

1 Characteristic polynomials of production matrices 2 for geometric graphs

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4 Abstract

5 An $n \times n$ production matrix for a class of geometric graphs has the
6 property that the numbers of these geometric graphs on up to n vertices
7 can be read off from the powers of the matrix. Recently, we obtained
8 such production matrices for non-crossing geometric graphs on point sets
9 in convex position [6]. In this note, we determine the characteristic poly-
10 nomials of these matrices. Then, the Cayley-Hamilton theorem implies
11 relations among the numbers of geometric graphs with different num-
12 bers of vertices. Further, relations between characteristic polynomials
13 of production matrices for geometric graphs and Fibonacci numbers are
14 revealed.

15 1 Introduction

16 A *geometric graph* on a point set \mathcal{S} is a graph with vertex set \mathcal{S} whose edges are
17 straight-line segments with endpoints in \mathcal{S} . It is called *non-crossing* if no two
18 edges intersect except at common endpoints. Here, we consider non-crossing ge-
19 ometric graphs on sets \mathcal{S} of n points in convex position for the following graph
20 classes: triangulations, matchings, spanning trees, forests, spanning paths, and
21 all geometric graphs on n vertices. The numbers of these graphs are well known,
22 see for instance the work of Flajolet and Noy [4]. Recently, in [6], we counted
23 such geometric graphs by using an $n \times n$ matrix A_n , called *production matrix*,
24 associated to the graph class. The numbers of these graphs on a certain number
25 of vertices are then given by (a column of) powers of A_n . In order to derive a
26 production matrix, first the graphs on $i \leq n$ vertices are partitioned according
27 to the degree of a specified root vertex. Each part is counted in the elements
28 of an n -element integer vector \vec{v}^i , and hence the sum of the elements gives the
29 number of geometric graphs on i vertices. The *production matrix* A_n is such
30 that $\vec{v}^{i+1} = A_n \vec{v}^i = A_n^{i+1-c} \vec{v}^c$, when starting with a vector \vec{v}^c for a constant
31 number of vertices, which will usually be $(1, 0, \dots, 0)^\top$. To find the matrix A_n ,

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$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$
(a) Triangulations	(b) Matchings	(c) Spanning trees
$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 0 & 2 & 4 & 4 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$
(d) Forests	(e) Geometric graphs	(f) Paths

Figure 1: Production matrices for six different graph classes, for $n = 6$. Matrix (f) is for paths on at most $n/2$ points.

32 the graphs are implicitly arranged in a tree structure (called *generating tree*),
33 s.t., for each graph on i vertices and with root degree j , the number of its de-
34 scendants on $i + 1$ vertices with root degree ℓ (for each ℓ) is known. Generating
35 trees are the basis of the *ECO method* [1], and have been used to obtain matrix
36 representations for combinatorial objects [3, 8]. Here we omit how the produc-
37 tion matrices for geometric graphs are obtained, and only refer to works by
38 Hurtado and Noy [7] for a generating tree of triangulations, and by Hernando
39 et al. [5] for a generating tree of spanning trees; and also to [6]. Figure 1 shows
40 the obtained production matrices for the studied graph classes, for $n = 6$. The
41 matrix for triangulations was well-known, see e.g. [3, 8]. All matrices except
42 for the matrix for paths are upper Hessenberg matrices. The different structure
43 of the production matrix for paths, which is formed by four blocks, is due to
44 a necessary distinction between paths with root vertex begin an endpoint of a
45 path or an interior point. Also, for paths, the degree of the root vertex is de-
46 fined in a different way, based on visibility (we omit the definition here). For all
47 other graph classes, the degree is the number of edges incident to the root vertex.

48
49 The aim of the present work is to analyze these matrices by finding their char-
50 acteristic polynomials. Previously, these polynomials were only known for trian-
51 gulations [2], matchings, and spanning trees [6]. Here we determine the charac-
52 teristic polynomial of the production matrices for geometric graphs, for forests,
53 and for paths, solving the problem which was left open in [6]. An application of
54 the characteristic polynomial of a production matrix follows from the Cayley-
55 Hamilton theorem, which then implies a relation among the numbers of graphs
56 with given root vertex degree. For example, via the production matrix for trian-
57 gulations and $t_n(\lambda)$ one obtains the relation $\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n-j+1}{j} (-1)^{n+j} C_{n-j} = 0$

$$\begin{aligned}
(a) \text{ triangulations} \quad & t_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \binom{n-k+1}{k} (-1)^{n+k} \lambda^{n-k} \\
(b) \text{ matchings} \quad & m_n(\lambda) = \sum_{k=0}^{\lceil n/2 \rceil} \sum_{j=0}^{n-k} \binom{n-k+1}{k} \binom{n-k}{j} (-1)^{n+k} \lambda^j \\
(c) \text{ spanning trees} \quad & s_n(\lambda) = \sum_{k=0}^n \binom{2k+2}{n-k} \lambda^k (-1)^k \\
(d) \text{ forests} \quad & f_n(\lambda) = \sum_{k=0}^n \lambda^k (-1)^k \sum_{\ell=0}^n \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell}{2\ell+j-n} \binom{2\ell+j-n}{k+\ell+j-n} \\
(e) \text{ geometric graphs} \quad & g_n(\lambda) = 2^{n-1} + \sum_{k=1}^n \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-j}{k} 2^{n-k} (-1)^k \lambda^k \\
(f) \text{ paths} \quad & p_n(\lambda) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n/2-1}{k} (-1)^{n/2+k+1} \lambda^k + \sum_{k=0}^{n/2} 2 \frac{n-k}{n} \binom{n/2}{k} (-1)^{n/2+k} \lambda^{n/2+k}
\end{aligned}$$

Figure 2: The characteristic polynomials of $n \times n$ production matrices for several graph classes.²

58 for the Catalan numbers C_i ; we only refer to [6] for the full example.

