Random Strategies are Nearly Optimal for Generalized van der Waerden Games

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Abstract

In a $(1:q)$ Maker-Breaker game, one of the central questions is to find (or at least estimate) the maximal value of q that allows Maker to win the game. Based on the ideas of Bednarska and Łuczak [1], who studied biased H-games, we prove general winning criteria for Maker and Breaker and a hypergraph generalization of their result. Furthermore, we study the biased version of a strong generalization of the van der Waerden games introduced by Beck [2] and apply our criteria to determine the threshold bias of these games up to constant factor. As in the result of [1], the random strategy for Maker is again the best known strategy.

Keywords: Maker-Breaker games, Generalised van der Waerden games, General Winning Criteria

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1 Introduction

A *k-term arithmetic progression* is a set of integers that can be written in the form $\{a, a+d, a+(k-1)d\}$ for some $a, d \in \mathbb{Z}$, $k \geq 3$ and $d \neq 0$. Beck introduced the *van der Waerden games* [2] as the Maker-Breaker positional games played on the board $[n] = \{1, ..., n\}$. Two players, Maker and Breaker, take turns occupying vertices that have not been previously occupied by their opponent. Maker wins if he covers a k-term arithmetic progression for some given $k \geq 3$ and Breaker's goal is to keep Maker from achieving his. The well-known theorem of van der Waerden [5] states that for every $k \geq 3$ there exists an integer $W(k)$ such that any two-colouring of $[W(k)]$ contains a monochromatic k-term arithmetic progression. Combining this with a standard strategy stealing argument (cf. [9]) it is not hard to see that Maker wins if $n \geq W(k)$. Beck defined $W^\star(k)$ to be the least integer, such that Maker has a winning strategy in the unbiased game when the board has size $n \geq W^\star(k)$, and established that $W^\star(k) = 2^{k(1+o(1))}$. This moderate growth is in strong contrast to the known bounds of the van der Waerden number (cf. [6,8,4]).

The *biased version* of positional games is a widely studied direction, with deep connections to the theory of random structures. The notion was first suggested by Chvátal and Erdős [7] while investigating the connectivity game and the triangle-building game played on the edges of the complete graph. Given a hypergraph $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ and a positive integer *bias* q, we define the *q-biased Maker-Breaker game* $\mathbf{G}(\mathcal{H}; q)$ as the game where *Maker* and *Breaker* again take turns occupying previously unoccupied vertices from $V(\mathcal{H})$, with Maker occupying one vertex in each round and Breaker occupying up to q. Maker wins if his selection completely covers an edge from $\mathcal{H}$ and Breaker wins otherwise. Note that by definition the game cannot end in a draw. For a more complete introduction to biased positional games, see [9].

Given a hypergraph $\mathcal{H}$, one is interested in determining the *threshold bias* $q(\mathcal{H})$, defined to be the smallest integer $q \in \mathbb{N}$ for which Breaker has a winning strategy in the q-biased game $\mathbf{G}(\mathcal{H}; q)$. In [1] the authors found the right order of the threshold bias for the game played on the edge set of the complete graph K_n, where Maker's goal is to occupy a copy of a given fixed graph H. Building on the ideas laid out in [1], we analyze the biased version of Beck's van der Waerden games, showing that the threshold bias satisfies $q(\mathcal{W}_k(n)) = \Theta\left(n^{1/(k-1)}\right)$. Here $\mathcal{W}_k(n)$ denotes the hypergraph of k-APs in $[n]$. We in fact obtain tight results for a much broader class of games, in which Maker's goal is to occupy a solution to an arbitrary given linear homogeneous system of equations. The most important aspects of these results will follow from two general winning criteria for Maker and Breaker, respectively. These criteria will also allow us to generalize the results of [1] to uniform hypergraphs.

