Ernesto Burattini

Fibonacci's Computation Methods vs Modern Algorithms

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Towards a critical edition of Fibonacci's *Liber Abaci*

ed. by Giuseppe Germano

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Fibonacci's Computation Methods vs Modern Algorithms

by Ernesto Burattini

1. *Introduction*

In computer science the concept of algorithm has many facets and different interpretations. The origin of the word "algorithm" is generally ascribed to the name of the place of birth al-Khwarizmi of the famous arab mathematician Muhammad ibn Musa al-Khwarizmi. In fact the word al-Khwarizmi was later on modified in algorism and then in algorithm. For those who are interested D.E. Knuth [1968][1] provides a short but interesting history of the term "algorithm" showing possible different genesis.

According to many researchers, Leonardo Fibonacci, of which in this paper we wish to show some computation procedures and their interpretation in terms of algorithms and software implementation, was also inspired in writing the *Liber Abaci*, by Mohammed ibn Musa al-Khwarizmi (see Rashed R. 2003)[2].

Also the definition of "algorithm" in computer science is debated, and also in this case we refer to Knuth who gave a list of five properties that are widely accepted as requirements for an algorithm:

- *Finiteness*: «An algorithm must always terminate after a finite number of steps».
- *Definiteness*: «Each step of an algorithm must be precisely defined; the actions to be carried out must be rigorously and unambiguously specified for each case».
- *Input*: «...quantities which are given to it initially before the algorithm begins. These inputs are taken from specified sets of objects».

[1] D.E. Knuth, *The art of computer programming. Fundamental Algorithms*, I, Reading (Mass.) 1968.
[2] R. Rashed, *Fibonacci et le prolongement latin des mathématiques arabes*, in «Bollettino di storia delle scienze matematiche», 23 (2003), 2, pp. 55-73.

- *Output*: «...quantities which have a specified relation to the inputs».
- *Effectiveness*: «... all of the operations to be performed in the algorithm must be sufficiently basic that they can in principle be done exactly and in a finite length of time by a man using pencil and paper».

Later on we will show that for some algorithms, derived from computation methods proposed by Fibonacci to perform arithmetic operations, the five properties required by Knuth are complied.

Indeed, the arithmetic operations are performed working sequentially on the number's single digit, and since they are finite also the proposed algorithms will end after a finite number of steps proportionally to the number of digit at issue. Therefore the *finiteness* property is satisfied. Later on, as an example, we will evaluate the number of steps requested to perform a multiplication by means of the "cross" algorithm.

The *definiteness* is a little more questionable since in some cases the Fibonacci's computation procedures are partially modified shifting from simple cases to more complex one. For instance in the chap. VI the author describes the product of two integer with fractions or parts of fractions. Discussing some more complicated cases, he introduces several simplification criteria notwithstanding no mention of it was done in the introduction of the computation method.

The *input* and the *output* are clearly defined since we deal with a finite number of arithmetic operations among integer number or fraction.

The *effectiveness* is surely satisfied since the examples accomplished by Fibonacci give evidence that it is possible to carry out the required computations with paper and pencil or better with hands and "tabula alba".

In this paper we try to show how the Fibonacci's computation methods to perform the arithmetic operations among integer numbers with or without fractions, or parts of fraction, may be interpreted as modern algorithms. Of these algorithms we will give an informal description and some example in pseudo-code. We implemented in C++ the algorithms and we will show also some examples of outputs. To exhibit the coherence of our algorithm to the computation methods proposed by Fibonacci in the *Liber Abaci* the outputs, *i.e.* the description of the computations, are written in Latin.

The *Liber Abaci* excerpts quoted in this paper originate from the work of our research group in particular by Giuseppe Germano, Eva Caianiello and Concetta Carotenuto[3]. Since the work is in progress no page references are available for these fragments. Moreover we will quote also the corresponding pages of the Boncompagni book[4].

[3] E. Burattini, E. Caianiello, C. Carotenuto, G. Germano and L. Sauro, *Per un'edizione critica del* Liber Abaci *di Leonardo Pisano, detto il Fibonacci*, in *Forme e modi delle lingue e dei testi tecnici antichi*, ed. by R. Grisolia and G. Matino, Napoli 2012.

[4] *Scritti di Leonardo Pisano matematico del secolo decimoterzo pubblicati da Baldassarre Boncompagni*, 1 (*Il* Liber abbaci *di Leonardo Pisano pubblicato secondo la lezione del Codice*

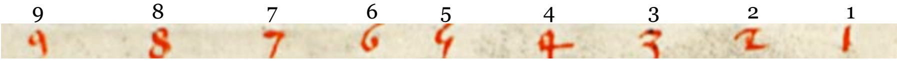
2. *The indian figure and some computation tricks*

Before to describe the Fibonacci's methods we wish remind that in the west countries he was one of the first people to introduce the Hindu numeral system based on the positional notation in a decimal system.

From *Liber Abaci* cap. 1 let us quote:

(1) The nine Indian figures are:

9 8 7 6 5 4 3 2 1



With these nine figures, and with the sign 0, so-called zephir by arabs, any number is written, as is demonstrated below. In fact a number is a collection of not specified units or so to speak a sum of units , which through its degree increases to infinity. The first of these degrees comprises the units from one up to ten. The second is composed by the tens from ten up to the hundred. The third is made by the hundreds from hundred up to thousand. The fourth is made by the thousands from thousand up to ten thousand, and in this way each of the subsequent degrees until the infinity consists of ten times its antecedent. (2) The first degree in the writing of the numbers begins at right. Then the second follows to the left. The third follows the second. The fourth the third, and the fifth the fourth and thus ever to the left, degree follows degree. And therefore the figure that is found in the first degree represents itself; that is, if in the first degree will be the figure of the unit, it represents one; if the figure two, it represents two; if the figure three, three, and thus in order those that follow up to the figure nine if there is the figure of nine. Then the figures in the second degree represent as many tens as in the first degree units: that is, if the unit figure occupies the second degree, it denotes ten; if the double twenty; if the triple, thirty; and so on until ninety. Moreover the figure which occupies the third degree denotes the number of hundreds , as that in the second degree tens, or in the first units; and if the figure is one, one hundred; if the double two hundred; if the triple three hundred and so on until nine hundred. Therefore the figure which is in the fourth degree denotes as many thousands as in the third, hundreds, and as in the second, tens, or in the first, units; and thus ever changing degree the number increases by tenfold[5].

Magliabechiano C.1., 2616, Badia Fiorentina, n. 73 da B.B.), Roma 1857 (henceforth Boncompagni, *Il* Liber abbaci *di Leonardo Pisano*).

[5] From G. Germano, E. Caianiello, C. Carotenuto, E. Burattini, work in progress: «(1) Novem figure Indorum he sunt 9 8 7 6 5 4 3 2 1. Cum his, itaque, novem figuris, et cum hoc signo 0, quod arabice zephirum appellatur, scribitur quilibet numerus, ut inferius demonstratur. Nam numerus est unitatum perfusa collectio sive congregatio unitatum, que per suos in infinitum ascendit gradus. Ex quibus primus ex unitatibus, que sunt ab uno usque in decem, constat. Secundus ex decenis, que sunt a decem usque[5] in centum, fit. Tertius fit ex centenis que sunt a centum usque in mille. Quartus fit ex millenis que sunt a mille usque in decem milia, et sic sequentium graduum in infinitum quilibet ex decuplo sui antecedentis constat. (2) Primus gradus in descriptione numerorum incipit a destera; secundus, vero, versus sinistram sequitur primum; tertius secundum sequitur; quartus tertium et quintus quartum, et semper sic versus sinistram gradus gradum sequitur. Figura, itaque, que in primo reperitur gradu se ipsam representat: hoc est, si in primo gradu fuerit figura unitatis, unum representat; si binarii, duo; si ternarii, tria, et ita per ordinem que secuntur usque, si novenarii, novem. Figure, quidem, que in secundo gradu fuerint tot decenas representant quot in primo unitates: hoc est, si figura unitatis secundum occupat gradum, denotat decem; si binarii, viginti; si ternarii, triginta; si novenarii,

All the computation methods proposed by Fibonacci refer to the this representation which briefly may be described as follows:

Given an integer A made up of N digit A= a_{N-1} a_{N-2} ....a_1a_0 it is equivalent to

$$A= a_{N-1}*10^{N-1}+a_{N-2}*10^{N-2}+ ....a_1*10^1+a_0*10^0$$

where the terms a_i and b_j are integer numbers between 0 and 1. For instance the number 1058 is given by

$$1*10^3+ 0*10^2+ 5*10^1+8*10^0=100+50+8=1058.$$

In a computer system an integer number is represented by the binary notation (i.e. the digits may be only 0 and 1 and the base is equal to 2). The number 1058 in the binary notation becomes 10000100010. All programming languages give the possibility of defining for each number the type of representation one want. In general the decimal representation is used. Since to represent a number in a computer we have a finite number of computing units (bit) available this imply a boundary on the maximum value we may represent. For instance the language C++ allows a maximum integer equal to 2147483647. Moreover the number are available in its entirety but digit by digit. The following algorithm provides the single digit of an integer A starting from its decimal representation.

