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A staggered grid finite difference method for solving the gravity wave-model equations

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Abstract. The gravity wave-model equations are considered. The equations govern shallow water flows, where the gravity effect is significant. The equations form a system of partial differential equations with the hyperbolic type. When the gravity wave-model equations are solved using a finite difference method with staggered grids, artificial oscillations may occur in the numerical solution. In this paper, we propose a numerical treatment of a staggered grid finite difference method for solving the gravity wave-model equations. Using our proposed treatment, artificial oscillations can be eliminated, so the resulting numerical solution mimics the physics of the problem.

1. Introduction

The gravity wave-model equations have been used to model water flows, such as dam break problems [1] and flood simulations [2]. These equations can also be used to model river flows at their steady states [3]. Gravity waves have also been researched by other authors [4-5] relating to internal water waves.

The gravity wave-model equations considered in this paper are simplifications of the Saint-Venant shallow water wave equations, where the convective term is assumed to be negligible. This assumption makes the model easier to be used for flow simulations, yet the conservation of mass is still enforced, and the conservation of momentum based on gravity term still exists in the system.

Solving the gravity wave-model equations numerically is our goal in this paper. We shall use a staggered grid finite difference method. This method is chosen, because fluxes do not need to be approximated. Using the staggered grid technique, fluxes are known. This method is applicable for solving smooth problems. We note that staggered grid numerical methods have had a great success in solving some problems relating to elasticity, water, seismic, and other wave models [6-10].

However, a staggered grid method may generate unphysical oscillations when we implement it to solve nonsmooth problems. In this paper, a numerical treatment is proposed, so that the unphysical oscillations can be eliminated for nonsmooth problems. Numerical tests confirm our claims.

The rest of this paper is written as follows. We recall the mathematical model and numerical methods in Section 2. We provide numerical results in Section 3. Conclusion is drawn in Section 4.

2. Model and method

The gravity wave-model equations are

$$\frac{\partial h}{\partial t}+\frac{\partial q}{\partial x}=0\,, \tag{1}$$

$$\frac{\partial q}{\partial t}+\frac{g}{2}\frac{\partial h^2}{\partial x}=0\,. \tag{2}$$

where $h(x,t)$ is water depth, $q(x,t)$ is unit-discharge, and g is the acceleration due to gravity. The gravity wave-model equations (1) and (2) are simplifications of shallow water equations (the Saint-Venant shallow water wave equations). Ancey et al. [11] and Mungkasi et al. [12] provide some forms of shallow water equations.

To solve equation (1), we use discrete forms as follows

$$\left.\frac{\partial h}{\partial t}\right|_{\substack{x=x_j\\ t=t^n}} \approx \frac{h_j^{n+1}-h_j^n}{\Delta t}\,, \tag{3}$$

$$\left.\frac{\partial q}{\partial x}\right|_{\substack{x=x_j\\ t=t^n}} \approx \frac{q_{j+1/2}^n-q_{j-1/2}^n}{\Delta x}\,. \tag{4}$$

Here x_j is the j-th spatial point, t^n is the n-th temporal point, $j=1,2,3,\ldots$ and $n=0,1,2,3,\ldots$, Δx is uniform spatial distance, Δt is the time step, and h_j^n is water depth at time t^n and point x_j. The notations h_j^{n+1}, $q_{j+1/2}^n$, $q_{j-1/2}^n$ are understood analogously. Substitution of equations (3) and (4) to equation (1) results in

$$h_j^{n+1}=h_j^n-\frac{\Delta t}{\Delta x}\left(q_{j+\frac{1}{2}}^n-q_{j-\frac{1}{2}}^n\right). \tag{5}$$

