OMEGA LIMITS, PROLONGATIONAL LIMITS AND ALMOST PERIODIC POINTS OF A CONTINUOUS FLOW VIA EXTERIOR SPACES

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Abstract. In this paper we analyse some applications of the category of exterior spaces to the study of dynamical systems (flows). The limit space and end space of an exterior space are used to construct different types of limit spaces and end spaces of a dynamical system. In this work we analyse the relationships between the notions and constructions given by the exterior structures of a continuous flow and the more usual notions of omega-limits, first prolongational limits and several types of almost periodic points (Poisson-stable points, non-wandering points) of a flow.

1. Introduction

Recently the use of the category of exterior spaces has been applied to give a new approach to the study of continuous and discrete dynamical systems, see [19, 20]. This paper contains new results about the relations and applications of the category of exterior spaces to the study of continuous flows.

The category of exterior spaces has been provided with a well developed homotopy theory ([12, 16–18, 21]). The study of the exterior and proper homotopy invariants has proved to be useful in the study of non-compact manifolds ([8, 30]), the study of the shape of some compact spaces ([22]), the L-S proper category ([14, 15]), et cetera.

On the one hand, this category of exterior spaces has its own notions and constructions and others inherited from the category of topological spaces and from categories of pro-spaces ([10]). For example, for categories of pro-spaces one can consider the inverse limit of a prospace and homotopy invariants such as homotopy pro-groups and Steenrod, Čech and Brown-Grossmann groups (we note that the zero-dimensional Čech invariant is the set of Freudenthal ends when we consider the inverse system of cocompact subsets in a topological space).

Let us recall the definition of exterior space and some basic constructions (in this paper the exterior concepts and constructions are typed in **boldface**): Let $(X, \mathbf{t}_X)$ be a topological space, where $\mathbf{t}_X$ denotes the topology of X. An **externology** on $(X, \mathbf{t}_X)$ is a non-empty collection $\varepsilon(X)$ of open subsets which is closed under finite intersections and such that if $E \in \varepsilon(X)$, $U \in \mathbf{t}_X$ and $E \subset U$, then $U \in \varepsilon(X)$. The members of $\varepsilon(X)$ are called **exterior open subsets**. An **exterior space** $X = (X, \varepsilon(X), \mathbf{t}_X)$ consists of a topological space $(X, \mathbf{t}_X)$ together with an externology $\varepsilon(X)$. The **limit space** of an exterior space X is the topological subspace $L(X) = \cap_{E \in \varepsilon(X)} E$ and the **bar-limit space** of X is the topological subspace $\bar{L}(X) = \cap_{E \in \varepsilon(X)} \overline{E}$, where $\overline{E}$ denotes the closure of E.

On the other hand, we have common dynamical notions (which are typed in *italics*) as the following: A *flow* on a topological space $X = (X, \mathbf{t}_X)$ is a continuous map $\varphi \colon \mathbb{R} \times X \to X$, $\varphi(t, x) = t \cdot x$, such that $0 \cdot x = x$, $\forall x \in X$ and $t \cdot (s \cdot x) = (t + s) \cdot x$, $\forall x \in X$, $\forall t, s \in \mathbb{R}$. A point $x \in X$ is said to be *periodic* if there is $t \in \mathbb{R}$, $t \neq 0$ such that $t \cdot x = x$. We denote by $P(X)$ the invariant subset of periodic points of X. For a flow $X = (X, \varphi)$, the Λ^+-*limit set* (or *omega-limit set*) of a point $x \in X$ is given as follows: $\Lambda^+(x) = \{y \in X \mid \exists$ a net $t_\delta \to +\infty$ such that $t_\delta \cdot x \to y\}$. The Λ^+-*limit set* of a flow X is the following invariant subset: $\Lambda^+(X) = \bigcup_{x \in X} \Lambda^+(x)$.

The main goal of this paper is to find the most important relationships between the exterior notions and constructions and dynamic concepts.

To find the connections between exterior and dynamic concepts we construct a hybrid one containing both notions. We proceed as follows: Let $(X, \mathbf{t}_X)$ be a topological space endowed with an additional topology τ satisfying that $\mathbf{t}_X \subset \tau \subset \mathbf{d}_X$, where $\mathbf{d}_X$ is the discrete topology. Denote by $\tau_x = \{U \in \tau \mid x \in U\}$, where $x \in X$. On $\mathbb{R}$ with the usual topology we consider three externologies $\{\mathbf{r}, \mathbf{l}, \mathbf{c}\}$ which are induced by the neighbourhoods at $+\infty$, $-\infty$, ∞ (the Alexandroff infinity), respectively. Now given a pair (ϵ, τ) with $\epsilon \in \{\mathbf{r}, \mathbf{l}, \mathbf{c}\}$ and $\mathbf{t}_X \subset \tau \subset \mathbf{d}_X$ and a flow $\varphi \colon \mathbb{R} \times X \to X$, an open $N \in \mathbf{t}_X$ is said to be (ϵ, τ)-***exterior*** if for any $x \in X$ there are $T^x \in \epsilon$ and $U_x \in \tau_x$ such that $\varphi(T^x \times U_x) \subset N$. It is easy to check that the family of (ϵ, τ)-exterior subsets is an externology, that will be denoted by $\varepsilon^{(\epsilon, \tau)}(X)$. Therefore, we

have the ***exterior flow*** $X^{(\epsilon,\tau)} = (X, \varepsilon^{(\epsilon,\tau)}(X), \varphi)$. For concepts related to an exterior flow we will use ***bold italics***.

In this paper we show the existence of many relationships between exterior and dynamic theories. For instance, taking $\tau = \mathbf{d}_X$, for a T_1 flow X, we have proved (Theorem 6.3) that the **limit space** is the subflow of *periodic points*:

$$L(X^{(\epsilon,\mathbf{d}_X)}) = P(X),$$

and Corollary 7.6 (see also Definition 4.1) shows that under some topological conditions the **bar-limit space** is the topological closure of the Λ^+-*limit set* of X:

$$\bar{L}(X^{(\mathbf{r},\mathbf{d}_X)}) = \overline{\Lambda^+(X)}.$$

We note that the **limit** and **bar-limit** constructions developed for exterior spaces, which appear on the left side of the equalities above, can be interpreted as *relevant subflows* of a dynamical system.

In our study we also analyse these types of relationships between other exterior concepts and the more usual notions of *omega-limits, first prolongational limits* and several types of *almost periodic points (Poisson-stable points, non-wandering points)* of a flow.

It is worth pointing out that when we take $\tau = \mathbf{t}_X$ the corresponding exterior notions are related to dynamical notions which depend on the local behavior of the dynamical system. In this paper we analyse six externologies which correspond to the six pairs $(\epsilon, \tau) \in \{\mathbf{r}, \mathbf{l}, \mathbf{c}\} \times \{\mathbf{t}_X, \mathbf{d}_X\}$. Given a point x in a flow X and an open neighbourhood U at x, the forward behavior of the trajectory at x $((t_0, +\infty) \cdot x)$ is related to the externology $\varepsilon^{(\mathbf{r},\mathbf{d}_X)}(X)$ and, in order to study the forward behavior of an open neighbourhood U at x $((t_0, +\infty) \cdot U, \ U \in \mathbf{t}_{X_x})$, we consider the externology $\varepsilon^{(\mathbf{r},\mathbf{t}_X)}(X)$. The backward behavior of a point x or an open neighbourhood U is connected with the properties of the externologies $\varepsilon^{(\mathbf{l},\mathbf{d}_X)}(X), \varepsilon^{(\mathbf{l},\mathbf{t}_X)}(X)$, respectively. In order to simultaneously analyse the past and the future of a trajectory of a point $((-\infty, -t_0) \cdot x \cup (t_0, +\infty) \cdot x, \ t_0 > 0)$ or an open neighbourhood $((-\infty, -t_0) \cdot U \cup (t_0, +\infty) \cdot U, \ t_0 > 0)$, we respectively consider the externologies $\varepsilon^{(\mathbf{c},\mathbf{d}_X)}(X), \varepsilon^{(\mathbf{c},\mathbf{t}_X)}(X)$.

For the dynamical notions we use the definitions and notations given in the Bhatia and Szego book [4], taking into account that we are working on topological spaces instead of metric spaces and, in some cases, one has to replace sequences by nets. For the notion of attractor used in our study we do not assume the compactness condition considered in [4]. In this paper we have also introduced the dynamic notion of (ϵ, τ)-agglomerative point (see Definition 6.5) and a notion of basin (see Definition 5.3) which are not contained in [4].

The final section "Conclusions and further work" contains a list with the main relationships (analized in this paper) between exterior and dynamic notions. We also suggest some future research lines: (1) the **higher homotopy**

invariant of exterior spaces (Steenrod, Čech and Brown-Grossmann groups) could be very useful tools in the analysis of *local and global stability properties of continuous flows* and in the study of *chaotic dynamical systems*; (2) we think that the **externologies** $\varepsilon^{(\epsilon, \tau_{\text{int}})}(X)$ will play an important role in the study of *repellers* and *atractors* of a dynamical system, where τ_{int} is the intrinsic topology of a flow X.

2. Preliminaries on exterior spaces and dynamical systems

2.1. *Exterior spaces: End Spaces and Limit Spaces.* A continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non-compact spaces and manifolds (see [30]). Nevertheless, one has the problem that **P** does not have enough limits and colimits and therefore it is not possible to develop the usual homotopy constructions such as loops, homotopy limits and colimits, et cetera. An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps, see [16, 17]. We refer to [9, 11, 12, 21, 22] for further properties and applications of exterior homotopy, and to [26] for a survey of proper homotopy.

DEFINITION 2.1. Let $(X, \mathbf{t}_X)$ be a topological space, where X is the underlying set and $\mathbf{t}_X$ its topology. An **externology** on $(X, \mathbf{t}_X)$ is a non-empty collection $\varepsilon(X)$ of open subsets which is closed under finite intersections and such that if $E \in \varepsilon(X)$, $U \in \mathbf{t}_X$ and $E \subset U$ then $U \in \varepsilon(X)$. The members of $\varepsilon(X)$ are called **exterior open subsets**. An **exterior space** $(X, \varepsilon(X), \mathbf{t}_X)$ consists of a space $(X, \mathbf{t}_X)$ together with an externology $\varepsilon(X)$. Given an exterior space $(X, \varepsilon(X), \mathbf{t}_X)$ it is useful to work with an **exterior basis** (or just a **basis** of the externology), which is nothing else than a subcollection $\beta \subset \varepsilon(X)$ such that for every $E \in \varepsilon(X)$ there exists $F \in \beta$ with $F \subset E$. A map $f : (X, \varepsilon(X), \mathbf{t}_X) \to (X', \varepsilon(X'), \mathbf{t}_{X'})$ is said to be an **exterior map** if it is continuous and $f^{-1}(E) \in \varepsilon(X)$, for all $E \in \varepsilon(X')$.

The category of exterior spaces and maps will be denoted by **E**. Given a space $(X, \mathbf{t}_X)$, we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ or the total exterior space if one takes $\varepsilon = \mathbf{t}_X$. An important example of externology on a given topological space X is the one constituted by the complements of all closed-compact subsets of X. It will be called the **cocompact externology** and usually written as $\varepsilon^{\mathbf{c}}(X)$. The category of spaces and proper maps can be considered as a full subcategory of the category of exterior spaces via the full embedding $(\cdot)^{\mathbf{c}} : \mathbf{P} \hookrightarrow \mathbf{E}$. The functor $(\cdot)^{\mathbf{c}}$ carries a space X to the exterior space $X^{\mathbf{c}}$ which is endowed with the topology

of X and the externology $\varepsilon^{\mathbf{c}}(X)$. A map $f \colon X \to Y$ is carried to the exterior map $f^{\mathbf{c}} \colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ given by $f^{\mathbf{c}} = f$. It is easy to check that a continuous map $f \colon X \to Y$ is proper if and only if $f = f^{\mathbf{c}} \colon X^{\mathbf{c}} \to Y^{\mathbf{c}}$ is exterior.

An important role in this paper will be played by the following construction: Let $(X, \varepsilon(X), \mathbf{t}_X)$ be an exterior space, and let $(Y, \mathbf{t}_Y, \tau)$ be a set Y endowed with two topologies $\mathbf{t}_Y, \tau$ such that $\mathbf{t}_Y \subset \tau$. For $y \in Y$ we denote by τ_y the family of open neighbourhoods of (Y, τ) at y. Given the topological spaces $(X, \mathbf{t}_X), (Y, \mathbf{t}_Y)$, we consider on $X \times Y$ the product topology which is denoted by $\mathbf{t}_{X \times Y}$ and the externology $\varepsilon(X \bar{\times}_\tau Y)$ given by those $E \in \mathbf{t}_{X \times Y}$ such that for each $y \in Y$ there exists $U_y \in \tau_y$ and $T^y \in \varepsilon(X)$ such that $T^y \times U_y \subset E$. The resulting exterior space $(X \times Y, \varepsilon(X \bar{\times}_\tau Y), \mathbf{t}_{X \times Y})$ will be denoted by $X \bar{\times}_\tau Y$.

In some contexts, for instance working with exterior homotopies, it is frequent to take $\tau = \mathbf{t}_Y$. For these cases, we will use the shorter notation

$$X \bar{\times} Y = (X \times Y, \varepsilon(X \bar{\times}_{\mathbf{t}_Y} Y), \mathbf{t}_{X \times Y}) = X \bar{\times}_{\mathbf{t}_Y} Y.$$

Therefore, for these cases we have an induced functor:

$$(\cdot) \bar{\times} (\cdot) \colon \mathbf{E} \times \mathbf{Top} \to \mathbf{E}, (X, Y) \to X \bar{\times} Y.$$

When Y is a compact space, we have that E is an exterior open subset of $X \bar{\times} Y$ if and only if it is an open subset and there exists $G \in \varepsilon(X)$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon(X) = \varepsilon^{\mathbf{c}}(X)$, then $\varepsilon(X \bar{\times} Y)$ coincides with $\varepsilon^{\mathbf{c}}(X \times Y)$ the externology of the complements of closed-compact subsets of $X \times Y$. We also note that if Y is a discrete space, then E is an exterior open subset of $X \bar{\times} Y$ if and only if it is open and for each $y \in Y$ there is $T^y \in \varepsilon(X)$ such that $T^y \times \{y\} \subset E$.

