N,p)^{\text{ét}}_{\overline{L}}, \mathcal{F})$. Then using corollary 3.8 and the same argument as in the proof of [2, Prop. 3.18] we obtain:

$$\mathrm{H}^1(\Gamma, D_\lambda) = \mathrm{H}^1(M(N,p)^{\text{ét}}_{\overline{L}}, \mathcal{D}_\lambda).$$

We use the same argument as in [2, Prop. 3.19] to prove the second isomorphism. As before, our case is easier because there are no cusps. Finally these isomorphisms are compatible with the Hecke operators. □

Remark 3.12. 1) These isomorphisms are compatible with specializations (see subsection 3.3). We can verify this statement following each step in the proof of the last proposition.

2) Moreover, the second isomorphism in the statement of proposition 3.11 is compatible with the action of the Galois group of L.

4. The morphism

In this section we carry out the construction of the analogous of the modular sheaves constructed in [5] on Faltings' sites. Moreover we give an explicit description of a map relating overconvergent cohomology with overconvergent modular forms. The construction of this map is the main step towards the proof of theorem 0.1.

4.1. **A torsor.** Let $r > 0$ be an integer and $w > 0$ such that $w < \frac{1}{p^{r-2}(p+1)}$ if $r > 1$ and $w \leq \frac{1}{p}$ if $r = 1$. We denote $v = \frac{w}{p-1}$. Moreover we suppose that the field L contains a primitive p^rth root of the unity and an element of valuation w.

Consider the functor $v : \mathcal{M}(w)^{\mathtt{ket}} \longrightarrow \mathfrak{M}(w)$ defined in section 1, then we define $\omega_{\mathfrak{M}(w)} = v^*(\underline{\omega})$ (see section 1 for the definition of $\underline{\omega}$). The sheaf $\omega_{\mathfrak{M}(w)}$ is a continuous sheaf on $\mathfrak{M}(w)$. In fact, it is an $\hat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}(w)}$-module.

Let $\mathcal{A} \to \mathcal{M}(w)$ be the universal object of $\mathcal{M}(w)$ and $\mathcal{C}_1 \to \mathcal{M}(w)$ the canonical subgroup of $\mathcal{A}[p]^1$. Moreover by hypothesis the canonical subgroup of $\mathcal{A}[p^r]^1$ exists and is denoted by $\mathcal{C}_r \to \mathcal{M}(w)$.

Let $n \in \mathbb{N}$ the $\mathcal{M}(w)$-scheme $(\mathcal{A}[p^n]^1)^\vee$ induces a sheaf on $M(w)^{\text{ét}}_{\overline{L}}$ which we denote by the same symbol: $(\mathcal{A}[p^n]^1)^\vee$. We consider the continuous sheaf $\mathcal{T} = ((\mathcal{A}[p^n]^1)^\vee)_{n\in\mathbb{N}} \in \mathbf{Sh}(M(w)^{\text{ét}}_{\overline{L}})^{\mathbb{N}}$. By considering the $\mathcal{M}(w)$-scheme $\mathcal{C}_1^\vee$ as a sheaf on $M(w)^{\text{ét}}_{\overline{L}}$ we obtain a morphism of sheaves $\pi : \mathcal{T} \to \mathcal{C}_1^\vee$. We define the sheaf $\mathcal{T}^0 \subset \mathcal{T}$ as the inverse image of $\mathcal{C}_1^\vee - \{0\}$ under π. Using the functor $u_* : \mathrm{Sh}(M(w)^{\text{ket}}_{\overline{L}}) \to \mathrm{Sh}(\mathfrak{M}(w))$ described in 1.4, we obtain sheaves on $\mathfrak{M}(w)$ denoted by the same symbols: $(\mathcal{A}[p^n]^1)^\vee, \mathcal{C}_1^\vee, \mathcal{T}$ and $\mathcal{T}^0$. Using the construction in [5, §3] and the definition of the site $\mathfrak{M}(w)$ we obtain a morphism of $\hat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules:

$$\mathrm{dlog}_{\mathfrak{M}(w)} : \mathcal{T} \otimes \hat{\mathcal{O}}_{\mathfrak{M}(w)} \to \omega_{\mathfrak{M}(w)} \otimes_{\hat{\mathcal{O}}^{\mathrm{un}}_{\mathfrak{M}(w)}} \hat{\mathcal{O}}_{\mathfrak{M}(w)}. \tag{2}$$

Let $\mathcal{F}_{\mathfrak{M}(w)} := \mathrm{Im}(\mathrm{dlog}_{\mathfrak{M}(w)})$. Considering the localization at each small affine object and using [5, Thm. 4.2], we deduce in the same way as in [2, Lemma 4.3] that $\mathcal{F}_{\mathfrak{M}(w)}$ is a locally free sheaf of $\hat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules of rank 1. Moreover using [5, Prop. 4.4] we obtain an isomorphism of $\hat{\mathcal{O}}_{\mathfrak{M}(w)}$-modules:

$$\mathcal{F}_{\mathfrak{M}(w)}/p^{r-v}\mathcal{F}_{\mathfrak{M}(w)} \xrightarrow{\sim} \mathcal{C}_r^\vee \otimes \mathcal{O}_{\mathfrak{M}(w)}/p^{r-v}\mathcal{O}_{\mathfrak{M}(w)} \tag{3}$$