To solve equation (2), we use discrete approximations as follows

$$\left.\frac{\partial q}{\partial t}\right|_{\substack{x=x_{j+1/2}\\ t=t^n}} \approx \frac{q_{j+1/2}^{n+1}-q_{j+1/2}^n}{\Delta t}\,, \tag{6}$$

$$\left.\frac{\partial h^2}{\partial x}\right|_{\substack{x=x_{j+\frac{1}{2}}\\ t=t^n}} \approx \frac{\left(h_{j+1}^n\right)^2-\left(h_j^n\right)^2}{\Delta x}. \tag{7}$$

Substitution of equations (6) and (7) to equation (2) leads to

$$q_{j+1/2}^{n+1}=q_{j+1/2}^n-\frac{g}{2}\frac{\Delta t}{\Delta x}\left((h_{j+1}^n)^2-(h_j^n)^2\right). \tag{8}$$

The system of equations (5) and (8) is the staggered grid numerical finite difference scheme. It is an explicit scheme. It is stable as long as the time step is taken sufficiently small.

3. Numerical results

In this section, we provide three test cases. The first is a problem having a smooth solution. The second is a problem having a discontinuous solution. The first and the second problems are solved using the standard staggered finite difference scheme. The third is a problem having a discontinuous solution, but is solved using a modified staggered finite difference scheme. The modification is the numerical treatment that has been aforementioned and shall be discussed. All quantities are assumed to have SI units with the MKS system. The acceleration due to gravity is 9,81.

3.1. Problem having smooth solution

Assume that at time $t=0$, water depth is given by

$$h(x,0)=\begin{cases}11+\cos(x)\,, & \text{if} -\pi\le x\le\pi,\\ 0, & \text{if } x<-\pi \text{ or } x>\pi,\end{cases}$$

and its velocity is zero everywhere. Water surface and velocity at the initial time are shown in Figure 1.

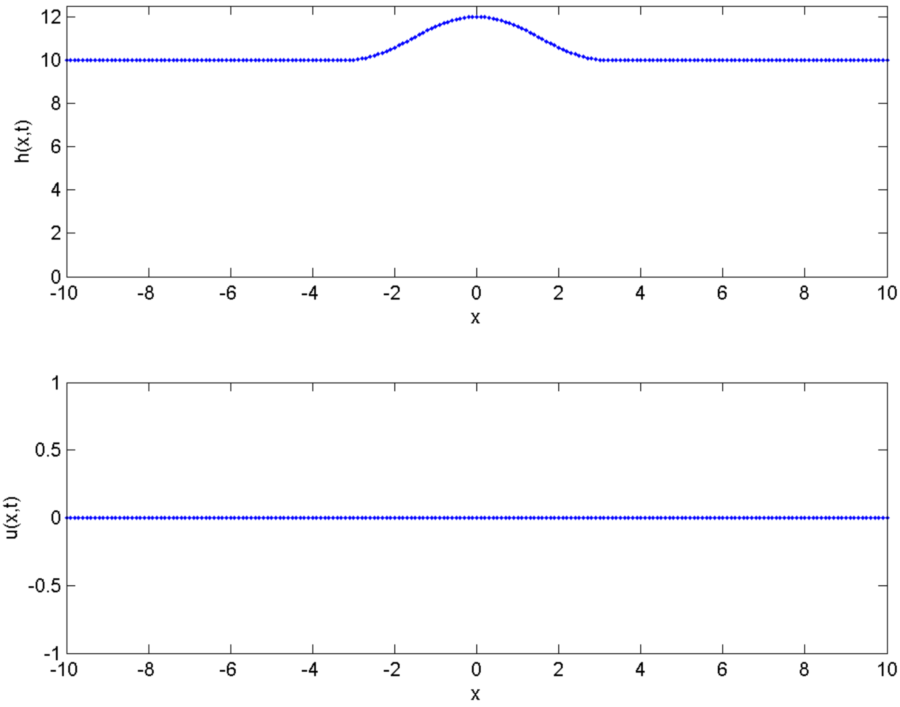

Figure 1. Water surface and velocity at time $t = 0$.