59 Figure 2 shows the characteristic polynomials of the production matrices for
60 the six studied graph classes. We give an outline of the proofs for geometric
61 graphs and for forests in the following two sections.

62 2 Geometric graphs

Theorem 2.1 *The characteristic polynomial $g_n(\lambda)$ of the $n \times n$ production matrix G_n of geometric graphs is*

$$g_n(\lambda) = 2^{n-1} + \sum_{k=1}^n \sum_{j=0}^{k-1} \binom{k-1}{j} \binom{n-j}{k} 2^{n-k} (-1)^k \lambda^k.$$

63 The proof is by induction on n , using the following two lemmas. First, we
64 develop the determinant of $G_n - \lambda I_n$, where I_n is the $n \times n$ identity matrix, to
65 obtain a recurrence equation for $g_n(\lambda)$, and then to make induction work, we
66 need to show a binomial identity.

67 **Lemma 2.2** *The characteristic polynomial $g_n(\lambda)$ of the matrix G_n satisfies the*
68 *recurrence relation*

$$69 \quad g_n(\lambda) = (2 - \lambda)g_{n-1}(\lambda) - 2\lambda g_{n-2}(\lambda). \quad (1)$$

Lemma 2.3 *For $\ell \geq 2$ and $n \in \mathbb{N}$,*

$$\sum_{j=0}^{\ell-2} \left(\binom{\ell-2}{j} \binom{n-2-j}{\ell-1} - \binom{\ell-2}{j-1} \binom{n-1-j}{\ell-1} \right) = \binom{n-\ell}{\ell-1}.$$

²The polynomials are also defined for $\lambda = 0$ by setting the indeterminate form $0^0 := 1$.

70 **3 Forests**

Theorem 3.1 *The characteristic polynomial $f_n(\lambda)$ of the $n \times n$ production matrix F_n of forests is*

$$f_n(\lambda) = \sum_{k=0}^n \lambda^k (-1)^k \sum_{\ell=0}^n \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell}{2\ell+j-n} \binom{2\ell+j-n}{k+\ell+j-n}.$$

71 **Lemma 3.2** *The characteristic polynomial $f_n(\lambda)$ of the matrix F_n satisfies the*
 72 *recurrence relation*

73
$$f_n(\lambda) = (1 - \lambda)f_{n-1}(\lambda) + (1 - 2\lambda)f_{n-2}(\lambda) - \lambda f_{n-3}(\lambda). \quad (2)$$

To prove Theorem 3.1, we apply Riordan arrays as described for instance in [8]. Consider the infinite matrix

$$M = \begin{pmatrix} 1 - \lambda & 1 - 2\lambda & -\lambda & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let \vec{w}^0 be the vector $(1, 0, 0, \dots)^\top$, and let $\vec{w}^i = (f_i(\lambda), f_{i-1}(\lambda), f_{i-2}(\lambda), \dots, f_1(\lambda), 1, 0, 0, \dots)^\top$. Then $\vec{w}^{i+1} = M \cdot \vec{w}^i$. Note that \vec{w}^n is the first column of M^n . Using the notation from [8], we have that the Z -sequence is $\{1 - \lambda, 1 - 2\lambda, -\lambda, 0, \dots\}$ and the A -sequence is $\{1, 0, \dots\}$; it follows that $h(t) = 1$ and

$$\begin{aligned} d(t) &= \frac{d_0}{1 - t \cdot Z(th(t))} = \frac{1}{1 - t(1 - \lambda + (1 - 2\lambda)t - \lambda t^2)} = \\ &= \sum_{k=0}^{\infty} t^k (t+1)^k (1 - \lambda - \lambda t)^k; \end{aligned}$$

74 the characteristic polynomial $f_n(\lambda)$ is then the coefficient of t^n in this expression.

75 **3.1 Relation to Fibonacci numbers**

76 Denote with $Fib(n)$ the n -th Fibonacci number, where $Fib(1) = Fib(2) = 1$,
 77 and $Fib(n) = Fib(n-1) + Fib(n-2)$ for $n > 2$.

78 **Corollary 3.3** *The determinant of F_n is the Fibonacci number $Fib(n+1)$.*

79 Since the determinant of F_n is $f_n(0)$, Corollary 3.3 follows immediately from
 80 Equation (2) and $f_0(0) = f_1(0) = 1$. Another family of matrices whose determi-
 81 nants are Fibonacci numbers can be found for instance in [9]. We remark that
 82 when we substitute $\lambda = 0$ in the formula of Theorem 3.1 we also obtain, after
 83 simplification, the following well known identity.

Corollary 3.4 *The Fibonacci numbers satisfy the equation*


$$Fib(n+1) = \sum_{k=0}^n \binom{k}{n-k}.$$

84 Also, the characteristic polynomial $f_n(\lambda)$ can be expressed recursively using
85 Fibonacci numbers.

Corollary 3.5

$$f_n(\lambda) + \lambda f_{n-1}(\lambda) = Fib(n+1) - \sum_{k=2}^n Fib(k+2)\lambda f_{n-k}(\lambda).$$

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