In subsection 1.4 we have described the functor $j_r : \mathfrak{M}(w) \to \mathfrak{M}^r(w)$. From $\mathcal{F}_{\mathfrak{M}(w)}$ we obtain a sheaf on $\mathfrak{M}^r(w)$ denoted by $\mathcal{F}_{\mathfrak{M}^r(w)}$. Using the same functor we consider $\mathcal{C}_r^\vee$ as a sheaf on $\mathfrak{M}^r(w)$, denoted by the same symbols. We have an isomorphism of $\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}$-modules:

$$\mathcal{F}_{\mathfrak{M}^r(w)}/p^{r-v}\mathcal{F}_{\mathfrak{M}^r(w)} \xrightarrow{\sim} \mathcal{C}_r^\vee \otimes \mathcal{O}_{\mathfrak{M}^r(w)}/p^{r-v}\mathcal{O}_{\mathfrak{M}^r(w)}. \tag{4}$$

Remark that over the site $\mathfrak{M}^r(w)$ the sheaf $\mathcal{C}_r^\vee$ is constant (see [5, §3]). We denote by $\mathcal{F}'_{\mathfrak{M}^r(w)}$ the inverse image of the sheaf $\mathcal{C}_r^\vee - \mathcal{C}_r^\vee[p^{r-1}]$ under the map $\mathcal{F}_{\mathfrak{M}^r(w)} \to \mathcal{C}_r^\vee \otimes \mathcal{O}_{\mathfrak{M}^r(w)}/p^{r-v}\mathcal{O}_{\mathfrak{M}^r(w)}$, which is obtained by composition of (4) and $\mathcal{F}_{\mathfrak{M}^r(w)} \to \mathcal{F}_{\mathfrak{M}^r(w)}/p^{r-v}\mathcal{F}_{\mathfrak{M}^r(w)}$. Then $\mathcal{F}'_{\mathfrak{M}^r(w)}$ is a sheaf of sets endowed with an action of the sheaf of groups $S_{\mathfrak{M}^r(w)} := \mathbb{Z}_p^\times(1 + p^{r-v}\hat{\mathcal{O}}_{\mathfrak{M}^r(w)})$. Following the proof of [2, Lemma 4.4] we obtain the following result:

Lemma 4.1. *We have that $\mathcal{F}'_{\mathfrak{M}^r(w)}$ is a $S_{\mathfrak{M}^r(w)}$-torsor. Moreover, it is trivial over a covering of the type $\{(\mathcal{U}_i, \mathcal{U}_i \times \mathrm{M}^r(w))\}_{i\in I}$, where $\{\mathcal{U}_i\}_{i\in I}$ is a covering of $\mathcal{M}(w)$ by small affine objects.*

4.2. **Modular sheaves.** Let (B, λ) be a couple as in subsection 3.1. In this subsection and in the rest of this paper we need to consider the w adapted to λ: we suppose that w satisfies the conditions at the beginning of subsection 4.1 and $w < (p-1)\left(\mathrm{ord}(b_\lambda) + r - \frac{1}{p-1}\right)$, recall that b_λ was introduced at the beginning of section 3.

Consider the following continuous sheaf on $\mathfrak{M}^r(w)$ defined by $\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B := ((\mathcal{O}_{\mathfrak{M}^r(w)}/\pi^n\mathcal{O}_{\mathfrak{M}^r(w)}) \otimes B/\mathfrak{m}^n)_{n\in\mathbb{N}}$. We denote by $\hat{\mathcal{O}}^\lambda_{\mathfrak{M}^r(w)}$ the continuous sheaf $\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B$ endowed with the action of $S_{\mathfrak{M}^r(w)}$ defined as follows. Let $(\mathcal{U}, W, u)$ be an object of $\mathfrak{M}^r(w)$. Let $ax \in S_{\mathfrak{M}^r(w)}(\mathcal{U}, W, u) = \mathbb{Z}_p^\times(1 + p^{r-v}\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}(\mathcal{U}, W, u))$ and $y \in (\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)(\mathcal{U}, W, u)$, then we put:

$$(ax) \cdot y := x^{b_\lambda}\lambda(a)y \in (\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes}B)(\mathcal{U}, W, u),$$

this action is well defined because w is adapted to r.

For each $n \in \mathbb{N}$ we denote by $\hat{\mathcal{O}}^{\lambda}_{\mathfrak{M}^r(w),n}$ the sheaf $(\mathcal{O}_{\mathfrak{M}^r(w)}/\pi^n\mathcal{O}_{\mathfrak{M}^r(w)}) \otimes B/\mathfrak{m}^n$ endowed with the action of $S_{\mathfrak{M}^r(w)}$ defined in the same way as above.

Let G_r be the Galois group of the étale covering $M^r(w) \to M(w)$. This group acts on the site $\mathfrak{M}^r(w)$. Let $\sigma \in G_r$ and an object $(\mathcal{U}, W, u)$ of $\mathfrak{M}^r(w)$, then we define $\sigma \cdot (\mathcal{U}, W, u) := (\mathcal{U}, W, \sigma \circ u)$. Acting by identity on the morphisms we obtain a continuous functor on $\mathfrak{M}^r(w)$. This induces an action of G_r on the sheaves of the site $\mathfrak{M}^r(w)$. Moreover, if $\mathcal{F}$ is a sheaf on $\mathfrak{M}^r(w)$ such that $\sigma \cdot \mathcal{F} = \mathcal{F}$ for each $\sigma \in G_r$ then we have a canonical action of G_r on the sheaf $j_{r,*}(\mathcal{F})$ on $\mathfrak{M}(w)$ (to prove this sentence we repeat the proof of [2, Lemma 4.5]).