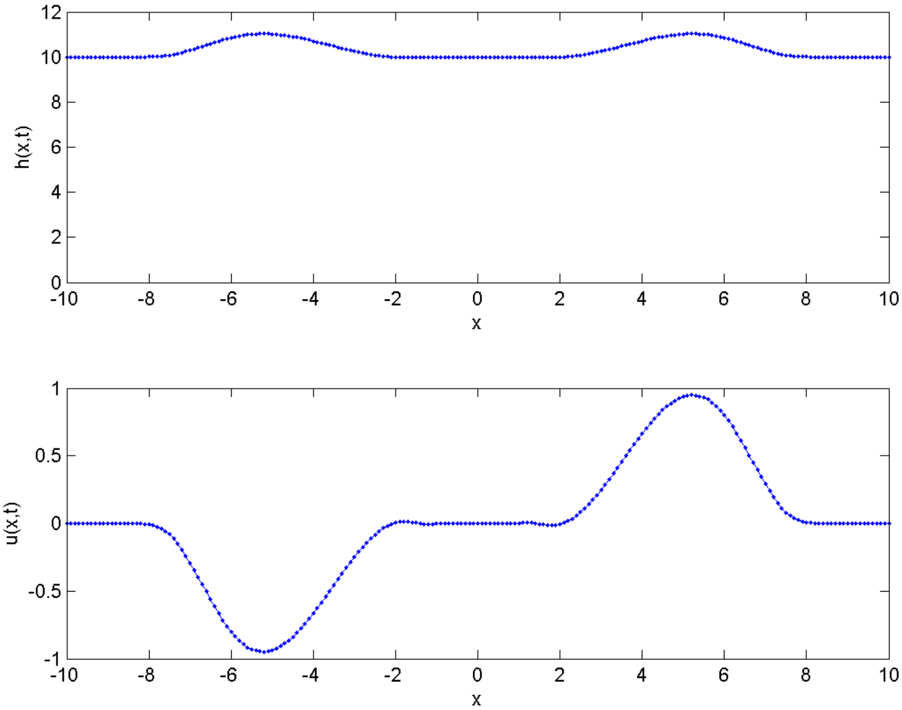

Figure 2. Water surface and velocity at time $t = 0.5$.

Figure 2 shows an illustration of the simulated fluid depth and fluid velocity for the considered problem. This simulation uses 201 grid points, $\Delta x = 0.1$, and $\Delta t = 0.005\Delta x$. The results of this simulation are correct with respect to its physics. That is, if the velocity is positive, then water wave moves to the right. If the velocity is negative, then water moves to the left.

3.2. Problem having nonsmooth solution without numerical treatment

Suppose that the water depth from the point $x = -10$ until $x = 0$ is 10 and the water depth from the point $x = 0$ until $x = 10$ is 5. Initially, water is still. These are illustrated in Figure 3.

