Regularization in fractional order Sobolev spaces for a parameter identification problem

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Abstract

In this work we aim the identification of an unknown parameter function in the main part of an elliptic partial differential equation. It is a well known fact, that identification problems are in general ill-posed. Our idea is to apply a Tichonov-type regularization in fractional order Sobolev spaces. For such a problem, we derive existence of solutions and first order necessary conditions. Under a size condition for the regularization parameter s, corresponding to the fractional order of differentiation, we are able to derive a second order sufficient condition as well.

Fractional order Sobolev norm are challenging to implement. We therefore prove their equivalence to a multilevel based operator norm, for $s \in [0, 3/2)$, which we can implement.

This operator norm and the second order sufficient condition enable us to show superlinear convergence of an SQP-method. In the end we present a numerical example.

Zusammenfassung

In dieser Arbeit beabsichtigen wir unbekannte Parameterfunktionen im Hauptteil von elliptischen partiellen Differentialgleichungen zu identifizieren. Es ist eine allgemein bekannte Tatsache, dass solche Identifikationsprobleme im Allgemeinen schlecht gestellte Probleme darstellen. Daher ist unsere Idee, das Problem mit einem Tichonovterm in Sobolevräumen von reellwertiger Ordnung zu regularisieren. Für dieses Problem leiten wir Existenz von Lösungen und notwendige Optimalitätsbedingungen erster Ordnung her. Setzen wir eine Bedingung an den Regularisierungsparameter s, der der Ordnung des Sobolevraums entspricht, voraus, können wir auch hinreichende Optimalitätsbedingungen zweiter Ordnung herleiten.

Normen zu Sobolevräumen reellwertiger Ordnung sind sehr schwierig zu implementieren. Daher führen wir eine Multilevel basierte Operatornorm ein, deren Äquivalenz zu Sobolevnormen wir für $s \in [0, 3/2)$ beweisen. Diese sind einfacher zu implementieren.

Die Operatornorm und die hinreichende Optimalitätsbedingung sind Zutaten mit deren Hilfe wir superlineare Konvergenz eines SQP-Verfahrens zeigen. Am Ende stellen wir ein numerisches Beispiel vor.

Contents

Chapter 1

Introduction

Partial differential equations (abbr.: PDEs) describe a variety of phenomena arising in natural sciences. The first descriptions of PDEs go back to the late 17th century where Newton, Leibnitz and John and James Bernoulli were among the first to describe and study them. Since then, a well-established theory has been developed. Nevertheless, they are still an important subject in nowadays research in many mathematical fields, as for example optimal control or inverse problems and many others. In this work we are concerned with an elliptic partial differential equation. This type of PDE is usually associated with steady-state behavior because it is not time dependent. They describe for instance steady, irrotational flow, electrostatic potential without charge, equilibrium temperature distribution in a material, etc. In this work, we consider elliptic PDEs as a part of a boundary value problem with homogeneous Dirichlet boundary conditions

$$\begin{cases} Ly = g & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \end{cases}$$

with an open, bounded set $\Omega \in \mathbb{R}^N$, an unknown function $y : \bar{\Omega} \to \mathbb{R}$ and a given function $g : \Omega \to \mathbb{R}$. Our main interest lies in elliptic differential operators with divergence structure, i.e.

$$Ly = -\sum_{i,j=1}^{n} (a^{i,j}(x)y_{x_i})_{x_j} + \sum_{i=1}^{n} b^i(x)y_{x_i} + c(x)y$$

with given coefficient functions $a^{i,j}$, b^i, c $(i, j = 1, \dots n)$. In general, one examines the terms and conditions that allow us to determine the unknown y.

A comprehensive introduction of partial differential equations in general and on elliptic equations in particular can be found for instance in [25]. Within the framework of this thesis, we are particularly interested in problems of the following type

$$\begin{cases} -\nabla \cdot (a\nabla y) = g & \text{in } \Omega, \\ y = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, the lower order terms and thus the coefficients b^i and c $(i = 1, \dots n)$ do not occur. In the second order term, there only appears one parameter function a. Some physical interpretation of this kind of problem could be:
Heat equation in equilibrium:

- y: heat distribution
- a: material property (e.g. heat conductivity)
- g: heat sources or anything of that kind.

Groundwater filtration:

- y: groundwater level
- a: transmissivity of the ground
- g: sinks and sources in the domain.

The approach in this work is a different one. We are not interested in finding the unknown state y of the PDE, but on the contrary, we want to identify the material parameter a from a given state y and a fixed right-hand side g. In other words, we want to solve the so called *parameter-to-state mapping* $S(a) = y$ backwards. It is a fairly natural assumption that one normally has to deal with measurements y_d rather than the exact state y. We assume that the measured noisy data y_d fulfill the estimate

$$\|y_d - y\| \leq \delta,$$

with some noise level $\delta > 0$. We thus imply that there exists a measurement of the function y in the domain Ω. In the case of groundwater filtration for example this assumption makes sense which is unfortunately often not the case. Typically, one can only expect to obtain measurements on the boundary. The parameter-to-state operator is nonlinear and also ill-posed. Thus, we are confronted with an amount of problems. We are in particular not able to invert the parameter-to-state operator. Therefore, one has to find remedies. One first thing that is always reasonable is to integrate additional information about the problem. In our case, we assume pointwise box-constraints on the parameter function a, i.e.

$$0 < a_{\min} \leq a(x) \leq a_{\max}.$$

This is a suitable approach as material parameters are likely to be bounded. Additional information that arise from physical or technical background knowledge are called *objective apriori information* (see [33], Chapt. 2.3). Such a kind of information is useful and important, but they do not solve the problem of ill-posedness on their own. In the theory of inverse problems and especially parameter identification problems, a lot of work has been done in the last decades to find regularization methods to overcome the problem of ill-posedness, see e.g. the textbooks [7] and [24]. A very established kind of regularization is the Tichonov regularization, see for example the well known works [4], [46]. In our work we use a type of Tichonov regularization in the following way,

$$\min J(y,a) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}R(a),$$

where $y = y(a)$ is always to be understood as a solution of (1.1). The minimization of only the first term is called *output-least-squares method.*
The choice of the regularization term is very important. There are different terms imaginable, some of which have already been investigated, see for example [37], [42], or [22], for matrix-valued parameter functions, or [52], where they used an additional L^1-regularization term for the control. In this work we made the choice of the following term for $0 < s < \frac{3}{2}$,

$$R(a) = \|a\|^2_{H^s(\Omega)}.$$

There are upsides and downsides to this choice. On the one hand it has a regularization effect, such that we can prove existence of solutions to the parameter identification problem and we are also able to derive first and second order optimality conditions, although we need an additional size condition for s for the second order condition. Also, the choice seems reasonable for applications.

Imagine a workpiece consisting of at least two different kinds of material. In that case, one would expect to have a possible jump in the material parameter at the interface. Those jumps are allowed for small regularization parameters $s \in]0, \frac{1}{2}[$. On the other hand, one could see a putative downside in the numerical implementation. The norm and inner product of Sobolev-Slobodeckii spaces are defined in such a way that is makes their implementation problematic, if not impossible. Our idea to overcome that difficulty is to work with another norm concept that is equivalent to the usual Sobolev-Slobodeckii norm.

Structure of the thesis

In chapter 2 we provide the mathematical background for the treatment of partial differential equations that is needed in this work, which mainly comes from functional analysis. Later in this chapter we introduce a regularity result that will play an important role in later chapters, especially for the derivation of second order sufficient optimality conditions and for the convergence of the SQP-method. After this we give a short introduction to our parameter identification problem with some first global assumptions.

Chapter 3 is devoted to the Finite Element method . We get acquainted with the idea the decomposition of a domain, trial spaces and the Galerkin method. In the end we state very important inverse inequalities.

In chapter 4 we introduce the so called multilevel operator and show that it induces an equivalent norm on the Sobolev space $H^s(\Omega)$. With this at hand, we can replace the H^s-norm in the regularization term of the objective functional by the multilevel based norm, when needed. This is a benefit for example for the numerical treatment as the multilevel operator can be implemented without further problems in contrast to other realizations of the H^s-norm like the Sobolev-Slobodeckii norm or a fourier approach.

In chapter 5 we discuss the parameter identification problem. We show existence of solutions and derive first order necessary and second order sufficient optimality conditions. For the derivation of the second order sufficient condition we have to overcome the so called two norm discrepancy and have to require a size condition for the regularization parameter s.

In chapter 6 we establish the sequential quadratic programming method (SQP-method) for the given parameter identification problem and show local superlinear convergence.

Chapter 7 is devoted to the numerical treatment. We introduce a primal-dual active set method for the quadratic subproblem of the SQP-method. At the end, we present a numerical example.

Chapter 2

Mathematical setting

In this chapter we provide the mathematical background for the treatment of our problem. For further and more specific information on functional analysis, see for example [51] or [56]. We start with an assumption on the underlying domain Ω. Next, we introduce function spaces on this domain and state further important results and properties.

2.1 The underlying domain Ω

In the theory of partial differential equations, one requires a certain smoothness of the boundary of a domain. We state one definition here, which can be found for instance in [55], [26] and [47].

Definition 2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\Gamma = \partial\Omega$. We say the domain Ω or its boundary Γ are of class $C^{k,1}$, $k \in \mathbb{N} \cup \{0\}$, if there exist finitely many local coordinate systems $S_1, \ldots, S_M$, functions $h_1, \ldots, h_M$ and constants a, $b > 0$, such that the following properties are fulfilled:*

1. *Each function h_i, $i = 1, \ldots, M$, is k-times continuously differentiable on the $(n-1)$-dimensional closed hypercube*

$$\bar{Q}_{n-1} = \{y = (y_1, \ldots, y_{n-1}) : |y_i| \leq a, \, i = 1, \ldots, n-1\}$$

and its derivatives of order k are Lipschitz continuous.

2. *For every* $x \in \Gamma$ *there is a* $i \in \{1, \dots, M\}$ *such that* x *has the representation* $x = (y, h_i(y))$, $y \in Q_{n-1}$ *with respect to the coordinate system* S_i.

3. *For the local coordinate system* S_i *there holds*

$$(y, y_n) \in \Omega \Leftrightarrow y \in \bar{Q}_{n-1},\ h_i(y) < y_n < h_i(y) + b$$
$$(y, y_n) \notin \Omega \Leftrightarrow y \in \bar{Q}_{n-1},\ h_i(y) - b < y_n < h_i(y).$$

Domains or boundaries of class $C^{0,1}$ *are called Lipschitz domain or Lipschitz boundary.*

This is a very technical definition. Roughly speaking one can think of the boundary as locally being a Lipschitz continuous function or k-times differentiable with k-th derivative being Lipschitz continuous, respectively. And the domain is locally situated only on one side of the boundary. Domains that we are interested in throughout this work are two or three dimensional polygonal or polyhedral domains that are convex, e.g. the unit square or unit cube. In [29], one finds a proposition saying that convex and polygonal or polyhedral domains in $\mathbb{R}^N$, $N = 2, 3$ are domains with Lipschitz boundary.

Remark 2.2. *The dimension of the underlying domain will be denoted by* N, *vectors* $x \in \Omega$ *in the domain will be given as* $x = (x_1, \dots, x_N)$ *and the Euclidean norm of* x *will be referred to as* $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$.

2.2 Function spaces

Let us start this small introduction on function spaces with continuous function spaces.

Definition 2.3. 1. *Let* $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$, $N \in \mathbb{N}$, *be a* multi-index *and* $|\alpha| := \alpha_1 + \cdots + \alpha_N$, *then* $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$, *where* $D_i = \frac{\partial}{\partial x_i}$ *denotes the* partial derivative with respect to the i-th component *of* $\mathbb{R}^N$ *and* D_i^0 *denotes the* 0-th derivative, *thus the identity. The number* $|\alpha|$ *is called the* order of the derivative D^α.

2. *We assume* $\Omega \in \mathbb{R}^N$ *to be an open connected domain and* $v : \Omega \to \mathbb{R}$ *a function on* Ω. *Then, we call* $supp(v) := \overline{\{x \in \Omega : v(x) \neq 0\}}$ *the*

support *of* v. *Then, we denote by* $C_0^k(\Omega)$, $0 \leq k \leq \infty$ *the space of* k*-times differentiable functions* $v : \Omega \to \mathbb{R}$ *with compact support. Likewise, one defines* $C^k(\Omega)$ *as the space of* k*-times differentiable functions* $v : \Omega \to \mathbb{R}$. *The same can be done on the closure of the domain* $\bar{\Omega}$. *Those spaces equipped with the norms* $\|v\|_{C^k(\Omega)} = \max_{x\in\Omega} \sum_{|\alpha|\leq k} |D^\alpha v(x)|$ *are Banach spaces. We denote by* $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ *the* space of test functions, *i.e. the space of infinitely differentiable functions* v *that have compact support in* Ω.

Next, we would like to introduce Lebesgue spaces. The following definitions and properties can be found in a lot of different references. Let us name for instance [[1] chapter 2].

Definition 2.4. *Let* Ω *be a domain in* $\mathbb{R}^N$ *and let* p *be a positive real number, i.e.* $1 \leq p < \infty$. *We denote by* $L^p(\Omega)$ *the class of all measurable functions* u *defined on* Ω *for which hold*

$$\int_\Omega |u(x)|^p dx < \infty.$$

We define $L^1_{loc}(\Omega)$ *to be the space of locally integrable functions, i.e., the set of all measurable functions* $u : \Omega \to \mathbb{R}$ *such that*

$$\int_K u(x)\, dx < \infty$$

for all compact subsets $K \subset \Omega$.

The elements of those spaces are equivalence classes. This means that two functions are equivalent if they are equal a.e. in Ω. Let us introduce the L^p-norm, which is the natural norm on $L^p(\Omega)$, as

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. Let us now have a look at the limit case where $p = \infty$.

Definition 2.5. *Let* $\Omega \in R^N$. *Then, we call a function* u *that is measurable in* Ω essentially bounded *on* Ω *if there is a constant* K *such that* $|u(x)| \leq K$ *a.e. in* Ω. *We denote by* $L^\infty(\Omega)$ *the vector space of all functions* u *that are essentially bounded in* Ω.

The elements of this space are again equivalence classes, in the same sense as above. We call the greatest lower bound of those constants K in the definition the *essential supremum* of $|u|$ in Ω and denote it by $\operatorname{ess\,sup}_{x\in\Omega} |u(x)|$. Thus, the functional $\|\cdot\|_{L^\infty(\Omega)}$ with

$$\|u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{x\in\Omega} |u(x)|$$

is a natural choice as a norm on $L^\infty(\Omega)$. Let us state a useful embedding theorem for L^p-spaces, see [1].

Theorem 2.6. *Suppose that* $\mathrm{vol}(\Omega) = \int_\Omega 1\, dx < \infty$ *and* $1 \le p \le q \le \infty$. *If* $u \in L^q(\Omega)$, *then* $u \in L^p(\Omega)$ *and*

$$\|u\|_{L^p(\Omega)} \le (\mathrm{vol}(\Omega))^{(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^q(\Omega)}.$$

Hence,

$$L^q(\Omega) \hookrightarrow L^p(\Omega).$$

The L^p-spaces with respect to the associated norms are Banach spaces for $1 \le p \le \infty$, i.e. they are complete metric spaces. The space $L^2(\Omega)$ is also a Hilbert space with respect to the inner product $(u,v)_{L^2(\Omega)} = \int_\Omega u(x)v(x)dx$, where $\|u\|^2_{L^2(\Omega)} = (u,u)_{L^2(\Omega)}$. The dual space to $L^p(\Omega)$, $1 < p < \infty$, is given by $L^q(\Omega)$, where the exponents p and q are connected via the equation $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, the dual pairing is defined by

$$\langle u, v\rangle_{L^q(\Omega),L^p(\Omega)} = \int_\Omega u(x)v(x)dx.$$

Before passing on to Sobolev spaces, let us introduce the concept of weak differentiability. The following definitions can be found for instance in [1] or [55].

Definition 2.7. *Let* $y \in L^1_{loc}(\Omega)$ *and* α *be a multi-index. If there exists a locally integrable function* $w \in L^1_{loc}(\Omega)$, *such that*

$$\int_\Omega y D^\alpha v(x)dx = (-1)^{|\alpha|} \int_\Omega w(x)v(x)dx$$

for all $v \in C_0^\infty(\Omega)$, *then* w *is called* α*-th weak derivative of* y, *and we write* $w = D^\alpha y$.

Definition 2.8. *Let $1 \le p < \infty$, $k \in \mathbb{N}$. We denote by $W^{k,p}(\Omega)$ the space of functions $y \in L^p(\Omega)$, whose weak derivatives $D^\alpha y$ exist for all α with $|\alpha| \le k$ and belong to $L^p(\Omega)$ as well. Those spaces are called* Sobolev spaces.

Sobolev spaces are equipped with the following norm,

$$\|y\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \le k} \int_\Omega |D^\alpha y|^p dx \right)^{\frac{1}{p}}.$$

The space $W^{k,\infty}(\Omega)$ is similarly defined with norm

$$\|y\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha| \le k} \|D^\alpha y\|_{L^\infty(\Omega)}.$$

Again, for $p = 2$, one obtains Hilbert spaces $W^{k,2}(\Omega) =: H^k(\Omega)$. Let us write down the specific norm and inner product for the space $H^1(\Omega) = \{y \in L^2(\Omega) : D_i y \in L^2(\Omega), i = 1, \dots n\}$ as an example.

$$\|y\|_{H^1(\Omega)} = \left(\int_\Omega (y^2 + |\nabla y|^2)\, dx \right)^{\frac{1}{2}} \tag{2.1}$$
$$(u, v)_{H^1(\Omega)} = \int_\Omega u\, v\, dx + \int_\Omega \nabla u \cdot \nabla v\, dx$$

Definition 2.9. *The closure of $C_0^\infty(\Omega)$ in the Sobolev space $W^{k,p}(\Omega)$ is called $W_0^{k,p}(\Omega)$. This space is equipped with the same norm as $W^{k,p}(\Omega)$ and is a closed subspace of $W^{k,p}(\Omega)$. In particular, $W_0^{k,2}(\Omega) =: H_0^k(\Omega)$.*

On the space $H_0^1(\Omega)$ one can introduce another norm by

$$|y|^2_{H_0^1(\Omega)} := \int_\Omega |\nabla y|^2 dx, \tag{2.2}$$

that is equivalent to the norm on $H^1(\Omega)$, (2.1). This is only true in the space $H_0^1(\Omega)$. In $H^1(\Omega)$ the norm (2.2) is only a seminorm. For $1 \le p < \infty$ one introduces the associated seminorms as follows,

$$|y|^p_{W^{k,p}(\Omega)} = \sum_{|\alpha| = k} \|D^\alpha y\|^p_{L^p(\Omega)}.$$

In this work we are particularly interested in an extended understanding of Sobolev spaces for noninteger s. There are several possible ways of defining fractional order Sobolev spaces, see [[1], 7.57-7.64]. Let us name for example the spaces of Bessel potential, that use Fourier transforms of the functions multiplied with a factor containing the order of smoothness. Another way to define fractional Sobolev spaces yields the so called Sobolev-Slobodeckii spaces. The following definition of the associated norms can be found for instance in [53]. This is a rather general definition as it includes not only Hilbert spaces, but is stated for $1 \leq p < \infty$.

Definition 2.10. *Let $1 \leq p < \infty$, $s \in \mathbb{R}$, $s > 0$, $s = k + \kappa$ and $k \in \mathbb{N}_0$, $\kappa \in (0,1)$. We denote by $W^{s,p}(\Omega)$ the space of functions $u \in W^{k,p}(\Omega)$ that fulfill*

$$|u|^p_{W^{s,p}(\Omega)} := \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p\kappa}} dx\, dy < \infty. \tag{2.3}$$

Then,

$$\|u\|_{W^{s,p}(\Omega)} := \left\{ \|u\|^p_{W^{k,p}(\Omega)} + |u|^p_{W^{s,p}(\Omega)} \right\}^{\frac{1}{p}} \tag{2.4}$$

is the Sobolev-Slobodeckii norm *and $|u|_{W^{s,p}(\Omega)}$ the associated seminorm.*

For $p = 2$ the spaces $W^{s,2}(\Omega)$ are Hilbert spaces with the inner product

$$(u,v)_{W^{k,2}(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha u(x) D^\alpha v(x)\, dx$$

for $s = k \in \mathbb{N}_0$ and

$$\begin{aligned}(u,v)_{W^{s,2}(\Omega)} &= (u,v)_{W^{k,2}(\Omega)} \\ &\quad + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha u(x) - D^\alpha u(y))(D^\alpha v(x) - D^\alpha v(y))}{|x-y|^{n+2\kappa}} dx\, dy\end{aligned}$$

for $s = k + \kappa$ and $k \in \mathbb{N}_0$, $\kappa \in (0,1)$. Let us now introduce the dual space of $W^{k,p}_0(\Omega)$.

Definition 2.11. *The dual space of $W^{k,p}_0(\Omega)$ with $k > 0$, where $p \in (1,\infty)$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ is given as*

$$W^{-k,q}(\Omega) = \left(W^{k,p}_0(\Omega) \right)^*$$

with the norm

$$\|u\|_{W^{-k,q}(\Omega)} := \sup_{v \in W^{k,p}_0(\Omega), v\neq 0} \frac{|\int_\Omega u(x)v(x)dx|}{\|v\|_{W^{k,p}(\Omega)}}.$$

These spaces are Banach spaces as well.

2.3 Sobolev's embedding theorem

It is essential when working with Sobolev spaces to consider inclusion relations between Sobolev spaces in order to arrive at some sort of ordering among them. For Sobolev spaces of integer order one derives directly from the definition the following inclusions, see [15].

Proposition 2.12. *Suppose that Ω is any domain, k and m are nonnegative integers satisfying $k \leq m$, and p is any real number satisfying $1 \leq p \leq \infty$. Then $W^{m,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$.*

Proposition 2.13. *Suppose that Ω is a bounded domain, k is a nonnegative integer, and p and q are real numbers satisfying $1 \leq p \leq q \leq \infty$. Then $W^{k,q}(\Omega) \hookrightarrow W^{k,p}(\Omega)$.*

There exist more general inclusions or embeddings that are not evident and they can even be shown for fractional spaces. The main embedding theorem is the following. In this fractional form it can be found for instance in [29] but without proof. A version for Sobolev spaces with integer order can be found in many references about sobolev spaces, for instance [1], [25], [15] etc.

Theorem 2.14. *Let $\Omega \subset \mathbb{R}^N$ be bounded and open with Lipschitz boundary $\partial\Omega$, $p > 1$. Then,*

$$W^{s,p}(\Omega) \hookrightarrow W^{t,q}(\Omega) \tag{2.5}$$

for $0 < t, s \in \mathbb{R}$, $p \leq q$ *such that* $s - N/p = t - N/q$ *and*

$$W^{s,p}(\Omega) \hookrightarrow C^{k,\alpha}(\Omega)$$

for $k < s - N/p < k + 1$, $\alpha = s - k - N/p$, k a non-negative integer.

A proof of the first inclusion can be found e.g. in [9] and of the second inclusion e.g. in [54].

2.4 $W^{1,p}$-regularity of the solution of the PDE

It is well known from the theory of elliptic boundary value problems that the Lax-Milgram lemma yields the existence and some regularity of the weak solution of an elliptic problem. In our case this means that for every $g \in L^2(\Omega)$ and $a \in L^\infty(\Omega)$ that fulfills the pointwise constraints $0 < a_{\min} \leq a(x) \leq a_{\max}$, where $a_{\min}$ and $a_{\max}$ are positive constants, there exists a unique solution y of the weak formulation

$$\int_\Omega a\nabla y \cdot \nabla v \, dx = \int_\Omega g\, v\, dx \qquad \forall v \in H_0^1(\Omega).$$

Furthermore we obtain H^1-regularity of y, namely

$$\|y\|_{H^1(\Omega)} \leq c\|G\|_{H^1(\Omega)^*}, \tag{2.6}$$

with $G(v) := (g,v)_{L^2(\Omega)}$. In our special case we even obtain uniform boundedness of y in $H^1(\Omega)$ for a fixed function $g \in L^2(\Omega)$, because the constant c does not depend on the parameter function a. Unfortunately, this regularity is not sufficient for our purposes. But one can find a remedy in the work of Meyers [[45], theorem 2] and for a more general case in the work of Gröger [30]. Both articles apply to problems on domains with Lipschitz boundary. Meyers considers elliptic problems of divergence structure with homogeneous Dirichlet boundary condition. He derives the existence of some constant $\bar{q} > 2$ such that given the existence of a solution $y \in W^{1,p_1}(\Omega)$ for a $p_1 \in]\bar{q}', \bar{q}[$ with $\frac{1}{\bar{q}} + \frac{1}{\bar{q}'} = 1$, then $y \in W^{1,p}(\Omega)$ for all $\bar{q}' < p < \bar{q}$. In two or three dimensions a right-hand side $g \in L^2(\Omega)$ is sufficiently smooth. The occurring constant $\bar{q}$ depends on the dimension of the domain Ω, on the smoothness of its boundary and on the pointwise bounds of the parameter function a, more precisely on their ratio, i.e.

$$\frac{a_{\min}}{a_{\max}} \to 0 \;\Rightarrow\; \bar{q} \to 2,$$
$$\frac{a_{\min}}{a_{\max}} \to 1 \;\Rightarrow\; \bar{q} \to \infty.$$

In the paper of Gröger this result is generalized to the case of mixed boundary conditions and to more general operators. With these results we obtain higher regularity of the solution y, namely the uniform boundedness of y in $W^{1,p}(\Omega)$ for $p \in (2, \bar{q}]$ and $\bar{q} > 2$.

Lemma 2.15. *Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary. Then, there exists a constant $\bar{q} = \bar{q}(\Omega, a_{\min}, a_{\max}) > 2$, such that*

$$\|y\|_{W_0^{1,p}(\Omega)} \leq c_p\|G\|_{W^{-1,p}(\Omega)}, \tag{2.7}$$

for $p \in [2, \bar{q}]$ with $c_p = c_p(\Omega, a_{\min}, a_{\max}, p)$. G only depends on the right-hand side of the PDE g.

Remark 2.16. *In the upper lemma we obtain uniform boundedness of y for the $W^{1,p}$ seminorm. Together with the estimate (2.6), we can derive uniform boundedness also in the full $W^{1,p}$ norm.*

2.5 Introduction to the problem

Now, let us introduce the minimization problem that we will discuss in this thesis. We consider a quadratic cost functional with a Tichonov type regularization term subject to a nonlinear elliptic partial differential equation.

$$\left.\begin{array}{lcl} \text{minimize} & J(y,a) := \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)} & \\ \text{subject to} & -\nabla \cdot (a\nabla y) = g & \text{in } \Omega \\ & y = 0 & \text{on } \Gamma \\ & 0 < a_{\min} \leq a(x) \leq a_{\max} & \text{a.e. in } \Omega \\ & y \in H^1_0(\Omega),\ a \in H^s(\Omega) & s > 0 \end{array}\right\} \quad (2.8)$$

We want to identify the parameter function a in the main part of the elliptic PDE. Let us shortly note that the terminology in optimal control theory, which is rather connected to this type of problems, too, is slightly different. The PDE is called state equation, the parameter a control. As it acts on the whole domain, we say that it is a distributed control. The function y is referred to as state in optimal control and data function in the parameter identification. Additionally to the PDE, we require some pointwise control constraints or pointwise boundaries for the unknown parameter function. Let us now clear some requirements for this problem.

- The underlying domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$ is required to be a Lipschitz domain, with Lipschitz boundary $\Gamma = \partial\Omega$. This strong requirement is made for simplicity and could be weakened for some of the results.

- The desired state or measured noisy data y_d and the right-hand side of the PDE g are both functions in $L^\infty(\Omega)$.

- The pointwise bounds are positive real numbers satisfying $a_{\min} < a_{\max}$.

- The regularization parameters α and s are both positive real numbers.

In view of the numerical treatment of the problem, let us already point out, that we will use a different norm concept for the fractional Sobolev space $H^s(\Omega)$, an operator based norm subject to a multilevel finite element approach. The usual Sobolev-Slobodeckii norm (2.4) is not useful for implementation and it also appeared to be reasonable to use this equivalent norm concept in the convergence proof of the Lagrange-Newton SQP-method.

Chapter 3

Finite Element Method

In this chapter we want to introduce the Finite Element Method (abbr.: FEM). This will be the basis for the following chapter about an equivalent norm for the Sobolev Space $H^s(\Omega)$.
As already said in the first chapter, partial differential equations describe physical or technical processes. Unfortunately, only in rare cases PDEs can be solved analytically. In the other cases one has to rely on numerical methods. Instead of the continuous problem one considers and solves a finite dimensional approximation of the problem. This is the idea of discretization methods. One of the most popular methods for solving PDEs is the finite element method.
In the literature one finds numerous textbooks about the finite element method, not to mention the number of articles. We basically focused on the books [36], [15] and [21].

3.1 Decomposition and basis functions

The basic idea of the FEM is to decompose the underlying domain and to define polynomial basis functions on the resulting grid. Let us be a bit more precise about this. We define decompositions $\{\mathcal{T}_h\}_{h>0}$ on the domain $\bar{\Omega}$ and restrict our considerations on domains $\Omega \in \mathbb{R}^2$ with a polygonal boundary Γ. Therefore, it is consistent to focus on decompositions with triangular elements T that have the following properties, see [12].

Definition 3.1. *1. A decomposition $\mathcal{T} = \{T_1, T_2, \ldots, T_N\}$ of Ω with triangular elements is called admissible if there holds*

- $\bar{\Omega} = \cup_{T \in \mathcal{T}} \bar{T}$, *i.e. the triangles* T *cover the domain* $\bar{\Omega}$ *exactly.*
- *The triangles have pairwise disjoint interiors.*
- *No vertex of any triangle lies in the interior of an edge of another triangle.*

2. *We write* $\mathcal{T}_h$ *instead of* $\mathcal{T}$*, if the diameter of every element is at most* $2h$*.*
3. *A decomposition* $\mathcal{T}_h$ *is called quasi-uniform, if there exists a* $\kappa > 0$*, such that every element* $T \in \mathcal{T}_h$ *contains a circle of radius* ρ_T *with*

$$\rho_T \geq h_T/\kappa,$$

where $2h_T$ *is the diameter of* T*.*

4. *We define the mesh size of the triangulation* T_h *as*

$$h := \max_{T \in \mathcal{T}_h} h_T.$$

5. *A decomposition* $\mathcal{T}_h$ *is called uniform, if there exists a* $\kappa > 0$*, such that every element* $T \in \mathcal{T}_h$ *contains a circle of radius* ρ_T *with*

$$\rho_T \geq h/\kappa.$$

We assume furthermore that the mesh size converges to zero as the number of elements of the triangulation tends to infinity. In the literature a uniform triangulation $\mathcal{T}_h$ is also called *regular* or *isotropic*. By a finite element decomposition of the underlying domain one obtains a certain number of elements. This number is to rise rapidly depending on the fineness of the discretization. For a reasonable treatment of all elements one introduces a so called reference element and its projection on the respective element. Such an approach is appropriate for affine families of finite element spaces, see [12], which includes our setting. In the case of triangular elements it is useful to set

$$\widehat{T} = \{\hat{x} = (\hat{x}_1, \hat{x}_2) : 0 \leq \hat{x}_1, \hat{x}_2 \leq 1, \hat{x}_1 + \hat{x}_2 \leq 1\}.$$

The transformation on an element T is then given as

$$x = x_T(\hat{x}) = B_T \hat{x} + x_1.$$

Consequently, the transformation of an element T onto $\widehat{T}$ is

$$\hat{x} = \hat{x}_T(x) = (B_T)^{-1}(x - x_1).$$

The transformation matrix is given as

$$B_T = \begin{pmatrix} x_{2,1} - x_{1,1} & x_{3,1} - x_{1,1} \\ x_{2,2} - x_{1,2} & x_{3,2} - x_{1,2} \end{pmatrix}.$$

Furthermore, by the volume meas(T) of an element T we understand

$$\text{meas}(T) = \int_T dx = \int_{\hat{T}} |\det B_T| d\hat{x} = |\det B_T| \int_0^1 \int_0^{1-\xi} d\xi_2 d\xi_1 = \frac{1}{2}|\det B_T|. \tag{3.1}$$

On such a grid one defines a finite dimensional trial space. In our case the space of piecewise linear and continuous functions $S_h^1(T)$ is the space of choice. Every basis function φ_j is equal to one at exactly one node of the grid. It decreases linearly to zero on neighboring elements and is equal to zero elsewhere, i.e. $\varphi_j(x_i) = \delta_{ij}$, $\forall i, j = 1 \dots n_h$, where n_h is the number of nodes of the triangulation. These functions have only a local support, which leads to sparse matrices in the numerical realization. On each element T there exist only three basis functions that are nonzero and we obtain the transformation

$$\varphi_\alpha(x) = p_\alpha(\hat{x}_T(x)) \quad \forall x \in T,$$

where $\alpha = 1, 2, 3$ is a local numbering of the nodes in T, and p_α, $\alpha = 1, 2, 3$ are the basis functions on the reference element. For those basis functions there exists the explicit representation

$$p_1(\hat{x}) = 1 - \hat{x}_1 - \hat{x}_2, \ \ p_2(\hat{x}) = \hat{x}_1, \ \ p_3(\hat{x}) = \hat{x}_2.$$

For the derivative there holds $D_x = (B_T)^{-\top} D_{\hat{x}}$ on element T. The trial space is then given as

$$V_h = \{v_h : v_h(x) = \sum_{j=1}^{n_h} v_j \varphi_j(x)\} = \text{span}\{\varphi_j : j = 1, \dots n_h\}. \tag{3.2}$$

With the transformation of T onto $\hat{T}$ we can write for functions $v \in H^m(T)$, with an integer $m \geq 0$,

$$v(x) = v(B_T\hat{x} + x_1) = \hat{v}(\hat{x})$$

and we obtain the existence of a constant $c = c(N, m)$ such that

$$|\hat{v}|_{H^m(\hat{T})} \leq c \cdot \|B_T\|^m \cdot |\det B_T|^{-1/2} |v|_{H^m(T)}, \tag{3.3}$$

analogously one obtains

$$|v|_{H^m(T)} \leq c \cdot \|B_T^{-1}\|^m \cdot |\det B_T|^{1/2} |\hat{v}|_{H^m(\hat{T})}. \tag{3.4}$$

By applying the chain rule we obtain for the gradient

$$D_{\hat{x}} \hat{v}(\hat{x}) = B_T^\top D_x v(x)$$

and also

$$D_x v(x) = B_T^{-\top} D_{\hat{x}} \hat{v}(\hat{x}).$$

3.2 Galerkin method

The starting point is a variational form

$$a(u, v) = g(v) \qquad \forall v \in V, \tag{3.5}$$

with a bilinear form $a(\cdot, \cdot)$ and a linear form g, where we want to find the solution u in a function space V. Furthermore, we claim the assumptions of the Lax-Milgram lemma to hold, i.e. there exist real nonnegative constants α_a and β_a, such that

$$|a(u, v)| \leq \alpha_a \|u\|_V \cdot \|v\|_V, \tag{3.6}$$

$$a(u, u) \geq \beta_a \|u\|_V^2. \tag{3.7}$$

Now, we choose finite dimensional subspaces $V_h \subset V$, e.g. the trial spaces (3.2) above, with fineness $h > 0$. The Galerkin method determines approximations $u_h \in V_h$ of the continuous solution $u \in V$ as

$$a(u_h, v_h) = g(v_h) \qquad \forall v_h \in V_h. \tag{3.8}$$

Because of the V-ellipticity of the bilinearform a we obtain via Lax-Milgram's lemma the existence of a solution $u_h \in V_h$ and Cea's lemma yields a quasi best approximation property.

Lemma 3.2. *[Cea] Assume that $a : V \times V \to \mathbb{R}$ is a bounded bilinear form which is V-elliptic. Given some $g \in V$, let u and u_h be the solutions of* (3.5) *and* (3.8)*, respectively. Then,*

$$\|u - u_h\|_V \leq \frac{\alpha_a}{\beta_a} \inf_{v_h \in V_h} \|u - v_h\|_V .$$

In the case of symmetric bilinear forms a, the element u_h is the orthogonal projection of u in V_h and also the best approximation in V_h. For the finite dimensional space V_h with $\dim(V_h) = n_h$ the functions $\{\varphi_1, \dots, \varphi_{n_h}\} \subset V_h$, as introduced in the last section, form a basis. Then, for any function $v_h \in V_h$ there exists a unique representation $v_h(x) = \sum_{i=1}^{N} v_i \varphi_i(x)$. If we insert these expansions into the variational form (3.8), we obtain a matrix $\mathbb{A} \in \mathbb{R}^{n_h \times n_h}$ and $\mathbf{G} \in \mathbb{R}^{n_h}$ representing the bilinear form a and g. Thus, we have a linear system,

$$\mathbb{A}\mathbf{u} = \mathbf{G}, \tag{3.9}$$

with $\mathbf{u} = [\mathbf{u}_1, \dots, \mathbf{u}_{n_h}]$. Since the bilinear form a is symmetric and V-elliptic, $\mathbb{A}$ is symmetric and positive definite. Hence, (3.9) is uniquely solvable.