This bar construction $(\bar{\times})$ provides a natural way to define **exterior homotopy** in $\mathbf{E}$. Indeed, if I denotes the usual closed unit interval $[0, 1]$, given exterior maps $f, g : X \to Y$, it is said that f **is exterior homotopic to** g if there exists an exterior map $H : X \bar{\times} I \to Y$ (called **exterior homotopy**) such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, for all $x \in X$. The corresponding homotopy category of exterior spaces will be denoted by $\pi \mathbf{E}$. Similarly, the usual homotopy category of topological spaces will be denoted by $\pi \mathbf{Top}$.

In this subsection we also deal with special limit constructions associated to any exterior space. In particular we recall the functors $L, \bar{L}, \check{\pi}_0, \mathring{\check{\pi}}_0 \colon \mathbf{E} \to \mathbf{Top}$ introduced in [19].

Given an exterior space $X = (X, \varepsilon(X))$, its externology $\varepsilon(X)$ can be considered as an inverse system of spaces $\{E\}_{E \in \varepsilon(X)}$. Recall that an inverse system of topological spaces is a functor $Z \colon I \to \mathbf{Top}$, where I is a directed set. The functor Z carries $i \geq j$, $i, j \in I$, to $Z_j^i \colon Z_i \to Z_j$. The inverse limit of $Z = \{Z_i\}$ is denoted by $\lim_{i \in I} Z_i$ (or just by $\lim Z_i$). An element of the inverse limit can be represented by an element $(z_i)_{i \in I}$ of the product $\prod_{i \in I} Z_i$

satisfying that $Z_j^i(z_i) = z_j$, $i \geq j$. For more results and properties about inverse systems, we refer the reader to [10].

For a topological space Y, $\pi_0(Y)$ denotes the set of path-components of Y and we have a canonical map $Y \to \pi_0(Y)$ which induces a quotient topology on $\pi_0(Y)$. We remark that if Y is locally path-connected, then $\pi_0(Y)$ is a discrete space. We also recall that given a subset A of a topological space Y the closure of A is denoted by $\overline{A}$.

Given an exterior space $X = (X, \varepsilon(X))$, for the inverse systems

$$\varepsilon(X) = \{E\}_{E \in \varepsilon(X)}, \quad \pi_0\varepsilon(X) = \{\pi_0(E)\}_{E \in \varepsilon(X)}, \quad \bar{\pi}_0\varepsilon(X) = \{\pi_0(\overline{E})\}_{E \in \varepsilon(X)},$$

we can take inverse limits to obtain the following notions:

DEFINITION 2.2. Let $X = (X, \varepsilon(X))$ be an exterior space. The **limit space** of X is the topological subspace:

$$L(X) = \lim \varepsilon(X) = \cap_{E \in \varepsilon(X)} E.$$

The **end space** of X is the inverse limit:

$$\check{\pi}_0(X) = \lim \pi_0\varepsilon(X) = \lim_{E \in \varepsilon(X)} \pi_0(E)$$

endowed with the inverse limit topology of the spaces $\pi_0(E)$.

The **bar-limit space** of X is the topological subspace:

$$\bar{L}(X) = \lim(\{\overline{E}\}_{E \in \varepsilon(X)}) = \cap_{E \in \varepsilon(X)} \overline{E}.$$

The **bar-end space** of X is the inverse limit:

$$\check{\bar{\pi}}_0(X) = \lim \bar{\pi}_0\varepsilon(X) = \lim_{E \in \varepsilon(X)} \pi_0(\overline{E})$$

endowed with the inverse limit topology of the spaces $\pi_0(\overline{E})$.

It is interesting to observe that if X is an exterior space and X is locally path-connected, then $\check{\pi}_0(X)$ is a prodiscrete space.

We note that $\check{\pi}_0(X)$ and $\check{\bar{\pi}}_0(X)$ are generalizations to exterior spaces of the space of Freudenthal end points of a topological space. Indeed, if for a topological space X we consider the externology $\varepsilon^{\mathbf{c}}(X)$, then one has that $\check{\pi}_0(X)$ is precisely the space of Freudenthal ends of the space X, see [13, 23]. This is the reason why, sometimes, the elements of $\check{\pi}_0(X), \check{\bar{\pi}}_0(X)$ are also called **Freudenthal end points** (or **Freudenthal ends**) of the exterior space X. We also mention that the set of Freudenthal end points of an exterior space is a zero dimensional analogue of the Čech invariant of a pro-space (see [10]) or of the Borsuk invariant (see [6, 7]) given in shape theory.

For every $E \in \varepsilon(X)$, we have a commutative diagram:

$$E \longrightarrow \overline{E}$$
$$\downarrow \qquad \downarrow$$
$$\pi_0(E) \longrightarrow \pi_0(\overline{E}).$$

Then, one has the natural transformations of the following commutative diagram:

$$L(X) \longrightarrow \bar{L}(X)$$
$$e \downarrow \qquad \downarrow \bar{e}$$
$$\check{\pi}_0(X) \longrightarrow \check{\bar{\pi}}_0(X).$$

DEFINITION 2.3. Given an exterior space $X = (X, \varepsilon(X))$, an end point $a \in \check{\pi}_0(X)$ is said to be e-**representable** if there is $x \in L(X)$ such that $e(x) = a$. Similarly, a point $b \in \check{\bar{\pi}}_0(X)$ is $\bar{e}$-**representable** if there is $x \in \bar{L}(X)$ such that $\bar{e}(x) = b$.

Notice that the maps $e: L(X) \to \check{\pi}_0(X)$, $\bar{e}: L(X) \to \check{\bar{\pi}}_0(X)$ induce an e-decomposition and an $\bar{e}$-decomposition

$$L(X) = \bigsqcup_{a \in \check{\pi}_0(X)} L_a(X), \quad \bar{L}(X) = \bigsqcup_{b \in \check{\bar{\pi}}_0(X)} \bar{L}_b(X)$$

where $L_a(X) = e^{-1}(a)$ and $\bar{L}_b(X) = \bar{e}^{-1}(b)$.

It is not difficult to check that the functors $L, \bar{L}$ preserve homotopies and the functors $\check{\pi}_0, \check{\bar{\pi}}_0$ are invariant by exterior homotopy. As a consequence of these facts one has:

PROPOSITION 2.4. *The functors* $L, \bar{L}, \check{\pi}_0, \check{\bar{\pi}}_0: \mathbf{E} \to \mathbf{Top}$ *induce functors*

$$L, \bar{L}: \pi\mathbf{E} \to \pi\mathbf{Top}, \qquad \check{\pi}_0, \check{\bar{\pi}}_0: \pi\mathbf{E} \to \mathbf{Top}.$$

REMARK 2.5. A more exhaustive study of these functors and their properties can be seen at [19, 20]. We can also consider other type of functors:

- Taking the set $c(E)$ $(c(\overline{E}))$ of connected components, for every $E \in \varepsilon(X)$, instead of path-connected components, one has some analogous functors: $\check{c}$ $\check{\bar{c}}: \pi\mathbf{E} \to \mathbf{Top}$, see [19].
- Some flow-completion functors, related to the Freudenthal compactification of a space (see [13]), are constructed in [20].

2.2. *Dynamical Systems, omega limits and prolongational omega limits.* Next we recall some elementary concepts about dynamical systems.

DEFINITION 2.6. A *dynamical system* (or *flow*) on a topological space X is a continuous map $\varphi: \mathbb{R} \times X \to X$, $\varphi(t, x) = t \cdot x$, such that

(i) $0 \cdot x = x$, $\forall x \in X$;

(ii) $t \cdot (s \cdot x) = (t + s) \cdot x$, $\forall x \in X$, $\forall t, s \in \mathbb{R}$.

A flow on X will be denoted by (X, φ) and, when no confusion is possible, we use X for short.

For a subset $A \subset X$, we denote $\mathrm{inv}(A) = \{x \in A \mid \mathbb{R} \cdot x \subset A\}$.

DEFINITION 2.7. A subset S of a flow X is said to be *invariant* if $\mathrm{inv}(S) = S$.

Given a flow (X, φ) one has a subgroup $\{\varphi_t \colon X \to X \mid t \in \mathbb{R}\}$ of homeomorphisms, $\varphi_t(x) = \varphi(t, x)$, and a family of trajectories (motions, orbits) $\{\varphi^x \colon \mathbb{R} \to X \mid x \in X\}$, $\varphi^x(t) = \varphi(t, x)$.

DEFINITION 2.8. Given two flows (X, φ), (Y, ψ), a *flow morphism* $f \colon (X, \varphi) \to (Y, \psi)$ is a continuous map $f \colon X \to Y$ such that $f(t \cdot x) = t \cdot f(x)$ for every $t \in \mathbb{R}$ and for every $x \in X$.

We note that if $S \subset X$ is invariant, S has a flow structure and the inclusion is a flow morphism. We denote by $\mathbf{F}$ the category of flows and flow morphisms.

DEFINITION 2.9. For a flow X, the *omega-limit set* (in the literature one can find the alternative terminology: Λ^+-limit, right-limit set or positive limit set) of a point $x \in X$ is given as follows:

$$\Lambda^+(x) = \{y \in X \mid \exists \text{ a net } t_\delta \to +\infty \text{ such that } t_\delta \cdot x \to y\}.$$

Note that the subset $\Lambda^+(x)$ admits the alternative definition

$$\Lambda^+(x) = \bigcap_{t \geq 0} \overline{[t, +\infty) \cdot x}$$

which has the advantage of showing that $\Lambda^+(x)$ is closed.

DEFINITION 2.10. The Λ^+-*limit set* of a flow X is the following invariant subset:

$$\Lambda^+(X) = \bigcup_{x \in X} \Lambda^+(x).$$

DEFINITION 2.11. For a flow X, the *prolongational omega-limit set* (or prolongational right limit set, or prolongational positive limit set) of a point $x \in X$ is given as follows:

$$J^+(x) = \{y \in X \mid \exists \text{ a net } x_\delta \to x \text{ and a net } t_\delta \to +\infty \text{ such that } t_\delta \cdot x_\delta \to y\}.$$

Note that the prolongational omega-limit set admits the alternative definition

$$J^+(x) = \bigcap_{t \geq 0, U \in \mathbf{t}_{X_x}} \overline{[t, +\infty) \cdot U}$$

which has the advantage of showing that $J^+(x)$ is closed.

DEFINITION 2.12. The J^+-*limit set* of a flow X is the following invariant subset:

$$J^+(X) = \bigcup_{x \in X} J^+(x).$$

REMARK 2.13. We note that for the set $J^+(x)$ Bhatia and Szego ([4]) use the longer expression "first positive prolongational limit set" of x.

Now we introduce the basic notions of critical, periodic and Poisson stable points.

Let X be a flow. A point $x \in X$ is said to be a *critical point* (or *fixed point*) if for every $t \in \mathbb{R}$, $t \cdot x = x$. We denote by $C(X)$ the invariant subset of critical points of X. A point $x \in X$ is said to be *periodic* if there is $t \in \mathbb{R}$, $t \neq 0$ such that $t \cdot x = x$. We denote by $P(X)$ the invariant subset of periodic points of X.

It is clear that any critical point is a periodic point. Then

$$C(X) \subset P(X).$$

If $x \in X$ is a periodic point but not critical, then there is a real number $t \neq 0$ such that $t \cdot x = x$ and t is called a *period* of x. The smallest positive period t_0 of x is called the *fundamental period* of x.

A point $x \in X$ is said to be *positively Poisson-stable* if there is a net $t_\delta \to +\infty$ such that $t_\delta \cdot x \to x$; that is, $x \in \Lambda^+(x)$. We will denote by Poisson$^+(X)$ the invariant subset of positively Poisson-stable points of X.

The reader can easily check that

$$P(X) \subset \text{Poisson}^+(X) \subset \Lambda^+(X).$$

The notions above can be dualized by considering the ones of *alpha-limit sets* $\Lambda^-(x), J^-(x)$ of a point x, *negatively Poisson-stable points*, et cetera. A point x is said to be *Poisson-stable* if x is positively Poisson-stable and negatively Poisson-stable. To have a more coherent notation, in some cases, we also use **r**-Poisson-stable instead of positively Poisson-stable and **l**-Poisson-stable instead of negatively Poisson-stable. For a study of the properties of omega-limits, alpha-limits, et cetera, we refer the reader to [4].

REMARK 2.14. Observe that when X satisfies the first axiom of countability (for instance, when X is metrizable) we can consider sequences instead of nets in the definitions above.

DEFINITION 2.15. A point $x \in X$ is said to be a *wandering point* if there is $U \in \mathbf{t}_{X_x}$ and there is $r_0 \in \mathbb{R}$ such that for every $r \geq r_0$, $\varphi_r(U) \cap U = \emptyset$. Denote by $W(X)$ the open subset of wandering points of X and by $N(X) = X \setminus W(X)$ the closed invariant subset of non-wandering points. Hence a point x is *non-wandering* if for every open neigbourhood U at x and for every $r_0 \in \mathbb{R}$, there is $r_1 \geq r_0$ such that $\varphi_{r_1}(U) \cap U \neq \emptyset$.

REMARK 2.16. The original definition of wandering point was given by G.D. Birkhoff ([5]).

It is easy to check that x is non-wandering if and only if $x \in J^+(x)$ if and only if $x \in J^-(x)$, see [4]. Therefore, we have:

PROPOSITION 2.17. *Let X be a flow, then*
$$N(X) \subset J^-(X) \cap J^+(X).$$

We use the notion of region of weak attraction and attraction given in [4], but we remove the compactness condition:

DEFINITION 2.18. Let X be a flow and let $S \subset X$. The *region $PA(S)$ of pseudo-attraction* of S is defined by
$$PA(S) = \{x \in X \mid \Lambda^+(x) \subset S\}.$$

The *region $WA(S)$ of weak-attraction* of S is defined by
$$WA(S) = \{x \in X \mid \Lambda^+(x) \cap S \neq \emptyset\}.$$

The *region $A(S)$ of attraction* of S is defined by
$$A(S) = \{x \in X \mid \Lambda^+(x) \subset S, \Lambda^+(x) \neq \emptyset\}.$$

Obviously, $A(S) \subset PA(S) \cap WA(S)$.

DEFINITION 2.19. Let X be a flow and let $S \subset T \subset X$. It is said that S is *an attractor of T* if T is a neighbourhood of S and $T \subset A(S)$ and S is said to be an *attractor* if $A(S)$ is a neighbourhood of S.

