Tables of subspace codes

Daniel Heinlein, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann*

December 12, 2017

The main problem of subspace coding asks for the maximum possible cardinality of a subspace code with minimum distance at least d over $\mathbb{F}_q^n$, where the dimensions of the codewords, which are vector spaces, are contained in $K \subseteq \{0, 1, \ldots, n\}$. In the special case of $K = \{k\}$ one speaks of constant dimension codes. Since this emerging field is very prosperous on the one hand side and there are a lot of connections to classical objects from Galois geometry it is a bit difficult to keep or to obtain an overview about the current state of knowledge. To this end we have implemented an on-line database of the (at least to us) known results at subspacecodes.uni-bayreuth.de. The aim of this technical report is to provide a user guide how this technical tool can be used in research projects and to describe the so far implemented theoretic and algorithmic knowledge.

Keywords: Galois geometry, subspace codes, partial spreads, constant dimension codes
MSC: 51E23; 05B40, 11T71, 94B25

1. Introduction

The seminal paper by Kötter and Kschischang [60] started the interest in subspace codes which are sets of subspaces of the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$. Two widely used distance measures for subspace codes (motivated by an information-theoretic analysis of the Kötter-Kschischang-Silva model, see e.g. [77]) are the *subspace distance*

$$d_S(U, W) := \dim(U + W) - \dim(U \cap W) = 2 \cdot \dim(U + W) - \dim(U) - \dim(W)$$

and the *injection distance*

$$d_I(U, W) := \max\{\dim(U), \dim(W)\} - \dim(U \cap W),$$

where U and W are subspaces of $\mathbb{F}_q^n$. The two metrics are equivalent, i.e., it is known that $d_I(U, W) \le d_S(U, W) \le 2d_I(U, W)$. Here, we restrict ourselves to the subspace distance.

The set of all k-dimensional subspaces of an $\mathbb{F}_q$-vector space V will be denoted by $\begin{bmatrix} V \\ k \end{bmatrix}_q$. For $n = \dim(V)$, its cardinality is given by the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{cases} \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)} & \text{if } 0 \le k \le n; \\ 0 & \text{otherwise.} \end{cases}$$

*All authors are with the Department of Mathematics, Physics, and Computer Science, University of Bayreuth, Bayreuth, GERMANY. email: firstname.lastname@uni-bayreuth.de
The work was supported by the ICT COST Action IC1104 and grants KU 2430/3-1, WA 1666/9-1 – “Integer Linear Programming Models for Subspace Codes and Finite Geometry” – from the German Research Foundation.

A set $\mathcal{C}$ of subspaces of V is called a *subspace code*. The *minimum distance* of $\mathcal{C}$ is given by $d = \min\{d_S(U,W) \mid U,W \in \mathcal{C}, U \neq W\}$. If the dimensions of the *codewords*, i.e., the elements of $\mathcal{C}$ are contained in some set $K \subseteq \{1,\dots,n\}$, $\mathcal{C}$ is called an $(n, \#\mathcal{C}, d; K)_q$ subspace code. In the unrestricted case $K = \{0,\dots,n\}$, also called mixed dimension case, we use the notation $(n, \#\mathcal{C}, d)_q$ subspace code. In the other extreme case $K = \{k\}$, we use the notation $(n, \#\mathcal{C}, d; k)_q$ and call $\mathcal{C}$ a *constant dimension code*.

For fixed ambient parameters q, n, K and d, the *main problem of subspace coding* asks for the determination of the maximum possible size $A_q(n,d;K) := M$ of an $(n, M, \geq d; K)_q$ subspace code and – as a refinement – the classification of all corresponding optimal codes up to isomorphism. Again, the simplified notations $A_q(n,d)$ and $A_q(n,d;k)$ are used for the unrestricted case $K = \{0,\dots,n\}$ and the constant dimension case $K = \{k\}$, respectively. Note that in the latter case $d_S(U,W) = 2 \cdot d_I(U,W) \in 2 \cdot \mathbb{N}$ is an even number.

In general, the exact determination of $A_q(n,d;K)$ is a hard problem, both on the theoretic and the algorithmic side. Therefore, lower and upper bounds on $A_q(n,d;K)$ have been intensively studied in the last years, see e.g. [20]. Since the underlying discrete structures arose under different names in different fields of discrete mathematics, it is even more difficult to get an overview of the state of the art. For example, geometers are interested in so-called partial $(k-1)$-spreads of $\mathrm{PG}(n-1,q)$. Following the track of partial spreads, one can end up with orthogonal arrays or (s,r,μ)-nets. Furthermore, q-analogs of Steiner systems provide optimal constant dimension codes. For some sets of parameters constant dimension codes are in one-to-one correspondence with so-called vector space partitions.

The aim of this report is to describe the underlying theoretical base of an on-line database, found at

http://subspacecodes.uni-bayreuth.de

and maintained by the authors that tries to collect up-to-date information on the best lower and upper bounds for subspace codes. Whenever the exact value $A_q(n,d;K)$ could be determined, we ask for a complete classification of all optimal codes up to isomorphism. Occasionally we list classifications for non-maximum codes, too. Since the overall task is rather comprehensive, we start by focusing on the special cases of constant dimension codes, $A_q(n,d;k)$, and (unrestricted) subspace codes, $A_q(n,d)$, using the subspace distance as metric. For a more comprehensive survey on network coding we refer the interested reader e.g. to [6]. For algorithmic aspects we refer the interested reader e.g. to [61].

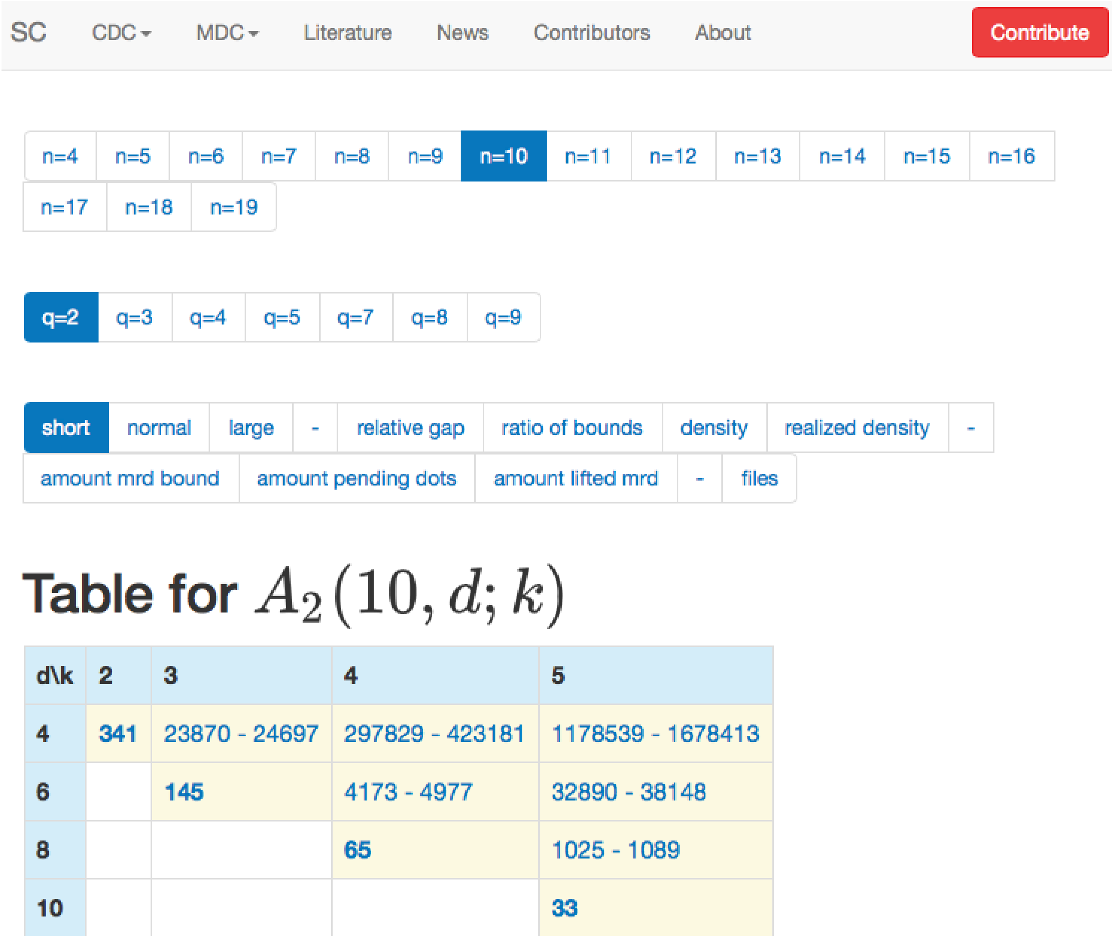
The remaining part of this report is structured as follows. In Section 2 we outline how to use the tables. Constant dimension codes (CDC) are treated in Section 3, where the currently implemented lower bounds, constructions, and upper bounds are described in Subsection 3.1 and Subsection 3.2, respectively. Mixed dimension codes (MDC) are treated in Section 4, where the implemented lower bounds, constructions, and upper bounds are described in Subsection 4.1 and Subsection 4.2, respectively. The application programming interface (API) is the topic of Section 5. Finally we draw a conclusion in Section 6 and list some explicit tables on upper and lower bounds in an appendix.

2. How to use the tables

On the website the two special cases $A_q(n,d;k)$ and $A_q(n,d)$ can be accessed via the menu items `CDC` (constant dimension code) and `MDC` (mixed dimension code), see Figure 1. Selecting the item `Table` yields the rough data that we will outline in this section. Selecting the item `Constraints` yields information about the so far implemented lower and upper bounds.

2.1. Constant dimension codes – CDC

For a constant dimension code the dimension n of the ambient space (first *selection row*) and the field size q (second *selection row*) can be chosen. The current limits are $2 \leq q \leq 9$ and $4 \leq n \leq 19$ (resp. in the *large* view $1 \leq n \leq 19$). For each chosen pair of those parameters a table with the information on lower and upper bounds on constant dimension codes over $\mathbb{F}_q^n$ is displayed.



d\k	2	3	4	5
4	341	23870 - 24697	297829 - 423181	1178539 - 1678413
6		145	4173 - 4977	32890 - 38148
8			65	1025 - 1089
10				33

Figure 1: Tables of constant dimension codes

The rows of those tables are labeled by the minimum distance $d = d_S(\star)$ and the columns are labeled by the dimension k of the codewords. In the third *selection row* several *views* can be picked. The first three options, `short`, `normal`, and `large`, specify the subset of possible values for the parameters d and k. In the most extensive view `large`, k can take all integers between 0 and n. For d the integers between 1 and n are considered. As

- $A_q(n, d; 0) = 1$ for all $1 \leq d \leq n$;
- $A_q(n, d; k) = A_q(n, d; n - k)$;
- $A_q(n, 2d' + 1; k) = A_q(n, 2d' + 2; k)$ for all $d' \in \mathbb{N}$;

one may assume $1 \leq k \leq \lfloor n/2 \rfloor$, $2 \leq d \leq n$, and $d \in 2\mathbb{N}$. These assumptions are implemented in the view `normal`. However, some exact values of $A_q(n, d; k)$ are rather easy to determine

- $A_q(n,2;k) = \begin{bmatrix} n \\ k \end{bmatrix}_q$, since any two different k-dimensional subspaces of $\mathbb{F}_q^n$ have a subspace distance of at least 2;
- if $d > 2k$, then we can have at most one codeword, i.e., $A_q(n,d;k) = 1$.

Thus, we may assume $2 \le k \le \lfloor n/2 \rfloor$, $4 \le d \le 2k$, and $d \in 2\mathbb{N}$. These assumptions are implemented in the view `short`. The standard selection is given by $n = 4$, $q = 2$ and the view `short`.

Given one of these three views, a table entry may consist of

- a range l–u: An example is given by the parameters $q = 2$, $n = 7$, $d = 4$, $k = 3$, where $l = 333$ and $u = 381$. The meaning is that for the corresponding maximum cardinality of a constant dimension code only the lower bound l and the upper bound u is known, i.e., $333 \le A_2(7,4;3) \le 381$ in the example.
- a **bold** number m: An example is given by the parameters $q = 2$, $n = 10$, $d = 8$, $k = 4$, where $m = 65$. The meaning is that the corresponding maximum cardinality of a constant dimension code is exactly determined, i.e., $A_2(10,8;4) = 65$ in the example.
- a **bold** number m with an asterisk and a number l in brackets: An example is given by the parameters $q = 2$, $n = 6$, $d = 4$, $k = 3$, where $m = 77$ and $l = 5$. The meaning is that the corresponding maximum cardinality of a constant dimension code is exactly determined and all optimal codes have been classified up to isomorphism, i.e., $A_2(6,4;3) = 77$ and there are exactly 5 isomorphism types in the example, see [51]. Another example is given for the parameters $q = 2$, $n = 6$, $d = 4$, and $k = 2$, where there are exactly $131,044$ isomorphism types of constant dimension codes attaining cardinality $A_2(6,4;2) = 21$, see [68].
- a **bold** number m with a lower bound $\ge l$ in brackets: An example is given by the parameters $q = 2$, $n = 13$, $d = 4$, $k = 3$, where $m = 1597245$ and $l = 512$. The meaning is that the corresponding maximum cardinality of a constant dimension code is exactly determined and there are at least l isomorphism classes of optimal codes.

Each nontrivial table entry is clickable and then yields further information on several lower and upper bounds, see Subsection 3.1 and Subsection 3.2 for the details.

In some cases, e.g., for the parameters $q = 2$, $n = 6$, $d = 4$, and $k = 3$, the corresponding codes are also available for download using the button called "file". The format of these codes is mostly GAP[1] or MAGMA[2].

Besides the views `short`, `normal`, and `large` for the selection of ranges for the parameters d and k, there are some additional views. The views `relative gap` and `ratio of bounds` condense the current lack of knowledge on the exact value of $A_q(n,d;k)$ to a single number. For the view `relative gap` this number is given by the formula

$$\frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}},$$

i.e., we obtain a non-negative real number. While principally any number in $\mathbb{R}_{\ge 0}$ can be obtained, the largest relative gap in our database is currently given by about 0.727 for the parameters $q = 2, n = 19, d = 4, k = 9$. A gap of 0.0 corresponds to the determination of the exact value $A_q(n,d;k)$. The mentioned formula is also displayed on the webpage, when you move your mouse over the word `relative gap`. For the view `ratio of bounds` the corresponding number is given by the formula

$$\frac{\text{lower bound}}{\text{upper bound}},$$

[1] http://www.gap-system.org

[2] http://magma.maths.usyd.edu.au

which may take any real number in $(0,1]$. The smallest ratio of bounds in our database is given by about 0.579 for the same parameters as above. Clearly, the largest relative gap yields the smallest ratio of bounds and vice versa as the function $x \mapsto \frac{1}{x} - 1$ is strictly decreasing in $(0,1]$. A ratio of bounds of 1.0 corresponds to the determination of the exact value $A_q(n,d;k)$. The mouse-over effect is also implemented in that case.

The views `density` and `realized density` compare the Anticode bound, see (Theorem 3.33), to the best known upper bound and best known lower bound, respectively, i.e.,

$$\frac{\text{best known upper bound}}{\text{Anticode bound}}, \qquad \text{and} \qquad \frac{\text{best known lower bound}}{\text{Anticode bound}}.$$

Hence, they are a measure how dense it is possible to fill the Grassmannian with codewords. Note that in the case of Steiner Systems, both bounds, the `density` and the `realized density`, are one since the size of a Steiner System is exactly the size of the Anticode bound.

Another type of view arose from some of the various constructions described in Subsection 3.1. They are labeled as `amount pending dots` and `amount lifted mrd` and condense the *strength* of a certain construction to a single number in $\mathbb{R}_{\geq 1}$. This number is always given as the quotient between the currently best known lower bound and the value obtained by the respective construction. Here, a value of one means that the currently best known code can be obtained by the respective construction. A value larger than 1 measures how much better a more tailored construction is for this specific set of parameters compared to the respective general construction method. We remark that `amount pending dots` is still experimental and in some cases there may still be better codes obtained from the underlying very general construction technique, which has quite some degrees of freedom. With respect to upper bounds the additional view `amount mrd bound` is introduced. Here the displayed single number is given by the currently best known lower bound divided by the so-called MRD bound, see Subsection 3.2.3.

The view `files` is like the view `short` but the background gets a green color if there is a downloadable file for these parameters.

2.1.1. Toplist

These statistics, see Figure 2, show how often a single constraint yields the best known bound for the parameters $2 \leq q \leq 9$, $4 \leq n \leq 19$, $2 \leq k \leq \lfloor n/2 \rfloor$, and $4 \leq d \leq 2k$, where d is even. For each set of parameters in which a single constraint yields the best known value, it scores a point. This score is then divided by the size of the set of parameters, i.e., all constant dimension code parameters in the database. Constraints are grouped into two categories: lower and upper bounds and then ordered by their normalized score. The special constraints that yield the exact code sizes appear in both categories and are denoted with an asterisk (*).

Currently the lower bound with the highest score is the `improved_linkage` construction, see Theorem 3.18, and it yields the best known lower bound in 69.1% of the constant dimension code parameters of the database.