3.3 Trial spaces

Let us be a bit more specific about the assumptions on the trial spaces and the underlying decompositions, particularly in view of the multilevel approach. Therefore, let us require for a family of triangulations $\{\mathcal{T}_{h_j}\}_{j \in \mathbb{N}_0}$ on the domain Ω to be nested. This is obtained by taking a uniform coarse grid $\mathcal{T}_{h_0}$ and applying a globally uniform refinement strategy. The resulting sequence $\{\mathcal{T}_{h_j}\}_{j \in \mathbb{N}_0}$ then fulfills

$$c_1 2^{-j} \leq h_j \leq c_2 2^{-j} \tag{3.10}$$

for all $j = 0, 1, 2, \dots$, where h_j is the global mesh size of the triangulation $\mathcal{T}_{h_j}$ and some constants c_1 and c_2. For the associated trial spaces of piecewise linear functions $V_j = S^1_{h_j}(\Omega)$ there holds

$$V_0 \subset V_1 \subset \cdots \subset V_j = S^1_{h_j}(\Omega) \subset V_{j+1} \subset \cdots \subset H^s(\Omega), \quad \text{for } s \in [0, 3/2). \tag{3.11}$$

For the last inclusion see for example [[53], p.221, p.307]. For one dimension we show as an example that trial spaces of piecewise linear continuous functions are contained in Sobolev spaces $H^s(\Omega)$ for $s \in [0, 3/2)$. Therefore, we have a look at the Heaviside function. Let $\Theta : \mathbb{R} \to \mathbb{R}$ be defined as

$$\Theta(x) := \begin{cases} 0, & x < 0 \\ 1, & x \geq 0. \end{cases}$$

We show that the Sobolev-Slobodeckii norm for $s \in [0, 1/2)$ exists for Θ with a value smaller than infinity. Then, Θ is an element of fractional Sobolev spaces $H^s(\Omega)$ for $0 \leq s < 1/2$. Let us recall the Sobolev-Slobodeckii norm for $s > 0$, with $s = k + \kappa$, $k \in \mathbb{N}_o$, $\kappa \in (0, 1)$:

$$\|u\|_{W_p^s(\Omega)} := \{\|u\|^p_{W_p^k(\Omega)} + |u|^p_{W_p^s(\Omega)}\}^{1/p}$$

with the associated seminorm

$$|u|^p_{W_p^s(\Omega)} = \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{d+2\kappa}} \, dx \, dy.$$

In our case we have $d = 1$ and the domain $\Omega = [-1, 1]$, $p = 2$, $s < 1$, thus $k = 0$ and $\kappa = s$. Hence, we find by inserting the definition of the Heaviside function

$$\begin{aligned}
|\Theta|^2_{H^s([-1,1])} &= \int_{-1}^{1} \int_{-1}^{1} \frac{|\Theta(x) - \Theta(y)|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&= \int_{-1}^{0} \int_{-1}^{1} \frac{|\Theta(x)|^2}{|x-y|^{1+2s}} \, dx \, dy + \int_{0}^{1} \int_{-1}^{1} \frac{|\Theta(x) - 1|^2}{|x-y|^{1+2s}} \, dx \, dy \\
&= \int_{-1}^{0} \int_{-1}^{0} 0 \, dx \, dy + \int_{-1}^{0} \int_{0}^{1} \frac{1}{|x-y|^{1+2s}} \, dx \, dy \\
&+ \int_{0}^{1} \int_{-1}^{0} \frac{1}{|x-y|^{1+2s}} \, dx \, dy + \int_{0}^{1} \int_{0}^{1} 0 \, dx \, dy \\
&= \int_{-1}^{0} \int_{0}^{1} \frac{1}{(x-y)^{1+2s}} \, dx \, dy + \int_{0}^{1} \int_{-1}^{0} \frac{1}{(y-x)^{1+2s}} \, dx \, dy.
\end{aligned}$$

Thus, there are two terms left for integration,

$$
\begin{aligned}
|\Theta|^2_{H^s([-1,1])} &= \int_{-1}^{0} \left[\frac{1}{-2s}(x-y)^{-2s}\right]_0^1 dy + \int_0^1 \left[\frac{1}{2s}(y-x)^{-2s}\right]_{-1}^0 dy \\
&= \int_{-1}^{0} \frac{1}{-2s}(1-y)^{-2s} - \frac{1}{-2s}(-y)^{-2s} dy + \int_0^1 \frac{1}{2s}y^{-2s} - \frac{1}{2s}(y+1)^{-2s} dy \\
&= \left[\frac{1}{2s}\cdot\frac{1}{-2s+1}(1-y)^{-2s+1} - \frac{1}{2s}\cdot\frac{1}{-2s+1}(-y)^{-2s+1}\right]_{-1}^0 \\
&+ \left[\frac{1}{2s}\cdot\frac{1}{-2s+1}y^{-2s+1} - \frac{1}{2s}\cdot\frac{1}{-2s+1}(y+1)^{-2s+1}\right]_0^1 .
\end{aligned}
$$

This norm exists and is nonnegative and bounded for $s \in (0, 1/2)$. The denominators $-2s+1$ and $2s$ are nonzero for $s \neq 1/2$ and $s \neq 0$.

$$
\begin{aligned}
|\Theta|^2_{H^s([-1,1])} &= \frac{1}{2s}\cdot\frac{1}{-2s+1}\left(1 - 2^{-2s+1} + 1 + 1 - 2^{-2s+1} + 1\right) \\
&= \frac{2 - 2^{-2s+1}}{-2s^2 + s}
\end{aligned}
$$

Thus, the Heaviside function is an element of $H^s(\Omega)$ for $s < 1/2$. This implies that piecewise linear continuous functions are elements of $H^s(\Omega)$ for $s < 3/2$. Their weak derivatives are piecewise constant functions with jumps that can be treated likewise the Heaviside function, thus, they have bounded norms in $H^s(\Omega)$ for $s < 1/2$. Note that we showed this only for the one dimensional case.

3.4 Inverse inequalities

An important tool when working in the finite element setting are inverse inequalities. With this kind of inequalities we compare different norms on a finite element space. A stronger norm, with respect to the order of differentiability, of a finite element function can be estimated by a weaker one. Therefore, we multiply the weaker norm with a factor depending on the mesh size to the power of the difference in differentiability. For a triangulation $\mathcal{T}_h$ of the domain Ω, the following theorem holds.

Theorem 3.3. *Let $\mathcal{T}_h$ be a quasi uniform triangulation of the domain Ω and let V_j be a piecewise linear and continuous trial space on the domain. Let*

$s \in [0, 3/2)$ and $t \in [0, s]$. Then, there exists a constant $c = c(s,t)$ such that for every $v \in V_j$

$$\|v\|_{H^s(\Omega)} \leq c \cdot h_j^{-t} \|v\|_{H^{s-t}(\Omega)}. \tag{3.12}$$

In particular, we obtain for $t = s$

$$\|v\|_{H^s(\Omega)} \leq c \cdot h_j^{-s} \|v\|_{L^2(\Omega)}. \tag{3.13}$$

For example in [15], inverse inequalities are treated and proved in a more general way in chapter 4.5. In particular, Theorem 3.3 is a consequence of [[15], Theorem 4.5.11] and [[15], Remark 4.5.20].

Chapter 4

Multiscale approach

For the regularization of the identification problem we choose fractional Sobolev spaces $H^s(\Omega)$, where the parameter s can be chosen freely between 0 and $\frac{3}{2}$. In chapter 2 we already introduced several ways of defining fractional Sobolev spaces, none of which are very practicable for numerical treatment. Hence, with this choice of the regularization term we are facing the challenge of finding a way to avoid the usual norms of $H^s(\Omega)$. In the literature we can find an interesting approach for this topic. It is presented for instance in [14], [53] and [57]. The authors introduce a so called multilevel operator that is essentially a weighted linear combination of L^2-projection operators onto trial spaces with different levels of refinement. In their work, they furthermore show equivalence of a multilevel operator based scalar product and the H^s-norm for $s = 1$. The same result is evident for $s = 0$ and thus by interpolation one obtains that is also holds for $s \in [0, 1]$. The authors state that the proof can be generalized to the case where $s \in (1, \frac{3}{2})$. This is what we are showing in the following section. For this purpose, we follow the lead of [53] and introduce step by step the ingredients needed for the multilevel approach.

4.1 Projection operators

There are several imaginable ways of projecting elements of function spaces onto finite-dimensional discretization spaces. At the beginning of this chapter, we introduce one kind of projections and get acquainted with some of their properties, especially their approximation properties, i.e. error estimates, and furthermore their stability for different norms. Let us start with the L^2-projection, that maps L^2 functions into the trialspace $V_j = S^1_{h_j}(\Omega)$.

Definition 4.1. *The operator $Q_j : L^2(\Omega) \to V_j$ is called L^2-projection operator and for any $v \in L^2(\Omega)$, $Q_j v$ is given as the unique solution of the variational problem*

$$\langle v, v_j \rangle_{L^2(\Omega)} = \langle Q_j v, v_j \rangle_{L^2(\Omega)}, \qquad \forall v_j \in V_j. \tag{4.1}$$

In the literature, e.g. in [53] one finds this definition for the L^2-projection operator and also for the H^1-projection operator. We are interested in a more general definition of projection operators that includes the H^1-projection operator if we set $s = 1$. Thus, we define the following.

Definition 4.2. *Let $s \in (0, 3/2)$. The operator $Q^s_j : H^s(\Omega) \to V_j$ is called H^s-projection operator and for any $v \in H^s(\Omega)$, $Q^s_j v$ is given as the unique solution of the variational problem*

$$\langle v, v_j \rangle_{H^s(\Omega)} = \langle Q^s_j v, v_j \rangle_{H^s(\Omega)}, \qquad \forall v_j \in V_j. \tag{4.2}$$

We can certainly extend the last definition to $s = 0$ and accordingly also include the L^2-projection operator into the same definition. But we chose to define them separately. Nevertheless, we show some properties of the projection operators for $s \in [0, 3/2)$ all at once. Let us set $Q_j = 0$ for $j = -1$.

Lemma 4.3. *The H^s-projection operator $Q^s_j : H^s(\Omega) \to V_j$ is self-adjoint for $0 \le s < 3/2$.*

Proof. Let $u, v \in H^s(\Omega)$. Hence, $Q^s_j u, Q^s_j v \in V_j$. Then, by definition we easily see that

$$\langle Q^s_j v, u \rangle_{H^s(\Omega)} = \langle Q^s_j v, Q^s_j u \rangle_{H^s(\Omega)} = \langle v, Q^s_j u \rangle_{H^s(\Omega)} \tag{4.3}$$

holds. □

The next properties also hold true for all parameters $s \in [0, \frac{3}{2})$.

Lemma 4.4. *Let $s \in [0, 3/2)$. Then, a sequence $\{Q_j\}_{j\in\mathbb{N}_0}$ of projection operators Q_j^s fulfills the following properties.*

- $Q_k^s Q_j^s = Q_{\min\{k,j\}}^s$,
- $(Q_k^s - Q_{k-1}^s)(Q_j^s - Q_{j-1}^s) = 0$, *for* $k \neq j$,
- $(Q_j^s - Q_{j-1}^s)^2 = Q_j^s - Q_{j-1}^s$

The proof of these properties is straight forward, see [53], for $s = 0$.

4.2 Multilevel operator

Definition 4.5. *For $s \in [0, 3/2)$ let us define the multilevel operator*

$$B^s := \sum_{k=0}^{\infty} h_k^{-2s}(Q_k - Q_{k-1}) \tag{4.4}$$

as a weighted linear combination of L^2-projection operators.

The multilevel operator (4.4) induces an equivalent norm on the Sobolev space $H^s(\Omega)$.

Theorem 4.6. *The multilevel operator B^s satisfies the spectral equivalence inequalities*

$$c_1\|v\|_{H^s(\Omega)}^2 \leq \langle B^s v, v\rangle_{L^2(\Omega)} \leq c_2\|v\|_{H^s(\Omega)}^2 \tag{4.5}$$

for $s \in [0, 3/2)$ and for $v \in H^s(\Omega)$.

This property has already been stated in [53] or [57]. Unfortunately, the proof has been carried out solely for the case $s = 1$. In particular for $s \in (1, 3/2)$ a generalization of this proof requires some effort. The proof of this theorem for $s \in [0, 3/2)$ is the main goal of this section. Let us first have a look at several other results.

4.3 Interpolation error estimates

For $s \in [0, \frac{3}{2})$, the projection operators fulfill various error estimates. The definition of the projection operators implies the so called Galerkin orthogonality for $v \in H^s(\Omega)$

$$\langle v - Q_j^s v, v_j \rangle_{H^s(\Omega)} = 0 \qquad \forall v_j \in V_j. \tag{4.6}$$

This directly leads to

$$\begin{aligned}
\|v - Q_j^s v\|^2_{H^s(\Omega)} &= \langle v - Q_j^s v, v - Q_j^s v \rangle_{H^s(\Omega)} \\
&= \langle v - Q_j^s v, v \rangle_{H^s(\Omega)} - \langle v - Q_j^s v, Q_j^s v \rangle_{H^s(\Omega)} \\
&= \langle v - Q_j^s v, v \rangle_{H^s(\Omega)} \leq \|v - Q_j^s v\|_{H^s(\Omega)} \|v\|_{H^s(\Omega)}.
\end{aligned}$$

This is the first interpolation error estimate we want to mention.

Lemma 4.7. *Let $s \in [0, \frac{3}{2})$. Then, the following interpolation error estimate holds true,*

$$\|v - Q_j^s v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\Omega)} \qquad \forall v \in H^s(\Omega). \tag{4.7}$$

The error estimate (4.7) stays true for the L^2-projection as well, i.e.

$$\|v - Q_j v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} \qquad \forall v \in L^2(\Omega).$$

Furthermore, we find the following estimate for the L^2-projection in the literature (e.g. in [12] [21], [53], etc.)

$$\|v - Q_j v\|_{L^2(\Omega)} \leq c h_j^2 \|v\|_{H^2(\Omega)} \qquad \forall v \in H^2(\Omega).$$

By an interpolation argument we obtain

$$\|v - Q_j v\|_{L^2(\Omega)} \leq c h_j^s \|v\|_{H^s(\Omega)} \qquad \forall v \in H^s(\Omega). \tag{4.8}$$

With this estimate, we are now able to proof another error estimate for the L^2-projection.

Lemma 4.8. *Let $s \in [0, 2]$, $t \in [0, s]$. Furthermore, let $v \in V_j$. Then, there holds*

$$\|v - Q_j v\|_{H^{s-t}(\Omega)} \leq c h_j^t \|v\|_{H^s(\Omega)} \qquad \forall v \in V_j. \tag{4.9}$$

Proof. With the inverse inequality (3.13) and (4.8) we obtain

$$\begin{aligned}\|v - Q_j v\|_{H^{s-t}(\Omega)} &\leq h_j^{-s+t} \|v - Q_j v\|_{L^2(\Omega)} \\ &\leq h_j^{-s+t} h_j^s \|v\|_{H^s(\Omega)} = h_j^t \|v\|_{H^s(\Omega)}.\end{aligned}$$

□

Let us put on hold the line of error estimates for an instant. It is well-known that orthogonal projections are best-approximation with respect to the particular norm. For the H^s-projection this means

$$\|v - Q_j^s v\|_{H^s(\Omega)} \leq \inf_{p \in S^1_{h_j}(\Omega)} \|v - p\|_{H^s(\Omega)}. \tag{4.10}$$

Having this in mind, let us state and discuss an important result from Dupont and Scott, see [23]. Therefore we introduce shortly their notation. Let D be a bounded set in $\mathbb{R}^n$ with diameter d. Let D be star-shaped with respect to every point in an open ball B. They define a function $\varphi \in C_0^\infty(B)$ with support in B and with $\int_B \varphi(y)dy = 1$. Then, they introduce Sobolev's Representation. i.e. if $f \in C^\infty(D)$, l a positive integer and $x \in D$, then

$$f(x) = \mathcal{Q}^l f(x) + \mathcal{R}^l f(x),$$

where

$$\mathcal{Q}^l f(x) = \sum_{|\alpha|<l} \int_B \varphi(y) f^{(\alpha)}(y) \frac{(x-y)^\alpha}{\alpha!} dy$$

is a polynomial of degree less than l and

$$\mathcal{R}^l f(x) = \sum_{|\alpha|=l} \int_D k_\alpha(x,y) f^{(\alpha)}(y) dy.$$

The kernels k_α are given by $k_\alpha(x,y) = (l/\alpha!)(x-y)^\alpha k(x,y)$, where $k(x,y) = \int_0^1 s^{-n-1}\varphi(x + s^{-1}(y-x))ds$. With these notations cleared we can state their result [[23], Theorem 6.1].

Theorem 4.9. *Suppose that $m = \bar{m} + \theta$, where $0 < \theta < 1$ and $\bar{m}$ is a nonnegative integer. Let $l = \bar{m} + 1$, and let $\mathcal{Q}^l$ be defined as above. Then, there exists a constant $C = C(n, \varphi, d, m)$ such that, for $1 \leq p \leq \infty$ and $f \in W_p^m(D)$, there holds*

$$\|f - \mathcal{Q}^l f\|_{W_p^m(D)} \leq C|f|_{W_p^m(D)}. \tag{4.11}$$

We will use the following consequence of the upper result.

Corollary 4.10. *Let $s = 1 + \bar{s}$, $\bar{s} \in (0, 1/2)$. Then, there holds*

$$\|(I - Q_j^s)v\|_{H^s(\Omega)} \leq c|v|_{H^s(\Omega)}. \tag{4.12}$$

Proof. As there holds $\mathcal{Q}^2 v \in S^1_{h_j}$, we can estimate with (4.10)

$$\|(I - Q_j^s)v\|_{H^s(\Omega)} \leq \|v - \mathcal{Q}^2 v\|_{H^s(\Omega)} \leq C|v|_{H^s(\Omega)},$$

which already finishes the proof. □

Let us now come to another important error estimate for the projection operator Q_j^s with $0 \leq s < \frac{3}{2}$.

Lemma 4.11. *Let $s \in [0, \frac{3}{2})$, $t \in [0, s]$ if $s \in [0, 1]$ and $t \in [0, \bar{s}]$ if $s = 1 + \bar{s} \in (1, \frac{3}{2})$. Furthermore, let $v \in V_j$. Then, the H^s-projection fulfills the following error estimate,*

$$\|v - Q_j^s v\|_{H^{s-t}(\Omega)} \leq c h_j^t \|v\|_{H^s(\Omega)} \qquad \forall v \in V_j. \tag{4.13}$$

The proof of this lemma is done in several steps. Let us show the estimate

$$\|v - Q_j^s v\|_{H^{s-t}(\Omega)} \leq c h_j^t \|v - Q_j^s v\|_{H^s(\Omega)} \leq c h_j^t \|v\|_{H^s(\Omega)},$$

where the the second inequality is a direct consequence of (4.7). For the first inequality we need the following auxiliary result.

Lemma 4.12. *Let $s \in (0, 3/2)$ and let $w \in H^s(\Omega)$ be the unique solution of the variational problem*

$$\langle w, u \rangle_{H^s(\Omega)} = \langle v - Q_j^s v, u \rangle_{H^{s-t}(\Omega)},$$

then, there holds

$$\|w\|_{H^{s+t}(\Omega)} \leq \|v - Q_j^s v\|_{H^{s-t}(\Omega)}.$$

Proof. For convex domains there holds for $s \in (0, 3/2)$

$$\|u\|_{H^s(\Omega)} \cong \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}.$$

Let $(\varphi_j, \rho_j)_j$ be eigenfunctions and eigenvalues of $-\Delta$. Then, due to $u = \sum_{j=1}^{\infty} \langle u, \varphi \rangle \varphi$ for $u \in H^s(\Omega)$, there holds

$$(-\Delta)^s \varphi_j = \rho_j^s \varphi_j \quad \text{and} \quad (-\Delta)^s u = \sum_{j=1}^{\infty} \rho_j^s \langle u, \varphi_j \rangle \varphi_j.$$

Thus, we find

$$\begin{aligned} &\langle w, u \rangle_{H^s(\Omega)} = \langle v - Q_j^s v, u \rangle_{H^{s-t}(\Omega)} \quad \forall u \in H^s(\Omega) \\ \Leftrightarrow\ &\langle \sum_{j=1}^{\infty} \rho_j^{s/2} \langle w, \varphi_j \rangle \varphi_j, \sum_{j=1}^{\infty} \rho_j^{s/2} \langle u, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)} \\ &\qquad = \langle \sum_{j=1}^{\infty} \rho_j^{(s-t)/2} \langle v - Q_j^s v, \varphi_j \rangle \varphi_j, \sum_{j=1}^{\infty} \rho_j^{(s-t)/2} \langle u, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)}. \end{aligned}$$

We choose as test function $u \in H^s(\Omega)$, such that $u = \sum_{j=1}^{m} \rho_j^t \langle w, \varphi_j \rangle \varphi_j$ holds. Then we see,

$$\begin{aligned} &\langle \sum_{j=1}^{\infty} \rho_j^{s/2} \langle w, \varphi_j \rangle \varphi_j, \sum_{j=1}^{\infty} \rho_j^{s/2+t} \langle w, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)} \\ &\qquad = \langle \sum_{j=1}^{\infty} \rho_j^{(s-t)/2} \langle v - Q_j^s v, \varphi_j \rangle \varphi_j, \sum_{j=1}^{\infty} \rho_j^{(s+t)/2} \langle w, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)} \\ \Leftrightarrow\ &\sum_{j=1}^{m} \rho_j^{s+t} \langle w, \varphi_j \rangle^2 = \sum_{j=1}^{m} \rho_j^{(s-t)/2} \rho_j^{(s+t)/2} \langle v - Q_j^s v, \varphi_j \rangle \langle w, \varphi_j \rangle \\ \Leftrightarrow\ &\langle \sum_{j=1}^{m} \rho_j^{(s+t)/2} \langle w, \varphi_j \rangle \varphi_j, \sum_{j=1}^{m} \rho_j^{(s+t)/2} \langle w, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)} \\ &\qquad = \langle \sum_{j=1}^{m} \rho_j^{(s-t)/2} \langle v - Q_j^s v, \varphi_j \rangle \varphi_j, \sum_{j=1}^{m} \rho_j^{(s+t)/2} \langle w, \varphi_j \rangle \varphi_j \rangle_{L^2(\Omega)}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \| \sum_{j=1}^{m} \rho_j^{(s+t)/2} \langle w, \varphi_j \rangle \varphi_j \|_{L^2(\Omega)} &\le \| \sum_{j=1}^{m} \rho_j^{(s-t)/2} \langle v - Q_j^s v, \varphi_j \rangle \varphi_j \|_{L^2(\Omega)} \\ &\le \| v - Q_j^s v \|_{H^{s-t}(\Omega)} \end{aligned}$$

and thus, letting m tend to infinity, we get

$$\| w \|_{H^{s+t}(\Omega)} \le \| v - Q_j^s v \|_{H^{s-t}(\Omega)}.$$

□

For the proof of the next lemma, we use furthermore the following estimate which is a consequence of well known interpolation error estimates, where I is the nodal interpolant.

$$\|w - Q_j^s w\|_{H^s(\Omega)} \leq \|w - Iw\|_{H^s(\Omega)} \leq ch^t \|w\|_{H^{s+t}(\Omega)}. \tag{4.14}$$

Lemma 4.13. *Let $w \in H^s(\Omega)$ be given as in the previous lemma. Then, there holds*

$$\|v - Q_j^s v\|_{H^{s+t}(\Omega)} \leq c \cdot h^t \|v - Q_j^s v\|_{H^s(\Omega)}.$$

Proof. We find

$$\begin{aligned}
\|v - Q_j^s v\|^2_{H^{s-t}(\Omega)} &= \langle v - Q_j^s v, v - Q_j^s v\rangle_{H^{s-t}(\Omega)} \\
&= \langle w, v - Q_j^s v\rangle_{H^{s-t}(\Omega)} \\
&= \langle w - Q_j^s w, v - Q_j^s v\rangle_{H^{s-t}(\Omega)} \\
&\leq \|w - Q_j^s w\|_{H^s(\Omega)} \|v - Q_j^s v\|_{H^s(\Omega)} \\
&\leq c \cdot h^t \|w\|_{H^{s+t}(\Omega)} \|v - Q_j^s v\|_{H^s(\Omega)}.
\end{aligned}$$

Here, we applied (4.14). We continue estimating,

$$c \cdot h^t \|w\|_{H^{s+t}(\Omega)} \|v - Q_j^s v\|_{H^s(\Omega)} \leq c \cdot h^t \|v - Q_j^s v\|_{H^{s-t}(\Omega)} \|v - Q_j^s v\|_{H^s(\Omega)}.$$

Hence, we get

$$\|v - Q_j^s v\|_{H^{s-t}(\Omega)} \leq c \cdot h^t \|v - Q_j^s v\|_{H^s(\Omega)},$$

which finishes the proof. □

The upper considerations yield the proof of lemma 4.11.

Let us state a last error estimate.

Lemma 4.14. *Let $s \in [0, \frac{3}{2})$, $t \in [0, s]$ if $s \in [0, 1]$ and $t \in [0, \bar{s}]$ if $s = 1 + \bar{s} \in (1, \frac{3}{2})$. Furthermore, let $v \in V_j$. Then, the H^s-projection fulfills the following error estimate,*

$$\|v - Q_j^s v\|_{H^{s-t}(\Omega)} \leq c\|v\|_{H^{s-t}(\Omega)} \qquad \forall v \in V_j. \tag{4.15}$$

Proof. The assertion is a direct result of lemma 4.11 and the inverse inequality (3.12).

□

4.4 Stability estimates

One first stability estimate is a direct consequence of definitions 4.1 and 4.2. By choosing $v_j = Q_j^s v$ and applying Cauchy-Schwarz inequality we obtain directly the following.

Lemma 4.15. *For all $v \in H^s(\Omega)$, the stability estimate*

$$\|Q_j^s v\|_{H^s(\Omega)} \leq \|v\|_{H^s(\Omega)} \tag{4.16}$$

holds for $s \in [0, \frac{3}{2})$.

Note that the stability estimate for the L^2-projection in the L^2-norm is included for $s = 0$. We further know from [13] that the L^2-projection is also stable in $H^1(\Omega)$, thus

$$\|Q_j v\|_{H^1(\Omega)} \leq c\|v\|_{H^1(\Omega)} \qquad \forall v \in H^1(\Omega). \tag{4.17}$$

We show a more general result for the stability of the L^2-projection.

Lemma 4.16. *Let $s \in [0, \frac{3}{2})$, then there holds for all $v \in H^s(\Omega)$*

$$\|Q_j v\|_{H^s(\Omega)} \leq c\|v\|_{H^s(\Omega)}. \tag{4.18}$$

Proof. The case $s \in [0, 1]$ can be done by an interpolation argument. Let us have a closer look at the case $s = 1 + \bar{s} \in (1, \frac{3}{2})$. For the estimation, we use $Q_j v = v$ for $v \in V_j$, lemma 4.15, theorem 3.3, (4.17) and lemma 4.11,

$$\begin{aligned}
\|Q_j v\|_{H^s(\Omega)} &= \|Q_j^s v + Q_j v - Q_j^s v\|_{H^s(\Omega)} \\
&\leq \|Q_j^s v\|_{H^s(\Omega)} + \|Q_j v - Q_j^s v\|_{H^s(\Omega)} \\
&= \|Q_j^s v\|_{H^s(\Omega)} + \|Q_j v - Q_j Q_j^s v\|_{H^s(\Omega)} \\
&\leq \|v\|_{H^s(\Omega)} + \|Q_j (v - Q_j^s v)\|_{H^s(\Omega)} \\
&\leq \|v\|_{H^s(\Omega)} + c h_j^{-\bar{s}} \|Q_j (v - Q_j^s v)\|_{H^1(\Omega)} \\
&\leq \|v\|_{H^s(\Omega)} + c h_j^{-\bar{s}} \|v - Q_j^s v\|_{H^1(\Omega)} \\
&\leq \|v\|_{H^s(\Omega)} + c h_j^{-\bar{s}} h_j^{\bar{s}} \|v\|_{H^s(\Omega)} \leq c\|v\|_{H^s(\Omega)}.
\end{aligned}$$

□

We still need one more stability estimate for the H^s-projection, $s \in (0, \frac{3}{2})$.

Lemma 4.17. *Let $s \in [0, \frac{3}{2})$. Then, the H^s-projection is stable in $H^{s-t}(\Omega)$ for*

$$t \in \begin{cases} [0, s], & \text{if } s \in [0,1] \\ [0, \bar{s}], & \text{if } s = 1 + \bar{s} \in (1, \frac{3}{2}), \end{cases}$$

i.e.

$$\|Q_j^s v\|_{H^{s-t}(\Omega)} \le c\|v\|_{H^{s-t}(\Omega)} \qquad \forall v \in H^{s-t}(\Omega). \tag{4.19}$$

Proof.

$$\begin{aligned} \|Q_j^s v\|_{H^{s-t}(\Omega)} &= \|Q_j^s v - v + v\|_{H^{s-t}(\Omega)} \\ &\le \|Q_j^s v - v\|_{H^{s-t}(\Omega)} + \|v\|_{H^{s-t}(\Omega)} \\ &\le c\|v\|_{H^{s-t}(\Omega)}. \end{aligned}$$

We used lemma 4.14. □

4.5 Auxiliary results

In this section, we want to introduce two results that are necessary in the following. The first result is the so called Schur lemma. Its proof can be found for instance in [53].

Lemma 4.18. *For a countable set I we consider the matrix $A = (A[l,k])_{k,l \in I}$ and the vector $\underline{u} = (u_k)_{k \in I}$. For an arbitrary $\alpha \in \mathbb{R}$ we then have*

$$\|A\underline{u}\|_2^2 \le \left[\sup_{l \in I} \sum_{k \in I} |A[l,k]| 2^{\alpha(k-l)}\right] \left[\sup_{l \in I} \sum_{k \in I} |A[l,k]| 2^{\alpha(k-l)}\right] \|\underline{u}\|_2^2.$$

Because of the consistency of induced matrix norms we obtain for an arbitrary $\alpha \in \mathbb{R}$

$$\|A\|_2 \le \left[\sup_{l \in I} \sum_{k \in I} |A[l,k]| 2^{\alpha(k-l)}\right]^{1/2} \cdot \left[\sup_{l \in I} \sum_{k \in I} |A[l,k]| 2^{\alpha(k-l)}\right]^{1/2}.$$

In the particular case of a symmetric matrix A and $\alpha = 0$ we have

$$\|A\|_2 \le \sup_{l \in I} \sum_{k \in I} |A[l,k]|. \tag{4.20}$$

The next result is a strengthened Cauchy-Schwarz inequality. For $s = 1$ it can be found in [53]. We prove that it holds true, also for $s \in (0, 3/2)$.

Lemma 4.19. *For the decompositions $\mathcal{T}_{N_j}$ may hold $c_1 2^{-j} \leq h_j \leq c_2 2^{-j}$ for all $j \in \mathbb{N}$. Then there exists a $q < 1$ such that*

$$|\langle (Q_i - Q_{i-1})v, (Q_j - Q_{j-1})v \rangle_{H^s(\Omega)}| \\ \leq c q^{|i-j|} \|(Q_i - Q_{i-1})v\|_{H^s(\Omega)} \|(Q_j - Q_{j-1})v\|_{H^s(\Omega)}$$

holds for all $v \in H^s(\Omega)$, $s \in (0, 3/2)$.

Proof. As already mentioned, the proof for $s = 1$ can be found in [53]. We generalize it here and introduce for that purpose a parameter t with respect to the parameter s in the following way,

$$t \in \begin{cases} (0, s), & \text{if } s \in (0, 1) \\ (1, s), & \text{if } s = 1 + \bar{s} \in (1, \frac{3}{2}). \end{cases}$$

Without loss of generality let $j < i$. For $v_j \in V_j$ there holds $Q_j^s v_j = v_j \in V_j$ and therefore

$$\begin{aligned} \langle (Q_i - Q_{i-1})v, (Q_j - Q_{j-1})v \rangle_{H^s(\Omega)} &= \langle (Q_i - Q_{i-1})v, Q_j^s (Q_j - Q_{j-1})v \rangle_{H^s(\Omega)} \\ &\leq \langle Q_j^s (Q_i - Q_{i-1})v, (Q_j - Q_{j-1})v \rangle_{H^s(\Omega)} \\ &\leq \|Q_j^s (Q_i - Q_{i-1})v\|_{H^s(\Omega)} \|(Q_j - Q_{j-1})v\|_{H^s(\Omega)}, \end{aligned}$$

where we used the self-adjointness of Q_j^s with respect to the H^s-norm and Cauchy-Schwarz inequality. Let us estimate the first term by using (3.12), (4.19), property 3 of lemma 4.4 and (4.9),

$$\begin{aligned} \|Q_j^s (Q_i - Q_{i-1})v\|_{H^s(\Omega)} &\leq c h_j^{-t} \|Q_j^s (Q_i - Q_{i-1})v\|_{H^{s-t}(\Omega)} \\ &\leq c h_j^{-t} \|(Q_i - Q_{i-1})v\|_{H^{s-t}(\Omega)} = c h_j^{-t} \|(Q_i - Q_{i-1})^2 v\|_{H^{s-t}(\Omega)} \\ &\leq c h_j^{-t} \big(\|(Q_i - I)(Q_i - Q_{i-1})v\|_{H^{s-t}(\Omega)} \\ &\quad + \|(I - Q_{i-1})(Q_i - Q_{i-1})v\|_{H^{s-t}(\Omega)} \big) \\ &\leq c h_j^{-t} [h_i^t + h_{i-1}^t] \|(Q_i - Q_{i-1})v\|_{H^s(\Omega)} \\ &\leq c 2^{-t[j-i]} \|(Q_i - Q_{i-1})v\|_{H^s(\Omega)}. \end{aligned}$$

All in all, with $q := 2^{-t}$, we obtain the desired result

$$|\langle (Q_i - Q_{i-1})v, (Q_j - Q_{j-1})v \rangle_{H^{1+\bar{s}}(\Omega)}| \\ \leq c\, q^{[i-j]} \|(Q_i - Q_{i-1})v\|_{H^{1+\bar{s}}(\Omega)} \|(Q_j - Q_{j-1})v\|_{H^{1+\bar{s}}(\Omega)}.$$

For $i < j$, we proceed likewise because of the symmetric structure to arrive at the same result. □

4.6 Spectral equivalence

Now we are able to prove theorem 4.6. For convenience, the assertion is divided into several lemmata. Note that we use a generic constant c again throughout this section. The proofs of the next lemmata reproduce the reflections in [53] and generalize them, when required. Let us start with the upper inequality

$$\sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \leq c\|v\|^2_{H^s(\Omega)}. \tag{4.21}$$

This inequality will be shown in two steps.

Lemma 4.20. *For all $v \in H^s(\Omega)$ the following inequality is fulfilled,*

$$\sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \leq c \sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)}.$$

Proof. We find with lemma 4.4

$$\begin{aligned}(*) := \sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} &= \sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})^2 v\|^2_{L^2(\Omega)} \\ &= \sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - I + I - Q_{k-1})(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \\ &\leq 2\sum_{k=0}^{\infty} h_k^{-2s} \Big[\|(Q_k - I)(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \\ &\quad + \|(I - Q_{k-1})(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}\Big].\end{aligned}$$

Next, we apply (4.8) and the fact that the grids are chosen to be nested, i.e. (3.11).

$$\begin{aligned}(*) &\leq 2\sum_{k=0}^{\infty} h_k^{-2s}(c_1 h_k^{2s} + c_2 h_{k-1}^{2s}) \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} \\ &\leq c\sum_{k=0}^{\infty} h_k^{-2s} h_k^{2s} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} = c\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)}.\end{aligned}$$

□

Lemma 4.21. *The estimate*

$$\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} \leq c\|v\|^2_{H^s(\Omega)} \tag{4.22}$$

is valid for all $v \in H^s(\Omega)$.

Proof. For all $v \in H^s(\Omega)$ there exists a representation with respect to the sequence of operators $\{Q_j^s\}_{j\in\mathbb{N}_0}$, with $Q_{-1}^s = 0$:

$$v = \sum_{j=0}^{\infty}(Q_j^s - Q_{j-1}^s)v = \sum_{j=0}^{\infty} v_j \quad \text{where } v_j := (Q_j^s - Q_{j-1}^s)v.$$

For $i < k$ there holds $v_i \in V_i \subseteq V_{k-1} \subset V_k$ and hence

$$(Q_k - Q_{k-1})v_i = Q_k v_i - Q_{k-1}v_i = v_i - v_i = 0. \tag{4.23}$$

This holds in particular for $v_i = (Q_i^s - Q_{i-1}^s)v$. Let us start the proof with investigating the term on the left-hand side of (4.22). We will make use of (4.23),

$$\begin{aligned}
\sum_{k=0}^{\infty}\|(Q_k - Q_{k-1})v\|_{H^s(\Omega)}^2 &= \sum_{k=0}^{\infty}\langle (Q_k - Q_{k-1})v, (Q_k - Q_{k-1})v\rangle_{H^s(\Omega)} \\
&= \sum_{k=0}^{\infty}\langle \sum_{i=0}^{\infty}(Q_k - Q_{k-1})v_i, \sum_{j=0}^{\infty}(Q_k - Q_{k-1})v_j\rangle_{H^s(\Omega)} \\
&= \sum_{k=0}^{\min\{i,j\}}\sum_{i,j=0}^{\infty}\langle (Q_k - Q_{k-1})v_i, (Q_k - Q_{k-1})v_j\rangle_{H^s(\Omega)} \\
&= \sum_{i,j=0}^{\infty}\sum_{k=0}^{\min\{i,j\}}\langle (Q_k - Q_{k-1})v_i, (Q_k - Q_{k-1})v_j\rangle_{H^s(\Omega)} \\
&\leq \sum_{i,j=0}^{\infty}\sum_{k=0}^{\min\{i,j\}}\|(Q_k - Q_{k-1})v_i\|_{H^s(\Omega)}\|(Q_k - Q_{k-1})v_j\|_{H^s(\Omega)}.
\end{aligned}$$

Next, we estimate the first term $\|(Q_k - Q_{k-1})v_i\|_{H^s(\Omega)}$. The second term is equivalent and can be estimated in the same way. To this end, we need (3.12), lemma 4.16, lemma 4.4 and lemma 4.11,

$$\begin{aligned}
\|(Q_k - Q_{k-1})v_i\|_{H^s(\Omega)} &\leq ch_k^{-t}\|(Q_k - Q_{k-1})v_i\|_{H^{s-t}(\Omega)} \leq ch_k^{-t}\|v_i\|_{H^{s-t}(\Omega)} \\
&= ch_k^{-t}\|(Q_i^s - Q_{i-1}^s)v\|_{H^{s-t}(\Omega)} = ch_k^{-t}\|(Q_i^s - Q_{i-1}^s)^2 v\|_{H^{s-t}(\Omega)} \\
&= ch_k^{-t}\|(Q_i^s - I + I - Q_{i-1}^s)(Q_i^s - Q_{i-1}^s)v\|_{H^{s-t}(\Omega)} \\
&\leq ch_k^{-t}\|(I - Q_i^s)v_i\|_{H^{s-t}(\Omega)} + ch_k^{-t}\|(I - Q_{i-1}^s)v_i\|_{H^{s-t}(\Omega)} \\
&\leq ch_k^{-t}\ h_i^t\|v_i\|_{H^s(\Omega)} + ch_k^{-t}\ h_{i-1}^t\|v_i\|_{H^s(\Omega)} \leq ch_k^{-t}\ h_i^t\|v_i\|_{H^s(\Omega)}.
\end{aligned}$$

Putting these estimates together, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} &\leq c \sum_{i,j=0}^{\infty} \sum_{k=0}^{\min\{i,j\}} h_k^{-2t} h_i^t h_j^t \|v_i\|_{H^s(\Omega)} \|v_j\|_{H^s(\Omega)} \\ &\leq c \sum_{i,j=0}^{\infty} 2^{-t|i-j|} \|v_i\|_{H^s(\Omega)} \|v_j\|_{H^s(\Omega)}.\end{aligned}$$

For the last estimate, we need the following consideration for some fixed $t \in (0, s]$ or $t \in (0, \bar{s}]$, depending on the size of s,

$$\begin{aligned} & h_k^{-2t} \leq c \cdot \left(2^{-k}\right)^{-2t} = c \left(2^{\min\{i,j\}-k}\right)^{-2t} 2^{2t\min\{i,j\}} \\ \Rightarrow \quad & \sum_{k=0}^{\min\{i,j\}} h_k^{-2t} \leq c \cdot 2^{2t\min\{i,j\}} \sum_{k=0}^{\min\{i,j\}} \left(2^{-2t}\right)^{\min\{i,j\}-k} \leq c\, 2^{2t\min\{i,j\}}.\end{aligned}$$

The sum $\sum_{k=0}^{\min\{i,j\}} \left(2^{-2t}\right)^{\min\{i,j\}-k}$ can be estimated from above by a constant, because it is a partial sum of a geometic series. Then, we obtain with a case by case analysis

$$\sum_{k=0}^{\min\{i,j\}} h_k^{-2t} h_i^t h_j^t \leq c\, 2^{2t\min\{i,j\}} 2^{-t(i+j)} = 2^{-t|i-j|}.$$

Let us define a matrix A by $A[j,i] = 2^{-t|i-j|}$. Then, we estimate

$$\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} \leq c\|A\|_2 \sum_{i=0}^{\infty} \|v_i\|^2_{H^s(\Omega)}.$$

Applying (4.20), a consequence of the Schur lemma, we obtain

$$\|A\|_2 \leq \sup_{j \in \mathbb{N}_0} \sum_{i=0}^{\infty} 2^{-t|i-j|}.$$

For $q := 2^{-t}$ and some $j \in \mathbb{N}_0$, this norm is bounded by

$$\sum_{i=0}^{\infty} q^{|i-j|} = \sum_{i=0}^{j-1} q^{j-i} + \sum_{i=j}^{\infty} q^{i-j} = \sum_{i=1}^{j} q^i + \sum_{i=0}^{\infty} q^i \leq 2 \sum_{i=0}^{\infty} q^i = \frac{2}{1-q}.$$

Hence, we obtain

$$\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} \leq c \sum_{i=0}^{\infty} \|v_i\|^2_{H^s(\Omega)}$$

and finally

$$\begin{aligned}\sum_{i=0}^{\infty} \|v_i\|^2_{H^s(\Omega)} &\leq \sum_{i=0}^{\infty} \langle (Q_i^s - Q_{i-1}^s)v, (Q_i^s - Q_{i-1}^s)v\rangle_{H^s(\Omega)} \\ &= \sum_{i=0}^{\infty} \langle (Q_i^s - Q_{i-1}^s)^2 v, v\rangle_{H^s(\Omega)} \\ &= \sum_{i=0}^{\infty} \langle (Q_i^s - Q_{i-1}^s)v, v\rangle_{H^s(\Omega)} \\ &= \langle v, v\rangle_{H^s(\Omega)} = \|v\|^2_{H^s(\Omega)},\end{aligned}$$

which finishes the proof. □

The previous two lemmata put together provide the upper estimate (4.21). It remains to show the lower estimate

$$\|v\|^2_{H^s(\Omega)} \leq c \sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}. \tag{4.24}$$

We divide it again into two steps. The first lemma directly follows from 3.12.

