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Vertex disjoint 4-cycles in bipartite tournaments

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ABSTRACT

Let $k \geq 2$ be an integer. Bermond and Thomassen conjectured that every digraph with minimum out-degree at least $2k-1$ contains k vertex-disjoint cycles. Recently Bai, Li and Li proved this conjecture for bipartite digraphs. In this paper we prove that every bipartite tournament with minimum out-degree at least $2k-2$, minimum in-degree at least 1 and partite sets of cardinality at least $2k$ contains k vertex-disjoint 4-cycles whenever $k \geq 3$. Finally, we show that every bipartite tournament with minimum degree $\delta = \min\{\delta^+, \delta^-\}$ at least $1.5k-1$ contains at least k vertex-disjoint 4-cycles.

1. Introduction and terminology

Bermond and Thomassen [5] posted the following conjecture, which relates the number of disjoint cycles in a digraph with the minimum out-degree.

Conjecture 1.1 ([5]). *Every digraph D with* $\delta^+(D) \geq 2k-1$ *has k disjoint cycles.*

This conjecture has been proved for general digraphs when $k = 2$, $k = 3$ and for tournaments [3,6,7,8]. Thomassen [8] established the existence of a finite integer $f(k)$ such that every digraph of minimum out-degree at least $f(k)$ contains k disjoint cycles. Alon [1] proved in 1996 that for every integer k, the value $64k$ is suitable for $f(k)$.

A bipartite tournament is an oriented complete bipartite graph. Observe that, the girth of any bipartite tournament containing a cycle is four. We denote a cycle of length four by C_4. Very recently, Bay, Li and Li [2], proved Conjecture 1.1 for bipartite tournaments as a consequence of another result related to the numbers of vertex disjoint cycles of a given length in bipartite tournaments with minimum out-degree at least $qr-1$, for $q \geq 2$ and $r \geq 1$ two integers. In this paper we will only consider bipartite tournaments. First, we present an alternative proof of this conjecture in a direct way for bipartite tournaments. We also prove that every bipartite tournament with minimum out-degree at least $2k-2$, minimum in-degree at least 1 and partite sets of cardinality at least $2k$ contains k disjoint 4-cycles whenever $k \geq 3$. Finally, we show that every bipartite tournament with both minimum out-degree and minimum in-degree at least $(3k-1)/2$, contains at least k disjoint cycles for all $k \geq 2$.

For terminology and notation we follow the book by Bang-Jensen and Gutin [4]. Through this work only finite digraphs without loops and multiple edges are considered. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. Two subgraphs D_1 and D_2 of D are disjoint if their vertex sets are disjoint. We denote by $\delta^+(D)$ the minimum out-degree of a vertex in D, by

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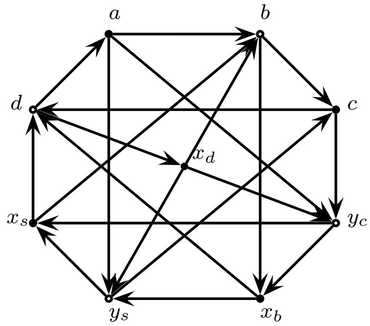

Fig. 1. Bipartite tournament with $\delta^+ = 2$ and $\delta^- = 1$ without two disjoint 4-cycles.

$\delta^-(D)$ the minimum in-degree of a vertex in D, and by $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ the minimum degree of D. Two vertices u and v are *twins* if $N^+(u) = N^+(v)$ and $N^-(u) = N^-(v)$. A vertex v with $d^-(v) = 0$ is called a *source*. Similarly, a vertex v with $d^+(v) = 0$ is called a *sink*. The minimum length of a cycle in D is called the girth of D. For a set $X \subseteq V(D)$, we use the notation $D[X]$ to denote the subdigraph of D induced by the vertices of X. Let uv be an arc of D. By *reversing the arc* uv, we mean that we replace the arc uv by the arc vu. The *converse* of a digraph D is the digraph H obtained from D by reversing all arcs.

1.1. Results

Conjecture 1.1 is proved for bipartite tournaments in [2]. In Theorem 1.1 we present an alternative proof of this result in a direct and short way. Our proof is the starting point for obtaining the rest of results contained in this paper.

Theorem 1.1. *Let $k \geq 2$ be an integer. If T is a bipartite tournament with $\delta^+(T) \geq 2k-1$ (or $\delta^-(T) \geq 2k-1$), then T has at least k disjoint cycles.*

By Theorem 1.1, Conjecture 1.1 holds for bipartite tournaments. Next we give sufficient conditions to prove that every bipartite tournament with minimum out-degree at least $2k-2$, minimum in-degree at least 1 and partite sets of cardinality at least $2k$ contains k disjoint 4-cycles whenever $k \geq 3$.

Theorem 1.2. *Let $k \geq 3$ be an integer. If T is a bipartite tournament with $\delta^+(T) \geq 2k-2$, $\delta^-(T) \geq 1$ and partite sets of cardinality at least $2k$, then T has at least k disjoint cycles.*

Remark 1.1. The following bipartite tournament (for $k = 2$) with $\delta^+ = 2$, $\delta^- = 1$, and partite sets of cardinality at least 4, has no two disjoint C_4, see Fig. 1. Hence, the condition $k = 3$ is necessary in Theorem 1.2.

Let T be the bipartite tournament with partite sets $X = \{a, c, x_b, x_s, x_d\}$ and $Y = \{b, d, y_s, y_c\}$. The arcs of T are the following:
$N^+(a) = \{b, y_s, y_c\} = N^+(x_d)$, $N^-(a) = \{d\} = N^-(x_d)$;
$N^+(c) = \{d, y_c\}$, $N^-(c) = \{b, y_s\}$;
$N^+(x_s) = \{b, d\}$, $N^-(x_s) = \{y_s, y_c\}$;
$N^+(x_b) = \{d, y_s\}$, $N^-(x_b) = \{b, y_c\}$.

Considering digraphs with a given girth, Bang-Jensen, Bessy and Thomassé [3] established the following conjecture.

Conjecture 1.2 ([3]). *Every digraph D with girth $g \geq 2$ and minimum degree $\delta^+(D) \geq \frac{g}{g-1}k$ contains k disjoint cycles.*

Clearly, in order to a bipartite digraph has k disjoint cycles it must have partite sets of cardinality at least $2k$. Hence, Conjecture 1.2 can be easily corrected adding this requirement on the cardinality of the partite sets.