A1 - Algorithm to extract the k-th Digit
Given the number A= a_{N-1} a_{N-2} ....a_1a_0 extract the digit of degree k
for i=0;i=k;i=i+1 [for all digit between 0 and k]
{ a_i = $mod_{10}(A_i)$; [a_i is the digit corresponding to the i degree]
$A_i = (A_i - mod_{10}(A_i))/10$ } [delete the digit a_i and go back]

where $mod_{10}(A_i)$ means the residue of the division of A_i by 10, that is the modulus base 10 of A_i.

If we wish to manipulate many times the digit composing an integer number, it doesn't pay to recall each time the previous algorithm, instead, it is better to store suitably the single digit in a vector. Therefore, from number A_i= a_{N-1} a_{N-2} ....a_0 we will extract, for instance, the digit a_i which will be stored in the i-th location of a vector Avett. The vector will appear as follows: Avett[a_0,a_1,....$_{N-2}$,a_{N-1}].The A1 algorithm will be modified as follows:

A2 - Algorithm to collect Digit in a Vector
Given the number *A*= a_{N-1} a_{N-2} ....a_1a_0
for i=0;i=N-1;i=i+1 [for all digit between 0 and N-1]

nonaginta. Figura, namque, que in tertio fuerit gradu tot centenas denotat, quot in secundo decenas vel in primo unitates, ut si figura unitatis centum; si binarii, ducenta; si ternarii, trecenta, et novenarii, nongenta. Ipsa, igitur, que fuerit in quarto grado tot millenas quot in tertio centenas, aut in secundo decenas, vel in primo unitates denotat, et sic semper, mutando gradum, numerus decuplando ascendit». Cfr. Firenze, BNC, ms. Conv. Soppr. C. 1. 2616, f. 2*v*; the indian numbers are written according to the arab use from right to left. See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 2.

{ $Avett[i] = mod_{10}(A_i)$;	[store the digit corresponding to the i-th degree into the location i-th of the vector Avett]
$A_i = (A_i - mod_{10}(A_i))/10$ }	[delete the digit a_i and go back]

Therefore, in the i-th location of the vector Avett[i] we will find the digit which in the positional notation is multiplied by 10^i. Hence the number 1058 will be represented as Avett [8, 5, 0, 1].

In several computation methods proposed by Fibonacci the opportunity of storing partial data arises. For this purpose he uses the fingers of both hands or a "tabula alba" where he writes or deletes data. In a computer system we will use the so called "support variable". In other words we assign a symbolic name to some memory area to which we will associate the data. The trick we will use in the following to replicate the Fibonacci's computation procedures will resort to the two previously described solutions: number representation by means of a vector, temporary storing of partial data by support variables.

There is another recurrent feature in the Fibonacci's computation procedures which call to mind some well known programming paradigms. More often, at the beginning, the description of the computation process is described by the author in a quite general way. Furthermore, he provides a lot of examples of increasing difficulties. In any case, however, the sequence of operations to be performed is almost always the same. This means that, if the algorithm is known, then it must be valid for any kind of input belonging to the set of values to which the algorithm concerns (for instance to all positive integers). Technically we call such an algorithm "function" which represents the kernel of the so called structured programming. Accordingly, the algorithms we will introduce in the following have to be interpreted as functions which may be recalled by other procedures when required. Later on we will show some examples.

Another problem arising in several computation procedures appearing in the *Liber Abaci* is the demonstration of their correctness.

For this purpose, Fibonacci for the arithmetic operations often uses the casting out nines test. This check is necessary but not sufficient, that is, if the test fails the operation is certainly wrong, if it is successful the correctness of the operation cannot be guaranteed.

Let us remind that, for instance, in the case of the addition the casting out nines envisages for each addend of crossing out all nines and pairs of digits that sum up nine, then add together what remains, the so called excesses. Add up leftover digits for each addend until one digit is reached. Now process the sum and the excesses to get a final excess. The excess from the sum should equal the final excess from the addends.

In the cap. III Fibonacci provides a geometrical demonstration of the casting out nines soundness:

> Chapter III - The check.
> .....(2) And lastly to show from where such a check proceeds; let .ab. and .bg. be the two numbers which we wish to add together; the sum of them will therefore be .ag. I indeed say that from the sum of the residue of the number .ab. and of the residue of

the number .bg. results the check of number ag. which gives the check of the sum. First, let each of the numbers .ab. and .bg. be divided integrally by 9; therefore will be 9 the common measure of the numbers .ab. and .bg. For this reason the total number .ag. is divided integrally by 9, will therefore be zephir its residue, as result from the addition of the residues of numbers .ab. and .bg. (3) Also be one of them integrally divided by 9, and the other is not, and be .ab. the number which is integrally divided by 9, indeed from the number .bg. divided by 9, there remains the number .dg.; .then the numbers bd. and .ab. are divided integrally by 9, and therefore the total number .ad. is divided by 9. And for the reason that the number .ag. exceeds the number .ad. by the number .bd., and the number .ad. is divided integrally by 9, there will therefore remain from the total number .ag., the number .dg., indivisible by 9, which results from the addition of the check of the number .ab. which is zephir with the check of the number .bd. which is the number .dg. which is indivisible. Again assume none of the numbers .ab. and .bg. is divided integrally by 9; but from the number .ab. remains the number .ae., and from the number .bg. remains the number .dg.. The rest, in fact, namely the numbers .eb. and .bd. are divide integrally by 9. And for the reason that the total .ed. is divisible, and being built of a multitude of nines, therefore remain indivisible, the numbers .ae. and .dg., out of the total number .ag., these are the checks of the numbers .ab. and .bg., from which addition results the residue of number .ag., as had to be shown[6].

In other words: let A and B be two integer numbers represented as segments: $\overline{ab}$ and $\overline{bg}$.

In the most general case (see Fig. 1) suppose the segments $\overline{ab}$ and $\overline{bg}$ represent two numbers not entirely divisible by nine. Let the segment represents the number nine.

We have:

$\overline{ab} = \overline{ae} + h * \overline{xy}$ $\overline{ae}$ =r1 (excess of $\overline{ab}$ compared to 9) $\overline{bg} = \overline{dg} + k * \overline{xy}$ $\overline{dg}$ =r2 (excess of $\overline{bg}$ compared to 9). Then

$$\overline{ag} = \overline{ab} + \overline{bg}$$

[6] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «Cap. III ...(2) Demum, ut ostendatur unde talis probatio procedat, sint duo numeri *.a.b.* et *.b.g.* quos insimul addere volumus: erit ergo coniunctus ex eis numerus *.a.g.* Dico, quidem, quod ex addita pensa numeri *.a.b.* cum pensa numeri *.b.g.* provenit pensa numeri *.a.g.* que sit probatio. Sit primum quod unusquisque numerorum *.a.b.* et *.b.g.* dividatur integraliter per 9, erunt itaque 9 communis mensura numerorum *.a.b.* et *.b.g.* Quare totus numerus *.a.g.* dividetur integraliter per 9, erit ergo pensa ipsius zephyrum ut habetur ex aditione probarum numerorum *.a.b.* et *.b.g.* (3) Item unus illorum dividatur integraliter per 9 alius non, et sit numerus *.ab.* ille qui integraliter dividitur per 9, et ex numero *.b.g.* diviso per 9 remaneat numerus *.d.g*: ergo numeri *.d.b.* et *.b.a.* dividuntur integraliter per 9, et totus ergo *.d.a.* numerus per 9 dividetur. Et quia numerus *.a.g.* superhabundat numerum *.a.d.* in numero *.d.g.*, et numerus *.a.d.* dividitur integraliter per 9, remanebit ergo ex toto *.a.g.* numerus *.g.d.*. indivisibilis per 9 qui provenit ex additione probe numeri ab que est zephyrum cum proba numeri *.b.d.* que est etiam zephirum et numero *.d.g.* que est indivisibilis. (4) Rursus nullus numerorum *a.b.* et *.b.g.* dividatur integraliter per 9. Sed ex numero *.a.b.* remaneat numerus *.a.e.* et ex numero *.b.g.* remaneat numerus *.d.g.* Residui quidem scilicet numeri *.e.b.* et *.b.d.* dividuntur integraliter per 9. Quare et totus *.e.d.* divisibilis est, cum sit ex aliqua multitudine novenariorum concretus: remanent ergo indivisibiles numeri *.a.e.* et *.d.g.* ex toto numero *.a.g.* qui sunt probe numerorum *.a.b.* et *.b.g.* ex quorum coniunctione provenit pensa numeri *a.g.* ut oportebat ostendere». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 20.