Lemma 4.22. *For all $v \in H^s(\Omega)$ we have*

$$\sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)} \leq c \sum_{k=0}^{\infty} h_k^{-2s} \|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}.$$

Let us come to the last ingredient of the proof of theorem 4.6.

Lemma 4.23. *The estimate*

$$\|v\|^2_{H^s(\Omega)} \leq c \sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)}$$

holds for all $v \in H^s(\Omega)$.

Proof. For all $v \in H^s(\Omega)$ there exists the following representation with respect to the sequence of operators $\{Q_j\}_{j\in\mathbb{N}_0}$

$$v = \sum_{j=0}^{\infty} (Q_j - Q_{j-1})v.$$

The estimate follows by applying lemma 4.19 and (4.20) by the same arguments as in the proof of lemma 4.21.

$$\begin{aligned}\|v\|^2_{H^s(\Omega)} &= \left\langle \sum_{i=0}^{\infty}(Q_i - Q_{i-1})v, \sum_{j=0}^{\infty}(Q_j - Q_{j-1})v \right\rangle_{H^s(\Omega)} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \langle (Q_i - Q_{i-1})v, (Q_j - Q_{j-1})v\rangle_{H^s(\Omega)} \\ &\leq c \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} q^{|i-j|} \|(Q_i - Q_{i-1})v\|_{H^s(\Omega)} \|(Q_j - Q_{j-1})v\|_{H^s(\Omega)} \\ &\leq c \sum_{k=0}^{\infty} \|(Q_k - Q_{k-1})v\|^2_{H^s(\Omega)}.\end{aligned}$$

□

Now we know that the term $\sum_{k=0}^{\infty} h_k^{-2s}\|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}$ is well defined and equivalent to the H^s-norm of v, i.e. that holds

$$c_1\|v\|^2_{H^s(\Omega)} \leq \sum_{k=0}^{\infty} h_k^{-2s}\|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)} \leq c_2\|v\|^2_{H^s(\Omega)}.$$

Hence, we find the following equality.

Lemma 4.24. *The following equality holds for* $v \in H^s(\Omega)$,

$$\langle B^s v, v\rangle_{L^2(\Omega)} = \sum_{k=0}^{\infty} h_k^{-2s}\|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}.$$

Proof. For all $v \in H^s(\Omega)$ there holds

$$\begin{aligned}\langle B^s v, v\rangle_{L^2(\Omega)} &= \left\langle \sum_{k=0}^{\infty} h_k^{-2s}(Q_k - Q_{k-1})v, v \right\rangle_{L^2(\Omega)} \\ &= \sum_{k=0}^{\infty} h_k^{-2s} \langle (Q_k - Q_{k-1})v, v\rangle_{L^2(\Omega)} \\ &= \sum_{k=0}^{\infty} h_k^{-2s} \left\langle (Q_k - Q_{k-1})v, (Q_k - Q_{k-1})v\right\rangle_{L^2(\Omega)} \\ &= \sum_{k=0}^{\infty} h_k^{-2s}\|(Q_k - Q_{k-1})v\|^2_{L^2(\Omega)}.\end{aligned}$$

□

Combining lemmata 4.20-4.24, we obtain the proof of theorem 4.6.

Lemma 4.25. *The multilevel operator B^s is self-adjoint with respect to the L^2-norm. Furthermore it satisfies $B^{2s} = (B^s)^2$ for $s \in [0, 3/4)$.*

Proof. The operator B^s is self-adjoint with respect to the L^2 norm if and only if

$$\langle B^s v, u\rangle_{L^2(\Omega)} = \langle v, B^s u\rangle_{L^2(\Omega)}. \tag{4.25}$$

We can interchange the summation and integration in (4.25) and use the self-adjointness of the projection operators (4.3) to show the self-adjointness for B^s. The second matter is a direct result of Lemma 4.4. □

Remark 4.26.

1. *In the case of piecewise constant basis functions the spectral equivalence inequalities still hold for $s \in [0, 1/2)$, see [53].*

2. *Due to Lemma 4.25 there holds*

$$\langle B^s v, v\rangle_{L^2(\mathcal{D})} = \langle B^{s/2} v, B^{s/2} v\rangle_{L^2(\mathcal{D})} = \|B^{s/2} v\|^2_{L^2(\mathcal{D})}. \tag{4.26}$$

Consequently, we have the norm equivalence

$$\|B^{s/2} v\|^2_{L^2(\mathcal{D})} \sim \|v\|^2_{H^s(\mathcal{D})}. \tag{4.27}$$

3. *In the particular case of $v_m \in V_m$, the infinite sum reduces to a finite sum*

$$B^s v_m = \sum_{k=0}^{\infty} h_k^{-2s}(Q_k - Q_{k-1}) v_m = \sum_{k=0}^{m} h_k^{-2s}(Q_k - Q_{k-1}) v_m.$$

This allows a numerical evaluation of the expression.

Chapter 5

Parameter Identification

In section 2.5 we already introduced our problem. Let us rewrite it once again here.

$$\left.\begin{array}{rcl} \text{minimize} & J(y,a) := \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)} & \\ \text{subject to} & -\nabla\cdot(a\nabla y) = g & \text{in } \Omega \\ & y = 0 & \text{on } \Gamma \\ & 0 < a_{\min} \le a(x) \le a_{\max} & \text{a.e. in } \Omega \end{array}\right\} \tag{5.1}$$

Let us start with some helpful notation. We split the objective into two functionals F and Q that only depend on either y or a. They are defined as $F(y) = \frac{1}{2}\|y-y_\delta\|^2_{L^2(\Omega)}$ and $Q(a) = \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)}$. With these definitions we can write the objective functional as $J(y,a) = F(y) + Q(a)$. Next, we define the set of admissible parameter functions as

$$A_{ad} = \{a \in H^s(\Omega) \;:\; 0 < a_{\min} \le a(x) \le a_{\max} \text{ a.e. in } \Omega\}.$$

This definition implies $a \in L^\infty(\Omega)$ due to the pointwise bounds $a_{\min}$ and $a_{\max}$. At first, we are interested in the analysis of the elliptic PDE

$$\begin{aligned} -\nabla\cdot(a\nabla y) &= g \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{5.2}$$

As it is usual the case in the theory of elliptic boundary value problems, one cannot expect the existence of classical solutions $y \in C^2(\Omega)\cap C(\bar{\Omega})$ here. The Lax-Milgram lemma provides the existence of a unique weak solution $y = S(a)$ for every $a \in A_{ad}$ and for every right-hand side $g \in L^2(\Omega)$, i.e. there exists a unique solution $y = y(a)$ of the variational form

$$\int_\Omega a\nabla y \cdot \nabla v \, dx = \int_\Omega gv \, dx =: G(v) \quad \forall v \in H_0^1(\Omega)$$

for all $a \in A_{ad}$. At this point, we do not necessarily need to restrict the parameter functions to the space $H^s(\Omega)$, we only need them to fulfill the pointwise bounds, thus to be in $L^\infty(\Omega)$.
We want to derive the variational form of the PDE and therefore we suppose for one moment that there exists a classical solution and thus that all integrals occurring in the following exist. We multiply each side of the PDE by an arbitrary test function $v \in H_0^1(\Omega)$ and consider the integral over Ω

$$-\int_\Omega \nabla \cdot (a\nabla y) v \, dx = \int_\Omega gv \, dx.$$

Integration by parts yields

$$\int_\Omega a\nabla y \cdot \nabla v \, dx = \int_\Omega gv \, dx \quad \Rightarrow \quad (a\nabla y, \nabla v)_{(L^2(\Omega))^N} = (g, v)_{L^2(\Omega)}.$$

The boundary terms vanish because test functions $v \in H_0^1(\Omega)$ have a compact support and thus they are zero on the boundary. The existence and uniqueness of solutions permit the definition of a parameter-to-state mapping (*control-to-state mapping* in the optimal control setting) $S : L^\infty(\Omega) \to H_0^1(\Omega)$, $a \mapsto y$ such that (1.1) holds. Then, the objective functional can be written in a reduced form

$$J(y, a) = J(S(a), a) = F(S(a)) + Q(a) =: f(a).$$

Additionally to the existence, the Lax-Milgram lemma also yields boundedness of the state in $H^1(\Omega)$, i.e. $\|y\|_{H^1(\Omega)} \le c\|G\|_{(H^1(\Omega))^*} \le c\|g\|_{L^2(\Omega)}$, where the constant c depends on the pointwise bounds $a_{\min}$ and $a_{\max}$. This implies uniform boundedness of all $y = S(a)$ in $H^1(\Omega)$, thus $\exists K > 0$, such that $\|y\|_{H^1(\Omega)} \le K$, for a fixed right-hand side $g \in L^\infty(\Omega) \hookrightarrow L^2(\Omega)$. In section 2.4 we introduced a result by Meyers [45] and a generalization of it by Gröger [30] that ensure a higher regularity of the solution y. The application to our problem yields the existence of a constant $\bar{q} > 2$, such that $\|y\|_{W^{1,p}(\Omega)} \le c\|G\|_{W^{-1,p}(\Omega)}$ holds for $2 < p < \bar{q}$. For a fixed right-hand side g, we also obtain uniform boundedness in $W^{1,p}(\Omega)$, i.e.

$$\|y\|_{W^{1,p}(\Omega)} \le K. \tag{5.3}$$

Those regularity results hold also for the adjoint state p.

5.1 Existence of solutions

The existence of a solution of (5.1) has been thoroughly discussed in my Diploma thesis [5] and we will only present the main ideas and difficulties that arise. However, there is a slight difference, as we considered the Tichonov term in the H^1-norm in [5] whereas we consider the more general H^s-norm here.

Lemma 5.1. *The problem* (5.1) *has at least one optimal solution $\bar{a}$ with optimal state $\bar{y} = y(\bar{a})$.*

Proof. The proof basically follows the same path as the standard proof of existence of solutions in the case of semilinear quadratic elliptic problems [[55] Theorem 4.13]. Nevertheless, one has to be careful and adapt some steps.

1. *Boundedness of the objective:* The objective functional $J(y, a)$ is bounded from below. Thus there exists a nonnegative real number j defined by $j := \inf_{a \in A_{ad}} J(y, a)$ and a minimizing sequence (y_n, a_n) with $a_n \in A_{ad}$, $y_n = S(a_n)$ such that $J(a_n, y_n) \to j$, $n \to \infty$. For every $n \in \mathbb{N}$ and every $v \in H_0^1(\Omega)$ there holds $(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N} = (g, v)_{L^2(\Omega)}$.

2. *Passing to the limit:* We examine the behavior of $(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N}$ for $n \to \infty$. We show the existence of $\bar{a}$, $\bar{y}$, such that $a_n \nabla v \to \bar{a} \nabla v$ in $(L^q(\Omega))^N$ and $\nabla y_n \rightharpoonup \nabla \bar{y}$ in $(L^p(\Omega))^N$ for a certain subsequence, where $p > 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we obtain convergence

$$(\nabla y_n, a_n \nabla v)_{(L^2(\Omega))^N} \to (\nabla \bar{y}, \bar{a} \nabla v)_{(L^2(\Omega))^N}.$$

   - *Convergence of $a_n \nabla v$ towards $\bar{a} \nabla v$:* The functional value $J(y_1, a_1)$ of the first element of the minimizing sequence is an upper bound for the functional values of the following elements. Thus there holds in particular $\frac{\alpha}{2} \|a_n\|^2_{H^s(\Omega)} \leq J(y_1, a_1)\ \forall n \in \mathbb{N}$, i.e. $\{a_n\}_{n=1}^{\infty}$ is uniformly bounded in $H^s(\Omega)$. Hence, for $s > 0$, the sequence is also precompact in $L^2(\Omega)$, i.e. for a subsequence there holds $\|a_n - \bar{a}\|_{L^2(\Omega)} \to 0$, as $n \to \infty$. Together with the boundedness in $L^\infty(\Omega)$ by the pointwise bounds $\|a_n - \bar{a}\|_{L^\infty(\Omega)} \leq a_{\max} - a_{\min}$, for $n \in \mathbb{N}$, we obtain

$$\|a_n - \bar{a}\|_{L^r(\Omega)} \to 0, \text{ as } n \to \infty$$

for an arbitrary $2 \leq r \leq \infty$. Hölder's inequality then yields convergence of $\|(a_n - \bar{a})\nabla v\|_{(L^q(\Omega))^N}$ for $q \in [1, 2)$.

- *Weak convergence of ∇y_n towards $\nabla \bar{y}$:* The sequence $\{y_n\}_{n=1}^{\infty}$ is uniformly bounded in $W^{1,p}(\Omega)$ for some $p > 2$, see (5.3). Thus, we have a weakly convergent subsequence satisfying $y_n \rightharpoonup \bar{y}$ in $W^{1,p}(\Omega)$ as $n \to \infty$ and there holds $\nabla y_n \rightharpoonup \nabla \bar{y}$ in $(L^p(\Omega))^N$ as $n \to \infty$.

Hence, $(\bar{y}, \bar{a})$ satisfies the variational form.

3. *Optimality of $(\bar{y}, \bar{a})$:* We know that $\bar{a} \in A_{ad}$, since the admissible set is weakly sequentially closed. The last step is to show optimality of $(\bar{y}, \bar{a})$, i.e. that holds $J(\bar{a}, \bar{y}) = j$. At first we use the continuity of $F(y) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)}$ and see that $\lim_{n\to\infty} F(y_n) = F(\bar{y})$ follows directly from $y_n \to \bar{y}$ in $L^2(\Omega)$. Secondly, we use continuity and convexity of $G(a) = \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)}$, thus its weak lower semi-continuity. Then, there holds $\liminf_{n\to\infty} Q(a_n) \geq Q(\bar{a})$. Hence, there holds

$$j = \lim_{n\to\infty} J(y_n, a_n) \geq \lim_{n\to\infty} F(y_n) + \liminf_{n\to\infty} Q(a_n) \geq F(\bar{y}) + Q(\bar{a}) = J(\bar{y}, \bar{a}).$$

Due to the definition of j as infimum, there also holds $j \leq J(\bar{y}, \bar{a})$. Hence, we get $j = J(\bar{y}, \bar{a})$.

□

5.2 Optimality conditions

5.2.1 Fréchet-differentiability

In the following we want to derive first and second order necessary optimality conditions. To this end, we need a first and second derivative of the parameter-to-state mapping. Let us recall the following definition of Fréchet differentiability, which can be found e.g. in [55].

Definition 5.2. *A mapping $F : U \to V$ is Fréchet-differentiable in $u \in U$ if there exists an operator $A : \mathcal{L}(U, V)$ and a mapping $r : U \times U \to V$ such that for all $h \in U$*

$$F(u + h) = F(u) + Ah + r(u, h),$$

where the remainder term r fulfills the following condition,

$$\frac{\|r(u,h)\|_V}{\|h\|_U} \to 0 \qquad \textit{for } \|h\|_U.$$

Let us now shortly restate the first and second order differentiability of the parameter-to-state mapping $S : L^\infty(\Omega) \to H^1(\Omega),\ S(a) = y$ corresponding to the partial differential equation

$$\begin{aligned} -\nabla \cdot (a\nabla y) &= g \quad \text{in } \ \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{5.4}$$

The proofs of the first and second order Fréchet differentiability are done in detail in [5] and [6]. Thus, we only explain the main steps here.

Lemma 5.3. *The parameter-to-state-mapping* $S : L^\infty(\Omega) \to H^1(\Omega)$ *is Fréchet-differentiable. Its derivative can be described by* $S'(a)a_1 = y_1'$*, where* $y_1' \in H^1(\Omega)$ *is the weak solution of the following problem*

$$\begin{aligned} -\nabla \cdot (a\nabla y_1') &= \nabla \cdot (a_1 \nabla y) \quad \textit{in } \Omega \\ y_1' &= 0 \qquad\qquad\quad \textit{on } \Gamma. \end{aligned} \tag{5.5}$$

Here, a is an admissible parameter function with respect to (2.8) *and y is the corresponding state* $y = S(a)$*.*

By means of the Lax-Milgram lemma $y_1' \in H^1(\Omega)$ is well-defined because $\nabla \cdot (a_1 \nabla y)$ is an element of $H^{-1}(\Omega)$.

Proof. We have to show the existence of a linear continuous operator $D : L^\infty(\Omega) \to H^1(\Omega)$, such that $S(a + a_1) - S(a) = Da_1 + r(a, a_1)$ holds for all $a_1 \in L^\infty(\Omega)$ satisfying the equation

$$\frac{\|r(a,a_1)\|_{H^1(\Omega)}}{\|a_1\|_{L^\infty(\Omega)}} \to 0, \ \text{for } \|a_1\|_{L^\infty(\Omega)} \to 0.$$

Then D is the Fréchet-derivative of S. Let us assume

$$-\nabla \cdot (a\nabla y_1') = \nabla \cdot (a_1 \nabla y) \tag{5.6}$$

to be the PDE associated with Da_1. We easily verify linearity and continuity of D. Next, we want to examine the term $r(a, a_1)$, thus $S(a+a_1) - S(a) - Da_1$.

In order to do so, we subtract the associated PDEs of $S(a)$ and Da_1, i.e. (5.4) and (5.6), from the PDE of $S(a+a_1) = y_1$, which is given as

$$-\nabla \cdot ((a+a_1)\nabla y_1) = g.$$

All partial differential equations mentioned before have homogeneous Dirichlet boundary conditions. A short computation gives

$$-\nabla \cdot (a\nabla(\underbrace{y_1 - y - y_1'}_{=:y_\delta})) = \nabla \cdot (a_1\nabla(y_1 - y)).$$

We now have to show $\dfrac{\|y_\delta\|_{H^1(\Omega)}}{\|a_1\|_{L^\infty(\Omega)}} \to 0$ for $\|a_1\|_{L^\infty(\Omega)} \to 0$. Therefore, we estimate $\|y_\delta\|_{H^1(\Omega)}$ by means of Cauchy-Schwarz inequality, Friedrich's inequality and Lax-Milgrams estimate. All in all we obtain $\|y_\delta\|_{H^1(\Omega)} \leq c\|a\|^2_{L^\infty(\Omega)}$, which implies the desired remainder term condition. □

Let us continue with the second order Fréchet differentiability of the operator S. We do this by showing first order Fréchet differentiability of the mapping $a \to S'(a)a_1$ for all $a_1 \in L^\infty(\Omega)$.

Lemma 5.4. *The mapping $a \to S'(a)a_1$ is Fréchet-differentiable from $L^\infty(\Omega)$ onto $H^1(\Omega)$ for all $a_1 \in L^\infty(\Omega)$. Its derivative is given by $S''(a)[a_1, a_2] = y''$ where y'' is the weak solution of the following problem*

$$\begin{aligned} -\nabla \cdot (a\nabla y'') &= \nabla \cdot (a_1\nabla y_2') + \nabla \cdot (a_2\nabla y_1') && \text{in } \Omega \\ y'' &= 0 && \text{on } \Gamma, \end{aligned} \tag{5.7}$$

with y_i', $i = 1, 2$ being defined as the weak solution of $-\nabla \cdot (a\nabla y_i') = \nabla \cdot (a_i\nabla y)$ and $y = S(a)$ being the solution of $-\nabla \cdot (a\nabla y) = g$.

The proof of this lemma can be done using the same techniques as in the proof of first order differentiability of the operator S.

5.2.2 First order necessary condition

An optimal parameter function $\bar{a} \in A_{ad}$ has to fulfill the following variational inequality

$$f'(\bar{a})(a - \bar{a}) \geq 0 \qquad \forall a \in A_{ad}. \tag{5.8}$$

Let us now compute the derivative of the objective in $\bar{a}$ which was given as $f(\bar{a}) := J(y(\bar{a}), \bar{a}) = F(S(\bar{a})) + Q(\bar{a})$,

$$\begin{aligned} f'(\bar{a})(a-\bar{a}) &= F'(S(\bar{a}))S'(\bar{a})(a-\bar{a}) + Q'(\bar{a})(a-\bar{a}) \\ &= F'(S(\bar{a}))y_1' + Q'(\bar{a})(a-\bar{a}) \\ &= (S(\bar{a}) - y_d, y_1')_{L^2(\Omega)} + (\alpha\bar{a}, (a-\bar{a}))_{H^s(\Omega)}. \end{aligned} \tag{5.9}$$

We introduce the adjoint state in order to transform the variational equation into the desired form. The weak solution $p \in H_0^1(\Omega)$ of the adjoint equation

$$\begin{aligned} -\nabla \cdot (a\nabla p) &= y - y_d && \text{in } \Omega \\ p &= 0 && \text{on } \Gamma \end{aligned}$$

is called adjoint state. We denote by $\bar{p}$ the adjoint state belonging to the optimal pairing $\bar{a}$, $\bar{y}$. Considering the weak formulations of the adjoint equation and (5.5) with y_1' and $\bar{p}$ as test functions, respectively, we easily see that

$$-\int_\Omega (a-\bar{a})\nabla\bar{y}\cdot\nabla\bar{p}\,dx = \int_\Omega (\bar{y}-y_d)y_1'\,dx$$

holds. Thus we obtain a first order necessary optimality condition:

Lemma 5.5. *An optimal parameter $\bar{a}$ together with the optimal state $\bar{y} = S(\bar{a})$ and the optimal adjoint state $\bar{p}$ necessarily fulfills the following condition*

$$-((a-\bar{a})\nabla\bar{y}, \nabla\bar{p})_{(L^2(\Omega))^N} + (\alpha\bar{a}, a-\bar{a})_{H^s(\Omega)} \geq 0, \tag{5.10}$$

for all $a \in A_{ad}$.

Before continuing with the second order sufficient conditions, let us shortly comment on the uniform boundedness of the adjoint state in the $W^{1,p}$ norm.

Lemma 5.6. *Let Ω be again a bounded domain in $\mathbb{R}^N$ with Lipschitz-boundary. Then, there exists a constant $\bar{q} = \bar{q}(\Omega, a_{\min}, a_{\max}) > 2$, such that*

$$\|p\|_{W_0^{1,p}(\Omega)} \leq C \tag{5.11}$$

for $p \in [2, \bar{q}]$ with $C = C(\Omega, a_{\min}, a_{\max}, p, y_d, g)$, and g beeing the right-hand side of the state equation (5.2).

Proof. Let us define $H(v) := (y - y_d, v)_{L^2(\Omega)}$. Then, we use lemma 2.15, and Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$ and obtain,

$$\begin{aligned}\|p\|_{W_0^{1,p}(\Omega)} \leq c\|H\|_{W^{-1,p}(\Omega)} &= c \sup_{v \in W^{1,q}(\Omega), v \neq 0} \frac{|(y - y_d, v)_{L^2(\Omega)}|}{\|v\|_{W^{1,q}(\Omega)}} \\ &\leq c \sup_{v \in W^{1,q}(\Omega), v \neq 0} \frac{\|y - y_d\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}}{\|v\|_{W^{1,q}(\Omega)}} \\ &\leq c \sup_{v \in W^{1,q}(\Omega), v \neq 0} \frac{\|y - y_d\|_{L^p(\Omega)} \|v\|_{W^{1,q}(\Omega)}}{\|v\|_{W^{1,q}(\Omega)}} \\ &\leq c\|y - y_d\|_{L^p(\Omega)} \leq c\|y\|_{L^p(\Omega)} + c\|y_d\|_{L^p(\Omega)}.\end{aligned}$$

Next, we apply the uniform boundedness of y in $H^1(\Omega)$ or $W^{1,p}(\Omega)$. Thus, we obtain the desired result, as $y_d \in L^\infty(\Omega)$. □

Remark 5.7. *In a similar manner we obtain via the Lax-Milgram estimate* (2.6) *uniform boundedness of the adjoint state p in $H^1(\Omega)$. Combining these estimates, we get uniform boundedness of p in the full $W^{1,p}$ norm.*

5.2.3 Second order sufficient condition

Second order sufficient conditions (abbr.: SSCs) play an important role for optimal control of PDEs. Over the last decades more and more articles concerning SSC have been published, e.g. [11], [17], [19], [20], [39], [48], [49]. Other than in finite dimensional spaces, SSCs are not primarily a tool to verify if a stationary solution is a local minimum. Indeed, it is in general difficult to verify such a condition for a given stationary point, see [50]. However, they play an important role in the theory of optimal control problems. A priori error estimates for different discretizations have been proved assuming a SSC holds, see [3], [16], [18], [31], [38]. SSC can also lead to Lipschitz stability [2], [32], [43], which is the main ingredient in the convergence analysis of SQP-methods (Sequential Quadratic Programming), [27], [28], see also chapter 6.

Two-norm-discrepancy

In finite dimensional spaces like $\mathbb{R}^N$, second order sufficient conditions are given in the following way. A point $\bar{u} \in \mathbb{R}^N$ is a local minimum of a function $f : \mathbb{R}^N \to \mathbb{R}$, if $f(\bar{u}) = 0$ is fulfilled and the Hesse matrix $f''(\bar{u})$ is positive definite, i.e. there exists a $\delta > 0$ such that for all $u \in \mathbb{R}^N$ holds

$$u^\top f''(\bar{u})u \geq \delta |u|^2.$$

In infinite dimensional spaces the situation is a different one. Already for simple problems, we see that we cannot transfer the second order sufficient conditions without further care. Let us consider the problem $\min_{u \in C} J(u)$, where U is a Banach space, $C \subset U$ a convex set and $J : U \to \mathbb{R}$ is a twice Fréchet differentiable functional. One could expect that verifying the following conditions $J'(\bar{u})(u - \bar{u}) = 0$ and $J''(\bar{u})v^2 \geq \delta \|v\|_U^2$, $\forall v \in U$ for some $\delta > 0$ would be a good way to find out if $\bar{u}$ was a strictly local optimal solution of the problem. Let us give an example where the second order condition is not fulfilled. It can be found in [55]. We consider the following problem,

$$\min_{0 \leq u(x) \leq 2\pi} J(u) = -\int_0^1 \cos(u(x))dx.$$

A global minimum is given by $\bar{u} \equiv 0$. It fulfills the first order necessary condition

$$J'(\bar{u})(u - \bar{u}) = \int_0^1 \sin(\bar{u}(x))(u(x) - \bar{u}(x))dx = \int_0^1 \sin(0)u(x)dx = 0.$$

For the second order condition we obtain formally

$$J''(\bar{u})u^2 = \int_0^1 cos(0)u^2(x)dx = \int_0^1 u^2(x)dx = \|u\|^2_{L^2(0,1)}. \tag{5.12}$$

The minimum is unique in a small L^∞-neighborhood, but if we consider any L^p-neighborhood, we have other points with the same value

$$u_\varepsilon(x) = \begin{cases} 2\pi, & 0 \leq x \leq \varepsilon \\ 0 & \varepsilon < x \leq 1 \end{cases}$$

since

$$\|u_\varepsilon(x) - \bar{u}\|^2_{L^2(0,1)} = \left(\int_0^1 u_\varepsilon^2(x)dx\right)^{1/2} = \left(\int_0^\varepsilon (2\pi)^2 dx\right)^{1/2} = 2\pi\sqrt{\varepsilon}.$$

On the one hand, we obtain the desired coercivity estimate (5.12) in the L^2-norm. And on the other hand, there is no local uniqueness of the minimum in an L^2-neighborhood. This is caused by the non-differentiability of the mapping

$$u(\cdot) \mapsto cos(u(\cdot))$$

as mapping from $L^2(0,1)$ to $L^2(0,1)$. We have differentiability in the L^∞-norm. However, we do not have the desired estimate in the L^∞-norm. This phenomenon is called the *two-norm discrepancy.* A functional J is twice differentiable with respect to one norm, but the coercivity estimate holds in a weaker norm in which the functional is not twice differentiable. This is a frequent situation for optimal control problems which was first mentioned in [34].

Second order sufficient condition

In the following we are going to proof that the following second order optimality condition

$$f''(\bar{a})(a-\bar{a})^2 \geq \delta \|a-\bar{a}\|^2_{H^s(\Omega)}$$

together with the first order necessary optimality condition (5.10) implies quadratic growth of the objective functional in the parameter $\bar{a}$ and thus the local optimality of this parameter. Let us start by calculating the second order derivative of the objective functional. To this end, let us restate the second order derivative of the control-to-state operator, i.e. $S''(a)[a_1,a_2] = y''$ with

$$\begin{aligned} -\nabla\cdot(a\nabla y'') &= \nabla\cdot(a_1\nabla y_2') + \nabla\cdot(a_2\nabla y_1') && \text{in } \Omega \\ y'' &= 0 && \text{on } \Gamma. \end{aligned} \tag{5.13}$$

Here again, the first derivatives y_i', $i=1,2$ are defined by $-\nabla\cdot(a\nabla y_i') = \nabla\cdot(a_i\nabla y)$, for $i=1,2$, respectively. With this at hand let us now have a look at the second order derivative of the objective functional

$$J(y,a) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)} = J(S(a),a) = F(S(a)) + Q(a) = f(a).$$

We find

$$\begin{aligned} f''(\bar{a})[a_1,a_2] &= F''(S(\bar{a}))[S'(\bar{a})a_1, S'(\bar{a})a_2] \\ &\quad + F'(S(\bar{a}))S''(\bar{a})[a_1,a_2] + Q''(\bar{a})[a_1,a_2] \\ &= (S'(\bar{a})a_2, S'(\bar{a})a_1)_{L^2(\Omega)} + \alpha(a_2,a_1)_{H^s(\Omega)} \\ &\quad + (S(\bar{a})-y_d, S''(\bar{a})[a_1,a_2])_{L^2(\Omega)}. \end{aligned}$$

We want to replace the last term, in which the second derivative of the state occurs. It can be done in the same way as before for the first order derivative. TFor that purpose, we again need the adjoint state p that is given by the adjoint equation $-\nabla(a\nabla p) = y - y_d$. Next, we consider the weak formulation of the adjoint equation with y' as test function and the weak formulation of (5.13) with p as test function,

$$\begin{aligned}
(p\nabla a, \nabla y'')_{(L^2(\Omega))^N} &= (y - y_d, y'')_{L^2(\Omega)} \\
(y''\nabla a, \nabla p)_{(L^2(\Omega))^N} &= -(a_1\nabla y_2', \nabla p)_{(L^2(\Omega))^N} - (a_2\nabla y_1', \nabla p)_{(L^2(\Omega))^N} \\
\Rightarrow (y - y_d, y'')_{L^2(\Omega)} &= -(a_1\nabla y_2', \nabla p)_{(L^2(\Omega))^N} - (a_2\nabla y_1', \nabla p)_{(L^2(\Omega))^N}.
\end{aligned}$$

The second order derivative of the objective functional now reads as

$$\begin{aligned}
f''(\bar{a})[a_1, a_2] = {} & (S'(\bar{a})a_2, S'(\bar{a})a_1)_{L^2(\Omega)} + \alpha(a_2, a_1)_{H^s(\Omega)} \\
& - (a_1\nabla y_2', \nabla p)_{(L^2(\Omega))^N} - (a_2\nabla y_1', \nabla p)_{(L^2(\Omega))^N}. \qquad (5.14)
\end{aligned}$$

Now we can come to the crucial point of this section. For the proof of the second order sufficient condition we need a Lipschitz estimate and we are going to prove it in several steps right now.

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^N$ be a Lipschitz domain. Then there exists a constant L belonging to the objective functional*

$$f(a) = J(y, a) = J(S(a), a) = \frac{1}{2}\|S(a) - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|a\|^2_{H^s(\Omega)},$$

that is independent from a, h, a_1, a_2, such that

$$|f''(a + h)[a_1, a_2] - f''(a)[a_1, a_2]| \leq L \cdot \|h\|_{L^\infty(\Omega)}\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)} \qquad (5.15)$$

for all a, h, a_1, $a_2 \in L^\infty(\Omega)$ and $s \geq \frac{N}{\bar{q}}$.

First of all, we split the left-hand side of the last inequality into six terms

$$\begin{aligned}
& |f''(a + h)[a_1, a_2] - f''(a)[a_1, a_2]| \\
& = |\alpha(a_1, a_2)_{H^s(\Omega)} + (y_{1,h}', y_{2,h}')_{L^2(\Omega)} - (a_2\nabla y_{1,h}', \nabla p_h)_{(L^2(\Omega))^N} \\
& - (a_1\nabla y_{2,h}', \nabla p_h)_{(L^2(\Omega))^N} - \alpha(a_1, a_2)_{H^s(\Omega)} - (y_1', y_2')_{L^2(\Omega)} \\
& + (a_2\nabla y_1', \nabla p)_{(L^2(\Omega))^N} + (a_1\nabla y_2', \nabla p)_{(L^2(\Omega))^N}| \\
& \leq |(y_{1,h}', y_{2,h}' - y_2')_{L^2(\Omega)}| + |(y_{1,h}' - y_1', y_2')_{L^2(\Omega)}| \\
& + |(a_2\nabla y_{1,h}', \nabla(p_h - p))_{(L^2(\Omega))^N}| + |(a_2\nabla(y_{1,h}' - y_1'), \nabla p)_{(L^2(\Omega))^N}| \\
& + |(a_1\nabla y_{2,h}', \nabla(p_h - p))_{(L^2(\Omega))^N}| + |(a_1\nabla(y_{2,h}' - y_2'), \nabla p)_{(L^2(\Omega))^N}| \\
& = T_1 + T_2 + T_3 + T_4 + T_5 + T_6. \qquad (5.16)
\end{aligned}$$

We have to estimate each of these six terms by the right-hand side of the Lipschitz estimate (5.15). The terms T_1 and T_2 can be treated together, just as T_3 and T_5 and also T_4 and T_6.

Estimates

Let us now prove the remaining estimates. Therefore we first clarify our notation. The functions $y_h = S(a+h)$, p_h, $y_i' = S'(a)a_i$ and $y_{i,h}' = S'(a+h)a_i$ are defined to be the weak solutions of the following partial differential equations

$$-\nabla \cdot ((a+h)\nabla y_h) = g \tag{5.17}$$

$$-\nabla \cdot ((a+h)\nabla p_h) = y_h - y_d \tag{5.18}$$

$$-\nabla \cdot (a\nabla y_i') = \nabla \cdot (a_i \nabla y) \tag{5.19}$$

$$-\nabla \cdot ((a+h)\nabla y_{i,h}') = \nabla \cdot (a_i \nabla y_h), \tag{5.20}$$

each with homogeneous Dirichlet boundary conditions, $i \in \{1,2\}$. In the following proofs, we will use a generic constant c.

Lemma 5.9. *Let $s \geq \frac{N}{\bar{q}}$. Then, the estimates*

$$a) \qquad \|y_i'\|_{H^1(\Omega)} \leq c \cdot \|a_i\|_{H^s(\Omega)} \tag{5.21}$$

and

$$b) \qquad \|y_{i,h}'\|_{H^1(\Omega)} \leq c \cdot \|a_i\|_{H^s(\Omega)} \tag{5.22}$$

are satisfied for $i \in \{1,2\}$.