Corollary 1.1. *Conjecture 1.2 holds for bipartite tournaments with $\delta^-(D) \geq 1$ and partite sets of cardinality at least $2k$ for $k = 2, 3, 4$.*

Theorem 1.3. *Every bipartite tournament with $\min\{\delta^+, \delta^-\} \geq 2$ has at least 2 disjoint cycles.*

Remark 1.2. Theorem 1.3, is best possible as shown for the bipartite tournament described in Remark 1.1, see Fig. 1.

Finally, we establish that if both the minimum out-degree and minimum in-degree are at least $(3k-1)/2$, then the bipartite tournament has at least k disjoint C_4.

Theorem 1.4. *Let $k \geq 2$ be an integer. If T is a bipartite tournament such that $\min\{\delta^+, \delta^-\} \geq (3k-1)/2$, then T has at least k disjoint cycles.*

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2. Proofs

Proof of Theorem 1.1. Note that if a bipartite tournament has r disjoint cycles, then it has r disjoint 4-cycles. Suppose that T has exactly r disjoint 4-cycles. Let $\mathcal{C}$ be a family of r disjoint 4-cycles and let T_1 be the subdigraph induced by $\mathcal{C}$. If $T_1 = T$, then every vertex $x \in V(T)$ is on a 4-cycle, yielding that $d^+(x) \leq 2(r-1)+1 = 2r-1$. By hypothesis $d^+(x) \geq \delta^+(T) \geq 2k-1$ yielding that $r \geq k$. If $T_1 \subset T$, let $T_2 = T - V(T_1)$, clearly T_2 is an acyclic digraph. Let $v \in V(T_2)$ such that $d^+_{T_2}(v) = 0$. Hence, $N^+(v) \subseteq V(T_1)$ and $d^+(v) \leq 2r$. By hypothesis $d^+(v) \geq \delta^+(T) \geq 2k-1$ yielding that $r \geq k$. ■

Proof of Theorem 1.2. Let $k \geq 3$ and let $T = (X, Y)$ be a bipartite tournament with $\delta^+(T) \geq 2k-2$ and $\delta^-(T) \geq 1$. By Theorem 1.1, we may assume that $\delta^+(T) = 2k-2$, and since $2k-2 > 2(k-1)-1$ and $k-1 \geq 2$, it follows by Theorem 1.1 that T has at least $k-1$ disjoint 4-cycles. Let us denote these cycles by $(a_i, b_i, c_i, d_i, a_i)$ for all $1 \leq i \leq k-1$ and let T_1 be the bipartite tournament induced by these $k-1$ cycles. Let $T_1 = (X_1, Y_1)$ with $X_1 = \{a_i, c_i : i = 1, \ldots, k-1\}$ and $Y_1 = \{b_i, d_i : i = 1, \ldots, k-1\}$. Let $T_2 = (X_2, Y_2)$ be the bipartite tournament induced by $V(T) \setminus V(T_1)$. Observe that T_2 is nonempty because by hypothesis the partite sets of T have cardinality at least $2k$. Clearly, if T_2 has a cycle we are done. Then we assume that T_2 is acyclic. In order to prove the existence of k disjoint cycles, we use the vertices of T_2 and the vertices of one of the cycles $(a_i, b_i, c_i, d_i, a_i)$ from T_1 to construct two new 4-cycles.

Without loss of generality, suppose that T_2 has a sink $x_s \in X_2$. Then $Y_2 = N^-(x_s)$ and $N^+(x_s) = Y_1$ because $d^+(x_s) \geq 2k-2 = |Y_1|$. Let $x \in X_1, y \in Y_1$. Since $|N^+(x) \cap Y_1|, |N^+(y) \cap X_1| \leq 2k-3$, and $d^+(x), d^+(y) \geq 2k-2$ it follows that for all $x \in X_1$ and for all $y \in Y_1$,

$$|N^+(x) \cap Y_2| \geq 1, \ \ |N^+(y) \cap (X_2 - x_s)| \geq 1. \tag{1}$$

Case 1. Suppose $x_s' \in X_2 - x_s$ is a sink of $T_2 - x_s$. Then $N^-(x_s') = Y_2$ and $N^+(x_s') = Y_1$, so that x_s and x_s' are twins in T. By (1), we can take $y_{a_i} \in N^+(a_i) \cap Y_2$ and $y_{c_i} \in N^+(c_i) \cap Y_2$. If there is $j \in \{1, \ldots, k-1\}$ such that are $y_{a_j} \neq y_{c_j}$, then $(a_j, y_{a_j}, x_s, d_j, a_j)$ and $(c_j, y_{c_j}, x_s', b_j, c_j)$ are two disjoint C_4 and we are done.

Let us assume that for all $i \in \{1, \ldots, k-1\}$,

$$N^+(a_i) \cap Y_2 = N^+(c_i) \cap Y_2 = \{y_i\} \text{ where } y_i \in Y_2. \tag{2}$$

Hence, $N^+(a_i) = (Y_1 - d_i) \cup \{y_i\}$, $N^+(c_i) = (Y_1 - b_i) \cup \{y_i\}$ for all $i \in \{1, \ldots, k-1\}$. This implies that $N^+(b_i) \cap X_1 = \{c_i\}$ and $N^+(d_i) \cap X_1 = \{a_i\}$ for all $i \in \{1, \ldots, k-1\}$ because T is a bipartite tournament. Thus, $N^+(b_i) - c_i, \ N^+(d_i) - a_i \subseteq X_2 \setminus \{x_s, x_s'\}$ for all $i \in \{1, \ldots, k-1\}$.

Let $x_{b_i} \in N^+(b_i) \cap X_2$ and $x_{d_i} \in N^+(d_i) \cap X_2$ (note that both are different from x_s and x_s'). Since $|N^+(x_{b_i}) \cap Y_1| \leq |Y_1 - b_i| = 2k-3$, there is $y_{b_i} \in N^+(x_{b_i}) \cap Y_2$ and similarly, there is $y_{d_i} \in N^+(x_{d_i}) \cap Y_2$. If $y_{b_j} \neq y_j$ for some $j \in \{1, \ldots, k-1\}$, then $(a_j, y_j, x_s, d_j, a_j)$ and $(x_{b_j}, y_{b_j}, x_s', b_j, x_{b_j})$ are two disjoint C_4. And if $y_{d_j} \neq y_j$ for some $j \in \{1, \ldots, k-1\}$, then $(c_j, y_j, x_s, b_j, c_j)$ and $(x_{d_j}, y_{d_j}, x_s', d_j, x_{d_j})$ are two disjoint C_4. In both cases we have k disjoint cycles. Therefore, assume that for all $i \in \{1, \ldots, k-1\}$,

$$N^+(x_{b_i}) \cap Y_2 = N^+(x_{d_i}) \cap Y_2 = N^+(a_i) \cap Y_2 = N^+(c_i) \cap Y_2 = \{y_i\}. \tag{3}$$