In a dual way, using $\Lambda^-(x)$ one has the corresponding repelling notions.

3. EXTERIOR DYNAMICAL SYSTEMS

Given a flow (X, φ), a point $x \in X$ and an open neighbourhood $U \in \mathbf{t}_{X_x}$, we want to analyse the following questions: On the one hand, the study of the limits $\Lambda^-(x), \Lambda^+(x)$ and the prolongational limits $J^-(x), J^+(x)$ gives an 'end panoramic' of the trajectories of x and neighbourhood points $y \in U$ looking backwards or forwards. On the other hand, the relative position of the points $\varphi_t(x)$ and $\varphi_t(y)$ when $t \to -\infty$ or $t \to +\infty$ with respect to some criterion (distance in a metric space, convergence to the same Freudenthal end, et cetera) induces different stability or sensitivity notions with respect to the initial position ($t = 0$) of the trajectories. All these properties can be analysed by using the constructions and results that have been developed for exterior spaces. The idea is quite simple: A flow is given by a continuous map $\varphi \colon \mathbb{R} \times X \to X$, and we can consider appropriate externologies on $\mathbb{R} \times X$ and on X such that $\varphi \colon \mathbb{R} \times X \to X$ becomes an exterior map. Using different externologies on $\mathbb{R} \times X$ we can study different dynamical properties related to recurrence and omega limits of a dynamical system. In order to consider

externologies on $\mathbb{R} \times X$, we take an externology ϵ on $\mathbb{R}$ and a topology τ in X such that τ is finer that the topology $\mathbf{t}_X$. In this work, we consider three externologies on $\mathbb{R}$, taking open neighbourhoods at $-\infty$, open neighbourhoods at $+\infty$ and the cocompact externology. Regarding τ, we take τ the discrete topology $\mathbf{d}_X$ or $\tau = \mathbf{t}_X$. However, many of the properties developed in this work remain for more general externologies ϵ on $\mathbb{R}$ and topologies τ on X.

3.1. *The category of (ϵ, τ)-exterior flows.* In order to compare exterior spaces with flows we introduce the notion of ***exterior flow***, which is a hybrid containing both the notions of an exterior space and of a flow. We use the convention that the first time that a concept related to an ***exterior flow*** is introduced is typed in bold italics.

We consider three externologies on $\mathbb{R}$ which are determined by the following exterior basis (see Definition 2.1):

A basis for $\mathbf{l}$ is
$$\mathcal{B}(\mathbf{l}) = \{(-\infty, -n) | n \in \mathbb{N}\}.$$

$\mathbf{l} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (-\infty, -n) \subset U\}.$

A basis for $\mathbf{r}$ is
$$\mathcal{B}(\mathbf{r}) = \{(n, +\infty) | n \in \mathbb{N}\}.$$

$\mathbf{r} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } (n, +\infty) \subset U\}.$

A basis for $\mathbf{c} = \varepsilon^{\mathbf{c}}$ is
$$\mathcal{B}(\mathbf{c}) = \{\mathbb{R} \setminus [-n, n] | n \in \mathbb{N}\}.$$

$\mathbf{c} = \{U | U \text{ is open and there is } n \in \mathbb{N} \text{ such that } \mathbb{R} \setminus [-n, n] \subset U\}.$

We denote the corresponding exterior spaces by $\mathbb{R}^{\mathbf{l}}, \mathbb{R}^{\mathbf{r}}, \mathbb{R}^{\mathbf{c}}$, respectively. We note that in these three cases all the canonical translations $(t \to t + t_0)$ on $\mathbb{R}$ are exterior maps.

Let ϵ be an externology on $\mathbb{R}$ such that $\epsilon \in \{\mathbf{l}, \mathbf{r}, \mathbf{c}\}$ and for a topological space $X = (X, \mathbf{t}_X)$ consider a topology $\tau \in \{\mathbf{d}_X, \mathbf{t}_X\}$, where $\mathbf{d}_X$ is the discrete topology on X. Recall that using the procedure given in subsection 2.1, the pair (ϵ, τ) induces a canonical externology $\varepsilon^{(\epsilon, \tau)}(\mathbb{R} \times X)$ on the product topological space $\mathbb{R} \times X$ as follows:

$N \in \varepsilon^{(\epsilon, \tau)}(\mathbb{R} \times X)$ if N is an open subset of the product space $\mathbb{R} \times X$ and for every $x \in X$ there is $T^x \in \epsilon$ and $U_x \in \tau_x$ such that $T^x \times U_x \subset N$.

We note that
$$\varepsilon^{(\mathbf{c}, \tau)}(\mathbb{R} \times X) = \varepsilon^{(\mathbf{l}, \tau)}(\mathbb{R} \times X) \cap \varepsilon^{(\mathbf{r}, \tau)}(\mathbb{R} \times X)$$

and $\varepsilon^{(\epsilon, \mathbf{t})}(\mathbb{R} \times X) \subset \varepsilon^{(\epsilon, \mathbf{d})}(\mathbb{R} \times X)$.

Recall that $\mathbb{R}^{\epsilon} \bar{\times}_\tau X$ denotes the exterior space $(\mathbb{R} \times X, \varepsilon^{(\epsilon, \tau)}(\mathbb{R} \times X))$ and in case that $\tau = \mathbf{t}_X$ the notation can be reduced to $\mathbb{R}^{\epsilon} \bar{\times} X$ (see subsection 2.1).

DEFINITION 3.1. Let $X = (X, \varepsilon(X), \mathbf{t}_X)$ be an exterior space and suppose that $(\epsilon, \tau) \in \{\mathbf{l}, \mathbf{r}, \mathbf{c}\} \times \{\mathbf{d}_X, \mathbf{t}_X\}$. An (ϵ, τ)-**exterior flow** is a continuous flow $\varphi \colon \mathbb{R} \times X \to X$ such that

(i) $\varphi \colon \mathbb{R}^\epsilon \bar{\times}_\tau X \to X$ is exterior and,
(ii) for any $t \in \mathbb{R}$, $F_t \colon X \bar{\times} I \to X$, $F_t(x, s) = \varphi(ts, x)$, $s \in I$, $x \in X$, is also exterior.

An (ϵ, τ)-**exterior flow morphism** of (ϵ, τ)-exterior flows $f \colon M \to N$ is a flow morphism such that f is exterior.

Denote by $\mathbf{E}^{(\epsilon, \tau)}\mathbf{F}$ the category of (ϵ, τ)-exterior flows and (ϵ, τ)-exterior flow morphisms. When $\tau = \mathbf{t}_X$ or $\tau = \mathbf{d}_X$ we have the categories $\mathbf{E}^{(\epsilon, \mathbf{t})}\mathbf{F}$ or $\mathbf{E}^{(\epsilon, \mathbf{d})}\mathbf{F}$, respectively.

REMARK 3.2. Observe that the second statement of the definition above establishes that F_t is an exterior homotopy between id_X and $\varphi_t \colon X \to X$.

The following six cases will be analysed: $(\epsilon, \tau) \in \{\mathbf{l}, \mathbf{r}, \mathbf{c}\} \times \{\mathbf{d}_X, \mathbf{t}_X\}$. We note that these six exterior spaces are related by the following diagram

$$
\begin{array}{ccccc}
\mathbb{R}^{\mathbf{l}} \bar{\times}_{\mathbf{d}} X & \longrightarrow & \mathbb{R}^{\mathbf{c}} \bar{\times}_{\mathbf{d}} X & \longleftarrow & \mathbb{R}^{\mathbf{r}} \bar{\times}_{\mathbf{d}} X \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R}^{\mathbf{l}} \bar{\times}_{\mathbf{t}} X & \longrightarrow & \mathbb{R}^{\mathbf{c}} \bar{\times}_{\mathbf{t}} X & \longleftarrow & \mathbb{R}^{\mathbf{r}} \bar{\times}_{\mathbf{t}} X.
\end{array}
$$

It is worth remarking that this diagram induces the following commutative diagram of forgetful functors:

$$
\begin{array}{ccccc}
\mathbf{E}^{(\mathbf{l}, \mathbf{d})}\mathbf{F} & \longleftarrow & \mathbf{E}^{(\mathbf{c}, \mathbf{d})}\mathbf{F} & \longrightarrow & \mathbf{E}^{(\mathbf{r}, \mathbf{d})}\mathbf{F} \\
\uparrow & & \uparrow & & \uparrow \\
\mathbf{E}^{(\mathbf{l}, \mathbf{t})}\mathbf{F} & \longleftarrow & \mathbf{E}^{(\mathbf{c}, \mathbf{t})}\mathbf{F} & \longrightarrow & \mathbf{E}^{(\mathbf{r}, \mathbf{t})}\mathbf{F}.
\end{array}
$$

We will use the following notation: if $X = (X, \varepsilon(X), \mathbf{t}_X)$ is an exterior space, then the underlying topological space $(X, \mathbf{t}_X)$ will be denoted by $X_{\mathbf{t}}$.

Since we are working with a new notion mixing the notions of a flow and an exterior space, we always have the following canonical forgetful functors:

Given an (ϵ, τ)-exterior flow $X = (X, \phi) \in \mathbf{E}^{(\epsilon, \tau)}\mathbf{F}$, one also has a flow $X_{\mathbf{t}} = (X_{\mathbf{t}}, \phi) \in \mathbf{F}$. This gives a forgetful functor:

$$(\cdot)_{\mathbf{t}} \colon \mathbf{E}^{(\epsilon, \tau)}\mathbf{F} \to \mathbf{F} \ , X \to X_{\mathbf{t}}.$$

For a given (ϵ, τ)-exterior flow $X = (X, \phi) \in \mathbf{E}^{(\epsilon, \tau)}\mathbf{F}$, we also have that X has the structure of an exterior space with externology $\varepsilon(X)$, then there is a canonical forgetful functor:

$$(\cdot)^\varepsilon \colon \mathbf{E}^{(\epsilon, \tau)}\mathbf{F} \to \mathbf{E} \ , (X, \phi) \to X^\varepsilon = (X, \varepsilon(X)).$$

This allows us to apply many constructions and properties of exterior spaces to (ϵ, τ)-exterior flows.

3.2. *Exterior flows associated with a flow.* The forgetful functor

$$(\cdot)_\mathbf{t} \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{F}$$

has a left adjoint that can be constructed as follows:

Given a flow $\varphi \colon \mathbb{R} \times X \to X$, an open $N \in \mathbf{t}_X$ is said to be (ϵ, τ)-**exterior** if for any $x \in X$ there are $T^x \in \epsilon$ and $U_x \in \tau_x$ such that $\varphi(T^x \times U_x) \subset N$.

It is easy to check that the family of (ϵ, τ)-exterior subsets is an externology, that will be denoted by $\varepsilon^{(\epsilon,\tau)}(X)$, and φ is an (ϵ, τ)-exterior flow for the exterior space $X^{(\epsilon,\tau)} = (X, \varepsilon^{(\epsilon,\tau)}(X))$.

We also note that, given a flow X, one has that $\varepsilon^{(\epsilon,\mathbf{t})}(X) \subset \varepsilon^{(\epsilon,\mathbf{d})}(X)$ and $\varepsilon^{(\mathbf{c},\tau)}(X) = \varepsilon^{(\mathbf{l},\tau)}(X) \cap \varepsilon^{(\mathbf{r},\tau)}(X)$.

DEFINITION 3.3. Given a flow (X, φ) and $(\epsilon, \tau) \in \{\mathbf{l}, \mathbf{r}, \mathbf{c}\} \times \{\mathbf{d}_X, \mathbf{t}_X\}$, the pair $(X^{(\epsilon,\tau)}, \varphi)$ is said to be the (ϵ, τ)-**exterior flow induced by** (X, φ).

When there is no possibility of confusion, $(X^{(\epsilon,\tau)}, \varphi)$ will be briefly denoted by $X^{(\epsilon,\tau)}$. It is easy to check that $\varepsilon^{(\epsilon,\tau)}(X)$ is the finest externology on X that makes exterior the continuous map $\varphi \colon \mathbb{R}^\epsilon \bar{\times}_\tau X \to X$. We also note that if $(X, \varepsilon(X), \varphi)$ is an (ϵ, τ)-exterior flow, one has that $\varepsilon(X) \subset \varepsilon^{(\epsilon,\tau)}(X)$.

The functor

$$(\cdot)^{(\epsilon,\tau)} \colon \mathbf{F} \to \mathbf{E}^{(\epsilon,\tau)}\mathbf{F},$$

which carries a flow X to the (ϵ, τ)-exterior flow $X^{(\epsilon,\tau)}$, permits us to apply to any flow all the constructions developed for exterior flows.

THEOREM 3.4. *The functor* $(\cdot)^{(\epsilon,\tau)} \colon \mathbf{F} \to \mathbf{E}^{(\epsilon,\tau)}\mathbf{F}$ *is left adjoint to the functor* $(\cdot)_\mathbf{t} \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{F}$. *Moreover* $(\cdot)_\mathbf{t}\, (\cdot)^{(\epsilon,\tau)} = \mathrm{id}$ *and* $\mathbf{F}$ *can be considered as a full subcategory of* $\mathbf{E}^{(\epsilon,\tau)}\mathbf{F}$ *via* $(\cdot)^{(\epsilon,\tau)}$.

PROOF. Let X be in $\mathbf{F}$ and Y be in $\mathbf{E}^{(\epsilon,\tau)}\mathbf{F}$. If $f \colon X^{(\epsilon,\tau)} \to Y$ is a morphism in $\mathbf{E}^{(\epsilon,\tau)}\mathbf{F}$, then it is clear that $f = f_\mathbf{t} \colon X = (X^{(\epsilon,\tau)})_\mathbf{t} \to Y_\mathbf{t}$ is a morphism in $\mathbf{F}$. Now if $g \colon X \to Y_\mathbf{t}$ is a morphism in $\mathbf{F}$, suppose that $E \in \varepsilon(Y)$. Given $x \in X$, since Y is an (ϵ, τ)-exterior flow, there are $T^{g(x)} \in \epsilon$ and $V_{g(x)} \in \tau_{Y_{g(x)}}$ such that $T^{g(x)} \cdot V_{g(x)} \subset E$. Since $g \colon X \to Y_\mathbf{t}$ is continuous, it follows that $g^{-1}(V_{g(x)}) \in \tau_{X_x}$ and $T^{g(x)} \cdot g^{-1}(V_{g(x)}) \subset g^{-1}(E)$. Therefore $g^{-1}(E) \in \varepsilon^{(\epsilon,\tau)}(X)$. So $g \colon X^{(\epsilon,\tau)} \to Y$ is (ϵ, τ)-exterior and $g_\mathbf{t} = g$. □

4. OMEGA-LIMITS OF EXTERIOR FLOWS

We consider the following notion of omega-limit for exterior flows:

Given a net t_δ in $\mathbb{R}$, if $\epsilon \in \{\mathbf{l}, \mathbf{c}, \mathbf{r}\}$ we say that t_δ is ϵ-divergent if for every $T \in \epsilon$ there is δ_T such that $t_\delta \in T$ for every $\delta \geq \delta_T$. Note that t_δ is $\mathbf{l}$-divergent if and only if $t_\delta \to -\infty$, t_δ is $\mathbf{c}$-divergent if and only if $t_\delta \to \infty$ (in the Alexandroff compactification) and t_δ is $\mathbf{r}$-divergent if and only if $t_\delta \to +\infty$.