Proof. a) We consider the partial differential equations

$$-\nabla \cdot (a\nabla y_i') = \nabla \cdot (a_i \nabla y)$$

with the weak solutions y_i', $i \in \{1,2\}$. The weak formulation is given as

$$(a\nabla y_i', \nabla v)_{L^2(\Omega)} = -(a_i \nabla y, \nabla v)_{L^2(\Omega)} =: K(v).$$

We use the Lax-Milgram lemma to obtain

$$\begin{aligned}\|y_i'\|_{H^1(\Omega)} \leq c\|K(v)\|_{(H_0^1(\Omega))^*} &= c \cdot \sup_{v \in H_0^1(\Omega)} \frac{|(a_i \nabla y, \nabla v)_{L^2(\Omega)}|}{\|v\|_{H^1(\Omega)}} \\ &\leq c \cdot \sup_{v \in H_0^1(\Omega)} \frac{\|a_i \nabla y\|_{(L^2(\Omega))^N} \|\nabla v\|_{(L^2(\Omega))^N}}{\|v\|_{H^1(\Omega)}} \\ &\leq c \cdot \|a_i \nabla y\|_{(L^2(\Omega))^N}.\end{aligned}$$

In the next estimate, we use Hölder's inequality with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|y_i'\|_{H^1(\Omega)} \le c\|a_i\|_{L^{2p}(\Omega)} \cdot \|\nabla y\|_{(L^{2p'}(\Omega))^N}.$$

With the choice $2p' := \bar{q}$, the state y is uniformly bounded in $W^{1,\bar{q}}(\Omega)$ due to (5.3). This choice yields

$$p' = \frac{1}{1-1/p} \quad \Leftarrow \quad \bar{q} = 2p' = \frac{1}{1/2 - 1/2p} \quad \Leftrightarrow \quad 2p = \frac{1}{1/2 - 1/\bar{q}}$$

and with Sobolev's embedding theorem (2.14) we obtain $\|a_i\|_{L^{2p}(\Omega)} \le \|a_i\|_{H^s(\Omega)}$ for $s \ge \frac{N}{\bar{q}}$, and thus

$$\|y_i'\|_{H^1(\Omega)} \le c\|a_i\|_{H^s(\Omega)} \cdot \|y\|_{W^{1,\bar{q}}(\Omega)} \le c\|a_i\|_{H^s(\Omega)}.$$

b) The proof of the second estimate is very similar to the proof of the first one. We consider a slightly different equation, i.e.

$$-\nabla \cdot ((a+h)\nabla y_{i,h}') = \nabla \cdot (a_i \nabla y_h).$$

Now we can do the same procedure as before up to the point where we have to show uniform boundedness of $\|y_h\|_{W^{1,2p'}(\Omega)} =: \|y_h\|_{W^{1,\bar{q}}(\Omega)}$. There again, with (5.3) we obtain uniform boundedness of the states y_h for all $h \in L^\infty(\Omega)$. Thus we obtain also in this case for $s \ge \frac{N}{\bar{q}}$

$$\|y_{i,h}'\|_{H^1(\Omega)} \le c\|a_i\|_{H^s(\Omega)} \cdot \|y_h\|_{W^{1,\bar{q}}(\Omega)} \le c\|a_i\|_{H^s(\Omega)}.$$

□

Lemma 5.10. *The estimate*

$$\|y_h - y\|_{W^{1,\tilde{q}}(\Omega)} \le c\|h\|_{L^\infty(\Omega)} \tag{5.23}$$

holds true for arbitrary $\tilde{q} \in [2, \bar{q}]$.

Proof. Let us recall the PDEs for y_h and y respectively, i.e.

$$-\nabla \cdot ((a+h)\nabla y_h) = g$$
$$-\nabla \cdot (a\nabla y) = g.$$

We subtract them

$$-\nabla\cdot((a+h)\nabla y_h)+\nabla\cdot(a\nabla y)=g-g \Leftrightarrow -\nabla\cdot(a\nabla(y_h-y))=\nabla\cdot(h\nabla y_h)$$

and obtain from the weak formulation

$$(a\nabla(y_h-y),\nabla v)_{(L^2(\Omega))^N}=-(h\nabla y_h,\nabla v)_{(L^2(\Omega))^N}=:F(v)$$

for all $v\in H_0^1(\Omega)$ with (2.7) the following estimate,

$$\begin{aligned}\|y_h-y\|_{W^{1,\tilde q}(\Omega)} &\le c_{\tilde q}\|F(v)\|_{W^{-1,\tilde q}(\Omega)}\le c\|h\nabla y_h\|_{(L^{\tilde q}(\Omega))^N}\\ &\le c\|h\|_{L^\infty(\Omega)}\cdot\|\nabla y_h\|_{(L^{\tilde q}(\Omega))^N}\\ &\le c\|h\|_{L^\infty(\Omega)}\cdot\|y_h\|_{W^{1,\tilde q}(\Omega)}\le c\|h\|_{L^\infty(\Omega)}.\end{aligned}$$

The last estimate is valid because y_h is uniformly bounded in $W^{1,\tilde q}(\Omega)$ for $\tilde q\in[2,\bar q]$ for all $h\in L^\infty(\Omega)$, due to (5.3). □

Lemma 5.11. *The estimate*

$$\|y'_{i,h}-y'_i\|_{H^1(\Omega)}\le c\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)},\qquad i\in 1,2 \tag{5.24}$$

holds true for all $h\in L^\infty(\Omega)$, for $s\ge\frac{N}{\bar q}$.

Proof. Analogously to the proof of the previous lemma, we subtract the PDEs corresponding to $y'_{i,h}$ and y'_i and obtain

$$\begin{aligned}&-\nabla\cdot((a+h)\nabla y'_{i,h})+\nabla\cdot(a\nabla y'_i)=\nabla\cdot(a_i\nabla y_h)-\nabla\cdot(a_i\nabla y)\\ &\Leftrightarrow -\nabla\cdot(a\nabla(y'_{i,h}-y'_i))=\nabla\cdot(a_i\nabla(y_h-y))+\nabla\cdot(h\nabla y'_{i,h}).\end{aligned}$$

With the Lax-Milgram lemma we obtain as before

$$\|y'_{i,h}-y'_i\|_{H^1(\Omega)}\le c\left(\|a_i\nabla(y_h-y)\|_{(L^2(\Omega))^N}+\|h\nabla y'_{i,h}\|_{(L^2(\Omega))^N}\right)=:(*).$$

Next, we apply Hölder's inequality with $p,p'>1$, $\frac{1}{p}+\frac{1}{p'}=1$,

$$\begin{aligned}(*)&\le c\left(\|a_i\|_{L^{2p}(\Omega)}\|\nabla(y_h-y)\|_{(L^{2p'}(\Omega))^N}+\|h\|_{L^\infty(\Omega)}\|\nabla y'_{i,h}\|_{(L^2(\Omega))^N}\right)\\ &\le c\left(\|a_i\|_{H^s(\Omega)}\|y_h-y\|_{W^{1,2p'}(\Omega)}+\|h\|_{L^\infty(\Omega)}\|y'_{i,h}\|_{H^1(\Omega)}\right).\end{aligned}$$

In the last step we chose $2p':=\bar q\Leftrightarrow 2p=\frac{1}{1/2-1/\bar q}$ and applied again Sobolev's embedding theorem. Hence, we obtain the size condition for the regularization parameter s, $s \geq \frac{N}{\bar{q}}$. From lemma 5.10 and 5.9 we know how to estimate the terms $\|y_h - y\|_{W^{1,\bar{q}}(\Omega)}$ and $\|y'_{i,h}\|_{H^1(\Omega)}$ and obtain

$$\begin{aligned}\|y'_{i,h} - y'_i\|_{H^1(\Omega)} &\leq c\left(\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}\|a_i\|_{H^s(\Omega)}\right)\\ &= c\left(\|a_i\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}\right).\end{aligned}$$

□

Lemma 5.12. *The estimate*

$$\|p_h - p\|_{W^{1,\bar{q}}(\Omega)} \leq c\|h\|_{L^\infty(\Omega)} \tag{5.25}$$

is valid.

Proof. We start this proof again by subtracting the PDEs belonging to p and p_h

$$-\nabla \cdot (a\nabla(p_h - p)) = y_h - y + \nabla \cdot (h\nabla p_h).$$

We estimate $\|p_h - p\|_{W^{1,\bar{q}}(\Omega)}$ with (2.7)

$$\begin{aligned}\|p_h - p\|_{W^{1,\bar{q}}(\Omega)} &\leq c(\|y_h - y\|_{L^{\bar{q}}(\Omega)} + \|h\nabla p_h\|_{(L^{\bar{q}}(\Omega))^N})\\ &\leq c(\|y_h - y\|_{W^{1,\bar{q}}(\Omega)} + \|h\|_{L^\infty(\Omega)}\|\nabla p_h\|_{(L^{\bar{q}}(\Omega))^N})\\ &\leq c(\|y_h - y\|_{W^{1,\bar{q}}(\Omega)} + \|h\|_{L^\infty(\Omega)}\|p_h\|_{W^{1,\bar{q}}(\Omega)}).\end{aligned}$$

The term $\|y_h - y\|_{W^{1,\bar{q}}(\Omega)}$ is bounded by $\|h\|_{L^\infty(\Omega)}$ and the adjoint states p_h are uniformly bounded in $H^1(\Omega)$ for all $h \in L^\infty(\Omega)$ as a result of the Lax-Milgram Lemma. Thus, we obtain

$$\|p_h - p\|_{W^{1,\bar{q}}(\Omega)} \leq c\|h\|_{L^\infty(\Omega)}.$$

□

Now that these estimates are available, let us continue the proof of the Lipschitz estimate (5.15).

Lemma 5.13. *The terms T_1 and T_2 can be estimated in the following way for $s \geq \frac{N}{\bar{q}}$,*

$$|(y'_{1,h}, y'_{2,h} - y'_2)_{L^2(\Omega)}| \leq c\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}, \tag{5.26}$$

$$|(y'_{1,h} - y'_1, y'_2)_{L^2(\Omega)}| \leq c\|a_1\|_{H^s(\Omega)}\|a_2\|_{H^s(\Omega)}\|h\|_{L^\infty(\Omega)}. \tag{5.27}$$

Proof. We estimate both terms

$$|(y'_{1,h}, y'_{2,h} - y'_2)_{L^2(\Omega)}| \leq \|y'_{1,h}\|_{H^1(\Omega)} \|y'_{2,h} - y'_2\|_{H^1(\Omega)},$$
$$|(y'_{1,h} - y'_1, y'_2)_{L^2(\Omega)}| \leq \|y'_{1,h} - y'_1\|_{H^1(\Omega)} \|y'_2\|_{H^1(\Omega)}.$$

With estimates (5.21) and (5.22) from lemma 5.9 and estimate (5.24) from lemma 5.11, i.e.

$$\|y'_i\|_{H^1(\Omega)} \leq c\|a_i\|_{H^s(\Omega)}$$
$$\|y'_{i,h}\|_{H^1(\Omega)} \leq c\|a_i\|_{H^s(\Omega)}$$
$$\|y'_{i,h} - y'_i\|_{H^1(\Omega)} \leq c\|a_i\|_{H^s(\Omega)} \|h\|_{L^\infty(\Omega)}$$

for $i, j \in 1, 2,\ i \neq j$, we obtain the desired results straight away. □

Lemma 5.14. *The following estimates for T_3 and T_5 are valid for $i, j \in \{1, 2\}$, $i \neq j$ and $s \geq \frac{N}{\bar{q}}$*

$$|(a_i \nabla y'_{j,h}, \nabla(p_h - p))_{(L^2(\Omega))^N}| \leq \|a_i\|_{H^s(\Omega)} \|a_j\|_{H^s(\Omega)} \|h\|_{L^\infty(\Omega)}. \tag{5.28}$$

Proof. For $i, j \in \{1, 2\},\ i \neq j$ and $p, p' > 1, \frac{1}{p} + \frac{1}{p'} = 1$ we get with Hölder's inequality

$$|(a_i \nabla y'_{j,h}, \nabla(p_h - p))_{(L^2(\Omega))^N}| \leq \|a_i \nabla y'_{j,h}\|_{(L^p(\Omega))^N} \cdot \|\nabla(p_h - p)\|_{(L^{p'}(\Omega))^N} =: (*).$$

We set $p' := \bar{q}$ and thus $p = \frac{1}{1-1/\bar{q}}$ and obtain with $q, q' > 1, \frac{1}{q} + \frac{1}{q'} = 1$ and a further application of Hölder's inequality,

$$\begin{aligned}(*) &\leq \|a_i\|_{L^{pq}(\Omega)} \cdot \|\nabla y'_{j,h}\|_{(L^{pq'}(\Omega))^N} \cdot \|p_h - p\|_{W^{1,\bar{q}}(\Omega)} \\ &\leq \|a_i\|_{L^{pq}(\Omega)} \cdot \|y'_{j,h}\|_{W^{1,pq'}(\Omega)} \cdot \|p_h - p\|_{W^{1,\bar{q}}(\Omega)}.\end{aligned}$$

Now we set $pq' := 2$ and compute

$$q' = \frac{1}{1 - 1/\bar{q}} \quad \Leftrightarrow \quad 2 = pq' = \frac{1}{1/p - 1/pq}.$$

We know that $p = \frac{1}{1-1/\bar{q}}$ holds, because we set $p' = \bar{q}$, thus we obtain

$$\frac{1}{2} = \frac{1}{p} - \frac{1}{pq} \quad \Leftrightarrow \quad \frac{1}{2} = 1 - \frac{1}{\bar{q}} - \frac{1}{pq} \quad \Leftrightarrow \quad pq = \frac{1}{1/2 - 1/\bar{q}}.$$

Now again, we apply Sobolev's embedding theorem (2.14) and obtain for $s \geq \frac{N}{\bar{q}}$

$$\|a_i\|_{L^{pq}(\Omega)} \leq \|a_i\|_{H^s(\Omega)}.$$

We accomplish the proof by applying estimates (5.22) and (5.25). □

Lemma 5.15. *The terms T_4 and T_6 can be estimated as follows*

$$|(a_i \nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| \leq c \cdot \|a_i\|_{H^s(\Omega)} \cdot \|a_j\|_{H^s(\Omega)} \cdot \|h\|_{L^\infty(\Omega)}$$

with $i, j \in \{1, 2\}$, $i \neq j$.

Proof. For $i, j \in \{1, 2\}$, $i \neq j$ and $p, p' > 1, \frac{1}{p} + \frac{1}{p'} = 1$ we get

$$|(a_i \nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| \leq \|a_i \nabla(y'_{j,h} - y'_j)\|_{(L^p(\Omega))^N} \cdot \|\nabla p\|_{(L^{p'}(\Omega))^N}.$$

As before we set $p' := \bar{q}$. With $q, q' > 1, \frac{1}{q} + \frac{1}{q'} = 1$ we get

$$\begin{aligned} &|(a_i \nabla(y'_{j,h} - y'_j), \nabla p)_{(L^2(\Omega))^N}| \\ &\leq \|a_i\|_{L^{pq}(\Omega)} \cdot \|y'_{j,h} - y'_j\|_{(W^{1,pq'}(\Omega))^N} \cdot \|p\|_{W^{1,\bar{q}}(\Omega)}. \end{aligned}$$

Again we set $pq' := 2$, thus $pq = \frac{1}{1/2 - 1/\bar{q}}$ and apply the embedding theorem (2.14) with $s \geq \frac{N}{\bar{q}}$ for the first term, estimate (5.24) for the second term and uniform boundedness of p in $W^{1,\bar{q}}(\Omega)$ to complete the proof. □

Finally we have all components for the proof of theorem 5.8.

Proof. [Theorem 5.8] In lemmata 5.13 -5.15 we have shown how the different terms in (5.16) can be estimated. All in all this yields to

$$\begin{aligned} |f''(a+h)[a_1, a_1] - f''(a)[a_1, a_2]| &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 \\ &\leq c \cdot \|a_i\|_{H^s(\Omega)} \cdot \|a_j\|_{H^s(\Omega)} \cdot \|h\|_{L^\infty(\Omega)} \end{aligned}$$

for all $a,\ h,\ a_1,\ a_2\ \in L^\infty(\Omega)$. □

Now, we can write down and prove a second order sufficient condition. Therefore, let us introduce the following cone.

Definition 5.16. *The cone $C(\bar{a})$ is given as*

$$C(\bar{a}) := \{a \in H^s(\Omega) : a(x) \geq 0, \textit{if } \bar{a}(x) = a_{\min},\ a(x) \leq 0, \textit{if } \bar{a}(x) = a_{\max}\}. \tag{5.29}$$

Now, we can write down the following second order sufficient condition (abbr.: SSC): There exists a constant $\sigma > 0$ such that

$$f''(\bar{a})(a)^2 \geq \sigma \|a\|^2_{H^s(\Omega)} \tag{5.30}$$

holds for all $a \in C(\bar{a})$ with the corresponding solution $y \in W^{1,\bar{q}}(\Omega)$ of the linearized equation

$$\begin{aligned} -\nabla \cdot (\bar{a}\nabla y) &= \nabla \cdot (a\nabla \bar{y}) \quad && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma. \end{aligned} \tag{5.31}$$

In the next lemma we deduce a different SSC.

Lemma 5.17. *Let* (5.30) *hold for all* $a \in C(\bar{a})$ *with the corresponding states* $y \in W^{1,\bar{q}}(\Omega)$ *as above. Then, there exist* $\varepsilon > 0$ *and* $\sigma > 0$*, such that*

$$f''(\bar{a})(a-\bar{a})^2 \geq \sigma \|a-\bar{a}\|^2_{H^s(\Omega)} \tag{5.32}$$

holds for all $a \in A_{ad}$ *with corresponding state* $y = S(a)$*, provided that* $\|a - \bar{a}\|_{L^\infty(\Omega)} + \|y - \bar{y}\|_{H_0^1(\Omega)} \leq \varepsilon$.

Proof. We use lemma A.8, see Appendix, for this proof. Let (y, a) satisfy the state equation $-\nabla \cdot (a\nabla y) = g$. Of course, $(\bar{y}, \bar{a})$ fulfill the state equation too, i.e. $-\nabla \cdot (\bar{a}\nabla \bar{y}) = g$. Now, let $\hat{y}$ be the solution of the linearized equation belonging to $\hat{a} := a - \bar{a}$, thus

$$-\nabla \cdot (\bar{a}\nabla \hat{y}) = \nabla \cdot (\hat{a}\nabla \bar{y}).$$

We assume the SSC (5.30) to hold for all directions satisfying the linearized equation, thus it holds also for $(\hat{y}, \hat{a})$. Combining all equations above, we obtain for the error $\delta y := y - \bar{y} - \hat{y}$,

$$-\nabla \cdot (\bar{a}\nabla \delta y) = \nabla \cdot (\underbrace{(a - \bar{a} - \hat{a})}_{=0}\nabla y) - \nabla \cdot (\underbrace{\hat{a}}_{=a-\bar{a}}\nabla(\bar{y} - y))$$

and thus

$$\begin{aligned} \|\delta y\|_{H_0^1(\Omega)} &\leq \|(a-\bar{a})\nabla(\bar{y}-y)\|_{L^2(\Omega)} \leq \|a-\bar{a}\|_{L^\infty(\Omega)}\|\bar{y}-y\|_{H_0^1(\Omega)} \\ &\leq \varepsilon \left(\|\delta y\|_{H_0^1(\Omega)} + \|\hat{y}\|_{H_0^1(\Omega)} \right). \end{aligned}$$

Now, we choose ε sufficiently small, such that $1 - \varepsilon > 0$, then there holds

$$\|\delta y\|_{H_0^1(\Omega)} \leq \frac{\varepsilon}{1-\varepsilon}\|\hat{y}\|_{H_0^1(\Omega)}.$$

Now, we can apply lemma A.8 directly and finish the proof. □

Let us come to the main theorem of this chapter.

Theorem 5.18. *Let $s \geq \frac{N}{\bar{q}}$ and $\bar{a} \in A_{ad}$, the associated state $\bar{y} = S(\bar{a})$ and the adjoint state p fulfill the necessary condition* (5.10)*. If in addition $\bar{a}$ and $\bar{y}$ satisfy the second order sufficient condition*

$$f''(\bar{a})(a-\bar{a})^2 \geq \delta \|a-\bar{a}\|^2_{H^s(\Omega)}$$

for some constant $\delta > 0$, and for all $a \in A_{ad}$, with associated state $y = S(a)$, then there are constants $\varepsilon > 0$ and $\sigma > 0$, such that the quadratic condition for growth

$$f(a) \geq f(\bar{a}) + \sigma \|a-\bar{a}\|^2_{H^s(\Omega)}$$

holds for all $a \in A_{ad}$ with $\|a-\bar{a}\|_{L^\infty(\Omega)} \leq \varepsilon$ and the belonging state $y = S(a)$. Thus, $\bar{a}$ is a locally optimal parameter.

Proof. We develop the Taylor expansion up to the term of second order

$$f(a) = f(\bar{a}) + f'(\bar{a})(a-\bar{a}) + \frac{1}{2} f''(\bar{a}+\theta(a-\bar{a}))(a-\bar{a})^2$$

with $\theta \in (0,1)$. The first order term is nonnegative due to the necessary condition. We now estimate the second order term. Observe therefore, that the directions $a-\bar{a}$ lie in the cone $C(\bar{a})$ for all $a \in A_{ad}$.

$$\begin{aligned} f''(\bar{a}+\theta(a-\bar{a}))(a-\bar{a})^2 &= f''(\bar{a})(a-\bar{a})^2 + [f''(\bar{a}+\theta(a-\bar{a})) - f''(\bar{a})](a-\bar{a})^2 \\ &\geq \delta\|a-\bar{a}\|^2_{H^s(\Omega)} - L \cdot \|a-\bar{a}\|_{L^\infty(\Omega)} \|a-\bar{a}\|^2_{H^s(\Omega)} \\ &\geq \frac{\delta}{2} \|a-\bar{a}\|^2_{H^s(\Omega)}. \end{aligned}$$

This is valid if ε is sufficiently small, namely $\varepsilon \leq \frac{\delta}{2L}$. For these estimates we used (5.15) and the sufficient optimality condition (5.30). At last we obtain

$$f(a) \geq f(\bar{a}) + \frac{\delta}{4}\|a-\bar{a}\|^2_{H^s(\Omega)} = f(\bar{a}) + \sigma\|a-\bar{a}\|^2_{H^s(\Omega)}$$

with $\sigma = \delta/4$, if $\|a-\bar{a}\|^2_{L^\infty(\Omega)} \leq \varepsilon$ and $\varepsilon \leq \frac{\delta}{2L}$. □

5.2.4 Optimality conditions via the Lagrange functional

The Lagrange functional or Lagrangian is given as

$$\mathcal{L}(y,a,p) = J(y,a) - \int_\Omega a\nabla y \cdot \nabla p \, dx + \int_\Omega gp \, dx.$$

We see that the Lagrange multiplier p corresponds to the adjoint state. We verify

$$\mathcal{L}_y(\bar{y},\bar{a},\bar{p})y = \int_\Omega (\bar{y}-y_d)y\,dx - \int_\Omega \bar{a}\nabla y\cdot\nabla\bar{p}\,dx = 0 \quad \forall y\in H_0^1(\Omega),$$

$$\mathcal{L}_p(\bar{y},\bar{a},\bar{p})p = -\int_\Omega \bar{a}\nabla\bar{y}\cdot\nabla p\,dx + \int_\Omega gp\,dx = 0 \quad \forall p\in H_0^1(\Omega),$$

$$\mathcal{L}_a(\bar{y},\bar{a},\bar{p})(a-\bar{a}) = \alpha\int_\Omega B^{s/2}\bar{a}B^{s/2}(a-\bar{a}) - \nabla\bar{y}\cdot\nabla\bar{p}(a-\bar{a})dx \ge 0 \;\; \forall a\in A_{ad}.$$

The first equation is the weak formulation of the adjoint equation applied to y, the second equation is the weak formulation of the state equation applied to p and the third equation is the first order necessary optimality condition (5.10). The second order derivative of the Lagrangian provides an equivalent representation of the second order sufficient condition. Therefore, we have to write down the first and the second order derivative,

$$\begin{aligned}\mathcal{L}'(y,a,p)(y_1,a_1) = &\int_\Omega (y-y_d)y_1\;dx + \alpha\int_\Omega B^{s/2}aB^{s/2}a_1\;dx\\ &-\int_\Omega a\nabla y_1\cdot\nabla p\;dx - \int_\Omega a_1\nabla y\cdot\nabla p\;dx,\end{aligned}$$

$$\begin{aligned}\mathcal{L}''(y,a,p)[(y_1,a_1),(y_2,a_2)] = &\int_\Omega y_1y_2\,dx + \int_\Omega \alpha B^{s/2}a_1B^{s/2}a_2\,dx\\ &-\int_\Omega a_1\nabla y_2\cdot\nabla p\;dx - \int_\Omega a_2\nabla y_1\cdot\nabla p\;dx.\end{aligned}$$

This second order derivative is equal to the second order derivative of the reduced functional, i.e.

$$\mathcal{L}''(y,a,p)[(y_1,a_1),(y_2,a_2)] = f''(a)[a_1,a_2],$$

where y and p are the state and adjoint state corresponding to a and $y_i = S'(a)a_i$, $i=1,2$ are the solutions of the linearized equation

$$\begin{aligned}-\nabla\cdot(a\nabla y_i) &= \nabla\cdot(a_i\nabla y) &&\text{in } \Omega\\ y_i &= 0 &&\text{on } \Gamma.\end{aligned}\tag{5.33}$$

Let us finish this section with an equivalent representation of the second order sufficient condition.

Lemma 5.19. *The second order sufficient condition* (5.30) *can be equivalently expressed in terms of the Lagrangian in the following way. There exists a constant* $\sigma > 0$*, such that*

$$\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(y, a)^2 \geq \sigma \|a\|^2_{H^s(\Omega)} \tag{5.34}$$

for all $a \in C(\bar{a})$ *and all* $y \in H^1_0(\Omega)$ *satisfying*

$$\begin{aligned} -\nabla \cdot (\bar{a}\nabla y) &= \nabla \cdot (a \nabla \bar{y}) \quad && \textit{in } \Omega \\ y &= 0 && \textit{on } \Gamma. \end{aligned} \tag{5.35}$$

Chapter 6

Superlinear convergence of the SQP-method

In [28], for example, local quadratic convergence of the SQP-method is shown. In this chapter we follow the lead of this work. However, one has to be very careful with the adaptation to our problem, because a lot has to be done differently. Special care has to be brought to the choice of the spaces. Furthermore, the concept of the generalized equation with the occurring cones cannot be applied here because of the lack of Lagrange multipliers in function space. Another difference is, that in the end we will arrive at superlinear, not quadratic, convergence.

Let us start and shortly recall the nonlinear problem. Note that we use the operator based multilevel norm in the regularization term instead of the Sobolev-Slobodeckii norm for the space $H^s(\Omega)$.

Remark 6.1. *In chapter 4 we have shown the equivalence of the norms*

$$\|B^{s/2}v\|_{L^2(\mathcal{D})}^2 \sim \|v\|_{H^s(\mathcal{D})}^2. \tag{6.1}$$

In the following, we use the terms H^s-norm and multilevel norm equivalently and always refer to the multilevel norm, which replaces the H^s-norm in the remaining chapters.

The objective functional is then given as

$$\min J(y,a) = \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|B^{s/2}a\|_{L^2(\Omega)}^2.$$

The elliptic PDE is given as follows

$$\begin{aligned} -\nabla \cdot (a\nabla y) &= g \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma, \end{aligned}$$

with pointwise box-constraints

$$0 < a_{\min} \leq a(x) \leq a_{\max}.$$

For the numerical realization, we later choose Ω to be the unit square in $\mathbb{R}^2$ with boundary $\Gamma = \partial\Omega$, but for the proof of quadratic convergence of the SQP-method we let $\Omega \in \mathbb{R}^N$, $N = 2, 3$ be a domain with Lipschitz boundary. Let the general assumptions for the problem hold. Then, one finds the corresponding Lagrange functional

$$\mathcal{L}(y, a, p) = J(y, a) - \int_\Omega a\nabla y \cdot \nabla p \, dx + \int_\Omega g \, p \, dx$$

and the system of first-order necessary optimality conditions

$$(OS) \begin{cases} -\nabla \cdot (a\nabla y) = g, \\ -\nabla \cdot (a\nabla p) = y - y_d, \\ \langle \alpha B^s a - \nabla y \cdot \nabla p, \tilde{a} - a \rangle_{H^{s*}, H^s} \geq 0, \quad \forall \tilde{a} \in A_{ad} \\ a_{\min} \leq a(x) \leq a_{\max} \end{cases}$$

Let us define a Lagrange-multiplier-like quantity $\mu \in H^{-s}(\Omega)$ as

$$\mu := \alpha B^s a - \nabla y \cdot \nabla p. \tag{6.2}$$

For the parameter function there holds $a \in H^s(\Omega)$ and thus $B^s a \in H^{-s}(\Omega)$ and for the second term on the right-hand side there holds $\nabla y \cdot \nabla p \in L^p(\Omega)$ for some $p > 1$. We need to ask the question for which p an embedding $L^p(\Omega) \hookrightarrow H^{-s}(\Omega)$ holds true. This is the case if and only if the opposite direction holds for the dual spaces, i.e. $H^s(\Omega) \hookrightarrow L^{p'}(\Omega)$, with $\frac{1}{p} + \frac{1}{p'} = 1$. Via the Sobolev embedding theorem we see that this is the case for $p' \geq 2$ and $s - \frac{N}{2} = -\frac{N}{p'}$. Hence, we obtain that $L^p(\Omega) \hookrightarrow H^{-s}(\Omega)$ is fulfilled for $p \leq 2$ and $p \geq \frac{2N}{2s+N}$. Then, in two or three dimensions, $p \geq \frac{2N}{2s-N}$ is fulfilled for $s \in (0, 3/2)$. In order to play it safe, we can require $\nabla y \cdot \nabla p \in L^2(\Omega)$. As we pointed out in section 2.4, the regularities of the state y and the adjoint state p in smooth domains Ω depend on the quotient of the pointwise bounds $a_{\min}/a_{\max}$. For $a_{\min}/a_{\max} \to 0$, we have $\bar{q} \to 2$. Conversely, we have for $a_{\min}/a_{\max} \to 1$ that $\bar{q} \to \infty$. For $\bar{q} \geq 4$ we obtain y, $p \in W^{1,4}(\Omega)$ and thus the requirement $\nabla y \cdot \nabla p \in L^2(\Omega)$ is fulfilled.

With this definition of μ, we see

$$\langle \mu, \tilde{a} - a \rangle_{H^{s*}, H^s} \geq 0, \quad \forall \tilde{a} \in A_{ad}. \tag{6.3}$$

Assumption 6.2. *For the rest of this chapter we require the following:*

1. *Let the regularization parameter s fulfill*

$$s \geq \max\left\{ \frac{N}{\bar{q}}, \frac{N}{\bar{q}} - \frac{N}{2} + \frac{N}{\hat{q}}, \frac{N}{\bar{q}} + \frac{N}{2} - \frac{N}{\hat{q}} \right\}.$$

2. *Let $\bar{q} > 4$ hold.*

6.1 Lagrange-Newton-SQP-method

Assumption 6.3. *We assume that $(\bar{y}, \bar{a}) \in W^{1,\bar{q}}(\Omega) \times A_{ad}$ together with the associated adjoint state $p \in W^{1,\bar{q}}(\Omega)$ fulfill the first order necessary and second order sufficient conditions,* (5.10) *and* (5.30).

In each step of the SQP-method, one solves a linear quadratic subproblem. Its objective functional consists of the first derivative of the objective function of the initial problem evaluated at the current solution (y_k, a_k) in the direction of $(y - y_k, a - a_k)$ and the second order derivative of the Lagrangian evaluated at the same point applied to $(y - y_k, a - a_k)^2$

$$\min F(y, a) := J'(y_k, a_k)(y - y_k, a - a_k) + \frac{1}{2}\mathcal{L}''(y_k, a_k, p_k)(y - y_k, a - a_k)^2$$

Therefore, let us compute

$$\begin{aligned} &J'(y_k, a_k)(y - y_k, a - a_k) \\ &\qquad = \int_\Omega (y_k - y_d)(y - y_k)\, dx + \alpha \int_\Omega (B^{s/2} a_k)(B^{s/2}(a - a_k))\, dx \end{aligned}$$

and the first and second order derivative of the Lagrangian

$$\begin{aligned} \mathcal{L}'(y_k, a_k, p_k)(y - y_k, a - a_k) &= J'(y_k, a_k)(y - y_k, a - a_k) \\ &\quad - \int_\Omega a_k \nabla(y - y_k) \cdot \nabla p_k\, dx - \int_\Omega (a - a_k) \nabla y_k \cdot \nabla p_k\, dx, \end{aligned}$$

$$\mathcal{L}''(y_k, a_k, p_k)(y - y_k, a - a_k)^2 = \int_\Omega (y - y_k)^2 \, dx + \alpha \int_\Omega (B^{s/2}(a - a_k))^2 \, dx$$
$$- \int_\Omega (a - a_k)\nabla(y - y_k) \cdot \nabla p_k \, dx - \int_\Omega (a - a_k)\nabla(y - y_k) \cdot \nabla p_k \, dx.$$

Thus, we obtain the following quadratic functional:

$$F(y,a) = \Big\{ (y_k - y_d, y - y_k)_{L^2(\Omega)} + \alpha(B^{s/2}a_k, B^{s/2}(a - a_k))_{L^2(\Omega)}$$
$$- ((a - a_k), \nabla(y - y_k) \cdot \nabla p_k)_{L^2(\Omega)} + \frac{1}{2}\|y - y_k\|^2_{L^2(\Omega)}$$
$$+ \frac{\alpha}{2}\|B^{s/2}(a - a_k)\|^2_{L^2(\Omega)} \Big\}$$

The PDE of the linear quadratic subproblem is the linearization of the initial state equation. The linearization is done according to the pattern $F(y) \approx F(y_k) + F'(y_k)(y - y_k)$.

$$-\nabla \cdot (a_k \nabla y_k) - g - \nabla \cdot ((a - a_k)\nabla y_k) - \nabla \cdot (a_k \nabla(y - y_k)) = 0$$
$$\Leftrightarrow -\nabla \cdot (a \nabla y_k) - \nabla \cdot (a_k \nabla y) = g - \nabla \cdot (a_k \nabla y_k)$$

6.1.1 Linear-quadratic subproblem

The linear-quadratic subproblem in step k is the following,

$$(QP)_k \begin{cases} \min \quad F(y,a) = (y_k - y_d, y - y_k)_{L^2(\Omega)} \\ \qquad\qquad +\alpha(B^{s/2}a_k, B^{s/2}(a - a_k))_{L^2(\Omega)} \\ \qquad\qquad -((a - a_k), \nabla(y - y_k) \cdot \nabla p_k)_{L^2(\Omega)} \\ \qquad\qquad +\frac{1}{2}\|y - y_k\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|B^{s/2}(a - a_k)\|^2_{L^2(\Omega)} \\ \text{s. t.} \quad -\nabla \cdot (a_k \nabla y) = \nabla \cdot (a \nabla y_k) - \nabla \cdot (a_k \nabla y_k) + g \quad \text{in } \Omega \\ \qquad\qquad y = 0 \quad \text{on } \Gamma \\ \qquad 0 < a_{\min} \le a(x) \le a_{\max} \quad \text{a.e. in } \Omega. \end{cases}$$

The corresponding adjoint state $p \in W_0^{1,\bar{q}}(\Omega)$ is the unique solution of

$$-\nabla \cdot (a_k \nabla p) = \nabla \cdot (a \nabla p_k) - \nabla \cdot (a_k \nabla p_k) + y - y_d.$$

Let us define as before $\mu \in H^{-s}(\Omega)$ as

$$\mu := \alpha B^s a - \nabla y \cdot \nabla p_k - \nabla y_k \cdot \nabla p + \nabla y_k \cdot \nabla p_k. \tag{6.4}$$

Thus, the optimality system of $(QP)_k$ is then given as

$$(OS)_k \begin{cases} -\nabla \cdot (a_k \nabla y) = \nabla \cdot (a \nabla y_k) - \nabla \cdot (a_k \nabla y_k) + g, \\ -\nabla \cdot (a_k \nabla p) = \nabla \cdot (a \nabla p_k) - \nabla \cdot (a_k \nabla p_k) + y - y_d, \\ \alpha B^s a - \nabla y \cdot \nabla p_k - \nabla y_k \cdot \nabla p + \nabla y_k \cdot \nabla p_k - \mu = 0, \\ \langle \mu, \tilde{a} - a \rangle_{H^{-s}, H^s} \geq 0 \quad \forall \tilde{a} \in A_{ad}, \\ a_{\min} \leq a(x) \leq a_{\max}. \end{cases} \tag{6.5}$$

This optimality system can also be obtained via the formal Lagrange method, as in chapter 5.2.4. We have to ensure the existence of a unique solution in every step of the subproblem $(QP)_k$. This result will not be true in general for any point (y_k, a_k). Thus, we require the current iterate to be close enough to the optimal solution, i.e. $\|x_k - \bar{x}\|_{W^{1,\bar{q}}(\Omega) \times L^\infty(\Omega)} \leq r$ for some appropriate $r > 0$ with $x_k = (y_k, a_k)$ and $\bar{x} = (\bar{y}, \bar{a})$. At the end of this subsection we show, that this requirement is no restriction if we choose the starting point x_0 to be close enough to $\bar{x}$.

The following procedure is based on [28]. At first we show an auxiliary result.