Since T_2 is acyclic, it follows that T_2 has a source. Suppose that $\hat{x} \in X_2 \setminus \{x_s, x_s'\}$ is a source of T_2. Since $d^-(\hat{x}) \geq 1$, it follows that there is some $y \in Y_1$ such that $y \in N^-(\hat{x})$. Then $y = b_i$ or $y = d_i$, so that $\hat{x} \in N^+(b_i) \cup N^+(d_i)$, yielding that $\hat{x} \in \{x_{b_i}, x_{d_i}\}$, which contradicts (3) because $Y_2 \subseteq N^+(\hat{x})$ and $|Y_2| \geq 2k - |Y_1| = 2k - 2k + 2 = 2$. Hence, every source of T_2 is $\hat{y} \in Y_2$, implying that $X_2 \subseteq N^+(\hat{y})$. Since $d^-(\hat{y}) \geq 1$, it follows that there is some $x \in X_1$ such that $x \in N^-(\hat{y})$ and $x = a_i$ or $x = c_i$. By (2), $\hat{y} = y_i$, which is a contradiction, because $x_{b_i}, x_{d_i} \in N^-(y_i)$. Thus, in Case 1, we have k disjoint cycles.

Case 2. Suppose that $y_s' \in Y_2$ is a sink of $T_2 - x_s$. Then $X_2 - x_s \subseteq N^-(y_s')$ and let $Z_s = N^-(y_s') \cap X_1$ with $|Z_s| \leq 1$, note that $N^+(y_s') = (X_1 - Z_s) \cup \{x_s\}$, because $d^+(y_s') \geq 2k-2$.

Case 2.1 Suppose that $|Z_s| = 1$. Without loss of generality, suppose that $Z_s = \{a_1\}$. By (1), let $x_{b_1} \in N^+(b_1) \cap (X_2 - x_s)$. Since $|N^+(x_{b_1}) \cap Y_1| \leq |Y_1 - b_1| = 2k-3$, there is $y_{b_1} \in N^+(x_{b_1}) \cap Y_2$. If $y_s' \neq y_{b_1}$, then $(y_s', c_1, d_1, a_1, y_s')$ and $(x_s, b_1, x_{b_1}, y_{b_1}, x_s)$ are two disjoint C_4, and we have k disjoint cycles. Therefore, we assume that

$$y_{b_1} = y_s' \text{ and } N^-(x_{b_1}) = (Y_2 - y_s') \cup \{b_1\}, \ \ N^+(x_{b_1}) = \{y_s'\} \cup (Y_1 - b_1). \tag{4}$$

By hypothesis $k \geq 3$, and by (1) we can take $y_{a_i} \in N^+(a_i) \cap Y_2$ and $x_{b_i} \in N^+(b_i) \cap (X_2 - x_s)$ for all $i = 2, \ldots, k-1$. Hence, by (4), we have $(a_i, y_{a_i}, x_{b_1}, d_i, a_i)$ and $(x_{b_i}, y_s', x_s, b_i, x_{b_i})$ are two disjoint C_4 because $y_{a_i} \neq y_s'$ since $y_s' \in N^-(a_i)$. Hence, we have k disjoint cycles.

Case 2.2 Suppose that $|Z_s| = 0$. Then $N^+(y_s') = X_1 \cup \{x_s\}$. If $x_s'' \in X_2 - x_s$ is a sink of $T_3 = T_2 - \{x_s, y_s'\}$, then $|N^-(x_s'') \cap Y_1| \leq 1$. Since $k \geq 3$ there exists a cycle $(a_i, b_i, c_i, d_i, a_i)$ such that $b_i, d_i \in N^+(x_s'')$. By (1) there is a vertex $y_{a_i} \in N^+(a_i) \cap Y_2$ and a vertex $x_{b_i} \in N^+(b_i) \cap X_2$. In this case $(d_i, a_i, y_{a_i}, x_s'', d_i)$ and $(b_i, x_{b_i}, y_s', x_s, b_i)$ are two disjoint cycles. Hence, T has k disjoint cycles.

Therefore, we may assume that any sink of T_3 is a vertex y_s'' of $Y_2 - y_s'$. Observe that y_s'' is a sink of $T_2 - \{x_s\}$, and by *Case* 2.1 we may assume that $N^+(y_s'') = X_1 \cup \{x_s\}$. Let $x_{b_i} \in N^+(b_i) \cap (X_2 - x_s)$ and $x_{d_i} \in N^+(d_i) \cap (X_2 - x_s)$. Then $(a_i, b_i, x_{b_i}, y_s', a_i)$ and $(c_i, d_i, x_{d_i}, y_s'', c_i)$ are two disjoint cycles. Hence, T has k disjoint cycles.

Therefore we conclude that in either case T must have at least k disjoint cycles and the theorem holds. ■

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Proof of Corollary 1.1. The girth of a bipartite tournament containing a cycle is $g = 4$. Suppose that $k = 2$, and let T be a bipartite tournament with $\delta^+(T) \geq \lceil 8/3 \rceil = 3$. From Theorem 1.1, it follows that T has at least 2 disjoint cycles. Suppose $k = 3$, and let T be a bipartite tournament with $\delta^+(T) \geq \lceil 12/3 \rceil = 4 = 2 \cdot 3 - 2$. From Theorem 1.2, T has at least 3 disjoint cycles. Analogously, for $k = 4$, T has at least 4 disjoint cycles. ■

Proof of Theorem 1.3. Let $T = (X, Y)$ be a bipartite tournament with $\delta(T) = \min\{\delta^+, \delta^-\} \geq 2$. Thus, T is not acyclic and T has a 4-cycle $C = (a, b, c, d, a)$. Let $T' = (X', Y')$ be the bipartite tournament induced by $V(T) \setminus V(C)$. If T' is not acyclic, then we are done. Assume that T' is an acyclic bipartite tournament. In order to prove the existence of 2 disjoint cycles, we use the vertices of T' and the vertices of C to construct two new 4-cycles. Moreover, $|X'|, |Y'| \geq 2$, because for all $x \in \{a, c\} \cup X'$, $d^-(x) + d^+(x) = |\{b, d\}| + |Y'| \geq 4$; and for all $y \in \{b, d\} \cup Y'$, $d^-(y) + d^+(y) = |\{a, c\}| + |X'| \geq 4$. Without loss of generality, we may assume that $\hat{x} \in X'$ is a source of T'. Hence, $N^+(\hat{x}) = Y'$ and $N^-(\hat{x}) = \{b, d\}$. Moreover, T' has also a sink, let us distinguish the following cases according to where the sink is placed.