Crossing out all segments equal to $\overline{xy}$ from $\overline{ab}$ and $\overline{bg}$ we claim:

$$\overline{ag} = \overline{ae} - \overline{eb} - \overline{bd} + \overline{dg} = r1 + r2$$

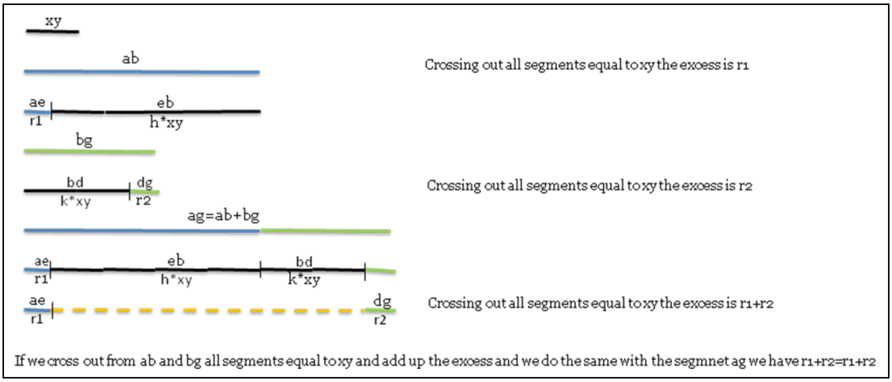


Fig. 1

An algorithm which does the casting out nines for the addition is the following:

A3 - Algorithm casting out nines for the addition
Let a and b be two vectors, with respectively Na and Nb elements containing the digit of numbers to be added. Let **z** a vector with M elements containing the digit of their sum.

```
for (int i=1;i<=Na;i++){              [for all digit of a]
    sumA=sumA+a[i];}                  [add the digit a_i]
   sumA=sumA%9;                       [cross out the 9 from sumA]
while (sumA>9)                        [ until the sumA is >9 cross out 9
  {   sumA=sumA%9;  }
for (int i=1;i<=Nb;i++){              [for all digit of b]
    sumB=sumB+b[i];}                  [add the digit b_i]
   sumB=sumB%9;                       [cross out the 9 from sumB]
while (sumB>9)                        [until the sumB is >9 cross out 9]
  {   sumB=sumB%9;  }
  sum_A_B=sumA+sumB;                  [add sumA+ sumB]
  excessS= sum_A_B %9                 [cross out 9 from sum _A_B]
for (int i=1;i<=M;i++){               [for all digit of z]
    sumZ=sumZ+z[i];}                  [add the digit z_i]
    sumZ=sumZ%9;                      [cross out 9 from sumZ]
  while (sumZ>9)                      [until the sumZ is >9 cross out 9]
  { excessZ =sumZ%9;  }
  if (excessZ== excessS              [if the excesses are equal the addition is
                                                correct];
      print («The addition is correct»)
 else
      print («The addition isn't correct»)
```

3. *The "cross" multiplication*

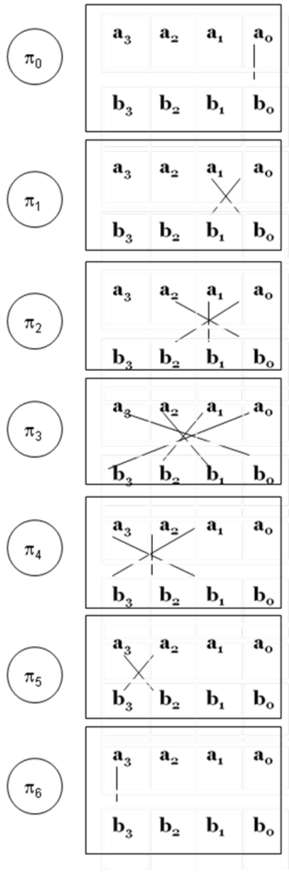
In the chap. II Fibonacci deals with the multiplication between two integer numbers introducing a method by the abacists denoted as "cross" multiplication. This computation method lies in multiplying set of digits, from now on called figures, and from time to time adding them.

In the case of two numbers each with four figure Fibonacci writes:

> Chapter II - On the Multiplication of Four Figures.
> If someone wants to multiply four figures by four, write the numbers and place in column and similar degree are located below similar degree; multiplies the first figure by the first and put the units, remembering however always to keep the tens after the units are put, and multiplies the first figure by the second plus the first by the second and put the units; and the first by the third, plus the first by the third plus the second by the second and put the units; and the first by the fourth plus the first by the fourth, plus the second by the third, plus the second by the third, and put the units; and the third by the fourth, plus the third by the fourth, and put the units; and the fourth by the fourth and put the units. And thus will be had the multiplication of any numbers of four figures whatever they are equal or unequa[7].

In other words we may describe the operations proposed by Fibonacci as follows. Suppose you wish to multiply two numbers of four figures: $(a_3a_2a_1a_0)$ by $(b_3b_2b_1b_0)$. Let $(p_8p_7p_6p_5p_4p_3p_2p_1p_0)$ be their product. The "cross" multiplication method may be explained by means the schema of Fig. 2.

[7] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «*Pars tertia de multiplicatione quattuor figurarum.*(1) Cum autem quattuor figuras contra quattuor quis multiplicare voluerit, describat numeros et, collocatis gradibus sub gradibus similibus, multiplicet primam per primam et ponat, reminiscendo tamen servare decenas semper cum posuerit unitates, et multiplicet primam per secundam et primam per secundam et ponat; [N, f.15r] et primam per tertiam, et primam per tertiam, et secundam per secundam, et ponat; et primam per quartam et primam per quartam et secundam per tertiam et secundam per tertiam, et ponat; et secundam per quartam et secundam per quartam et tertiam per tertiam, et ponat; et tertiam per quartam et tertiam per quartam, et ponat; et quartam per quartam, et ponat. Et sic habebit multiplicationem quorumlibet numerorum quattuor figurarum sive equales sive inequales extiterint». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 13.



For N=4 figures we have 7 schema. In general we have 2N-1 schema. In fig. 2 each schema shows by means of segments the figures which must be multiplied each other. To every product any residue left by previous product must be added.

Let us call:
a_i , b_i the figures of the multiplicands of i-th degree;
r the residue left by product at step i-1;
m the product corresponding to the schema π_i of fig. 2;
p_j the figure to be inserted in the j-th degree of the final product.

In the follow the algorithm derived by the *Liber Abaci.*

A4 - Algorithm for the 'cross' multiplication.

```
{           The first N figures p0 .... pN-1 are obtained as follows
r = 0                           [set residue at step 0 equal 0]
m = a0 *b0                      [set product at step 0 equal 0]
p0 =mod10 (m)                   [first product figure]
for (h=1;h<=N-1;h++)            [for all figures from 1 to N-1
                                compute]

{ k=h;
r=(m-m%10)/10;                  [residue at step h]
m=0;
{for (int i=0;i<=h;i++)         [for all h figures add their products

{m=m+a[i]*b[k];                 [following the schema πi]
                 k--;}
                 }
         m=m+r;                 [add the residue to the final product ]
         p[h]=m%10; }           [h-th product figure]

            The other N figures pN .... p2N-2 are obtained as follows
r=(m-m%10)/10;                  [residue at step N]
for (int z=1;z<=N-1;z++)        [for all h figures from N to 2N-2]
   { k=N-1;
   m=0;
   for (int i=z;i<=N-1;i++)     [for all h figures from z to N-1
                                compute]
         { m=m+a[i]*b[k];       [add their products
         k--;}..                following the schema πi]
         m=m+r;                 [add the residue to the final
                                product]
         p[h]=m%10;             [h-th product figure]
         h++;
         r=(m-m%10)/10; [residue at step h+1]
   }
   p[h]=r                         [last product figure]
}
```

Fig. 2

For the "cross" multiplication the number of operations which are performed to multiply two integers with N figures each is of order of N^2. Indeed, considering the schema (Fig. 2) from π_0 to π_{N-1} the number of multiplication to be performed are

$$\sum_{i=1}^{i=N} i = \frac{N*(N+1)}{2}$$

from π_N to π_{2N-2} the number of multiplication to be performed are

$$\sum_{i=1}^{i=N-1} i = \frac{N*(N-1)}{2}$$

Adding the two expressions we have the total number of multiplications

$$\frac{N*(N+1)}{2} + \frac{N*(N-1)}{2} = \frac{N*(2*N)}{2} = N^2$$

To this number we must add N^2 additions. For N large enough we have at least $\approx N^2$ operations and then the algorithm will stop after a reasonable finite number of steps.

Fibonacci provides the check of the product correctness by means of the casting out of nines.

> Let a*b=p, then we sum up the figures of both numbers a and b and for each of them we compute the casting out of nines or, in the modern mathematic, their value modulo 9. Hence, we multiply the obtained two casting out of nines and we calculate the casting out of nines of their product. Similarly we compute the casting of nines of the product p. If the two last obtained values are equal then the multiplication has been correctly performed.

The resulting algorithm is shown below:

```
A5 - Algorithm for the casting out nines of the multiplication.
{
 Let Avett, Bvett, and Pvett be the vectors holding the figures of numbers A, B and of
their product P.
 Without loss of generality let N be the number of figure of both numbers.
  for (int i=0;i<N;i++)
    { sum_A=sum_A+a[i];                [add the figures of A]
     sum_B=sum_B+b[i];                 [add the figures of B]
     sum_P=sum_P+p[i]+p[Na+i];         [add the figures of P]
    }
  modA=sum_A%9;                        [casting out of nines of A]
  modB=sum_B%9;                        [casting out of nines of B]
  modP=sum_P%9;                        [casting out of nines of P]
  prodMod=modA*modB;                   [product of casting out of nines of A and B]
  modProd=prodMod%9;                   [casting out of nines of the product of
                                       casting out of nines of A and B]

  if (modP==modProd)
  print "The product is correct";
```

```
 else
 print "The product isn't correct"
}
```

A last remark on the check of casting out of nines. Suppose we wish multiply 9 by 50 and suppose that the result be 540 instead of 450, as should be. The casting out of nines of the first term, i.e. 9, is equal to 0, of 50 it is 5, then the product of the excesses, i.e. 0*5, is equal to 0; moreover also the casting out of nines of 540 is equal to 0, the check suggests that the result is correct while it isn't.

