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CUBIC STRUCTURE

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Abstract. In this paper we examine the relationships between cubic structures, totally symmetric medial quasigroups, and commutative groups. We prove that the existence of a cubic structure on the given set is equivalent to the existence of a totally symmetric medial quasigroup on this set, and it is equivalent to the existence of a commutative group on this set. We give also some interesting geometric examples of cubic structures. By means of these examples, each theorem that can be proved for an abstract cubic structure has a number of geometric consequences. In the final part of the paper, we prove also some simple properties of abstract cubic structures.

By studying different curves of the third and fourth order and some other similar geometric problems one often finds an abstract structure (see [1–3, 10, 11]). For this reason, we consider that it is useful to study these structures, which will be called *cubic structures*. In this paper we mention a number of geometric models of this structures. Cubic structures could be also very useful for studying some types of configurations of points and lines (see [4–7]).

1. Cubic structures, TSM-quasigroups and commutative groups

Let Q be a nonempty set, whose elements are called points, and let $[\ ] \subseteq Q^3$ be a ternary relation on Q. Such a relation and the ordered pair $(Q, [\ ])$ will be called a *cubic relation* and a *cubic structure*, respectively, if the following properties are satisfied:

C1. For any two points $a, b \in Q$ there is a unique point $c \in Q$ such that $[a, b, c]$, i.e., $(a, b, c) \in [\ ]$.

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C2. The relation $[\ ]$ is totally symmetric, i.e., $[a,b,c]$ implies $[a,c,b]$, $[b,a,c]$, $[b,c,a]$, $[c,a,b]$ and $[c,b,a]$.

C3. $[a,b,c]$, $[d,e,f]$, $[g,h,i]$, $[a,d,g]$ and $[b,e,h]$ imply $[c,f,i]$, which can be clearly written in the form of the following table

$$\begin{array}{cc|c|} \cline{3-3} a & b & c \\ d & e & f \\ g & h & i \\ \cline{3-3} \end{array}$$

In property C2, it is enough to require that $[a,b,c]$ implies e.g. $[b,a,c]$ and $[c,b,a]$.

Let Q be a nonempty set and $\cdot$ a binary operation on Q. The ordered pair $(Q,\cdot)$ is a *quasigroup* if for each $a,b\in Q$ there exist unique elements x and y such that $ax=b$ and $ya=b$. The quasigroup $(Q,\cdot)$ is *medial* (in the older literature it was also termed *entropic, alternative, bisymmetric, Abelian*) if the following identity is valid

$$ab\cdot cd = ac\cdot bd, \tag{1.1}$$

and *totally symmetric* if it satisfies the identities

$$ab\cdot b = a, \tag{1.2}$$

$$a\cdot ab = b, \tag{1.3}$$

where e.g. $ab\cdot cd$ is a shorter notation for $(ab)(cd)$. A totally symmetric quasigroup is commutative, i.e., the identity

$$ab = ba \tag{1.4}$$

holds, because we get

$$ab \overset{(1.3)}{=} (ba)(ba\cdot a)\cdot b \overset{(1.2)}{=} (ba\cdot b)b \overset{(1.2)}{=} ba.$$

A totally symmetric medial quasigroup will be called a TSM-quasigroup for short. According to (1.1) and (1.4), it follows that a TSM-quasigroup satisfies the identity $ab\cdot cd = ef\cdot gh$, where (e,f,g,h) is any permutation of the set $\{a,b,c,d\}$.

THEOREM 1.1. *If the ternary relation $[\ ]$ and the binary operation $\cdot$ on the set Q are connected by the equivalence*

$$[a,b,c] \Leftrightarrow ab=c, \tag{1.5}$$

then $(Q,[\ ])$ is a cubic structure if and only if $(Q,\cdot)$ is a TSM–quasigroup.