Case 1. Suppose T' has a sink $x_s \in X' - \hat{x}$. Then $N^-(x_s) = Y'$ and $N^+(x_s) = \{b, d\}$.

If there exists $y_0 \in Y'$ such that the vertices $\{c, y_0, a\}$ induce a path of length 2 in T, then (a, b, c, y_0, a) and $(\hat{x}, y, x_s, d, \hat{x})$, for $y \in Y' - y_0$ (or (c, d, a, y_0, c) and $(\hat{x}, y, x_s, b, \hat{x})$, for $y \in Y' - y_0$) are 2 disjoint 4-cycles in T, and we are done. Therefore, assume that

$$N^+(c) \cap N^-(a) \cap Y' = \emptyset \text{ and } N^-(c) \cap N^+(a) \cap Y' = \emptyset. \quad (5)$$

In this case, $|X'| \geq 3$, else $d^+(y) = 1$ or $d^-(y) = 1$ for every $y \in Y'$ which is a contradiction. Let us consider the acyclic bipartite tournament $\hat{T} = T' - \{\hat{x}, x_s\}$.

Case 1.1. Suppose that $\hat{T}$ has a source $\hat{x}' \in X' \setminus \{\hat{x}, x_s\}$. Then $N^+(\hat{x}') = Y'$ and $N^-(\hat{x}') = \{b, d\}$, that is, $\hat{x}$ and $\hat{x}'$ are twins in T.

If $x'_s \in X' \setminus \{\hat{x}, x_s, \hat{x}'\}$ is a sink of $\hat{T}$, then $N^-(x'_s) = Y'$ and $N^+(x'_s) = \{b, d\}$, yielding that $(\hat{x}, y, x_s, d, \hat{x})$ for $y \in Y'$ and $(\hat{x}', y', x'_s, b, \hat{x}')$ for $y' \in Y' - y$, are two disjoint 4-cycles, and we are done. Therefore any sink of $\hat{T}$ must be some $y'_s \in Y'$, so that $X' - x_s \subseteq N^-(y'_s)$. Let us show that $N^+(y'_s) = \{a, c\} \cup \{x_s\}$. Indeed, if $a \notin N^+(y'_s)$, by (5), $c \notin N^+(y'_s)$, yielding that $d^+(y'_s) \leq 1$ which is a contradiction. Then $N^+(y'_s) = \{a, c\} \cup \{x_s\}$. Thus, $(\hat{x}, y'_s, c, d, \hat{x})$ and $(\hat{x}', y, x_s, b, \hat{x}')$ are two disjoint 4-cycles for all $y \in Y' - y'_s$, and we are done.

Case 1.2. Suppose that $\hat{T}$ has a source $\hat{y}' \in Y'$. Then $N^+(\hat{y}') = X' - \hat{x}$ and $N^-(\hat{y}') = \{a, c\} \cup \{\hat{x}\}$ because (5) and $\delta^-(T) \geq 2$. If $x'_s \in X' \setminus \{\hat{x}, x_s\}$ is a sink of $\hat{T}$, then $(a, \hat{y}', x'_s, d, a)$ and $(\hat{x}, y, x_s, b, \hat{x})$ for $y \in Y' - \hat{y}'$, are two disjoint 4-cycles, and we are done. Hence any sink of $\hat{T}$ must be some $y'_s \in Y' - \hat{y}'$. Suppose $Y' = \{\hat{y}', y'_s\}$. Then for every $x \in X' \setminus \{\hat{x}, x_s\}$, $|N^+(x) \cap \{b, d\}| = |N^-(x) \cap \{b, d\}| = 1$. If $b \in N^+(x)$, then $(x, b, c, \hat{y}', x)$ and $(x_s, d, \hat{x}, y'_s, x_s)$ are two disjoint 4-cycles; and if $d \in N^+(x)$, then $(x, d, a, \hat{y}', x)$ and $(x_s, b, \hat{x}, y'_s, x_s)$ are two disjoint 4-cycles. Hence, we may assume that $|Y'| \geq 3$, then the 4-cycle $(a, \hat{y}', x, y'_s, a)$, for $x \in X' \setminus \{\hat{x}, x_s\}$, and the 4-cycle $(b, \hat{x}, y, x_s, b)$, for $y \in Y' \setminus \{\hat{y}', y'_s\}$, are two disjoint cycles, and we are done.

Case 2. Suppose that $y_s \in Y'$ is a sink of T'. Then $N^-(y_s) = X'$ and $N^+(y_s) = \{a, c\}$. Consider the bipartite tournament $\hat{T} = T' - \{\hat{x}, y_s\}$, which is clearly acyclic.

Case 2.1. Some vertex $\hat{x}' \in X' - \hat{x}$ is a source of $\hat{T}$. Then $N^+(\hat{x}') = Y'$ and $N^-(\hat{x}') = \{b, d\}$.

If $y'_s \in Y' - y_s$ is a sink of $\hat{T}$, then it is also a sink of T', yielding that $N^-(y'_s) = X'$ and $N^+(y'_s) = \{a, c\}$. In this case $(\hat{x}, y_s, a, b, \hat{x})$ and $(\hat{x}', y'_s, c, d, \hat{x}')$ are two disjoint 4-cycles in T, and we are done. Therefore, any sink of $\hat{T}$ is some $x'_s \in X' \setminus \{\hat{x}, \hat{x}'\}$. Then $Y' - y_s \subseteq N^-(x'_s)$ and $|N^+(x'_s) \cap \{b, d\}| \geq 1$ since $y_s \in N^+(x'_s)$ and $d^+(x'_s) \geq 2$. Let $v' \in \{b, d\} \cap N^+(x'_s)$, $v \in \{b, d\} - v'$ and $\alpha \in \{a, c\} \cap N^-(v)$. Then $(\hat{x}, y_s, \alpha, v, \hat{x})$ and $(\hat{x}', y, x'_s, v', \hat{x}')$ for $y \in Y' - y_s$, are two disjoint 4-cycles. This gives that T has at least 2 disjoint cycles, and we are done.