In Fig. 3 we show the output of a code which mixes both the 'cross' multiplication and the casting out

```
                    MULTIPLICATIONES INTEGRORUM NUMERORUM
Si multiplicare vis 456 per 123
    multiplicentur 6 per 3
    erunt 18 que iungantur cum 0 erunt 18
    ponantur 8 et serventur 1

    multiplicentur 6 per 2
    multiplicentur 5 per 3
    erunt 27 que iungantur cum 1 erunt 28
    ponantur 8 et serventur 2

    multiplicentur 6 per 1
    multiplicentur 5 per 2
    multiplicentur 4 per 3
    erunt 28 que iungantur cum 2 erunt 30
    ponantur 0 et serventur 3

    multiplicentur 6 per 0
    multiplicentur 5 per 1
    multiplicentur 4 per 2
    multiplicentur 0 per 3
    erunt 13 que iungantur cum 3 erunt 16
    ponantur 6 et serventur 1

    multiplicentur 6 per 0
    multiplicentur 5 per 0
    multiplicentur 4 per 1
    multiplicentur 0 per 2
    multiplicentur 0 per 3
    erunt 4 que iungantur cum 1 erunt 5
    ponantur 5 et serventur 0

    multiplicentur 6 per 0
    multiplicentur 5 per 0
    multiplicentur 4 per 0
    multiplicentur 0 per 1
    multiplicentur 0 per 2
    multiplicentur 0 per 3
    erunt 0 que iungantur cum 0 erunt 0
    ponantur 0 et serventur 0
     et sic habebis pro summa dicte multiplicationis
```

Modo videamus si hec multiplicatio recta est Iuganture figure de superiori 456 et demantur 15 de quibus extrahantur omnes novene que sunt in eisdem 15 remanebit pro pensa 6 Vel, aliter: iugantur figure que sunt in predictis 123 erunt 6 de quibus demantur 9 remanebit 6 et multiplicentur 6 per 6 erunt 36 de quibus demantur 9 remanebit 0 Postea, colligantur figure que sunt in summa multiplicationis 56088 erunt 27 de quibus demantur 9 remanebit 0 remanebit 0 pro pensa, sicuti remanere oportebat

Fig. 3

4. *The fractions*

In the *Liber Abaci*, Fibonacci worked not only on problems concerning the properties of integer numbers, see for instance the case of perfect numbers, but also on actual problems concerning the business and the markets world. In these cases not always it is possible to deal with integer numbers whereas it is necessary to use also fractions or parts of fractions. In fact at the Fibonacci's time the decimal numbers were still unknown.[8] On this ground Fibonacci deals with fractions and explains several operations which can be done with them. We remark that the Fibonacci's fractions aren't only represented by one numerator and one denominator but they may have more terms as numerator and as denominator. As a matter of fact he wrote:

> Chapter V - On the Divisions of Integral Numbers.
> ...(2) When over any number a fraction line is put, and above it another number is written, then the number put over denotes the part or parts of the number placed under it: indeed the inferior is said denominator and the superior numerator. (3) Truly if over the number 2 will put a fraction line and above it will be written the number 1, this number 1 state one part of the two parts of an integer; that is the half, that is $\frac{1}{2}$ ...Furthermore if under the same fraction line will be put more numbers and over each of them will be written other numbers, the number which will be put over the number placed at the begin of fraction line right part will designate, as we said, the part or the parts of the number put under. That, then, over the second denotes the parts of such second of the parts of the first number put under. That, further, over the third express the parts of such third of the parts of the second of the part of the first, and so on those

[8] *The decimal numbers were introduced by Simone Stevino (1600).*

> following above the fraction line specify the parts of the parts of all numbers preceding under the fraction line. Thus if under certain fraction line one puts 2 and 7 over the 2 is 1, and over the 7 is 4, as here is displayed $\frac{1}{2}\frac{4}{7}$, four sevenths plus one half of one seventh are denoted. However if over the 7 is the zephir, thus $\frac{1}{2}\frac{0}{7}$, one half of one seventh will be denoted. Also if under another fraction line are 2, 6 and 10 and over the 2 is 1, and over the 6 is 5, and over the 10 is 7, as is here displayed $\frac{1}{2}\frac{5}{6}\frac{7}{10}$ the seven that is over the 10 at the head of the fraction line represents seven tenths, and the 5 that is over the sixths denotes five sixths of one tenth, and the 1 which is over the 2 denotes one half of one sixth of one tenth, and thus singly, one at a time, they are understood… [9].

In general, Fibonacci writes a fraction with many fraction parts as follows:

$$\frac{a_n \ldots\ldots a_3\, a_2\, a_1}{b_n \ldots\ldots b_3\, b_2\, b_1}$$

which in the modern mathematical language may be expressed as:

$$\frac{\ldots(a_n * b_{n-1} + a_{n-1})*b_{n-2} + a_{n-2})*b_{n-3} + \ldots\ldots\ldots) + a_1}{b_n * b_{n-1} * \ldots\ldots * b_1} \quad (1)$$

[9] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «Cap. V (2). Cum super quemlibet numerum quedam virgula protracta fuerit, et super ipsam quilibet alius numerus descriptus fuerit, superior numerus partem vel partes inferioris numeri affirmat: nam inferior denominatus, et superior denominans appellatur. Ut si super binarium protracta fuerit virgula, et super ipsam unitas descripta sit, ipsa unitas unam partem de duabus partibus unius integri affirmat, hoc est medietatem, sic $\frac{1}{2}$,… (3) Item si sub una eadem virgula plures numeri positi fuerint, et super unum quemque ipsorum alii numeri describentur, numerus qui in capite virgule dextere partis super numerum positus fuerit ipsius sub positi numeri partem vel partes ut prediximus denotabit. Qui vero super secundum ipsius secundi partes de partibus primi sub positi numeri declarat. Qui autem super tertium, ipsius tertii partes partium secundi de partibus primi affirmat: et sic semper qui sequentur super virgulam partes partium cunctorum antecedentium sub virgula denotant. (3) Ut si sub quadam virgula fiat 2 et 7 et super 2 sit 1 et super 7 sint 4, ut hic cernitur, $\frac{1}{2}\frac{4}{7}$ denotantur quattuor septime, et medietas unius septime. Si autem super 7 esset[9] zephyrum, sic $\frac{1}{2}\frac{0}{7}$, medietas tantum unius septime denotaretur. Item si sub quadam alia virgula sint 2 et 6 et 10; et super 2 sit 1; et super 6 sint 5 et super 10 sint 7, ut hic ostenditur, $\frac{1}{2}\frac{5}{6}\frac{7}{10}$ septem que sunt super 10 in capite virgule representant septem decimas, et 5 que sunt super 6 denotant quinque sextas unius decime partis, et 1 quod est super 2 denotat medietatem sexte unius decime partis et sic singulariter de singulis intelligatur». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 24.

Besides that, Fibonacci introduces other kinds of fractions with many terms which we leave out here. From an informatics viewpoint to turn a fraction with many fraction parts into a fraction with one numerator and one denominator by means of expression (1) is trivial. Anyway, we remark that also in this case we used the vectors to represent fractions. In fact, in a first vector the numerators are neatly recorded, in a second vector correspondingly the denominators, and in a third vector the partial products of denominators as the computation went along. From this representation it is easy to transform a fraction with many fraction parts into a fraction with one numerator and one denominator following the rule (1).

5. *The division*

Fibonacci suggests some computation methods to divide two numbers. He starts from simple cases moving on more complex examples. After having suggested a method to divide an integer number by a number with one figure he provides also a method to compute by mind these ratios. Afterwards a method to divide integer numbers by prime numbers is described. The choice of discussing first of all the division by the prime numbers is due to the fact that any other number is composed by prime factor and then the division may be reduced to a succession of division by prime numbers. At last, Fibonacci suggests how to divide a number by a non prime number introducing also a method for the factor decomposition of the denominator.

Straight afterwards we will show the algorithm derived from the method to divide a number by a prime numbers with two figures, furthermore a method for the factor decomposition and at last a method to divide a number by a composite number.