Lemma 6.4. *There exist constants $r > 0$ and $\delta' > 0$ such that*

$$\mathcal{L}''(y_k, a_k, p_k)[(y, a), (y, a)] \geq \delta' \|a\|^2_{H^s(\Omega)} \tag{6.6}$$

holds true for all $(y, a) \in H^1(\Omega) \times H^s(\Omega)$ that fulfill

$$\begin{aligned} -\nabla \cdot (a_k \nabla y) &= \nabla \cdot (a \nabla y_k) \quad \text{in } \Omega \\ y &= 0 \qquad\qquad\quad \text{on } \Gamma, \end{aligned} \tag{6.7}$$

given that

$$\|y_k - \bar{y}\|_{W^{1,\bar{q}}(\Omega)} + \|a_k - \bar{a}\|_{L^\infty(\Omega)} \leq r. \tag{6.8}$$

Proof. Let us start estimating the left-hand side of (6.6),

$$\begin{aligned} &\mathcal{L}''(y_k, a_k, p_k)[(y, a), (y, a)] \\ &= \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})[(y, a), (y, a)] - [\mathcal{L}''(\bar{y}, \bar{a}, \bar{p}) - \mathcal{L}''(y_k, a_k, p_k)] [(y, a), (y, a)] \\ &\geq \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})[(y, a), (y, a)] - \|a_k - \bar{a}\|_{L^\infty(\Omega)} \|a\|^2_{H^s(\Omega)} \\ &\geq \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})[(y, a), (y, a)] - r \|a\|^2_{H^s(\Omega)}. \end{aligned}$$

Here, we used Lipschitz-estimate (5.15) and assumption (6.8). It remains to estimate $\mathcal{L}''(\bar{y},\bar{a},\bar{p})[(y,a),(y,a)]$ suitably from below. Unfortunately, the second order sufficient condition that we derived in chapter 5.2.3 does not imply this term. The directions (y,a) do not fulfill the linearized equation (5.35), which is mandatory for the second order sufficient condition to hold. Therefore, we want to apply lemma A.8. Let (y,a) be a solution of the following equation

$$\begin{aligned} -\nabla\cdot(a_k\nabla y) &= \nabla\cdot(a\nabla y_k) \quad &&\text{in } \Omega \\ y &= 0 &&\text{on } \Gamma. \end{aligned} \tag{6.9}$$

In chapter 5 we developed the second order sufficient condition that is fulfilled for solutions of the linearized equation (5.35). Let $\hat{y}$ be the solution of the linearized equation belonging to the same parameter function a as above, i.e. $(\hat{y},a)$ shall fulfill

$$\begin{aligned} -\nabla\cdot(\bar{a}\nabla\hat{y}) &= \nabla\cdot(a\nabla\bar{y}) \quad &&\text{in } \Omega \\ \hat{y} &= 0 &&\text{on } \Gamma. \end{aligned} \tag{6.10}$$

Then, we know

$$\mathcal{L}''(\bar{y},\bar{a},\bar{p})[(\hat{y},a),(\hat{y},a)] \geq c\|a\|^2_{H^s(\Omega)}.$$

Combining (6.9) and (6.10), we obtain

$$-\nabla\cdot(a_k\nabla(y-\hat{y})) = \nabla\cdot(a\nabla y_k) + \nabla\cdot(a_k\nabla\hat{y}) - \nabla\cdot(\bar{a}\nabla\hat{y}) - \nabla\cdot(a\nabla\bar{y}).$$

Now, the lemma of Lax-Milgram yields

$$\begin{aligned} \|y-\hat{y}\|_{H^1_0(\Omega)} &\leq \|a\nabla(y_k-\bar{y}) + (a_k-\bar{a})\nabla\hat{y}\|_{L^2(\Omega)} \\ &\leq \|y_k-\bar{y}\|_{W^{1,\bar{q}}(\Omega)}\|a\|_{H^s(\Omega)} + \|a_k-\bar{a}\|_{L^\infty(\Omega)}\|\hat{y}\|_{H^1_0(\Omega)} \\ &\leq r\|a\|_{H^s(\Omega)} + r\|\hat{y}\|_{H^1_0(\Omega)}, \end{aligned}$$

if $\|y_k-\bar{y}\|_{W^{1,\bar{q}}(\Omega)}$ and $\|a_k-\bar{a}\|_{L^\infty(\Omega)}$ are small enough. Thus, we can apply lemma A.8, which finishes the proof. □

Lemma 6.5. *There exists a constant $r>0$, such that problem $(QP)_k$ possesses a unique solution, provided that $\|y_k-\bar{y}\|_{W^{1,\bar{q}}(\Omega)} + \|a_k-\bar{a}\|_{L^\infty(\Omega)} \leq r$ holds.*

Proof. The feasible set

$$M^k := \{(y,a) \in H_0^1(\Omega) \times H^s(\Omega) : 0 < a_{\min} \le a(x) \le a_{\max},$$
$$-\nabla \cdot (a_k \nabla y) = \nabla \cdot (a \nabla y_k) - \nabla \cdot (a_k \nabla y_k) + g\}$$

is non-empty, closed and convex. Due to lemma 6.4, for the quadratic part of the objective functional there exist $r > 0$ and $\delta' > 0$, such that

$$\mathcal{L}''(y_k, a_k, p_k)[(y,a),(y,a)] \ge \delta' \|a\|^2_{H^s(\Omega)}$$

holds true for all $(y,a) \in H^1(\Omega) \times H^s(\Omega)$ that fulfill

$$\begin{aligned} -\nabla \cdot (a_k \nabla y) &= \nabla \cdot (a \nabla y_k) && \text{in } \Omega \\ y &= 0 && \text{on } \Gamma, \end{aligned}$$

provided that $\|y_k - \bar{y}\|_{W^{1,\bar{q}}(\Omega)} + \|a_k - \bar{a}\|_{L^\infty(\Omega)} \le r$. Thus, the objective functional is uniformly convex and continuous. Hence, the problem $(QP)_k$ possesses a unique solution. □

In the next lemma we show that in any step of the Lagrange-Newton SQP method, the solution of $(OS)_k$ is uniformly bounded with respect to a particular norm, provided that the starting point is uniformly bounded with respect to the same norm. In the following we use c as a generic constant.

Lemma 6.6. *Given a starting point $x_0 = (y_0, a_0)$ with the corresponding adjoint state p_0 that is uniformly bounded in $W^{1,\bar{q}}(\Omega) \times L^\infty(\Omega) \times W^{1,\bar{q}}(\Omega)$, i.e.*

$$\|y_0\|_{W^{1,\bar{q}}(\Omega)} + \|a_0\|_{L^\infty(\Omega)} + \|p_0\|_{W^{1,\bar{q}}(\Omega)} \le r_0,$$

for some appropriately chosen $r_0 > 0$. Then, there exists a constant $r > 0$, such that for every solution (y,a,p) of the optimality system $(OS)_k$ there holds

$$\|y\|_{W^{1,\bar{q}}(\Omega)} + \|a\|_{L^\infty(\Omega)} + \|p\|_{W^{1,\bar{q}}(\Omega)} \le r. \tag{6.11}$$

Proof. 1. *Estimation of the parameter function a:* The uniform boundedness of a with respect to the L^∞-norm is a direct consequence of the box constraints, i.e.

$$\|a\|_{L^\infty(\Omega)} \le a_{\max} - a_{\min}.$$

Naturally, this holds true for every optimal parameter function in every step of the iteration.

2. *Estimation of the state y:* We proof the uniform boundedness of y in $W^{1,\bar{q}}(\Omega)$ by induction. For the first state y_0 we require a uniform boundedness to hold. Let us assume that we already showed the boundedness condition for y_k, i.e.

$$\|y_k\|_{W^{1,\bar{q}}(\Omega)} \leq K, \tag{6.12}$$

for some constant $K > 0$. Let us have a look at the linearized state equation of the quadratic subproblem,

$$-\nabla \cdot (a_k \nabla y) = \nabla \cdot (a \nabla y_k) - \nabla \cdot (a_k \nabla y_k) + g.$$

We can estimate the state y with the Gröger estimate, analogously to 2.7,

$$\begin{aligned}\|y\|_{W^{1,\bar{q}}(\Omega)} \leq c \big(&\|\nabla \cdot (a \nabla y_k)\|_{W^{-1,\bar{q}}(\Omega)} + \|\nabla \cdot (a_k \nabla y_k)\|_{W^{-1,\bar{q}}(\Omega)} \\ &+ \|g\|_{W^{-1,\bar{q}}(\Omega)}\big).\end{aligned}$$

This is consistent because $g, a, a_k \in L^\infty(\Omega)$, and because of the assumption $y_k \in W^{1,\bar{q}}(\Omega)$. Thus, we can estimate with $\frac{1}{\bar{q}} + \frac{1}{p} = 1$

$$\begin{aligned}\|y\|_{W^{1,\bar{q}}(\Omega)} &\leq c \left(\sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{|(a \nabla y_k, \nabla v)_{L^2(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} \right. \\ &\left. + \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{|(a_k \nabla y_k, \nabla v)_{L^2(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} + \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{|(g, v)_{L^2(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} \right) \\ &\leq c \left(\sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{\|a \nabla y_k\|_{L^{\bar{q}}(\Omega)} \|\nabla v\|_{L^p(\Omega)}}{\|v\|_{W^{1,p}(\Omega)}} \right. \\ &\left. + \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{\|a_k \nabla y_k\|_{L^{\bar{q}}(\Omega)} \|\nabla v\|_{L^p(\Omega)}}{\|v\|_{W^{1,p}(\Omega)}} + \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{\|g\|_{L^\infty(\Omega)} \|v\|_{L^p(\Omega)}}{\|v\|_{W^{1,p}(\Omega)}} \right) \\ &\leq c \left(\|a \nabla y_k\|_{L^{\bar{q}}(\Omega)} + \|a_k \nabla y_k\|_{L^{\bar{q}}(\Omega)} + \|g\|_{L^\infty(\Omega)} \right) \\ &\leq c \left(\|a\|_{L^\infty(\Omega)} \|y_k\|_{W^{1,\bar{q}}(\Omega)} + \|a\|_{L^\infty(\Omega)} \|y_k\|_{W^{1,\bar{q}}(\Omega)} + \|g\|_{L^\infty(\Omega)} \right) \\ &\leq c.\end{aligned}$$

The last estimate is valid due to assumption (6.12) and due to $g, a \in L^\infty(\Omega)$. Thus, y is uniformly bounded in $W^{1,\bar{q}}(\Omega)$.

3. *Estimation of the adjoint state p:* This part of the proof is done by the same induction as for the state y. We start the iteration with an adjoint state p_0 that is uniformly bounded. The next step is that we assume, that we showed uniform boundedness already for p_k, i.e.

$$\|p_k\|_{W^{1,\bar{q}}(\Omega)} \leq K', \tag{6.13}$$

for some constant $K' > 0$. Then we estimate the linearized adjoint equation as follows:

$$\begin{aligned}\|p\|_{W^{1,\bar{q}}(\Omega)} \leq c \big(&\|\nabla \cdot (a \nabla p_k)\|_{W^{-1,\bar{q}}(\Omega)} + \|\nabla \cdot (a_k \nabla p_k)\|_{W^{-1,\bar{q}}(\Omega)} \\ &+ \|y - y_d\|_{W^{-1,\bar{q}}(\Omega)} \big) .\end{aligned}$$

The first and second term on the right hand side are estimated in the same way as above, where we use (6.13). The third term can be handled easily, too.

$$\begin{aligned}\|y - y_d\|_{W^{-1,\bar{q}}(\Omega)} &\leq \|y\|_{W^{-1,\bar{q}}(\Omega)} + \|y_d\|_{W^{-1,\bar{q}}(\Omega)} \\ &\leq \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{|(y,v)_{L^2(\Omega)}|}{\|v\|_{W^{1,p}(\Omega)}} + \|y_d\|_{L^\infty(\Omega)} \leq \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\| \neq 0}} \frac{\|y\|_{L^{\bar{q}}(\Omega)} \|v\|_{L^p(\Omega)}}{\|v\|_{W^{1,p}(\Omega)}} + c \\ &\leq \|y\|_{W^{1,\bar{q}}(\Omega)} + c \leq c.\end{aligned}$$

Here, we used the uniform boundedness of y that we showed above. Thus, we obtain uniform boundedness of the adjoint state p in $W^{1,\bar{q}}(\Omega)$.

All together, there hence exists a constant $r > 0$, such that

$$\|y\|_{W^{1,\bar{q}}(\Omega)} + \|a\|_{L^\infty(\Omega)} + \|p\|_{W^{1,\bar{q}}(\Omega)} \leq r.$$

□

Remark 6.7. *The result* (6.11) *is also valid in* $W^{1,\hat{q}}(\Omega) \times L^\infty(\Omega) \times W^{1,\hat{q}}(\Omega)$, *for* $\hat{p} \in (2, \bar{q}]$.

6.1.2 Auxiliary linear-quadratic problem

Let us define the following spaces,

$$W := W_0^{1,\hat{q}}(\Omega) \times H^s(\Omega) \times W_0^{1,\hat{q}}(\Omega) \times H^{-s}(\Omega)$$

$$\text{and} \quad Z := W^{-1,\hat{q}}(\Omega) \times L^2(\Omega) \times W^{-1,\hat{q}}(\Omega),$$

for a $\hat{q} \in (2, \bar{q})$. Let $\bar{w} = (\bar{y}, \bar{a}, \bar{p}, \bar{\mu}) \in W$ fulfill the optimality system (OS) and the second order sufficient condition (5.30), i.e.

$$f''(\bar{a})a^2 \geq \sigma \|a\|^2_{H^s(\Omega)},$$

for all $a \in C(\bar{a})$, together with the corresponding solution y of the linearized equation $-\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla \bar{y})$, where $C(\bar{a})$ is given as the cone $C(\bar{a}) := \{a \in H^s(\Omega) : a(x) \geq 0, \text{if } \bar{a}(x) = a_{\min},\ a(x) \leq 0, \text{if } \bar{a}(x) = a_{\max}\}$, see (5.29).

Let us introduce an auxiliary quantity $\delta = (\delta_1, \delta_2, \delta_3) \in Z$, where δ_i, $i = 1, 2, 3$ are supposed to be perturbations in the following linear quadratic problem $(QP)(\delta)$,

$$(QP)(\delta) \begin{cases} \min \quad F(y,a) = (\bar{y} - y_d, y - \bar{y})_{L^2(\Omega)} + \alpha(B^{s/2}\bar{a}, B^{s/2}(a - \bar{a}))_{L^2(\Omega)} \\ \qquad\qquad -((a-\bar{a}), \nabla(y - \bar{y}) \cdot \nabla \bar{p})_{L^2(\Omega)} \\ \qquad\qquad -\langle \delta_1, y - \bar{y}\rangle_{W^{-1,\hat{q}},W^{1,\hat{q}}} - (\delta_2, a - \bar{a})_{L^2(\Omega)} \\ \qquad\qquad +\frac{1}{2}\|y - \bar{y}\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|B^{s/2}(a - \bar{a})\|^2_{L^2(\Omega)} \\ \text{s. t.} \quad -\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla \bar{y}) - \nabla \cdot (\bar{a}\nabla \bar{y}) + g + \delta_3 \quad \text{in } \Omega \\ \qquad\qquad y = 0 \quad \text{on } \Gamma \\ \qquad 0 < a_{\min} \leq a(x) \leq a_{\max} \quad \text{a.e. in } \Omega. \end{cases}$$

The corresponding system of first order necessary conditions reads as

$$(OS)(\delta) \begin{cases} -\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla \bar{y}) - \nabla \cdot (\bar{a}\nabla \bar{y}) + g + \delta_3, \\ -\nabla \cdot (\bar{a}\nabla p) = \nabla \cdot (a\nabla \bar{p}) - \nabla \cdot (\bar{a}\nabla \bar{p}) + y - y_d - \delta_1, \\ \alpha B^s a - \nabla y \cdot \nabla \bar{p} - \nabla \bar{y} \cdot \nabla p + \nabla \bar{y} \cdot \nabla \bar{p} - \delta_2 - \mu = 0, \\ \langle \mu, \tilde{a} - a\rangle_{H^{-s},H^s} \geq 0 \quad \forall \tilde{a} \in A_{ad}, \\ a_{\min} \leq a(x) \leq a_{\max}. \end{cases}$$

Here, μ is defined as in (6.4). A comparison of the systems (OS) and $(OS)(\delta)$ yields a further representation of the perturbations δ_i, $i = 1, \ldots, 3$,

$$\delta_3 = -\nabla \cdot ((\bar{a} - a_k)\nabla y) - \nabla \cdot (a\nabla(\bar{y} - y_k)) + \nabla \cdot (\bar{a}\nabla \bar{y}) - \nabla \cdot (a_k \nabla y_k), \tag{6.14}$$

$$\delta_1 = -\nabla \cdot ((a_k - \bar{a})\nabla p) - \nabla \cdot (a\nabla(p_k - \bar{p})) + \nabla \cdot (a_k \nabla p_k) - \nabla \cdot (\bar{a}\nabla \bar{p}), \tag{6.15}$$

$$\delta_2 = \nabla y \cdot \nabla(p_k - \bar{p}) + \nabla(y_k - \bar{y})\nabla p - \nabla y_k \cdot \nabla p_k + \nabla \bar{y} \cdot \nabla \bar{p}. \tag{6.16}$$

The perturbations play an important role in the proof of the convergence result at the end of this chapter. In preparation of this, let us show two estimates subject to δ, one from above and one from below. The proofs of these estimates are done in two very different ways, they are using either one of the above representations of δ. Now, let us start with an existence result for problem $(QP)(\delta)$.

Lemma 6.8. *The problem $(QP)(\delta)$ possesses a unique solution.*

Proof. The objective functional of $(QP)(\delta)$ is continuous and uniformly convex due to the second order sufficient condition. Thus, it is radially unbounded, too. Let us be more precise and show the uniform convexity of the objective. We do this for the reduced functional and denote the parameter-to-state operator by S_δ. Let us have a look at the first and second order derivatives of the objective functional F.

$$\begin{aligned} F'(S_\delta(a), a)h = & (\bar{y} - y_d, S_\delta'(a)h)_{L^2(\Omega)} + \alpha(B^{s/2}\bar{a}, B^{s/2}h)_{L^2(\Omega)} \\ & - (h, \nabla(y - \bar{y}) \cdot \nabla\bar{p})_{L^2(\Omega)} - ((a - \bar{a}), \nabla S_\delta'(a)h \cdot \nabla\bar{p})_{L^2(\Omega)} \\ & - \langle \delta_1, S_\delta'(a)h \rangle_{W^{-1,\hat{q}},W^{1,\hat{q}}} - (\delta_2, h)_{L^2(\Omega)} + (S_\delta'(a)h, y - \bar{y})_{L^2(\Omega)} \\ & + \alpha(B^{s/2}h, B^{s/2}(a - \bar{a}))_{L^2(\Omega)}. \end{aligned}$$

Because S_δ is linear we get

$$\begin{aligned} F''(S_\delta(a), a)h^2 = & \underbrace{(\bar{y} - y_d, S_\delta''(a)h^2)_{L^2(\Omega)}}_{=0} - 2(h, \nabla S_\delta'(a)h \cdot \nabla\bar{p})_{L^2(\Omega)} \\ & - \underbrace{\langle \delta_1, S_\delta''(a)h^2 \rangle_{W^{-1,\hat{q}},W^{1,\hat{q}}}}_{=0} + \underbrace{(S_\delta''(a)h^2, y - \bar{y})_{L^2(\Omega)}}_{=0} \\ & + (S_\delta'(a)h, S_\delta'(a)h)_{L^2(\Omega)} + \alpha(B^{s/2}h, B^{s/2}h)_{L^2(\Omega)} \\ = & -2(h, \nabla S_\delta'(a)h\nabla\bar{p})_{L^2(\Omega)} + (S_\delta'(a)h, S_\delta'(a)h)_{L^2(\Omega)} + \alpha(B^{s/2}h, B^{s/2}h)_{L^2(\Omega)}. \end{aligned}$$

We see that $(S_\delta'(a)h, a)$ is the solution of

$$-\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla\bar{y}),$$

which is the linearization of the perturbed linearized equation $-\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla\bar{y}) - \nabla \cdot (\bar{a}\nabla\bar{y}) + g + \delta_3$. Thus, the above is exactly the second order condition, (5.34). Hence, there holds

$$F''(S_\delta(a), a)h^2 \geq \kappa\|a\|^2_{H^s(\Omega)},$$

i.e. the objective functional is uniformly convex. Furthermore, the admissible set

$$M^\delta := \{(y,a) \in H_0^1(\Omega) \times H^s(\Omega) : 0 < a_{\min} \leq a(x) \leq a_{\max},$$
$$-\nabla \cdot (\bar{a}\nabla y) = \nabla \cdot (a\nabla \bar{y}) - \nabla \cdot (\bar{a}\nabla \bar{y}) + g + \delta_3\}$$

is non-empty, closed and convex. Thus, there exists a unique solution of the linear quadratic problem $(QP)(\delta)$, which follows from standard arguments. □

Remark 6.9. *The solution of* $(QP)(\delta)$ *is uniformly bounded in* $W^{1,\hat{q}}(\Omega) \times L^\infty(\Omega) \times W^{1,\hat{q}}(\Omega)$ *for* $\hat{q} \in (2, \bar{q})$ *and there exists a constant* $r > 0$*, such that*

$$\|y_\delta\|_{W^{1,\hat{q}}(\Omega)} + \|a_\delta\|_{L^\infty(\Omega)} + \|p_\delta\|_{W^{1,\hat{q}}(\Omega)} \leq r. \tag{6.17}$$

We can adapt the proof of lemma 6.6, as $\delta_1, \delta_3 \in W^{-1,\hat{q}}(\Omega)$.

6.2 Stability of $(QP)(\delta)$ with respect to perturbations

The first important step on the way to show superlinear convergence of the Lagrange-Newton-SQP-method is to show stability of the $(QP)(\delta)$ with respect to perturbations δ. This will be the estimate from below of the perturbations δ.

Theorem 6.10. *Let* $w_\delta = (y_\delta, a_\delta, p_\delta, \mu_\delta)$ *be the solution of* $(QP)(\delta)$ *with respect to* $\delta = (\delta_1, \delta_2, \delta_3) \in Z$ *and* $w_{\delta'} = (y_{\delta'}, a_{\delta'}, p_{\delta'}, \mu_{\delta'})$ *be the solution of* $(QP)(\delta)$ *with respect to a perturbation* $\delta' = (\delta_1', \delta_2', \delta_3') \in Z$*. Then, there exists a constant* $L > 0$*, such that*

$$\|w_\delta - w_{\delta'}\|_W \leq L\,\|\delta - \delta'\|_Z \tag{6.18}$$

holds for all $\delta, \delta' \in Z$*.*

Proof. Let $\delta w = (\delta y, \delta a, \delta p, \delta \mu)$ denote the difference $w_\delta - w_{\delta'}$.

1. *Assembly of equations for* $w_\delta - w_{\delta'}$*:* We write down the corresponding equations and start with δy. Let us consider the weak perturbed lineari-

zed state equation for δ and δ' with δp as test function.

$$\begin{aligned}
(\bar{a}\nabla y_\delta, \nabla\delta p)_{(L^2(\Omega))^N} &= -(a_\delta\nabla\bar{y}, \nabla\delta p)_{(L^2(\Omega))^N} + (\bar{a}\nabla\bar{y}, \nabla\delta p)_{(L^2(\Omega))^N} \\
&\quad + (g, \delta p)_{L^2(\Omega)} + \langle\delta_3, \delta p\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}} \\
(\bar{a}\nabla y_{\delta'}, \nabla\delta p)_{(L^2(\Omega))^N} &= -(a_{\delta'}\nabla\bar{y}, \nabla\delta p)_{(L^2(\Omega))^N} + (\bar{a}\nabla\bar{y}, \nabla\delta p)_{(L^2(\Omega))^N} \\
&\quad + (g, \delta p)_{L^2(\Omega)} + \langle\delta_3', \delta p\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}} \\
\Rightarrow (\bar{a}\nabla\delta y, \nabla\delta p)_{(L^2(\Omega))^N} &= -(\delta a\nabla\bar{y}, \nabla\delta p)_{(L^2(\Omega))^N} \\
&\quad + \langle\delta_3 - \delta_3', \delta p\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}}. \qquad (6.19)
\end{aligned}$$

We find the analog for the adjoint equation by using δy as test function.

$$\begin{aligned}
(\bar{a}\nabla p_\delta, \nabla\delta y)_{(L^2(\Omega))^N} &= -(a_\delta\nabla\bar{p}, \nabla\delta y)_{(L^2(\Omega))^N} + (\bar{a}\nabla\bar{p}, \nabla\delta y)_{(L^2(\Omega))^N} \\
&\quad + (y_\delta, \delta y)_{L^2(\Omega)} - (y_d, \delta y)_{L^2(\Omega)} + \langle\delta_1, \delta y\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}} \\
(\bar{a}\nabla p_{\delta'}, \nabla\delta y)_{(L^2(\Omega))^N} &= -(a_{\delta'}\nabla\bar{p}, \nabla\delta y)_{(L^2(\Omega))^N} + (\bar{a}\nabla\bar{p}, \nabla\delta y)_{(L^2(\Omega))^N} \\
&\quad + (y_\delta', \delta y)_{L^2(\Omega)} - (y_d, \delta y)_{L^2(\Omega)} + \langle\delta_1', \delta y\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}} \\
\Rightarrow (\bar{a}\nabla\delta p, \nabla\delta y)_{(L^2(\Omega))^N} &= -(\delta a\nabla\bar{p}, \nabla\delta y)_{(L^2(\Omega))^N} + (\delta y, \delta y)_{L^2(\Omega)} \\
&\quad + \langle\delta_1 - \delta_1', \delta y\rangle_{(W_0^{1,\hat{q}})^*, W_0^{1,\hat{q}}}. \qquad (6.20)
\end{aligned}$$

Finally, for the variational equation we obtain with δa as test function,

$$\begin{aligned}
\alpha(B^{s/2}a_\delta, B^{s/2}\delta a)_{L^2(\Omega)} - (\nabla\bar{y}\cdot\nabla p_\delta, \delta a)_{L^2(\Omega)} - (\nabla y_\delta\cdot\nabla\bar{p}, \delta a)_{L^2(\Omega)} \\
+(\nabla\bar{y}\cdot\nabla\bar{p}, \delta a)_{L^2(\Omega)} + \langle -\mu_\delta, \delta a\rangle_{H^{s*}, H^s} - (\delta_2, \delta a)_{L^2(\Omega)} = 0 \\
\alpha(B^{s/2}a_{\delta'}, B^{s/2}\delta a)_{L^2(\Omega)} - (\nabla\bar{y}\cdot\nabla p_{\delta'}, \delta a)_{L^2(\Omega)} - (\nabla y_{\delta'}\cdot\nabla\bar{p}, \delta a)_{L^2(\Omega)} \\
+(\nabla\bar{y}\cdot\nabla\bar{p}, \delta a)_{L^2(\Omega)} + \langle -\mu_{\delta'}, \delta a\rangle_{H^{s*}, H^s} - (\delta_2', \delta a)_{L^2(\Omega)} = 0 \\
\Rightarrow \alpha(B^{s/2}\delta a, B^{s/2}\delta a)_{L^2(\Omega)} - (\nabla\bar{y}\cdot\nabla\delta p, \delta a)_{L^2(\Omega)} - (\nabla\delta y\cdot\nabla\bar{p}, \delta a)_{L^2(\Omega)} \\
+\langle -\delta\mu, \delta a\rangle_{H^{s*}, H^s} - (\delta_2 - \delta_2', \delta a)_{L^2(\Omega)} = 0. \\
(6.21)
\end{aligned}$$

Combining these three equations, we obtain after a little computation

$$
\begin{aligned}
&\|\delta y\|^2_{L^2(\Omega)} + \alpha\|B^{s/2}\delta a\|^2_{L^2(\Omega)} - 2(\delta a\nabla\delta y, \nabla\bar{p})_{(L^2(\Omega))^N} + \langle -\delta\mu, \delta a\rangle_{H^{s*},H^s}\\
&= \langle \delta_3 - \delta_3', \delta p\rangle_{(W_0^{1,\hat{q}})^*,W_0^{1,\hat{q}}} - \langle \delta_1 - \delta_1', \delta y\rangle_{(W_0^{1,\hat{q}})^*,W_0^{1,\hat{q}}} + (\delta_2 - \delta_2', \delta a)_{L^2(\Omega)}\\
&\le \|\delta_3 - \delta_3'\|_{(W_0^{1,\hat{q}}(\Omega))^*}\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} + \|\delta_1 - \delta_1'\|_{(W_0^{1,\hat{q}}(\Omega))^*}\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)}\\
&+ \|\delta_2 - \delta_2'\|_{L^2(\Omega)}\|\delta a\|_{H^s(\Omega)}\\
&\le \|\delta_3 - \delta_3'\|_{W^{-1,\hat{q}}(\Omega)}\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} + \|\delta_1 - \delta_1'\|_{W^{-1,\hat{q}}(\Omega)}\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)}\\
&+ \|\delta_2 - \delta_2'\|_{L^2(\Omega)}\|\delta a\|_{H^s(\Omega)}.
\end{aligned} \tag{6.22}
$$

The term on the left-hand side depending on μ can be estimated using the fourth line of the optimality system $(OS)(\delta)$.

$$
\begin{aligned}
\langle -\delta\mu, \delta a\rangle_{H^{s*},H^s} &= \langle -\mu_\delta + \mu_{\delta'}, a_\delta - a_{\delta'}\rangle_{H^{s*},H^s}\\
&= \langle \mu_\delta, a_{\delta'} - a_\delta\rangle_{H^{s*},H^s} + \langle \mu_{\delta'}, a_\delta - a_{\delta'}\rangle_{H^{s*},H^s} \ge 0,
\end{aligned}
$$

as $a_\delta, a_{\delta'} \in A_{ad}$. Then, with Young's inequality (A.3), (6.22) becomes

$$
\begin{aligned}
&\|\delta y\|^2_{L^2(\Omega)} + \alpha\|B^{s/2}\delta a\|^2_{L^2(\Omega)} - 2(\delta a\nabla\delta y, \nabla\bar{p})_{(L^2(\Omega))^N}\\
&\quad\le \|\delta_3 - \delta_3'\|_{W^{-1,\hat{q}}(\Omega)}\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} + \|\delta_1 - \delta_1'\|_{W^{-1,\hat{q}}(\Omega)}\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)}\\
&\quad+ \|\delta_2 - \delta_2'\|_{L^2(\Omega)}\|\delta a\|_{H^s(\Omega)}\\
&\quad\le \gamma\left(\|\delta p\|^2_{W_0^{1,\hat{q}}(\Omega)} + \|\delta y\|^2_{W_0^{1,\hat{q}}(\Omega)} + \|\delta a\|^2_{H^s(\Omega)}\right)\\
&\quad+ C(\gamma)\left(\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)} + \|\delta_1 - \delta_1'\|^2_{W^{-1,\hat{q}}(\Omega)} + \|\delta_2 - \delta_2'\|^2_{L^2(\Omega)}\right),
\end{aligned} \tag{6.23}
$$

where the constant γ can be chosen arbitrarily small.

2. *Application of the SSC:* The next step of the proof is to estimate the left-hand side

$$
\|\delta y\|^2_{L^2(\Omega)} + \alpha\|B^{s/2}\delta a\|^2_{L^2(\Omega)} - 2(\delta a\nabla\delta y, \nabla\bar{p})_{(L^2(\Omega))^N} = \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y, \delta a)^2
$$

from below. To this end, we split δy into

$$
\delta y = \delta y_{\text{lin}} + \delta y_{\text{rest}},
$$

where δy_{lin} is the solution of the linearized state equation,

$$
-\nabla\cdot(\bar{a}\nabla\delta y_{\text{lin}}) = \nabla\cdot(\delta a\nabla\bar{y}) \tag{6.24}
$$

and δy_{rest} is the solution of

$$-\nabla \cdot (\bar{a} \nabla \delta y_{\text{rest}}) = \delta_3 - \delta_3'. \tag{6.25}$$

Both solutions, δy_{lin} and δy_{rest}, are bounded. By (2.7) we know that

$$\|\delta y_{\text{rest}}\|_{W_0^{1,\hat{q}}(\Omega)} \leq c\|\delta_3 - \delta_3'\|_{W^{-1,\hat{q}}(\Omega)} \tag{6.26}$$

holds. The boundedness of δy_{lin} follows with the same argument and the definition of the duality product, where $\frac{1}{\hat{q}} + \frac{1}{\hat{q}'} = 1$, namely

$$\begin{aligned}
\|\delta y_{\text{lin}}\|_{W_0^{1,\hat{q}}(\Omega)} &\leq c\|\delta a \nabla \bar{y}\|_{W^{-1,\hat{q}}(\Omega)} \leq c \sup_{v \in W_0^{1,\hat{q}'}(\Omega)} \frac{(\delta a \nabla \bar{y}, \nabla v)_{L^2(\Omega)}}{\|v\|_{W_0^{1,\hat{q}'}(\Omega)}} \\
&\leq c \sup_{v \in W_0^{1,\hat{q}'}(\Omega)} \frac{\|\delta a \nabla \bar{y}\|_{(L^{\hat{q}}(\Omega))^N} \|\nabla v\|_{L^{\hat{q}'}(\Omega)}}{\|v\|_{W_0^{1,\hat{q}'}(\Omega)}} \leq \|\delta a \nabla \bar{y}\|_{(L^{\hat{q}}(\Omega))^N} \\
&\leq \|\delta a\|_{L^\infty(\Omega)} \|\bar{y}\|_{W_0^{1,\hat{q}}(\Omega)} \leq c.
\end{aligned} \tag{6.27}$$

For the last estimate we used $\|\delta a\|_{L^\infty(\Omega)} = \|a_\delta - a_{\delta'}\|_{L^\infty(\Omega)} \leq 2(a_{\max} - a_{\min})$ and the uniform boundedness of $\|\bar{y}\|$ in $W^{1,\hat{q}}(\Omega)$. The second order sufficient optimality condition is not directly applicable to the term $\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y, \delta a)^2$ because δy is not a solution of the linearized equation. But for δy_{lin} we have the second order optimality condition

$$\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y_{\text{lin}}, \delta a)^2 \geq \sigma \|B^{s/2} \delta a\|^2_{L^2(\Omega)}, \tag{6.28}$$

see (5.34). Thus, we split the term $\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y, \delta a)^2$, such that we can apply (6.28)

$$\begin{aligned}
\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y, \delta a)^2 &= \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y_{\text{lin}} + \delta y_{\text{rest}}, \delta a + 0)^2 \\
&= \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y_{\text{lin}}, \delta a)^2 + 2\mathcal{L}''(\bar{y}, \bar{a}, \bar{p})((\delta y_{\text{lin}}, \delta a), (\delta y_{\text{rest}}, 0)) \\
&\quad + \mathcal{L}''(\bar{y}, \bar{a}, \bar{p})(\delta y_{\text{rest}}, 0)^2.
\end{aligned}$$

Let us estimate the remaining terms from below. Therefore we use $\delta y = \delta y_{\text{lin}} + \delta y_{\text{rest}}$ and thus $\|\delta y\|^2_{L^2(\Omega)} = \|\delta y_{\text{lin}} + \delta y_{\text{rest}}\|^2_{L^2(\Omega)} = \|\delta y_{\text{lin}}\|^2_{L^2(\Omega)} +$

$2(\delta y_{\text{lin}}, \delta y_{\text{rest}})_{L^2(\Omega)} + \|\delta y_{\text{rest}}\|^2_{L^2(\Omega)}$.

$$\begin{aligned}
&2\mathcal{L}''(\bar{y},\bar{a},\bar{p})((\delta y_{\text{lin}},\delta a),(\delta y_{\text{rest}},0)) + \mathcal{L}''(\bar{y},\bar{a},\bar{p})(\delta y_{\text{rest}},0)^2\\
&\quad= \mathcal{L}''(\bar{y},\bar{a},\bar{p})(\delta y,\delta a)^2 - \mathcal{L}''(\bar{y},\bar{a},\bar{p})(\delta y_{\text{lin}},\delta a)^2\\
&\quad= \|\delta y\|^2_{L^2(\Omega)} - \|\delta y_{\text{lin}}\|^2_{L^2(\Omega)} - 2(\delta a\nabla\delta y,\nabla\bar{p})_{(L^2(\Omega))^N}\\
&\quad+ 2(\delta a\nabla\delta y_{\text{lin}},\nabla\bar{p})_{(L^2(\Omega))^N}\\
&\quad= 2(\delta y_{\text{lin}},\delta y_{\text{rest}})_{L^2(\Omega)} + \|\delta y_{\text{rest}}\|^2_{L^2(\Omega)} - 2(\delta a\nabla\delta y_{\text{rest}},\nabla\bar{p})_{(L^2(\Omega))^N}\\
&\quad\geq \sigma\|B^{s/2}\delta a\|^2_{L^2(\Omega)} - c\|\delta_3-\delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)}
\end{aligned}$$

with $\frac{1}{\hat{q}'}+\frac{1}{\hat{q}}=1$, and $2<\hat{q}<\bar{q}$, coming from Gröger's estimate (2.6). Let us have a closer look at the last estimate. We start with the first term and apply Gröger's estimate

$$(\delta a\delta y_{\text{rest}},\nabla\bar{p})_{(L^2(\Omega))^N} \leq \|\delta a\nabla\delta y_{\text{rest}}\|_{L^p(\Omega)}\|\nabla\bar{p}\|_{L^{\bar{q}}(\Omega)} \leq c\|\delta a\nabla\delta y_{\text{rest}}\|_{L^p(\Omega)},$$

with $p=\frac{1}{1-1/\bar{q}}$. We continue estimating

$$\|\delta a\nabla\delta y_{\text{rest}}\|_{L^p(\Omega)} \leq \|\delta a\|_{L^{pq}(\Omega)}\|\nabla\delta y_{\text{rest}}\|_{L^{pq'}(\Omega)},$$

with $\frac{1}{q}+\frac{1}{q'}=1$. We set $pq' := \hat{q}$ and obtain $pq=\frac{1}{1/p-1/\hat{q}}$. We require $s\geq -\frac{N}{2}+\frac{N}{\bar{q}}+\frac{N}{\hat{q}}$ and apply Sobolev's embedding theorem to obtain

$$\begin{aligned}
(\delta y_{\text{lin}},\delta y_{\text{rest}})_{L^2(\Omega)} &\leq c\|\delta a\|_{H^s(\Omega)}\|\delta y_{\text{rest}}\|_{W^{1,\hat{q}}(\Omega)} \leq c\|\delta a\|_{H^s(\Omega)}\|\delta_3-\delta_3'\|_{W^{-1,\hat{q}}(\Omega)}\\
&\leq c\cdot\gamma\|\delta a\|^2_{H^s(\Omega)} + c\cdot C(\gamma)\|\delta_3-\delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)},
\end{aligned}$$

with Young's inequality. Let us come to the next term.

$$\begin{aligned}
(\delta y_{\text{lin}},\delta y_{\text{rest}})_{L^2(\Omega)} &\leq \|\delta y_{\text{lin}}\|_{W^{1,\hat{q}}(\Omega)}\|\delta y_{\text{rest}}\|_{W^{1,\hat{q}}(\Omega)}\\
&\leq c\|\delta_3-\delta_3'\|_{W^{-1,\hat{q}}(\Omega)}\|\delta y_{\text{lin}}\|_{W^{1,\hat{q}}(\Omega)}.
\end{aligned}$$

Then we see with Gröger's estimate and $\frac{1}{p}+\frac{1}{p'}$

$$\begin{aligned}
\|\delta y_{\text{lin}}\|_{W^{1,\hat{q}}(\Omega)} &\leq c\|\nabla(\delta a\nabla\bar{y})\|_{W^{-1,\hat{q}}(\Omega)} \leq c\|\delta a\nabla\bar{y}\|_{L^{\hat{q}}(\Omega)}\\
&\leq \|\delta a\|_{L^{\hat{q}p}(\Omega)}\|\nabla\bar{y}\|_{L^{\hat{q}p'}(\Omega)}.
\end{aligned}$$

We set $\hat{q}p' := \bar{q}$ and thus obtain $\hat{q}p=\frac{1}{1/\hat{q}-1/\bar{q}}$. For $s\geq\frac{N}{\bar{q}}-\frac{N}{\hat{q}}+\frac{N}{2}$ we apply Sobolev's embedding theorem

$$\|\delta a\|_{L^{\hat{q}p}(\Omega)}\|\nabla\bar{y}\|_{L^{\hat{q}p'}(\Omega)} \leq c\|\delta a\|_{H^s(\Omega)}\|\nabla\bar{y}\|_{L^{\bar{q}}(\Omega)} \leq c\|\delta a\|_{H^s(\Omega)}.$$

Hence, we get

$$(\delta y_{\text{lin}}, \delta y_{\text{rest}})_{L^2(\Omega)} \le c\|\delta a\|_{H^s(\Omega)}\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)} \le c\cdot\gamma\|\delta a\|^2_{H^s(\Omega)} + c\cdot C(\gamma)\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)}.$$

In both estimates we chose δ arbitrarily small.
With (6.23) we then obtain

$$\begin{aligned}&\sigma\|B^{s/2}\delta a\|^2_{L^2(\Omega)} - c\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)}\\ &\le \gamma\left(\|\delta p\|^2_{W_0^{1,\hat q}(\Omega)} + \|\delta y\|^2_{W_0^{1,\hat q}(\Omega)} + \|\delta a\|^2_{H^s(\Omega)}\right)\\ &+ C(\gamma)\left(\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)} + \|\delta_1 - \delta_1'\|^2_{W^{-1,\hat q}(\Omega)} + \|\delta_2 - \delta_2'\|^2_{L^2(\Omega)}\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\Leftrightarrow\quad &\sigma\|B^{s/2}\delta a\|^2_{L^2(\Omega)}\\ &\le \gamma\left(\|\delta p\|^2_{W_0^{1,\hat q}(\Omega)} + \|\delta y\|^2_{W_0^{1,\hat q}(\Omega)} + \|\delta a\|^2_{H^s(\Omega)}\right) + c\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)}\\ &+ C(\gamma)\left(\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat q}(\Omega)} + \|\delta_1 - \delta_1'\|^2_{W^{-1,\hat q}(\Omega)} + \|\delta_2 - \delta_2'\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{6.29}$$

Thus, the proof is almost done.