Definition 4.2. **1)** We define the following sheaves on $\mathfrak{M}^r(w)$:

$$\Omega^{\lambda}_{\mathfrak{M}^r(w)} := \mathrm{Hom}_{S_{\mathfrak{M}^r(w)}}(\mathcal{F}'_{\mathfrak{M}^r(w)}, \hat{\mathcal{O}}^{\lambda^{-1}}_{\mathfrak{M}^r(w)}) \ \text{ and } \ \omega^{\lambda}_{\mathfrak{M}^r(w)} := \Omega^{\lambda}_{\mathfrak{M}^r(w)}[1/p].$$

2) We define the sheaves on $\mathfrak{M}(w)$:

$$\Omega^{\lambda}_{\mathfrak{M}(w)} := j_{r,*}(\Omega^{\lambda}_{\mathfrak{M}^r(w)})^{G_r} \ \text{ and } \ \omega^{\lambda}_{\mathfrak{M}(w)} := j_{r,*}(\Omega^{\lambda}_{\mathfrak{M}^r(w)}[1/p])^{G_r}.$$

Remark 4.3. We can verify that we have the following isomorphism of sheaves on $\mathfrak{M}^r(w)$:

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes} B}(\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)}, \mathcal{O}_{\mathfrak{M}^r(w)}\hat{\otimes} B) \cong \Omega^{\lambda}_{\mathfrak{M}^r(w)}.$$

When $(B, \lambda) = (\mathcal{O}_L, \lambda)$ we denote by $\omega^{\lambda}_{M(w)}$ the invertible sheaf on $M(w)$ constructed in [5, Prop. 5.12]. Let $(B, \lambda) = (\Lambda_{\mathcal{U}}, \lambda_{\mathcal{U}})$, then using [5, Prop. 5.17] we obtain an invertible sheaf on $\mathcal{U} \times M(w)$. We can consider this sheaf as a sheaf of $\mathcal{O}_{M(w)}\hat{\otimes} B$-modules on $M(w)$ and we denote it by $\omega^{\lambda}_{M(w)}$. We have:

Proposition 4.4. *The sheaf $\omega^{\lambda}_{\mathfrak{M}(w)}$ is a locally free $\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes} B[1/p]$-module of rank 1. We have $\omega^{\lambda}_{\mathfrak{M}(w)} \cong \omega^{\lambda}_{M(w)}\hat{\otimes}_{\hat{\mathcal{O}}_{\mathcal{M}(w)}}\hat{\mathcal{O}}_{\mathfrak{M}(w)}$. Moreover, we have a Hecke-equivariant isomorphism of $B\hat{\otimes}\mathbb{C}_p$-modules with a semi-linear action of G_L:*

$$\mathrm{H}^1(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)) \simeq \mathrm{H}^0(M(w), \omega^{\lambda+2}_{M(w)})\hat{\otimes}_L \mathbb{C}_p\,.$$

Proof. From lemma 4.1 we deduce that the sheaf $\omega^{\lambda}_{\mathfrak{M}(w)}$ is a locally free $\mathcal{O}_{\mathfrak{M}(w)}\hat{\otimes} B[1/p]$-module of rank 1. To prove the isomorphism $\omega^{\lambda}_{\mathfrak{M}(w)} \cong \omega^{\lambda}_{M(w)}\hat{\otimes}_{\hat{\mathcal{O}}_{\mathcal{M}(w)}}\hat{\mathcal{O}}_{\mathfrak{M}(w)}$ we localize at connected small affine objects and use lemma 4.1.

To prove the last statement we use the Leray spectral sequence (see [1, §2.2.7]):

$$\mathrm{H}^p(\mathcal{M}(w)^{\mathtt{ket}}, R^q v_{\mathfrak{M}(w),*}(\omega^{\lambda}_{\mathfrak{M}(w)}(1))) \Rightarrow \mathrm{H}^{p+q}(\mathfrak{M}(w), \omega^{\lambda}_{\mathfrak{M}(w)}(1)).$$

Following [1, Prop. 2.12] and [1, Lemma 2.24], we prove that $R^q v_{\mathfrak{M}(w),*}(\omega^{\lambda}_{\mathfrak{M}(w)}(1))$ is the sheaf associated to the presheaf on $\mathcal{M}(w)^{\mathtt{ket}}$: $\mathcal{U} = (Spf(R), N) \mapsto \mathrm{H}^q(\mathcal{G}_U, \omega^{\lambda}_{\mathfrak{M}(w)}(1)(\overline{R}, \overline{N}))$, here $\mathcal{G}_U$ is the Kummer étale geometric fundamental group of $\mathcal{U}$ for a base point.

Using the identification $\omega^{\lambda}_{\mathfrak{M}(w)} \cong \omega^{\lambda}_{M(w)} \hat{\otimes}_{\hat{\mathcal{O}}_{\mathcal{M}(w)}} \hat{\mathcal{O}}_{\mathfrak{M}(w)}$, [12, Thm. 3], [12, Rmk. page 138] and the Kodaira-Spencer isomorphism we deduce that:

$$R^q v_{\mathfrak{M}(w),*}(\omega^{\lambda}_{\mathfrak{M}(w)}(1)) = \begin{cases} 0 & \text{if } q > 1 \\ \omega^{\lambda+2}_{M(w)} \otimes_K \mathbb{C}_p & \text{if } q = 1 \\ \omega^{\lambda}_{M(w)} \otimes_K \mathbb{C}_p(1) & \text{if } q = 0 \end{cases}$$

Finally, the theorem follows from the fact that M(w) is an affinoid and then Kiehl's theorem imply that:

$$\mathrm{H}^1(\mathcal{M}(w)^{\mathtt{ket}}, \omega^{\lambda}_{M(w)} \otimes_K \mathbb{C}_p(1)) = \mathrm{H}^1(\mathrm{M}(w), \omega^{\lambda}_{M(w)} \otimes_K \mathbb{C}_p(1)) = 0.$$