DEFINITION 4.1. Let X be an (ϵ, τ)-exterior flow. Given $x \in X$, the (ϵ, τ)-**omega limit** of x is the invariant subspace

$$\Omega^{(\epsilon,\tau)}(x) = \bigcap_{T \in \epsilon, U \in \tau_x} \overline{T \cdot U}.$$

Notice that $y \in \Omega^{(\epsilon,\tau)}(x)$ if and only if there are nets t_δ in $\mathbb{R}$ and x_δ in X such that t_δ is ϵ-divergent, $x_\delta \to x$ in X_τ and $t_\delta \cdot x_\delta \to y$ in $X_{\mathbf{t}}$.

The (ϵ, τ)-**omega limit of a subspace** $S \subset X$ is the following invariant subset:

$$\Omega^{(\epsilon,\tau)}(S) = \bigcup_{x \in S} \Omega^{(\epsilon,\tau)}(x).$$

Notice that we have the following particular cases of omega-limits:

- $\Omega^{(\mathbf{l},\mathbf{d})}(x) = \Lambda^-(x), \quad \Omega^{(\mathbf{l},\mathbf{t})}(x) = J^-(x);$
- $\Omega^{(\mathbf{r},\mathbf{d})}(x) = \Lambda^+(x), \quad \Omega^{(\mathbf{r},\mathbf{t})}(x) = J^+(x);$
- $\Omega^{(\mathbf{c},\mathbf{d})}(x) = \Lambda^-(x) \cup \Lambda^+(x), \quad \Omega^{(\mathbf{c},\mathbf{t})}(x) = J^-(x) \cup J^+(x).$

It is interesting to note that if we change $\Lambda^+(x)$ by $\Omega^{(\epsilon,\tau)}(x)$ in Definition 2.18 of region of (pseudo, weak) attraction, we can introduce the notion of region of (ϵ, τ)-**(pseudo, weak) attraction** $A^{(\epsilon,\tau)}(S)$ of $S \subset X$, $(PA^{(\epsilon,\tau)}(S)$, $WA^{(\epsilon,\tau)}(S))$.

The use of the notation $\Omega^{(\epsilon,\tau)}(x)$ permits us to unify some common properties of omega-limits, prolongational limits, attractors and repellers (see subsection 2.2). For example, for $S \subset X$, the region of (ϵ, τ)-**attraction of** S is given by

$$A^{(\epsilon,\tau)}(S) = \{x \in X \mid \emptyset \neq \Omega^{(\epsilon,\tau)}(x) \subset S\}.$$

In this case, one has that $A^{(\mathbf{r},\mathbf{d})}(S) = A(S)$ and $A^{(\mathbf{l},\mathbf{d})}(S)$ is the region of repulsion of S.

5. LIMIT SPACES, END SPACES AND BASINS OF AN EXTERIOR FLOW

5.1. *Limit spaces and end spaces of an exterior flow.* Using the forgetful functor

$$(\cdot)^\varepsilon \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{E}$$

and the functors $L, \bar{L}, \breve{\pi}_0, \check{\pi}_0 \colon \mathbf{E} \to \mathbf{Top}$ we have the corresponding composites that will be equipped with the same notation:

$$L, \bar{L}, \breve{\pi}_0, \check{\pi}_0 \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{Top}.$$

PROPOSITION 5.1. *If X is an (ϵ, τ)-exterior flow, then:*

(i) *The spaces $L(X)$, $\bar{L}(X)$ are invariant;*
(ii) *There are trivial induced flows on $\breve{\pi}_0(X)$ and $\check{\pi}_0(X)$.*

PROOF. (i) We have that $L(X) = \cap_{E \in \varepsilon(X)} E$ and $\bar{L}(X) = \cap_{E \in \varepsilon(X)} \overline{E}$. It is easy to check that for any $r \in \mathbb{R}$, $\varphi_r(E) \in \varepsilon(X)$ if and only if $E \in \varepsilon(X)$. Therefore

$$\varphi_r(L(X)) = \varphi_r(\cap_{E \in \varepsilon(X)} E) = \cap_{E \in \varepsilon(X)} \varphi_r(E) = \cap_{E \in \varepsilon(X)} E = L(X).$$

Now taking into account that $\overline{\varphi_r(E)} = \varphi_r(\overline{E})$, a similar result is obtained for $\bar{L}(X)$.

(ii) For each $r \in \mathbb{R}$, consider the exterior homotopy $F_r \colon X \bar{\times} I \to X$, $F_r(x,t) = \varphi(rt, x)$, from id_X to φ_r. By Proposition 2.4, it follows that $\mathrm{id} = \check{\pi}_0(\varphi_r)$ and similarly $\mathrm{id} = \check{\pi}_0(\varphi_r)$. Therefore the induced actions are trivial. $\square$

As a consequence of Proposition 5.1, one also has the corresponding canonical functors

$$L, \bar{L}, \check{\pi}_0, \check{\pi}_0 \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{F}.$$

5.2. *The beginning and end of a trajectory.* This subsection is devoted to the functor $\check{\pi}_0$. However similar notions and results are obtained for the functor $\check{\pi}_0$. We leave the formulation of the corresponding results to the reader.

Let X be an (ϵ, τ)-exterior flow and suppose that X is τ-locally path-connected (that is, (X, τ) is locally path-connected). Then, given $p \in X$ and $E \in \varepsilon(X)$, there are $T^p \in \epsilon$ and $U_p \in \tau$ such that $p \in U_p$, U_p is path-connected and $T^p \cdot U_p \subset E$. If $\epsilon \in \{\mathbf{l}, \mathbf{r}\}$ we can also suppose that T^p is path-connected, so $T^p \cdot U_p$ is path-connected and there is a unique $\omega_\epsilon(p, E)$ path-component of E such that $T^p \cdot U_p \subset \omega_\epsilon(p, E) \subset E$. This gives τ-continuous maps $\omega_\epsilon(\cdot, E) \colon X \to \pi_0(E)$, $E \in \varepsilon(X)$. Note that if $E_1 \subset E_2$, we have a commutative diagram:

Therefore, by the universal property of the inverse limit, there is an induced map $\omega_\epsilon \colon X \to \check{\pi}_0(X)$ making commutative the following diagram:

If $\epsilon = \mathbf{c}$, then we can suppose that $T^p = T^p_{\mathbf{l}} \cup T^p_{\mathbf{r}}$ has two path-connected components, so that $T^p \cdot U_p = T^p_{\mathbf{l}} \cdot U_p \cup T^p_{\mathbf{r}} \cdot U_p$ has two path-connected

components and there are two unique path-components $\omega_{\mathbf{l}}(p, E), \omega_{\mathbf{r}}(p, E)$ of E such that

$$T_{\mathbf{l}}^p \cdot U_p \subset \omega_{\mathbf{l}}(p, E) \subset E, \qquad T_{\mathbf{r}}^p \cdot U_p \subset \omega_{\mathbf{r}}(p, E) \subset E.$$

This gives τ-continuous maps $\omega_{\mathbf{l}}(\cdot, E), \omega_{\mathbf{r}}(\cdot, E) \colon X \to \pi_0(E)$, $E \in \varepsilon(X)$ and induced maps $\omega_{\mathbf{l}}, \omega_{\mathbf{r}} \colon X \to \check{\pi}_0(X)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
L(X) & & \\
\downarrow & \searrow^{(e,e)} & \\
X & \xrightarrow[(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})]{} & \check{\pi}_0(X) \times \check{\pi}_0(X).
\end{array}
$$

PROPOSITION 5.2. *Let X be an (ϵ, τ)-exterior flow and suppose that X is τ-locally path-connected.*

   (i) If $\epsilon = \mathbf{l}$, then $\omega_{\mathbf{l}} \colon X \to \check{\pi}_0(X)$ is τ-continuous.
   (ii) If $\epsilon = \mathbf{r}$, then $\omega_{\mathbf{r}} \colon X \to \check{\pi}_0(X)$ is τ-continuous.
   (iii) If $\epsilon = \mathbf{c}$, then $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}}) \colon X \to \check{\pi}_0(X) \times \check{\pi}_0(X)$ is τ-continuous.

   5.3. *Basins of an exterior flow.*

DEFINITION 5.3. Let X be an (ϵ, τ)-exterior flow and suppose that X is τ-locally path-connected:
   If $\epsilon \in \{\mathbf{l}, \mathbf{c}\}$, denote

$$X_{(\mathbf{l},b)} = \omega_{\mathbf{l}}^{-1}(b), \quad b \in \check{\pi}_0(X).$$

   If $\epsilon \in \{\mathbf{c}, \mathbf{r}\}$ denote

$$X_{(\mathbf{r},b)} = \omega_{\mathbf{r}}^{-1}(b), \quad b \in \check{\pi}_0(X)$$

and if $\epsilon = \mathbf{c}$ denote

$$X_{(a,b)} = (\omega_{\mathbf{l}}, \omega_{\mathbf{r}})^{-1}((a, b)), \quad (a, b) \in \check{\pi}_0(X) \times \check{\pi}_0(X).$$

   The invariant space $X_{(\mathbf{l},b)}$ will be called the **l-*basin*** at b, $X_{(\mathbf{r},b)}$ the **r-*basin*** at b and $X_{(a,b)}$ the $(\mathbf{l}, \mathbf{r})$-***basin*** at (a, b).
   The induced partitions of X in simpler flows

$$X = \bigsqcup_{b \in \check{\pi}_0(X)} X_{(\mathbf{l},b)}, \quad X = \bigsqcup_{b \in \check{\pi}_0(X)} X_{(\mathbf{r},b)}, \quad X = \bigsqcup_{(a,b) \in \check{\pi}_0(X) \times \check{\pi}_0(X)} X_{(a,b)}$$

will be called $\omega_{\mathbf{l}}$-***decomposition***, $\omega_{\mathbf{r}}$-***decomposition*** and $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})$-***decomposition*** of the exterior flow X, respectively.

REMARK 5.4. Since the category of flows can be considered as a full subcategory of this type of categories of exterior flows, the $\omega_{\mathbf{l}}$-decomposition

$$X = \bigsqcup_{b \in \check{\pi}_0(X)} X_{(\mathbf{l},b)}$$

can be considered as a generalization for a continuous flow of the decomposition as a disjoint union of "unstable" submanifolds of a differentiable flow. Dually, the $\omega_{\mathbf{r}}$-decomposition divides a flow in "stable" subflows (see [27–29, 31]).

5.4. *Representable, attracting and repelling end points of an exterior flow.* Recall that in Definition 2.3 the notion of e-representable end point has been introduced for any Freudenthal end of an exterior space. In particular, we also have this notion for exterior flows. Moreover, we can give the following omega-representable notion for end points of exterior flows:

DEFINITION 5.5. Let X be an (ϵ, τ)-exterior flow and suppose that X is τ-locally path-connected.

If $\epsilon \in \{\mathbf{l}, \mathbf{r}\}$, a Freundenthal end $b \in \check{\pi}_0(X)$ is said to be ω_ϵ-***representable*** if $\omega_\epsilon^{-1}(b) \neq \emptyset$; and b is said to be $(\omega_\epsilon \setminus e)$-***representable*** if $\omega_\epsilon^{-1}(b) \setminus e^{-1}(b) \neq \emptyset$.

If $\epsilon = \mathbf{c}$, given a pair of end points $(a, b) \in \check{\pi}_0(X) \times \check{\pi}_0(X)$, (a, b) is said to be $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})$-***representable*** if $X_{(a,b)} = (\omega_{\mathbf{l}}, \omega_{\mathbf{r}})^{-1}((a, b)) \neq \emptyset$. An end point $a \in \check{\pi}_0(X)$ is said to be $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})$-***representable*** if the pair (a, a) is $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})$-representable; finally a is said to be $((\omega_{\mathbf{l}}, \omega_{\mathbf{r}}) \setminus e)$-***representable*** if $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})^{-1}((a, a)) \setminus e^{-1}(a) \neq \emptyset$.

DEFINITION 5.6. Let X be an (ϵ, τ)-exterior flow and suppose that X is τ-locally path-connected.

If $\epsilon = \mathbf{l}$, an end point $b \in \check{\pi}_0(X)$ is said to be a ***repelling end point*** if b is $\omega_{\mathbf{l}}$-representable and the $\mathbf{l}$-basin $X_{(\mathbf{l},b)}$ is an open subset.

If $\epsilon = \mathbf{r}$, an end point $b \in \check{\pi}_0(X)$ is said to be an ***attracting end point*** if b is $\omega_{\mathbf{r}}$-representable and the $\mathbf{r}$-basin $X_{(\mathbf{r},b)}$ is an open subset.

If $\epsilon = \mathbf{c}$, a pair of end points $(a, b) \in \check{\pi}_0(X) \times \check{\pi}_0(X)$ is said to be a ***repelling-attracting pair*** if (a, b) is $(\omega_{\mathbf{l}}, \omega_{\mathbf{r}})$-representable and the $(\mathbf{l}, \mathbf{r})$-basin $X_{(a,b)}$ is an open subset.