Fibonacci describes the method to divide a number by a prime number as follows[10]:

[10] From G. Germano, E. Caianiello, C. Carotenuto, E. Burattini, work in progress: «Cap. V. *Incipiunt divisiones numerorum per numeros incompositos secundi gradus.* (…) Numerum quidam sunt incompositi, et sunt illi qui in arismetrica et in geometrica primi appellantur. Ideo quia a nullis numeris minoribus existentibus ipsis, preter quam ab unitate, metiuntur vel numerantur. Arabes ipsos hasam appellant. Greci coris canon, nos autem sine regulis eos appellamus; ex quibus illi qui sunt infra centum, in quadam tabula in sequentibus describuntur. Alios vero primos, qui sunt ultra centum per regulam invenire docebo. Reliqui vero compositi, vel epipedi, idest superficiales, peritissimo geometriae Euclide appellantur. Ideo quia componuntur ex multiplicatione ali quorum numerorum, ut duodecim que componuntur ex multiplicatione binarii in 6, vel ternarii in 4, nos autem ipsos regulares numeros appellamus. Et cum dividendi doctrina per primos et compositos non sit eadem, in primis, scilicet per eos qui sunt sine regulis infra centum, quoslibet numeros ipsis maiores existentes dividere ostendamus. Cum autem quemlibet numerum per aliquem prescriptorum, qui sit sine regula, quis dividere voluerit, describat numerum in tabula, et sub ipso ponat ipsum primum numerum, per quem dividere voluerit, collocans siquidem similem gradum sub simili et videat, si due ultime numeri dividendi figure maiorem numerum facient, vel equalem vel minorem ipso primo numero, per quam numerus dividetur. Et si maiorem vel equalem numerum fecerint, incipiendus est ultimus

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……………………………………
Division of Numbers by incomposite Numbers of two places.
(…) (4). On the other hand when one will wish to divide any number by any other number written above which be without rule, then he writes the number in the table and under it he puts the prime number by which he will wish to divide, placing in column similar degrees below similar degrees, and he sees whether the two last figures of the number to be divided make a greater, equal, or smaller number than the prime number by which the number is to be divided. (5) And if a greater or equal number is made, then the last degree of the quotient must begin under the degree following the last degree of the dividend, that is below the penultimate figure, and he puts there, according with the evaluation, a figure such that, multiplied by the number itself or divisor provide as result the number of the aforesaid last two figures, or nearly so. And then he will multiply by the last figure of the prime number itself, namely the divisor, and he must subtract the product from the last figure. And if something will exceed, then he writes the excess above the figure itself. And he will multiply this figure by the first of the same prime number, namely the divisor, and the multiplication of the said union he subtracts from the penultimate figure, and the remainder if it makes a number of two figures that is greater than 10, then he will put the first degree of that number above the penultimate figure, and the last above the last. And if the excess will be of first degree, namely less than 10, then he puts the figure of that number above the penultimate, and he couples the excess with the third figure from the last. And below the third figure he puts according to the evaluation such a figure that multiplied by the same divisor provides as result the number of the said couple, or nearly so. (6)

gradus exeuntis numeri sub sequenti ultimo gradu dividendi numeri, hoc est sub penultima, et ponat ibidem arbitrio talem figuram, que multiplicata per ipsum divisorem numerum, faciat numerum duarum figurarum ultimarum predictarum, vel fere. Et tunc multiplicabis ipsam per ultimam figuram ipsius primi numeri, scilicet divisoris, et exeuntem summam de ultima figura extrahat. Et si aliquid super habundaverit, describat habundantiam super ipsam figuram. Et multiplicet eamdem positam figuram per primam eiusdem primi numeri, scilicet divisoris, et multiplicationem de copulatione dicte super habundantie et penultime figure extrahat, et residuum si fuerit numerus duarum figurarum hoc est quod sit amplius de 10, ponat primum gradum ipsius numeri super penultimam figuram, et ultimum super ultimam. Si autem primi gradus ipsum superfluum extiterit, scilicet minus 10, ponat figuram ipsius super penultimam et copulet ipsum superfluum cum tertia figura ab ultima. Et sub ipsa tertia figura ponat arbitrio talem figuram que multiplicata per eundem divisorem, faciat numerum dicte copulationis, vel fere: quod arbitrium qualiter ex arte habeatur; in sequentibus divisionibus, secundum differentiam ipsorum, ostendere procurabo. Et tunc multiplicet ipsam positam figuram sub tertia per ultimam divisoris, et summam extrahat, si possibile fuerit, ex ultimo gradu dicti superhabundantis et coniuncti numeri. Sin autem extrahet eam de copulatione ultime et sequentis et superfluum ponat super eundem gradum. Et multiplicet iterum ipsam per primam divisoris et summam extrahat de remanenti numero et superfluum ponat desuper. Et sic semper copulando superflua cum figuris per gradus sequentes et sub ipsis gradibus figuras ponendo arbitrio et secundum prescriptum ordinem multiplicando usquequo ad finem numeri devenerit procedere studeat. Verum cum sepe contigerit quod de copulatione superflui et antecedentis figure numerus divisor extrahi non poterit tunc scribendum erit zephirum sub eadem antecedente figura et copulabit eos scilicet antecedenti vel sequenti et superfluo aliam vel sequentem antecedentem figuram et sub ipsa ponat illam figuram que multiplicata per divisorem numerum faciat numerum illarum dictarum trium figurarum scilicet ipsarum que exibunt ex copulatione superhabundantis figure et duarum antecedentium vel sequentium figurarum. Undde si due ultime figure dividendi numeri minorem numerum divisore ut prediximus fecerit incipiendus erit ultimus grads exeuntis numeri sb tertia figura ab ultima; et ita quoslibet numeros per predictos primos numeros dividere poteris». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 30.

How such a judgment may be based on an art; I will manage to show in the succeeding divisions according to their differences. (7) And then he multiplies the figure put below the third degree by the last of the divisor, and the product he subtracts if possible, from the last degree of the said excess after it will be joined. If then he will subtract from the union of the last figure and the successive and he will put the excess above the same degree. And he again multiplies this figure by the first of the divisor, and he will subtract the product from the remaining number, and the excess he will put above. And thus ever coupling the excess with the following figure degree by degree, and putting according to the evaluation the figures beneath these degrees and according to the prescribed order proceeds to multiply until the end of the number is reached up. (8) Truly, from the moment that it often happens that from the coupling of the excess and of the preceding figure he cannot subtract the divisor number; then there will be written a zephir below the preceding figure, and one will couple it, namely the preceding or the following and to the excess the other figure the preceding or the following and beneath he will put the figure that multiplied by the divisor number makes as result the number of the said three figures, namely those that will appear from the coupling of the exceeding figure and the two preceding or the following figures. Whence if the two last figures of the divide number are less than the divisor, as we said before, then the last degree of the quotient will begin below the third figure. And thus any numbers can be divided by the given prime number.

The algorithm providing the division of a number by a prime numbers with two figure may summarized as follows.

A6 - Algorithm to divide a number by a prime numbers with two figure.
Let $N=n_1\ n_{2\ldots}\ n_k$ be a number with k figures which must be divided by the prime number $D= d_1\ d_2$ of two figures.
If N=D return 1 and stop, otherwise
Looking through the figures of N we start from the two first figures namely $n_1\ n_2$.
In general we could divide two o three figures by the divisor $d_1\ d_2$.
if we have a number with two figures $n_i\ n_{i+1}$ then
    if it is greater than $d_1\ d_2$ then
        calculate the quotient $q_j= n_i\ n_{i+1}/d_1\ d_2$
        calculate the difference $n_i\ n_{i+1}- q_j\ d_1\ d_2$
        join this difference with n_{i+2}
        take a new figure of N into account
    otherwise
        let $q_j=0$
        join the difference $n_in_{i+1}- q_j\ d_1d_2$ with n_{i+2}
        take a new figure of N into account
    otherwise (the dividend is composed by three figures)
    calculate the ratio $n_in_{i+1}n_{i+2}/d_1d_2$
    if $d_2 <=5$
        let $q_j= n_in_{i+1}/d_1d_2 – 1$
    otherwise
        let $q_j= n_in_{i+1}/d_1d_2 + 1$
    multiply q_j by d_1d_2
    join $(n_in_{i+1}n_{i+2} - q_j*d_1d_2)$ with n_{i+3}
    take a new figure of N into account.
When we get the last figure of the dividend
    multiply the last figure of the last quotient by d_1, i.e. $q_j* d_1$
    subtract this product from the last computed difference $(n_in_{i+1}n_{i+2} - q_j)- q_j * d_1$
    join this value with the last figure of N, i.e. compute Num $=(n_in_{i+1}n_{i+2} - q_j - q_j * d_1)n_k$.
The residue is given by (Num - $q_j *d_{2)}$/D and the final result is equal to (Num - $q_j *d_{2)}$/D q_j

We remark that when Fibonacci performs the subtraction $(n_i n_{i+1} n_{i+2} - q_j)- q_j * d_1$ he applies the following approximation rule: if q_j is an integer with a fraction part he deletes the fraction part otherwise subtract 1 from the integer number.

As an example, in the following, we show the output obtained dividing 18456 by 17.