□

4.3. **The morphism.** In subsection 3.5 we have constructed the sheaves $\mathcal{A}^{\circ}_{\lambda,h}, \mathcal{D}^{\circ}_{\lambda,h}, \mathcal{A}^{\circ}_{\lambda}, \mathcal{D}^{\circ}_{\lambda}, \mathcal{A}_{\lambda}$ and $\mathcal{D}_{\lambda}$ on $\mathfrak{M}(N,p)$. We use the functor $\nu^* : \mathrm{Sh}(\mathfrak{M}(N,p)) \to \mathrm{Sh}(\mathfrak{M}(w))$ (see subsection 1.4) to obtain sheaves on $\mathfrak{M}(w)$ denoted respectively by $\mathcal{A}^{\circ}_{\lambda,h}(w)$, $\mathcal{D}^{\circ}_{\lambda,h}(w)$, $\mathcal{A}^{\circ}_{\lambda}(w)$, $\mathcal{D}^{\circ}_{\lambda}(w)$, $\mathcal{A}_{\lambda}(w)$ and $\mathcal{D}_{\lambda}(w)$. Remark that we have a morphism of sheaves on $\mathfrak{M}(w)$, $\nu^*(\mathcal{O}_{\mathfrak{M}(N,p)}) \to \mathcal{O}_{\mathfrak{M}(w)}$, then we deduce in the way as at the end of [5, §3] that for each $n \in \mathbb{N}$ we have a morphism:

$$\mathrm{H}^1(\mathfrak{M}(N,p), \mathcal{D}_{\lambda,n} \otimes \mathcal{O}_{\mathfrak{M}(N,p)}/\pi^n) \to \mathrm{H}^1(\mathfrak{M}(w), \mathcal{D}_{\lambda,n}(w) \otimes \mathcal{O}_{\mathfrak{M}(w)}/\pi^n). \tag{5}$$

The rest of this subsection is devoted to constructing the morphism from the overconvergent cohomology to the overconvergent modular forms. To do that we proceed in three steps:

Step 1) By definition the map (2) induces a map of sheaves on $\mathfrak{M}^r(w)$, $j_r^*\mathcal{T}_0 \to \mathcal{F}'_{\mathfrak{M}^r(w)}$, then we obtain a morphism:

$$\alpha : \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)} \to \mathrm{Hom}_{\mathbb{Z}_p^*}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^{\lambda}_{\mathfrak{M}^r(w)}). \tag{6}$$

Let $n \in \mathbb{N}$, we have an inclusion of $\mathcal{O}_L$-modules

$$A^{\circ}_{\lambda}/\pi^n A^{\circ}_{\lambda} \hookrightarrow Hom_{\mathbb{Z}_p^{\times}}(\mathbb{Z}_p^{\times} \times \mathbb{Z}_p, (B/\pi^n)^{\lambda}),$$

here $\mathbb{Z}_p^{\times}$ acts on $\mathbb{Z}_p^{\times} \times \mathbb{Z}_p$ by multiplication on each coordinate and on B/π^n by λ. From this map we obtain a map of sheaves on $\mathfrak{M}(w)$: $\mathcal{A}^{\circ}_{\lambda,n}(w) \otimes (\hat{\mathcal{O}}_{\mathfrak{M}(w)}/\pi^n \hat{\mathcal{O}}_{\mathfrak{M}(w)}) \to \mathrm{Hom}_{\mathbb{Z}_p^{\times}}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^{\lambda}_{\mathfrak{M}^r(w),n})$. Applying the functor j_r^* and varying n we obtain a morphism of continuous sheaves on $\mathfrak{M}^r(w)$:

$$\beta : j_r^*\mathcal{A}^{\circ}_{\lambda}(w) \otimes \hat{\mathcal{O}}_{\mathfrak{M}(w)} \to \mathrm{Hom}_{\mathbb{Z}_p^{\times}}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^{\lambda}_{\mathfrak{M}^r(w)}). \tag{7}$$

Lemma 4.5. *There exists a morphism $\gamma : \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)} \to j_r^*\mathcal{A}^{\circ}_{\lambda}(w) \otimes \hat{\mathcal{O}}_{\mathfrak{M}^r(w)}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)} & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p^{\times}}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^{\lambda}_{\mathfrak{M}^r(w)}) \\ \downarrow \gamma & \nearrow \beta & \\ j_r^*\mathcal{A}^{\circ}_{\lambda}(w) \otimes \hat{\mathcal{O}}_{\mathfrak{M}^r(w)} & & \end{array}$$

Proof. We write $\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w)} := (\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n})_{n\in\mathbb{N}}$, then it is enough to prove that for each $n\in\mathbb{N}$ there exists a map $\gamma_n : \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} \to j_r^*\mathcal{A}^\circ_\lambda(w)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}$ such that we have the following commutative diagram:

$$\begin{array}{ccc} \Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n} & \xrightarrow{\alpha_n} & \mathrm{Hom}_{\mathbb{Z}_p^\times}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^\lambda_{\mathfrak{M}^r(w),n}) \\ \downarrow{\scriptstyle \gamma_n} & \nearrow{\scriptstyle \beta_n} & \\ j_r^*\mathcal{A}^\circ_\lambda(w)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n\hat{\mathcal{O}}_{\mathfrak{M}^r(w)} & & \end{array}$$