5.5. *Limit flows, bar-limit flows and Freudenthal-end spaces of a flow.* If we consider the functor given in subsection 3.2

$$(\cdot)^{(\epsilon,\tau)} \colon \mathbf{F} \to \mathbf{E}^{(\epsilon,\tau)}\mathbf{F}$$

and the functors developed in subsection 5.1,

$$L, \bar{L}, \check{\pi}_0, \check{\bar{\pi}}_0 \colon \mathbf{E}^{(\epsilon,\tau)}\mathbf{F} \to \mathbf{F},$$

we can consider the composites

$$L^{(\epsilon,\tau)} = L(\cdot)^{(\epsilon,\tau)}, \ \bar{L}^{(\epsilon,\tau)} = \bar{L}(\cdot)^{(\epsilon,\tau)}, \check{\pi}_0^{(\epsilon,\tau)} = \check{\pi}_0(\cdot)^{(\epsilon,\tau)}, \check{\bar{\pi}}_0^{(\epsilon,\tau)} = \check{\bar{\pi}}_0(\cdot)^{(\epsilon,\tau)}$$

that will be taken as functors from $\mathbf{F}$ to $\mathbf{F}$.

6. LIMIT FLOW OF AN EXTERIOR FLOW, PERIODIC POINTS AND AGGLOMERATIVE POINTS

In this section we analyse the relation between exterior closed singletons and periodic and agglomerative points.

6.1. *Exterior $(\epsilon, \mathbf{d})$-closed singletons and periodic points.* In this subsection, for a point x in a flow X we see that the dynamic property of being a *non-periodic point* can be reformulated by using exterior terminology by the fact that $X \setminus \{x\}$ is an $(\epsilon, \mathbf{d})$-**exterior open subset**. Moreover, the *subflow of periodic points* is interpreted as the **limit space** of the exterior space $X^{(\epsilon, \mathbf{d})}$.

LEMMA 6.1. *Let X be an $(\epsilon, \mathbf{d})$-exterior flow and suppose that $x \in X$. If x is a periodic point, then for every $E \in \varepsilon(X)$, $x \in E$. Consequently, $P(X) \subset L(X)$.*

PROOF. Suppose that x is a periodic point. If $E \in \varepsilon(X)$, then there is $T \in \epsilon$ such that $T \cdot x \subset E$. Since x is periodic, $T \cdot x = \mathbb{R} \cdot x$; taking into account that $x \in \mathbb{R} \cdot x$, we have that that $x \in E$. $\qquad\square$

THEOREM 6.2. *Let X be a T_1 flow. If $x \in X$, then the following statements are equivalent:*

(i) x is a non-periodic point,
(ii) $X \setminus \{x\}$ is an $(\epsilon, \mathbf{d})$-exterior subset.

PROOF. Suppose that x is a non-periodic point. Take y in X. If y is not in the trajectory of x, then for every $T \in \epsilon$, $T \cdot y \subset X \setminus \{x\}$. If y is in the trajectory of x, taking into account that x is not periodic, there is $T \in \epsilon$ such that $T \cdot y \subset X \setminus \{x\}$. Then $X \setminus \{x\}$ is $(\epsilon, \mathbf{d})$-exterior.

Conversely, suppose that x is a periodic point. Then, by Lemma 6.1 above, $X \setminus \{x\}$ is not $(\epsilon, \mathbf{d})$-exterior. $\qquad\square$

THEOREM 6.3. *Let X be a T_1 flow. Then, $L^{(\epsilon, \mathbf{d})}(X) = P(X)$ is precisely the set of periodic points of X.*

PROOF. Let $x \in X \setminus P(X)$. By Theorem 6.2, $X \setminus \{x\}$ is an $(\epsilon, \mathbf{d})$-exterior subset and this implies that $x \notin L^{(\epsilon, \mathbf{d})}(X)$. Then, $L^{(\epsilon, \mathbf{d})}(X) \subset P(X)$ and, by Lemma 6.1, we have that $L^{(\epsilon, \mathbf{d})}(X) = P(X)$. $\qquad\square$

REMARK 6.4. As a consequence of this result, one has that for T_1 flows X the construction $L^{(\epsilon, \mathbf{d})}$ is idempotent:

$$L^{(\epsilon, \mathbf{d})} L^{(\epsilon, \mathbf{d})}(X) = L^{(\epsilon, \mathbf{d})}(X).$$

6.2. *Exterior* $(\epsilon, \mathbf{t})$-*closed singletons and agglomerative points.* Recall that in Definition 2.9 we have considered the omega-limit (Λ^+-limit) set $\Lambda^+(x)$ of a point x in a flow X and after Definition 4.1 we have seen that $\Omega^{(\mathbf{r},\mathbf{d})}(x) = \Lambda^+(x)$.

In this subsection, for a point x in a flow X we see that the dynamic property $\Lambda^+(x) = \emptyset$ can be reformulated by using exterior terminology by the fact that $X \setminus \{x\}$ is an $(\mathbf{l}, \mathbf{t})$-**exterior open subset**. Moreover, the *subflow* $\{x \in X | \Lambda^+(x) \neq \emptyset\}$ is interpreted as the **limit space** of the exterior space $X^{(\mathbf{l},\mathbf{t})}$.

We consider the following notion of agglomerative point for exterior flows:

DEFINITION 6.5. Let X be an (ϵ, τ)-exterior flow. Given $x \in X$, x is said to be an (ϵ, τ)-**agglomerative** point if

$$\Omega^{(\epsilon,\tau)}(x) \neq \emptyset.$$

We denote by $Ag^{(\epsilon,\tau)}(X)$ the invariant subspace of (ϵ, τ)-agglomerative points of X. Note that $Ag^{(\epsilon,\tau)}(X) = A^{(\epsilon,\tau)}(X)$ is the region of (ϵ, τ)-attraction of X (see Definition 4.1). In particular, $Ag^{(\mathbf{r},\mathbf{d})}(X) = A(X)$ is the region of attraction of X, see Definition 2.18. If X is a flow, given $x \in X$, x is said an (ϵ, τ)-*agglomerative* point if x is (ϵ, τ)-agglomerative in $X^{(\epsilon,\tau)}$. $Ag^{(\epsilon,\tau)}(X)$ denotes the subflow of (ϵ, τ)-agglomerative points of X.

PROPOSITION 6.6. Let X be an (ϵ, τ)-exterior flow where $\epsilon \in \{\mathbf{l}, \mathbf{r}\}$. If $x_1 \in \Omega^{(\epsilon,\mathbf{d})}(x_0)$ and $x_2 \in \Omega^{(\epsilon,\mathbf{d})}(x_1)$, then $x_2 \in \Omega^{(\epsilon,\mathbf{d})}(x_0)$.

PROOF. In this proof we suppose that $\epsilon = \mathbf{r}$. Similarly, one has an analogous proof for $\epsilon = \mathbf{l}$.

Taking into account that $x_2 \in \Omega^{(\mathbf{r},\mathbf{d})}(x_1)$, for each $U \in \mathbf{t}_{X_{x_2}}$ and $t \geq 0$ there is $r_U^t \in \mathbb{R}_+$ such that $r_U^t \geq t$ and $r_U^t \cdot x_1 \in U$. Since $x_1 \in \Omega^{(\mathbf{r},\mathbf{d})}(x_0)$, we also have that for each $V \in \mathbf{t}_{X_{x_1}}$ and $s \geq 0$ there is $r_V^s \in \mathbb{R}_+$ such that $r_V^s \geq s$ and $r_V^s \cdot x_0 \in V$.

Given $U \in \mathbf{t}_{X_{x_2}}$ and $t \geq 0$, since $r_U^t \cdot x_1 \in U$ and the flow is continuous, there is $V_U^t \in \mathbf{t}_{X_{x_1}}$ such that $r_U^t \cdot V_U^t \subset U$. For $V_U^t \in \mathbf{t}_{X_{x_1}}$ and $s = 0$, there are $r_{V_U^t}^0 \in \mathbb{R}_+$ such that $r_{V_U^t}^0 \cdot x_0 \in V_U^t$. This implies that

$$(r_U^t + r_{V_U^t}^0) \geq t + 0 = t$$

and

$$(r_U^t + r_{V_U^t}^0) \cdot x_0 = r_U^t \cdot (r_{V_U^t}^0 \cdot x_0) \in r_U^t \cdot V_U^t \subset U.$$

Therefore $x_2 \in \overline{[t, +\infty) \cdot x_0}$ for every $t \geq 0$, so $x_2 \in \Omega^{(\mathbf{r},\mathbf{d})}(x_0)$. □

For the following results we use the notation $-\mathbf{l} = \mathbf{r}$, $-\mathbf{r} = \mathbf{l}$ and $-\mathbf{c} = \mathbf{c}$.

PROPOSITION 6.7. Let X be an $(\epsilon, \mathbf{t})$-exterior flow. If $x_1 \in \Omega^{(\epsilon,\mathbf{t})}(x_0)$, then $x_0 \in \Omega^{(-\epsilon,\mathbf{t})}(x_1)$.

PROOF. Suppose that there are nets x_δ and t_δ such that $x_\delta \to x_0$, t_δ is ϵ-divergent and $t_\delta \cdot x_\delta \to x_1$. Then, we have that $-t_\delta$ is $(-\epsilon)$-divergent and $(-t_\delta) \cdot (t_\delta \cdot x_\delta) \to x_0$. Therefore $x_0 \in \Omega^{(-\epsilon,\mathbf{t})}(x_1)$.   □

COROLLARY 6.8. *Let X be a flow. Then*

$$Ag^{(-\epsilon,\mathbf{d})}(X) \subset Ag^{(-\epsilon,\mathbf{t})}(X) \subset \Omega^{(\epsilon,\mathbf{t})}(X).$$

PROOF. Given $x \in X$, we have that $\Omega^{(-\epsilon,\mathbf{d})}(x) \subset \Omega^{(-\epsilon,\mathbf{t})}(x)$. If $\emptyset \neq \Omega^{(-\epsilon,\mathbf{d})}(x)$, then $\emptyset \neq \Omega^{(-\epsilon,\mathbf{t})}(x)$. This implies that

$$Ag^{(-\epsilon,\mathbf{d})}(X) \subset Ag^{(-\epsilon,\mathbf{t})}(X).$$

Now, take $x \in Ag^{(-\epsilon,\mathbf{t})}(X)$, then $\Omega^{(-\epsilon,\mathbf{t})}(x) \neq \emptyset$. Take $y \in \Omega^{(-\epsilon,\mathbf{t})}(x)$, by Proposition 6.7, $x \in \Omega^{(\epsilon,\mathbf{t})}(y) \subset \Omega^{(\epsilon,\mathbf{t})}(X)$. Therefore, $x \in \Omega^{(\epsilon,\mathbf{t})}(X)$.   □

THEOREM 6.9. *Let X be a T_1 flow. Then the following statements hold:*

(i) *Given $x, y \in X$, there is an open neighbourhood V_y at y and there is $T \in \epsilon$ such that $x \notin T \cdot V_y$ if and only if $y \notin \Omega^{(-\epsilon,\mathbf{d})}(x)$,*

(ii) *$X \setminus \{x\}$ is an $(\epsilon,\mathbf{t})$-exterior subset if and only if $\Omega^{(-\epsilon,\mathbf{d})}(x) = \emptyset$.*

PROOF. (i) Suppose that there is an open neighbourhood V_y at y and there is $T \in \epsilon$ such that $x \notin T \cdot V_y$. If we assume that $y \in \Omega^{(-\epsilon,\mathbf{d})}(x)$, then there is an $(-\epsilon)$-divergent net t_δ such that $t_\delta \cdot x \in V_y$ and $-t_\delta \in T$ for every $\delta \geq \delta_0$. If we denote $v_{\delta_0} = t_{\delta_0} \cdot x \in V_y$, one has that $x = (-t_{\delta_0}) \cdot v_{\delta_0} \in T \cdot V_y$, which is a contradiction. Therefore, $y \notin \Omega^{(-\epsilon,\mathbf{d})}(x)$.

Conversely, suppose that $y \notin \Omega^{(-\epsilon,\mathbf{d})}(x)$ and for every V_y and every $T \in \epsilon$, $x \in T \cdot V_y$. Note that

$$\Delta = \{(T, V_y) \mid t \in \epsilon, \, V_y \in \mathbf{t}_{X_y}\}$$

is a directed set, $\delta = (T, V_y) \leq \delta' = (T', V_y')\}$ if $T' \subset T$ and $V_y' \subset V_y$. Then there are an ϵ-divergent net (t_δ) and a net (v_δ) such that $v_\delta \to y$ and $t_\delta \cdot v_\delta = x$. Thus the net $(-t_\delta)$ verifies that $(-t_\delta) \cdot x = v_\delta \to y$. Therefore $y \in \Omega^{(-\epsilon,\mathbf{d})}(x)$ which is a contradiction. Consequently, there is an open neigbourhood V_y at y and there is $T \in \epsilon$ such that $x \notin T \cdot V_y$.

(ii) follows from (i).   □

COROLLARY 6.10. *If X is a T_1 flow, then*

$$L^{(\epsilon,\mathbf{t})}(X) = Ag^{(-\epsilon,\mathbf{d})}(X).$$

PROOF. If $\Omega^{(-\epsilon,\mathbf{d})}(x) = \emptyset$, then, by Theorem 6.9 (ii), $X \setminus \{x\}$ is an $(\epsilon,\mathbf{t})$-exterior subset and this implies that $x \notin L^{(\epsilon,\mathbf{t})}(X)$. If $x \notin L^{(\epsilon,\mathbf{t})}(X)$, then there is an $(\epsilon,\mathbf{t})$-open subset E such that $x \notin E$. Then, $E \subset X \setminus \{x\}$. Since X is T_1, we have that $X \setminus \{x\}$ is an $(\epsilon,\mathbf{t})$-exterior subset. By Theorem 6.9 (ii), it follows that $\Omega^{(-\epsilon,\mathbf{d})}(x) = \emptyset$.   □

COROLLARY 6.11. *If X is a T_1 flow, then*

(i) $L^{(\mathbf{l},\mathbf{t})}(X) = \{x \in X | \Omega^{(\mathbf{r},\mathbf{d})}(x) \neq \emptyset\} = Ag^{(\mathbf{r},\mathbf{d})}(X)$,

(ii) $L^{(\mathbf{r},\mathbf{t})}(X) = \{x \in X | \Omega^{(\mathbf{l},\mathbf{d})}(x) \neq \emptyset\} = Ag^{(\mathbf{l},\mathbf{d})}(X)$,

(iii) $L^{(\mathbf{c},\mathbf{t})}(X) = L^{(\mathbf{l},\mathbf{t})}(X) \cup L^{(\mathbf{r},\mathbf{t})}(X)$.