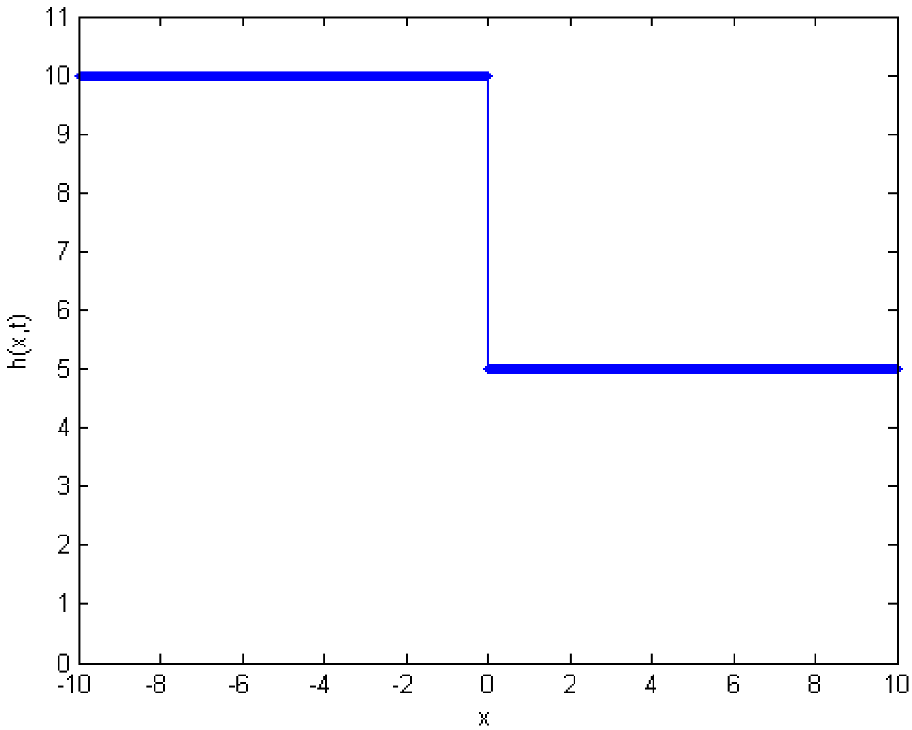

Figure 3. Initial water surface of the dam break problem.

Figure 4 shows an illustration of simulated water depth and velocity at time $t = 0.1$ using the scheme (5) and (8). This simulation uses 2001 spatial points, $\Delta x = 0.01$, and $\Delta t = 0.005\Delta x$. The simulation shows that artificial oscillations occur in the numerical results. To avoid oscillation, numerical treatment is used, as follows: h_j^n in equation (5) is changed to the average $(h_{j+1}^n + h_j^n + h_{j-1}^n)/3$, and q_j^n in equation (8) is replaced by the average $(q_{j+3/2}^n + q_{j+1/2}^n + q_{j-1/2}^n)/3$, so we obtain the following alternative scheme

$$h_j^{n+1} = \frac{h_j^{n-1} + h_j^n + h_j^{n+1}}{3} - \frac{\Delta t}{\Delta x}\left(q_{j+\frac{1}{2}}^n - q_{j-\frac{1}{2}}^n\right), \quad (9)$$

$$q_{j+1/2}^{n+1} = \frac{q_{j+3/2}^n + q_{j+1/2}^n + q_{j-1/2}^n}{3} - \frac{g}{2}\frac{\Delta t}{\Delta x}\left((h_{j+1}^n)^2 - (h_j^n)^2\right). \quad (10)$$