2 General Winning Criteria

In order to simplify notation we often identify the hypergraph $\mathcal{H}$ with its edge set $E(\mathcal{H})$. We denote the number of vertices of a hypergraph $\mathcal{H}$ by $v(\mathcal{H})$, the number of edges by $e(\mathcal{H})$ and its *density* by $d(\mathcal{H}) = e(\mathcal{H})/v(\mathcal{H})$. Given a subset $S \subseteq V(\mathcal{H})$ of vertices, let $d(S) = |\{e \in \mathcal{H} : S \subset e\}|$. For any integer $\ell \in \mathbb{N}$ the *maximum ℓ-degree* is given by $\Delta_\ell(\mathcal{H}) = \max\{d(S) : S \subseteq V(\mathcal{H}), |S| = \ell\}$. Note that if $\mathcal{H}$ is k-uniform for some integer $k \in \mathbb{N}$, then $\Delta_k(\mathcal{H}) = 1$ and $\Delta_\ell(\mathcal{H}) = 0$ for all integers $\ell > k$. In order to state our general winning criteria we also introduce the function

$$f(\mathcal{H}) = \min_{2 \leq \ell \leq k} \left(\frac{d(\mathcal{H})}{\Delta_\ell(\mathcal{H})} \right)^{\frac{1}{\ell-1}} \tag{1}$$

for any given k-uniform hypergraph $\mathcal{H}$. The first statement gives a winning criterion for Maker in $\mathbf{G}(\mathcal{H}; q)$ and its proof employs a random strategy.

Theorem 2.1 (Maker Win Criterion) *For every $k \geq 2$ and every positive $c_1 \geq k$ there exists $c = c(k, c_1) > 0$ and $\tilde{c} = \tilde{c}(k, c_1) > 0$ such that the following holds. If $\mathcal{H}$ is a k-uniform hypergraph satisfying*

$$\textit{(i)}\ \Delta_1(\mathcal{H}) \leq c_1\, d(\mathcal{H}), \qquad \textit{(ii)}\ 1 < f(\mathcal{H}), \qquad \textit{(iii)}\ \frac{v(\mathcal{H})}{f(\mathcal{H})} \left(1 - \frac{1}{f(\mathcal{H})} \right) \geq \tilde{c}$$

then Maker has a winning strategy in $\mathbf{G}(\mathcal{H}; q)$ provided

$$q \leq c\, f(\mathcal{H}) - 1. \tag{2}$$

The second statement now gives a winning criterion for Breaker.

Theorem 2.2 (Breaker Win Criterion) *For every integer $k \geq 2$ and $0 < \epsilon < 1$ there exists $v_0 = v_0(k, \epsilon)$ and a constant $C_1 = C_1(k) > 0$ such that the following holds. If $\mathcal{H}$ is a k-uniform hypergraph on $v(\mathcal{H}) \geq v_0$ vertices, then Breaker has a winning strategy in $\mathbf{G}(\mathcal{H}; q)$ provided that*

$$q \geq C_1 \max\left(\Delta_1(\mathcal{H})^{\frac{1}{k-1}},\ \max_{2 \leq \ell \leq k-1} \left(\Delta_\ell(\mathcal{H})^{\frac{1}{k-\ell}} \right) v(\mathcal{H})^{\epsilon} \right). \tag{3}$$

Using these two criteria we can state the following corollary describing sequences of hypergraphs for which we know the right growth of the threshold bias. Note that the applications described in the next two parts only partially fall under this corollary.

Corollary 2.3 *For every integer $k \geq 2$ the following holds. If $\mathcal{H} = (\mathcal{H}_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform hypergraphs satisfying*

$$\text{(i) } \Delta_1(\mathcal{H}_n) = O(d(\mathcal{H}_n)) \qquad \text{and} \qquad \text{(ii) } \omega(1) = f(\mathcal{H}_n) = o(v(\mathcal{H}_n))$$

$$\text{and} \qquad \text{(iii) } \Delta_\ell(\mathcal{H}_n)^{\frac{1}{k-\ell}}\, v(\mathcal{H}_n)^{o(1)} \leq \Delta_1(\mathcal{H}_n)^{\frac{1}{k-1}} \text{ for all } 2 \leq \ell \leq k-1$$

then the threshold bias of the game on $\mathcal{H}_n$ satisfies $q(\mathcal{H}_n) = \Theta(d(\mathcal{H}_n)^{1/(k-1)}) = \Theta(f(\mathcal{H}_n))$.

3 Generalized van der Waerden Games

It is well-known that k-term arithmetic progressions can be described as non-trivial solutions to a certain system of linear homogenous equations. This motivates the following generalization of the van der Waerden game to any system of linear homogenous equations.