COROLLARY 6.12. *If X is a T_1 flow, then*

$$L^{(\epsilon,\mathbf{t})}(X) \subset \Omega^{(\epsilon,\mathbf{t})}(X).$$

PROOF. It follows from Corollary 6.8 and Corollary 6.10. □

COROLLARY 6.13. *Let X be a T_1 compact flow. Then,*

$$L^{(\mathbf{l},\mathbf{t})}(X) = L^{(\mathbf{r},\mathbf{t})}(X) = L^{(\mathbf{c},\mathbf{t})}(X) = X.$$

PROOF. It follows from the property of finite intersection of the compact flow X and Corollary 6.10. □

THEOREM 6.14. *Let X be a T_2 flow. Suppose that for every $x \in L^{(\epsilon,\mathbf{t})}(X)$, $\Omega^{(-\epsilon,\mathbf{d})}(x)$ is compact. Then,*

(i) *For every $x \in L^{(\epsilon,\mathbf{t})}(X)$, $\emptyset \neq \Omega^{(-\epsilon,\mathbf{d})}(x) \subset L^{(\epsilon,\mathbf{t})}(X)$;*

(ii) $L^{(\epsilon,\mathbf{t})}(L^{(\epsilon,\mathbf{t})}(X)) = L^{(\epsilon,\mathbf{t})}(X)$;

(iii) $\Omega^{(-\epsilon,\mathbf{d})}(X) \subset L^{(\epsilon,\mathbf{t})}(X)$ *and the region of $(-\epsilon,\mathbf{d})$-attraction of $\Omega^{(-\epsilon,\mathbf{d})}(X)$ is $L^{(\epsilon,\mathbf{t})}(X)$.*

PROOF. (i) Suppose $x \in L^{(\epsilon,\mathbf{t})}(X)$ and take $y \in \Omega^{(-\epsilon,\mathbf{d})}(x)$. Since $\Omega^{(-\epsilon,\mathbf{d})}(x)$ is a closed compact invariant subset of X, one has that

$$\{\overline{T \cdot y} \mid T \in (-\epsilon)\}$$

is a family of closed subsets of $\Omega^{(-\epsilon,\mathbf{d})}(x)$ with non-empty intersection $\bigcap_{T \in (-\epsilon)} \overline{T \cdot y} = \Omega^{(-\epsilon,\mathbf{d})}(y)$. Note that $\Omega^{(-\epsilon,\mathbf{d})}(y) \subset \Omega^{(-\epsilon,\mathbf{d})}(x)$. (ii) and (iii) follow from (i) and Corollary 6.10. □

Given an exterior space $X = (X, \varepsilon(X))$, $\mathcal{B}$ is a **neighbourhood basis** of $\varepsilon(X)$ if for every $E \in \varepsilon(X)$ there is $K \in \mathcal{B}$ such that $K \subset E$ and its interior $\text{Int}(K) \in \varepsilon(X)$.

DEFINITION 6.15. An exterior space $X = (X, \varepsilon(X))$ is said to be **locally compact at infinity** (or $\varepsilon(X)$ is locally compact at infinity) if $\varepsilon(X)$ has a neighbourhood basis $\mathcal{B}$ such that every $K \in \mathcal{B}$ is a compact subset.

COROLLARY 6.16. *Let X be a T_2 flow. If $\varepsilon^{(\mathbf{r},\mathbf{d})}(X)$ is locally compact at infinity, then*

(i) $L^{(\mathbf{l},\mathbf{t})}(X) = X = A(X)$,

(ii) $\Omega^{(\mathbf{r},\mathbf{d})}(X)$ *is an attractor of X.*

PROOF. Take $x \in X$. Since $X^{(\mathbf{r},\mathbf{d})}$ is locally compact at infinity, there are a compact subset K of X and $T \in \mathbf{r}$ such that $T \cdot x \subset \mathrm{Int}(K) \subset K$. Note that K is a closed subset because X is T_2, and therefore $\overline{T \cdot x} \subset K$. Applying the finite intersection property we obtain that $\Omega^{(\mathbf{r},\mathbf{d})}(x)$ is a non-empty compact subset. $\square$

COROLLARY 6.17. *Let X be a T_2 flow. If $\varepsilon^{(\mathbf{c},\mathbf{d})}(X)$ is locally compact at infinity, then*

(i) $L^{(\mathbf{l},\mathbf{t})}(X) = X = L^{(\mathbf{r},\mathbf{t})}(X)$,
(ii) $\Omega^{(\mathbf{r},\mathbf{d})}(X)$ *is an attractor of X,*
(iii) $\Omega^{(\mathbf{l},\mathbf{d})}(X)$ *is a repeller of X.*

In the following, we use the notation $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$.

DEFINITION 6.18. Let X be a flow. A point $x \in X$ is said to be *positively Lagrange-stable* if $\overline{\mathbb{R}_+ \cdot x}$ is compact, x is said to be *negatively Lagrange-stable* if $\overline{\mathbb{R}_- \cdot x}$ is compact and x is said *Lagrange-stable* if x is positively and negatively Lagrange-stable (note that in this case the closure of the trajectory is compact). A flow X is said to be *positively (resp., negatively) Lagrange-stable* if for every $x \in X$, x is positively Lagrange-stable (resp., x is negatively Lagrange-stable). X is said to be *Lagrange-stable* if X is positively and negatively Lagrange-stable.

COROLLARY 6.19. *Let X be a T_2 flow. If X is positively Lagrange-stable, then*

(i) $L^{(\mathbf{l},\mathbf{t})}(X) = X$,
(ii) $\Omega^{(\mathbf{r},\mathbf{d})}(X)$ *is an attractor of X.*

REMARK 6.20. Dually, one has that if $\varepsilon^{(\mathbf{l},\mathbf{d})}(X)$ is locally compact at infinity or X is negatively Lagrange-stable, then $\Omega^{(\mathbf{r},\mathbf{d})}(X)$ is a repeller of X. We also have that if $\varepsilon^{(\mathbf{c},\mathbf{d})}(X)$ is locally compact at infinity or X is Lagrange-stable, then $\Omega^{(\mathbf{c},\mathbf{d})}(X)$ is an attractor and a repeller of X.

REMARK 6.21. For a Morse function ([25]) $f \colon X \to \mathbb{R}$, where X is a compact T_2 Riemannian manifold, one has that the opposite of the gradient of f induces a flow with a finite number of critical points. In this case, we have that X is locally path-connected and the (ϵ, τ)-exterior flow $X^{(\epsilon,\tau)}$ is locally compact at infinity. Then, one has that

$$L^{(\epsilon,\mathbf{d})}(X) = \Omega^{(\epsilon,\mathbf{d})}(X) = P(X) = C(X), \qquad L^{(\epsilon,\mathbf{t})}(X) = \Omega^{(\epsilon,\mathbf{t})} = X$$

and $L^{(\epsilon,\mathbf{d})}(X)$ is an attractor and a repeller of X.

7. Bar-limit space of an exterior flow, omega-limits and almost periodic points

7.1. *Bar-limit space of an exterior flow and omega-limits.* We recall that the omega-limit set $\Lambda^+(x)$ and the prolongational omega-limit set $J^+(x)$ of a point x in a flow X have been defined in Definition 2.9 and Definition 2.11, respectively, and after Definition 4.1 we have seen that $\Omega^{(\mathbf{r},\mathbf{d})}(x) = \Lambda^+(x)$, $\Omega^{(\mathbf{r},\mathbf{t})}(x) = J^+(x)$. Recall the notation $\Omega^{(\epsilon,\tau)}(X) = \cup_{x\in X}\Omega^{(\epsilon,\tau)}(x)$.

In this subsection we prove that under some topological conditions, the *subflow closure* of $\Omega^{(\mathbf{r},\mathbf{d})}(X)$ is the **bar-limit space** of the exterior space $X^{(\mathbf{r},\mathbf{d})}$ and similarly the *subflow closure* of $\Omega^{(\mathbf{r},\mathbf{t})}(X)$ is the **bar-limit space** of the exterior space $X^{(\mathbf{r},\mathbf{t})}$. Analogous results are obtained for

$$\Omega^{(\mathbf{l},\mathbf{d})}(x) = \Lambda^-(x), \qquad \Omega^{(\mathbf{l},\mathbf{t})}(x) = J^-(x).$$

Then, these *relevant subflows* can be described as **bar-limit spaces** of exterior spaces.

LEMMA 7.1. *If X is an (ϵ,τ)-exterior flow, then*

$$\Omega^{(\epsilon,\tau)}(X) \subset \bar{L}(X).$$

PROOF. If $E \in \varepsilon(X)$, for every $x \in X$ there exist $T \in \epsilon$ and $U \in \tau_x$ such that $T \cdot U \subset E$. Taking the closure, we have $\overline{T \cdot U} \subset \overline{E}$. This implies that $\Omega^{(\epsilon,\tau)}(x) \subset \bar{L}(X)$ for every $x \in X$ and therefore $\Omega^{(\epsilon,\tau)}(X) \subset \bar{L}(X)$. □

PROPOSITION 7.2. *If X is a flow and $\epsilon \in \{\mathbf{l},\mathbf{c},\mathbf{r}\}$, then*
(i) $\Omega^{(\epsilon,\mathbf{d})}(x) \subset \Omega^{(\epsilon,\mathbf{t})}(x)$, for every $x \in X$,
(ii) the following diagram is commutative

$$
\begin{array}{ccc}
\Omega^{(\epsilon,\mathbf{d})}(X) & \longrightarrow & \bar{L}^{(\epsilon,\mathbf{d})}(X) \\
\downarrow & & \downarrow \\
\Omega^{(\epsilon,\mathbf{t})}(X) & \longrightarrow & \bar{L}^{(\epsilon,\mathbf{t})}(X)
\end{array}
$$

where all the arrows are inclusions.

PROOF. (i) It follows from the fact that for any $U \in \tau_x$ and $T \in \epsilon$, $T \cdot x \subset T \cdot U$. The rest of the proof is straightforward. □

In the following results, it is interesting to take into account that if $X = (X, \varepsilon(X))$ is an (ϵ,τ)-exterior flow, then $\varepsilon(X) \subset \varepsilon^{(\epsilon,\tau)}(X_\mathbf{t})$.

PROPOSITION 7.3. *Let $X = (X, \varepsilon(X))$ be an (ϵ,τ)-exterior flow and $x \in X$. Then there exists $V \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V} \in \varepsilon(X)$ if and only if $x \notin \bar{L}(X)$.*

PROOF. Suppose that there exists $V \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V} \in \varepsilon(X)$. Since $V \cap (X \setminus \overline{V}) = \emptyset$, then $x \notin \overline{X \setminus \overline{V}}$. Since $X \setminus \overline{V} \in \varepsilon(X)$, it follows that

$$x \notin \bigcap_{E \in \varepsilon(X)} \overline{E} = \bar{L}(X).$$

Conversely, if $x \notin \bar{L}(X)$, there is $E \in \varepsilon(X)$ such that $x \notin \overline{E}$; then taking $V = X \setminus \overline{E} = \mathrm{Int}(X \setminus E)$, we have that $V \in (\mathbf{t}_X)_x$ and $E \subset X \setminus \overline{V}$, so $X \setminus \overline{V} \in \varepsilon(X)$. $\qquad\square$

LEMMA 7.4. *Let* $X = (X, \varepsilon(X))$ *be an* (ϵ, τ)-*exterior flow and* $x \in X$. *If there exists* $V \in (\mathbf{t}_X)_x$ *such that* $X \setminus \overline{V} \in \varepsilon(X)$, *then* $x \notin \overline{\Omega^{(\epsilon,\tau)}(X)}$.

PROOF. Suppose there exists $V \in (\mathbf{t}_X)_x$ such that $X \setminus \overline{V} \in \varepsilon(X)$; then by Proposition 7.3, it follows that $x \notin \bar{L}(X)$. By Lemma 7.1, one has that

$$\Omega^{(\epsilon,\tau)}(X) \subset \overline{\Omega^{(\epsilon,\tau)}(X)} \subset \bar{L}(X),$$

so $x \notin \overline{\Omega^{(\epsilon,\tau)}(X)}$. $\qquad\square$

LEMMA 7.5. *Let* X *be a locally compact regular flow. If* $x \notin \overline{\Omega^{(\epsilon,\tau)}(X)}$, *then there exists* $V \in (\mathbf{t}_X)_x$ *such that* $X \setminus \overline{V}$ *is* (ϵ, τ)-*exterior.*

PROOF. Suppose that $x \notin \overline{\Omega^{(\epsilon,\tau)}(X)}$. Since X is locally compact, there is a compact neighbourhood K at x such that $K \cap \Omega^{(\epsilon,\tau)}(X) = \emptyset$. Take $y \in X$ and suppose that for every open neighbourhood U at y and for every $T \in \epsilon$, $(T \cdot U) \cap K \neq \emptyset$. Therefore there are a net $y_\delta \to y$ and an ϵ-divergent net t_δ such that $t_\delta \cdot y_\delta \in K$. Since K is compact, there are a subnet $y_{\delta_i} \to y$ and a divergent subnet t_{δ_i} such that $t_{\delta_i} \cdot y_{\delta_i} \to u \in K$. This implies that $u \in K \cap \Omega^{(\epsilon,\tau)}(y) \subset K \cap \Omega^{(\epsilon,\tau)}(X)$, which contradicts the fact that $K \cap \Omega^{(\epsilon,\tau)}(X) = \emptyset$. Then, for every $y \in X$ there are $U \in \mathbf{t}_{X_y}$ and $T \in \epsilon$ such that $T \cdot U \cap K = \emptyset$. Since X is regular, there is $V \in \mathbf{t}_{X_x}$ such that $\overline{V} \subset K$; then $X \setminus \overline{V}$ is (ϵ, τ)-exterior. $\qquad\square$

In this way, we obtain the following relation between bar-limits and omega-limits:

COROLLARY 7.6. *If* X *is a locally compact regular flow, then*

$$\bar{L}^{(\epsilon,\tau)}(X) = \overline{\Omega^{(\epsilon,\tau)}(X)}.$$

PROOF. By Lemma 7.1, one has $\overline{\Omega^{(\epsilon,\tau)}(X)} \subset \bar{L}^{(\epsilon,\tau)}(X)$. Besides, if $x \notin \overline{\Omega^{(\epsilon,\tau)}(X)}$, by Lemma 7.5, there exists $V \in \mathbf{t}_{X_x}$ such that $X \setminus \overline{V}$ is (ϵ, τ)-exterior. By Proposition 7.3 it follows that $x \notin \bar{L}^{(\epsilon,\tau)}(X)$. $\qquad\square$