The upper bound `improved_johnson` has currently the highest score for upper bounds with 90.2%.

2.1.2. Views for single CDCs

Each constant dimension code entry provides multiple level of details which are also called views, see Figure 3.

The view `all` shows all constraints.

The view `short` is the default and displays only the best instances for the same constraint but with different parameters.

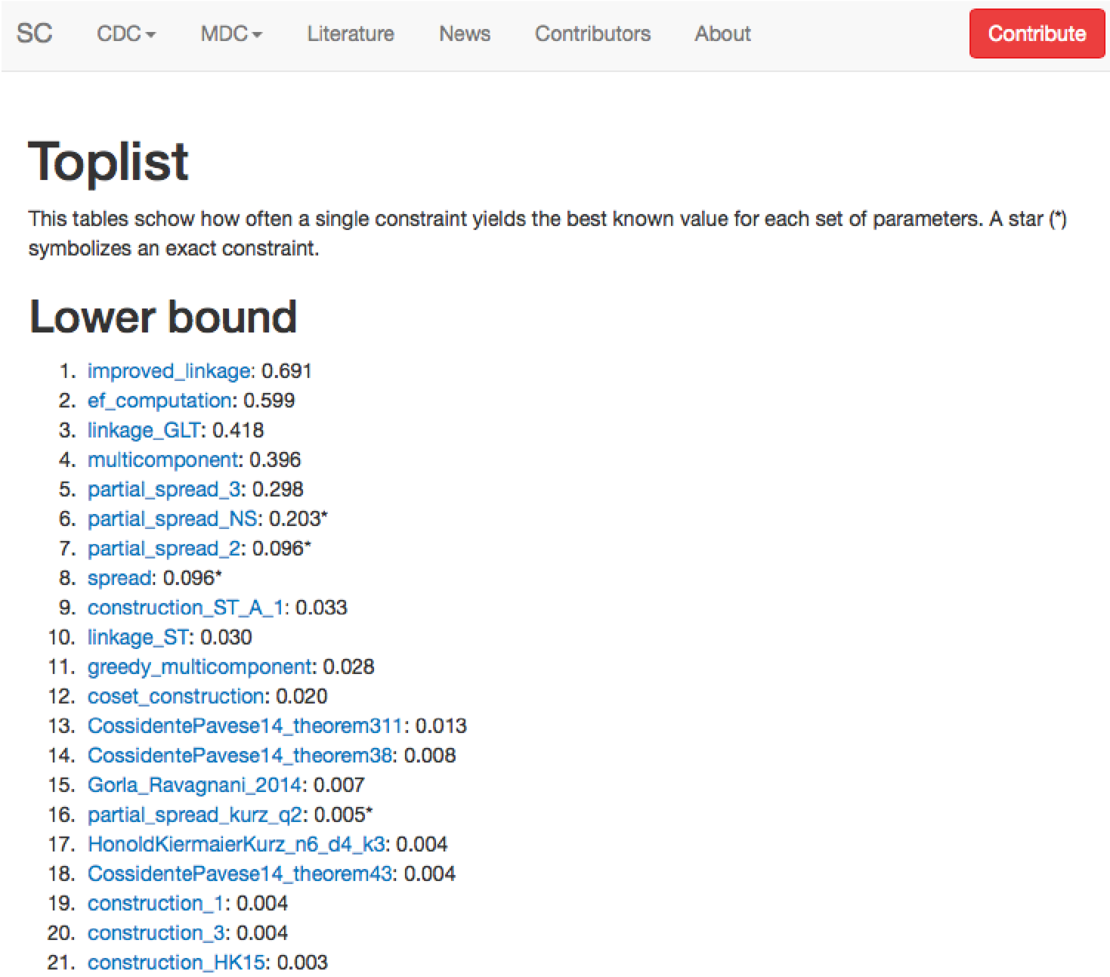


Figure 2: Toplist of constant dimension codes

The third view, `dominance`, performs like `short` but incorporates an additional filtering due to known relationships between constraints. Hence, if a constraint is always worse than another constraint that is present, it is omitted.

Currently `dominance` respects for the parameters of this database the following relations between upper bounds:

`sphere_packing` $\leq$ `all_subs`
`anticode` $\leq$ `sphere_packing`
`anticode` $\leq$ `singleton`
`johnson_1` $\leq$ `johnson_2`
`johnson_1` $\leq$ `anticode`
`johnson_1` $\leq$ `ilp_1`
`ilp_1` $\leq$ `ilp_2`
`ilp_4` $\leq$ `ilp_3`
`johnson_2` $\leq$ `ilp_4`

Ahlswede_Aydinian ≤ johnson_1
Ahlswede_Aydinian ≤ johnson_2

And between lower bounds:

sphere_covering ≤ trivial_1
echelon_ferrers ≤ lin_poly
ef_computation ≤ echelon_ferrers
improved_linkage ≤ linkage_GLT
improved_linkage ≤ linkage_ST

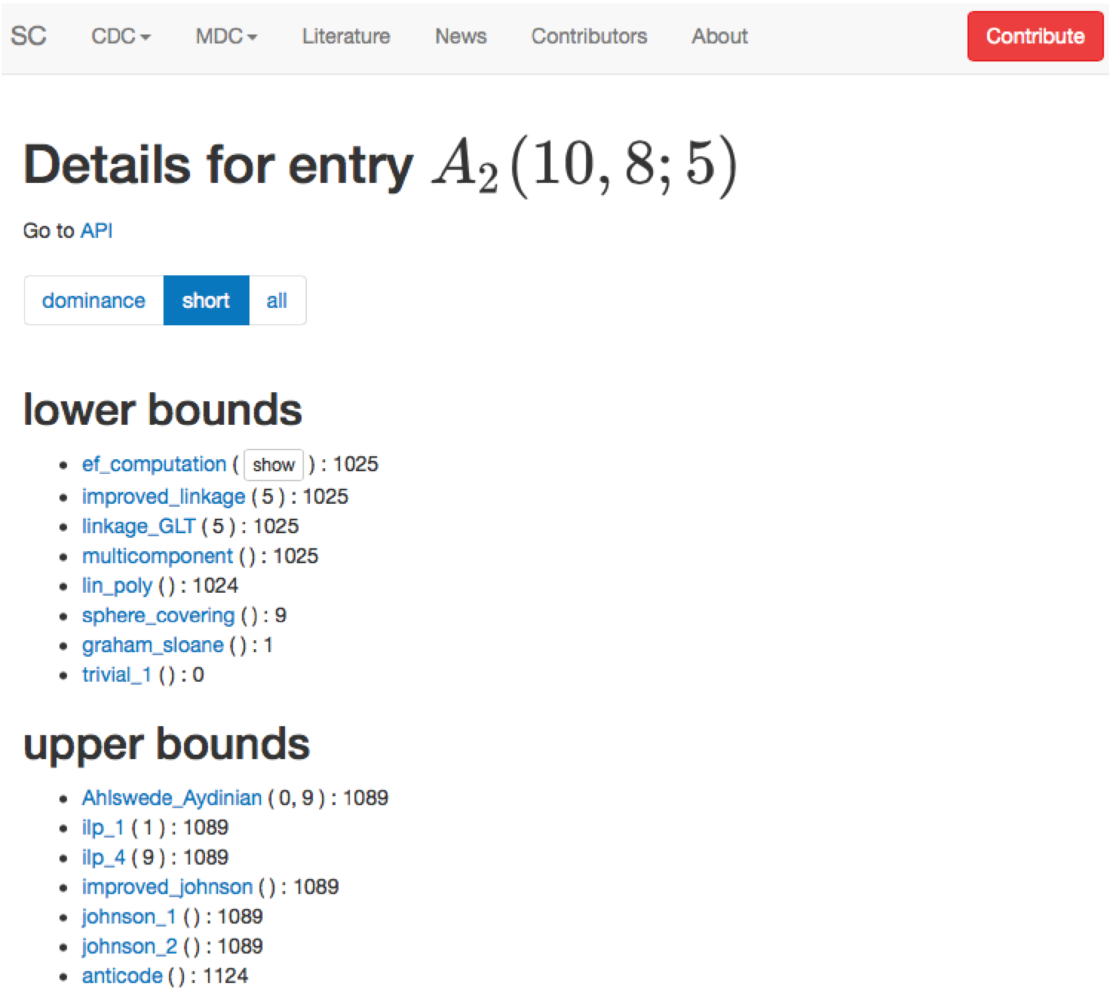
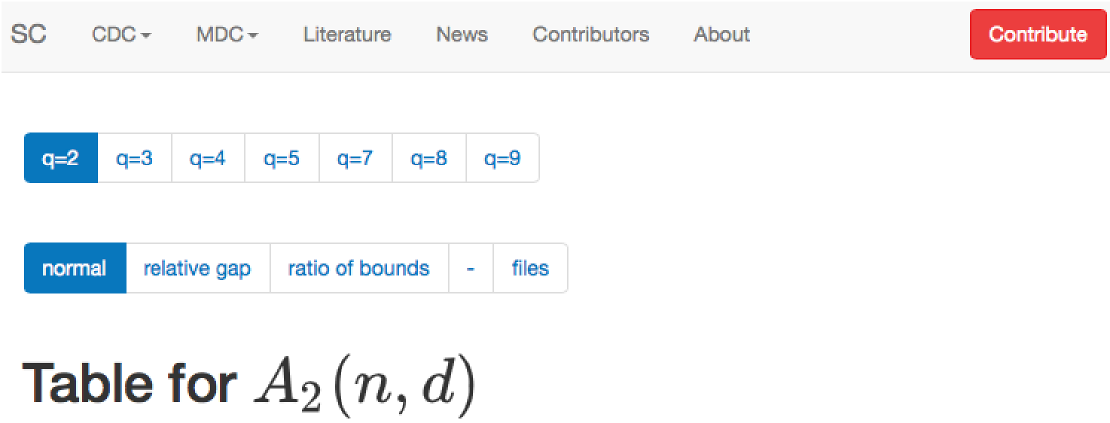


Figure 3: Views of constant dimension code entries

2.2. Mixed dimension codes – `MDC`

For a subspace code with mixed dimensions the field size q (*selection row* number one) can be chosen. The current limits are given by $2 \le q \le 9$. For each chosen parameter a table with the information on lower and upper bounds on $A_q(n, d)$ over $\mathbb{F}_q^n$ ($n \le 19$) is displayed, see Figure 4.



d\n	1	2	3	4	5	6	7	8	9	10	11	1
1	2 * (1)	5 * (1)	16 * (1)	67 * (1)	374 * (1)	2825 * (1)	29212 * (1)	417199 * (1)	8283458 * (1)	229755605 * (1)	8933488744 * (1)	4 *
2		3 * (1)	8 * (1)	37 * (1)	187 * (1)	1521 * (1)	14606 * (1)	222379 * (1)	4141729 * (1)	121919127 * (1)	4466744372 * (1)	2 *
3			2 * (2)	5 * (3)	18 * (24298)	108 - 118	614 - 776	5687 - 9268	71427 - 107419	1242813 - 2532929	37307418 - 62828622	1 2
4				5 * (1)	9 * (7)	77 * (4)	334 - 463	4803 - 9635	36947 - 114472	1178541 - 2574853	18766341 - 62828622	1 2

Figure 4: Tables of (mixed dimension) subspace codes

The rows of those tables are labeled by the distance $d = d_S(\star)$ and the columns are label by the dimension n of the ambient space $\mathbb{F}_q^n$. In the second *selection row* several *views* can be picked. The view `normal`, c.f. Subsection 2.1, already incorporates the restriction to $1 \le d \le n \le 19$. The views `relative gap` and `ratio of bounds` condense the current lack of knowledge on the exact value of $A_q(n, d)$ to a single number. For the view `relative gap` this number is given by the formula

$$\frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}},$$

i.e. we obtain a non-negative real number. While principally any number in $\mathbb{R}_{\ge 0}$ can be obtained, the largest relative gap in our database is currently given by about 2.493 for the parameters $q = 2, n = 19, d = 4$. A relative gap of 0.0 corresponds to the determination of the exact value $A_q(n, d)$. The mentioned formula is also displayed on the webpage, when you move your mouse over the word `relative gap`. For the view `ratio of bounds` the corresponding number is given by the formula

$$\frac{\text{lower bound}}{\text{upper bound}},$$

which may take any real number in $(0, 1]$. The smallest ratio of bounds in our database is given by about 0.29 for the same parameters as above. Clearly the largest relative gap yields the smallest ratio of bounds and vice versa as the function $x \mapsto \frac{1}{x} - 1$ is strictly decreasing in $(0, 1]$. A ratio of bounds of 1.0 corresponds to the determination of the exact value $A_q(n, d)$. The mouse-over effect is also implemented in that case.

The view `files` is like the view `normal` but the background gets a green color if there is a downloadable file for these parameters.

2.2.1. Toplist

These statistics show how often a single constraint yields the best known bound for the parameters $2 \le q \le 9$, $4 \le n \le 19$, and $1 \le d \le n$. For each set of parameters in which a single constraint yields the best known value, it scores a point. This score is then divided by the size of the set of parameters, i.e., all mixed dimension code parameters in the database. Constraints are grouped into two categories: lower and upper bounds and then ordered by their normalized score. The special constraints that yield the exact code sizes appear in both categories and are denoted with an asterisk (*).

Currently the lower bound with the highest score is `layer_construction` (Theorem 4.5) and it yields the best known lower bound in 78.7% of the mixed dimension code parameters of the database.

The upper bound `improved_cdc_upper_bound` (Theorem 4.14) has currently the highest score for upper bounds with 73.1%, see Figure 5.

2.2.2. MDC table for arbitrary q

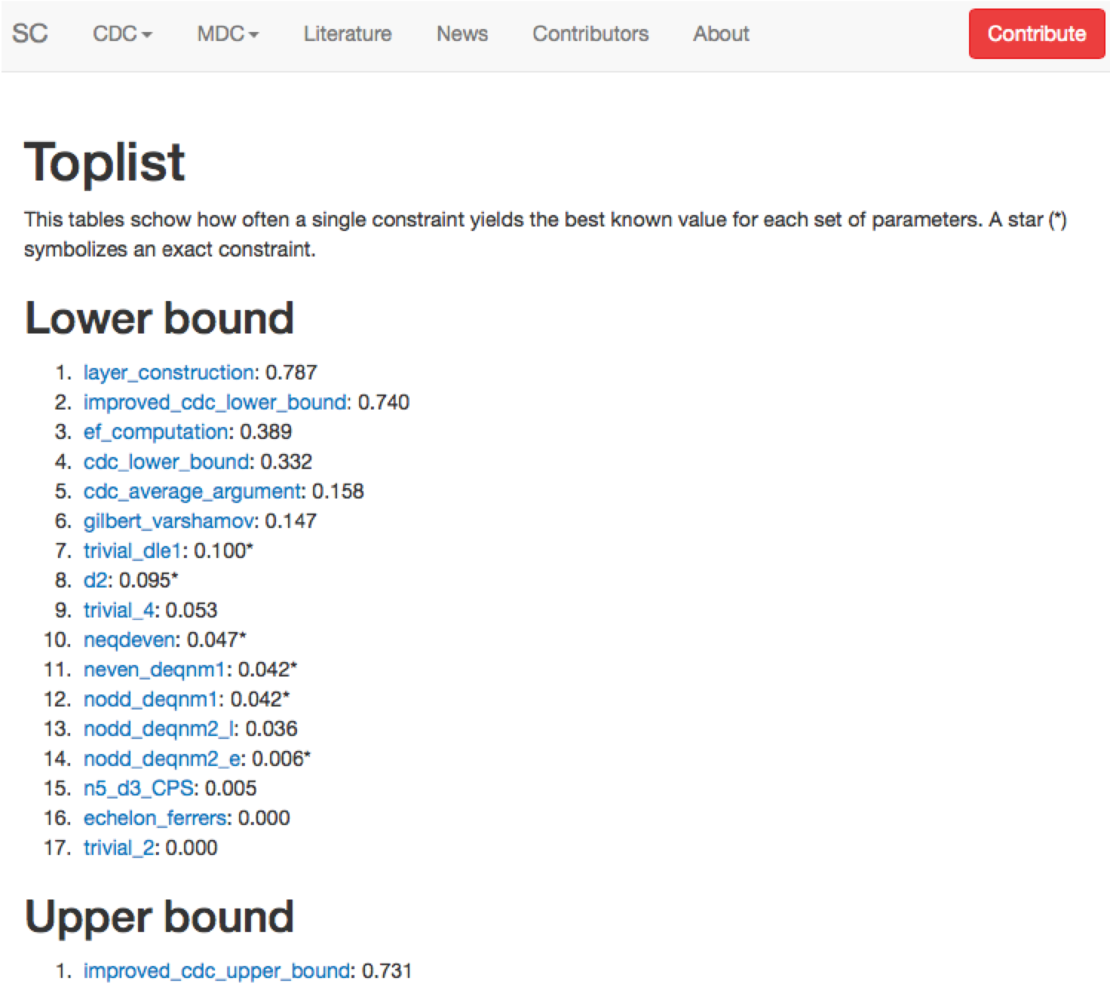
This table has the same layout as the other MDC tables. The benefit is that the optimal sizes for $A_q(n, d)$ is known for all q and $n \le 5$, as well as the number of isomorphism types in some of these cases. This table, see Figure 6, show the sizes in terms of q-polynomials that may even be evaluated for specific q values by entering them in the input box and pressing the "Compute" button or the "Enter" key. It is possible to leave the input field blank to get the q-polynomials back or to enter non prime power, even negative, numbers.