Proof. Let $(Q, [\,])$ be a cubic structure and let $a, b \in Q$ be any elements. If $ab = c$, then (1.5) implies $[a, b, c]$. By C2, we get $[c, b, a]$ and $[a, c, b]$, and, according to (1.5), this gives $cb = a$ and $ac = b$, i.e., $ab \cdot b = a$, $a \cdot ab = b$, so we get the identities (1.2) and (1.3). According to C1 and C2, for each $a, b \in Q$ there are unique elements $x, y \in Q$ such that $[a, x, b]$ and $[y, a, b]$, i.e., owing to (1.5), such that $ax = b$ and $ya = b$. Let $a, b, c, d \in Q$ be any elements and let

$$ab = e, \quad cd = f, \quad ac = g, \quad bd = h, \quad gh = i. \tag{1.6}$$

Based on (1.5), we get that $[a, b, e]$, $[c, d, f]$, $[a, c, g]$, $[b, d, h]$, and $[g, h, i]$, and from the table

$$\begin{array}{cc|c|} \cline{3-3} a & b & e \\ c & d & f \\ g & h & i \\ \cline{3-3} \end{array}$$

we acquire $[e, f, i]$, and by (1.5), the equality $ef = i$. So, because of (1.6), we obtain

$$ab \cdot cd = ef = i = gh = ac \cdot bd,$$

showing that $(Q, \cdot)$ is a TSM–quasigroup.

Conversely, assume that $(Q, \cdot)$ is a TSM–quasigroup. Then the property C1 follows from the fact that $\cdot$ is a binary operation. For the proof of property C2 it is enough to show that $[a, b, c]$ implies $[b, a, c]$ and $[c, b, a]$. Based upon (1.5), this means that it is necessary to prove that $ab = c$ implies $ba = c$ and $cb = a$, but this follows from the identities (1.4) and (1.2). For the proof of property C3, owing to (1.5), it is necessary to prove that $ab = c$, $de = f$, $gh = i$, $ad = g$, and $be = h$ imply $cf = i$. However, because of (1.1), we have

$$cf = ab \cdot de = ad \cdot be = gh = i.$$

∎

In [1], it is proved that a TSM-quasigroup $(Q, \cdot)$ can be obtained from a commutative group $(Q, +)$ by defining

$$ab = k - a - b, \tag{1.7}$$

where $k \in Q$ is a fixed element, while the commutative group $(Q, +)$ can be obtained from a TSM-quasigroup by defining $a + b = ab \cdot o$, where $o \in Q$ is a fixed element, which becomes the neutral element of this group. So, each TSM-quasigroup can be obtained from a commutative group. There is a connection between the elements k and o, which is given by the relation $k = oo$.

The equivalence (1.5) and definition (1.7) imply the equivalence

$$[a, b, c] \Leftrightarrow a + b + c = k, \tag{1.8}$$

where $k \in Q$ is a fixed element. These results immediately imply the following theorem.

Theorem 1.2. *Let* $[\ ]$ *be a ternary relation on the set* Q. $(Q, [\ ])$ *is a cubic structure if and only if there is a commutative group* $(Q, +)$ *and an element* $k \in Q$ *such that the equivalency* (1.8) *holds.*

Let us mention that in [11], a medial quasigroup $(Q, \cdot)$ is studied as a C-structure in which, instead of the total symmetry property, a weaker property of semi-symmetry given by identities $ab \cdot a = b$ and $a \cdot ba = b$ are satisfied, and with no commutative property.

2. Examples of cubic structures

There is a number of geometrical examples of cubic structures. We shall mention some of them.

Example 2.1. Let Q be the set of all nonsingular points of the planar cubic curve Γ, and for three given points $a, b, c \in Q$ let the statement $ab = c$ mean that the points a, b, and c lie on the same line, while in the case when two of these points coincide, we take that this line touches the cubic curve Γ at this point taken twice, and if the considered line is tangent to the curve Γ at its inflection point a, then we take that $aa = a$. In [1], it is proved that $(Q, \cdot)$ is a TSM-quasigroup. It is a natural consequence of the well-known fact (see, e.g., [10, pp. 89–94]) that in the set Q, point addition can be defined such that $(Q, +)$ is a commutative group and then collinearity of three points from the set Q is equivalent to the equality $a + b + c = k$, where $k \in Q$ is a fixed point. If we define the ternary relation $[\ ]$ such that $[a, b, c]$ means that the points $a, b, c \in Q$ are collinear (with the above convention about the touching line and curve), then Theorem 1.2 implies that $(Q, [\ ])$ is a cubic structure.

By duality of this example we get one more example, in which there is a curve of the third class with the concurrency of triples of the tangents to this curve.