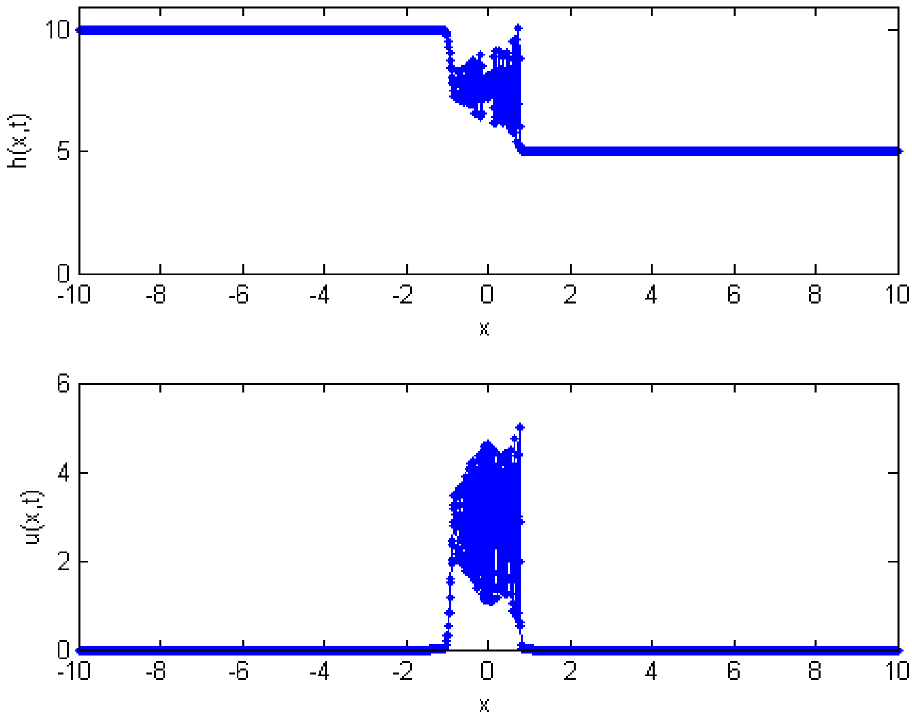

Figure 4. Staggered grid solution without scheme modification.

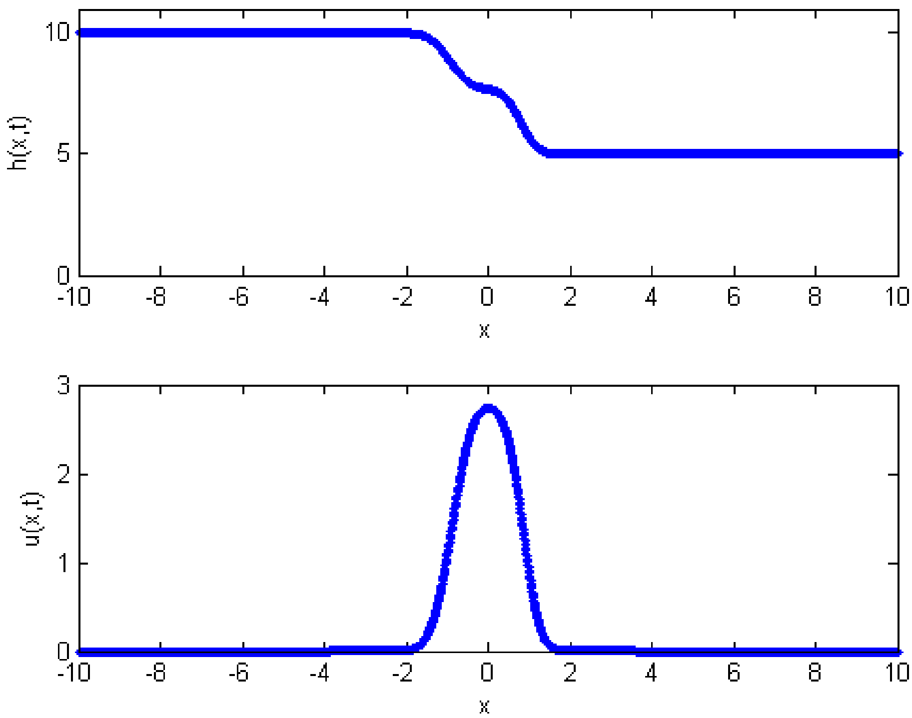

Figure 5. Staggered grid solution with scheme modification as a numerical treatment.

Figure 5 shows an illustration of simulated water depth and velocity at time $t = 0.1$ using the scheme (9) and (10). This simulation uses the same numerical setting as before, that is, 2001 spatial points, $\Delta x = 0.01$, and $\Delta t = 0.005\Delta x$. We observe that artificial oscillations do not occur in these results, as we expect. The solution is slightly diffusive.

4. Conclusion

We have solved the gravity wave-model equations using a staggered grid finite difference method. Our numerical tests involve smooth and nonsmooth solutions. A numerical treatment has been proposed, so that artificial oscillations do not occur in the solution, especially when we deal with nonsmooth problems. Future direction could be about minimising the diffusion around the discontinuity of the nonsmooth solution.

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