3. Bounds for CDCs

For constant dimension codes much more bounds are known than for mixed dimension codes. We state lower bounds, i.e., constructions, in Subsection 3.1 and upper bounds in Subsection 3.2.

3.1. Lower bounds and constructions for CDCs

Any subspace code is a set and hence its size is at least zero. This most trivial bound $A_q(n, d; k) \ge 0$ is `trivial_1`. Lifted MRD codes, see Subsection 3.1.1, are one type of building blocks of the Echelon-Ferrers construction, see Subsection 3.1.2. The latter is a nice interplay between the subspace distance, the rank distance and the Hamming distance. Another construction based on similar ideas is the so-called Coset construction, see Subsection 3.1.3. The most effective general recursive construction is the so-called linkage construction and its generalization, see Subsection 3.1.4. The expurgation-augmentation method, starting from a lifted MRD code and then adding and removing codewords, is


SC CDC MDC Literature News Contributors About Contribute

Toplist

This tables schow how often a single constraint yields the best known value for each set of parameters. A star (*) symbolizes an exact constraint.

Lower bound

1. layer_construction: 0.787
2. improved_cdc_lower_bound: 0.740
3. ef_computation: 0.389
4. cdc_lower_bound: 0.332
5. cdc_average_argument: 0.158
6. gilbert_varshamov: 0.147
7. trivial_dle1: 0.100*
8. d2: 0.095*
9. trivial_4: 0.053
10. neqdeven: 0.047*
11. neven_deqnm1: 0.042*
12. nodd_deqnm1: 0.042*
13. nodd_deqnm2_l: 0.036
14. nodd_deqnm2_e: 0.006*
15. n5_d3_CPS: 0.005
16. echelon_ferrers: 0.000
17. trivial_2: 0.000

Upper bound

1. improved_cdc_upper_bound: 0.731

Figure 5: Toplist of (mixed dimension) subspace codes

briefly describe in Subsection 3.1.5. Constant dimension codes with prescribed automorphisms are the topic of Subsection 3.1.6. Also the non-constructive lower bounds for classical codes in the Hamming metric can be transferred, see Subsection 3.1.7. Last but not least, also geometrical ideas can be employed in order to obtain good constructions for constant dimension codes, see Subsection 3.1.8.

3.1.1. Lifted MRD codes

For matrices $A, B \in \mathbb{F}_q^{m\times n}$ the rank distance is defined via $d_R(A,B) := \operatorname{rk}(A-B)$. It is indeed a metric, as observed in [29].

Theorem 3.1. *(see [29]) Let $m, n \ge d$ be positive integers, q a prime power, and $\mathcal{C} \subseteq \mathbb{F}_q^{m\times n}$ be a rank-metric code with minimum rank distance d. Then, $\#\mathcal{C} \le q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)}$.*

Codes attaining this upper bound are called maximum rank distance (MRD) codes. They exist for all (suitable) choices of parameters. If $m < d$ or $n < d$, then only $\#\mathcal{C} = 1$ is possible, which may be

SC CDC▾ MDC▾ Literature News Contributors About Contribute

Complete table for $A_q(n, d)$ where $n \leq 5$

q Compute

d\n	1	2	3	4	5
1	2 * (1)	$q+3$ * (1)	$2q^2+2q+4$ * (1)	$q^4+3q^3+4q^2+3q+5$ * (1)	$2q^6+2q^5+6q^4+6q^3+6q^2+4$ * (1)
2		$q+1$ * (1)	q^2+q+2 * (1)	$q^4+q^3+2q^2+q+3$ * (1)	$q^6+q^5+3q^4+3q^3+3q^2+2q$ (1)
3			2 * (2)	q^2+1	$2q^3+2$
4				q^2+1	q^3+1
5					2 * (3)

For more information and proofs we refer to HonoldKiermaierKurz20152.

Figure 6: Mixed dimension subspace codes for arbitrary q

summarized to the single upper bound $\#C \leq \lceil q^{\max\{n,m\}\cdot(\min\{n,m\}-d+1)} \rceil$. Using an $m \times m$ identity matrix as a prefix one obtains the so-called lifted MRD codes.

Theorem 3.2. *(see [77]) For positive integers k, d, n with $k \leq n$, $d \leq 2\min\{k, n-k\}$, and $d \equiv 0 \pmod 2$, the size of a lifted MRD code in $\begin{bmatrix} \mathbb{F}_q^n \\ k \end{bmatrix}$ with subspace distance d is given by*

$$M(q, k, n, d) := q^{\max\{k,n-k\}\cdot(\min\{k,n-k\}-d/2+1)}.$$

If $d > 2\min\{k, n-k\}$, then we have $M(q, k, n, d) = 1$.

As MRD codes can be obtained from linearized polynomials, we have the very same bound implemented as `lin_poly`:

Theorem 3.3. *(Linearized polynomials, see [60] and Section 3.1.1)*

$$A_q(n, d; k) \geq q^{(n-k)(k-d/2+1)}$$

3.1.2. Echelon-Ferrers or multilevel construction

In [22] a generalization, the so-called multi-level construction, based on lifted MRD codes was presented. Let $1 \leq k \leq n$ be integers and $v \in \mathbb{F}_2^n$ a binary vector of weight k. By $\mathrm{EF}_q(v)$ we denote the set of all $k \times n$ matrices over $\mathbb{F}_q$ that are in row-reduced echelon form, i.e., the Gaussian algorithm had been applied, and the pivot columns coincide with the positions where v has a 1-entry.

Theorem 3.4. *(see [22]) For integers k, n, δ with $1 \le k \le n$ and $1 \le \delta \le \min\{k, n-k\}$, let $\mathcal{B}$ be a binary constant weight code of length n, weight k, and minimum Hamming distance 2δ. For each $b \in \mathcal{B}$ let $\mathcal{C}_b$ be a code in $\operatorname{EF}_q(b)$ with minimum rank distance at least δ. Then, $\cup_{b\in\mathcal{B}} \mathcal{C}_b$ is a constant dimension code of dimension k having a subspace distance of at least 2δ.*

The code $\mathcal{B}$ is also called *skeleton code*. For $\mathcal{C}_b$ we have the following upper bound:

Theorem 3.5. *(see [22]) Let $\mathcal{F}$ be the Ferrers diagram of $\operatorname{EF}_q(v)$ and $\mathcal{C} \subseteq \operatorname{EF}_q(v)$ be a subspace code having a subspace distance of at least 2δ, then*

$$\#\mathcal{C} \le q^{\min\{\nu_i \,:\, 0 \le i \le \delta-1\}},$$

where ν_i is the number of dots in $\mathcal{F}$, which are neither contained in the first i rows nor contained in the rightmost $\delta - 1 - i$ columns.

The authors of [22] conjecture that Theorem 3.5 is tight for all parameters q, $\mathcal{F}$, and δ, which is still unrebutted. Constructions settling the conjecture in several cases are given in [21].

There is one rather obvious skeleton code that needs to be considered. Taking binary vectors with k consecutive ones we are in the classical MRD case. So, taking binary vectors v_i, where the ones are located in positions $(i-1)k+1$ to ik for all $1 \le i \le \lfloor n/k \rfloor$, clearly gives a binary constant weight code of length n, weight k, and minimum Hamming distance $2k$.

Observation 3.6. *(see e.g. [64]) For positive integers k, n with $n > 2k$ and $n \not\equiv 0 \pmod{k}$, there exists a constant dimension code in $\begin{bmatrix}\mathbb{F}_q^n\\ k\end{bmatrix}$ with subspace distance $2k$ having cardinality*

$$1 + \sum_{i=1}^{\lfloor n/k \rfloor - 1} q^{n-ik} = 1 + q^{k+(n \bmod k)} \cdot \frac{q^{n-k-(n \bmod k)} - 1}{q^k - 1} = \frac{q^n - q^{k+(n \bmod k)} + q^k - 1}{q^k - 1}.$$

The observation is implemented as `multicomponent`. We remark that a more general construction, among similar lines and including explicit formulas for the respective cardinalities, has been presented in [78]. This lower bound for partial spreads, i.e., $d = 2k$, is exactly the same as:

Theorem 3.7. *(Partial spreads, see [26]) If $d = 2k$ then:*

$$A_q(n, d; k) \ge \frac{q^n - q^k(q^{(n \bmod k)} - 1) - 1}{q^k - 1}$$

This lower bound is implemented as `partial_spread_3` and equals the size of the construction of Beutelspacher, see [8].

We remark that the general Echelon-Ferrers or multilevel construction contains the mentioned observation as a very easy special case. However, our knowledge on the size of an MRD code over $\operatorname{EF}_q(v)$ is still very limited. As mentioned, there is an explicit conjecture, which so far is neither proven nor disproved. Let the field size q, the constant dimension k, and the minimum subspace distance d be fix, in order to ease the notation. By V we denote the set of binary vectors of weight k in $\{0,1\}^n$. Let $c(v)$ denote the maximum size of a known MRD code over $\operatorname{EF}_q(v)$ matching distance d. The optimal Echelon-Ferrers construction can be modeled as an ILP:

$$\begin{aligned} \max \sum_{v \in \mathbb{F}_2^n} & c(v) \cdot x_v \\ \text{s.t.}\, & x_a + x_b \le 1 && \forall a \ne b \in \mathbb{F}_2^n : d_{\mathrm{H}}(a, b) < d \\ & x_v \in \{0, 1\} && \forall v \in \mathbb{F}_2^n. \end{aligned}$$

This is implemented as echelon_ferrers. However, the evaluation of this ILP is only feasible for rather moderate sized parameters. More sophisticated algorithmic considerations, unfortunately still unpublished, give bounds for the exact optimum of the Echelon-Ferrers construction, which is implemented as ef_computation. A greedy-type approach has been considered by Alexander Shishkin, see [74] and also [75]. It is implemented as greedy_multicomponent. (However, we have not checked that all corresponding MRD codes for the involved Ferrers diagrams exist.) In [30, 31] the authors considered block designs as skeleton codes.

By choosing some explicit skeleton code and constructing the corresponding MRD codes in $\mathrm{EF}_q(v)$, one can obtain explicit lower bounds:

Theorem 3.8 ([38, Example 59]). *For* $q > 2$*:*
$A_q(10,6,5) \ge q^{15} + q^6 + 2q^2 + q + 1$,
$A_q(11,6,5) \ge q^{18} + q^9 + q^6 + q^4 + 4q^3 + 3q^2$,
$A_q(14,6,4) \ge q^{20} + q^{14} + q^{10} + q^9 + q^8 + 2(q^6 + q^5 + q^4) + q^3 + q^2$,
$A_q(14,8,5) \ge q^{18} + q^{10} + q^3 + 1$, *and*
$A_q(15,10,6) \ge q^{18} + q^5 + 1$

This is implemented as Gorla_Ravagnani_2014.

The Echelon-Ferrers construction has even been fine-tuned to the so-called pending dots [23] implemented as pending_dots, and the so-called pending blocks [76] constructions. Of course, these variants have even more degrees of freedom, so that a general solution of the best codes within these classes of constructions is out of sight.

Explicit series of constructions using pending dots are given by:

Theorem 3.9 ([23, Construction 1, see chapter IV, Theorem 16]).

$$A_q(n, 2(k-1); k) \ge q^{2(n-k)} + A_q(n-k, 2(k-2), k-1)$$

if $q^2 + q + 1 \ge s$ *with* $s = n - 4$ *if n is odd and* $s = n - 3$ *else*

This is implemented as construction_1.

Theorem 3.10 ([23, Construction 2, see chapter IV, Theorem 17]).

$$A_q(n, 4; 3) \ge q^{2(n-3)} + \sum_{i=1}^{\alpha} q^{2(n-3-(q^2+q+2)i)}$$

if $q^2 + q + 1 < s$ *with* $s = n - 4$ *if n is odd and* $s = n - 3$ *else and* $\alpha = \left\lfloor \frac{n-3}{q^2+q+2} \right\rfloor$

This is implemented as construction_2.

Explicit series of constructions using pending blocks are given by:

Theorem 3.11 ([76, Construction A, see chapter III, Theorem 19, Corollary 20]). *Let* $n \ge \frac{k^2+3k-2}{2}$ *and* $q^2 + q + 1 \ge \ell$*, where* $\ell = n - \frac{k^2+k-6}{2}$ *for odd* $n - \frac{k^2+k-6}{2}$ *(or* $\ell = n - \frac{k^2+k-4}{2}$ *for even* $n - \frac{k^2+k-6}{2}$*). Then* $A_q(n, 2k-2; k) \ge q^{2(n-k)} + \sum_{j=3}^{k-1} q^{2(n-\sum_{i=j}^{k} i)} + \left[\begin{smallmatrix} n-\frac{k^2+k-6}{2} \\ 2 \end{smallmatrix}\right]_q$.

This is implemented as construction_ST_A_1.

Theorem 3.12 ([76, Construction B, see chapter IV, theorem 26, Corollary 27]). *Let* $n \ge 2k + 2$*. Then* $A_q(n, 4; k) \ge \sum_{i=1}^{\lfloor \frac{n-2}{k} \rfloor - 1} \left(q^{(k-1)(n-ik)} + \frac{(q^{2(k-2)}-1)(q^{2(n-ik-1)}-1)}{(q^4-1)^2} q^{(k-3)(n-ik-2)+4} \right)$.

This is implemented as construction_ST_B.

3.1.3. Coset construction

The so-called Coset construction, see [46], grounds, similar as the Echelon-Ferrers construction, on the interplay between the subspace distance, the rank distance and the Hamming distance. Another way to look at it is that it generalizes the construction from [23, Theorem 18], which yields $A_2(8,4;4) \ge 4797$. Implemented as `construction_3`, we have for general prime powers q:

Theorem 3.13 ([23, Construction 3, see chapter V, Theorem 18, Remark 6]). $A_q(8,4;4) \ge q^{12} + \begin{bmatrix}4\\2\end{bmatrix}_q (q^2+1)q^2 + 1$

Even more than the Echelon-Ferrers construction, it is a rather general approach that restricts the general optimization problem of determining the subspace codes with the maximum cardinality to the best combination of some structured building blocks. Here the building blocks are even more sophisticated than the MRD codes over $\mathrm{EF}_q(v)$, so that in general only lower and upper bounds for their sizes are known. Two explicit parameterized constructions are given by;

Theorem 3.14 ([46, Section V-A, Theorem 11]). *For all q, we have* $A_q(8,4;4) \ge q^{12} + \begin{bmatrix}4\\2\end{bmatrix}_q (q^2+1)q^2+1$.

For each $k \ge 4$ *and arbitrary q we have* $A_q(3k-3, 2k-2; k) \ge q^{4k-6} + \frac{q^{2k-3}-q}{q^{k-2}-1} - q + 1$.

This is implemented as `coset_construction`.

Theorem 3.15 ([46, Theorem 9]). *If* $\begin{bmatrix}\mathbb{F}_q^{n_i}\\k_i\end{bmatrix}$ *admit parallelisms, i.e., a partition into spreads, for* $i = 1, 2$ *then* $A_q(n_1+n_2, 4; k_1+k_2) \ge s_1 \cdot s_2 \cdot \min\{p_1, p_2\} \cdot m$*, where* $s_i = \frac{q^{n_i}-1}{q^{k_i}-1}$ *is the size of a spread and* $p_i = \frac{\begin{bmatrix}n_i\\k_i\end{bmatrix}_q}{s_i}$ *is the size of a parallelism in* $\begin{bmatrix}\mathbb{F}_q^{n_i}\\k_i\end{bmatrix}$ *for* $i = 1, 2$*, and* $m = \lceil q^{\max\{k_1, n_2-k_2\}(\min\{k_1, n_2-k_2\}-1)} \rceil$ *is the size of an MRD code with shape* $k_1 \times (n_2-k_2)$ *and rank distance 2 over* $\mathbb{F}_q$.

Unfortunately, the existence question for parallelisms in $\begin{bmatrix}\mathbb{F}_q^{n}\\k\end{bmatrix}$ is still open in general. They are known to exist for:

1. $q = 2, k = 2$ and n even;
2. $k = 2$, all q and $n = 2^m$ for $m \ge 2$;
3. $n = 4$, $k = 2$, and $q \equiv 2 \pmod 3$;
4. $q = 2, k = 3, n = 6$,

see e.g. [24]. All applicable parameter combinations for (n_1, k_1) and (n_2, k_2) are implemented as `coset_construction_parallelism_part`.

Based on a packing of a $(6, 77, 4; 3)$ code into several subcodes with minimum subspace distance 6, a construction for $A_2(10,4;4) \ge 4173$ was obtained in [46, Theorem 13]. This is still the best known code for these parameters and can be downloaded as a file. For $q \ge 3$ it remains unknown whether a similar construction can improve upon the best known construction obtained from the Echelon-Ferrers construction.

3.1.4. Linkage constructions

A powerful construction to obtain large codes from a given code $\mathcal{C}$ is to append all possible choices of an MRD code to the matrices in row-echelon form of the codewords of $\mathcal{C}$. This resulting size of the constructed code is the size of $\mathcal{C}$ times the size of the MRD code. This approach is called Construction D in [76], see [76, Theorem 37] and also [36, Theorem 5.1].