We fix $n\in\mathbb{N}$. It is enough to prove this statement after applying the localization functors (see 1.5). Let $\mathcal{U} = (\mathrm{R}, N)$ be a connected small affine object of $\mathfrak{M}(w)$, $g\in\Upsilon_{\mathcal{U}}$ and $\eta := \mathrm{Spec}(\mathbb{K})$ be a geometric generic point of $\mathrm{Spm}(R_L)$. In the same way as in 1.5 we consider the $\mathbb{Z}_p$-module $T := \varprojlim_n (\mathcal{A}_{L,\eta}[p^n]^1)^\vee$ and $T_0 \subset T$ the inverse image of $\mathcal{C}^\vee_{1,L,\eta} - \{0\}$ under the map $\pi : T \to \mathcal{A}^\vee_{1,\eta} \to \mathcal{C}^\vee_{1,L,\eta}$. Moreover, we fix a basis $\{\epsilon_0, \epsilon_1\}$ of T such that $\pi(\epsilon_1) = 0$. Let $x, y : T \to \mathbb{Z}_p$ defined by $x(a\epsilon_0 + b\epsilon_1) = a$ and $y(a\epsilon_0 + b\epsilon_1) = b$ here $a, b\in\mathbb{Z}_p$. If we denote $D = \hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}(\overline{R}, \overline{N}, g)$ then we have:

$$j_r^*\mathcal{A}^\circ_\lambda(w)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}(\overline{R}, \overline{N}, g) = \oplus_{i=0}^{p^r-1}\oplus_{h=0}^{\infty} D\lambda(x)\left(\frac{y/x - i}{p^r}\right)^h 1_{i+p^r\mathbb{Z}_p}(y/x),$$

$$\mathrm{Hom}_{\mathbb{Z}_p^\times}(j_r^*\mathcal{T}_0, \hat{\mathcal{O}}^\lambda_{\mathfrak{M}^r(w),n})(\overline{R}, \overline{N}, g) = \{f : T_0 \to D \mid f \text{ is continuous}, f(cx) = \lambda(c)f(x), \forall c\in\mathbb{Z}_p^\times, x\in T_0\}.$$

Let $\{\varepsilon_0, \varepsilon_1\}$ be a $\hat{\overline{R}}$-basis of $T\otimes_{\mathbb{Z}_p}\hat{\overline{R}}$ such that $\mathrm{dlog}_{\mathcal{A}}(\varepsilon_0) \equiv \epsilon_0 \bmod p^{r-v}$ and $\varepsilon_1 \equiv \epsilon_1 \bmod p^{r-v}$, here $\mathrm{dlog}_{\mathcal{A}}$ is the map defined in [5, §3]. Moreover we denote by $X, Y : T\otimes_{\mathbb{Z}_p}\hat{\overline{R}} \to \hat{\overline{R}}$ the maps defined by $X(u\varepsilon_0 + v\varepsilon_1) = u$ and $Y(u\varepsilon_0 + v\varepsilon_1) = v$, then we have:

$$\Omega^{\lambda^{-1}}_{\mathfrak{M}^r(w),n}(\overline{R}, \overline{N}, g) = \mathrm{Hom}_S(Sw_0, D^\lambda) = D\lambda(X),$$

here $S = \mathbb{Z}_p^\times(1 + p^{r-v}\hat{\overline{R}})$, $w_0 = \mathrm{dlog}_{\mathcal{A}}(\varepsilon_0)$, D^λ means D considered with the action given by λ and $\lambda(X) : Sw_0 \to D$ is defined by $\lambda(X)(axw_0) = \lambda(a)x^{b_\lambda}w_0$. From these descriptions it is clear that we need to verify that $\alpha_n(\lambda(X)) \in \mathrm{Image}(\beta_n)$. We can write $X = ux + vy$ for $u, v\in\hat{\overline{R}}$ such that $u \equiv 1 \bmod p^{r-v}\hat{\overline{R}}$ and $v \equiv 0 \bmod p^{r-v}\hat{\overline{R}}$, then we have $\alpha_n(\lambda(X)) = \lambda(x)(u + vy/x)^{b_\lambda}$. From the conditions on u and v we deduce that $(u + vy/x)^{b_\lambda} \in 1 + p^{r-v}\hat{\overline{R}}\langle y/x\rangle$, then we deduce that $\alpha_n(\lambda(X)) \in \mathrm{Image}(\beta_n)$. □

Remark 4.6. It is natural to consider $\mathcal{T}_r \subset \mathcal{T}$ defined as the inverse image of $\mathcal{C}_r^\vee - \mathcal{C}_r^\vee[p^{r-1}]$ under the map $\mathcal{T} \to \mathcal{C}_r^\vee$. But it is not difficult to prove that we have $\mathcal{T}_r = \mathcal{T}_0$.

Step 2) From remark 4.3, the identification $\mathrm{Hom}_{\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}}(j_r^*\mathcal{A}^\circ_\lambda(w)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}, \hat{\mathcal{O}}_{\mathfrak{M}^r(w)}) \cong \mathrm{Hom}_B(j_r^*\mathcal{A}^\circ_\lambda(w), B)\otimes_B \hat{\mathcal{O}}_{\mathfrak{M}^r(w)}$ and the last lemma we obtain the morphism:

$$\delta : \mathrm{Hom}_B(j_r^*\mathcal{A}^\circ_\lambda(w), B)\otimes_{\mathcal{O}_L}\hat{\mathcal{O}}_{\mathfrak{M}^r(w)} \to \Omega^\lambda_{\mathfrak{M}^r(w)}.$$