Case 2.2. Every source of $\hat{T}$ is some $\hat{y}' \in Y' - y_s$. Then $X' - \hat{x} \subseteq N^+(\hat{y}')$ and $|N^-(\hat{y}') \cap \{a, c\}| \geq 1$ because $\hat{x} \in N^-(\hat{y}')$ and $d^-(\hat{y}') \geq 2$. Hence, there is $t \in N^-(\hat{y}') \cap \{a, c\}$, implying that $(t, \hat{y}', x, y_s, t)$ for all $x \in X' - \hat{x}$ is a 4-cycle in T.

If $y'_s \in Y' \setminus \{y_s, \hat{y}'\}$ is a sink of $\hat{T}$, then it is also a sink of T', yielding that $N^-(y'_s) = X'$ and $N^+(y'_s) = \{a, c\}$. Then $(\hat{x}, y'_s, z, w, \hat{x})$ where $z \in \{a, c\} - t$, $w \in \{b, d\}$ and $zw \in A(T)$, is a 4-cycle disjoint with $(t, \hat{y}', x, y_s, t)$ for $x \in X' - \hat{x}$. Thus, T has at least 2 disjoint cycles, and we are done.

If $x'_s \in X' - \hat{x}$ is a sink of $\hat{T}$, then $Y' - y_s \subseteq N^-(x'_s)$, and $|N^+(x'_s) \cap \{b, d\}| \geq 1$ because $y_s \in N^+(x'_s)$ and $d^+(x'_s) \geq 2$. If $|Y'| \geq 3$, $(\hat{x}, y, x'_s, w, \hat{x})$ for $y \in Y' \setminus \{y_s, \hat{y}'\}$ and $w \in \{b, d\} \cap N^+(x'_s)$, is a 4-cycle disjoint with $(t, \hat{y}', x, y_s, t)$ for all $x \in X' \setminus \{\hat{x}, x'_s\}$, and we are done. Thus, assume that $Y' = \{y_s, \hat{y}'\}$. If $N^-(\hat{y}') \cap \{a, c\} = \{c\}$, then $N^+(\hat{y}') \cap \{a, c\} = \{a\}$, yielding that $N^+(a) = \{b\}$ which is a contradiction. Therefore $N^-(\hat{y}') \cap \{a, c\} = \{a, c\}$. Similarly, if $|N^+(x'_s) \cap \{b, d\}| = 2$, then $N^-(x'_s) = \{\hat{y}'\}$ which is a contradiction. If $x'_s b, dx'_s \in A(T)$, then (x'_s, b, c, d, x'_s) and $(a, \hat{y}', x, y_s, a)$ are two disjoint C_4 and we are done. If $bx'_s, x'_s d \in A(T)$, then (x'_s, d, a, b, x'_s) and $(c, \hat{y}', x, y_s, c)$ for $x \in X' \setminus \{\hat{x}, x'_s\}$, are two disjoint C_4.

Therefore we conclude that T must have at least 2 disjoint cycles. ■

Proof of Theorem 1.4. If $k = 2$ the result holds by Theorem 1.3. Let $k = 3, 4$, and observe that $\lceil (3k - 1)/2 \rceil = 2k - 2$ for these two values. Let $T = (X, Y)$ and note that for all $x \in X$, $d(x) = d^-(x) + d^+(x) = |Y| \geq 2(2k - 2) > 2k$; and for all $y \in Y$, $d(y) = d^-(y) + d^+(y) = |X| \geq 2(2k - 2) > 2k$. Hence, by Theorem 1.2 the theorem holds for $k = 3, 4$. Thus, assume $k \geq 5$. We reason by induction on k, so assume that the theorem holds for any value less than or equal to $k - 1$, that is, T has $k - 1$ disjoint cycles by the induction hypothesis. Let us denote these cycles by $(a_i, b_i, c_i, d_i, a_i)$ for all $i = 1, 2, \ldots, k - 1$ and let

$T_1 = (X_1, Y_1)$, $T_2 = (X_2, Y_2)$, and $T = (X_1 \cup X_2, Y_1 \cup Y_2)$ be the same as in Theorem 1.2. Without loss of generality, assume that $\hat{x} \in X_2$ is a source of T_2, that is, $Y_2 \subseteq N^+(\hat{x})$. Let $\hat{V} = N^+(\hat{x}) \cap Y_1$, then $Y_1 \setminus \hat{V} = N^-(\hat{x})$ and $|\hat{V}| \le (k-3)/2$, because $|Y_1| = 2k-2$ and $\delta(T) \ge (3k-1)/2$. Observe that $|X_2| > 2$, because if $|X_2| = 2$, then $|\hat{V}| \ge (3k-1)/2 - 2 > (k-3)/2$, which is a contradiction. As in the proof of Theorem 1.2, we will find two disjoint C_4 using vertices of just one cycle $(a_i, b_i, c_i, d_i, a_i)$ and vertices of $X_2 \cup Y_2$. Hence, T will have k disjoint cycles. Since T_2 is acyclic, it has also a sink. Let us distinguish the following cases according the location of a sink of T_2.

Case 1. T_2 has a sink $x_s \in X_2 - \hat{x}$. Then $Y_2 \subseteq N^-(x_s)$ and let $V_s = N^-(x_s) \cap Y_1$. Therefore $Y_1 \setminus V_s = N^+(x_s)$ and $|V_s| \le (k-3)/2$, because $|Y_1| = 2k-2$ and $\delta(T) \ge (3k-1)/2$. Let us consider the acyclic bipartite tournament $T_3 = T_2 - \{\hat{x}, x_s\}$.

Case 1.1. T_3 has a source $\hat{x}' \in X_2 \setminus \{\hat{x}, x_s\}$. Then $Y_2 \subseteq N^+(\hat{x}')$ and let $\hat{V}' = N^+(\hat{x}') \cap Y_1$. Therefore $Y_1 \setminus \hat{V}' = N^-(\hat{x}')$ and $|\hat{V}'| \le (k-3)/2$.