Performing a tighter analysis of the occurring subspace distances one notices that one can add further codewords from a code in a smaller ambient space to Construction D. This gives:

Theorem 3.16. *(linkage by Silberstein and (Horlemann-)Trautmann, see [76, Corollary 39]) For $3k \le n$ and $k \le \Delta \le n$ we have:*

$$A_q(n, d; k) \ge q^{\Delta(k-d/2+1)} A_q(n - \Delta, d; k) + A_q(\Delta, d; k)$$

This bound is implemented as `linkage_ST`. Without the assumption $3k \le n$, the same bound is independently obtained in:

Theorem 3.17. *(linkage by Gluesing-Luerssen and Troha [37, Theorem 2.3]) For $k \le m \le n - k$ we have:*

$$A_q(n, d; k) \ge A_q(m, d; k) \cdot \left\lceil q^{(n-m)(k-d/2+1)} \right\rceil + A_q(n - m, d; k)$$

We remark that for $n < 3k$ better constructions are known, see e.g. [45, Footnote 2]. An improved analysis of the involved distances yields:

Theorem 3.18. *(improved linkage) For $k \le m \le n - d/2$ we have:*

$$A_q(n, d; k) \ge A_q(m, d; k) \cdot \left\lceil q^{\max\{n-m,k\}(\min\{n-m,k\}-d/2+1)} \right\rceil + A_q(n - m + k - d/2, d; k)$$

This bound is implemented as `improved_linkage`. The description of the application of all three constraints contains Δ respective m in brackets.

3.1.5. The expurgation-augmentation method

The success of the Echelon–Ferrers and the coset construction is mainly given by the fact that lifted MRD codes have a quite large cardinality, which is asymptotically optimal in a certain sense. While both two methods try to append some additional subcodes, the linkage constructions employ the MRD codes in a product type construction. Another approach is to start from a lifted MRD code, remove some codewords in order to add more codewords again. This approach is coined *expurgation-augmentation* and invented by Thomas Honold.

The starting point is possible a computer–free construction for the lower bound $A_2(7, 4; 3) \ge 329$, see [40], which was previously obtained by a computer search using prescribed automorphisms, see [11]. Successors are:

Theorem 3.19 ([51, Theorem 2])**.** $A_q(6, 4; 3) \ge q^6 + 2q^2 + 2q + 1$ *for* $3 \le q$

This is implemented as `HonoldKiermaierKurz_n6_d4_k3`. Note that the right hand side in Theorem 3.19 is larger for all $q \ge 3$ than the right hand side in Theorem 3.30.

Theorem 3.20 ([49])**.** $A_q(7, 4; 3) \ge q^8 + q^5 + q^4 - q - 1$

This is implemented as `construction_honold` and superseded by `construction_HK15`.

Theorem 3.21 ([50, Theorem 4])**.** $A_2(7, 4; 3) \ge 329$*,* $A_3(7, 4; 3) \ge 6977$*,* $A_q(7, 4; 3) \ge q^8 + q^5 + q^4 + q^2 - q$

This is implemented as `construction_HK15`.

While the sketched idea of the expurgation-augmentation method is rather general, several theoretical insights are possible. Prescribing automorphisms in the constructions also helps to obtain optimization problems that are more structured and computationally feasible. A whole theoretical framework is introduced in [2]. As a purely analytical result we have:

Theorem 3.22 ([2, Main Theorem]). $A_2(v, 4; 3) \geq 2^{2(v-3)} + \frac{9}{8}\begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2$ *for* $v \equiv 7 \pmod 8$
$A_2(v, 4; 3) \geq 2^{2(v-3)} + \frac{81}{64}\begin{bmatrix} v-3 \\ 2 \end{bmatrix}_2$ *for* $v \equiv 3 \pmod 8$ *and* $v \geq 11$

This is implemented as `expurgation_augmentation_general`.
Explicit computer calculations allow further improvements:

Theorem 3.23 ([2, Table 1]). $A_2(7, 4; 3) \geq 2^8 + 45$,
$A_2(8, 4; 3) \geq 2^{10} + 93$,
$A_2(9, 4; 3) \geq 2^{12} + 756$,
$A_2(10, 4; 3) \geq 2^{14} + 2540$,
$A_2(11, 4; 3) \geq 2^{16} + 13770$,
$A_2(12, 4; 3) \geq 2^{18} + 47523$,
$A_2(13, 4; 3) \geq 2^{20} + 239382$,
$A_2(14, 4; 3) \geq 2^{22} + 775813$,
$A_2(15, 4; 3) \geq 2^{24} + 3783708$, *and*
$A_2(16, 4; 3) \geq 2^{26} + 12499466$

This is implemented as `expurgation_augmentation_special_cases`.

3.1.6. Codes with prescribed automorphisms

The computational complexity of the general optimization problem for $A_q(n, d; k)$ can be reduced if one assumes that the desired constant dimension code $\mathcal{C}$ admits some automorphisms, see [61]. So, the idea is to prescribe some subgroup G of the automorphism group. If G is cyclic, then some authors speak of cyclic orbit codes, see e.g. [12, 35, 36, 55, 79]. For these objects one can utilize the theory of subspace polynomials, see [7, 71], and Sidon spaces, see [72]. The Singer cycle is one prominent example since it acts transitively on the one-dimensional subspaces of $\mathbb{F}_q^n$. We restate the computational results from [61] for $A_2(n, 4; 3)$:

n	k	l	# orbits	# codewords	d
6	3	1	19	$1 \cdot 63 = 63$	4
7	3	2	93	$2 \cdot 127 = 254$	4
8	3	5	381	$5 \cdot 255 = 1275^*$	4
9	3	11	1542	$11 \cdot 511 = 5621^*$	4
10	3	21	6205	$21 \cdot 1023 = 21483^*$	4
11	3	39	24893	$39 \cdot 2047 = 79833^*$	4
12	3	77	99718	$77 \cdot 4095 = 315315^*$	4
13	3	141	399165	$141 \cdot 8191 = 1154931$	4
14	3	255	1597245	$255 \cdot 16383 = 4177665$	4

Here l denotes the number of chosen orbits from the total number of orbits. Those code sizes that were the best known lower bound at that time are marked with an asterisk. We remark that the stated values correspond to the optimal solutions of the corresponding ILP for $6 \leq n \leq 8$. For $n = 9$ it was reported that $l = 12$ might be possible, which would be larger than the best known code for $A_2(9, 4; 3) \geq 5986$ found in [10]. Later on the following, partially weaker, results have been obtained using the normalizer of the Singer cycle:

Theorem 3.24 ([5, Example 2.7 and 2.8]). $A_2(n, 4; 3) \geq n \cdot (2^n - 1)$ *for* $12 \leq n \leq 20$,
$A_2(8, 4; 3) \geq 2 \cdot (2^8 - 1)$,
$A_2(9, 4; 3) \geq 9 \cdot (2^9 - 1)$,
$A_2(13, 6; 4) \geq 13 \cdot (2^{13} - 1)$, *and*
$A_2(17, 6; 4) \geq 17 \cdot (2^{17} - 1)$

This is implemented as Bardestani_Iranmanesh.

A slight variation of cyclic subspace codes was considered in [33, 39].

3.1.7. Transferred *classical* non-constructive lower bounds

The classical Gilbert-Varshamov lower bound, based on sphere coverings, has been transferred to constant dimension codes:

Theorem 3.25. *(Sphere covering, see [60])*

$$A_q(n,d;k) \ge \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \left(\sum_{i=0}^{(d/2-1)+1} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q \cdot q^{i^2} \right)$$

This lower bound is implemented as sphere_covering.

A Graham-Sloane type bound was obtained in [80]:

Theorem 3.26. *(Graham, Sloane, see [80])*

$$A_q(n,d;k) \ge \frac{(q-1)\begin{bmatrix} n \\ k \end{bmatrix}_q}{(q^n-1)q^{n(d/2-2)}}$$

This lower bound is implemented as graham_sloane. For minimum subspace distance $d = 4$ is yields a strictly larger lower bound than Theorem 3.25.

3.1.8. Geometric constructions

Geometric concepts like the Segre variety and the Veronese variety where also used to obtain constructions for constant dimension codes:

Theorem 3.27 ([13, Theorem 3.11])**.** *If $n \ge 5$ is odd, then*
$A_q(2n,4;n) \ge q^{n^2-n} + \sum_{r=2}^{n-2} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)}-1) + \prod_{i=1}^{n-1}(q^i+1) - q^{\frac{n(n-1)}{2}} - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \left(q^{\frac{(n-1)(n-2)}{2}} - q^{\frac{(n-1)(n-3)}{4}} \prod_{i=1}^{\frac{n-1}{2}} (q^{2i-1}-1) \right) + y(y-1)+1$, *using* $y := q^{n-2}+q^{n-4}+\cdots+q^3+1$.

This is implemented as CossidentePavese14_theorem311.

Theorem 3.28 ([13, Theorem 3.8])**.** *If $n \ge 4$ is even, then*

$$A_q(2n,4;n) \ge q^{n^2-n} + \sum_{r=2}^{n-2} \begin{bmatrix} n \\ r \end{bmatrix}_q \sum_{j=2}^{r} (-1)^{(r-j)} \begin{bmatrix} r \\ j \end{bmatrix}_q q^{\binom{r-j}{2}} (q^{n(j-1)}-1)$$
$$+(q+1)\left(\prod_{i=1}^{n-1}(q^i+1) - 2q^{\frac{n(n-1)}{2}} + q^{\frac{n(n-2)}{4}} \prod_{i=1}^{\frac{n}{2}} (q^{2i-1}-1) \right) - q\cdot|G| + \begin{bmatrix} \frac{n}{2} \\ 1 \end{bmatrix}_{q^2} \left(\begin{bmatrix} \frac{n}{2} \\ 1 \end{bmatrix}_{q^2} - 1 \right) + 1$$

using $|G| = 2\prod_{i=1}^{n/2-1}(q^{2i}+1) - 2q^{(n(n-2)/4)}$ *if $n/2$ is odd and* $|G| = 2\prod_{i=1}^{n/2-1}(q^{2i}+1) - 2q^{(n(n-2)/4)} + q^{n(n-4)/8}\prod_{i=1}^{n/4}(q^{4i-2}-1)$ *if $n/2$ is even.*

This is implemented as CossidentePavese14_theorem38.

Theorem 3.29 ([13, Theorem 4.3]). $A_q(8,4;4) \ge q^{12} + q^2(q^2+1)^2(q^2+q+1)+1$

This is implemented as `CossidentePavese14_theorem43`.

Theorem 3.30 ([14, Corollary 7.4]). $A_q(6,4;3) \ge q^3(q^2-1)(q-1)/3 + (q^2+1)(q^2+q+1)$

This is implemented as `CossidentePavese_n6_d4_k3`.

3.2. Upper bounds for CDCs

Surveys and partial comparisons of upper bounds for constant dimension codes can e.g. be found in [4, 45, 58].

Assuming $0 \le k \le n$ we always have $A_q(n,d;k) \ge 1$. Since we can take no more than all subspaces of a given dimension, we obtain the trivial upper bound $A_q(n,d;k) \le \begin{bmatrix} n \\ k \end{bmatrix}_q$ which is implemented as `all_subs`. Transferred bounds from classical coding theory are stated in Subsection 3.2.1. Of special importance is the Johnson bound, so that implications are treated in Subsection 3.2.2. In our description of known constructions we have seen that the lifted MRD codes play a major role in many constructions. For those codes tighter upper bounds are known, see Subsection 3.2.3. As the Johnson bound recurs back to bounds for partial spreads we state the corresponding bounds in Subsection 3.2.4. Everything else that does not fit into the previous categories is collected in Subsection 3.2.5.

Nevertheless there is a large variety of upper bounds for constant dimension codes, the picture for the currently tightest known bounds is pretty clear. Besides the exact values $A_2(6,4;3) = 77$ and $A_2(8,6;4) = 257$, obtained with integer linear programming techniques, see Subsection 3.2.5, all upper bounds are given by formula (4), which refers to partial spreads. For partial spreads, Theorem 3.42 (the construction for spreads), Theorem 3.54, Theorem 3.55, and Theorem 3.58 are sufficient. The latter three results are implications of the Delsarte linear programming method for projective linear divisible codes with respect to the Hamming metric.

3.2.1. *Classical* coding theory bounds

Theorem 3.31. *(Singleton bound, see [60])*

$$A_q(n,d;k) \le \begin{bmatrix} n-d/2+1 \\ k-d/2+1 \end{bmatrix}_q$$

This upper bound is implemented as `singleton`.

Theorem 3.32. *(Sphere packing bound, see [60])*

$$A_q(n,d;k) \le \left\lfloor \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \left(\sum_{i=0}^{\lfloor (d/2-1)/2 \rfloor} \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q \cdot q^{i^2} \right) \right\rfloor$$

This upper bound is implemented as `sphere_packing`.

Theorem 3.33. *(Anticode bound, see [26])*

$$A_q(n,d;k) \le \left\lfloor \begin{bmatrix} n \\ k \end{bmatrix}_q \Big/ \begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q \right\rfloor$$

This upper bound is implemented as `anticode`.

In 1962 Johnson obtained several bounds for constant weight codes, see [57]. All of them could be transferred to constant dimension codes:

Theorem 3.34 ([81, Theorem 2]). $A_q(n,d;k) \le \left\lfloor \frac{(q^k-q^{k-d/2})(q^n-1)}{(q^k-1)^2-(q^n-1)(q^{k-d/2}-1)} \right\rfloor$ *if* $(q^k-1)^2-(q^n-1)(q^{k-d/2}-1) > 0$

This is implemented as `XiaFuJohnson1`. We remark that requested condition can be simplified considerably, see [45] and also [32].

Proposition 3.35. *For $0 \le k < v$, the bound in Theorem 3.34 is applicable iff $d = 2\min\{k, v-k\}$ and $k \ge 1$. Then, it is equivalent to*

$$A_q(v,d;k) \le \frac{q^v-1}{q^{\min\{k,v-k\}}-1}.$$

In other words, Theorem 3.34 is a, rather weak, bound for partial spreads obtained by dividing the number of points of the ambient space by the number of points of the codewords.

Theorem 3.36. *(Johnson bounds, see [26])*

$$A_q(n,d;k) \le \left\lfloor \frac{(q^n-1)\cdot A_q(n-1,d;k-1)}{q^k-1} \right\rfloor \tag{1}$$

and

$$A_q(n,d;k) \le \left\lfloor \frac{(q^n-1)\cdot A_q(n-1,d;k)}{q^{n-k}-1} \right\rfloor \tag{2}$$

These upper bounds are implemented as `johnson_1` and `johnson_2`, respectively. Note that for $d = 2k$ Inequality (1) gives $A_q(v,2k;k) \le \left\lfloor \frac{q^v-1}{q^k-1} \right\rfloor$ since we have $A_q(v-1,2k;k-1) = 1$ by definition. Similarly, for $d = 2(v-k)$, Inequality (2) gives $A_q(v,2v-2k;k) \le \left\lfloor \frac{q^v-1}{q^{v-k}-1} \right\rfloor$. Some sources like [81, Theorem 3] list just Inequality 1 and omit Inequality 2. This goes in line with the treatment of the classical Johnson type bound II for binary error-correcting codes, see e.g. [67, Theorem 4 on page 527], where the other bound is formulated as Problem (2) on page 528 with the hint that ones should be replaced by zeros. Analogously, we can consider orthogonal codes:

Proposition 3.37 ([45, Proposition 2], cf. [27, Section III, esp. Lemma 13]). *Inequality (1) and Inequality (2) are equivalent using orthogonality.*

3.2.2. Implications and generalizations of the Johnson bounds

The constraints of the binary linear program

$$\begin{aligned}
\max \sum_{U \in \left[\mathbb{F}_q^n \atop k\right]} x_U & & \\
\text{s.t.} \sum_{U \ge W} x_U &\le A_q(n-w,d;k-w) & \forall W \in \begin{bmatrix}\mathbb{F}_q^n \\ w\end{bmatrix}\ \forall w \in \{1,\ldots,k-1\} \\
\sum_{U \le A} x_U &\le A_q(a,d;k) & \forall A \in \begin{bmatrix}\mathbb{F}_q^n \\ a\end{bmatrix}\ \forall a \in \{k+1,\ldots,n-1\} \\
x_U &\in \{0,1\} & \forall U \in \begin{bmatrix}\mathbb{F}_q^n \\ k\end{bmatrix}
\end{aligned}$$

can be combined to get:

$$A_q(n,d;k) \le \frac{\begin{bmatrix} n \\ w \end{bmatrix}_q}{\begin{bmatrix} k \\ w \end{bmatrix}_q} A_q(n-w,d;k-w) \ \forall w \in \{1,\ldots,k-d/2\} \qquad \texttt{ilp\_1}$$

$$A_q(n,d;k) \le \frac{\begin{bmatrix} n \\ w \end{bmatrix}_q}{\begin{bmatrix} k \\ w \end{bmatrix}_q} \ \forall w \in \{k-d/2+1,\ldots,k-1\} \qquad \texttt{ilp\_2}$$

$$A_q(n,d;k) \le \frac{\begin{bmatrix} n \\ a \end{bmatrix}_q}{\begin{bmatrix} n-k \\ a-k \end{bmatrix}_q} \ \forall a \in \{k+1,\ldots,k+d/2-1\} \qquad \texttt{ilp\_3}$$

$$A_q(n,d;k) \le \frac{\begin{bmatrix} n \\ a \end{bmatrix}_q}{\begin{bmatrix} n-k \\ a-k \end{bmatrix}_q} A_q(a,d;k) \ \forall a \in \{k+d/2,\ldots,n-1\} \qquad \texttt{ilp\_4}$$

Note that ilp_2 is ilp_1 using $A_q(n-w,d;k-w)=1$ for $w \in \{k-d/2+1,\ldots,k-1\}$. The same is true for ilp_3 and ilp_4 using $A_q(a,d;k)=1$ for $a \in \{k+1,\ldots,k+d/2-1\}$. Also, note that ilp_1 is for $w=1$ johnson_1, ilp_2 is for $w=k-d/2+1$ anticode, and ilp_4 is for $a=n-1$ johnson_2. In general, all these upper bounds are obtained from iterative applications of the Johnson bound from Theorem 3.36. As it turns out that this bound is one of the tightest known bounds, we look at it in more detail. In the classical Johnson space the optimal combination of the corresponding two inequalities in a recursive application is unclear, see e.g. [67, Research Problem 17.1]. For constant dimension codes there is an easy criterion for the optimal choice:

Proposition 3.38 ([45, Proposition 3])**.** *For $k \le n/2$ we have*

$$\left\lfloor \frac{q^n-1}{q^k-1} A_q(n-1,d;k-1) \right\rfloor \le \left\lfloor \frac{q^n-1}{q^{n-k}-1} A_q(n-1,d;k) \right\rfloor,$$

where equality holds iff $n=2k$.