3. *Incorporation of $\|\delta y\|_{W^{1,\hat q}(\Omega)}$, $\|\delta p\|_{W^{1,\hat q}(\Omega)}$ and $\|\delta\mu\|_{(H^s(\Omega))^*}$:* On the left-hand side of (6.29) there are still the terms $\|\delta y\|_{W^{1,\hat q}(\Omega)}$, $\|\delta p\|_{W^{1,\hat q}(\Omega)}$ and $\|\delta\mu\|_{(H^s(\Omega))^*}$ missing. This problem can be solved by estimating these terms in the following way. For δp we use Gröger's estimate (2.6) and obtain via the perturbed linearized adjoint equation (6.20)

$$\begin{aligned}\|\delta p\|_{W_0^{1,\hat q}(\Omega)} &\le c\|\delta a\nabla\bar p\|_{(W^{-1,\hat q}(\Omega))^N} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat q}(\Omega)} + c\|\delta y\|_{W^{-1,\hat q}(\Omega)}\\ &\le c\|\delta a\nabla\bar p\|_{(L^2(\Omega))^N} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat q}(\Omega)} + c\|\delta y\|_{W_0^{1,\hat q}(\Omega)}\\ &\le c\|\delta a\|_{L^{2p}(\Omega)}\|\nabla\bar p\|_{(L^{2p'}(\Omega))^N} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat q}(\Omega)} + c\|\delta y\|_{W_0^{1,\hat q}(\Omega)}\\ &\le c\|B^{s/2}\delta a\|_{L^2(\Omega)}\|\bar p\|_{W_0^{1,\hat q}(\Omega)} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat q}(\Omega)} + c\|\delta y\|_{W_0^{1,\hat q}(\Omega)}\\ &\le cK\|B^{s/2}\delta a\|_{L^2(\Omega)} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat q}(\Omega)} + c\|\delta y\|_{W_0^{1,\hat q}(\Omega)}.\end{aligned}$$

We used again Hölder's inequality (A.1) with $\frac{1}{p} + \frac{1}{p'} = 1$. We then chose $2p' := \hat q$ and thus obtained $2p = \frac{1}{1/2 - 1/\hat q}$ and used the uniform boundedness of $\bar p$, (5.11). Finally, we applied Sobolev's embedding theorem with $s \ge \frac{N}{\hat q}$ and replaced the H^s-norm by the multilevel norm.

For $\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)}$ we obtain equivalently

$$\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)} \leq cK\|B^{s/2}\delta a\|_{L^2(\Omega)} + c\|\delta_3 - \delta_3'\|_{W^{-1,\hat{q}}(\Omega)} \tag{6.30}$$

and thus,

$$\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} \leq cK\|B^{s/2}\delta a\|_{L^2(\Omega)} + c\|\delta_1 - \delta_1'\|_{W^{-1,\hat{q}}(\Omega)} + c\|\delta_3 - \delta_3'\|_{W^{-1,\hat{q}}(\Omega)} \tag{6.31}$$

Estimates (6.30) and (6.31) can easily be integrated into (6.29), as we see later. For the estimation of $\|\delta\mu\|_{(H^s(\Omega))^*}$ let us have a look at the gradient equation.

$$\delta\mu = \alpha B^s\delta a - \nabla\bar{y}\cdot\nabla\delta p - \nabla\delta y\cdot\nabla\bar{p} - (\delta_2 - \delta_2')$$

We estimate

$$\begin{aligned}\|\delta\mu\|_{(H^s(\Omega))^*} &\leq \|\alpha B^s\delta a\|_{(H^s(\Omega))^*} + \|\nabla\bar{y}\cdot\nabla\delta p\|_{(H^s(\Omega))^*}\\ &\quad + \|\nabla\delta y\cdot\nabla\bar{p}\|_{(H^s(\Omega))^*} + \|\delta_2 - \delta_2'\|_{(H^s(\Omega))^*}\\ &\leq \|\alpha B^{s/2}\delta a\|_{L^2(\Omega)} + \|\nabla\bar{y}\cdot\nabla\delta p\|_{(H^s(\Omega))^*}\\ &\quad + \|\nabla\delta y\cdot\nabla\bar{p}\|_{(H^s(\Omega))^*} + \|\delta_2 - \delta_2'\|_{L^2(\Omega)}.\end{aligned}$$

Here we used for the first term

$$\begin{aligned}\|B^s\delta a\|_{(H^s(\Omega))^*} &= \sup_{v\in H^s(\Omega)}\frac{|(B^s\delta a, v)_{L^2(\Omega)}|}{\|v\|_{H^s(\Omega)}} = \sup_{v\in H^s(\Omega)}\frac{|(B^{s/2}\delta a, B^{s/2}v)_{L^2(\Omega)}|}{\|v\|_{H^s(\Omega)}}\\ &\leq \sup_{v\in H^s(\Omega)}\frac{\|B^{s/2}\delta a\|_{L^2(\Omega)}\|B^{s/2}v\|_{L^2(\Omega)}}{\|v\|_{H^s(\Omega)}}\\ &\leq \sup_{v\in H^s(\Omega)} c\frac{\|B^{s/2}\delta a\|_{L^2(\Omega)}\|v\|_{H^s(\Omega)}}{\|v\|_{H^s(\Omega)}} = c\|B^{s/2}\delta a\|_{L^2(\Omega)}.\end{aligned}$$

Now, let us estimate the terms $\|\nabla\bar{y}\cdot\nabla\delta p\|_{(H^s(\Omega))^*}$ and $\|\nabla\delta y\cdot\nabla\bar{p}\|_{(H^s(\Omega))^*}$

as follows,

$$\begin{aligned}
\|\nabla\bar{y}\cdot\nabla\delta p\|_{(H^s(\Omega))^*} &= \sup_{v\in H^s(\Omega)} \frac{|(\nabla\bar{y}\cdot\nabla\delta p, v)_{L^2(\Omega)}|}{\|v\|_{H^s(\Omega)}} \\
&\le \sup_{v\in H^s(\Omega)} \frac{\|\nabla\delta p\|_{L^2(\Omega)}\|\nabla\bar{y}\, v\|_{L^2(\Omega)}}{\|v\|_{H^s(\Omega)}} \\
&\le \|\delta p\|_{H^1(\Omega)} \sup_{v\in H^s(\Omega)} \frac{\|\nabla\bar{y}\|_{L^{2p}(\Omega)}\|v\|_{L^{2p'}(\Omega)}}{\|v\|_{H^s(\Omega)}} \\
&\le \|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} \sup_{v\in H^s(\Omega)} \frac{\|\bar{y}\|_{W_0^{1,\hat{q}}(\Omega)}\|v\|_{L^{2p'}(\Omega)}}{\|v\|_{H^s(\Omega)}} \\
&\le \|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} \sup_{v\in H^s(\Omega)} \frac{\|\bar{y}\|_{W_0^{1,\hat{q}}(\Omega)}\|v\|_{H^s(\Omega)}}{\|v\|_{H^s(\Omega)}} = K\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)}.
\end{aligned}$$

Here, we used Cauchy-Schwarz inequality (A.2) and Hölder's inequality (A.1) with $\frac{1}{p}+\frac{1}{p'}=1$. Then we set $2p =: \hat{q}$ and thus $2p' = \frac{1}{1/2-1/\hat{q}}$. Then, we applied Sobolev's embedding theorem for $s \ge \frac{N}{\hat{q}}$. The analog holds true for $\|\nabla\delta y\cdot\nabla\bar{p}\|_{(H^s(\Omega))^*}$, i.e. $\|\nabla\delta y\cdot\nabla\bar{p}\|_{(H^s(\Omega))^*} \le K\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)}$. We apply again the uniform boundedness of $\bar{y}$ and $\bar{p}$ in $W^{1,\hat{p}}$, i.e. (2.7) and (5.11). Thus, we continue estimating

$$\begin{aligned}
\|\delta\mu\|_{(H^s(\Omega))^*} \le& \|\alpha B^{s/2}\delta a\|_{L^2(\Omega)} + K\|\delta p\|_{W_0^{1,\hat{q}}(\Omega)} \\
&+ K\|\delta y\|_{W_0^{1,\hat{q}}(\Omega)} + \|\delta_2-\delta_2'\|_{L^2(\Omega)}.
\end{aligned} \tag{6.32}$$

4. *Combination of estimates:* Let us at the end of the proof bring everything together. The estimates for $\|\delta y\|_{W^{1,\hat{q}}(\Omega)}$, $\|\delta p\|_{W^{1,\hat{q}}(\Omega)}$ and $\|\delta\mu\|_{(H^s(\Omega))^*}$, i.e. (6.30), (6.31) and (6.32), that we just derived, have to be squared. The left-hand sides do not present a problem. The right-hand sides can be estimated by using $(a+b)^2 \le 2a^2+2b^2$. With a generic constant c this yields

$$\begin{aligned}
\|\delta y\|^2_{W_0^{1,\hat{q}}(\Omega)} &\le c\|B^{s/2}\delta a\|^2_{L^2(\Omega)} + c\|\delta_3-\delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)} \\
\|\delta p\|^2_{W_0^{1,\hat{q}}(\Omega)} &\le c\|B^{s/2}\delta a\|^2_{L^2(\Omega)} + c\|\delta_1-\delta_1'\|^2_{W^{-1,\hat{q}}(\Omega)} + c\|\delta_3-\delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)} \\
\|\delta\mu\|^2_{(H^s(\Omega))^*} &\le c\|\alpha B^{s/2}\delta a\|^2_{L^2(\Omega)} + c\|\delta p\|^2_{W_0^{1,\hat{q}}(\Omega)} + c\|\delta y\|^2_{W_0^{1,\hat{q}}(\Omega)} \\
&\quad + c\|\delta_2-\delta_2'\|^2_{L^2(\Omega)}.
\end{aligned}$$

The term $c\|B^{s/2}\delta a\|^2_{L^2(\Omega)}$ that occurs in each of these estimates can be estimated by (6.29). Now, we integrate these estimates into the main estimate (6.29). Then we obtain all in all

$$\|B^{s/2}\delta a\|^2_{L^2(\Omega)} + \|\delta y\|^2_{W^{1,\hat{q}}(\Omega)} + \|\delta p\|^2_{W^{1,\hat{q}}(\Omega)} + \|\delta\mu\|^2_{(H^s(\Omega))^*}$$
$$\leq c\,\gamma\left(\|\delta p\|^2_{W_0^{1,\hat{q}}(\Omega)} + \|\delta y\|^2_{W_0^{1,\hat{q}}(\Omega)} + \|\delta a\|^2_{H^s(\Omega)}\right)$$
$$+\, c\,C(\gamma)\left(\|\delta_3 - \delta_3'\|^2_{W^{-1,\hat{q}}(\Omega)} + \|\delta_1 - \delta_1'\|^2_{W^{-1,\hat{q}}(\Omega)} + \|\delta_2 - \delta_2'\|^2_{L^2(\Omega)}\right).$$

Let us recall that γ can be chosen arbitrarily small. Hence, the proof is done.

□

6.3 Superlinear convergence

The next step is to show that the perturbations $\delta = (\delta_1, \delta_2, \delta_3)$ can be estimated from above. For that purpose, we use their representations (6.14)-(6.16). Let us start and introduce of the following space,

$$W' := W_0^{1,\bar{q}}(\Omega) \times H^s(\Omega) \times W_0^{1,\bar{q}}(\Omega) \times (H^s(\Omega))^*.$$

This space will be used for the neighborhoods in the following lemmata. For a shorter presentation, let us in the following proofs drop the domain Ω in the notation and write only i.e. $W_0^{1,\bar{q}}$, as we are always referring to the same domain.

Lemma 6.11. *For any $r_1 > 0$, $r_2 > 0$ there exists $L > 0$ such that for $w_k \in B_{r_1}^{W'}(\bar{w})$ and for $w \in B_{r_2}^{W'}(\bar{w})$ there holds the Lipschitz condition*

$$\|\delta\|_Z \leq L\|w_k - \bar{w}\|_W. \tag{6.33}$$

Proof. Let $\bar{w} = (\bar{y}, \bar{a}, \bar{p}, \bar{\mu})$, $w = (y,\, a,\, p, \mu) \in B_{r_1}^{W'}(\bar{w})$ and $w_k = (y_k, a_k, p_k, \mu_k)$ $\in B_{r_2}^{W'}(\bar{w})$, for some r_1, $r_2 > 0$. In Lemma 6.6 we showed that such constants exist. We estimate the occurring equations one by one. Let us start with the first one (6.14)

$$
\begin{aligned}
&\| -\nabla\cdot(a\nabla(\bar y - y_k)) - \nabla\cdot((\bar a - a_k)\nabla y) + \nabla\cdot(\bar a\nabla\bar y) - \nabla\cdot(a_k\nabla y_k)\\
&\quad - \nabla\cdot(\bar a\nabla y_k) + \nabla\cdot(\bar a\nabla y_k)\|_{W^{-1,\hat q}}\\
&= \| -\nabla\cdot(a\nabla(\bar y - y_k)) - \nabla\cdot((\bar a - a_k)\nabla y)\\
&\quad + \nabla\cdot(\bar a\nabla(\bar y - y_k)) + \nabla\cdot((\bar a - a_k)\nabla y_k)\|_{W^{-1,\hat q}}\\
&\le \|\nabla\cdot((a-\bar a)\nabla\cdot(\bar y - y_k))\|_{W^{-1,\hat q}} + \|\nabla\cdot((\bar a - a_k)\nabla(y - y_k))\|_{W^{-1,\hat q}}\\
&\le \|a-\bar a\|_{L^\infty}\|\bar y - y_k\|_{W^{1,\hat q}} + \|\bar a - a_k\|_{H^s}\|y - y_k\|_{W^{1,\bar q}}.
\end{aligned}
$$

Let us have a closer look at the last estimate. For the first term we apply Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$ and obtain

$$
\begin{aligned}
&\|\nabla\cdot((a-\bar a)\nabla(\bar y - y_k))\|_{W^{-1,\hat q}} = \sup_{v\in W_0^{1,\hat q'}} \frac{|\langle (a-\bar a)\nabla(\bar y - y_k), \nabla v\rangle|}{\|v\|_{W_0^{1,\hat q'}}}\\
&\quad\le \sup_{v\in W_0^{1,\hat q'}} \frac{\|((a-\bar a)\nabla(\bar y - y_k)\|_{L^{\hat q}}\|\nabla v\|_{L^{\hat q'}}}{\|v\|_{W_0^{1,\hat q'}}}\\
&\quad\le \|(a-\bar a)\nabla(\bar y - y_k)\|_{L^{\hat q}} \le \|a-\bar a\|_{L^\infty}\|\bar y - y_k\|_{W^{1,\hat q}}.
\end{aligned}
$$

For the second term we equally see with $\frac{1}{p}+\frac{1}{p'}=1$ the following:

$$
\begin{aligned}
&\|\nabla((\bar a - a_k)\cdot\nabla(y - y_k))\|_{W^{-1,\hat q}} = \sup_{v\in W_0^{1,\hat q'}} \frac{|\langle (\bar a - a_k)\nabla(y - y_k), \nabla v\rangle|}{\|v\|_{W_0^{1,\hat q'}}}\\
&\quad\le \sup_{v\in W_0^{1,\hat q'}} \frac{\|(\bar a - a_k)\nabla(y - y_k)\|_{L^{\hat q}}\|\nabla v\|_{L^{\hat q'}}}{\|v\|_{W_0^{1,\hat q'}}}\\
&\quad\le \|(\bar a - a_k)\nabla(y - y_k)\|_{L^{\hat q}} \le \|\bar a - a_k\|_{L^{\hat q p}}\|y - y_k\|_{L^{\hat q p'}}.
\end{aligned}
$$

We choose $\bar q := \hat q p'$ and consequently get $\hat q p = \frac{1}{1/\hat q - 1/\bar q}$. With Sobolev's embedding theorem we then obtain for $s \ge \frac{N}{\bar q} + \frac{N}{2} - \frac{N}{\hat q}$

$$
\|\bar a - a_k\|_{L^{\hat q p}}\|\nabla(y - y_k)\|_{L^{\hat q p'}} \le \|\bar a - a_k\|_{H^s}\|y - y_k\|_{W^{1,\bar q}}.
$$

Thus, there follows

$$
\begin{aligned}
&\|\nabla\cdot(\bar a\nabla\bar y) - \nabla\cdot(a\nabla\bar y) - \nabla\cdot(\bar a\nabla y)\\
&\quad - \nabla\cdot(a_k\nabla y_k) + \nabla\cdot(a\nabla y_k) + \nabla\cdot(a_k\nabla y)\|_{W^{-1,\hat q}}\\
&\le r_2\|\bar y - y_k\|_{W^{1,\hat q}} + (r_2 + r_1)\|\bar a - a_k\|_{H^s}
\end{aligned}
$$

due to $\|a-\bar{a}\|_{L^\infty} \leq r_2$ and $\|y-y_k\|_{W^{1,\bar{q}}} \leq \|y-\bar{y}\|_{W^{1,\bar{q}}} + \|\bar{y}-y_k\|_{W^{1,\bar{q}}} \leq r_2+r_1$. The second equation (6.15) can be estimated in the same way,

$$
\begin{aligned}
&\| -\nabla\cdot(a\nabla(p_k-\bar{p})) - \nabla\cdot((a_k-\bar{a})\nabla p) + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar{a}\nabla\bar{p})\|_{W^{-1,\hat{q}}} \\
&\leq \|a-\bar{a}\|_{L^\infty}\|\bar{p}-p_k\|_{W^{1,\hat{q}}} + \|\bar{a}-a_k\|_{H^s}\|p-p_k\|_{W^{1,\bar{q}}} \\
&\leq r_2\|\bar{p}-p_k\|_{W^{1,\hat{q}}} + (r_2+r_1)\|\bar{a}-a_k\|_{H^s}.
\end{aligned}
$$

The third equation (6.16) can be estimated without further problems, as well.

$$
\begin{aligned}
&\|\nabla y\cdot\nabla(\bar{p}-p_k) + \nabla(\bar{y}-y_k)\cdot\nabla p - \nabla\bar{y}\cdot\nabla\bar{p} + \nabla y_k\cdot\nabla p_k \\
&\quad + \nabla\bar{y}\cdot\nabla p_k - \nabla\bar{y}\cdot\nabla p_k\|_{L^2} \\
&= \|\nabla y\cdot\nabla(\bar{p}-p_k) + \nabla(\bar{y}-y_k)\cdot\nabla p - \nabla\bar{y}\cdot\nabla(\bar{p}-p_k) \\
&\quad - \nabla(\bar{y}-y_k)\cdot\nabla p_k\|_{L^2} \\
&\leq \|\nabla(y-\bar{y})\cdot\nabla(\bar{p}-p_k)\|_{L^2} + \|\nabla(\bar{y}-y_k)\cdot\nabla(p-p_k)\|_{L^2} \\
&\leq \|y-\bar{y}\|_{W^{1,\bar{q}}}\|\bar{p}-p_k\|_{W^{1,\hat{q}}} + \|\bar{y}-y_k\|_{W^{1,\hat{q}}}\|p-p_k\|_{W^{1,\bar{q}}} \\
&\leq r_2\|\bar{p}-p_k\|_{W^{1,\hat{q}}} + (r_1+r_2)\|\bar{y}-y_k\|_{W^{1,\hat{q}}}.
\end{aligned}
$$

Let us have a short look at the estimation of the term $\|\nabla(y-\bar{y})\cdot\nabla(\bar{p}-p_k)\|_{L^2}$. We use again Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$,

$$
\|\nabla(y-\bar{y})\cdot\nabla(\bar{p}-p_k)\|_{L^2} \leq \|\nabla(y-\bar{y})\|_{L^{2p}}\|\nabla(\bar{p}-p_k)\|_{L^{2q}}.
$$

We set $2p := \bar{q} > 4$ and obtain $2q = \frac{1}{1/2-1/\bar{q}} < 4 < \bar{q}$. Thus, we can estimate $2p \leq \hat{q}$, for a $\hat{q} \in (2,\bar{q})$. This yields

$$
\begin{aligned}
\|\nabla(y-\bar{y})\cdot\nabla(\bar{p}-p_k)\|_{L^2} &\leq \|\nabla(y-\bar{y})\|_{L^{2p}}\|\nabla(\bar{p}-p_k)\|_{L^{2q}} \\
&\leq \|\nabla(y-\bar{y})\|_{L^{\bar{q}}}\|\nabla(\bar{p}-p_k)\|_{L^{\hat{q}}} \\
&\leq \|y-\bar{y}\|_{W^{1,\bar{q}}}\|\bar{p}-p_k\|_{W^{1,\hat{q}}} \\
&\leq r_2\|\bar{p}-p_k\|_{W^{1,\hat{q}}},
\end{aligned}
$$

due to $w=(y,a,p,\mu)\in B_{r_2}^{W'}(\bar{w})$.

The second term $\|\nabla(\bar{y}-y_k)\cdot\nabla(p-p_k)\|_{L^2}$ can be handled in a similar way. □

We just proved the estimate $\|\delta\|_Z \leq L\|w_k-\bar{w}\|_W$. This is one ingredient to show that the iterates of the Lagrange-Newton-SQP-method stay within a neighborhood. For the proof of superlinear convergence it is necessary to estimate δ from above a little differently. Therefore, we split the right hand sides of the equations (6.14)-(6.16) and treat the resulting parts separately,

$$\|\delta\|_Z = \|\delta_3\|_{W^{-1,\hat{q}}} + \|\delta_1\|_{W^{-1,\hat{q}}} + \|\delta_2\|_{L^2}.$$

More precisely, we estimate $\|\delta_3\|_{W^{-1,\hat{q}}}$, $\|\delta_1\|_{W^{-1,\hat{q}}}$ and $\|\delta_2\|_{L^2}$ by two terms, respectively. Then we combine the three first terms together and the three second terms.
Let us start with,

$$\begin{aligned}
\|\delta_3\|_{W^{-1,\hat{q}}} &= \| -\nabla\cdot((\bar{a}-a_k)\nabla y) - \nabla\cdot(a\nabla(\bar{y}-y_k)) \\
&\quad + \nabla\cdot(\bar{a}\nabla\bar{y}) - \nabla\cdot(a_k\nabla y_k)\|_{W^{-1,\hat{q}}} \\
&\le \| -\nabla\cdot(\bar{a}\nabla\bar{y}) - \nabla\cdot(a_k\nabla y_k) + \nabla\cdot(\bar{a}\nabla y_k) + \nabla\cdot(a_k\nabla\bar{y})\|_{W^{-1,\hat{q}}} \\
&\quad + \|2\cdot\nabla\cdot(\bar{a}\nabla\bar{y}) - \nabla\cdot(\bar{a}\nabla y) - \nabla\cdot(a\nabla\bar{y}) + \nabla\cdot(a\nabla y_k) - \nabla\cdot(\bar{a}\nabla y_k) \\
&\qquad + \nabla\cdot(a_k\nabla y) - \nabla\cdot(a_k\nabla\bar{y})\|_{W^{-1,\hat{q}}}.
\end{aligned}$$

The second term can be estimated as follows

$$\begin{aligned}
\|\delta_1\|_{W^{-1,\hat{q}}} &= \| -\nabla\cdot((a_k-\bar{a})\nabla p) - \nabla\cdot(a\nabla(p_k-\bar{p})) \\
&\quad + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar{a}\nabla\bar{p})\|_{W^{-1,\hat{q}}} \\
&\le \|\nabla\cdot(\bar{a}\nabla\bar{p}) + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar{a}\nabla p_k) - \nabla\cdot(a_k\nabla\bar{p})\|_{W^{-1,\hat{q}}} \\
&\quad + \| -2\cdot\nabla\cdot(\bar{a}\nabla\bar{p}) + \nabla\cdot(\bar{a}\nabla p) + \nabla\cdot(a\nabla\bar{p}) - \nabla\cdot(a\nabla p_k) \\
&\qquad + \nabla\cdot(\bar{a}\nabla p_k) - \nabla\cdot(a_k\nabla p) + \nabla\cdot(a_k\nabla\bar{p})\|_{W^{-1,\hat{q}}}.
\end{aligned}$$

Equally, the last term yields

$$\begin{aligned}
\|\delta_2\|_{L^2} &= \|\nabla y\cdot\nabla(p_k-\bar{p}) + \nabla(y_k-\bar{y})\nabla p - \nabla y_k\cdot\nabla p_k + \nabla\bar{y}\cdot\nabla\bar{p}\|_{L^2} \\
&\le \| -\nabla\bar{y}\cdot\nabla\bar{p} - \nabla y_k\nabla p_k + \nabla\bar{y}\cdot\nabla p_k + \nabla y_k\cdot\nabla\bar{p}\|_{L^2} \\
&\quad + \|2\cdot\nabla\bar{y}\cdot\nabla\bar{p} - \nabla y\cdot\nabla\bar{p} - \nabla\bar{y}\cdot\nabla p - \nabla y\cdot\nabla p_k - \nabla\bar{y}\cdot\nabla p_k \\
&\qquad + \nabla y_k\cdot\nabla p - \nabla y_k\cdot\nabla\bar{p}\|_{L^2}.
\end{aligned}$$

Next, we regroup the terms in two parts,

$$\begin{aligned}
\|\delta^{(1)}\|_Z :=& \| -\nabla\cdot(\bar{a}\nabla\bar{y}) + \nabla\cdot(a_k\nabla y_k) + \nabla\cdot(\bar{a}\nabla y_k) + \nabla\cdot(a_k\nabla\bar{y})\|_{W^{-1,\hat{q}}} \\
&+ \|\nabla\cdot(\bar{a}\nabla\bar{p}) + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar{a}\nabla p_k) - \nabla\cdot(a_k\nabla\bar{p})\|_{W^{-1,\hat{q}}} \\
&+ \| -\nabla\bar{y}\cdot\nabla\bar{p} - \nabla y_k\nabla p_k + \nabla\bar{y}\cdot\nabla p_k + \nabla y_k\cdot\nabla\bar{p}\|_{L^2} \qquad (6.34)
\end{aligned}$$

and

$$\begin{aligned}\|\delta^{(2)}\|_Z :=& \|2\cdot\nabla\cdot(\bar a\nabla\bar y) - \nabla\cdot(\bar a\nabla y) - \nabla\cdot(a\nabla\bar y) + \nabla\cdot(a\nabla y_k)\\ &\quad - \nabla\cdot(\bar a\nabla y_k) + \nabla\cdot(a_k\nabla y) - \nabla\cdot(a_k\nabla\bar y)\|_{W^{-1,\hat q}}\\ &+ \|-2\cdot\nabla\cdot(\bar a\nabla\bar p) + \nabla\cdot(\bar a\nabla p) + \nabla\cdot(a\nabla\bar p) - \nabla\cdot(a\nabla p_k)\\ &\quad + \nabla\cdot(\bar a\nabla p_k) - \nabla\cdot(a_k\nabla p) + \nabla\cdot(a_k\nabla\bar p)\|_{W^{-1,\hat q}}\\ &+ \|2\cdot\nabla\bar y\cdot\nabla\bar p - \nabla y\cdot\nabla\bar p - \nabla\bar y\cdot\nabla p - \nabla y\cdot\nabla p_k\\ &\quad - \nabla\bar y\cdot\nabla p_k + \nabla y_k\cdot\nabla p - \nabla y_k\cdot\nabla\bar p\|_{L^2}.\end{aligned} \tag{6.35}$$

Thus, we have shown

$$\|\delta\|_Z \le \|\delta^{(1)}\|_Z + \|\delta^{(2)}\|_Z. \tag{6.36}$$

In the following lemmata we derive estimates for $\delta^{(1)}$ and $\delta^{(2)}$ subject to two different norms. By interpolation we will then derive estimates subject to the space Z.

Lemma 6.12. *Consider $Z_2 = W^{-1,2}(\Omega)\times L^2(\Omega)\times W^{-1,2}(\Omega)$. Let $\bar w\in W$ fulfill the optimal system (OS) and the second order sufficient condition* (5.30) *and let w_k be an iterate generated by* (6.5)*, then there holds*

$$\|\delta^{(1)}\|_{Z_2} \le c\|w_k - \bar w\|_W^2. \tag{6.37}$$

Proof. We reorganize the occuring terms, which yields

$$\begin{aligned}\|\delta^{(1)}\|_{Z_2} =& \|-\nabla\cdot(\bar a\nabla\bar y) - \nabla\cdot(a_k\nabla y_k) + \nabla\cdot(\bar a\nabla y_k) + \nabla\cdot(a_k\nabla\bar y)\|_{H^{-1}}\\ &+ \|\nabla\cdot(\bar a\nabla\bar p) + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar a\nabla p_k) - \nabla\cdot(a_k\nabla\bar p)\|_{H^{-1}}\\ &+ \|-\nabla\bar y\cdot\nabla\bar p - \nabla y_k\cdot\nabla p_k + \nabla\bar y\cdot\nabla p_k + \nabla y_k\cdot\nabla\bar p\|_{L^2}\\ =& \|-\nabla\cdot((\bar a - a_k)\nabla(\bar y - y_k))\|_{H^{-1}} + \|\nabla\cdot((\bar a - a_k)\nabla(\bar p - p_k))\|_{H^{-1}}\\ &+ \|-\nabla(\bar y - y_k)\cdot\nabla(\bar p - p_k)\|_{L^2}.\end{aligned}$$

Next, we estimate

$$\begin{aligned}\|\delta^{(1)}\|_{Z_2} \le& \|\bar a - a_k\|_{H^s}\|\bar y - y_k\|_{W^{1,\hat q}} + \|\bar a - a_k\|_{H^s}\|\bar p - p_k\|_{W^{1,\hat q}}\\ &+ \|\bar y - y_k\|_{W^{1,\hat q}}\|\bar p - p_k\|_{W^{1,\hat q}}\\ \le& \|w_k - \bar w\|_W^2.\end{aligned}$$

Let us have a look at the term $\| -\nabla \cdot ((\bar{a}-a_k)\nabla(\bar{y}-y_k))\|_{H^{-1}}$. These considerations can then be applied also to the second term.

$$\begin{aligned}\| -\nabla \cdot ((\bar{a}-a_k)\nabla(\bar{y}-y_k)\|_{H^{-1}} &= \sup_{v\in H^1(\Omega)} \frac{|((\bar{a}-a_k)\nabla(\bar{y}-y_k),\nabla v)_{L^2}|}{\|v\|_{H^1}}\\ &\leq \sup_{v\in H^1(\Omega)} \frac{\|(\bar{a}-a_k)\nabla(\bar{y}-y_k)\|_{(L^2)^N}\|v\|_{H^1}}{\|v\|_{H^1}}\\ &\leq \|(\bar{a}-a_k)\nabla(\bar{y}-y_k)\|_{(L^2)^N}\\ &\leq \|\bar{a}-a_k\|_{L^{2p}}\|\nabla(\bar{y}-y_k)\|_{(L^{2p'})^N}\end{aligned}$$

We used Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$. We set $2p' := \hat{q}$ and obtain via Sobolev's embedding theorem $\|\bar{a}-a_k\|_{L^{2p}} \leq \|\bar{a}-a_k\|_{H^s(\Omega)}$ for $s \geq \frac{N}{\hat{q}}$. Thus,

$$\| -\nabla \cdot ((\bar{a}-a_k)\nabla(\bar{y}-y_k)\|_{H^{-1}} \leq \|\bar{a}-a_k\|_{H^s}\|\bar{y}-y_k\|_{W^{1,\hat{q}}}.$$

We still have to discuss the term $\|\nabla(\bar{y}-y_k)\cdot\nabla(\bar{p}-p_k)\|_{L^2}$. Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$ yields

$$\begin{aligned}\|\nabla(\bar{y}-y_k)\cdot\nabla(\bar{p}-p_k)\|_{L^2} &\leq \|\nabla(\bar{y}-y_k)\|_{L^{2p}}\|\nabla(\bar{p}-p_k)\|_{L^{2q}}\\ &\leq \|\bar{y}-y_k\|_{W^{1,2p}}\|\bar{p}-p_k\|_{W^{1,2q}} \leq \|\bar{y}-y_k\|_{W^{1,\hat{q}}}\|\bar{p}-p_k\|_{W^{1,\hat{q}}},\end{aligned}$$

which holds true for $\hat{q} \in (4,\bar{q})$. □

Lemma 6.13. *Consider now* $Z_{\bar{q}} = W^{-1,\bar{q}}(\Omega) \times L^2(\Omega) \times W^{-1,\bar{q}}(\Omega)$. *Let* $\bar{w} \in W$ *and* $w_k \in W$ *be chosen as in Lemma 6.12, then there holds*

$$\|\delta^{(1)}\|_{Z_{\bar{q}}} \leq c. \tag{6.38}$$

Proof. We rearrange and estimate

$$\begin{aligned}\|\delta^{(1)}\|_{Z_{\bar{q}}} &= \| -\nabla\cdot(\bar{a}\nabla\bar{y}) - \nabla\cdot(a_k\nabla y_k) + \nabla\cdot(\bar{a}\nabla y_k) + \nabla\cdot(a_k\nabla\bar{y})\|_{W^{-1,\bar{q}}}\\ &+ \|\nabla\cdot(\bar{a}\nabla\bar{p}) + \nabla\cdot(a_k\nabla p_k) - \nabla\cdot(\bar{a}\nabla p_k) - \nabla\cdot(a_k\nabla\bar{p})\|_{W^{-1,\bar{q}}}\\ &+ \| -\nabla\bar{y}\cdot\nabla\bar{p} - \nabla y_k\cdot\nabla p_k + \nabla\bar{y}\cdot\nabla p_k + \nabla y_k\cdot\nabla\bar{p}\|_{L^2}\\ &= \| -\nabla\cdot((\bar{a}-a_k)\nabla(\bar{y}-y_k))\|_{W^{-1,\bar{q}}(\Omega)} + \|\nabla\cdot((\bar{a}-a_k)\nabla(\bar{p}-p_k))\|_{W^{-1,\bar{q}}}\\ &+ \| -\nabla(\bar{y}-y_k)\cdot\nabla(\bar{p}-p_k)\|_{L^2}\\ &\leq \|\bar{a}-a_k\|_{L^\infty}\|\bar{y}-y_k\|_{W^{1,\bar{q}}} + \|\bar{a}-a_k\|_{L^\infty}\|\bar{p}-p_k\|_{W^{1,\bar{q}}}\\ &+ \|\bar{y}-y_k\|_{W^{1,\bar{q}}}\|\bar{p}-p_k\|_{W^{1,\bar{q}}}\\ &\leq c.\end{aligned}$$

Let us also discuss the first term in more detail:

$$
\begin{aligned}
&\| - \nabla \cdot ((\bar{a} - a_k)\nabla(\bar{y} - y_k))\|_{W^{-1,\bar{q}}(\Omega)} \\
&\quad = \sup_{v \in (W^{-1,\bar{q}}(\Omega))^*} \frac{|((\bar{a} - a_k)\nabla(\bar{y} - y_k), \nabla v)_{L^2}|}{\|v\|_{(W^{-1,\bar{q}})^*}} \\
&\quad \leq \sup_{v \in (W^{-1,\bar{q}}(\Omega))^*} \frac{\|(\bar{a} - a_k)\nabla(\bar{y} - y_k)\|_{(L^{\bar{q}})^N} \|\nabla v\|_{((L^{\bar{q}})^*)^N}}{\|v\|_{(W^{-1,\bar{q}})^*}} \\
&\quad \leq \|(\bar{a} - a_k)\nabla(\bar{y} - y_k)\|_{(L^{\bar{q}})^N} \\
&\quad \leq \|\bar{a} - a_k\|_{L^\infty} \|\nabla(\bar{y} - y_k)\|_{(L^{\bar{q}})^N} \\
&\quad \leq \|\bar{a} - a_k\|_{L^\infty} \|\bar{y} - y_k\|_{W^{1,\bar{q}}}.
\end{aligned}
$$

The second term can be estimated in the same way. The third term can be estimated as in the last proof, for $\bar{q} > 4$. □

Lemma 6.14. *Again, let $\bar{w} \in W$ and $w_k \in W$ be chosen as in Lemma 6.12, then there holds*

$$
\|\delta^{(2)}\|_{Z_2} \leq c\|w_k - \bar{w}\|_W \|w - \bar{w}\|_W. \tag{6.39}
$$

Proof. For better readability, we estimate the three terms of $\|\delta^{(2)}\|_{Z_{\bar{q}}}$, see (6.35), separately. The first yields

$$
\begin{aligned}
&\|2 \cdot \nabla \cdot (\bar{a}\nabla\bar{y}) - \nabla \cdot (\bar{a}\nabla y) - \nabla \cdot (a\nabla\bar{y}) + \nabla \cdot (a\nabla y_k) - \nabla \cdot (\bar{a}\nabla y_k) \\
&\qquad + \nabla \cdot (a_k\nabla y) - \nabla \cdot (a_k\nabla\bar{y})\|_{H^{-1}} \\
&\quad \leq \|\nabla((a - \bar{a}) \cdot \nabla(\bar{y} - y_k))\|_{H^{-1}} + \|\nabla((\bar{a} - a_k) \cdot \nabla(y - \bar{y}))\|_{H^{-1}} \\
&\quad \leq \|a - \bar{a}\|_{H^s} \|\bar{y} - y_k\|_{W^{1,\hat{q}}} + \|\bar{a} - a_k\|_{H^s} \|y - \bar{y}\|_{W^{1,\hat{q}}}.
\end{aligned}
$$