Case 1.1.1. $x'_s \in X_2 \setminus \{\hat{x}, x_s, \hat{x}'\}$ is a sink of T_3. Then $Y_2 \subseteq N^-(x'_s)$ and $Y_1 \setminus V'_s = N^+(x'_s)$ where $V'_s = N^-(x'_s) \cap Y_1$ and $|V'_s| \le (k-3)/2$. If there exists $i \in \{1, \dots, k-1\}$ such that $|\{b_i, d_i\} \cap (\hat{V} \cup \hat{V}' \cup V_s \cup V'_s)| = 0$, then $(\hat{x}, y, x_s, d_i, \hat{x})$ for $y \in Y_2$, and $(\hat{x}', y', x'_s, b_i, \hat{x}')$ for $y' \in Y_2 - y$, are two disjoint 4-cycles and we are done. Thus, we assume for all $i \in \{1, \dots, k-1\}$ that $|\{b_i, d_i\} \cap (\hat{V} \cup \hat{V}' \cup V_s \cup V'_s)| \ge 1$. For $h = 1, 2$, let $R_h = \{i \in \{1, \dots, k-1\} : |\{b_i, d_i\} \cap (\hat{V} \cup \hat{V}' \cup V_s \cup V'_s)| = h\}$. We have

$$|\hat{V} \cup \hat{V}' \cup V_s \cup V'_s| = 2|R_2| + |R_1| = |R_2| + k - 1.$$

Moreover, let $I = (V_s \cup \hat{V}) \cap (V'_s \cup \hat{V}')$, then $|\hat{V} \cup \hat{V}' \cup V_s \cup V'_s| \le 2(k-3) - |I|$, which implies that $|R_2| \le k - 5 - |I|$. Therefore, $|R_1| = k - 1 - |R_2| \ge k - 1 - (k - 5 - |I|) = 4 + |I|$. Hence, there exists $i \in R_1$ such that $|\{b_i, d_i\} \cap I| = 0$. Without loss of generality, suppose $b_i \notin \hat{V} \cup \hat{V}' \cup V_s \cup V'_s$. As $d_i \notin I$ then $d_i \notin V_s \cup \hat{V}$ or $d_i \notin V'_s \cup \hat{V}'$. Without loss of generality, suppose that $d_i \notin V_s \cup \hat{V}$, then $(\hat{x}, y, x_s, d_i, \hat{x})$ for $y \in Y_2$, and $(\hat{x}', y', x'_s, b_i, \hat{x}')$ for $y' \in Y_2 - y$, are two disjoint 4-cycles and we are done.

Case 1.1.2. $y'_s \in Y_2$ is a sink of T_3. Thus, $X_2 - x_s \subseteq N^-(y'_s)$, and $N^+(y'_s) = (X_1 \setminus Z'_s) \cup \{x_s\}$ where $Z'_s = N^-(y'_s) \cap X_1$ with $|Z'_s| \le (k-1)/2$ because $\delta(T) \ge (3k-1)/2$. Let $I = (V_s \cup \hat{V}) \cap \hat{V}'$ and $R = Y_1 \setminus (\hat{V} \cup \hat{V}' \cup V_s)$. For $h = 1, 2$, let $R_h = \{j \in \{1, \dots, k-1\} : |\{b_j, d_j\} \cap R| = h\}$ and $L_h = \{j \in \{1, \dots, k-1\} : |\{a_j, c_j\} \cap Z'_s| = h\}$. Then $2|R_2| + |R_1| = |R|$ and $2|L_2| + |L_1| = |Z'_s|$. Suppose that there is $j \in R_2$ such that $|\{a_j, c_j\} \cap Z'_s| \le 1$. Without loss of generality, suppose that $a_j \notin Z'_s$, then $(\hat{x}', y'_s, a_j, b_j, \hat{x}')$, and $(\hat{x}, y, x_s, d_j, \hat{x})$ for $y \in Y_2 - y'_s$ are two disjoint C_4, and we are done. Therefore we suppose that for all $j \in R_2$, $|\{a_j, c_j\} \cap Z'_s| = 2$, that is,

$$|R_2| \le |L_2|. \tag{6}$$

Since $|Y_1| = |\hat{V} \cup \hat{V}' \cup V_s \cup R| \le 3(k-3)/2 - |I| + |R|$, and $|Y_1| = 2k - 2$, it follows that $|R| \ge (k+5)/2 + |I|$, and by (6), $(k+5)/2 + |I| \le |R| = 2|R_2| + |R_1| \le 2|L_2| + |R_1|$. Let $W = \{j \in R_1 \setminus L_1 : |\{b_j, d_j\} \cap I| = 0\}$. If $W = \emptyset$, then $|R_1 \setminus L_1| \le |I|$ yielding that $(k+5)/2 + |I| \le 2|L_2| + |R_1| \le 2|L_2| + |L_1| + |I| = |Z'_s| + |I| \le (k-1)/2 + |I|$, which is a contradiction. Therefore $W \ne \emptyset$. Suppose that $W \subset L_2$. Then $|W| + |R_2| \le |L_2|$ because $W \cap R_2 = \emptyset$ by definition of W, and by (6). As $|W| = |R_1| - |L_1| - |I|$ we have $|R_2| + |R_1| \le |L_2| + |L_1| + |I|$. Adding $|R_2|$ on both sides of this inequality we have $|R| \le |R_2| + |L_2| + |L_1| + |I| \le 2|L_2| + |L_1| + |I| = |Z'_s| + |I| \le (k-1)/2 + |I|$, which is a contradiction because $|R| \ge (k+5)/2 + |I|$. It follows that there exists $\ell \in W \setminus L_2$, that is, $|Z'_s \cap \{a_\ell, c_\ell\}| = 0$, $|\{b_\ell, d_\ell\} \cap R| = 1$ and $|\{b_\ell, d_\ell\} \cap I| = 0$. Without loss of generality, suppose $b_\ell \in R$ and $d_\ell \notin R$. Since $d_\ell \notin I$ we have $d_\ell \notin V_s \cup \hat{V}$ or $d_\ell \notin \hat{V}'$. Thus, if $d_\ell \notin V_s \cup \hat{V}$, then $(\hat{x}', y'_s, a_\ell, b_\ell, \hat{x}')$, and $(\hat{x}, y, x_s, d_\ell, \hat{x})$ for $y \in Y_2 - y'_s$ are two disjoint C_4, and we are done. If $d_\ell \notin \hat{V}'$, then $(\hat{x}, y, x_s, b_\ell, \hat{x})$ and $(\hat{x}', y'_s, c_\ell, d_\ell, \hat{x}')$ for $y \in Y_2 - y'_s$ are two disjoint C_4, and we are done.

Case 1.2. Any source of T_3 is some $\hat{y}' \in Y_2$. Then $X_2 - \hat{x} \subseteq N^+(\hat{y}')$ and let $\hat{Z}' = X_1 \cap N^+(\hat{y}')$ with $|\hat{Z}'| \le (k-1)/2$ such that $(X_1 \setminus \hat{Z}') \cup \{\hat{x}\} = N^-(\hat{y}')$, because $\delta(T) \ge (3k-1)/2$. Observe that $|Y_2| > 2$ because otherwise $|\hat{Z}'| \ge (3k-1)/2 - 2$ which is a contradiction.

Case 1.2.1. $x'_s \in X_2 \setminus \{\hat{x}, x_s\}$ is a sink of T_3. This case is the same as *Case 1.1.2.* by considering the converse digraph of T.