With this the following non-recursive upper bound can be obtained:

$$A_q(n,d;k) \le \left\lfloor \frac{q^n-1}{q^k-1} \cdot \left\lfloor \frac{q^{n-1}-1}{q^{k-1}-1} \cdot \left\lfloor \cdots \left\lfloor \frac{q^{n'+1}-1}{q^{d'+1}-1} \cdot A_q(n',d;d') \right\rfloor \cdots \right\rfloor \right\rfloor \right\rfloor, \qquad (3)$$

where $d'=d/2$ and $n'=n-k+d'$, i.e., the unknown value on the right hand side corresponds to the case of a partial spread. Some authors plug in Theorem 3.44 in order to obtain an explicit upper bound. However, for partial spreads tighter bounds are available, see Subsection 3.2.4.

There is a recent improvement of Inequality (3) or Theorem 3.36. The idea behind Inequality (1) is that we can recursively determine an upper bound Λ on the number of codewords that can be incident with a given point. With this the number of codewords is at most $\Lambda \begin{bmatrix} n \\ 1 \end{bmatrix}_q / \begin{bmatrix} k \\ 1 \end{bmatrix}_q$. If this number is not an integer it can be rounded down. In that case it means that some points are not incident to Λ codewords. Consider the multiset of points with multiplicity Λ minus the number of incidences with codewords. This multiset is equivalent to a linear code over $\mathbb{F}_q$, whose Hamming weights are divisible by q^{k-1}, see [59]. Actually, this is a generalization of the concept of holes and linear projective divisible codes, see Subsection 3.2.4.

Theorem 3.39 ([59, Theorem 3 and Theorem 4])**.** *Let*

$$m = \begin{bmatrix} v \\ 1 \end{bmatrix}_q \cdot A_q(n-1,d;k-1) - \begin{bmatrix} k \\ 1 \end{bmatrix}_q \cdot \left\lfloor \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q \cdot A_q(n-1,d;k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor + \begin{bmatrix} k \\ 1 \end{bmatrix}_q \cdot \delta$$

for some $\delta \in \mathbb{N}_0$. If no q^{k-1}-divisible multiset of points in $\mathbb{F}_q^v$ of cardinality m exists, then

$$A_q(n,d;k) \le \left\lfloor \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q \cdot A_q(n-1,d;k-1)}{\begin{bmatrix} k \\ 1 \end{bmatrix}_q} \right\rfloor - \delta - 1.$$

Moreover, there exists a q^r-divisible multiset of points of cardinality n if and only if there are non-negative integers $a_0,\ldots,a_r$ with $n=\sum_{i=0}^r a_i s_{i,r}^q$, where $s_{i,r}^q = q^{r-i} \cdot \frac{q^{i+1}-1}{q-1}$.

This is implemented as `improved_johnson`. The iterated version is given by:

$$A_q(n,d;k) \le \left\{ \frac{q^n-1}{q^k-1} \left\{ \frac{q^{n-1}-1}{q^{k-1}-1} \left\{ \ldots \left\{ \frac{q^{n'+1}-1}{q^{d'+1}-1} A_q(n',d;d') \right\}_{d'+1} \ldots \right\}_{k-2} \right\}_{k-1} \right\}_k , \quad (4)$$

where $d' = d/2$, $n' = v - k + d'$, and $\left\{ a / \begin{bmatrix} k \\ 1 \end{bmatrix}_q \right\}_k := b$ with maximal $b \in \mathbb{N}$ permitting a representation of $a - b \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q$ as non-negative integer combination of the summands $q^{k-1-i} \cdot \frac{q^{i+1}-1}{q-1}$ for $0 \le i \le k-1$.[3]

3.2.3. MRD bound

Since the size of the lifted MRD code, see Theorem 3.1, is quite competitive, it is interesting to compare the best known constructions with this very general explicit construction. Even more, lifted MRD codes are the basis for more involved constructions, see Subsection 3.1.2. From this point of view it is very interesting that an upper bound for the cardinality of constant dimension codes containing the lifted MRD code (of shape $k \times (n-k)$ and rank distance $d/2$) can be stated:

Theorem 3.40. *(see [20, Theorem 10 and 11]) Let $\mathcal{C}$ be a constant dimension code with given parameters q, n, d, and k that contains a lifted MRD code. Then:*

- *if $d = 2(k-1)$ and $k \ge 3$ then $|\mathcal{C}| \le q^{2(n-k)} + A_q(n-k, 2(k-2); k-1)$;*
- *if $d = 2k$ then $|\mathcal{C}| \le q^{(n-2k)(k+1)} + \begin{bmatrix} n-2k \\ k \end{bmatrix}_q \frac{q^n - q^{n-2k}}{q^{2k}-q^k} + A_q(n-2k, 2k; 2k)$.*

Theorem 3.41 ([42, Proposition 1])**.** *For $2 \le d/2 \le k \le v-k$ let C be a $(v, \#C, d; k)_q$ CDC that contains an LMRD code.*

If $k < d \le 2/3 \cdot v$ we have

$$\#C \le q^{(v-k)(k-d/2+1)} + A_q(v-k, 2(d-k); d/2).$$

If additionally $d = 2k$, $r \equiv v \mod k$, $0 \le r < k$, and $\begin{bmatrix} r \\ 1 \end{bmatrix}_q < k$, then the right hand side is equal to $A_q(v,d;k)$ and achievable in all cases.

If $(n,d,k) \in \{(6+3l, 4+2l, 3+l), (6l, 4l, 3l) \mid l \ge 1\}$, then there is a CDC containing an LMRD with these parameters whose cardinality achieves the bound.

If $k < d$ and $v < 3d/2$ we have

$$\#C \le q^{(v-k)(k-d/2+1)} + 1$$

and this cardinality is achieved.

If $d \le k < 3d/2$ we have

$$\#C \le q^{(v-k)(k-d/2+1)} + A_q(v-k, 3d-2k; d) + \begin{bmatrix} v-k \\ d/2 \end{bmatrix}_q \begin{bmatrix} k \\ d-1 \end{bmatrix}_q q^{(k-d+1)(v-k-d/2)} / \begin{bmatrix} k-d/2 \\ d/2-1 \end{bmatrix}_q .$$

---

[3] As an example we consider $A_2(9;6;4) \le \left\{ \begin{bmatrix} 9 \\ 1 \end{bmatrix}_q A_2(8,6;3) / \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \right\}_4 = \left\{ \frac{17374}{15} \right\}_4$ using $A_2(8,6;3) = 34$. We have $\left\lfloor \frac{17374}{15} \right\rfloor = 1158$, $17374 - 1158 \cdot 15 = 4$, $17374 - 1157 \cdot 15 = 19$, and $17374 - 1156 \cdot 15 = 34$. Since 4 and 19 cannot be written as a non-negative linear combination of 8, 12, 14, and 15, but $34 = 14 + 12 + 8$, we have $A_2(9;6;4) \le 1156$, which improves upon the iterative Johnson bound by two. We remark that [59] contains an easy and fast algorithm to check the presentability as non-negative integer combination as specified above.

3.2.4. Bounds for partial spreads

Partial spreads attain the maximum possible subspace distance $d = 2k$ for constant dimension codes with codewords of dimension k. So, it does not surprise that good bounds are known for this special case. In this context it makes sense to write $n = tk + r$, where $0 \le r < k$. The cases $r = 0$ and $r = 1$ are completely resolved:

Theorem 3.42. *([73]; see also [3], [17, p. 29], Result 2.1 in [8]) $\mathbb{F}_q^n$ contains a k-spread if and only if k divides n, where we assume $1 \le k \le n$ and $k, n \in \mathbb{N}$.*

The corresponding exact value is implemented as upper bound `spread`.

Theorem 3.43. *([8]; see also [48] for the special case $q = 2$) For positive integers $1 \le k \le n$ be positive integers with $n \equiv 1 \pmod{k}$ we have $A_q(n, 2k; k) = \frac{q^n-q}{q^k-1} - q + 1 = q \cdot \frac{q^{n-1}-1}{q^k-1} - q + 1 = \frac{q^n-q^{k+1}+q^k-1}{q^k-1}$.*

The corresponding exact value is implemented as upper bound `partial_spread_2`.

Since $\mathbb{F}_q^n$ contains $\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}$ points and each k-dimensional codeword contains $\begin{bmatrix} k \\ 1 \end{bmatrix}_q = \frac{q^k-1}{q-1}$ point, we have:

Theorem 3.44. $A_q(n, 2k; k) \le \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor$

This is implemented as `spread_bound` and is equivalent to Theorem 3.34. It is tight if and only if $r = 0$, where it then matches Theorem 3.42.

Theorem 3.45 ([26])**.** $d = 2k \wedge k \nmid n \Rightarrow A_q(n, d; k) \le \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor - 1$

This is implemented as `partial_spread_5`. We remark that tighter bounds are known if either $r > 1$ or $q > 2$, i.e., it is tight for $(r, q) = (1, 2)$, where it matches Theorem 3.43. Given the trivial upper bound of Theorem 3.44, one defines $A_q(n, 2k; k) = \left\lfloor \frac{q^n-1}{q^k-1} \right\rfloor - \sigma$, where σ is called the *deficiency*. In these terms, we have $\sigma = 0$ iff $r = 0$ and $\sigma = q - 1$ if $r = 1$.

For $q = 2$ and $k = 3$, then requiring $r \in \{0, 1, 2\}$, the value of $A_2(n, 6; 3)$ can be determined exactly:

Theorem 3.46. *(see [19]) For each integer $m \ge 2$ we have*

(a) $A_2(3m, 6; 3) = \frac{2^{3m}-1}{7}$;

(b) $A_2(3m + 1, 6; 3) = \frac{2^{3m+1}-9}{7}$;

(c) $A_2(3m + 2, 6; 3) = \frac{2^{3m+2}-18}{7}$.

The corresponding upper bound is implemented as `partial_spread_1`. We remark that it is sufficient to construct a code matching $A_2(8, 6; 3) = 34$, which was found by a computer search in [19], to conclude $A_2(3m+2, 6; 3) = \frac{2^{3m+2}-18}{7}$ for all $m \ge 2$, since σ is a non-increasing function in n, see [54, Lemma 4]. The other cases are special instances of $r = 0$ and $r = 1$.

For $r = 2$ there are the following results:

Theorem 3.47. *(Theorem 4.3 in [64]) For each pair of integers $t \ge 1$ and $k \ge 4$ we have $A_2(k(t + 1) + 2, 2k; k) = \frac{2^{k(t+1)+2}-3 \cdot 2^k-1}{2^k-1}$.*

The corresponding upper bound is implemented as `partial_spread_kurz_q2`.

Lemma 3.48. *(Lemma 4.6 in [64]) For integers $t \ge 1$ and $k \ge 4$ we have $A_3(k(t + 1) + 2, 2k; k) \le \frac{3^{k(t+1)+2}-3^2}{3^k-1} - \frac{3^2+1}{2}$.*

The corresponding upper bound is implemented as partial_spread_kurz_q3. We remark that the above two theorems are improvements over the following general upper bound from 1979:

Theorem 3.49. *(Corollary 8 in [18]) If $n = k(t+1) + r$ with $0 < r < k$, then*

$$A_q(n, 2k; k) \le \sum_{i=0}^{t} q^{ik+r} - \lfloor\theta\rfloor - 1 = q^r \cdot \frac{q^{k(t+1)} - 1}{q^k - 1} - \lfloor\theta\rfloor - 1,$$

where $2\theta = \sqrt{1 + 4q^k(q^k - q^r)} - (2q^k - 2q^r + 1)$.

We remark that this theorem is also restated as Theorem 13 in [20] and as Theorem 44 in [25] with the small typo of not rounding down θ (Ω in their notation). The corresponding upper bound is implemented as DrakeFreeman.

Not too long ago it was shown that the construction of Observation 3.6, i.e., the Echelon-Ferrers construction with a skeleton code of disjoint codewords, gives the optimal value if $r \ge 1$ and k is *large enough*, i.e., it is asymptotically optimal:

Theorem 3.50. *(see [70, Theorem 5]) For $r = n \pmod k$ and $k > \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ we have:*

$$A_q(n, 2k; k) = \frac{q^n - q^{k+r}}{q^k - 1} + 1$$

This is implemented as partial_spread_NS. Theorem 3.47 is just a very special case of it. If $k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q$ similar techniques allow to obtain an improved upper bound:

Theorem 3.51 ([70, Lemma 10 and Remark 11])**.** $r = n \pmod k \wedge k = \begin{bmatrix} r \\ 1 \end{bmatrix}_q < n \wedge r \ge 2 \Rightarrow A_q(n, 2k; k) \le lq^k + \min\{q, \lceil q^r/2 \rceil\}$ *where* $l = \frac{q^{n-k} - q^r}{q^k - 1}$

This is implemented as partial_spread_NS_upper_bound.

Invoking a result on the existence of so-called *vector space partitions*, see [41, Theorem 1], the authors of [70] obtained the following tightenings:

Theorem 3.52 ([69, Theorem 6])**.** $r = n \pmod t \wedge 2 \le r < t \le \begin{bmatrix} r \\ 1 \end{bmatrix}_q \Rightarrow A_q(n, 2t; t) \le \frac{q^n - q^{t+r}}{q^t - 1} + q^r - (q-1)(t-2) - c_1 + c_2$ *where* $c_1 = 2 - t \pmod q$ *and* $c_2 = \begin{cases} q & q^2 \mid (q-1)(t-2) + c_1 \\ 0 & \textit{else} \end{cases}$ *such that* $-q + 1 \le -c_1 + c_2 \le q$

This is implemented as partial_spread_NS_2_Theorem6.

Theorem 3.53 ([69, Theorem 7])**.** $r = n \pmod t \wedge 2 \le r < t \le 2^r - 1 \Rightarrow A_2(n, 2t; t) \le \frac{2^n - 2^{t+r}}{2^t - 1} + 2^r - t + 1 + c$ *where* $c = \begin{cases} 1 & 4 \mid t - 1 \\ 0 & \textit{else} \end{cases}$

This is implemented as partial_spread_NS_2_Theorem7.