We note that some of the main results (Theorem 6.13 and Corollary 6.14) proved in [19] are consequences of Corollary 7.6. Moreover, if we denote

$$\text{Poisson}^{\mathbf{l}}(X) = \text{Poisson}^{-}(X),$$
$$\text{Poisson}^{\mathbf{r}}(X) = \text{Poisson}^{+}(X),$$
$$\text{Poisson}^{\mathbf{c}}(X) = \text{Poisson}^{+}(X) \cup \text{Poisson}^{-}(X),$$

we obtain the following results:

COROLLARY 7.7. *If X is a locally compact T_3 flow, then*

(i) $L^{(\epsilon,\mathbf{d})}(X) = P(X) \subset Poisson^{\epsilon}(X) \subset \Omega^{(\epsilon,\mathbf{d})}(X) \subset \overline{\Omega^{(\epsilon,\mathbf{d})}(X)} = \bar{L}^{(\epsilon,\mathbf{d})}(X),$

(ii) $L^{(\epsilon,\mathbf{t})}(X) = Ag^{(-\epsilon,\mathbf{d})}(X) \subset Ag^{(-\epsilon,\mathbf{t})}(X) \subset \Omega^{(\epsilon,\mathbf{t})}(X) \subset \overline{\Omega^{(\epsilon,\mathbf{t})}(X)} = \bar{L}^{(\epsilon,\mathbf{t})}(X).$

PROOF. (i) It follows from Theorem 6.3 and Corollary 7.6.
(ii) It is a consequence of Corollary 6.8, Corollary 6.10 and Corollary 7.6.
□

REMARK 7.8. If X is a locally compact T_3 flow, then the following diagram is commutative, where all the arrows are inclusions:

$$
\begin{array}{ccccc}
Ag^{(\mathbf{r},\mathbf{d})}(X) & \longrightarrow & Ag^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & Ag^{(\mathbf{l},\mathbf{d})}(X) \\
\| & & \| & & \| \\
L^{(\mathbf{l},\mathbf{t})}(X) & \longrightarrow & L^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & L^{(\mathbf{r},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
Ag^{(\mathbf{r},\mathbf{t})}(X) & \longrightarrow & Ag^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & Ag^{(\mathbf{l},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{(\mathbf{l},\mathbf{t})}(X) & \longrightarrow & \Omega^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \Omega^{(\mathbf{r},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\Omega^{(\mathbf{l},\mathbf{t})}(X)} & \longrightarrow & \overline{\Omega^{(\mathbf{c},\mathbf{t})}(X)} & \longleftarrow & \overline{\Omega^{(\mathbf{r},\mathbf{t})}(X)} \\
\| & & \| & & \| \\
\bar{L}^{(\mathbf{l},\mathbf{t})}(X) & \longrightarrow & \bar{L}^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \bar{L}^{(\mathbf{r},\mathbf{t})}(X)
\end{array}
$$

This diagram relates $(\epsilon,\mathbf{d})$-agglomerative points, $(-\epsilon,\mathbf{t})$-limits, $(\epsilon,\mathbf{t})$-agglomerative points, prolongational limits and $(-\epsilon,\mathbf{t})$-bar-limits.

REMARK 7.9. Given a flow X and $x \in X$, we can say that x is (ϵ, τ)-dispersive if $\Omega^{(\epsilon,\tau)}(x) = \emptyset$. We point out that $X \setminus Ag^{(\epsilon,\tau)}(X)$ is the set of (ϵ, τ)-dispersive points of X. For $\tau = \mathbf{t}_X$ and $\epsilon = \mathbf{r}$, we can apply Theorems 1.8 and 2.6 in Chapter IV of [4] to prove that, under suitable conditions, $X \setminus Ag^{(\epsilon,\tau)}(X)$ is a parallelizable subflow and $X \setminus L^{(-\epsilon,\tau)}(X)$ is an open parallelizable subflow.

REMARK 7.10. We have seen in Remark 6.4 and Theorem 6.14 that the functors $L^{(\epsilon,\tau)}$ are idempotent under some topological conditions. In general the idempotent property for bar-limits $\bar{L} = \bar{L}^{(\epsilon,\tau)}$ could be more difficult to prove. However, we can always consider for the ordinals $1, 2, \cdots, \alpha_0, \alpha_0 + 1, \cdots$ the corresponding injective maps:

$$\cdots \to \bar{L}^{\alpha_0+1} \to \bar{L}^{\alpha_0} \to \cdots \to \bar{L}^2 \to \bar{L}$$

where $\bar{L}^{\beta+1}(X) = \bar{L}(\bar{L}^\beta(X))$ and if β is a limit ordinal $\bar{L}^\beta(X) = \bigcap_{\alpha<\beta} \bar{L}^\alpha(X)$. It is clear that for any flow X there is an ordinal β such that

$$\bar{L}^{\beta+1}(X) = \bar{L}^\beta(X).$$

7.2. *Limits, bar-limits and (ϵ, τ)-periodic points.*

DEFINITION 7.11. Let $(X, \varepsilon(X))$ be an (ϵ, τ)-exterior flow. A point $x \in X$ is said to be (ϵ, τ)-**periodic** if for every $U \in \mathbf{t}_{X_x}$, for every $T \in \epsilon$ and for every $V \in \tau_x$, $(T \cdot V) \cap U \neq \emptyset$. Denote by $P^{(\epsilon,\tau)}(X)$ the τ-closed subset of points of X which are (ϵ, τ)-periodic.

Note that x is an (ϵ, τ)-periodic point if and only if $x \in \Omega^{(\epsilon,\tau)}(x)$. Then, $P^{(\epsilon,\tau)}(X) \subset \Omega^{(\epsilon,\tau)}(X)$.

If X is a flow, given $x \in X$, x is an (ϵ, τ)-*periodic* point if x is an (ϵ, τ)-periodic point in $X^{(\epsilon,\tau)}$. $P^{(\epsilon,\tau)}(X)$ denotes the subflow of (ϵ, τ)-periodic points of X.

EXAMPLE 7.12. Let $X = \mathbb{R}$ with the topology

$$\mathbf{t}_X = \{(t, +\infty) | t \in \mathbb{R}\} \cup \{\emptyset, X\}.$$

Consider the flow

$$\varphi \colon \mathbb{R} \times X \to X, \qquad \varphi(r, s) = r + s.$$

Now it is easy to check that every $s \in X$ is an $(\mathbf{r}, \mathbf{d})$-periodic point and X does not have $(\mathbf{l}, \mathbf{d})$-periodic points.

LEMMA 7.13. *Let X be a flow and $x \in X$. Then, x is $(\mathbf{l}, \mathbf{t})$-periodic if and only if x is $(\mathbf{r}, \mathbf{t})$-periodic if and only x is $(\mathbf{c}, \mathbf{t})$-periodic if and only if x is a non-wandering point.*

PROOF. First, we prove that a non-$(\mathbf{l}, \mathbf{t})$-periodic point is a wandering point: Indeed, suppose that x is a non-$(\mathbf{l}, \mathbf{t})$-periodic point; then there is $U \in \mathbf{t}_{X_x}$ and there is $T \in \mathbf{l}$ such that $(T \cdot U) \cap U = \emptyset$. Take $r_0' \in \mathbb{R}$ such that $(-\infty, r_0'] \subset T$; then, for every $r' \leq r_0'$, $\varphi_{r'}(U) \cap U = \emptyset$. Since $\varphi_{r'}$ is

a homeomorphism, it follows that $U \cap \varphi_{-r'}(U) = \emptyset$. Taking $r_0 = -r'_0$ one has that for $r \geq r_0$, $\varphi_r(U) \cap U = \emptyset$. Therefore x is a wandering point. The converse and other implications can be proved similarly. $\quad\square$

Consequently, one has that for any flow X:

$$N(X) = P^{(\mathbf{l},\mathbf{t})}(X) = P^{(\mathbf{c},\mathbf{t})}(X) = P^{(\mathbf{r},\mathbf{t})}(X).$$

PROPOSITION 7.14. *Suppose that X is a flow. Then,*

(i) $P(X) \subset P^{(\mathbf{l},\mathbf{d})}(X) \cap P^{(\mathbf{r},\mathbf{d})}(X)$,
(ii) $P^{(\mathbf{c},\mathbf{d})}(X) = P^{(\mathbf{l},\mathbf{d})}(X) \cup P^{(\mathbf{r},\mathbf{d})}(X)$,
(iii) $P^{(\epsilon,\mathbf{d})}(X) = Poisson^\epsilon(X)$,
(iv) $P^{(\epsilon,\mathbf{d})}(X) \subset P^{(\epsilon,\mathbf{t})}(X)$,
(v) $P^{(\epsilon,\mathbf{t})}(X) = N(X)$.

PROOF. (i) is easy to check. To prove (ii), suppose that x is $(\mathbf{c},\mathbf{d})$-periodic and x is not $(\mathbf{l},\mathbf{d})$-periodic. Since x is not $(\mathbf{l},\mathbf{d})$-periodic, if U is an open neighbourhood at x and $T \in \mathbf{r}$, there are $T' \in \mathbf{l}$ and $U' \in \mathbf{t}_{X_x}$, satisfying that $U' \subset U$ and $(T' \cdot x) \cap U' = \emptyset$. Taking into account that $T'' = T' \cup T \in \mathbf{c}$ and x is $(\mathbf{c},\mathbf{d})$-periodic, one has that $(T'' \cdot x) \cap U' \neq \emptyset$. This implies that $(T \cdot x) \cap U \neq \emptyset$; then x is $(\mathbf{r},\mathbf{d})$-periodic. Applying Lemma 7.13, we obtain (v) and the rest of the proof is straightforward. $\quad\square$

REMARK 7.15. For a flow on a compact metric space, we have that the recurrent points in the sense of Birkoff (see Remark 3.6 of Chapter III in [4]) are $(\mathbf{c},\mathbf{d})$-periodic points.

PROPOSITION 7.16. *Suppose that X is a regular flow. If a point x is not $(\epsilon,\mathbf{t})$-periodic, then there is $V \in \mathbf{t}_{X_x}$ such that $X \setminus \overline{V}$ is $(\epsilon,\mathbf{d})$-exterior. Moreover, there is $T \in \epsilon$ such that*

$$(T \cdot V) \subset X \setminus \overline{V}.$$

PROOF. If a point x is not $(\epsilon,\mathbf{t})$-periodic, then there is an open neighbourhood U at x and there is $T \in \epsilon$ such that $(T \cdot U) \cap U = \emptyset$. Since X is regular, there is $V \in \mathbf{t}_{X_x}$ such that $x \in V \subset \overline{V} \subset U$. Then $T \cdot V \subset X \setminus \overline{V}$. Let us check that $X \setminus \overline{V}$ is $(\epsilon,\mathbf{d})$-exterior:

Indeed, for a given $y \in X$, if $(\mathbb{R} \cdot y) \cap U \neq \emptyset$, then there is $r \in \mathbb{R}$ such that $r \cdot y \in U$. Since $(T \cdot U) \cap U = \emptyset$, it follows that $(r + T) \cdot (-r \cdot U) \cap U = \emptyset$. Taking $T^y = r + T$ and $U_y = -r \cdot U$, we have that $(T^y \cdot U_y) \cap U = \emptyset$. This implies that

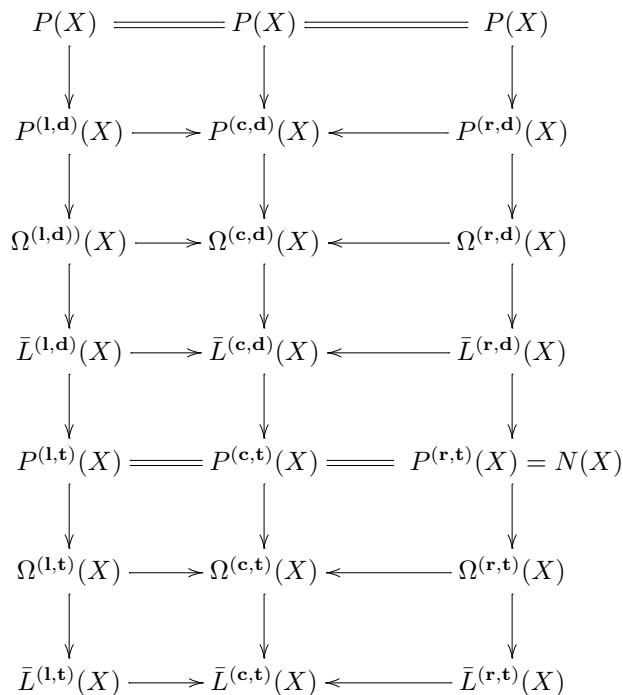$$T^y \cdot y \subset T^y \cdot U_y \subset X \setminus \overline{V}.$$

Otherwise, one has that $(\mathbb{R} \cdot y) \cap U = \emptyset$. Therefore $\mathbb{R} \cdot y \subset X \setminus \overline{V}$. Taking into account the definition of $(\epsilon,\mathbf{d})$-exterior, it follows that $X \setminus \overline{V}$ is $(\epsilon,\mathbf{d})$-exterior. $\quad\square$

THEOREM 7.17. *If X is a regular flow, then*

$$\bar{L}^{(\epsilon,\mathbf{d})}(X) \subset N(X).$$

PROOF. By part (v) of Proposition 7.14, $P^{(\epsilon,\mathbf{t})}(X) = N(X)$. By Proposition 7.16, if $x \notin P^{(\epsilon,\mathbf{t})}(X)$, there is $V \in \mathbf{t}_{X_x}$ such that $X \setminus \overline{V}$ is $(\epsilon, \mathbf{d})$- exterior. This implies that $x \notin \bar{L}^{(\epsilon,\mathbf{d})}(X)$. $\qquad\qquad\square$

REMARK 7.18. If X is a T_3 flow, then the following diagram is commutative, where all the arrows are inclusions:

$$
\begin{array}{ccccc}
P(X) & \!=\!=\!=\! & P(X) & \!=\!=\!=\! & P(X) \\
\downarrow & & \downarrow & & \downarrow \\
P^{(\mathbf{l},\mathbf{d})}(X) & \longrightarrow & P^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & P^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{(\mathbf{l},\mathbf{d}))}(X) & \longrightarrow & \Omega^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & \Omega^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L}^{(\mathbf{l},\mathbf{d})}(X) & \longrightarrow & \bar{L}^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & \bar{L}^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
P^{(\mathbf{l},\mathbf{t})}(X) & \!=\!=\! & P^{(\mathbf{c},\mathbf{t})}(X) & \!=\!=\! & P^{(\mathbf{r},\mathbf{t})}(X) = N(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{(\mathbf{l},\mathbf{t})}(X) & \longrightarrow & \Omega^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \Omega^{(\mathbf{r},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L}^{(\mathbf{l},\mathbf{t})}(X) & \longrightarrow & \bar{L}^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \bar{L}^{(\mathbf{r},\mathbf{t})}(X)
\end{array}
$$

This diagram relates positively (negatively) Poisson-stable points, omega-limits, $(\epsilon, \mathbf{d})$-bar-limits, non-wandering points, first prolongational limits and $(\epsilon, \mathbf{t})$-bar-limits.