Case 1.2.2. $y'_s \in Y_2 - \hat{y}'$ is a sink of T_3. Thus $X_2 - x_s \subseteq N^-(y'_s)$ and $N^+(y'_s) = (X_1 \setminus Z'_s) \cup \{x_s\}$ where $Z'_s = N^-(y'_s) \cap X_1$ with $|Z'_s| \le (k-1)/2$. For $h = 0, 1, 2$, let $L_h = \{i \in \{1, \dots, k-1\} : |\{a_i, c_i\} \cap (\hat{Z}' \cup Z'_s)| = h\}$ and $R_h = \{i \in \{1, \dots, k-1\} : |\{b_i, d_i\} \cap (\hat{V} \cup V_s)| = h\}$. If there is $i \in (L_0 \cup L_1) \cap (R_0 \cup R_1)$, then without loss of generality we may assume that $a_i \notin \hat{Z}' \cup Z'_s$ and $b_i \notin \hat{V} \cup V_s$. Hence, $(\hat{y}', x, y'_s, a_i, \hat{y}')$, for $x \in X_2 \setminus \{\hat{x}, x_s\}$, and $(\hat{x}, y, x_s, b_i, \hat{x})$, for $y \in Y_2 \setminus \{\hat{y}', y'_s\}$, are two disjoint cycles, and we are done. Thus, we must suppose that $(L_0 \cup L_1) \cap (R_0 \cup R_1) = \emptyset$ or equivalently, $L_0 \cup L_1 \subseteq R_2$ and $R_0 \cup R_1 \subseteq L_2$. Since $|\hat{Z}' \cup Z'_s| \le k - 1$ it follows that $|X_1 \setminus (\hat{Z}' \cup Z'_s)| = 2|L_0| + |L_1| = 2k - 2 - |\hat{Z}' \cup Z'_s| \ge k - 1 = |L_0| + |L_1| + |L_2|$ yielding that $|L_0| \ge |L_2|$ and so $|L_0| + |L_1| \ge (k-1)/2$, $|L_2| \le (k-1)/2$, and $|R_0| + |R_1| \le (k-1)/2$ because $R_0 \cup R_1 \subseteq L_2$. Furthermore, since $|\hat{V} \cup V_s| \le k - 3$, it follows that $|Y_1 \setminus (\hat{V} \cup V_s)| = 2|R_0| + |R_1| = 2k - 2 - |\hat{V} \cup V_s| \ge k + 1 = |R_0| + |R_1| + |R_2| + 2$, yielding $|R_0| + |R_1| \ge (k-1)/2$, and therefore $|R_0| + |R_1| = (k-1)/2$. Hence, $2|R_0| + |R_1| = |R_0| + (k-1)/2 \ge k + 1$, and so $|R_0| \ge (k+1)/2$, which is a contradiction.

Case 2. T_2 has a sink $y_s \in Y_2$. Then $X_2 \subseteq N^-(y_s)$ and let $Z_s = X_1 \cap N^-(y_s)$ with $|Z_s| \le (k-3)/2$ such that $N^+(y_s) = X_1 \setminus Z_s$. Let us consider the bipartite tournament $T_3 = T_2 - \{\hat{x}, y_s\}$ which is clearly acyclic.

Case 2.1. Some vertex $\hat{x}' \in X_2 - \hat{x}$ is a source of T_3. Then $Y_2 \subseteq N^+(\hat{x}')$ and $N^-(\hat{x}') = Y_1 \setminus \hat{V}'$ where $\hat{V}' = N^+(\hat{x}') \cap Y_1$ with $|\hat{V}'| \le (k-3)/2$.

Case 2.1.1. If some $y'_s \in Y_2 - y_s$ is a sink of T_3. Then $X_2 \subseteq N^-(y'_s)$ and $N^+(y'_s) = X_1 \setminus Z'_s$ where $Z'_s = N^-(y'_s) \cap X_1$ with $|Z'_s| \le (k-3)/2$. For $h = 0, 1, 2$, let $L_h = \{i \in \{1, \dots, k-1\} : |\{a_i, c_i\} \cap (Z_s \cup Z'_s)| = h\}$. Then $2|L_0| + |L_1| = |X_1 \setminus (Z_s \cup Z'_s)| \ge 2k - 2 - (k - 3 - |Z_s \cap Z'_s|) = k + 1 + |Z_s \cap Z'_s|$.

Suppose that there is $i \in L_0$, that is $|\{a_i, c_i\} \cap (Z_s \cup Z'_s)| = 0$, such that $|\{b_i, d_i\} \cap (\hat{V} \cap \hat{V}')| = 0$. Without loss of generality, suppose that $b_i \notin \hat{V}$ and $d_i \notin \hat{V}'$. Then $(\hat{x}', y_s, c_i, d_i, \hat{x}')$ and $(\hat{x}, y'_s, a_i, b_i, \hat{x})$ are disjoint 4-cycles in T and we are done. Therefore we assume that for all $i \in L_0$, $|\{b_i, d_i\} \cap (\hat{V} \cap \hat{V}')| \geq 1$ yielding that $|L_0| \leq |\hat{V} \cap \hat{V}'| \leq (k-3)/2$. Therefore $|L_1| + |L_0| \geq k + 1 + |Z_s \cap Z'_s| - |L_0| \geq (k+5)/2 + |Z_s \cap Z'_s|$. Hence, there is $i \in L_0 \cup L_1$ (i.e. $|\{a_i, c_i\} \cap (Z_s \cup Z'_s)| \leq 1$) such that $|\{a_i, c_i\} \cap (Z_s \cap Z'_s)| = 0$, and $|\{b_i, d_i\} \cap (\hat{V} \cap \hat{V}')| = 0$ because $|\hat{V} \cap \hat{V}'| \leq (k-3)/2$. Without loss of generality, suppose that $a_i \notin Z_s \cup Z'_s$ and $c_i \notin Z_s$. Then if $b_i \notin \hat{V}$ and $d_i \notin \hat{V}'$, then $(\hat{x}', y_s, c_i, d_i, \hat{x}')$ and $(\hat{x}, y'_s, a_i, b_i, \hat{x})$ are disjoint 4-cycles in T and we are done. If $b_i \notin \hat{V}'$ and $d_i \notin \hat{V}$, then $(\hat{x}, y_s, c_i, d_i, \hat{x})$ and $(\hat{x}', y'_s, a_i, b_i, \hat{x}')$ are disjoint 4-cycles in T. Hence, we are done.