The four bounds mentioned above can possibly be best explained using the concept of divisible codes, see [54]. To this end we call a point that is not contained in any k-dimensional codeword of a partial spread a *hole*. Taking the set of holes as columns of a generator matrix, we obtain a projective linear code over $\mathbb{F}_q$ of dimension k and the number of holes as length. It turns out that the Hamming weights of all codewords are divisible by q^{k-1}, see [54, Theorem 8]. Those codes have to satisfy the famous *MacWilliams Identities*, see [66],

$$\sum_{j=0}^{n-i} \binom{n-j}{i} A_j = q^{k-i} \cdot \sum_{j=0}^{i} \binom{n-j}{n-i} A_j^{\perp} \quad \text{for } 0 \le i \le n, \tag{5}$$

where A_j denotes the number of codewords of Hamming weight j and $A_j^{\perp}$ denotes the number of codewords of weight j of the dual code. We have $A_0 = A_0^{\perp} = 1$ and $A_1^{\perp} = 0$. Projectivity of the code is equivalent to $A_2^{\perp} = 0$ and the divisibility conditions says that $A_j = 0$ for all indices j that are not divisible by q^{k-1}. Moreover, the *residual codes* are q^{k-2}-divisible, which can be applied recursively. The first two MacWilliams Identities can be used to exclude the existence of quite some lengths of projective linear q^r-divisible codes. Translated back to partial spreads, this gives:

Theorem 3.54 ([62, 65]). *$r \ge 1 \wedge k \ge 2 \wedge z, u \ge 0 \wedge t = \left[{r \atop 1}\right]_q + 1 - z + u > r \Rightarrow A_q(n, 2t; t) \le lq^t + 1 + z(q-1)$ where $l = \frac{q^{n-t}-q^r}{q^t-1}$ and $n = kt + r$*

This is implemented as `partial_spread_kurz16_28` and contains Theorem 3.50 and Theorem 3.51 as a special case. If the infeasibility of the first three MacWilliams Identities is used, one obtains:

Theorem 3.55 ([65, Theorem 2.10],[54, Theorem 10]). *$r \ge 1 \wedge t \ge 2 \wedge y \ge \max\{r, 2\} \wedge z \ge 0 \wedge r, t, y, z \in \mathbb{Z} \wedge u = q^y \wedge y \le k \wedge k = \left[{r \atop 1}\right]_q + 1 - z > r \wedge v = kt + r \wedge l = \frac{q^{v-k}-q^r}{q^k-1} \Rightarrow A_q(v, 2k; k) \le lq^k + \lceil u - 1/2 - 1/2\sqrt{1 + 4u(u - (z + y - 1)(q-1) - 1)} \rceil$ Note that the description contains the value of y in brackets*

This is implemented as `partial_spread_HKK16_T10`. Setting $y = k$ in Theorem 3.55 gives Theorem 3.49, i.e., the *classical* result of Drake and Freeman. We remark that the combination of Theorem 3.54 and Theorem 3.55 is at least as tight as the combination of Theorem 3.50, Theorem 3.51, Theorem 3.52 and Theorem 3.53 and in several cases the first mentioned two theorems are strictly tighter. This statement was numerically verified for all $2 \le q \le 9$, $1 \le n, k \le 100$ in [54]. There is also a conceptual reason: The result of Heden on the existence of vector space partitions, see [41, Theorem 1], can be improved by using the implications of the first three MacWilliams Identities for divisible codes, see [54, Theorem 12], which classifies the possible length n of q^r-divisible codes for all $n \le rq^{r+1}$. For larger n some partial numerical results are obtained in [44]. A further, more direct, improvement of Heden's result can be found in [63].

Excluding codes by showing that the Equation (5) has no non-negative real solution is known as the linear programming method, which generally works for association schemes, see [15]. For Theorem 3.54 and Theorem 3.55 only a first few equations are taken into account and an analytical solution was obtained. For the first four equations of (5) the following analytical criterion was stated in [54]:

Lemma 3.56. *Let $\mathcal{C}$ be Δ-divisible over $\mathbb{F}_q$ of cardinality $n > 0$ and $t \in \mathbb{Z}$. Then $\sum_{i \ge 1} \Delta^2 (i-t)(i-t-1) \cdot (g_1 \cdot i + g_0) \cdot A_{i\Delta} + qhx = n(q-1)(n - t\Delta)(n - (t+1)\Delta) g_2$, where $g_1 = \Delta q h$, $g_0 = -n(q-1) g_2$, $g_2 = h - (2\Delta q t + \Delta q - 2nq + 2n + q - 2)$ and $h = \Delta^2 q^2 t^2 + \Delta^2 q^2 t - 2\Delta n q^2 t - \Delta n q^2 + 2\Delta n q t + n^2 q^2 + \Delta n q - 2n^2 q + n^2 + nq - n$.*

Corollary 3.57. *If there exists $t \in \mathbb{Z}$, using the notation of Lemma 3.56, with $n/\Delta \notin [t, t+1]$, $h \ge 0$, and $g_2 < 0$, then there is no Δ-divisible set over $\mathbb{F}_q$ of cardinality n.*

Numerically evaluating this criterion or numerically solving the corresponding linear programs, and taking into account that the deficiency σ is non-increasing in n, gives:

Theorem 3.58 ([62, 65]). *$A_2(4k+3, 8; 4) \le 2^4 l + 4$, where $l = \frac{2^{4k-1}-2^3}{2^4-1}$ and $k \ge 2$,*
$A_2(6k+4, 12; 6) \le 2^6 l + 8$, where $l = \frac{2^{6k-2}-2^4}{2^6-1}$ and $k \ge 2$,

$A_2(6k+5, 12; 6) \le 2^6 l + 18$, *where* $l = \frac{2^{6k-1}-2^5}{2^6-1}$ *and* $k \ge 2$,
$A_3(4k+3, 8; 4) \le 3^4 l + 14$, *where* $l = \frac{3^{4k-1}-3^3}{3^4-1}$ *and* $k \ge 2$,
$A_3(5k+3, 10; 5) \le 3^5 l + 13$, *where* $l = \frac{3^{5k-2}-3^5}{3^3-1}$ *and* $k \ge 2$,
$A_3(5k+4, 10; 5) \le 3^5 l + 44$, *where* $l = \frac{3^{5k-1}-3^4}{3^5-1}$ *and* $k \ge 2$,
$A_3(6k+4, 12; 6) \le 3^6 l + 41$, *where* $l = \frac{3^{6k-2}-3^4}{3^6-1}$ *and* $k \ge 2$,
$A_3(6k+5, 12; 6) \le 3^6 l + 133$, *where* $l = \frac{3^{6k-1}-3^5}{3^6-1}$ *and* $k \ge 2$,
$A_3(7k+4, 14; 7) \le 3^7 l + 40$, *where* $l = \frac{3^{7k-3}-3^4}{3^7-1}$ *and* $k \ge 2$,
$A_4(5k+3, 10; 5) \le 4^5 l + 32$, *where* $l = \frac{4^{5k-2}-4^3}{4^5-1}$ *and* $k \ge 2$,
$A_4(6k+3, 12; 6) \le 4^6 l + 30$, *where* $l = \frac{4^{6k-3}-4^3}{4^6-1}$ *and* $k \ge 2$,
$A_4(6k+5, 12; 6) \le 4^6 l + 548$, *where* $l = \frac{4^{6k-1}-4^5}{4^6-1}$ *and* $k \ge 2$,
$A_4(7k+4, 14; 7) \le 4^7 l + 128$, *where* $l = \frac{4^{7k-3}-4^4}{4^7-1}$ *and* $k \ge 2$,
$A_5(5k+2, 10; 5) \le 5^5 l + 7$, *where* $l = \frac{5^{5k-3}-5^2}{5^5-1}$ *and* $k \ge 2$,
$A_5(5k+4, 10; 5) \le 5^5 l + 329$, *where* $l = \frac{5^{5k-1}-5^4}{5^5-1}$ *and* $k \ge 2$,
$A_7(5k+4, 10; 5) \le 7^5 l + 1246$, *where* $l = \frac{7^{5k-1}-7^2}{7^5-1}$ *and* $k \ge 2$,
$A_8(4k+3, 8; 4) \le 8^4 l + 264$, *where* $l = \frac{8^{4k-1}-8^3}{8^4-1}$ *and* $k \ge 2$,
$A_8(5k+2, 10; 5) \le 8^5 l + 25$, *where* $l = \frac{8^{5k-3}-8^2}{8^5-1}$ *and* $k \ge 2$,
$A_8(6k+2, 12; 6) \le 8^6 l + 21$, *where* $l = \frac{8^{6k-4}-8^2}{8^6-1}$ *and* $k \ge 2$,
$A_9(3k+2, 6; 3) \le 9^3 l + 41$, *where* $l = \frac{9^{3k-1}-9^2}{9^3-1}$ *and* $k \ge 2$, *and*
$A_9(5k+3, 10; 5) \le 9^5 l + 365$, *where* $l = \frac{9^{5k-2}-9^3}{9^5-1}$ *and* $k \ge 2$

This is implemented as `partial_spread_kurz16_additional`. We remark that we are not aware of a set of parameters, where considering more than four equations from (5) yields an improvement.

3.2.5. Further upper bounds

Theorem 3.59. *(see [1, Theorem 3]) For* $0 \le t < r \le k$, $k - t \le m \le v$, *and* $t \le v - m$ *we have:*

$$A_q(n, 2r; k) \le \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q A_q(m, 2r-2t; k-t)}{\sum_{i=0}^{t} q^{i(m+i-k)} \begin{bmatrix} m \\ k-i \end{bmatrix}_q \begin{bmatrix} n-m \\ i \end{bmatrix}_q}$$

Note that the description of the application of the constraint contains t, m, and an optional o, indicating the application on the parameters of the orthogonal code, in brackets. This bound is implemented as `Ahlswede_Aydinian`. We remark that there are typos in the formulation in [1, 58]. The corrected stated version can also be found in [45]. The authors of [1] have observed that Theorem 3.59 contains Theorem 3.36, i.e., the Johnson bound, as a special case. In [45] it was numerically checked that Theorem 3.59 does not give strictly tighter bounds than Theorem 3.36 for all $2 \le q \le 9$, $4 \le v \le 100$, and $4 \le d \le 2k \le v$.

The Delsarte linear programming bound for the q-Johnson scheme, which is an association scheme, was obtained in [16]. However, numerical computations indicate that it is not better than the Anticode bound, see [4]. In [82] it was shown that the Anticode bound is implied by the Delsarte linear programming bound. In [4] it was shown that a semidefinite programming formulation, that is equivalent to the Delsarte linear programming bound, implies the Anticode bound of Theorem 3.33, the sphere-packing of Theorem 3.32, the *weak* Johnson bound of Theorem 3.34, and the Johnson bound of Theorem 3.36 (without rounding). This makes perfectly sense, since Theorem 3.32 and Theorem 3.34 are implied by Theorem 3.33 and the iteration of Theorem 3.36 without rounding gives exactly Theorem 3.33. Using `Maple` and exact arithmetic, we have checked that for all $2 \le q \le 9$, $4 \le n \le 19$, $2 \le k \le n/2$, $4 \le d \le 2k$ the optimal value of the Delsarte linear programming bound is indeed the Anticode bound. Given the result from [82] it remains to construct a feasible solution of the Delsarte linear programming formulation whose target value equals the Anticode bound. Such a feasible solution can also be constructed recursively. To this end, let $x_0, \dots, x_{k-1}$ denote a primal solution for the parameters of $A_q(n-1, d, k-1)$, then $z_0, \dots, z_k$ is a feasible solution for the parameters of $A_q(n, d, k)$ setting $z_i = x_i \cdot \begin{bmatrix} k \\ 1 \end{bmatrix}_q \begin{bmatrix} k-i \\ 1 \end{bmatrix}_q$ for all $0 \le i \le k-1$ and $z_k = \begin{bmatrix} n \\ k \end{bmatrix}_q / \begin{bmatrix} n-k+d/2-1 \\ d/2-1 \end{bmatrix}_q - z_0 - \dots - z_{k-1}$. For the mentioned parameter space this conjectured primal solution is feasible with the Anticode bound as target value. Due to the property of the symmetry group of $(\mathbb{F}_q^n, d_S)$, i.e., two-point homogeneous, the symmetry reduced version of the semidefinite programming formulation of the maximum clique problem formulation collapses the Delsarte linear programming bound for the q-Johnson scheme.

Another rather general technique to obtain upper bounds for the maximum cliques size of a graph is to p-ranks, see e.g. [56, Lemma 1.3].

Lemma 3.60. *Let G be a graph with adjacency matrix A and Y be a clique of G, then*

$$|Y| \le \begin{cases} \operatorname{rank}_p(A) + 1 & \text{if } p \text{ divides } |Y| - 1, \\ \operatorname{rank}_p(A) & \text{otherwise.} \end{cases}$$

Some numerical experiments suggest that the resulting upper bounds are rather weak, e.g., $A_2(4, 4; 2) \le 5$, $A_2(5, 4; 2) \le 19$, $A_2(6, 4; 2) \le 49$, $A_2(6, 4; 3) \le 223$, and $A_2(6, 6; 3) \le 19$.

We close this section by upper bounds obtained from tailored integer linear programming computations. The five optimal isomorphism types for $A_2(6, 4; 3) = 77$ have been determined in [51]. The upper bound $A_2(8, 6; 4) \le 272$ was obtained in [47] and is implemented as `special_case_2_8_6_4`. A little later the exact value $A_2(8, 6; 4) = 257$ and its two optimal isomorphism types were determined, see [43].

4. Bounds for MDCs

For mixed dimension subspace codes the choice between the subspace distance d_S and the injection distance d_I really makes a difference. Here we consider the subspace distance only. Subsection 4.1 is devoted to constructions and upper bounds are presented in Subsection 4.2. In general, mixed dimension subspace codes have obtained much less attention than constant dimension codes. Obtaining bounds seems to be more challenging. For surveys we refer e.g. to [4, 53, 58].

4.1. Lower bounds and constructions

Of course the empty set is a mixed dimension code for any dimension and subspace distance. $A_q(n, d) \ge 0$ is implemented as `trivial_2`. If $d \le 2n$ and $n \ge 1$, then $\{\langle 0 \rangle, \mathbb{F}_q^n\}$ is a mixed dimension codes, so that $A_q(n, d) \ge 2$. This is implemented as `trivial_4`. We structure the following lower bounds and constructions into nonrecursive, see Subsection 4.1.1, and recursive lower bounds, see Subsection 4.1.2. Some constraints leading to exact values are also collected in Subsection 4.3.

4.1.1. Nonrecursive lower bounds

The Echelon-Ferrers construction also works for mixed dimension codes, see e.g. [22, 38]. Also the stated ILP formulation directly transfers, which is implemented as `echelon_ferrers`. A more sophisticated search for the optimal construction within this setting is implemented as `ef_computation`, cf. Subsection 3.1.2.

Similar to the sphere covering bound for constant dimension codes in Theorem 3.25, there exists a version for mixed dimension codes.:

Theorem 4.1 ([26, Theorem 9])**.**

$$A_q(n,d) \geq \frac{\sum_{k=0}^{n}\sum_{j=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}n\\j\end{bmatrix}_q}{\sum_{k=0}^{n}\sum_{j=0}^{d-1}\sum_{i=0}^{j}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}k\\i\end{bmatrix}_q\begin{bmatrix}n-k\\j-i\end{bmatrix}_q q^{i(j-i)}}$$

This is implemented as `gilbert_varshamov`.

Theorem 4.2 ([53, Theorem 3.3.ii])**.** $A_q(v, v-2) \geq 2q^{k+1} + 1$ *for* $v = 2k+1 \geq 5$

This is implemented as `nodd_deqnm2_1`.

4.1.2. Recursive lower bounds

Theorem 4.3 ([22])**.** $\left\lceil \max_{k=0}^{n} \frac{q^{n+1-k}+q^k-2}{q^{n+1}-1} \cdot A_q(n+1, d+1; k) \right\rceil \leq A_q(n,d)$

This is implemented as `cdc_average_argument`.

Theorem 4.4. $\max_{k=0}^{n} A_q(n,d;k) \leq A_q(n,d)$

This is implemented as `cdc_lower_bound`.

Theorem 4.5 ([53, Lower bound of Theorem 2.5])**.** $\sum_{k=0 \wedge k \equiv \lfloor v/2 \rfloor \pmod{d}}^{v} A_q(v, 2\lceil d/2 \rceil; k) \leq A_q(v,d) \leq 2 + \sum_{k=\lceil d/2 \rceil}^{v-\lceil d/2 \rceil} A_q(v, 2\lceil d/2 \rceil; k)$

This is implemented as `improved_cdc_lower_bound`.

Theorem 4.6 ([53])**.** *The bound is* $A_q(n,d) \geq \max\{\sum_{k \in K} A_q(n,d;k) \mid K \subseteq \{0,\ldots,n\} : |k_1 - k_2| \geq d \ \forall k_1 \neq k_2 \in K\}$*. This is computed using dynamic programming and the function* $L(N) := \max\{\sum_{k \in K} A_q(n,d;k) \mid K \subseteq \{0,\ldots,N\} : |k_1 - k_2| \geq d \ \forall k_1 \neq k_2 \in K\} = \max\{L(N-1), L(N-d) + A_q(n,d;N)\}$ *for all* $N = 0,\ldots,n$*.*

This is implemented as `layer_construction`.

4.2. Implemented upper bounds

Quoting [4], bounds for mixed dimension codes are much harder to obtain than for constant dimension codes, since, for example, the size of balls in this space depends not only on their radius, but also on the dimension of their center. We structure the upper bound into nonrecursive, see Subsection 4.2.1, and recursive bound, see Subsection 4.2.2. Some constraints leading to exact values are also collected in Subsection 4.3.

4.2.1. Nonrecursive upper bounds

Theorem 4.7. *If* $d = n$ *then the whole vector space is the direct sum of each pair of codewords. If a code had three codewords, then* $2k = n$ *which is impossible for* n *odd.*

This is implemented as `nodd_deqn`.

Theorem 4.8 ([53, Upper bound of Theorem 3.3.ii])**.** $A_q(v, v-2) \leq 2q^{k+1} + 2$ *for* $v = 2k+1 \geq 5$.

This is implemented as `nodd_deqnm2_u`.