DE DIVISIONIBUS INTEGRORUM NUMERORUM

Si voluerit dividere 18456
per 17

dividat 1 per 17
cum non possit dividere 1 per 17
dividat 18 per 17
exibunt 1 et remanet 1
ponat 1 et copulet ipsa 1 cum 4 erunt 14

dividat 14 per 17
cum non possit dividere 14 per 17 ponat zephirum sub 17
dividat 145 per 17
exibunt 8 et remanent 9
ponat 8 et copulet ipsa 9 cum 6 erunt 96

dividat 96 per 17
exibunt 5 et remanent 11
ponat 5
remanentia 11 ponat super virgulam de 17 ex parte servata
et ante ipsa ponat numerum exeuntem ex divisione,
et scilicet 1085 pro quesita divisione

et sic habebitur 11/17 1085

Fig. 4

6. *Prime factor decomposition*

Before to deal with divisors formed of composed numbers, Fibonacci suggests the following algorithm for the factor decomposition later on used to decompose the denominators. He made a distinction between odd and even numbers, suggesting two algorithms very similar. Here we show that for the odd numbers.

> Chapter V - On the Divisions of Integral Numbers.
>
> ………………………………
>
> Factor decomposition of Odd Numbers. A Universal Rule.
>
> (1) Moreover when anyone … will wish to find the rules, that is the factor decomposition of any number with three or more figures, and he will wish to know if it is a prime number, that is without 'rule', then he will write down the number in the table, and after that, he will look whether the number will be even or odd. (2) In fact, if it is even, then he recognizes that it is composed. Indeed if it is odd, then it will be composite or prime. In fact, even numbers are indeed composed from evens and odds, or from evens alone. Therefore the 'rules' must be investigated starting from even numbers, as will be demonstrated at the opportune place. Odd numbers truly are composed of odds alone, whence we must look for the components only among the odds which are at the beginning of the number. (3) Therefore when the figure of first degree of any odd number be the number 5, one will know that it is composed by 5, that is the number is divided integrally by 5. (4) However if another odd figure will appear in the first degree this imply that the whole number is odd, then evaluate the casting out of nines; and if will result zephir, then $\frac{1}{9}$ will be in the factor decomposition, and if 3 or 6 then $\frac{1}{3}\frac{1}{9}$ will be in the factor decomposition, (5) if indeed the residue will show none of this, then one divides by 7; and if there will be an excess, the one again divides the number by 11; and if there is an excess then he divides again by 13, and always he goes on dividing in order by prime numbers according to what is written above in the table until he will find a prime number by which the proposed number can be divided without excess, or until he will come to the square root of the number itself; (6) if he will be able to divide by none of them, then one will judge the number to be prime. (7) However if he will be able to divide it by some given prime number without excess, he divides again by the same number the quotient of the division and again divides the number which results from the division by the same prime number; this is that from which ones will begin to seek the components in order of the all other prime numbers up to the square root if it doesn't have components: and thus doing he proceeds degree by degree, so long as all of its components were obtained. (8) After this were perfectly obtained, one strives

zealously to collate the lesser instead of the grater under a fraction line. And thus one will have the 'rule' that is the composition of any old number...[11].

Suppose we have a table containing the list of the first thousand prime numbers in increasing order. First of all we look for the factors of the number to be decomposed in the interval from 10 to 1. Afterwards, if the product of the found factors is less than the number to be decomposed this means that there are other factors for which it is divisible. These factors are prime numbers greater than 10. Then we look in the prime numbers table a prime number which divides exactly the number to be decomposed until the products of all found factors is less than the square root of the former number. At last we will provide the list of factors in increasing order.

[11] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «Cap. V. *Regula universalis de reperiendis compositionibus imparium numerorum.* Cum autem regulas prescriptorum numerorum in tabulis ex frequenti usu quis sciverit, et voluerit regulas, idest compositiones cuiuslibet numeri aliorum numerorum trium vel plurium figurarum reperire, vel qui primus numerus, idest secundum regulam extiterit, cognoscere voluerit, describat numerum in tabula, et descripto provideat si numerus par fuerit vel impar. Nam si par fuerit ipsum compositum esse cognoscat. Si impar autem compositus, aut primus erit. Sunt enim numeri pares compositi aut ex paribus et imparibus, aut ex paribus tantum. Quare regule ipsorum primo investigande sunt a paribus numeris, ut in suo demonstrabitur loco. Inpares vero numeri componuntur ex imparibus tantum. Unde componentes ipsos per impares tantum investigatur a quibus sumamus initium. Cum itaque figura primi gradus cuiuslibet imparis numeri 5 extiterit numerus, a 5 compositum esse cognoscat, hoc est quod per 5 integraliter dividetur. Si autem alia figura impar in primo gradu extiterit que facit totum numerum esse imparem, accipiat siquidem pensam ipsius per novenarium, que si fuerit zephyrum, tunc $\frac{1}{9}$ et si 3 vel 6 pensa fuerit, tunc $\frac{1}{3}$ in sua erit compositione: si autem pensa nulla istarum extiterit, dividat ipsum per 7; et si aliquid inde superfuerit, dividat iterum numerum per 11; et si aliquid superfuerit, dividet ipsum per 13 et semper eat dividendo per primos numeros ordinate, secundum quod scribuntur in tabula superius descripta donec aliquem primum numerum invenerit, per quem propositum numerum absque aliqua superatione possit dividere vel donec ad eiusdem venerit radicem: si per nullum ipsorum dividi potuerit, tunc ipsum primum esse udicabit. Si autem per aliquem predictorum primorum numerorum ipsum dividere absque superatione potuerit quod ex divisione provenerit, dividat iterum per ipsum; et numerus qui ex divisione extiterit, iterum per eumdem primum numerum dividat, hoc est quod ab eodem incipiet querere componentes ipsius per ordinem per reliquos primos numeros usque ad ipsius radicem, si ipse non habuerit compositionem: et sic semper faciendo egrediatur, donec omnes ipsum habuit componentes. Quibus perfecte habitis, ipsas sub quadam virgula minores per maiores summo studio studeat collocare. Et sic habebit regulam, idest compositionem cuiuslibet imparis numeri». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 30.

A7 - Algorithm for the factor decomposition of odd numbers

```
{   Let us call D the number to be decomposed
    for all numbers "i" from 10 to 1, provided the product of already found factors be
    less then D
            if D is exactly divisible by "i"
                    add "i" to the list of factors of D,
            otherwise decrease "i";
    if the product of the factors yet found is equal to D
            print the list of factors
    otherwise
            until the product of all factors yet found is less than the square root of D
                    look in the table of prime numbers the first value for which D
                    is exactly divisible
                    and add it to the list of factors
}.
```

To divide a number N by a composite number D, Fibonacci proposes the following method:

> Chapter V - On the Divisions of Integral Numbers.
> ....
> On the Divisions of Integral Numbers
> ...and although to divide by composite numbers is just as to divide by prime numbers; we multiply still easily and subtly; in the following the doctrine will be shown that is how the composition rules of numbers are found namely how the 'rules' are found, namely the numbers which they are composed and how they are put under a certain fraction line, so that the lesser will follow always the greater towards the left as taught previously in this chapter. After this, one divides the number one wish to divide by the smallest of the components of the divisor, that is by numeric quantity that is the smallest figure which is below the fraction line; and if some excess will appear, then he puts it above the same figure or above the number. And the quotient of the division be divided by the preceding number or figure in the line fraction. If there will be any excess he puts it over the preceding number or figure. And thus always apply oneself to divide the quotients of division by the preceding components until their end, and putting the excess above the components, one accustoms itself to put the quotient of the division of the last component that is the last number existing under the fraction line one puts it before. And thus will be had the solution of the division of any number by any composite number of any degree...[12].

[12] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «et quamvis per conpositos numeros tanquam per primos omnes numeros dividere multiplicamus tamen levius et subtilius in sequenti ostendit doctrina, scilicet ut reperiantur ipsorum regule. Scilicet numeri ex quibus componuntur et ponantur sub quadam virgula ut semper minores sequantur maiores versus sinistram, ut supra in hoc eodem capitulo edocetur. Post hoc dividat numerum quem dividere per minorem ex componentibus divisorem, hoc est per minorem numerum; vel figura que fuerit sub virgula; et si aliquid super habundaverit, ponat ipsum super eadem figura vel numerum et exiens numerus ex divisione dividatur per antecedentem numerum vel figuram. Et sic semper per ordinem per antecedentes conponentes numeros exeuntes ex divisione donec ad

Therefore, first of all Fibonacci calculates the factors decomposition of the denominator as previously described. Let this decomposition be:

$$\frac{10............0}{b_n b_{n-1}......b_1}$$

The algorithm goes on dividing the number N by the number D as follows:

Divide N by the smaller factor b_n. Let q_{n-1} be the quotient and a_n the excess. We continue by dividing by b_{n-1}, achieving the quotient q_{n-2} and the excess a_{n-1} and so on until we divide q_1 by b_1 getting the quotient Q and the excess a_1. Thereby we have that N/D is equal to

$$\frac{a_n a_{n-1}......a_1}{b_n b_{n-1}......b_1}\,\mathrm{Q}$$

A6 - Algorithm to divide a number by a composite number with two figure.
Let N be the numerator;
Let D be the denominator;
Let Num be a vector in which the numerators of fraction part, coming out from division, are stored ;
Let Fatt be a vector in which the NumFatt factors are stored;

{ decompose (Fatt,D, NumFatt);	[decompose the denominator D in factors and store the NumFatt factors in the vector Fatt]
Q=N;	[store in the temporary variable Q the value of N]
for (int k=1;k<= NumFatt;k++)	[for all factors stored in Fatt calculate the numerator]
{ Num[k]=Q%Fatt[k];	[store the excess of division of Q by the factor in the vector Num]
Q=Q/Fatt[k];}	[Calculate the new value for Q]
}.	