We start by giving a classification of the possible solutions of a homogeneous system of linear equations. Given an integer-valued matrix $A \in \mathbb{Z}^{r\times m}$ with r rows and m columns, let

(i) $\mathcal{S}(A) = \{\mathbf{x} \in \mathbb{Z}^m : A \cdot \mathbf{x}^T = \mathbf{0}^T\}$ denote the set of *all* solutions,

(ii) $\mathcal{S}_0(A) = \{\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{S}(A) : x_i \neq x_j \text{ for } i \neq j\}$ denote the set of all *proper* solutions.

Our analysis will allow for solutions with some repeated components which one may still consider non-trivial, that is their repetitions do not imply a loss of complexity in the system. In order to give a formal definition, we will need to introduce some notation. Given $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{S}(A)$ let

$$\mathfrak{p}(\mathbf{x}) = \big\{\{1 \leq j \leq m : x_i = x_j\} : 1 \leq i \leq m\big\} \tag{4}$$

denote the set partition of the column indices $[m]$ indicating the repeated entries in $\mathbf{x}$. Note that for $\mathbf{x} \in \mathcal{S}_0(A)$ we have $\mathfrak{p}(\mathbf{x}) = \{\{1\}, \ldots, \{m\}\}$. Given some set partition $\mathfrak{p}$ of $\{1, \ldots, m\}$, let $A_\mathfrak{p}$ denote the matrix obtained by summing up the columns of A according to $\mathfrak{p}$, that is for $\mathfrak{p} = \{T_1, \ldots, T_s\}$

such that $\min(T_1) < \cdots < \min(T_s)$ for some $1 \leq s \leq m$ and $\mathbf{c}_i$ the i-th column vector of A for every $1 \leq i \leq m$, we have

$$A_{\mathfrak{p}} = \left(\sum_{i \in T_1} \mathbf{c}_i \;\middle|\; \sum_{i \in T_2} \mathbf{c}_i \;\middle|\; \cdots \;\middle|\; \sum_{i \in T_s} \mathbf{c}_i \right). \tag{5}$$

Note that the assumption $\min(T_1) < \cdots < \min(T_s)$ ensures that this notion is well-defined and that $A_{\mathfrak{p}} = A$ for $\mathfrak{p} = \{\{1\}, \ldots, \{m\}\}$. A solution $\mathbf{x} \in \mathcal{S}(A)$ is now defined to be *non-trivial* if $\mathrm{rk}(A_{\mathfrak{p}(\mathbf{x})}) = \mathrm{rk}(A)$ where rk denotes the rank of a matrix. This definition is due to Rué et al. [10] and extends a previous definition for single-line equations due to Ruzsa [11]. We now let

(iii) $\mathcal{S}_1(A) = \{\mathbf{x} \in \mathcal{S}(A) : \mathrm{rk}(A_{\mathfrak{p}(\mathbf{x})}) = \mathrm{rk}(A)\}$ denote the set of all *non-trivial* solutions

and remark that $\mathcal{S}(A) \supseteq \mathcal{S}_1(A) \supseteq \mathcal{S}_0(A)$.

Next, we need to give a classification of the possible matrices. For $\emptyset \subseteq Q \subseteq [m]$ let A^Q denote the matrix obtained from A by keeping only the columns indexed by Q, where $A^\emptyset$ is the empty matrix. We call a given matrix $A \in \mathbb{Z}^{r \times m}$

(i) *positive* if $\mathcal{S}(A) \cap \mathbb{N}^m \neq \emptyset$, that is, there are solutions whose entries lie in the integers,
(ii) *irredundant* if $\mathcal{S}_0(A) \neq \emptyset$, that is, there are proper solutions,
(iii) *abundant* if $\mathrm{rk}(A^Q) = \mathrm{rk}(A)$ for all $Q \subseteq [m]$ satisfying $|Q| = m - 2$, that is, every submatrix obtained from A by deleting two columns must be of the same rank as A.

The importance of the first two notions should be immediately clear. Non-abundant systems turn out to be "degenerate" in some sense and in particular, Breaker wins with a bias of just 2. For readers familiar with the notion of *partition* and *density regular* (or *invariant*) matrices, note that they are trivially irredundant and positive. See [12] for an easy proof that they are also abundant.