Lemma 4.7. *For each $n \in \mathbb{N}$ there exists $m_n \geq n$ and a morphism $j_r^*\mathcal{D}^\circ_{\lambda,m_n}(w) \otimes \hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n\hat{\mathcal{O}}_{\mathfrak{M}^r(w)} \to \Omega^\lambda_{\mathfrak{M}^r(w),n}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_B(j_r^*\mathcal{A}^\circ_{\lambda,n}(w),B)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n & \longrightarrow & \Omega^\lambda_{\mathfrak{M}^r(w),n} \\ \downarrow & & \uparrow \\ \mathrm{Hom}_B(j_r^*\mathcal{A}^\circ_{\lambda,m_n}(w),B)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^{m_n} & \longrightarrow & j_r^*\mathcal{D}^\circ_{\lambda,m_n}(w)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^{m_n}. \end{array}$$

Proof. Fix $n \in \mathbb{N}$. Using notations as in the proof of lemma 4.5 we have:

$$\mathrm{Hom}_B(j_r^*\mathcal{A}^\circ_{\lambda,n}(w),B)\otimes\hat{\mathcal{O}}_{\mathfrak{M}^r(w)}/\pi^n(\overline{R},\overline{N},g) = \oplus_{i=0}^{p^r-1}\oplus_{h=0}^{\infty} D(\lambda(x)\left(\frac{y/x-i}{p^r}\right)^h 1_{i+p^r\mathbb{Z}_p}(y/x))^\vee,$$

$$\Omega^\lambda_{\mathfrak{M}^r(w),n}(\overline{R},\overline{N},g) = D\lambda(X)^\vee.$$

From the proof of lemma 4.5 we can write $\alpha_n(\lambda(X)) = \sum_{h=0}^\infty a_h\lambda(x)(y/x)^h$. Any $m \in \mathbb{N}$ such that $m > n + \max\{h \in \mathbb{N} \mid a_h \not\equiv 0 \bmod \pi^n\}$, satisfy conditions of lemma. ☐

Step 3) Let $n \in \mathbb{N}$, then from step 2) we obtain a map:

$$\mathcal{D}^\circ_{\lambda,m_n}(w) \to (j_{r,*}j_r^*\mathcal{D}^\circ_{\lambda,m_n}(w))^{G_r} \to \Omega^\lambda_{\mathfrak{M}(w),n}$$

This map induces a morphism $(\mathcal{D}^\circ_{\lambda,m_n}(w))_{n\in\mathbb{N}}[1/p] \to \omega^\lambda_{\mathfrak{M}(w)}$ of ind-continuous sheaves on $\mathfrak{M}(w)$. Using proposition 3.11 and the morphism (5) we obtain a morphism:

$$\mathrm{H}^1(\mathfrak{M}(N,p),\mathcal{D}_\lambda\otimes\hat{\mathcal{O}}_{\mathfrak{M}(N,p)}) \to \mathrm{H}^1(\mathfrak{M}(w),\omega^\lambda_{\mathfrak{M}(w)}) \tag{8}$$

Remark 4.8. Consider the case $(B,\lambda) = (\Lambda_U,\lambda_U)$. Proposition 3.11, the map (8) and proposition 4.4 allow us to define a morphism:

$$\mathrm{H}^1(\Gamma,D_U)\hat{\otimes}_L\mathbb{C}_p(1) \to \mathrm{H}^0(M(w),\omega^{\lambda_U+2}_{M(w)})\hat{\otimes}_L\mathbb{C}_p,$$

where $\omega^{\lambda_U+2}_{M(w)}$ is the modular sheaf constructed in [5, Prop. 5.17] (in fact using the characters λ_U+2 instead of λ_U). If $(B,\lambda) = (\mathcal{O}_L,\lambda)$ and $\lambda \in U(L)$, then in the same way as before we construct a morphism:

$$\mathrm{H}^1(\Gamma,D_\lambda)\hat{\otimes}_L\mathbb{C}_p(1) \to \mathrm{H}^0(M(w),\omega^{\lambda+2}_{M(w)})\hat{\otimes}_L\mathbb{C}_p\,.$$

These morphisms are equivariant for the actions of the Galois group G_L and the Hecke operators. Moreover, it is straightforward from the definitions that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma,D_U)\hat{\otimes}_L\mathbb{C}_p(1) & \longrightarrow & \mathrm{H}^0(M(w),\omega^{\lambda_U+2}_{M(w)})\hat{\otimes}_L\mathbb{C}_p \\ \downarrow & & \downarrow \\ \mathrm{H}^1(\Gamma,D_\lambda)\hat{\otimes}_L\mathbb{C}_p(1) & \longrightarrow & \mathrm{H}^0(M(w),\omega^{\lambda+2}_{M(w)})\hat{\otimes}_L\mathbb{C}_p \end{array}$$

5. Main Results

We fix first an integer $r > 0$, $w \in \mathbb{Q}$, $0 < w < \frac{1}{p^{r-2}(p+1)}$ and a finite extension of $\mathbb{Q}_p$, L, satisfying conditions at the beginning of subsection 4.2. Fix $U \subset \mathcal{W}_r$, a wide open disk defined over L and let $\lambda_U : \mathbb{Z}_p^\times \to \Lambda_U^\times$ denote the associated universal weight, i.e., if $t \in \mathbb{Z}_p^\times$ then $t^{\lambda_U}(x) = t^x$ for all $x \in U(L)$. Define $B_U := \Lambda_U \otimes_{\mathcal{O}_L} L$. Then B_U is a Banach $L-$algebra and moreover it is a principle ideal domain. Let us then fix a slope $h \in \mathbb{Q}$, $h \geq 0$. We suppose that U satisfies:

(1) There exists a classical weight $\lambda_0 \in U(L)$ such that if k_0 is the integer attached to λ_0, then $k_0 > h-1$;
(2) Both $\mathrm{H}^1(\Gamma, D_U)$ and $\mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})$ have slope h decompositions.

Remark 5.1. Condition (2) is satisfied for U small enough. This is a consequence of proposition 3.10 and [5, §6] (see also [6, Prop. 7.9] and discussion before [6, Prop. 7.10]).