The last value of Q must be written before the fraction line.

Next we show an example of output already calculated by Fibonacci in the *Liber Abbaci*.

finem ipsarum devenerit, dividere studeat; et superflua super eas ponenda et exeuntem numerum ex divisione ultime conpositionis, idest ultimi numeri sub virgula existentis ante ipsam ponere consuescat. Et sic habebit divisionem quorumlibet numerorum factam per quemlibet compositum numerum quorumlibet numerorum factam per quemlibet compositum numerum quorumlibet graduum». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 36.

DIVISIONES CONPOSITOS NUMEROS

Numerator = 67898
Denominator = 1760

Reperta regula de 1760 que est

$\frac{1\ 0\ 0\ 0}{2\ 8\ 10\ 11}$

dividat 67898 per 2
exibunt 33949 et remanet 0
quod 0 ponat super 2
dividat 33949 per 8
exibunt 4243 et remanet 5
quod 5 ponat super 8
dividat 4243 per 10
exibunt 424 et remanet 3
quod 3 ponat super 10
dividat 424 per 11
exibunt 38 et remanet 6
quod 6 ponat super 11
et 38 ponat ante virgulam

$\frac{0\ 5\ 3\ 6}{2\ 8\ 10\ 11}\ 38$

Fig. 5

7. *The multiplication of two integers with fraction lines and fraction parts*

One of the most interesting chapters, from an informatics point of view, is that pertaining to the multiplication of two integer numbers with one or more fraction lines with fraction parts. At the beginning the author describes, in a rough way, the method he will apply. Therefore he illustrates the application of this method to several cases more and more complex. He begins by multiplying two integers with only one fraction part, then he multiplies two numbers with more line of fraction and more fraction parts. In other word first of all Fibonacci shows the method to calculate an expression of the following type

$$\frac{n_1}{d_1} + A$$

up to expressions of the type

$$\frac{n_k \cdots .. n_{h+2}\, n_{h+1}}{d_k \ldots \ldots d_{h+2}\, d_{h+1}} + \cdots \frac{n_h \cdots .. n_{2i+}\, n_{i+1}}{d_h \ldots \ldots d_{i+2}\, d_{i+1}} + \frac{n_i \cdots .. n_2\, n_1}{d_i \ldots \ldots d_2\, d_1} + A$$

In this case our purpose was to look for a generalization of the Fibonacci's method, that is an algorithm which solves any product between two numeric expressions each consisting of an integer and some fraction lines each with more fraction parts.

Fibonacci describes the computation method as follows:

> Chapter VI - On the Multiplication of Integral Numbers with Fractions.
> ...Therefore if you will wish to multiply a number of any degree plus a fraction of one or several parts by a number plus a fraction of one or several parts, then you write the greater number and its fraction part beneath the smaller number and its fraction parts, namely number beneath number, and fraction part beneath fraction part. And you will take the upper number and its fraction parts, and calculate the fraction obtained by adding the given fraction plus its number. And similarly with the lower number you calculate its fraction. And you will multiply the computed fraction of the upper number by that computed of the lower number. And you divide the obtained numerator by denominators of both numbers under the fraction line, after you have them appropriately arranged and you will have the product of a number plus fractions[13].

In order to transform a number with one or more fraction lines and fraction parts in one fraction with one numerator and one denominator, Fibonacci gives the following method.

A number with one fraction with one numerator and one denominator *i.e.*

$$\frac{n_1}{d_1} + A$$

may be written as

$$\frac{(Ad_1 + n_1)}{d_1} \quad (2)$$

A number with one fraction line and two fraction parts as the following

$$\frac{n_2 n_1}{d_2 d_1} + A$$

[13] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «(1) Cum, autem, quemlibet numerum cuiuslibet gradus cum quolibet rupto vel ruptis per quemlibet numerum cum quolibet rupto vel ruptis multiplicare volueris, describe maiorem numerum cum suo rupto - vel ruptis - sub minori numero cum suis minutis, scilicet numerum sub numero, et minuta sub minutis. (2) Et accipe superiorem numerum cum suis minutis, et fac inde talia minuta qualia sunt illa que sunt cum ipso numero. Et similiter de inferiori facies sua minuta. (3) Et multiplicabis facta minuta superioris numeri per facta minuta inferioris. Et summam divides per minuta utriusque numeri sub una virgula, scilicet coaptata, et habebis cuiuslibet numerorum cum minutis multiplicationes». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 36.

may be written as

$$\frac{(Ad_1 + n_1)d_2 + n_2 d_1}{d_1 d_2}$$

repeating the former computation process. A general expression containing one fraction line with several parts as

$$\frac{n_i \cdots .. n_2\, n_1}{d_i \ldots \ldots d_2 d_1} + A$$

becomes

$$\frac{((Ad_1 + n_1)d_2 + \cdots .. n_{i-1})d_i}{d_1 d_2 \ldots \ldots d_i} \quad (3)$$

When we have many fraction lines with several fraction parts, as for instance

$$\frac{n_k \cdots .. n_{h+2}\, n_{h+1}}{d_k \ldots \ldots d_{h+2} d_{h+1}} + \cdots \frac{n_h \cdots .. n_2\, n_{i+1}}{d_h \ldots \ldots d_2 d_{i+1}} + \frac{n_i \cdots .. n_2\, n_1}{d_i \ldots \ldots d_2 d_1} + A \qquad (4)$$

It formally becomes

$$((n_\downarrow(h+1))d_\downarrow(h+2) + \cdots .. n_\downarrow(k-1))d_\downarrow k)/(d_\downarrow(h+1)\ d_\downarrow(h+2) \ldots \ldots d_\downarrow k\ ) +$$
$$\ldots ..((n_\downarrow(i+1))d_\downarrow(i+2) + \cdots .. n_\downarrow(i+j-1))d_\downarrow(i+j))/(d_\downarrow(i+1)\ d_\downarrow(i+2) \ldots \ldots d_\downarrow(i+j)\ ) +$$
$$\frac{((Ad_1 + n_1)d_2 + \cdots .. n_{i-1})d_i}{d_1 d_2 \ldots \ldots d_i} \quad (5)$$

In fact we found, in the examples proposed by Fibonacci, a general method to evaluate expressions like (4). Indeed, it is sufficient to apply the algorithm found for an expression with one fraction and several terms (3) to all fraction lines, after the first, assigning zero to the variable A. Fibonacci doesn't describe its method in this terms and for every example he performs its computation but some special cases.

According to the previous remarks an algorithm has been designed and applied to the examples suggested by Fibonacci, controlling the correctness of the results.

In many examples, described in the *Liber Abbaci*, the author in order to simplify the computation of the division, applies the Greatest Common Divisor Algorithm, as proposed by Euclid, to the numerator and denominator of the final fraction, so that he accomplishes the division between smaller numbers. Let us quote Fibonacci:

Chapter VI - On the Multiplication of Integral Numbers with Fractions.

……………………………….

Reduction.

… And you remark that when the numerator shares some factors with the denominator, namely the number which is above the fraction line with the number

> which is below the fraction line; then they must be simplified by dividing them by the largest number starting from which they have a common factor...[14].

Moreover, another trick has been introduced by Fibonacci to modify the factors of the denominator. As previously said we must decompose the denominator in its factors and then to divide the numerator by each factor. Clearly, the factors may be joined in many different ways provided the total product doesn't change. Therefore, Fibonacci claims that it is more 'beautiful' or 'elegant' to group factors from right to left in decreasing order starting from the prime number greater than 10, if any, and then to merge and to multiply the others factors among them provided the product be at most equal to 10.

> Chapter VI - On the Multiplication of Integral Numbers with Fractions.
>
> ....
>
> ...And if the number can be divided integrally neither by the 4, nor by the 5, then we habituate ourselves to divide with $\frac{1}{2}\frac{0}{10}$, because four fives make 20, for which the composition rule is $\frac{1}{2}\frac{0}{10}$. And this we make closer to a more beautiful expression because it is more elegant to say $\frac{1}{2}\frac{0}{10}$ than $\frac{1}{4}\frac{0}{5}$ although they are equal. Similarly you must understand the same of other numbers, namely when you have to divide some number by 3 and by 4, that is with $\frac{1}{3}\frac{0}{4}$; and the number is divided integrally by none of them, the you divide them with $\frac{1}{2}\frac{0}{6}$ which is more elegant. Again when you will have to divide by 4, and by 4, that is with $\frac{1}{4}\frac{0}{4}$, you divide it with $\frac{1}{2}\frac{0}{8}$. And when you will have to divide by 3 and by 6, that is with $\frac{1}{3}\frac{0}{6}$, you divide with $\frac{1}{2}\frac{0}{9}$ because the multiplication of the 2 by the 9 is the same as by the 3 and the 6. But we chose the most extreme numbers, with ten and less, in the composition of the number because $\frac{1}{4}\frac{0}{9}$, is more elegant than $\frac{1}{6}\frac{0}{6}$[15].