Finally, we introduce a parameter for abundant matrices originally due to Rödl and Ruciński [13]. To state it, let $r_Q = \mathrm{rk}(A) - \mathrm{rk}(A^{\overline{Q}})$ for any set of column indices $\emptyset \neq Q \subseteq [m]$ where we set $\mathrm{rk}(A^\emptyset) = 0$. The *maximum* 1-*density* of a given abundant matrix $A \in \mathbb{Z}^{r \times m}$ is defined as

$$m_1(A) = \max_{\substack{Q \subseteq [m] \\ 2 \leq |Q|}} \frac{|Q| - 1}{|Q| - r_Q - 1}. \tag{6}$$

It can be shown that this is indeed well-defined, that is $|Q| - r_Q - 1 > 0$ for all $Q \subseteq [m]$ satisfying $|Q| \geq 2$ if A is abundant. Rödl and Ruciński previously showed this for partition regular matrices.

For a given matrix $A \in \mathbb{Z}^{r \times m}$ and positive integers $q, n \in \mathbb{N}$ let $\mathcal{H}_n(A) = \left\{\{x_1, \ldots, x_m\} : \mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{S}_1(A) \cap [n]^m\right\}$ be the hypergraph of all non-trivial solutions in $[n]$ and refer to $\mathbf{G}(\mathcal{H}_n(A); q)$ as the $(1:q)$ *Maker-Breaker A-game on* $[n]$, that is the Maker-Breaker positional game played on the board $[n]$ where Maker wins if he covers all (not necessarily distinct) elements $x_1, \ldots, x_m \in [m]$ of some non-trivial solution $\mathbf{x} = (x_1, \ldots, x_m) \in \mathcal{S}_1(A) \cap [n]^m$. The following statement is the first central result of our paper.

Theorem 3.1 *For all integers $r \geq 1$, $m \geq 3$ and irredundant, positive and abundant matrices $A \in \mathbb{Z}^{r \times m}$ the threshold bias of the Maker-Breaker A-game on $[n]$ satisfies $q(\mathcal{H}_n(A)) = \Theta\left(n^{1/m_1(A)}\right)$.*

This result covers most cases of interest. The matrices associated to arithmetic progressions and Schur triples, that is solutions to the equation $x_1 + x_2 = x_3$, are for example irredundant, positive and abundant. In these cases the only non-trivial solutions are the proper ones. The matrix associated with the Sidon equation $x_1 + x_2 = x_3 + x_4$ is also irredundant, positive and abundant, but one can see that 3-term arithmetic progressions are non-trivial solutions with repeated entries.

4 Fixed Hypergraph Games

Given some r-uniform hypergraph $\mathcal{G}$ on at least $r+1$ non-isolated vertices, the Maker-Breaker $\mathcal{G}$-game is played on the edge set of the complete r-uniform hypergraph on n vertices, denoted $\mathcal{K}_n^{(r)}$. We define the *r-density* of $\mathcal{G}$ to be

$$m_r(\mathcal{G}) = \max_{\substack{\mathcal{F} \subseteq \mathcal{G} \\ v(\mathcal{F}) \geq r+1}} \frac{e(\mathcal{F}) - 1}{v(\mathcal{F}) - r}. \tag{7}$$

Note that this is an obvious generalisation of the 2-density of a graph. As previously mentioned, Bednarska and Łuczak [1] showed that the bias threshold in the Maker-Breaker game played on the edge set of K_n where Maker tries to cover a copy of G satisfies $q(n; G) = \Theta(n^{1/m_2(G)})$.
Let $\mathcal{H}_n(\mathcal{G})$ denote the hypergraph of all copies of $\mathcal{G}$ in $\mathcal{K}_n^{(r)}$ and refer to $\mathbf{G}(\mathcal{H}_n(\mathcal{G}); q)$ as the $(1:q)$ *Maker-Breaker $\mathcal{G}$-game on* $\mathcal{K}_n^{(r)}$. Using the general winning criteria for Maker and Breaker, we will generalize the result of [1] to the following statement.

Theorem 4.1 *For any integer $r \geq 2$ the following holds. If $\mathcal{G}$ is an r-uniform hypergraph on at least $r+1$ non-isolated vertices, then the threshold bias of the Maker-Breaker $\mathcal{G}$-game on $\mathcal{K}_n^{(r)}$ satisfies $q(\mathcal{H}_n(\mathcal{G})) = \Theta\left(n^{1/m_r(\mathcal{G})}\right)$.*

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