THEOREM 7.19. *If X is a T_3 locally compact flow and every non-wandering point is a Lagrange-stable point, then the following diagram is commutative, where all the arrows are inclusions (and we also have $\overline{\Omega^{(\epsilon,\tau)}(X)} = \bar{L}^{(\epsilon,\tau)}(X)$):*

$$
\begin{array}{ccccc}
L^{(\mathbf{l},\mathbf{d})}(X) & \!\!=\!\!\!=\!\!\!=\!\! & L^{(\mathbf{c},\mathbf{d})}(X) & \!\!=\!\!\!=\!\! & L^{(\mathbf{r},\mathbf{d})}(X) = P(X) \\
\downarrow & & \downarrow & & \downarrow \\
P^{(\mathbf{l},\mathbf{d})}(X) & \longrightarrow & P^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & P^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{(\mathbf{l},\mathbf{d})}(X) & \longrightarrow & \Omega^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & \Omega^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L}^{(\mathbf{l},\mathbf{d})}(X) & \longrightarrow & \bar{L}^{(\mathbf{c},\mathbf{d})}(X) & \longleftarrow & \bar{L}^{(\mathbf{r},\mathbf{d})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
N(X) & \!\!=\!\!\!=\!\!\!=\!\! & N(X) & \!\!=\!\!\!=\!\!\!=\!\! & N(X) \\
\downarrow & & \downarrow & & \downarrow \\
L^{(\mathbf{l},\mathbf{t})}(X) \cap L^{(\mathbf{r},\mathbf{t})}(X) & \longrightarrow & L^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & L^{(\mathbf{l},\mathbf{t})}(X) \cap L^{(\mathbf{r},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^{(\mathbf{l},\mathbf{t})}(X) \cap \Omega^{(\mathbf{r},\mathbf{t})}(X) & \longrightarrow & \Omega^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \Omega^{(\mathbf{l},\mathbf{t})}(X) \cap \Omega^{(\mathbf{r},\mathbf{t})}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L}^{(\mathbf{l},\mathbf{t})}(X) \cap \bar{L}^{(\mathbf{r},\mathbf{t})}(X) & \longrightarrow & \bar{L}^{(\mathbf{c},\mathbf{t})}(X) & \longleftarrow & \bar{L}^{(\mathbf{l},\mathbf{t})}(X) \cap \bar{L}^{(\mathbf{r},\mathbf{t})}(X)
\end{array}
$$

Proof. Notice that the results of Remark 7.18 can be completed by using Corollary 7.6 to obtain that $\overline{\Omega^{(\epsilon,\tau)}(X)} = \bar{L}^{(\epsilon,\tau)}(X)$. Moreover, from the condition that every non-wandering point is Lagrange-stable, applying Corollary 6.19 we also have that $N(X) \subset L^{(\mathbf{l},\mathbf{t})}(X) \cap L^{(\mathbf{r},\mathbf{t})}(X)$. $\qquad\square$

Remark 7.20. We note that if X is a T_2 compact flow, then X is a T_3 locally compact and every point of X is a Lagrange-stable point. In this case we have that $X = L^{(\epsilon,\mathbf{t})}(X) = \Omega^{(\epsilon,\mathbf{t})}(X) = \bar{L}^{(\epsilon,\mathbf{t})}(X)$ and all the subflows of the last three rows of diagram above are equal to X.

8. Relations between limit spaces and Freudenthal end spaces

Given a flow X, for $(\epsilon, \tau) \in \{\mathbf{l}, \mathbf{r}, \mathbf{c}\} \times \{\mathbf{d}, \mathbf{t}\}$ the corresponding (ϵ, τ)-limit and (ϵ, τ)-end spaces are related by the following commutative diagram:
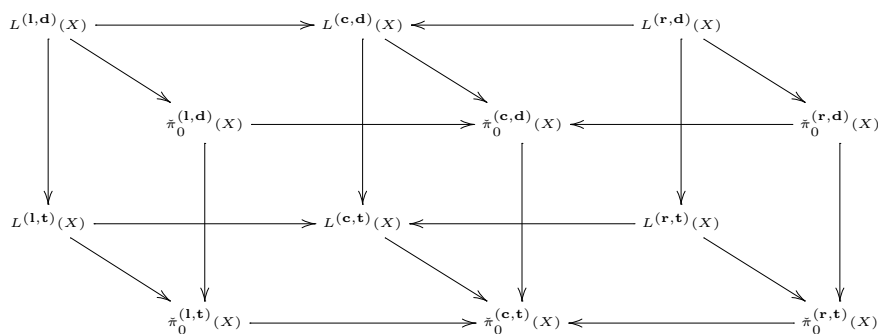
We have a similar diagram for (ϵ, τ)-bar-limit and (ϵ, τ)-bar-end spaces.

We note that this diagram gives decompositions of the different limits sub-flows as disjoint union of the fibres of the different canonical maps. Moreover, these various sets of Freudenthal ends can be used to divide the global flow as a union of disjoint basins associated with the end points of these Freudenthal sets.

It is interesting to observe the relationship between the limit of sets of path-components of exterior open subsets of an externology and the set of connected components of its limit. We can apply the properties of exterior spaces proved in Theorem 3.17 in [19] to the externologies $\varepsilon^{(\epsilon, \mathbf{d})}(X)$ (resp., $\varepsilon^{(\epsilon, \mathbf{t})}(X)$) of a flow X to establish an isomorphism between the set of connected components of periodic points (resp., agglomerative points) and the profinite spaces of Freudenthal ends. Consequently, we have:

COROLLARY 8.1. *Let X be a non-empty flow and suppose that X is a connected, locally path-connected space and $X^{(\epsilon, \tau)}$ is locally compact at infinity. Then,*

(i) *$L^{(\epsilon, \tau)}(X) = \bar{L}^{(\epsilon, \tau)}(X)$ is compact and $\check{\pi}_0^{(\epsilon, \tau)}(X) = \check{\check{\pi}}_0^{(\epsilon, \tau)}(X)$,*

(ii) *the canonical map $L^{(\epsilon, \tau)}(X) \to \check{\pi}_0^{(\epsilon, \tau)}(X)$ is surjective (i.e., any Freudenthal end is representable),*

(iii) *$\check{\pi}_0^{(\epsilon, \tau)}(X)$ is a profinite compact space,*

(iv) *if $a \in \check{\pi}_0^{(\epsilon, \tau)}(X)$, then $L_a^{(\epsilon, \tau)}(X)$ is a non-empty continuum,*

(v) *if $\tau = \mathbf{t}_X$, then $\check{\pi}_0^{(\epsilon, \mathbf{t})}(X) = \{*\}$ has only one end and*

$$L_*^{(\epsilon, \mathbf{t})}(X) = L^{(\epsilon, \mathbf{t})}(X) = \bar{L}_*^{(\epsilon, \mathbf{t})}(X)$$

is a non-empty continuum.

Given a topological space Y, we denote by $c(Y)$ the set of connected components of Y. Note that as a consequence of Corollary 8.1 we have:

COROLLARY 8.2. *Let X be a non-empty flow and suppose that X is a connected, locally path-connected space and $X^{(\epsilon, \tau)}$ is locally compact at infinity.*

Then,

$$c(L^{(\epsilon,\tau)}(X)) \cong \check{\pi}_0^{(\epsilon,\tau)}(X).$$

9. Conclusions and further work

In this paper we have seen that there are very nice connections between concepts of the theory of exterior spaces and the theory of continuous flows. These facts permit us to apply constructions and properties developed for exterior spaces to the study of dynamical systems.

On the one hand, for an exterior space, we have, among others, the following notions and constructions:

- **notion of open and closed exterior subsets (Definition 2.1),**
- **the limit space (Definition 2.2),**
- **the bar-limit space (Definition 2.2),**
- **the set (or space) of Freudenthal end points (Subsection 2.1),**
- **the set (or space) of Freudenthal bar-end points (Subsection 2.1),**
- **the notion of locally compact at infinity (Definition 6.15),**
- **higher homotopy invariant of exterior spaces ([12,21]) (Steenrod, Čech and Brown-Grossmann groups, see [10]).**

On the other hand, for a continuous flow, we have common dynamical notions as the following:

- *the omega-limit of a given point and the omega-limit of a subset of points* (Definition 2.9),
- *the first prolongational limit space of a given point and the first prolongational limit space of a subset of points* (Definition 2.11 and Remark 2.13),
- *periodic point* (Subsection 2.2),
- *positively (negatively) Poisson-stable point* (Subsection 2.2),
- *non wandering point* (Definition 2.15),
- *the region of attraction of a subset* (Definition 2.18),
- *attractor of a region* (Definition 2.19),
- *decomposition of a flow as disjoint union of basins* (Definition 5.3),
- *positively (negatively) agglomerative point* (Definition 6.5),
- *positively (negatively) Lagrange-stable point* (Definition 6.18),
- *dispersive and parallelizable flows* (Remark 7.9), et cetera.

In our study we have found many interesting results relating the exterior notions and constructions given in the first list and the dynamic notions given in the second list. For example:

- Theorem 6.14 proves that the $(-\epsilon, \mathbf{t})$-**limit** of a T_2 flow is the region of attraction of the $(\epsilon, \mathbf{d})$-*omega limit*.

- Theorem 7.19 summarizes many of the relations between the **subflows associated to canonical exterior constructions** and the *more important subflows of a continuous dynamical system*. In this Theorem we give some important relationships between $(\epsilon, \mathbf{d})$-**limit,** $(\epsilon, \mathbf{d})$-**bar-limits,** $(\epsilon, \mathbf{t})$-**limits,** $(\epsilon, \mathbf{t})$-**bar limits** and *periodic points, Poison-stable points, omega-limits, non-wandering points, agglomerative points and first prolongational limits.*

In addition, for a continuous flow X, the following important relationships and results have been proved in this paper:

- Given a point x of a T_1 flow X, $\{x\}$ is an $(\epsilon, \mathbf{d})$-**closed subset** if and only if x is a *non-periodic point* (Theorem 6.2);
- Given $x \in X$, $\{x\}$ is an $(\epsilon, \mathbf{t})$-**closed subset** if and only if x is not a $(-\epsilon, \mathbf{d})$-*agglomerative point* (Theorem 6.9 and Definition 6.5);
- For $\epsilon \in \{\mathbf{l}, \mathbf{c}, \mathbf{r}\}$, the $(\epsilon, \mathbf{d})$-**limit** of a T_1 flow is the subset of *periodic points* (Theorem 6.3);
- For $\epsilon \in \{\mathbf{l}, \mathbf{c}, \mathbf{r}\}$, the $(\epsilon, \mathbf{t})$-**limit** of a T_1 flow is the subset of $(-\epsilon, \mathbf{d})$-*agglomerative points* of X (Corollary 6.10);
- Results about the relation of the **locally compact at infinity** condition and *attractors and repellers* (Corollaries 6.16, 6.17 and Remark 6.20);
- Properties of $(\mathbf{l}, \mathbf{t})$-**limit** of a flow and the *positively Lagrange-stable* condition for a flow (Corollary 6.19);
- The $(\epsilon, \mathbf{d})$-**bar-limit** of a flow as the closure of the *omega-limit* of the flow (Corollary 7.6);
- The $(\epsilon, \mathbf{t})$-**bar-limit** of a flow as the closure of the *first prolongational limit* of the flow (Corollary 7.6);
- Remark 7.8 contains a diagram which relates $(\epsilon, \mathbf{d})$-*agglomerative points*, $(-\epsilon, \mathbf{t})$-**limits**, $(\epsilon, \mathbf{t})$-*agglomerative points*, *prolongational limits* and $(\epsilon, \mathbf{t})$-**bar-limits**;
- The complement of the $(\epsilon, \mathbf{t})$-**limit** of a flow and *dispersive and parallelizable* subflows (Remark 7.9);
- For a T_3 flow, the diagram given in Remark 7.18 relates *positively (negatively) Poisson-stable points, omega-limits*, $(\epsilon, \mathbf{d})$-**bar-limits**, *non-wandering points, first prolongational limits* and $(\epsilon, \mathbf{t})$-**bar-limits**;
- **Freudenthal ends of an exterior space** and decomposition of a flow as *disjoint union of basins* (Proposition 5.2 and Definition 5.3);
- The condition of **locally compact at infinity** is related to the decomposition of set of *periodic points* or the set of *agglomerative points* as disjoint union of continuum components (Corollary 8.1);
- The isomorphism obtained (under the conditions of Corollary 8.2) between the space of connected components of the subflow of *periodic*

points or the set of connected components of the subflow of *agglomerative points* and the corresponding **set of Freudenthal ends**.

With respect to further work and new research lines relating exterior spaces and continuous flows we suggest the following:

- To use **higher homotopy invariant of exterior spaces (Steenrod, Čech and Brown-Grossmann groups)** as new tools in the analysis of *local and global stability properties of continuous flows* and in the study of *chaotic dynamical systems*.
- To study the role of the **externologies** $\varepsilon^{(\epsilon,\tau_{\mathrm{int}})}(X)$, where τ_{int} is the intrinsic topology, in the study of *repellers* and *atractors* of a dynamical system. In particular, our techniques based on **exterior flows** can be used to generalize some results obtained in [1–3].
- To study the role of the **externologies** $\varepsilon^{(\epsilon,\tau_{\mathrm{tr}})}(X)$ with respect to the dynamical properties of a *flow* (without *fixed points*) in a Riemannian manifold, where τ_{tr} is the topology of the foliation transverse to the flow.
- To discover the existence of new relationships between the **theory of exterior spaces** and the *theories of discrete and continuous dynamical systems*.

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