The second can be treated in the following way

$$
\begin{aligned}
&\| - 2 \cdot \nabla \cdot (\bar{a}\nabla\bar{p}) + \nabla \cdot (\bar{a}\nabla p) + \nabla \cdot (a\nabla\bar{p}) - \nabla \cdot (a\nabla p_k) + \nabla \cdot (\bar{a}\nabla p_k) \\
&\qquad - \nabla \cdot (a_k\nabla p) + \nabla \cdot (a_k\nabla\bar{p})\|_{H^{-1}} \\
&\quad \leq \|\nabla((a - \bar{a}) \cdot \nabla(\bar{p} - p_k))\|_{H^{-1}} + \|\nabla((\bar{a} - a_k) \cdot \nabla(p - \bar{p}))\|_{H^{-1}} \\
&\quad \leq \|a - \bar{a}\|_{H^s} \|\bar{p} - p_k\|_{W^{1,\hat{q}}} + \|\bar{a} - a_k\|_{H^s} \|p - \bar{p}\|_{W^{1,\hat{q}}}.
\end{aligned}
$$

We estimate the third term and obtain

$$
\begin{aligned}
&\|2 \cdot \nabla\bar{y} \cdot \nabla\bar{p} - \nabla y \cdot \nabla\bar{p} - \nabla\bar{y} \cdot \nabla p - \nabla y \cdot \nabla p_k - \nabla\bar{y} \cdot \nabla p_k + \nabla y_k \cdot \nabla p \\
&\qquad - \nabla y_k \cdot \nabla\bar{p}\|_{L^2} \\
&\quad \leq \|\nabla(y - \bar{y}) \cdot \nabla(\bar{p} - p_k)\|_{L^2} + \|\nabla(\bar{y} - y_k) \cdot \nabla(p - \bar{p})\|_{L^2} \\
&\quad \leq \|y - \bar{y}\|_{W^{1,\hat{q}}} \|\bar{p} - p_k\|_{W^{1,\hat{q}}} + \|\bar{y} - y_k\|_{W^{1,\hat{q}}} \|p - \bar{p}\|_{W^{1,\hat{q}}}.
\end{aligned}
$$

Combined, this yields

$$\|\delta^{(2)}\|_{Z_2} \leq \|w_k - \bar{w}\|_W \|w - \bar{w}\|_W.$$

The estimates are done similarly as in the proof of Lemma 6.12. For the estimation of the terms $\|\nabla(y - \bar{y}) \cdot \nabla(\bar{p} - p_k)\|_{L^2}$ and $\|\nabla(\bar{y} - y_k) \cdot \nabla(p - \bar{p})\|_{L^2}$, we again have to require $\hat{q} \in (4, \bar{q})$. □

Lemma 6.15. *Again, let $\bar{w} \in W$ and $w_k \in W$ be chosen as in Lemma 6.12, then there holds*

$$\|\delta^{(2)}\|_{Z_{\bar{q}}} \leq c. \tag{6.40}$$

Proof. As in the last proof, we estimate $\|\delta^{(2)}\|_{Z_{\bar{q}}}$ in three parts.

$$\begin{aligned}
&\|2 \cdot \nabla \cdot (\bar{a}\nabla\bar{y}) - \nabla \cdot (\bar{a}\nabla y) - \nabla \cdot (a\nabla\bar{y}) + \nabla \cdot (a\nabla y_k) - \nabla \cdot (\bar{a}\nabla y_k) \\
&\quad + \nabla \cdot (a_k\nabla y) - \nabla \cdot (a_k\nabla\bar{y})\|_{W^{-1,\bar{q}}} \\
&\quad \leq \|\nabla \cdot ((a - \bar{a})\nabla(\bar{y} - y_k))\|_{W^{-1,\bar{q}}} + \|\nabla \cdot ((\bar{a} - a_k)\nabla(y - \bar{y}))\|_{W^{-1,\bar{q}}} \\
&\quad \leq \|a - \bar{a}\|_{L^\infty}\|\bar{y} - y_k\|_{W^{1,\bar{q}}} + \|\bar{a} - a_k\|_{L^\infty}\|y - \bar{y}\|_{W^{1,\bar{q}}}
\end{aligned}$$

$$\begin{aligned}
&\| - 2 \cdot \nabla \cdot (\bar{a}\nabla\bar{p}) + \nabla \cdot (\bar{a}\nabla p) + \nabla \cdot (a\nabla\bar{p}) - \nabla \cdot (a\nabla p_k) + \nabla \cdot (\bar{a}\nabla p_k) \\
&\quad - \nabla \cdot (a_k\nabla p) + \nabla \cdot (a_k\nabla\bar{p})\|_{W^{-1,\bar{q}}} \\
&\quad \leq \|\nabla \cdot ((a - \bar{a})\nabla(\bar{p} - p_k))\|_{W^{-1,\bar{q}}} + \|\nabla \cdot ((\bar{a} - a_k)\nabla(p - \bar{p}))\|_{W^{-1,\bar{q}}} \\
&\quad \leq \|a - \bar{a}\|_{L^\infty}\|\bar{p} - p_k\|_{W^{1,\bar{q}}} + \|\bar{a} - a_k\|_{L^\infty}\|p - \bar{p}\|_{W^{1,\bar{q}}}
\end{aligned}$$

$$\begin{aligned}
&\|2 \cdot \nabla\bar{y} \cdot \nabla\bar{p} - \nabla y \cdot \nabla\bar{p} - \nabla\bar{y} \cdot \nabla p - \nabla y \cdot \nabla p_k - \nabla\bar{y} \cdot \nabla p_k + \nabla y_k \cdot \nabla p \\
&\quad - \nabla y_k \cdot \nabla\bar{p}\|_{L^2} \\
&\quad \leq \|\nabla(y - \bar{y}) \cdot \nabla(\bar{p} - p_k)\|_{L^2} + \|\nabla(\bar{y} - y_k) \cdot \nabla(p - \bar{p})\|_{L^2} \\
&\quad \leq \|y - \bar{y}\|_{W^{1,\bar{q}}}\|\bar{p} - p_k\|_{W^{1,\bar{q}}} + \|\bar{y} - y_k\|_{W^{1,\bar{q}}}\|p - \bar{p}\|_{W^{1,\bar{q}}}
\end{aligned}$$

Thus, this yields

$$\|\delta^{(2)}\|_{Z_{\bar{q}}} \leq c.$$

Again, the estimates for $\|\nabla(y - \bar{y}) \cdot \nabla(\bar{p} - p_k)\|_{L^2}$ and $\|\nabla(\bar{y} - y_k) \cdot \nabla(p - \bar{p})\|_{L^2}$ hold true for $\bar{q} > 4$. □

Lemma 6.16. *Let $\bar{w} \in W$ and $w_k \in W$ be chosen as in Lemma 6.12. Then, there holds for an $\alpha \in (0,1)$*

$$\|\delta^{(1)}\|_Z \leq c\|w_k - \bar{w}\|_W^{2\alpha}, \tag{6.41}$$

$$\|\delta^{(2)}\|_Z \leq c\|w_k - \bar{w}\|_W^{\alpha}\|w - \bar{w}\|_W^{\alpha}. \tag{6.42}$$

The first estimate follows from $\|\delta^{(1)}\|_{Z_2} \leq c\|w_k - \bar{w}\|_W^2$, i.e. (6.37), and $\|\delta^{(1)}\|_{Z_{\bar{q}}} \leq c$, i.e. (6.38), with an interpolation argument. The second estimate is obtained by the same argument with estimates $\|\delta^{(2)}\|_{Z_2} \leq c\|w_k - \bar{w}\|_W\|w - \bar{w}\|_W$, i.e. (6.39), and $\|\delta^{(2)}\|_{Z_{\bar{q}}} \leq c$, i.e. (6.40).

Remark 6.17. *We can choose $\hat{q} \in (2, \bar{q})$ in such a way, that $\alpha \in (\frac{1}{2}, 1)$, i.e. $2\alpha > 1$. From now on we assume that $2\alpha > 1$ holds.*

Now, let us state the main result of this section.

Theorem 6.18. *Let $\bar{w}$ fulfill the optimal system (OS) and the second order sufficient condition* (5.30)*. Then, there exists a constant $r > 0$ and a constant $C > 0$, such that for each starting point $w^0 \in B_r^W(\bar{w})$, the sequence of iterates w_k generated by* (6.5) *is well-defined in $B_r^W(\bar{w})$ and satisfies*

$$\|w_{k+1} - \bar{w}\|_W \leq C\|w_k - \bar{w}\|_W^{2\alpha}. \tag{6.43}$$

Proof. Suppose the iterate $w_k \in B_r^W(\bar{w})$ is given. With (6.36), (6.41) and (6.42) we estimate

$$\|\delta\|_Z \leq \|\delta^{(1)}\|_Z + \|\delta^{(2)}\|_Z \leq c_1\|w_k - \bar{w}\|_W^{2\alpha} + c_2\|w_k - \bar{w}\|^{\alpha}\, \|w_{k+1} - \bar{w}\|_W^{\alpha}.$$

Moreover, due to the results of theorem 6.11 and theorem 6.10 and the assumption $w_k \in B_r^W(\bar{w})$ there holds

$$\|\delta\|_Z \leq L\|w_k - \bar{w}\|_W \leq Lr$$

$$\text{and} \qquad \|w_{k+1} - \bar{w}\|_W \leq L_\delta\|\delta\|_Z.$$

At first, let us show that the iterate w_{k+1} belongs to the neighborhood $B_r^W(\bar{w})$,

$$\begin{aligned}\|w_{k+1} - \bar{w}\|_W &\leq L_\delta\|\delta\|_Z \leq L_\delta(c_1\|w_k - \bar{w}\|_W^{2\alpha} + c_2\|w_k - \bar{w}\|_W^{\alpha}\|w_{k+1} - \bar{w}\|_W^{\alpha}) \\ &\leq L_\delta(c_1 r^{2\alpha} + c_2 r^{\alpha}\|w_{k+1} - \bar{w}\|_W^{\alpha}) \\ &\leq L_\delta(c_1 r^{2\alpha} + c_2 L_\delta^{\alpha} L^{\alpha} r^{2\alpha}) = L_\delta r^{2\alpha}(c_1 + c_2 L^{\alpha} L_\delta^{\alpha}).\end{aligned}$$

We know there holds $2\alpha > 1$, see Remark 6.17. Thus, we can choose $r > 0$, such that $L_\delta r^{2\alpha-1}(c_1 + c_2 L^\alpha L_\delta^\alpha) \in (0,1)$, then $\|w_{k+1} - \bar{w}\|_W < r$. With this choice of the radius r, we obtain by induction for a starting point $w^0 \in B_r^W(\bar{w})$ a sequence of iterates that lies in $B_r^W(\bar{w})$ as well. Next, we prove superlinear convergence and use Young's inequality to obtain

$$\begin{aligned}\|w_{k+1}-\bar{w}\|_W &\leq L_\delta c_1 \|w_k - \bar{w}\|_W^{2\alpha} + L_\delta c_2 \|w_k - \bar{w}\|_W^\alpha \|w_{k+1} - \bar{w}\|_W^\alpha \\ &\leq L_\delta c_1 \|w_k - \bar{w}\|_W^{2\alpha} + L_\delta c_2 C(\gamma) \|w_k - \bar{w}\|_W^{2\alpha} + L_\delta c_2 \gamma \|w_{k+1} - \bar{w}\|_W^{2\alpha}.\end{aligned}$$

Thus,

$$\|w_{k+1} - \bar{w}\|_W - L_\delta c_2 \gamma \|w_{k+1} - \bar{w}\|_W^{2\alpha} \leq (L_\delta c_1 + L_\delta c_2 C(\gamma)) \|w_k - \bar{w}\|_W^{2\alpha}.$$

Since there holds $2\alpha > 1$ we can estimate $\|w_{k+1} - \bar{w}\|_W^{2\alpha} \leq \|w_{k+1} - \bar{w}\|_W$ for a sufficiently small radius $r > 0$. Then we conclude

$$\|w_{k+1} - \bar{w}\|_W \leq \frac{L_\delta c_1 + L_\delta c_2 C(\gamma)}{1 - L_\delta c_2 \gamma} \|w_k - \bar{w}\|_W^{2\alpha}.$$

We choose $\gamma > 0$ in such a way that $1 - L_\delta c_2 \gamma > 0$. This finishes the proof. □

As a conclusion let us state, that the SQP-method has at least superlinear order of convergence for problems, that fulfill assumptions 6.2.

Chapter 7

Numerical example

7.1 The continuous problem

Let us restate the parameter identification problem that we want to solve. The objective functional is given as

$$\min J(y,a) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|B^{s/2}a\|^2_{L^2(\Omega)}, \tag{7.1}$$

with the multilevel norm as regularization term. The elliptic differential equation is given as

$$\begin{aligned} -\nabla\cdot(a\nabla y) &= g \quad \text{in } \Omega \\ y &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{7.2}$$

Furthermore we deal with box constraints

$$0 < a_{\min} \leq a(x) \leq a_{\max}. \tag{7.3}$$

We only treat the problem in two dimensions with domain Ω being the unit square and the boundary $\Gamma = \partial\Omega$.

7.2 SQP-method

We implement an SQP-method, where we solve in every step the following quadratic subproblem with a primal-dual active set strategy (PDAS).

$$
(QP)_k \begin{cases} \min & F(y,a) = (y_k - y_d, y - y_k)_{L^2(\Omega)} + \alpha(B^{s/2}a_k, B^{s/2}(a-a_k))_{L^2(\Omega)} \\ & \quad -((a-a_k), \nabla(y-y_k)\cdot\nabla p_k)_{L^2(\Omega)} \\ & \quad +\frac{1}{2}\|y-y_k\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|B^{s/2}(a-a_k)\|^2_{L^2(\Omega)} \\ \text{s. t.} & -\nabla\cdot(a_k\nabla y) = \nabla\cdot(a\nabla y_k) - \nabla\cdot(a_k\nabla y_k) + g \quad \text{in } \Omega \\ & y = 0 \quad \text{on } \Gamma \\ & 0 < a_{\min} \le a(x) \le a_{\max} \quad \text{a.e. in } \Omega. \end{cases}
$$

We already discussed in detail the SQP-method in the last chapter, therefore, we directly state the algorithm.

Algorithm 7.1 (SQP-Algorithm)**.**

1. *Select a starting point* (a_0, y_0, p_0)*, set k=0;*
2. *If* (a_k, y_k, p_k) *is a solution of* (7.1)-(7.3)*, then STOP;*
3. *Solve* $(QP)_k$ *to obtain the next iterate* $(a_{k+1}, y_{k+1}, p_{k+1})$*;*
4. *Set* $k = k+1$ *and go to (2).*

The SQP-method, without globalization strategies like linesearch, is only locally convergent. Another possibility is to do some steps of the gradient method first to obtain a starting point that is close enough to the solution.

7.3 Primal-dual active set strategy

In every step of the SQP-method we have to solve the quadratic subproblem $(QP)_k$. This problem has a quadratic objective functional and linear equality and inequality constraints. We will solve it by applying a primal-dual active set strategy, which is a widely used method, see for example [10], [41]. For a general overview over semi-smooth Newton methods and in particular over the primal dual active set strategy we refer to e.g. [35] or [40]. Let us restate the optimality system with a $\mu \in H^{-s}(\Omega)$, that is defined as

$$
\mu := \alpha B^s \bar{a} - \nabla\bar{y}\cdot\nabla p_k - \nabla y_k\cdot\nabla\bar{p} + \nabla y_k\cdot\nabla p_k. \tag{7.4}
$$

$$
(OS)_k \begin{cases} -\nabla \cdot (a_k \nabla \bar{y}) = \nabla \cdot (\bar{a} \nabla y_k) - \nabla \cdot (a_k \nabla y_k) + g, \\ -\nabla \cdot (a_k \nabla \bar{p}) = \nabla \cdot (\bar{a} \nabla p_k) - \nabla \cdot (a_k \nabla p_k) + \bar{y} - y_d, \\ \alpha B^s \bar{a} - \nabla \bar{y} \cdot \nabla p_k - \nabla y_k \cdot \nabla \bar{p} + \nabla y_k \cdot \nabla p_k - \mu = 0, \\ \langle \mu, a - \bar{a} \rangle_{H^{-s}, H^s} \geq 0 \quad \forall a \in A_{ad}, \\ a_{\min} \leq \bar{a}(x) \leq a_{\max}. \end{cases}
$$

A standard procedure in order to derive an active set method is to consider the gradient equation pointwise and look at a projection formula for the parameter function. For our problem at hand, we would have to face further difficulties due to the nonlocal structure of the multilevel operator. Let us therefore derive the PDAS strategy for the discrete case. As a start we have a look at the discretized optimality system. We apply the Galerkin method, that we introduced shortly in section 3.2. The assembly of the necessary matrices is thoroughly discussed in the appendix B, let us just shortly define them here.

- Mass matrix corresponding to the L^2 scalar product: $\mathbb{M}$;
- Stiffness matrix for a fixed parameter a_h^k for $(a_h^k \nabla y_h, \nabla v_h)_{(L^2(\Omega))^N}$: $\mathbb{KA}$;
- Stiffness matrix for a fixed state y_h^k for $(a_h \nabla y_h^k, \nabla v_h)_{(L^2(\Omega))^N}$: $\mathbb{KY}$;
- Stiffness matrix for a fixed adjoint state p_h^k for $(a_h \nabla p_h^k, \nabla v_h)_{(L^2(\Omega))^N}$: $\mathbb{KP}$;
- Stiffness matrix for a fixed state y_h^k for $(\nabla y_h^k \nabla p_h, v_h)_{L^2(\Omega)}$: $\mathbb{KY}^\top$;
- Stiffness matrix for a fixed adjoint state p_h^k for $(\nabla p_h^k \nabla y_h, v_h)_{L^2(\Omega)}$: $\mathbb{KP}^\top$;
- Multilevel operator for $B^s = \sum_{k=0}^{L} h_k^{-2s} (Q_k - Q_{k-1}) a_L$: $\mathbb{B}$.

Furthermore, every function $v_h \in V_h$ has a unique representation $\sum_{i=1}^{n_h} \mathbf{v_i} \varphi_i$, where $\mathbf{v_i}$ is the value of v_h at the corresponding node. The vectors containing the values of the functions in the nodes of the triangulation will also be written in bold letters, i.e. $\mathbf{v} = (\mathbf{v_1}, \dots, \mathbf{v_{n_h}})^\top$. Then we find the discrete optimal system,

$$
(DOS)_k \begin{cases} \mathbb{KA}\bar{\mathbf{y}} + \mathbb{KY}\bar{\mathbf{a}} = \mathbb{KA}\mathbf{y_k} + \mathbb{M}\mathbf{g}, \\ \mathbb{KA}\bar{\mathbf{p}} + \mathbb{KP}\bar{\mathbf{a}} - \mathbb{M}\bar{\mathbf{y}} = \mathbb{KA}\mathbf{p_k} - \mathbb{M}\mathbf{yd}, \\ (\alpha \mathbb{MB}\bar{\mathbf{a}} - \mathbb{KY}^\top \bar{\mathbf{p}} - \mathbb{KP}^\top \bar{\mathbf{y}} + \mathbb{KY}^\top \mathbf{p_k}, \mathbf{a_h} - \bar{\mathbf{a}}) \geq 0 \quad \forall \mathbf{a_h} \in \mathcal{A}_{ad}^h. \end{cases}
$$

with vectors $\mathbf{g} = (\mathbf{g_i})_{i=1}^{n_h}$, $\mathbf{g_i} = g(x_i)$, $i = 1, \dots, n_h$ and $\mathbf{y_d} = (\mathbf{y_{d,i}})_{i=1}^{n_h}$, $\mathbf{y_{d,i}} = y_d(x_i)$, $i = 1, \dots, n_h$ and the discrete feasible set

$$\mathcal{A}_{ad} = \{\mathbf{a} \in \mathbb{R}^{n_h} : 0 < a_{\min} \leq \mathbf{a_i} \leq a_{\max}\}.$$

We denote by $\mathbf{a}_{\min}$ and $\mathbf{a}_{\max}$ vectors that have n_h times the entry $a_{\min}$ or $a_{\max}$, respectively. Now, let us have a closer look at the discrete gradient equation. There holds $\bar{\mathbf{a}} = \mathbf{a}_{\min}$, if and only if

$$\alpha \mathbb{M}\mathbb{B}\bar{\mathbf{a}} - \mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} - \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} + \mathbb{K}\mathbb{Y}^\top \mathbf{p_k} > 0,$$

$\bar{\mathbf{a}} = \mathbf{a}_{\max}$ holds if and only if

$$\alpha \mathbb{M}\mathbb{B}\bar{\mathbf{a}} - \mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} - \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} + \mathbb{K}\mathbb{Y}^\top \mathbf{p_k} < 0$$

and $\bar{\mathbf{a}} \in [\mathbf{a}_{\min}, \mathbf{a}_{\max}]$ holds if and only if

$$\begin{aligned} & \alpha \mathbb{M}\mathbb{B}\bar{\mathbf{a}} - \mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} - \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} + \mathbb{K}\mathbb{Y}^\top \mathbf{p_k} = 0 \\ \Leftrightarrow \quad & \bar{\mathbf{a}} = \alpha^{-1}(\mathbb{M}\mathbb{B})^{-1} \left(\mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} + \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} - \mathbb{K}\mathbb{Y}^\top \mathbf{p_k}\right). \end{aligned}$$

Thus, we can combine the upper thoughts via a projection relation for the discrete parameter $\bar{\mathbf{a}}$,

$$\bar{\mathbf{a}} = \mathbb{P}_{[\mathbf{a}_{\min}, \mathbf{a}_{\max}]}\{\alpha^{-1}(\mathbb{M}\mathbb{B})^{-1} \left(\mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} + \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} - \mathbb{K}\mathbb{Y}^\top \mathbf{p_k}\right)\}.$$

We define

$$\boldsymbol{\mu} := \alpha^{-1}(\mathbb{M}\mathbb{B})^{-1} \left(\mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} + \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} - \mathbb{K}\mathbb{Y}^\top \mathbf{p_k}\right) - \bar{\mathbf{a}}$$

where we denote its entries by $\boldsymbol{\mu}_\mathbf{i}$, $i = 0, \dots, n_h$ and obtain with the projection relation

$$\bar{\mathbf{a}} = \begin{cases} \mathbf{a}_{\min}, & \text{if } \boldsymbol{\mu}_\mathbf{i} < 0, \\ \alpha^{-1}(\mathbb{M}\mathbb{B})^{-1} \left(\mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} + \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} - \mathbb{K}\mathbb{Y}^\top \mathbf{p_k}\right), & \text{if } \boldsymbol{\mu}_\mathbf{i} = 0, \\ \mathbf{a}_{\max}, & \text{if } \boldsymbol{\mu}_\mathbf{i} > 0. \end{cases}$$

Let us rewrite this in another way to derive the definition of the active and inactive sets,

$$\bar{\mathbf{a}} = \begin{cases} \mathbf{a}_{\min}, & \text{if } \bar{\mathbf{a}}_\mathbf{i} + \boldsymbol{\mu}_\mathbf{i} < \mathbf{a}_{\min}, \\ \alpha^{-1}(\mathbb{M}\mathbb{B})^{-1} \left(\mathbb{K}\mathbb{Y}^\top \bar{\mathbf{p}} + \mathbb{K}\mathbb{P}^\top \bar{\mathbf{y}} - \mathbb{K}\mathbb{Y}^\top \mathbf{p_k}\right), & \text{if } \bar{\mathbf{a}}_\mathbf{i} + \boldsymbol{\mu}_\mathbf{i} \in [\mathbf{a}_{\min}, \mathbf{a}_{\max}], \\ \mathbf{a}_{\max}, & \text{if } \bar{\mathbf{a}}_\mathbf{i} + \boldsymbol{\mu}_\mathbf{i} > \mathbf{a}_{\max}. \end{cases} \tag{7.5}$$

Now, we can write down the algorithm for the primal dual active set strategy.

Algorithm 7.2 (PDAS-Algorithm)**.**

1. *Select a starting point* $(\mathbf{a_0}, \mathbf{y_0}, \mathbf{p_0})$ *and find*

$$\boldsymbol{\mu_0} := \alpha^{-1}(\mathbb{MB})^{-1}\left(\mathbb{KY}^\top \mathbf{p_0} + \mathbb{KP}^\top \mathbf{y_0} - \mathbb{KY}^\top \mathbf{p_k}\right) - \mathbf{a_0}.$$

   Set $n = 1$*.*

2. *Determine active and inactive sets,*

$$\begin{aligned}
\mathcal{A}_n^{\min} &= \{i \in \{1, \dots, n_h\} : \mathbf{a_{n-1,i}} + \boldsymbol{\mu}_{\mathbf{n-1,i}} < \mathbf{a}_{\min}\},\\
\mathcal{A}_n^{\max} &= \{i \in \{1, \dots, n_h\} : \mathbf{a_{n-1,i}} + \boldsymbol{\mu}_{\mathbf{n-1,i}} > \mathbf{a}_{\max}\},\\
\mathcal{I}_n &= \{1, \dots, n_h\} \setminus (\mathcal{A}_n^{\max} \cup \mathcal{A}_n^{\min});
\end{aligned}$$

3. *If* $\mathcal{A}_n^{\max} = \mathcal{A}_{n-1}^{\max}$ *and* $\mathcal{A}_n^{\min} = \mathcal{A}_{n-1}^{\min}$*: STOP;*

4. *Find the next iterate* $(\mathbf{a_n}, \mathbf{y_n}, \mathbf{p_n})$ *by solving*

$$\begin{aligned}
\mathbb{KA}\mathbf{y} + \mathbb{KY}\mathbf{a} &= \mathbb{KA}\mathbf{y_k} + \mathbb{M}\mathbf{g},\\
\mathbb{KA}\mathbf{p} + \mathbb{KP}\mathbf{a} - \mathbb{M}\mathbf{y} &= \mathbb{KA}\mathbf{p_k} - \mathbb{M}\mathbf{yd}
\end{aligned}$$

   and (7.5)*.*

5. *Set* $\boldsymbol{\mu_n} = \alpha^{-1}(\mathbb{MB})^{-1}\left(\mathbb{KY}^\top \mathbf{p_n} + \mathbb{KP}^\top \mathbf{y_n} - \mathbb{KY}^\top \mathbf{p_k}\right) - \mathbf{a_n}$*,* $n{=}n{+}1$ *and go to 2.*

We implement (7.5) by means of characteristic functions $\chi_n^{\min}$ and $\chi_n^{\max}$ associated to $\mathcal{A}_n^{\min}$ and $\mathcal{A}_n^{\min}$, respectively, more precisely by the following diagonal matrices,

$$\mathbb{A}_n^{\min} := \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{A}_n^{\min},\\ 0, & \text{else,} \end{cases}$$

$$\mathbb{A}_n^{\max} := \begin{cases} 1, & \text{if } i = j \text{ and } i \in \mathcal{A}_n^{\max},\\ 0, & \text{else.} \end{cases}$$

Then, the matrix corresponding to the inactive set results as $\mathbb{I}_n = \mathbb{I} - \mathbb{A}_n^{\min} - \mathbb{A}_n^{\max}$, where $\mathbb{I}$ denotes the $n_h \times n_h$ identity matrix. This yields

$$\mathbf{a} - \mathbb{I}_n \alpha^{-1}(\mathbb{MB})^{-1}\left(\mathbb{KY}^\top \mathbf{p_n} + \mathbb{KP}^\top \mathbf{y_n} - \mathbb{KY}^\top \mathbf{p_k}\right) = \mathbb{A}_n^{\min}\mathbf{a}_{\min} + \mathbb{A}_n^{\max}\mathbf{a}_{\max}.$$

7.4 Example and results

Let us present a numerical example. Therefore, we set $\Omega := [0,1] \times [0,1]$. We chose the right hand sides of the state and adjoint equation as

$$g = \pi^2 \sin(\pi x1) \sin(\pi x2), \qquad y_d = (1 - \pi^2) \sin(\pi x1) \sin(\pi x2).$$

The pointwise bounds for the parameter are set as

$$a_{\min} = 0.01, \qquad a_{\max} = 10.$$

We tested this problem for several values of s and for different refinements and set the Tichonov parameter to $\alpha = 0.1$. In the following tables, we see the behavior for two different choices of s and Nint. (The number of elements is given as 2Nint^2).
At first we found on a coarser grid and $s = 0.15$:

k	$\|a_k - a_{11}\|_{H^s(\Omega)}$	$\|y_k - y_{11}\|_{H^1(\Omega)}$	$\|p_k - p_{11}\|_{H^1(\Omega)}$	$\|w_k - w_{11}\|$	$\frac{\|w_k - w_{11}\|}{\|w_{k-1} - w_{11}\|}$
1	6.36e-02	9.1e-03	8.5e-03	8.12e-02	–
2	3.97e-02	5.5e-03	5.1e-03	5.04e-02	6.26e-01
3	2.45e-02	3.3e-03	3.1e-03	3.09e-02	6.13e-01
4	1.50e-02	2.0e-03	1.8e-03	1.88e-02	6.08e-01
5	9.0e-03	1.2e-03	1.1e-03	1.13e-02	6.01e-01
6	5.4e-03	6.90e-04	6.39e-04	6.7e-03	5.92e-01
7	3.1e-03	3.65e-04	3.65e-04	3.9e-03	5.82e-01
8	1.7e-03	1.99e-04	1.99e-04	2.1e-03	5.38e-01
9	8.61e-04	9.88e-05	9.88e-05	1.1e-03	5.23e-01
10	3.29e-04	3.74e-05	3.74e-05	4.07e-04	3.70e-01

Tabelle 7.1: $\alpha = 0.1$, $s = 0.15$, Nint = 32

The next table corresponds to a finer grid and $s = 0.3$:

k	$\|a_k - a_{11}\|_{H^s(\Omega)}$	$\|y_k - y_{11}\|_{H^1(\Omega)}$	$\|p_k - p_{11}\|_{H^1(\Omega)}$	$\|w_k - w_{11}\|$	$\frac{\|w_k - w_{11}\|}{\|w_{k-1} - w_{11}\|}$
1	3.34e-01	4.62e-02	4.32e-02	4.23e-01	–
2	1.93e-01	2.40e-02	2.23e-02	2.40e-01	5.66e-01
3	1.02e-01	1.19e-02	1.11e-02	1.25e-01	5.20e-01
4	5.02e-02	5.7e-03	5.3e-03	6.11e-02	4.88e-01
5	2.36e-02	2.6e-03	2.4e-03	2.87e-02	4.69e-01
6	1.08e-02	1.2e-03	1.1e-03	1.31e-02	4.56e-01
7	4.8e-03	5.259e-04	4.88e-04	5.9e-03	4.50e-01
8	2.1e-03	2.25e-04	2.09e-04	2.5e-03	4.23e-01
9	8.35e-04	8.91e-05	8.27e-05	1.0e-03	4.00e-01
10	2.61e-04	2.77e-05	2.57e-05	3.15e-04	3.15e-01

Tabelle 7.2: $\alpha = 0.1$, $s = 0.3$, Nint $= 64$

In each last column we see, that the values are decreasing. Let us have a look at some plots. For the setting of the first table, we obtain

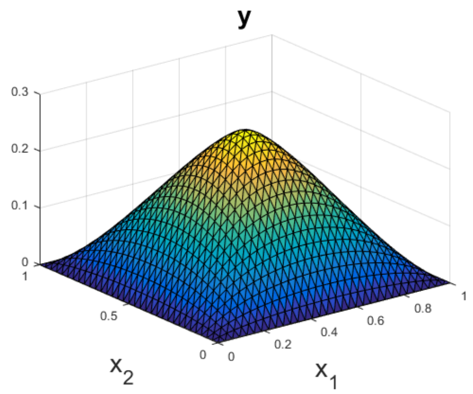

Figure 7.1: State y, $s = 0.15$

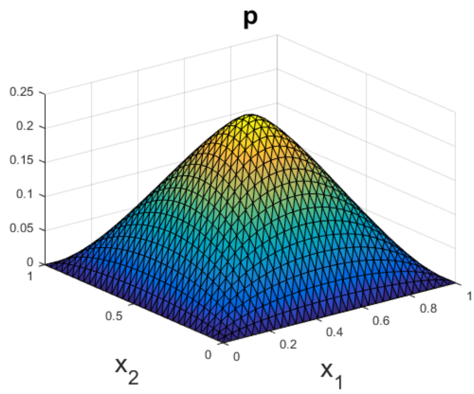

Figure 7.2: Adjoint state p, $s = 0.15$

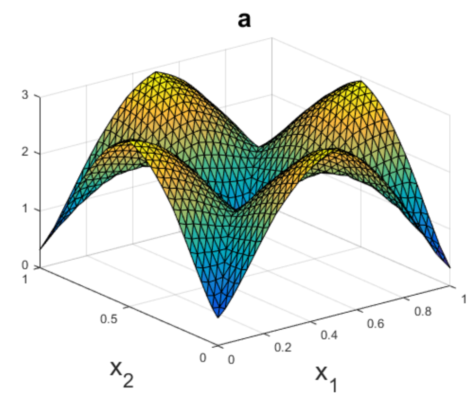

Figure 7.3: Parameter a, $s = 0.15$

The plots for the setting of the second table are the following:

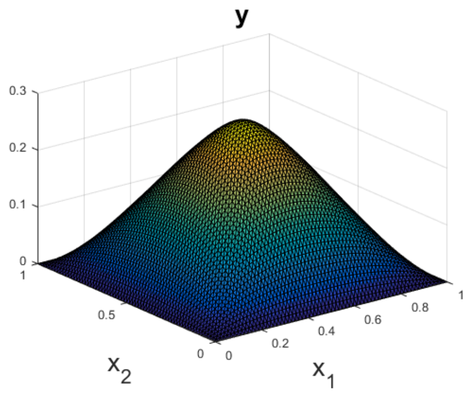

Figure 7.4: State y, $s = 0.3$

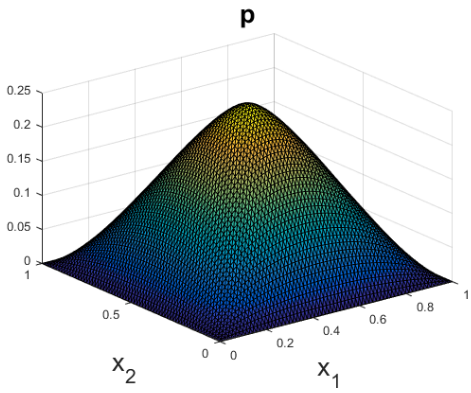

Figure 7.5: Adjoint state p, $s = 0.3$

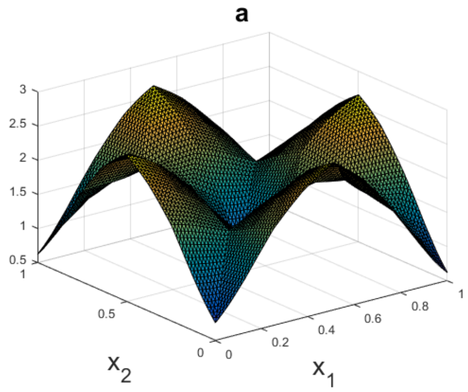

Figure 7.6: Parameter a, $s = 0.3$

There is something in the plots, that attracts our attention. We can already see, that for bigger sizes of s, the structure of the multilevel operator and the underlying grids gain more importance. This is even better to see, when s is further increasing. For other values of the regularization parameter s, we see the following plots of the parameter function:

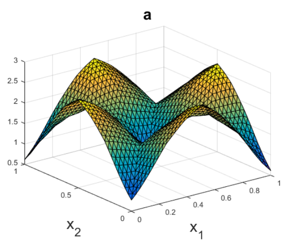

Figure 7.7: Parameter a, $s = 0.3$

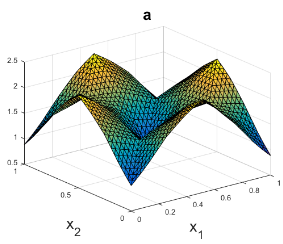

Figure 7.8: Parameter a, $s = 0.5$

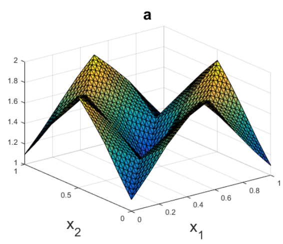

Figure 7.9: Parameter a, $s = 0.8$

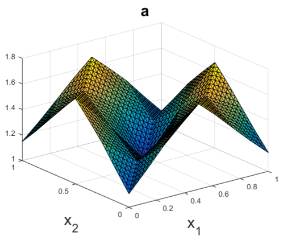

Figure 7.10: Parameter a, $s = 1$

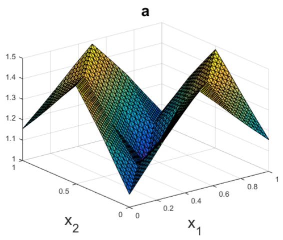

Figure 7.11: Parameter a, $s = 1.2$

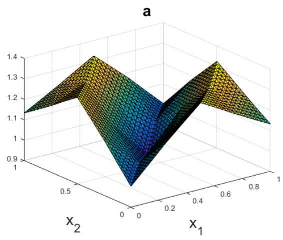

Figure 7.12: Parameter a, $s = 1.4$

This behavior is very unexpected, because we anticipated the results to become smoother for a growing parameter s. The last plots show the solutions for a fix s calculated on three grids that have different finenesses.

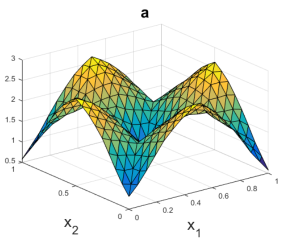

Figure 7.13: $s = 0.3$, Nint$= 16$

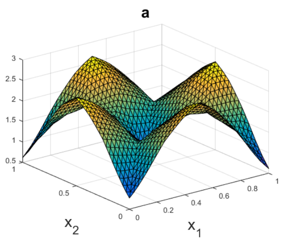

Figure 7.14: $s = 0.3$, Nint$= 32$

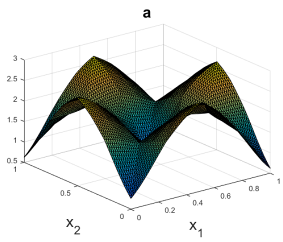

Figure 7.15: $s = 0.3$, Nint$= 64$

Appendix A

Useful estimates

The following well known estimates can be found in a lot of textbooks, e.g. [51], [56].