[14] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «Cap. VI - *De evitatione* (...) (2) Et nota cum numerus denominans comunicat cum denominato scilicet numerus qui est super virgam cum numero qui est sub virga tunc debent aptari dividendo eos per maiorem n.umerum qui est comunis utrisque a quo ipsi sunt comunicantes». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., pp. 36 and 51.

[15] From Germano, Caianiello, Carotenuto, Burattini, work in progress: «Cap.VI. *De evitatione*. (...). Et si numerus ille nec per 4 nec per 5 integraliter dividi possit, consuevimus ipsum dividere per $\frac{1}{2}\frac{0}{10}$, ideo quia quattuor et quinque faciunt 20, quorum regula est $\frac{1}{2}\frac{0}{10}$. (7) Et hoc facimus propter pulchriorem locutionem: quia pulchrius est dicere $\frac{1}{2}\frac{0}{10}$ quam $\frac{1}{4}\frac{0}{5}$, quamvis [A, f.14r] idem sint. Similiter debes intelligere de quibusdam aliis numeris. Scilicet, cum debueris dividere aliquem numerum per 3 et per 4, hoc est per $\frac{1}{3}\frac{0}{4}$; qui numerus non dividatur per aliquem ipsorum integraliter, divides eum per $\frac{1}{2}\frac{0}{6}$ quod est pulchrius. Item cum debueris dividere per 4 et per 4 hoc est per $\frac{1}{4}\frac{0}{4}$, divides eum per $\frac{1}{2}\frac{0}{8}$. Et cum debueris dividere per 3 et per 6, hoc est per $\frac{1}{3}\frac{0}{6}$, divides per $\frac{1}{2}\frac{0}{9}$, ideo quia tantum faciet multiplicatio de 2 in 9 quantum de 3 in 6.......

For instance, the number 1584000 has the following factors: 11, 5, 5, 5, 3, 3, 2, 2, 2, 2, 2, 2, 2. Fibonacci suggests to merge these factors in this way: 11, 5*2, 5*2, 5*2, 3*3, 2*2*2, 2 i.e. 11, 10, 10, 10, 9, 8, 2.

The algorithm generating the factors reorganization may be described as follows. Since the factors are obtained applying the A7 algorithm which generates the factors in increasing order then, starting from the last factor we check if it is a prime greater than 10. In this case we leave it unchanged. Once a factor is less than 10 we check if it is equal 5. If so we look for a 2 in the factors left over. If there is at least one, we substitute the 5 and 2 by 10, otherwise we multiply the factor we have on hand by the next. If the new product is less than 10 we replace the two factors by it otherwise we leave it unchanged and go on until the first but one factor has been achieved.

Finally, the product of two numbers with more fraction lines each with more parts may be obtained by the following algorithm:

A9 - Algorithm for the multiplication of two numbers with more fraction lines each with more parts

```
{ "FIRST TERM";
Let Nt1 be the number of fraction lines of the first term;
Let T1 a matrix of two rows and N>Nt1 columns.
Put in the first row at column i the numerator of i-th fraction line
Put in the second row at column i the denominator of i-th fraction line
 { reduce_to_one_fraction(N1,D1,i);     [for all the Nt1 fraction lines ask for the
                                         number of parts and then apply the
                                         formula (2)]
      T1[0][i]=N1;                      [put in T1[0][i] the numerator of i-th
                                         fraction line]
      T1[1][i]=D1;                      [put in T1[1][i] the denominator of i-th
                                         fraction line]
      DT1=DT1*D1; ;                     [calculate the product of all denominators]
 }
 for (int i=0;i<Nt1;i++)
 {                                      [calculate the numerator applying the
                                         formula (2)]
  T1[0][i]=T1[0][i]*DT/T1[1][i];
  NumTot1=NumTot1+T1[0][i];
 }

 { "SECOND TERM";
Let Nt2 be the number of fraction lines of the SECOND term;
Let T2 a matrix of two rows and N>Nt2 columns.
Put in the first row at column i the numerator of i-th fraction line
Put in the second row at column i the denominator of i-th fraction line

 for (int i=0;i<Nt2;i++)
```

(8) Sed nos diligimus plus extremos numeros qui sunt a decem et infra in compositionibus numerorum et ideo pulchrius est $\frac{1}{4}\frac{0}{9}$, quam $\frac{1}{6}\frac{0}{6}$. Et hoc idem intelligas de precedentibus». See also Boncompagni, *Il* Liber abbaci *di Leonardo Pisano* cit., p. 50.

{ reduce_to_one_fraction(N2,D2,i);	[for all the Nt2 fraction lines ask for the number of parts and then apply the formula (2)]
T2[0][i]=N2;	[put in T2[0][i] the numerator of i-th fraction line]
T2[1][i]=D2;	[put in T2[1][i] the denominator of i-th fraction line]
*DT2=DT2*D2; ;*	[calculate the product of all denominators]
}	
for (int i=0;i<Nt2;i++)	
{	[calculate the numerator applying the formula (2)]
*T2[0][i]=T2[0][i]*DT/T2[1][i];*	
NumTot2=NumTot2+T2[0][i];	
}	
simplify(NumTot1,Dt1);	[calculate the Greater Common Divisor between NumTot1 and Dt1 and simplify]
simplify (NumTot2,Dt2);	[calculate the Greater Common Divisor between NumTot2,Dt2 and simplify]
N= NumTot1 NumTot2*	[final fraction Numerator]
D= Dt1 Dt2*	[final fraction Denominator]
factor_decomposition(D, Fatt)	[decompose the denominator D putting the factors in the vector Fatt]
more_beatiful(Fatt)	[modify the factors arrangement according to the 'more beautiful' criterion]
division(N,Fatt)	[perform the division by algorithm A8 obtaining the final number and its fraction part]

In this algorithm we refer to the functions: *simplify*(NumTot,Dt), *factor_decomposition*(D, Fatt), *more_beatifull*(Fatt), *division*(N,Fatt), previously discussed.

In the following examples we will show the output of Algorithm A9 for two cases, already calculated by Fibonacci.

1. Multiply $\frac{1}{3}\frac{1}{4}\frac{1}{5}$ 21 by $\frac{3}{7}\frac{2}{9}\frac{1}{8}$ 32

2. Multiply $\frac{1}{5}\frac{2}{9}\ \frac{1}{2}\frac{5}{8}$ 17 by $\frac{2}{5}\frac{1}{5}\ \frac{3}{8}\frac{4}{11}$ 28

In this case the output aren't printed in latin since the algorithm we have implemented is quite different from the method proposed by Fibonacci, in the sense that the 'reductions' introduced by him throughout the VI chapter are collected all together. This has been useful to check the correctness of the results obtained by Fibonacci who sometimes apply the reduction some other time he doesn't apply it.

8. *Conclusions*

The study of several Fibonacci's computation methods highlights the capacity of the author of explaining by clear examples a lot of computation method based on, at his time, an absolutely new numbers representation. Moreover also its great didactic capabilities are stressed. The former aspect allowed us to traduce easily each method in an algorithm implementable on a modern computer by means of a C++ program. The possibility of quickly checking by means of a computer program the correctness of the results described in several different versions of the *Liber Abbaci* may give hints to identify the version closer to the original one. In appendix A examples of some errors are reported.

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Appendix

Since the work of our group is yet in progress to give a reference we quote the page of books by Boncompagni[16] and Sigler[17] who converted from Latin to English the text of Boncompagni in which these errors may be found.

Chap. V (Boncompagni, *Il* Liber Abbaci *di Leonardo Pisano* cit., p. 40; Sigler, Sigler, *Fibonacci’s* Liber Abaci. *A Translation* cit., p. 69)

The number to be decomposed is reported as 4644 while the number Fibonacci decomposes is 4664. However, in the rest of the Boncompagni's text, the calculation is done with respect to 4664.

Chap. VI (Boncompagni, *Il* Liber Abbaci *di Leonardo Pisano* cit., pp. 58-59; Sigler, *Fibonacci’s* Liber Abaci. *A Translation* cit., pp. 91-92)

The product $\frac{121}{355}\frac{123}{2910}\frac{116}{2717}11 \text{x} \frac{251}{367}\frac{122}{579}\frac{133}{2810}22$ in Fibonacci (Boncompagni and Sigler) is reported as $\frac{1441259972}{277899101017}274$ instead the correct value is $\frac{1210139504}{27789910101017}274$

Chap. VI (Boncompagni, *Il* Liber Abbaci *di Leonardo Pisano* cit., pp. 64-65; Sigler, *Fibonacci’s* Liber Abaci. *A Translation* cit., p. 97)

The product $\frac{25}{79}33\frac{13}{54} \text{x} \frac{15}{116}244\frac{13}{47} = \frac{355366}{4677911}3628$ in Fibonacci (Boncompagni and Sigler) is reported as $\frac{260144}{3778911}3628$ instead the correct value is $\frac{355366}{4677911}3628$.

[16] Boncompagni, *Il* Liber Abbaci *di Leonardo Pisano* cit., p. 1-459.
[17] L.E. Sigler, *Fibonacci’s* Liber Abaci. *A Translation into Modern English of Leonardo Pisano’s Book of Calculation*, New York-Berlin-Heidelberg 2002.