For the mixed dimension case the acting symmetry group is not 2-point homogeneous, so that the semidefinite programming formulation of the maximum clique problem after symmetrization does not collapse to a linear program. Numerical evaluations of this SLP are given by:

Theorem 4.9 ([4]). *$A_2(4,3) \le 6$, $A_2(5,3) \le 20$, $A_2(6,3) \le 124$, $A_2(7,3) \le 776$, $A_2(7,5) \le 35$, $A_2(8,3) \le 9268$, $A_2(8,5) \le 360$, $A_2(9,3) \le 107419$, $A_2(9,5) \le 2485$, $A_2(10,3) \le 2532929$, $A_2(10,5) \le 49394$, $A_2(10,7) \le 1223$, $A_2(11,5) \le 660285$, $A_2(11,7) \le 8990$, $A_2(12,7) \le 323374$, $A_2(12,9) \le 4487$, $A_2(13,7) \le 4691980$, $A_2(13,9) \le 34306$, $A_2(14,9) \le 2334086$, $A_2(14,11) \le 17159$, $A_2(15,11) \le 134095$, and $A_2(16,13) \le 67079$.*

This is implemented as `semidefinite_programming`.

Theorem 4.10 ([53]). *$A_2(6,3) \le 118$ and $A_2(7,4) \le 407$*

This is implemented as `special_cases_upper_notderived`.

Theorem 4.11. *A subspace code is a subset of the subspaces of $\mathbb{F}_q^n$, i.e., $A_q(n,d) \le \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$.*

This is implemented as `trivial_3`.

4.2.2. Recursive upper bounds

Theorem 4.12. *$A_q(n,d) \le \sum_{k=0}^{n} A_q(n,d;k)$*

This is implemented as `cdc_upper_bound`.

The following approach generalizes the sphere-packing bound for constant dimension codes facing the fact that the spheres have different sizes. To that end let $B(V,e)$ denote the ball with center V and radius e. Those balls around codewords are pairwise disjoint.

Theorem 4.13. *[26, Theorem 10] Denoting the number of k-dimensional subspaces contained in $B(V,e)$ with $\dim(V) = i$ by $c(i,k,e)$, we have*

$$c(i,k,e) = \sum_{j=\lceil \frac{i+k-e}{2} \rceil}^{\min\{k,i\}} \begin{bmatrix} i \\ j \end{bmatrix}_q \begin{bmatrix} n-i \\ k-j \end{bmatrix}_q q^{(i-j)(k-j)}.$$

Thus, $A_q(n, 2e+1)$ is at most as large as the target value of:

$$\begin{aligned} \max \sum_{i=0}^{n} a_i & & (6) \\ \text{subject to } a_i \le A_q(n, 2e+2; i) & \quad \forall 0 \le i \le n \\ \sum_{i=0}^{n} c(i,k,e) \cdot a_i \le \begin{bmatrix} n \\ k \end{bmatrix}_q & \quad \forall 0 \le k \le n \\ a_i \in \mathbb{N} & \quad \forall 0 \le i \le n \end{aligned}$$

This is implemented as `Etzion_Vardy_ilp`.

[58, Theorem 10] refers to another LP upper bound by Ahlswede and Aydinian, see [1].

Theorem 4.14 ([53, Upper bound of Theorem 2.5]). $\sum_{k=0 \wedge k \equiv \lfloor v/2 \rfloor \pmod{d}}^{v} A_q(v, 2\lceil d/2 \rceil; k) \le A_q(v,d) \le 2 + \sum_{k=\lceil d/2 \rceil}^{v-\lceil d/2 \rceil} A_q(v, 2\lceil d/2 \rceil; k)$

This is implemented as `improved_cdc_upper_bound`.

Theorem 4.15. $2 \nmid d \Rightarrow A_q(n,d) \le A_q(n,d-1)$

This is implemented as `relax_d`. This innocent and trivial looking inequality produces the tightest known upper bound in our database since e.g. Theorem 4.9 and Theorem 4.13 are not evaluated for all parameters.

4.3. Further constraints which determine an exact value

Theorem 4.16 ([53, Theorem 3.4])**.** $A_q(v,2) = \sum_{0\le i\le v \wedge i\equiv k \pmod 2} \begin{bmatrix} v \\ i \end{bmatrix}_q$

This is implemented as `d2`.

Theorem 4.17 ([53, Theorem 3.1.ii])**.** $A_q(v,v) = q^k + 1$ *for* $v = 2k$

This is implemented as `neqdeven`.

Theorem 4.18 ([53, Theorem 3.2.i])**.** $A_q(v,v-1) = q^k + 1$ *for* $v = 2k \ge 4$

This is implemented as `neven_deqnm1`.

Theorem 4.19 ([53, Theorem 3.2.ii])**.** $A_q(v,v-1) = q^{k+1} + 1$ *for* $v = 2k+1 \ge 5$

This is implemented as `nodd_deqnm1`.

Theorem 4.20 ([53, Theorem 3.3.ii]; see also [34] and footnote 44 in [53] referring to independent (still unpublished) work of Cossidente, Pavese and Storme)**.** $A_q(5,3) = 2q^3 + 2$

This is implemented as `n5_d3_CPS`.

Theorem 4.21 ([53, Theorem 3.3.ii])**.** $A_q(5,3) = 2q^3 + 2$ *for all* q *and* $A_2(7,5) = 34$

This is implemented as `nodd_deqnm2_e`. We remark that the 20 isomorphism types of all latter optimal codes have been classified in [52].

Theorem 4.22. *If the distance is 0 or 1 then the optimal subspace code consists of all subspaces of* $\mathbb{F}_q^n$*, i.e.,* $d \le 1 \Rightarrow A_q(n,d) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q$.

This is implemented as `trivial_dle1`.

5. Application programming interface

There is also an API available to access most data of the database. It is inspired by the REST (representational state transfer) style and only GET queries are supported. In order to access the data for the constant dimension case with parameters q, n, d and k, you query the URL

http://subspacecodes.uni-bayreuth.de/api/*q*/*n*/*d*/*k*/

Similarly in the mixed dimension case, the URL is

http://subspacecodes.uni-bayreuth.de/api/*q*/*n*/*d*/

The result is a JSON file which contains a subset of the following attributes:

- request = contains your specified q, n, d and k
- {lower,upper}_bound = lower or upper bound for the value $A_q(n, d; k)$
- comments = commentaries to this entry
- nondeduced = if the parameters are no parameters that are also viewable in the “short” mode, then they are trivial or computed using other parameters. nondeduced lists these other parameters.
- {lower,equal,upper}_bound_constraints = list of tuples which contain name, parameter and value of the applied constraints
- classified = boolean that is true if $A_q(n, d; k)$ is classified up to isomorphism
- known codes = list of tuples of size, details, file (to enable automatic downloads) and nrisotypes (the number of isomorphism types of this entry)
- liftedmrdsizebound = the bound for codes that contains the lifted MRD code as described in Subsection 3.2.3

In order to download the codes, you have to use the attribute file above and the URL

http://subspacecodes.uni-bayreuth.de/codes/*file*

We want to remark that the API (as well as the whole homepage) is still in an early evolutionary phase and therefore changes are likely to occur. As an example, the URL

`http://subspacecodes.uni-bayreuth.de/api/2/6/4/3/`

yields the output:

{”upper_bound_constraints”: [{”parameter”: ””, ”name”: ”all_subs”, ”value”: 1395}, {”parameter”: ””, ”name”: ”singleton”, ”value”: 155}, {”parameter”: ”2”, ”name”: ”ilp_2”, ”value”: 93}, {”parameter”: ”4”, ”name”: ”ilp_3”, ”value”: 93}, {”parameter”: ””, ”name”: ”anticode”, ”value”: 93}, {”parameter”: ””, ”name”: ”linear_programming_bound”, ”value”: 93}, {”parameter”: ””, ”name”: ”sphere_packing”, ”value”: 1395}, {”parameter”: ”1”, ”name”: ”ilp_1”, ”value”: 81}, {”parameter”: ”5”, ”name”: ”ilp_4”, ”value”: 81}, {”parameter”: ””, ”name”: ”johnson_1”, ”value”: 81}, {”parameter”: ””, ”name”: ”johnson_2”, ”value”: 81}, {”parameter”: ”0, 3”, ”name”: ”Ahlswede_Aydinian”, ”value”: 1395}, {”parameter”: ”0, 4”, ”name”: ”Ahlswede_Aydinian”, ”value”: 93}, {”parameter”: ”0, 5”, ”name”: ”Ahlswede_Aydinian”, ”value”: 81}, {”parameter”: ”1, 2”, ”name”: ”Ahlswede_Aydinian”, ”value”: 93}, {”parameter”: ”1, 3”, ”name”: ”Ahlswede_Aydinian”, ”value”: 98}, {”parameter”: ”1, 4”, ”name”: ”Ahlswede_Aydinian”, ”value”: 112}, {”parameter”: ”1, 5”, ”name”: ”Ahlswede_Aydinian”, ”value”: 155}, {”parameter”: ”0, 3, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 1395}, {”parameter”: ”0, 4, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 93}, {”parameter”: ”0, 5, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 81}, {”parameter”: ”1, 2, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 93}, {”parameter”: ”1, 3, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 98}, {”parameter”: ”1, 4, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 112}, {”parameter”: ”1, 5, o”, ”name”: ”Ahlswede_Aydinian”, ”value”: 155}, {”parameter”: ””, ”name”: ”improved_johnson”, ”value”: 81}], ”known_codes”: [{”nrisotypes”: ”5”, ”details”: ””, ”file”: ”code_2_6_4_3_optimal_size_77.zip”, ”size”: 77}], ”upper_bound”: 77, ”classified”: true, ”lower_bound”: 77, ”lower_bound_constraints”: [{”parameter”: ””, ”name”: ”trivial_1”, ”value”: 0}, {”parameter”: ””, ”name”: ”lin_poly”, ”value”: 64}, {”parameter”: ””, ”name”: ”sphere_covering”, ”value”: 15}, {”parameter”: ””, ”name”: ”graham_sloane”, ”value”: 23}, {”parameter”: ””, ”name”: ”construction_1”, ”value”: 71}, {”parameter”: ””, ”name”: ”multicomponent”, ”value”: 65}, {”parameter”: ””, ”name”: ”HonoldKiermaierKurz_n6_d4_k3”, ”value”: 77}, {”parameter”: ”[(0, 1, 2), (0, 3, 4), (1, 3, 5), (2, 4, 5)]”, ”name”: ”ef_computation”, ”value”: 71}, {”parameter”: ””, ”name”: ”CossidentePavese_n6_d4_k3”, ”value”: 43}, {”parameter”: ”3”, ”name”: ”linkage_GLT”, ”value”: 65}, {”parameter”: ”3”, ”name”: ”improved_linkage”, ”value”: 65}, {”parameter”: ”4”, ”name”: ”improved_linkage”, ”value”: 9}], ”request”: [2, 6, 4, 3], ”liftedmrdsizebound”: 71, ”comments”: ””, ”equal_bound_constraints”: []}

6. Conclusion

The collection of the known results on lower and upper bounds for subspace codes is an ongoing project. So far we have merely implemented the tip of the iceberg of the available knowledge. Even for upper bounds for constant dimension codes, which is the most advanced part of our summary, there are several pieces of work left. We still hope that the emerging on-line data base and the accompanying user's guide is already valuable for researchers in the field at this stage. One yardstick for our knowledge is the fraction between the best known lower bound and the best known upper bound for constant dimension codes. To be able to state some parametrical results, we compare the size of the lifted MRD code with the Singleton or the Anticode bound as done in [45]. To this end we utilize the so called q-Pochhammer symbol $(a;q)_n := \prod_{i=0}^{n-1}\left(1-aq^i\right)$ and its specialization $(1/q;1/q)_n = \prod_{i=1}^{n}\left(1-1/q^i\right)$.

Proposition 6.1. *[45, Proposition 7] For $k \le n-k$ the ratio of the size of an LMRD code divided by the size of the Singleton bound converges for $n \to \infty$ monotonically decreasing to $(1/q;1/q)_{k-d/2+1} \ge (1/2;1/2)_\infty > 0.288788$.*

Proposition 6.2. *[45, Proposition 8] For $k \le n-k$ the ratio of the size of an LMRD code divided by the size of the Anticode bound converges for $n \to \infty$ monotonically decreasing to $\frac{(1/q;1/q)_k}{(1/q;1/q)_{d/2-1}} \ge \frac{q}{q-1} \cdot (1/q;1/q)_k \ge 2 \cdot (1/2;1/2)_\infty > 0.577576$.*

The largest gap of this estimate is attained for $d = 4$ and $k = \lfloor n/2 \rfloor$. We remark that for this special case none of the mentioned upper bounds yields an asymptotic improvement over the Anticode bound and none of the described constructions yields an asymptotic improvement over the LMRD code construction. If k does not vary with n (or does increase very slowly), then the Anticode bound can be asymptotically be attained by an optimal code, see [28, Theorem 4.1] and also [9].

For mixed dimension codes comparatively little is known and more research is sorely needed. If you want to support us in our task, please let us know any known constructions, bounds or papers that we have missed so far via daniel.heinlein@uni-bayreuth.de or the *Contribute*-button in the upper right corner of the webpage `subspacecodes.uni-bayreuth.de`.

Tracing back results to their original source is a task on its own. We want to work on that issue more intensively in the future. If you observe possible enhancements in that direction, please let us know. Critique, suggestions for improvements and feature requests are also highly welcome.

7. Acknowledgement

The authors want to thank the contributors Alexander Shishkin, Ivan David Molina Naizir, and Francesco Pavese.

References

[1] Rudolf Ahlswede and Harout Aydinian. On error control codes for random network coding. In *Network Coding, Theory, and Applications, 2009. NetCod'09. Workshop on*, pages 68–73. IEEE, 2009.

[2] Jingmei Ai, Thomas Honold, and Haiteng Liu. The expurgation-augmentation method for constructing good plane subspace codes. *arXiv preprint 1601.01502*, 2016.

[3] Johannes André. Über nicht-desarguessche Ebenen mit transitiver Translationsgruppe. *Mathematische Zeitschrift*, 60(1):156–186, 1954.

[4] Christine Bachoc, Alberto Passuello, and Frank Vallentin. Bounds for projective codes from semidefinite programming. *Advances in Mathematics of Communications*, 7(2):127–145, 2013.

[5] F. Bardestani and Ali Iranmanesh. Cyclic orbit codes with the normalizer of a Singer subgroup. *J. Sci. Islam. Repub. Iran*, 26(1):49–55, 2015.

[6] Riccardo Bassoli, Hugo Marques, Jonathan Rodriguez, Kenneth W. Shum, and Rahim Tafazolli. Network coding theory: A survey. *Communications Surveys & Tutorials, IEEE*, 15(4):1950–1978, 2013.

[7] Eli Ben-Sasson, Tuvi Etzion, Ariel Gabizon, and Netanel Raviv. Subspace polynomials and cyclic subspace codes. *IEEE Transactions on Information Theory*, 62(3):1157–1165, 2016.

[8] Albrecht Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Mathematische Zeitschrift*, 145(3):211–229, 1975.

[9] Simon R Blackburn and Tuvi Etzion. The asymptotic behavior of grassmannian codes. *IEEE Transactions on Information Theory*, 58(10):6605–6609, 2012.

[10] Michael Braun, Patric R. J. Östergård, and Alfred Wassermann. New lower bounds for binary constant-dimension subspace codes. *Experimental Mathematics*, 0(0):1–5, 2016.

[11] Michael Braun and Jan Reichelt. q-analogs of packing designs. *Journal of Combinatorial Designs*, 22(7):306–321, 2014.

[12] Joan-Josep Climent, Verónica Requena, and Xaro Soler-Escrivà. A construction of orbit codes. In *International Castle Meeting on Coding Theory and Applications*, pages 72–83. Springer, 2017.

[13] Antonio Cossidente and Francesco Pavese. Subspace codes in $PG(2n-1,q)$. *Combinatorica*, pages 1–23, 2016.

[14] Antonio Cossidente and Francesco Pavese. Veronese subspace codes. *Des. Codes Cryptogr.*, 81(3):445–457, 2016.

[15] Philippe Delsarte. *An algebraic approach to the association schemes of coding theory*. PhD thesis, Philips Research Laboratories, 1973.

[16] Philippe Delsarte. Hahn polynomials, discrete harmonics, and t-designs. *SIAM Journal on Applied Mathematics*, 34(1):157–166, 1978.

[17] P. Dembowski. *Finite Geometries: Reprint of the 1968 Edition*. Springer Science & Business Media, 2012.

[18] David A. Drake and J.W. Freeman. Partial t-spreads and group constructible (s,r,μ)-nets. *Journal of Geometry*, 13(2):210–216, 1979.

[19] Saad El-Zanati, Heather Jordon, George F. Seelinger, Papa Sissokho, and Lawrence Spence. The maximum size of a partial 3-spread in a finite vector space over $GF(2)$. *Designs, Codes and Cryptography*, 54(2):101–107, 2010.

[20] Tuvi Etzion. Problems on q-analogs in coding theory. *arXiv preprint: 1305.6126, 37 pages*, 2013.

[21] Tuvi Etzion, Elisa Gorla, Alberto Ravagnani, and Antonia Wachter-Zeh. Optimal Ferrers diagram rank-metric codes. *IEEE Trans. Inform. Theory*, 62(4):1616–1630, 2016.

[22] Tuvi Etzion and Natalia Silberstein. Error-correcting codes in projective spaces via rank-metric codes and Ferrers diagrams. *IEEE Transactions on Information Theory*, 55(7):2909–2919, 2009.