Lemma A.1 (Hölder's inequality)**.** *Let* $\Omega \in \mathbb{R}^n$ *and* $1 \leq p, q \leq \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *(where we understand* $\frac{1}{\infty} = 0$*). If* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then* $fg \in L^1(\Omega)$ *and there holds*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}. \tag{A.1}$$

Lemma A.2 (Cauchy-Schwarz inequality:)**.** *For the special case where* $p = q = 2$*, the inequality* (A.1) *is also known as Cauchy-Schwarz inquality, i.e. for* $f \in L^2(\Omega)$ *and* $g \in L^2(\Omega)$ *there holds* $fg \in L^1(\Omega)$ *and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)}. \tag{A.2}$$

A consequence of Hölder's inequality is the following:

Lemma A.3 (Interpolation inequality: Log convexity of L^p norms)**.** *Whenever* $0 < p, q < \infty$, $0 < \theta < 1$ *and* $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$*, there holds*

$$\|f\|_r \leq \|f\|_p^{1-\theta} \|f\|_q^{\theta}.$$

Lemma A.4 (Young's inequality)**.** *Let* $1 < p, p' < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$*. Then*

$$ab \leq \gamma a^p + C(\gamma) b^{p'} \qquad (a, b \geq 0, \gamma > 0) \tag{A.3}$$

for $C(\gamma) = (\gamma p)^{-p'/p} p'^{-1}$.

The following Lemma can be found in [[53], Lemma 9.11].

Lemma A.5. *For $s \in (0,1]$ let $w \in H_0^1(\mathcal{T}_h)$ be the uniquely determined solution of the variational problem*

$$\langle w, v\rangle_{H^1(\mathcal{T}_h)} = \langle u - Q_h^1 u, v\rangle_{H^{1-s}(\mathcal{T}_h)} \quad \text{for all } v \in H^1(\mathcal{T}_h).$$

If we assume $w \in H^{1+s}(\mathcal{T}_h)$ satisfying

$$\|w\|_{H^{1+s}(\mathcal{T}_h)} \leq c\|u - Q_h^1 u\|_{H^{1-s}(\mathcal{T}_h)},$$

then there holds the error estimate

$$\|u - Q_h^1 u\|_{H^{1-s}(\mathcal{T}_h)} \leq ch^s\|u - Q_h^1 u\|_{H^1(\mathcal{T}_h)}.$$

We know there holds the error estimate

$$\|u - Q_h^1 u\|_{H^1(\mathcal{T}_h)} \leq \|u\|_{H^1(\mathcal{T}_h)},$$

as a direct consequence of the Galerkin orthogonality

$$\langle u - Q_h^1 u, v_h\rangle_{H^1(\mathcal{T}_h)} = 0 \quad \text{for all } v_h \in S_h^1(\mathcal{T}_h).$$

Thus there also holds

$$\|u - Q_h^1 u\|_{H^{1-s}(\mathcal{T}_h)} \leq ch^s\|u\|_{H^1(\mathcal{T}_h)}. \tag{A.4}$$

Let us have a look at the transformation from an arbitrary element T to the reference element $\hat{T}$. Let B_T be an invertible matrix and $\hat{x} \mapsto x = B_T\hat{x} + b$. In the following lemma we show that the spectral norm of B_T can be estimated from below and from above by h_j^2, where h_j is the global mesh size of the decomposition $\mathcal{T}_j$.

Lemma A.6. *For $N = 2$ there holds*

$$\underline{c} \cdot h_j^2 \leq \|B_T\|^2 \leq \bar{c} \cdot h_j^2 \tag{A.5}$$

and

$$\underline{C} \cdot h_j^{-2} \leq \|B_T^{-1}\|^2 \leq \overline{C} \cdot h_j^{-2}. \tag{A.6}$$

Proof. The spectral norm to the power of two of B_T is given as $\|B_T\|^2 = \lambda_{\max}(B_T^\top B_T)$. Let us evaluate that. To that end, we denote the corners of an element T by x_1, x_2 and x_3. Then, the transformation matrix B_T is given by

$$B_T = \begin{pmatrix} x_{2,1} - x_{1,1} & x_{3,1} - x_{1,1} \\ x_{2,2} - x_{1,2} & x_{3,2} - x_{1,2} \end{pmatrix}.$$

For convenience we shorten the terms and set

$$a := |x_2 - x_1|, \quad b := |x_3 - x_1|, \quad \alpha := \sphericalangle(x_3 - x_1,\ x_2 - x_1).$$

Then, we obtain

$$B_T^\top B_T = \begin{pmatrix} a^2 & ab\cos(\alpha) \\ ab\cos(\alpha) & b^2 \end{pmatrix},$$

and the corresponding eigenvalues of $B_T^\top B_T$

$$\lambda_{1,2} = \frac{1}{2}[a^2 + b^2 \pm \sqrt{(a^2 - b^2)^2 + 4a^2b^2\cos^2(\alpha)}].$$

We easily see,

$$\lambda_{\max}(B_T^\top B_T) \leq a^2 + b^2 \leq 2\ d_j^2 \leq 2\ \overline{c}\ r_j^2 \leq \frac{2\ \overline{c}}{\pi}\Delta_j = \overline{c}\ h_j^2.$$

Furthermore we know there holds $\lambda_{\min}(B_T^\top B_T) \cdot \lambda_{\max}(B_T^\top B_T) = \det(B_T^\top B_T) = |\det B_T|^2 = 4\Delta_j^2 = 4h_j^4$, thus $\lambda_{\min}(B_T^\top B_T) = \frac{4h_j^4}{\lambda_{\max}(B_T^\top B_T)} \geq \underline{c}\ h_j^2$. All in all we obtain the result

$$\underline{c}\ h_j^2 \leq \lambda_{\min}(B_T^\top B_T) \leq \lambda_{\max}(B_T^\top B_T) = \|B_T\|^2 \leq \overline{c}\ h_j^2.$$

The same can be done for B_T^{-1}. □

In [8] the authors develop some nonstandard finite element estimates in fractional order Sobolev spaces. Therefore, they introduce a technique to handle globally defined fractional order Sobolev norms in some local way. The following lemma is taken from this article.

Lemma A.7. *Let Ω be a bounded polyhedral domain in $\mathbb{R}^N$ for $N = 1, 2, 3$. Let T_j be a regular triangulation of Ω, where h_j is the global mesh size. Let furthermore k be a nonnegative integer, $\lambda \in (0,1)$ and $w \in H^{k+\lambda}(\Omega)$. Then, the following error estimate holds:*

$$|w|^2_{H^{k+\lambda}(\Omega)} \leq c_\lambda \sum_{|\alpha|=k} \sum_{T \in \mathcal{T}_j} \left(|\partial^\alpha w|^2_{H^\lambda(T)} + \int_T \frac{[\partial^\alpha w(x)]^2}{\rho(x, \partial T)^{2\lambda}} dx \right), \tag{A.7}$$

where $\rho(x, \partial T) = \inf_{y \in \partial T} |x - y|$ is the distance from x to the boundary of T.

Furthermore, they provide an important estimate on the reference element,

$$\int_{\hat{T}} \frac{u^2(x)}{\rho(x, \partial\hat{T})^{2\lambda}} dx \leq C_{\hat{T},\lambda} \|u\|^2_{H^\lambda(\hat{T})} \quad \forall u \in H^\lambda(\hat{T}) \text{ and } 0 < \lambda < \frac{1}{2}. \tag{A.8}$$

The proof of the lemma is given in the above-named article and the proof of the last estimate is given in the references therein.
Let us state a well-known result published by Maurer and Zowe, see [Lemma 5.5, [44]]

Lemma A.8. *Let B be a continuous symmetric bilinear form on $X \times X$, H a subset of X and $\delta > 0$ with*

$$B(h, h) \geq \delta \|h\|^2 \quad \textit{for all } h \in H.$$

Then, there are $\delta_0 > 0$ and $\gamma > 0$ such that

$$B(h + z, h + z) \geq \delta_0 \|h + z\|^2 \quad \textit{for all } h \in H,\ z \in X \textit{ and } \|z\| \leq \gamma \|h\|.$$

Appendix B

Implementation of the FE-Method

B.1 Assumptions

- Domain: $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$
- Decomposition: uniform triangulations and uniform refinement strategy, $\mathcal{T}_h, h = 0, \ldots, L$.
- Trial space: For both, the parameter function and the state we use as trial spaces the spaces of piecewise linear continuous functions with different refinements that we call V_h with mesh size h. There holds

$$V_0 \subset V_1 \subset \cdots \subset V_L \subset \cdots \subset H^s(\Omega) \text{ for } 0 \leq s < 3/2.$$

- Notation:
  - Let n_h be the number of nodes.
  - Let N_h be the number of finite elements of the triangulation $\mathcal{T}_h$.
  - Let $(\cdot, \cdot)_{n_h}$ denote the Euclidean scalar product in $\mathbb{R}^{n_h}$.
  - Let Nint denote the number of elements between zero and one on the x-axis, thus Nint_h^{-1} is the length of the elements of decomposition $\mathcal{T}_h$.
  - Local numbering: We denote the node at the right angle of every element by one and continue in clockwise direction.

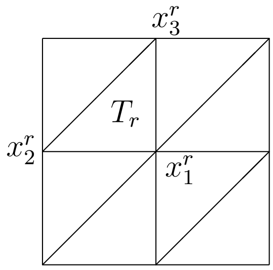

Figure B.1: Example for local numbering of nodes

We can write down a function y_h in the trial space V_h as follows:

$$y_h(x) = \sum_{j=1}^{n_h} \mathbf{y_j} \varphi_j(x)$$

with real unknowns $\mathbf{y_j},\ j = 1, \ldots, n_h$, that we combine in a vector $\mathbf{y} = (\mathbf{y_j})_{j=1}^{n_h}$. The parameter-to-state equation in weak form

$$\int_\Omega a \nabla y_k \cdot \nabla v \; dx + \int_\Omega a_k \nabla y \cdot \nabla v \; dx = \int_\Omega g \; v \; dx + \int_\Omega a_k \nabla y_k \cdot \nabla v \; dx,$$

has the following aspect subject to the trial space (for $j = 1, ...n_h$)

$$\int_\Omega \sum_{m=1}^{n_h} \mathbf{a_m} \varphi_m \left(\sum_{i=1}^{n_h} \mathbf{y_i^k} \nabla \varphi_i \right)^\top \nabla \varphi_j \; dx + \int_\Omega \sum_{m=1}^{n_h} \mathbf{a_m^k} \varphi_m \left(\sum_{i=1}^{n_h} \mathbf{y_i} \nabla \varphi_i \right)^\top \nabla \varphi_j \; dx$$

$$= \int_\Omega g \; \varphi_j \; dx + \int_\Omega \sum_{m=1}^{n_h} \mathbf{a_m^k} \varphi_m \left(\sum_{i=1}^{n_h} \mathbf{y_i^k} \nabla \varphi_i \right)^\top \cdot \nabla \varphi_j \; dx.$$

All n equations together compose a linear system for $\mathbf{y} = (\mathbf{y_1}, \ldots, \mathbf{y_{n_h}})^\top$ and $\mathbf{a} = (\mathbf{a_1}, \ldots, \mathbf{a_{n_h}})^\top$.

$$\mathbb{KY}_h \cdot \mathbf{a} + \mathbb{KA}_h \cdot \mathbf{y} = \mathbf{G_h} + \mathbb{KA}_h \cdot \mathbf{y_k}$$

with matrices

$$\mathbb{KY}_h = [\mathbb{KY}_{h,ij}]_{i,j=1}^{n_h} = \left[\int_\Omega \varphi_i \left(\sum_{m=1}^{n_h} \mathbf{y_m^k} \nabla \varphi_m \right)^\top \nabla \varphi_j \; dx \right]_{i,j=1}^{n_h},$$

$$\mathbb{KA}_h = [\mathbb{KA}_{h,ij}]_{i,j=1}^{n_h} = \left[\int_\Omega \sum_{m=1}^{n_h} \mathbf{a_m^k} \varphi_m \nabla \varphi_i^\top \nabla \varphi_j \; dx \right]_{i,j=1}^{n_h}$$

and the load vector

$$\mathbf{G_h} = \left[\int_\Omega g \varphi_i dx \right]_{i=1}^{n_h}.$$

B.2 Assembly of matrices

B.2.1 Load vector

We want to implement $\mathbf{G_h} = \left[\int_\Omega g\varphi_i dx\right]_{i=1}^{n_h}$ for a right-hand side g . Therefore we determine local load vectors for each element.

$$\mathbf{G^r} = \left[\int_{T_r} g\varphi_i dx\right]_{i=1}^{3} = \left[|\det B_{T_r}| \int_{\hat{T}} gp_\alpha d\hat{x}\right]_{\alpha=1}^{3}$$

Using a node quadrature rule we obtain with $|\det B_{T_r}| = 2\text{meas}(T_r) = \frac{1}{\text{Nint}_h^2}$

$$\mathbf{G^r} = \frac{1}{6}\frac{1}{\text{Nint}_h^2}\left[g(\hat{x}_1); g(\hat{x}_2); g(\hat{x}_3)\right],$$

where $\hat{x}_1, \hat{x}_2$ and $\hat{x}_3$ are the local nodes of T_r. Then we assemble the load vector by adding the entries of the local load vectors to the corresponding entries of the global load vector.

B.2.2 Mass matrix

The purpose is to calculate the L^2 scalar product in the trial spaces V_h, for $h = 1, \ldots, L$. For functions $u_h(x) = \sum_{i=1}^{n_h} \mathbf{u_i^h}\varphi_i(x)$ and $v_h(x) = \sum_{j=1}^{n_h} \mathbf{v_j^h}\varphi_j(x)$ there holds

$$\int_\Omega u_h(x)v_h(x)\, dx = \sum_{i=1}^{n_h} \mathbf{u_i^h} \sum_{j=1}^{n_h} \mathbf{v_j^h} \int_\Omega \varphi_i(x)\varphi_j(x)\, dx = (\mathbb{M}\mathbf{v_h}, \mathbf{u_h})_{n_h}.$$

where $\mathbf{u_h}$ and $\mathbf{v_h}$ are the vectors consisting of the node values belonging to u_h and v_h, respectively. Let us determine the local element mass matrices,

$$\mathbb{M}^r = \left[\int_{T_r} \varphi_\alpha\varphi_\beta dx\right]_{\alpha,\beta=1}^{3} = \left[\int_{\widehat{T}} p_\alpha p_\beta |\det B_{T_r}| d\hat{x}\right]_{\alpha,\beta=1}^{3}.$$

For $p_1 = 1 - \hat{x}_1 - \hat{x}_2$, $p_2 = \hat{x}_1$, $p_3 = \hat{x}_2$ we calculate

$$\int_{\widehat{T}} p_1 p_1 d\hat{x} = \int_0^1 \int_0^{1-\hat{x}_1} (1-\hat{x}_1-\hat{x}_2)(1-\hat{x}_1-\hat{x}_2) d\hat{x}_2 d\hat{x}_1 = \frac{1}{12}$$

and likewise

$$\int_{\widehat{T}} p_2 p_2 d\hat{x} = \int_{\widehat{T}} p_3 p_3\, d\hat{x} = \frac{1}{12},$$

$$\int_{\widehat{T}} p_2 p_1 d\hat{x} = \int_{\widehat{T}} p_1 p_2 d\hat{x} = \int_{\widehat{T}} p_3 p_1 d\hat{x} =$$
$$\int_{\widehat{T}} p_1 p_3 d\hat{x} = \int_{\widehat{T}} p_2 p_3 d\hat{x} = \int_{\widehat{T}} p_3 p_2 d\hat{x} = \frac{1}{24}.$$

Thus, the local mass matrices are given as

$$\mathbb{M}^r = \frac{\mathrm{Nint}_h^{-2}}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

With the map $\alpha \leftrightarrow j = j(r, \alpha), \quad \alpha \in \{1, 2, 3\}, j \in \{1, \dots, n_h\}$ between the local and global numbering of nodes we know where to add the entries of the local mass matrix in the mass matrix $\mathbb{M}$. It remains to incorporate the homogeneous Dirichlet boundary conditions. To that end, we set equal to zero all entries in rows and columns belonging to a node on the boundary, except for the entries on the diagonal, which are set equal to one.

B.2.3 Stiffness matrices

The stiffness matrices $\mathbb{KA}_k$ and $\mathbb{KY}_k$ occur in the parameter-to-state equation. They differ fundamentally from each other and have to be treated seperately. Let us start with $\mathbb{KA}_k$. Let $a_k = \sum_{m=1}^{n_h} \mathbf{a^k_m} \varphi_m \in V_h$ be given. For arbitrary $y_h,\ v_h \in V_h$ we want to calculate

$$\begin{aligned}
(a_k \nabla y_h, \nabla v_h)_{(L^2(\Omega))^{n_h}} &= \int_\Omega \sum_{m=1}^{n_h} \mathbf{a^k_m} \varphi_m \left(\sum_{j=1}^{n_h} \mathbf{v^h_j} \nabla \varphi_j \right)^\top \sum_{i=1}^{n_h} \mathbf{y^h_i} \nabla \varphi_i \, dx \\
&= \sum_{r=1}^{N_h} \sum_{\beta=1}^{3} \sum_{\alpha=1}^{3} \mathbf{y^h_\beta v^h_\alpha} \int_{T_r} \sum_{\gamma=1}^{3} \mathbf{a^k_\gamma} \varphi_\gamma \nabla \varphi_\alpha^\top \nabla \varphi_\beta \, dx \\
&= \sum_{r=1}^{N_h} (\mathbb{KA}_k^r \mathbf{y^r_h}, \mathbf{v^r_h}).
\end{aligned}$$

The vectors $\mathbf{y^r_h}$ and $\mathbf{v^r_h}$ are 3-dimensional and consist of the values of y_h and v_h at the nodes of T_r. $\mathbb{KA}_k^r$ is the element stiffness matrix of T_r, which is given as

$$\mathbb{KA}_k^r = [\mathbb{KA}_k^r(\alpha, \beta)]_{\alpha,\beta=1}^3 = \left[\int_{T^r} \sum_{\gamma=1}^{3} \mathbf{a^k_\gamma} \varphi_\gamma (\nabla \varphi_\alpha^r)^\top \nabla \varphi_\beta^r \, dx \right]_{\alpha,\beta=1}^3.$$

We transform everything onto the refence element and obtain

$$\mathbb{KA}_k^r(\alpha,\beta) = \int_{T^r} \sum_{\gamma=1}^{3} \mathbf{a}_{\boldsymbol{\gamma}}^{\mathbf{k}} \varphi_\gamma(x)(\nabla_x \varphi_\alpha^r(x))^\top \nabla_x \varphi_\beta^r(x)\, dx$$
$$= \int_{\widehat{T}} \sum_{\gamma=1}^{3} \mathbf{a}_{\boldsymbol{\gamma}}^{\mathbf{k}} p_\gamma(\hat{x})(\nabla_{\hat{x}} p_\alpha^r(\hat{x}))^\top (B_{T^r})^{-1}(B_{T^r})^{-\top} \nabla_{\hat{x}} p_\beta^r(\hat{x})\, |\det B_{T^r}|\, d\hat{x}.$$

A short computation yields

$$|\det B_{T^r}|(B_{T^r})^{-1}(B_{T^r})^{-\top} = \frac{|\det B_{T^r}|}{(\det B_{T^r})^2} \begin{pmatrix} x_{3,2}^r - x_{1,2}^r & -(x_{3,1}^r - x_{1,1}^r) \\ -(x_{2,2}^{(r)} - x_{1,2}^{(r)}) & x_{2,1}^{(r)} - x_{1,1}^{(r)} \end{pmatrix}$$
$$\cdot \begin{pmatrix} x_{3,2}^r - x_{1,2}^r & -(x_{2,2}^r - x_{1,2}^r) \\ -(x_{3,1}^r - x_{1,1}^r) & x_{2,1}^r - x_{1,1}^r \end{pmatrix}$$
$$= \frac{1}{|\det B_{T^r}|} \begin{pmatrix} \frac{1}{\mathrm{Nint}_h} & 0 \\ 0 & -\frac{1}{\mathrm{Nint}_h} \end{pmatrix} \begin{pmatrix} \frac{1}{\mathrm{Nint}_h} & 0 \\ 0 & -\frac{1}{\mathrm{Nint}_h} \end{pmatrix}$$
$$= \mathrm{Nint}_h^2 \begin{pmatrix} \frac{1}{\mathrm{Nint}_h^2} & 0 \\ 0 & \frac{1}{\mathrm{Nint}_h^2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There follows

$$\mathbb{KA}_k^r = [\mathbb{KA}_k^r(\alpha,\beta)]_{\alpha,\beta=1}^3 = [\int_{\widehat{T}} \sum_{\gamma=1}^{3} \mathbf{a}_{\boldsymbol{\gamma}}^{\mathbf{k}} p_\gamma (\nabla_{\hat{x}} p_\alpha)^\top \nabla_{\hat{x}} p_\beta d\hat{x}]_{\alpha,\beta=1}^3$$
$$= \int_{\widehat{T}} \sum_{\gamma=1}^{3} \mathbf{a}_{\boldsymbol{\gamma}}^{\mathbf{k}} p_\gamma \begin{bmatrix} (\nabla_{\hat{x}} p_1)^\top \nabla_{\hat{x}} p_1 & (\nabla_{\hat{x}} p_2)^\top \nabla_{\hat{x}} p_1 & (\nabla_{\hat{x}} p_3)^\top \nabla_{\hat{x}} p_1 \\ (\nabla_{\hat{x}} p_1)^\top \nabla_{\hat{x}} p_2 & (\nabla_{\hat{x}} p_2)^\top \nabla_{\hat{x}} p_2 & (\nabla_{\hat{x}} p_3)^\top \nabla_{\hat{x}} p_2 \\ (\nabla_{\hat{x}} p_1)^\top \nabla_{\hat{x}} p_3 & (\nabla_{\hat{x}} p_2)^\top \nabla_{\hat{x}} p_3 & (\nabla_{\hat{x}} p_3)^\top \nabla_{\hat{x}} p_3 \end{bmatrix} d\hat{x}.$$

The basis functions on the reference element are given as $p_1 = 1 - \hat{x}_1 - \hat{x}_2$, $p_2 = \hat{x}_1$, $p_3 = \hat{x}_2$, with $\nabla p_1 = \binom{-1}{-1}$, $\nabla p_2 = \binom{1}{0}$, $\nabla p_3 = \binom{0}{1}$. Thus

$$\mathbb{KA}_k^r = \int_{\widehat{T}} \sum_{\gamma=1}^{3} \mathbf{a}_{\boldsymbol{\gamma}}^{\mathbf{k}} p_\gamma \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} d\hat{x}$$
$$= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \int_{\widehat{T}} \mathbf{a}_{\mathbf{1}}^{\mathbf{k}}(1 - \hat{x}_1 - \hat{x}_2) + \mathbf{a}_{\mathbf{2}}^{\mathbf{k}} \hat{x}_1 + \mathbf{a}_{\mathbf{3}}^{\mathbf{k}} \hat{x}_2\, d\hat{x}$$
$$= \frac{1}{6}(\mathbf{a}_{\mathbf{1}}^{\mathbf{k}} + \mathbf{a}_{\mathbf{2}}^{\mathbf{k}} + \mathbf{a}_{\mathbf{3}}^{\mathbf{k}}) \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

This is due to $\int_{\widehat{T}} 1-\hat{x}_1-\hat{x}_2 \; d\hat{x} = \int_{\widehat{T}} \hat{x}_1 \; d\hat{x} = \int_{\widehat{T}} \hat{x}_2 \; d\hat{x} = \frac{1}{6}$.
Next, we assemble the matrix and incorporate the Dirichlet boundary values as before.
For the matrix $\mathbb{KY}_k$ we fix $y_k = \sum_{m=1}^{n_h} \mathbf{y^k_m}\varphi_m \in V_h$ and evaluate for arbitrary $a_h,\ v_h \ \in V_h$ the following

$$
\begin{aligned}
(a_h \nabla y_k, \nabla v_h)_{(L^2(\Omega))^{n_h}} &= \int_\Omega \left(\sum_{m=1}^{n_h} \mathbf{y^k_m}\nabla\varphi_m\right)^\top \sum_{i=1}^{n_h} \mathbf{a^h_i}\varphi_i \sum_{j=1}^{n_h} \mathbf{v^h_j}\nabla\varphi_j \; dx \\
&= \sum_{r=1}^{N_h}\sum_{\beta=1}^{3}\sum_{\alpha=1}^{3} \mathbf{a^h_{\boldsymbol\beta}}\mathbf{v^h_{\boldsymbol\alpha}} \int_{T^r} \left(\sum_{\gamma=1}^{3} \mathbf{y^k_\gamma}\nabla\varphi_\gamma\right)^\top \nabla\varphi_\alpha\varphi_\beta \; d\hat{x} \\
&= \sum_{r=1}^{N_h} (\mathbb{KY}^r_k \mathbf{a^r_h}, \mathbf{v^r_h}).
\end{aligned}
$$

Again, $\mathbf{a^r_h}$ and $\mathbf{v^r_h}$ are vectors with values of a_h und v_h at the nodes of T^r as entries. $\mathbb{KY}^r_k$ denotes the element stiffness matrix.

$$
\mathbb{KY}^r_k = [\mathbb{KY}^r_k(\alpha,\beta)]^3_{\alpha,\beta=1} = \left[\int_{T^r} \left(\sum_{\gamma=1}^{3} \mathbf{y^k_\gamma}\nabla\varphi_\gamma\right)^\top \nabla\varphi_\alpha\varphi_\beta dx\right]^3_{\alpha,\beta=1}.
$$

Transformation on refenence element

$$
\mathbb{KY}^r_k(\alpha,\beta) = \int_{\widehat{T}} \left(\sum_{\gamma=1}^{3} \mathbf{y^k_\gamma}\nabla_{\hat{x}}p_\gamma\right)^\top (B_{T^r})^{-1}(B_{T^r})^{-\top}(\nabla_{\hat{x}}p_\alpha)p_\beta |\det B_{T^r}| d\hat{x}.
$$

Again, we see $|\det B_{T^r}|(B_{T^r})^{-1}(B_{T^r})^{-\top} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and hence

$$
\mathbb{KY}^r_k(\alpha,\beta) = \int_{\widehat{T}} \left(\sum_{\gamma=1}^{3} \mathbf{y^k_\gamma}\nabla_{\hat{x}}p_\gamma\right)^\top (\nabla_{\hat{x}}p_\alpha)p_\beta \; d\hat{x}.
$$

Thus, with $\mathbf{Y} := \sum_{\gamma=1}^{3} \mathbf{y^k_\gamma}\nabla_{\hat{x}}p_\gamma$ we obtain

$$
\begin{aligned}
\mathbb{KY}^r_k &= [\mathbb{KY}^r_k(\alpha,\beta)]^3_{\alpha,\beta=1} \\
&= \int_{\widehat{T}} \begin{bmatrix} (\mathbf{Y}^\top\nabla_{\hat{x}}p_1)p_1 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_1)p_2 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_1)p_3 \\ \mathbf{Y}^\top(\nabla_{\hat{x}}p_2)p_1 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_2)p_2 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_2)p_3 \\ \mathbf{Y}^\top(\nabla_{\hat{x}}p_3)p_1 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_3)p_2 & \mathbf{Y}^\top(\nabla_{\hat{x}}p_3)p_3 \end{bmatrix} d\hat{x}.
\end{aligned}
$$

With the basis functions $p_1 = 1-\hat{x}_1-\hat{x}_2$, $p_2 = \hat{x}_1$, $p_3 = \hat{x}_2$ and with $\nabla p_1 = \binom{-1}{-1}$, $\nabla p_2 = \binom{1}{0}$, $\nabla p_3 = \binom{0}{1}$ we deduce

$$\mathbb{KY}_k^r = \int_{\widehat{T}} \begin{bmatrix} \mathbf{Y}^\top \binom{-1}{-1}(1-\hat{x}_1-\hat{x}_2) & \mathbf{Y}^\top \binom{-1}{-1}\hat{x}_1 & \mathbf{Y}^\top \binom{-1}{-1}\hat{x}_2 \\ \mathbf{Y}^\top \binom{1}{0}(1-\hat{x}_1-\hat{x}_2) & \mathbf{Y}^\top \binom{1}{0}\hat{x}_1 & \mathbf{Y}^\top \binom{1}{0}\hat{x}_2 \\ \mathbf{Y}^\top \binom{0}{1}(1-\hat{x}_1-\hat{x}_2) & \mathbf{Y}^\top \binom{0}{1}\hat{x}_1 & \mathbf{Y}^\top \binom{0}{1}\hat{x}_2 \end{bmatrix} d\hat{x}$$

and with $\mathbf{Y}^\top = \left(\mathbf{y_1^k}\binom{-1}{-1} + \mathbf{y_2^k}\binom{1}{0} + \mathbf{y_3^k}\binom{0}{1}\right)^\top = \left(-\mathbf{y_1^k}+\mathbf{y_2^k},\ -\mathbf{y_1^k}+\mathbf{y_3^k}\right)$ and $\int_{\widehat{T}} 1-\hat{x}_1-\hat{x}_2\, d\hat{x} = \int_{\widehat{T}} \hat{x}_1\, d\hat{x} = \int_{\widehat{T}} \hat{x}_2\, d\hat{x} = \frac{1}{6}$ we conclude

$$\mathbb{KY}_k^r = \frac{1}{6}\begin{bmatrix} 2\mathbf{y_1^k}-\mathbf{y_2^k}-\mathbf{y_3^k} & 2\mathbf{y_1^k}-\mathbf{y_2^k}-\mathbf{y_3^k} & 2\mathbf{y_1^k}-\mathbf{y_2^k}-\mathbf{y_3^k} \\ -\mathbf{y_1^k}+\mathbf{y_2^k} & -\mathbf{y_1^k}+\mathbf{y_2^k} & -\mathbf{y_1^k}+\mathbf{y_2^k} \\ -\mathbf{y_1^k}+\mathbf{y_3^k} & -\mathbf{y_1^k}+\mathbf{y_3^k} & -\mathbf{y_1^k}+\mathbf{y_3^k} \end{bmatrix},$$

due to $\int_{\widehat{T}} 1-\hat{x}_1-\hat{x}_2\ d\hat{x} = \int_{\widehat{T}} \hat{x}_1\ d\hat{x} = \int_{\widehat{T}} \hat{x}_2\ d\hat{x} = \frac{1}{6}$. We assembly and incorporate the Dirichlet boundary values. The same assembly can be done for a fixed p_k to obtain $\mathbb{KP}_k$.
The transposed matrices ${\mathbb{KY}_k}^\top$ and ${\mathbb{KP}_k}^\top$ are needed for the gradient equation to evaluate the term $(\nabla y_k^\top \nabla p_h, v_h)_{L^2(\Omega)}$ for a given $y_k \in V_h$ and the term $(\nabla y_h^\top \nabla p_k, v_h)_{L^2(\Omega)}$ for a given $p_k \in V_h$.

B.3 Implementation of multilevel operator

B.3.1 Representation of coarse functions in refined grids

In the following we implement matrices that map functions on coarse grids onto functions on finer grids, as it is done in [[53], pp. 317-319]. Certainly, the functions remain the same and the further node values are obtained via interpolation. We represent the coarse basis functions by a linear combination of fine basis functions and store the corresponding coefficients in a matrix. Let us have a look at trial spaces V_{h-1} and V_h, with $V_{h-1} \subset V_h$. For $\varphi_i^{h-1} \in V_{h-1}$, $i = 1, \dots, n_{h-1}$ we obtain

$$\varphi_i^{h-1} = \sum_{j=1}^{n_h} \mathbf{r_{i,j}^h} \varphi_j^h, \quad \text{for all } i = 1, \dots, n_{h-1},$$

see Figure B.2.

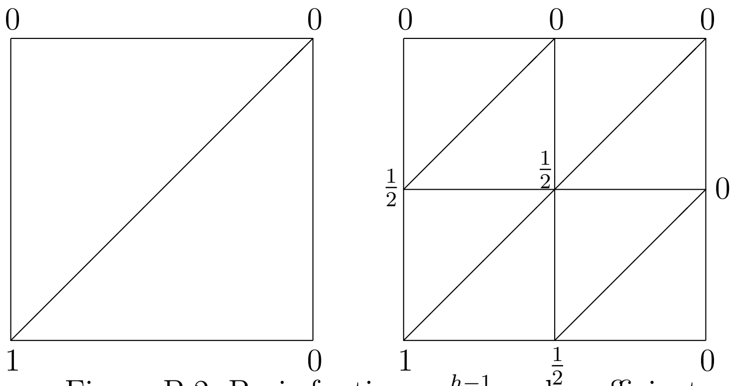

Figure B.2: Basis funtions φ_i^{h-1} and coefficients $\mathbf{r}_{\mathbf{i,j}}^{\mathbf{h}}$ in two dimensions

Thus, we obtain a matrix for the transition from V_{h-1} to V_h,

$$\mathbb{R}_{h-1,h}[i,j] = \mathbf{r}_{\mathbf{i,j}}^{\mathbf{h}} \qquad \text{für } j = 1,\ldots,l_h,\ i = 1,\ldots,n_{h-1}.$$

For a transition from V_h to the finest trial space V_L we apply them consecutively

$$\mathbb{R}_h := \mathbb{R}_{L-1,L}\ldots\mathbb{R}_{h,h+1} \quad \text{für } h = 0,\ldots,L-1, \quad \mathbb{R}_L := \mathbb{I},$$

where $\mathbb{I}$ is a $n_h \times n_h$ identity matrix. Thus, we are able to blow up coarser functions such that they all have the same number of entries as functions on the finest grid. It is important to mention that the shape of the function does not change , see B.3. For $a_h \in V_h$ there holds

$$a_h = \sum_{i=1}^{n_h} \mathbf{a}_{\mathbf{h}}^{\mathbf{i}} \varphi_i^h = \sum_{j=1}^{n_L} \mathbf{a}_{\mathbf{L}}^{\mathbf{j}} \varphi_j^L.$$

This procedure is important, because we have to sum up functions from differently refined grids. Therefore, they need to have the same dimension.

B.3.2 L^2-projection

Let $a_L \in V_L$, where V_L is the finest trial space. We want to find the L^2-projection of a_L, i.e. $a_k = Q_k a_L$ onto coarser trial spaces V_k, $k = 1\ldots,L-1$, which is defined by the variational equations

$$\langle a_k, \varphi_l^k \rangle_{L^2(\Omega)} = \langle a_L, \varphi_l^k \rangle_{L^2(\Omega)} \qquad \forall \varphi_l^k \in V_k. \tag{B.1}$$

Here, φ_l^k are again the basis functions belonging to the trial space V_k. This is equivalent to the linear system

$$\mathbb{M}_k \mathbf{a_k} = \mathbf{F_k}$$

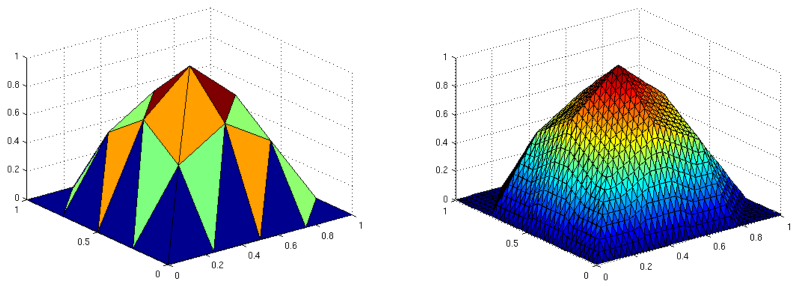

Figure B.3: Coarse function on coarse grid and fine grid

with mass matrix and load vector

$$\mathbb{M}_k = \mathbb{M}_k[l, j] = \langle \varphi_j^k, \varphi_l^k \rangle_{L^2(\Omega)}, \qquad \mathbf{F_{k,l}} = \langle a_L, \varphi_l^k \rangle_{L^2(\Omega)}.$$

We invert $\mathbb{M}_k$ and obtain

$$\mathbf{a_k} = \mathbb{M}_k^{-1} \mathbf{F_k} \qquad \text{for all } k = 1, \dots, L.$$

For the load vector there holds for $k \in \{1, \dots, L-1\}$

$$\mathbf{F_{k,l}} = \langle a_L, \varphi_l^k \rangle_{L^2(\Omega)} = \sum_{j=1}^{M_{k+1}} r_{l,j}^{k+1} \langle a_L, \varphi_j^{k+1} \rangle_{L^2(\Omega)} = \sum_{j=1}^{M_{k+1}} r_{l,j}^{k+1} \mathbf{F_{k+1,j}}$$

and thus

$$\mathbf{F_k} = \mathbb{R}_{k,k+1}^\top \mathbf{F_{k+1}} = \mathbb{R}_k^\top \mathbf{F_L}.$$

We know that $\mathbf{F_{L,l}}$ is given as

$$\mathbf{F_{L,l}} = \langle a_L, \varphi_l^L \rangle_{L^2(\Omega)} = \langle \sum_{i=1}^{L} \mathbf{a_{L,i}} \varphi_i^L, \varphi_l^L \rangle_{L^2(\Omega)}.$$

Hence, there holds

$$\mathbf{F_L} = \mathbb{M}_L \cdot \mathbf{a_L}.$$

Thus, the L^2-projection is given as

$$\mathbf{a_k} = \mathbb{M}_k^{-1} \mathbb{R}_k^\top \mathbb{M}_L \cdot \mathbf{a_L}$$

and we set

$$\mathbb{P}_k := \mathbb{M}_k^{-1} \mathbb{R}_k^\top \mathbb{M}_L.$$

B.3.3 Multilevel operator

We introduced the multilevel operator for $s \in [0, \frac{3}{2})$ as

$$B^s := \sum_{k=0}^{\infty} h_k^{-2s}(Q_k - Q_{k-1}).$$

It fulfills the spectral equivalence inequalities,

$$c_1\|v\|^2_{H^s(\Omega)} \leq \langle B^s v, v\rangle_{L^2(\Omega)} \leq c_2\|v\|^2_{H^s(\Omega)} \quad \text{for all } v \in H^s(\Omega),$$

see chapter 4. For a function $v_L \in V_L$ we want to implement

$$B^s v_L = \sum_{k=0}^{\infty} h_k^{-2s}(Q_k - Q_{k-1})v_L = \sum_{k=0}^{L} h_k^{-2s}(Q_k - Q_{k-1})v_L \quad \in V_L$$

The sum reduces to a finite sum because there holds $Q_k v_L = v_L$ for $k \geq L$. We restructure the sum and get

$$B^s v_L = \sum_{k=0}^{L} h_k^{-2s}(Q_k - Q_{k-1})v_L = h_L^{-2s} Q_L v_L + \sum_{k=0}^{L-1} (h_k^{-2s} - h_{k+1}^{-2s}) Q_k v_L.$$

Let $\mathbf{v_L}$ denote the vector of node values of v_L. We implement the multilevel operator as the following matrix $\mathbb{B}$

$$\mathbb{B}\mathbf{v_L} = H_L^{-2s}\mathbb{P}_L\mathbf{v_L} + \sum_{k=0}^{L-1} \left(H_k^{-2s} - H_{k+1}^{-2s}\right) \mathbb{P}_k\mathbf{v_L}$$

with the L^2 projection $\mathbb{P}_k, k = 1, \ldots, L$ and $H := \frac{1}{\mathrm{Nint}_h}$.

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