Recall that we have a $(B_U \hat{\otimes} \mathbb{C}_p)$-linear homomorphism

$$\Psi_U : \mathrm{H}^1(\Gamma, D_U)\hat{\otimes}_L \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})\hat{\otimes}_L \mathbb{C}_p,$$

which is equivariant for the actions of G_L and Hecke operators. Moreover it is compatible with specializations, i.e., we have:

Proposition 5.2. *If $\lambda \in U(L)$ is a classical weight, then the following diagram commutes:*

$$\begin{array}{ccc}
\mathrm{H}^1(\Gamma, D_U)\hat{\otimes}_L \mathbb{C}_p(1) & \xrightarrow{\Psi_U} & \mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})\hat{\otimes}_L \mathbb{C}_p \\
\downarrow & & \downarrow \alpha \\
\mathrm{H}^1(\Gamma, D_\lambda) \otimes_L \mathbb{C}_p(1) & \xrightarrow{\Psi_\lambda} & \mathrm{H}^0(M(w), \omega_w^{\lambda+2}) \otimes_L \mathbb{C}_p \\
\downarrow & & \uparrow \beta \\
\mathrm{H}^1(\Gamma, V_\lambda(1)) \otimes_L \mathbb{C}_p & \xrightarrow{\Psi_\lambda^{cl}} & \mathrm{H}^0(M(N,p), \omega^{\lambda+2}) \otimes_L \mathbb{C}_p,
\end{array}$$

where the left vertical maps and the map α are induced by the specializations (see 3.3 for details), the map β is a restriction and Ψ_λ^{cl} is the first projection obtained from proposition 2.1.

In fact, Ψ_U induces a $(B_U \hat{\otimes} \mathbb{C}_p)$-linear map

$$\Psi_U^{\leq h} : \mathrm{H}^1(\Gamma, D_U)^{\leq h}\hat{\otimes}_L \mathbb{C}_p(1) \longrightarrow \mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})^{\leq h}\hat{\otimes}_L \mathbb{C}_p,$$

equivariant for the actions of G_L and Hecke operators and compatible with specializations. In other words, the diagram

$$\begin{array}{ccc} \mathrm{H}^1(\Gamma, D_U)^{\leq h}\hat{\otimes}_L\mathbb{C}_p(1) & \xrightarrow{\Psi_U^{\leq h}} & \mathrm{H}^0(M(w), \omega_w^{\lambda_U+2})^{\leq h}\hat{\otimes}_L\mathbb{C}_p \\ \downarrow & & \downarrow \alpha \\ \mathrm{H}^1(\Gamma, D_\lambda)^{\leq h}\otimes_L\mathbb{C}_p(1) & \xrightarrow{\Psi_\lambda^{\leq h}} & \mathrm{H}^0(M(w), \omega_w^{\lambda+2})^{\leq h}\otimes_L\mathbb{C}_p \\ \downarrow & & \uparrow \beta \\ \mathrm{H}^1(\Gamma, V_\lambda(1))^{\leq h}\otimes_L\mathbb{C}_p & \xrightarrow{\Psi_\lambda^{cl}} & \mathrm{H}^0(M(N,p), \omega^{\lambda+2})^{\leq h}\otimes_L\mathbb{C}_p \end{array}$$

is also commutative.

For any wide open disk $V \subset U$, we denote by χ_V^{univ} the universal cyclotomic character attached to V, which is defined by the following composition:

$$G_L \xrightarrow{\chi} \mathbb{Z}_p^\times \xrightarrow{\lambda_V} B_V^\times \longrightarrow (B_V\hat{\otimes}\mathbb{C}_p)^\times,$$

where χ is the cyclotomic character of L. We prove the following theorem:

Theorem 5.3. *There exists a finite subset of weights $Z \subset U(\mathbb{C}_p)$ such that:*

(a) For each $\lambda \in U(L) - Z$ there is a finite dimensional $\mathbb{C}_p-$vector space $S_\lambda^{\leq h}$ endowed with trivial semilinear G_L-action and Hecke operators, such that we have natural G_L and Hecke equivariant isomorphisms

$$\mathrm{H}^1(\Gamma, D_\lambda)^{\leq h}\otimes_L\mathbb{C}_p(1) \cong \Big(\mathrm{H}^0(M(w), \omega_w^{\lambda+2})^{\leq h}\otimes_L\mathbb{C}_p\Big) \oplus \Big(S_\lambda^{\leq h}(\lambda+1)\Big),$$

where the first projection is $\Psi_\lambda^{\leq h}$.

(b) For every wide open disk $V \subset U$ defined over L such that $V(\mathbb{C}_p)\cap Z = \emptyset$, there is a finite free $B_V\hat{\otimes}_L\mathbb{C}_p-$module $S_V^{\leq h}$ endowed with trivial semilinear G_L-action and Hecke operators, for which we have a G_L and Hecke equivariant exact sequence

$$0 \longrightarrow S_V^{\leq h}(\chi\cdot\chi_V^{univ}) \longrightarrow \mathrm{H}^1(\Gamma, D_V)^{\leq h}\hat{\otimes}_L\mathbb{C}_p(1) \xrightarrow{\Psi_V^{\leq h}} \mathrm{H}^0(M(w), \omega_w^{\lambda_V+2})^{\leq h}\hat{\otimes}_L\mathbb{C}_p \longrightarrow 0.$$

Moreover, for any such open disk V, there is finite subset $Z' \subset V$ with the property that for any wide open disk $V' \subset V$ with $V'(\mathbb{C}_p)\cap Z' = \emptyset$, we have a natural G_L and Hecke equivariant isomorphism

$$\mathrm{H}^1(\Gamma, D_{V'})^{\leq h}\hat{\otimes}_L\mathbb{C}_p(1) \cong \Big(\mathrm{H}^0(M(w), \omega_w^{\lambda_{V'}+2})^{\leq h}\hat{\otimes}_L\mathbb{C}_p\Big) \oplus \Big(S_{V'}^{\leq h}(\chi_{V'}^{univ}\cdot\chi)\Big),$$

where the first projection is determined by $\Psi_V^{\leq h}$.