[23] Tuvi Etzion and Natalia Silberstein. Codes and designs related to lifted MRD codes. *IEEE Trans. Inform. Theory*, 59(2):1004–1017, 2013.

[24] Tuvi Etzion and Leo Storme. Galois geometries and coding theory. *Designs, Codes and Cryptography*, pages 1–40, 2015.

[25] Tuvi Etzion and Leo Storme. Galois geometries and coding theory. *Designs, Codes and Cryptography*, 78(1):311–350, to appear.

[26] Tuvi Etzion and Alexander Vardy. Error-correcting codes in projective space. *IEEE Transactions on Information Theory*, 57(2):1165–1173, 2011.

[27] Tuvi Etzion and Alexander Vardy. Error-correcting codes in projective space. *IEEE Transactions on Information Theory*, 57(2):1165–1173, 2011.

[28] P. Frankl and V. Rödl. Near perfect coverings in graphs and hypergraphs. *European Journal of Combinatorics*, 6(4):317–326, 1985.

[29] Ernst M. Gabidulin. Theory of codes with maximum rank distance. *Problemy Peredachi Informatsii*, 21(1):3–16, 1985.

[30] Ernst M. Gabidulin and Nina I. Pilipchuk. New multicomponent network codes based on block designs. In *Proc. Int. Mathematical Conf.50 years of IITP, Moscow*, 2011.

[31] Ernst M. Gabidulin and Nina I. Pilipchuk. Rank subcodes in multicomponent network coding. *Problems of Information Transmission*, 49(1):40–53, 2013.

[32] Ernst M. Gabidulin and Nina I. Pilipchuk. New constructions of multicomponent codes. In *International conference ACCT-2016*, 2016.

[33] Ismael Gutierrez Garcia and Ivan Molina Naizir. On quasi-cyclic subspace codes. *arXiv preprint 608.03215*, 2016.

[34] Anirban Ghatak. Optimal binary $(5, 3)$ projective space codes from maximal partial spreads. *arXiv preprint 1701.07245*, 2017.

[35] Heide Gluesing-Luerssen, Katherine Morrison, and Carolyn Troha. On the cardinality and distance of cyclic orbit codes based on stabilizer subfields. In *21st International Symposium on Mathematical Theory of Networks and Systems, July 7-11, 2014. Groningen, The Netherlands*, pages 1406–1408, 2014.

[36] Heide Gluesing-Luerssen, Katherine Morrison, and Carolyn Troha. Cyclic orbit codes and stabilizer subfields. *Advances in Mathematics of Communications*, 9(2):177–197, 2015.

[37] Heide Gluesing-Luerssen and Carolyn Troha. Construction of subspace codes through linkage. *Advances in Mathematics of Communications*, 10(3):525–540, 2016.

[38] Elisa Gorla and Alberto Ravagnani. Subspace codes from Ferrers diagrams. *Journal of Algebra and Its Applications*, 16(7):23p., 2017. arXiv preprint 1405.2736.

[39] Ismael Gutierrez and Ivan Molina. Some constructions of cyclic and quasi-cyclic subspaces codes. *arXiv preprint 1504.04553*, 2015.

[40] Liu Haiteng and Thomas Honold. Poster: A new approach to the main problem of subspace coding. In *Communications and Networking in China (CHINACOM), 2014 9th International Conference on*, pages 676–677. IEEE, 2014.

[41] Olof Heden. On the length of the tail of a vector space partition. *Discrete Mathematics*, 309(21):6169–6180, 2009.

[42] Daniel Heinlein. LMRD bounds for constant dimension codes. *in preparation*, 2017.

[43] Daniel Heinlein, Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. Classifying optimal binary subspace codes of length 8, constant dimension 4 and minimum distance 6. *arXiv preprint 1711.06624*, 2017.

[44] Daniel Heinlein, Thomas Honold, Michael Kiermaier, Sascha Kurz, and Alfred Wassermann. Projective divisible binary codes. The Tenth International Workshop on Coding and Cryptography, 2017. arXiv preprint 1703.08291.

[45] Daniel Heinlein and Sascha Kurz. Asymptotic bounds for the sizes of constant dimension codes and an improved lower bound. In Ángela I. Barbero, Vitaly Skachek, and Øyvind Ytrehus, editors, *Coding Theory and Applications: 5th International Castle Meeting, ICMCTA 2017, Vihula, Estonia, August 28-31, 2017, Proceedings*, volume 10495 of *Lecture Notes in Computer Science*, pages 163–191, Cham, 2017. Springer International Publishing. arXiv preprint 1703.08712.

[46] Daniel Heinlein and Sascha Kurz. Coset construction for subspace codes. *IEEE Transactions on Information Theory*, 63(12):7651–7660, 2017. arXiv preprint 1512.07634.

[47] Daniel Heinlein and Sascha Kurz. An upper bound for binary subspace codes of length 8, constant dimension 4 and minimum distance 6. The Tenth International Workshop on Coding and Cryptography, 2017. arXiv preprint 1705.03835.

[48] Se June Hong and A.M. Patel. A general class of maximal codes for computer applications. *IEEE Transactions on Computers*, 100(12):1322–1331, 1972.

[49] Thomas Honold. Remarks on constant-dimension subspace codes, 2015. Talk at ALCOMA15 on March 16, 2015.

[50] Thomas Honold and Michael Kiermaier. On putative q-analogues of the Fano plane and related combinatorial structures. In *Dynamical systems, number theory and applications*, pages 141–175. World Sci. Publ., Hackensack, NJ, 2016.

[51] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Optimal binary subspace codes of length 6, constant dimension 3 and minimum distance 4. *Contemp. Math.*, 632:157–176, 2015.

[52] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Classification of large partial plane spreads in PG(6, 2) and related combinatorial objects. *arXiv preprint 1606.07655*, 2016.

[53] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Constructions and bounds for mixed-dimension subspace codes. *Advances in Mathematics of Communications*, 10(3):649–682, 2016.

[54] Thomas Honold, Michael Kiermaier, and Sascha Kurz. Partial spreads and vector space partitions. *arXiv preprint 1611.06328*, 2016.

[55] Anna-Lena Horlemann-Trautmann, Felice Manganiello, Michael Braun, and Joachim Rosenthal. Correction to cyclic orbit codes. *IEEE Transactions on Information Theory*, 63(11):7616–7616, 2017.

[56] Ferdinand Ihringer, Peter Sin, and Qing Xiang. New bounds for partial spreads of $h(2d-1, q^2)$ and partial ovoids of the ree-tits octagon. *arXiv preprint 1604.06172*, 2016.

[57] Selmer Johnson. A new upper bound for error-correcting codes. *IRE Transactions on Information Theory*, 8(3):203–207, 1962.

[58] Azadeh Khaleghi, Danilo Silva, and Frank R. Kschischang. Subspace codes. In *IMA International Conference on Cryptography and Coding*, pages 1–21. Springer, 2009.

[59] Michael Kiermaier and Sascha Kurz. An improvement of the johnson bound for subspace codes. *arXiv preprint 1707.00650*, 2017.

[60] Ralf Koetter and Frank R. Kschischang. Coding for errors and erasures in random network coding. *IEEE Transactions on Information Theory*, 54(8):3579–3591, 2008.

[61] Axel Kohnert and Sascha Kurz. Construction of large constant dimension codes with a prescribed minimum distance. In *Mathematical methods in computer science*, pages 31–42. Springer, 2008.

[62] Sascha Kurz. Upper bounds for partial spreads. *arXiv preprint 1606.08581*, 2016.

[63] Sascha Kurz. Heden's bound on the tail of a vector space partition. *arXiv preprint 1708.01113*, 2017.

[64] Sascha Kurz. Improved upper bounds for partial spreads. *Designs, Codes and Cryptography*, 85(1):97–106, 2017.

[65] Sascha Kurz. Packing vector spaces into vector spaces. *The Australasian Journal of Combinatorics*, 68(1):122–130, 2017.

[66] Florence J. MacWilliams. A theorem on the distribution of weights in a systematic code. *The Bell System Technical Journal*, 42(1):79–94, 1963.

[67] Florence J. MacWilliams and Neil J. A. Sloane. *The theory of error-correcting codes. II*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. North-Holland Mathematical Library, Vol. 16.

[68] Zlatka T. Mateva and Svetlana T. Topalova. Line spreads of $\mathrm{PG}(5, 2)$. *J. Combin. Des.*, 17(1):90–102, 2009.

[69] Esmeralda Năstase and Papa Sissokho. The maximum size of a partial spread II: Upper bounds. *The maximum size of a partial spread II: Upper bounds. Discrete Math.*, 340:1481–1487, 2017.

[70] Esmeralda Năstase and Papa Sissokho. The maximum size of a partial spread in a finite projective space. *J. Combin. Theory Ser. A*, 152:353–362, 2017.

[71] Kamil Otal and Ferruh Özbudak. Cyclic subspace codes via subspace polynomials. *Designs, Codes and Cryptography*, 85(2):191–204, 2017.

[72] Ron M. Roth, Netanel Raviv, and Itzhak Tamo. Construction of Sidon spaces with applications to coding. *arXiv preprint 1705.04560*, 2017.

[73] Beniamino Segre. Teoria di galois, fibrazioni proiettive e geometrie non desarguesiane. *Annali di Matematica Pura ed Applicata*, 64(1):1–76, 1964.

[74] Alexander Shishkin. A combined method of constructing multicomponent network codes. *MIPT Proceedings*, 6(2):188–194, 2014. (in Russian), https://mipt.ru//upload/medialibrary/4fe/188-194.pdf.

[75] Alexander Shishkin, Ernst M. Gabidulin, and Nina I. Pilipchuk. On cardinality of network subspace codes. In *Proceeding of the Fourteenth Int. Workshop on Algebraic and Combinatorial Coding Theory (ACCT-XIV)*, volume 7, 2014.

[76] Natalia Silberstein and Anna-Lena Trautmann. Subspace codes based on graph matchings, Ferrers diagrams, and pending blocks. *IEEE Trans. Inform. Theory*, 61(7):3937–3953, 2015.

[77] Danilo Silva, Frank R. Kschischang, and Ralf Kötter. A rank-metric approach to error control in random network coding. *IEEE Transactions on Information Theory*, 54(9):3951–3967, 2008.

[78] Vitaly Skachek. Recursive code construction for random networks. *IEEE transactions on Information Theory*, 56(3):1378–1382, 2010.

[79] Anna-Lena Trautmann, Felice Manganiello, Michael Braun, and Joachim Rosenthal. Cyclic orbit codes. *IEEE Transactions on Information Theory*, 59(11):7386–7404, 2013.

[80] Shu-Tao Xia. A Graham-Sloane type construction of constant dimension codes. In *Network Coding, Theory and Applications, 2008. NetCod 2008. Fourth Workshop on*, pages 1–5. IEEE, 2008.

[81] Shu-Tao Xia and Fang-Wei Fu. Johnson type bounds on constant dimension codes. *Designs, Codes and Cryptography*, 50(2):163–172, 2009.

[82] Zong-Ying Zhang, Yong Jiang, and Shu-Tao Xia. On the linear programming bounds for constant dimension codes. In *Network Coding (NetCod), 2011 International Symposium on*, pages 1–4. IEEE, 2011.

A. Tables for binary constant dimension codes

$n=6$	2	3
4	21 * (131044)	77 * (5)
6		9 * (1)

$n=7$	2	3
4	41	333 - 381
6		17 * (715)

$n=8$	2	3	4
4	85	1326 - 1493	4801 - 6477
6		34 (≥ 624)	257 * (2)
8			17

$n=9$	2	3	4
4	169	5986 - 6205	36945 - 50861
6		73	1033 - 1156
8			33

$n=10$	2	3	4	5
4	341	23870 - 24697	297829 - 423181	1178539 - 1678413
6		145	4173 - 4977	32890 - 38148
8			65	1025 - 1089
10				33

$n=11$	2	3	4	5
4	681	97526 - 99717	2383041 - 3370315	18728043 - 27943597
6		290	16669 - 19785	262996 - 328641
8			129 - 132	4097 - 4289
10				65

$n=12$	2	3	4	5	6
4	1365	385515 - 398385	19664917 - 27222741	299769965 - 445207739	1212491081 - 1816333805
6		585	66680 - 79170	2104384 - 2613533	16813481 - 21361665
8			273	16401 - 17436	262165 - 278785
10				129	4097 - 4225
12					65

$n=13$	2	3	4	5	6
4	2729	1597245 (≥ 512)	157319501 - 217544769	4794061075 - 7192950693	38325127529 - 57884072859
6		1169	266891 - 319449	16835124 - 20918754	269057345 - 339800773
8			545	65793 - 72131	2097225 - 2266956
10				257 - 259	16385 - 16769
12					129

B. Tables for ternary constant dimension codes

$n=6$	2	3
4	91	754 - 784
6		28 * (7)

$n=7$	2	3
4	271	6978 - 7651
6		82

$n=8$	2	3	4
4	820	60259 - 68374	543142 - 627382
6		244 - 248	6562 - 6724
8			82

$n=9$	2	3	4
4	2458	549667 - 620740	14581540 - 16821712
6		757	59077 - 61010
8			244

$n = 10$	2	3	4	5
4	7381	5086963 - 5582305	394061122 - 458168194	3554720608 - 4104497728
6		2269	532183 - 558739	14349660 - 14886440
8			730 - 732	59050 - 59536
10				244

$n = 11$	2	3	4	5
4	22141	45782686 - 50289022	10639658410 - 12361037515	286680643528 - 335382904522
6		6805 - 6809	4789648 - 5024299	387447165 - 409001563
8			2188 - 2201	531442 - 535824
10				730

C. Tables for quaternary constant dimension codes

$n = 6$	2	3
4	273	4137 - 4225
6		65

$n = 7$	2	3
4	1089	66828 - 70993
6		257

$n = 8$	2	3	4
4	4369	1054373 - 1132817	16874321 - 18245201
6		1025 - 1033	65537 - 66049
8			257

$n = 9$	2	3	4
4	17473	16945153 - 18179409	1078530305 - 1164549201
6		4161	1048641 - 1061929
8			1025

$n = 10$	2	3	4	5
4	69905	273727489 - 290821441	69038576145 - 74754799185	1105471620389 - 1193662931025
6		16641	16781333 - 17110273	1073745960 - 1088477225
8			4097 - 4102	1048577 - 1050625
10				1025

$n = 11$	2	3	4	5
4	279617	4379639873 - 4654011921	4418468947289 - 4783502911565	282679561437637 - 306494895880785
6		66561 - 66569	268501329 - 273715273	68719805936 - 70152169473
8			16385 - 16418	16777217 - 16818202
10				4097

D. Table for (unrestricted) binary subspace codes

$q = 2$	1	2	3	4	5	6	7	8	9
1	2 * (1)	5 * (1)	16 * (1)	67 * (1)	374 * (1)	2825 * (1)	29212 * (1)	417199 * (1)	8283458 * (1)
2		3 * (1)	8 * (1)	37 * (1)	187 * (1)	1521 * (1)	14606 * (1)	222379 * (1)	4141729 * (1)
3			2 * (2)	5 * (3)	18 * (24298)	108 - 118	614 - 776	5687 - 9268	71427 - 107419
4				5 * (1)	9 * (7)	77 * (4)	334 - 463	4803 - 9635	36947 - 114472
5					2 * (3)	9 * (4)	34 * (20)	263 - 327	1994 - 2460
6						9 * (1)	17 * (928)	257 - 327	1034 - 2460
7							2 * (4)	17	65 - 66
8								17 * (7)	33
9									2 * (5)

E. Table for (unrestricted) ternary subspace codes

$q=3$	1	2	3	4	5	6	7	8	9
1	2 * (1)	6 * (1)	28 * (1)	212 * (1)	2664 * (1)	56632 * (1)	2052656 * (1)	127902864 * (1)	13721229088 * (1)
2		4 * (1)	14 * (1)	132 * (1)	1332 * (1)	34608 * (1)	1026328 * (1)	77705744 * (1)	6860614544 * (1)
3			2 * (2)	10	56	764 - 968	13248 - 15846	544431 - 765772	29137055 - 34889822
4				10 * (2)	28	754 - 968	6979 - 15846	543144 - 765772	14581542 - 34889822
5					2 * (3)	28	163 - 164	6574 - 7222	117621 - 123536
6						28 * (7)	82	6562 - 7222	59078 - 123536
7							2 * (4)	82	487 - 488
8								82	244
9									2 * (5)

F. Table for (unrestricted) quaternary subspace codes

$q=4$	1	2	3	4	5	6	7	8
1	2 * (1)	7 * (1)	44 * (1)	529 * (1)	12278 * (1)	565723 * (1)	51409856 * (1)	9371059621 * (1)
2		5 * (1)	22 * (1)	359 * (1)	6139 * (1)	379535 * (1)	25704928 * (1)	6269331761 * (1)
3			2 * (2)	17	130	4154 - 4773	131318 - 144166	16881731 - 20519575
4				17 * (3)	65	4137 - 4773	66829 - 144166	16874323 - 20519575
5					2 * (3)	65	513 - 514	65557 - 68117
6						65	257	65537 - 68117
7							2 * (4)	257
8								257