Case 2.1.2. Any sink of T_3 is $x'_s \in X_2 \setminus \{\hat{x}, \hat{x}'\}$. Thus, $Y_2 - y_s \subset N^-(x'_s)$ and let $V'_s = N^-(x'_s) \cap Y_1$ with $|V'_s| \leq (k-1)/2$ such that $(Y_1 \setminus V'_s) \cup \{y_s\} = N^+(x'_s)$. (Observe that this case is similar to Case 1.1.2 but now $|Z_s| \leq (k-3)/2$ and $|V'_s| \leq (k-1)/2$). Let $I = \hat{V} \cap (\hat{V}' \cup V'_s)$ and $R = Y_1 \setminus (\hat{V} \cup \hat{V}' \cup V'_s)$. For $h = 1, 2$, let $R_h = \{j \in \{1, \dots, k-1\} : |\{b_j, d_j\} \cap R| = h\}$ and $L_h = \{j \in \{1, \dots, k-1\} : |\{a_j, c_j\} \cap Z'_s| = h\}$. Then $2|R_2| + |R_1| = |R|$ and $2|L_2| + |L_1| = |Z_s|$. Suppose that there is $j \in R_2$ such that $|\{a_j, c_j\} \cap Z_s| \leq 1$. Without loss of generality, suppose that $a_j \notin Z_s$, then $(\hat{x}', y_s, a_j, b_j, \hat{x}')$, and $(\hat{x}, y, x'_s, d_j, \hat{x})$ for $y \in Y_2 - y_s$ are two disjoint C_4, and we are done. Therefore we suppose that for all $j \in R_2$, $|\{a_j, c_j\} \cap Z_s| = 2$, that is,

$$|R_2| \leq |L_2|. \tag{7}$$

Since $|Y_1| = |\hat{V} \cup \hat{V}' \cup V'_s \cup R| \leq (3k-7)/2 - |I| + |R|$, and $|Y_1| = 2k - 2$ it follows that $|R| \geq (k+3)/2 + |I|$ and by (7), $(k+3)/2 + |I| \leq |R| = 2|R_2| + |R_1| \leq 2|L_2| + |R_1|$. Let $W = \{j \in R_1 \setminus L_1 : |\{b_j, d_j\} \cap I| = 0\}$. If $W = \emptyset$, then $|R_1 \setminus L_1| \leq |I|$ yielding that $(k+3)/2 + |I| \leq 2|L_2| + |R_1| \leq 2|L_2| + |L_1| + |I| = |Z'_s| + |I| \leq (k-1)/2 + |I|$, which is a contradiction. Therefore $W \neq \emptyset$. If $W \subset L_2$, then $|W| + |R_2| \leq |L_2|$ because $W \cap R_2 = \emptyset$ by definition of W, and by (7). As $|W| = |R_1| - |L_1| - |I|$ we have $|R_2| + |R_1| \leq |L_2| + |L_1| + |I|$. Adding $|R_2|$ to both sides of the inequality we have $|R| \leq |R_2| + |L_2| + |L_1| + |I| \leq 2|L_2| + |L_1| + |I| = |Z_s| + |I| \leq (k-3)/2 + |I|$, which is a contradiction because $|R| \geq (k+3)/2 + |I|$. It follows that there exists $\ell \in W \setminus L_2$, that is, $|Z_s \cap \{a_\ell, c_\ell\}| = 0$, $|\{b_\ell, d_\ell\} \cap R| = 1$ and $|\{b_\ell, d_\ell\} \cap I| = 0$. Without loss of generality, suppose $b_\ell \in R$. Since $d_\ell \notin I$ we have $d_\ell \notin V'_s \cup \hat{V}'$ or $d_\ell \notin \hat{V}$. Thus, if $d_\ell \notin V'_s \cup \hat{V}'$, then $(\hat{x}, y_s, a_\ell, b_\ell, \hat{x})$, and $(\hat{x}', y, x'_s, d_\ell, \hat{x}')$ for $y \in Y_2 - y_s$ are two disjoint C_4, and we are done. If $d_\ell \notin \hat{V}$, then $(\hat{x}', y, x'_s, b_\ell, \hat{x}')$ and $(\hat{x}, y_s, c_\ell, d_\ell, \hat{x})$ for $y \in Y_2 - y_s$ are two disjoint C_4, and we are done.

Case 2.2. Every source of T_3 is a vertex $\hat{y}' \in Y_2 - y_s$. Therefore, $X_2 - \hat{x} \subset N^+(\hat{y}')$ and $N^-(\hat{y}') = (X_1 \setminus \hat{Z}') \cup \{\hat{x}\}$ where $\hat{Z}' = N^+(\hat{y}') \cap X_1$ with $|\hat{Z}'| \leq (k-1)/2$. Observe that $|Y_2| > 2$.

Case 2.2.1. Some $y'_s \in Y_2 \setminus \{y_s, \hat{y}'\}$ is a sink of T_3. This case is the same as *Case 2.1.2.* by considering the converse digraph of T.

Case 2.2.2. Any sink of T_3 is a vertex $x'_s \in X_2 - \hat{x}$. Then $Y_2 - y_s \subset N^-(x'_s)$, and let $V'_s = N^-(x'_s) \cap Y_1$ with $|V'_s| \leq (k-1)/2$ such that $N^+(x'_s) = (Y_1 \setminus V_s) \cup \{y_s\}$. Since $|\hat{Z}' \cup Z_s| \leq k-2$ and $|\hat{V} \cup V'_s| \leq k-2$, $|X_1 \setminus (\hat{Z}' \cup Z_s)| \geq 2k - 2 - (k-2) = k$ and $|Y_1 \setminus (\hat{V} \cup V'_s)| \geq k$. Hence, there exists $\ell \in \{1, \dots, k-1\}$, such that $|(\hat{Z}' \cup Z_s) \cap \{a_\ell, c_\ell\}| \leq 1$ and $|(\hat{V} \cup V'_s) \cap \{b_\ell, d_\ell\}| \leq 1$. Without loss of generality, suppose that $a_\ell \notin \hat{Z}' \cup Z_s$ and $d_\ell \notin \hat{V} \cup V'_s$. Then $(\hat{x}, y, x'_s, d_\ell, \hat{x})$ for $y \in Y_2 \setminus \{y_s, \hat{y}'\}$, is a C_4 disjoint with $(\hat{y}, x, y_s, a_\ell, \hat{y})$ for all $x \in X_2 \setminus \{\hat{x}, x'_s\}$, and we are done.

Therefore, we conclude that T must have at least k disjoint cycles. ■

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