(c) Let V be as in (b) and $\lambda \in V(L)$, let π_λ be a uniformizer of B_V at λ. Then $S_V^{\leq h}/\pi_\lambda S_V^{\leq h} \cong S_\lambda^{\leq h}$ as Hecke modules.

The proof of the above theorem is similar to the proof of [2, Theorem 6.1] so we provide a sketch of the proof:

(1) First, let $\lambda \in U(L)$ and $\pi_\lambda \in B_U$ a uniformizer at λ. We prove that the specialization maps $D_U \to D_\lambda$ and $\omega_w^{\lambda_U} \to \omega_w^\lambda$ induce exact sequences:

$$\mathrm{H}^1(\Gamma, D_U) \xrightarrow{\pi_\lambda} \mathrm{H}^1(\Gamma, D_U) \longrightarrow \mathrm{H}^1(\Gamma, D_\lambda) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathrm{H}^0(M(w), \omega_w^{\lambda_U}) \xrightarrow{\pi_\lambda} \mathrm{H}^0(M(w), \omega_w^{\lambda_U}) \longrightarrow \mathrm{H}^0(M(w), \omega_w^\lambda) \longrightarrow 0.$$

(2) Then we show that there is a nonzero element $b \in (B_U \hat{\otimes} \mathbb{C}_p)$ such that $b \cdot \mathrm{Coker}(\Psi_U^{\leq h}) = 0$. To do that we use proposition 5.2, classicality results for modular symbols (see subsection 3.4) and for overconvergent mudular forms (see [17]).

(3) Let $Z_1 \subset U(\mathbb{C}_p)$ be the finite set of zeros of b and let $V \subset U$ be a wide open disk defined over L such that $V(L)$ contains a classical weight λ with attached integer k satisfying $k > h-1$ and $V(\mathbb{C}_p) \cap Z_1 = \emptyset$. We show that the restriction to V gives an exact sequence

$$0 \longrightarrow T_V^{\leq h} \longrightarrow \mathrm{H}^1(\Gamma, D_V)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \xrightarrow{\Psi_V^{\leq h}} \mathrm{H}^0(M(w), \omega_w^{\lambda_V+2})^{\leq h} \hat{\otimes}_L \mathbb{C}_p \longrightarrow 0,$$

(4) For each wide open V in (3), let $S_V^{\leq h} := T_V^{\leq h}(\chi^{-1} \cdot (\chi_V^{univ})^{-1})$. Using Sen's theory in families, we prove that $S_V^{\leq h}$ is a finite free $(B_V \hat{\otimes} \mathbb{C}_p)$-module with trivial semilinear G_L-action.

(5) Then we show that for each V as in (3) and $S_V^{\leq h}$ as in (4), there is a nonzero element $c \in B_V$ such that the localized exact sequence

$$0 \longrightarrow \left(S_V^{\leq h}(\chi \cdot \chi_V^{univ})\right)_c \longrightarrow \left(\mathrm{H}^1(\Gamma, D_V)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1)\right)_c \longrightarrow \left(\mathrm{H}^0(M(w), \omega_w^{\lambda_V+2})^{\leq h} \hat{\otimes}_L \mathbb{C}_p\right)_c \longrightarrow 0,$$

uniquely splits as a sequence of G_L-modules.

(6) Now let $Z' \subset V(\mathbb{C}_p)$ be the finite set of zeros of c in (5) and let $V' \subset V$ be a wide open disk defined over L such that $V'(L)$ contains a classical weight λ with attached integer k satisfying $k > h-1$ and $V'(\mathbb{C}_p) \cap Z' = \emptyset$. We have a canonical splitting of the exact sequence of G_L-modules:

$$0 \longrightarrow S_{V'}^{\leq h}(\chi \cdot \chi_{V'}^{univ}) \longrightarrow \mathrm{H}^1(\Gamma, D_{V'})^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \xrightarrow{\Psi_{V'}^{\leq h}} \mathrm{H}^0(M(w), \omega_w^{\lambda_{V'}+2})^{\leq h} \hat{\otimes}_L \mathbb{C}_p \longrightarrow 0.$$

This proves part (b) of the theorem.

(7) Now let Z_1 and V be as in (3) and $S_V^{\leq h}$ be as in (4). For any $\lambda \in V(L)$, specializing the exact sequence in (5) we obtain an exact sequence of $\mathbb{C}_p$-vector spaces with continuous, semilinear G_L-action

$$0 \longrightarrow S_\lambda^{\leq h}(\lambda+1) \longrightarrow \mathrm{H}^1(\Gamma, D_\lambda)^{\leq h} \hat{\otimes}_L \mathbb{C}_p(1) \xrightarrow{\Psi_\lambda^{\leq h}} \mathrm{H}^0(M(w), \omega_w^{\lambda+2})^{\leq h} \hat{\otimes}_L \mathbb{C}_p \longrightarrow 0.$$

Now let $Z_2 := \{\lambda \in U(L) - Z_1 \mid \lambda = (s,i), s = -1\}$ and let $Z := Z_1 \cup Z_2$. Then part (a) follows from the main result of [21].

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