Example 2.2. Let Γ be a given conic in the Pappian projective plane, $U, V \notin \Gamma$ two given points on a given straight line p and $W \in \Gamma \setminus p$ a given point. Let Ω_2 be the two-parametric family of all conics which contain the points U, V, and W. In [8], it is proved that there is a commutative group $(\Gamma \setminus p, +)$ such that three points $A, B, C \in \Gamma \setminus p$ lie on a curve from the family Ω_2 if and only if the equality $A + B + C = K$ holds, where $K \in \Gamma \setminus p$ is a fixed point. Because of Theorem 1.2, it follows that $(\Gamma \setminus p, [\ ])$ is a cubic structure if $[A, B, C]$ means that the points $A, B, C \in \Gamma \setminus p$ lie on a curve belonging to the family Ω_2.

Example 2.3. Let Γ be a rational circular cubic in the (real or complex) plane and Ω_3 the three-parametric family of all circles not containing the singular point O of Γ, or let Γ be a rational bicircular quartic and Ω_3 the three-parametric family of all circles and straight lines not containing the (proper) double point O of Γ. In [9], it is proved that there is a commutative group $(\Gamma \setminus \{O\}, +)$ such that four points $A, B, C, D \in \Gamma \setminus \{O\}$ lie on a curve from the family Ω_3 if and only if the equality $A + B + C + D = K$ holds, where $K \in \Gamma \setminus \{O\}$ is a fixed point.

Now, let $D \in \Gamma \setminus \{O\}$ be a fixed point and let Ω_2 be the two-parametric family of all circles (resp. all circles and straight lines) from the family Ω_3 which contain the point D. If $[A, B, C]$ means that the points $A, B, C \in \Gamma \setminus \{O\}$ lie on a curve from the family Ω_2, then $(\Gamma \setminus \{O\}, [\ ])$ is again a cubic structure.

Examples 2.2 and 2.3 cover many interesting models of different circular cubics or bicircular quartics, and concyclity or collinearity of points on these curves.

Prior to giving some more interesting examples of cubic structure, let us prove a few (mostly well-known) lemmas.

LEMMA 2.4. *Let Q be the set of points of the rectangular hyperbola Γ with the equation $xy = 1$. The points $A, B, C, D \in Q$ form an orthocentric quadruple of points (i.e., each of these points is the orthocenter of the triangle formed by the remaining points) if and only if $abcd = -1$, where a, b, c, and d are the abscissas of the points A, B, C, and D.*

PROOF. Let

$$A = \left(a, \frac{1}{a}\right), \quad B = \left(b, \frac{1}{b}\right), \quad C = \left(c, \frac{1}{c}\right)$$

be any three points on the hyperbola Γ. The height from the vertex A of the triangle ABC has the equation

$$y - \frac{1}{a} = bc(x - a)$$

and it meets again the hyperbola Γ at the point $D = \left(d, \frac{1}{d}\right)$ satisfying

$$\frac{1}{d} - \frac{1}{a} = bc(d - a),$$

which implies $abcd = -1$. The symmetry in a, b, and c of this equality implies that D is the orthocenter of the triangle ABC, and because of the symmetry in a, b, c, and d, we get the statement of the lemma. ∎

LEMMA 2.5. *Let Q be the set of points of the parabola Γ with the parametric equation*

$$x = t, \quad y = t^2. \tag{2.1}$$

Normal lines to the parabola Γ at the points $A, B, C \in Q$ have a common point if and only if $a + b + c = 0$, where a, b, and c are the values of parameter t associated with points A, B, and C.

PROOF. The equation of the normal line to the parabola (2.1) at the point with parameter t is $x + 2ty - (2t^3 + t) = 0$. Normal lines at the points A, B, and C have a common point (p, q) if and only if a, b, and c are the solutions in t of the equation $2t^3 - (2q+1)t - p = 0$, i.e., if and only if $a+b+c = 0$. ◻

LEMMA 2.6. *Let Q be the set of points of the deltoid (i.e., a hypocycloid with three cusps) Γ with parametric equations*

$$x = 2\,\cos t + \cos 2t, \quad y = 2\,\sin t - \sin 2t. \tag{2.2}$$

Tangent lines to the curve Γ at points $A, B, C \in Q$ have a common point if and only if

$$a + b + c \equiv 0 \pmod{2\pi}, \tag{2.3}$$

where a, b, and c are values of the parameter t associated with the points A, B, and C.

PROOF. If $x' = \frac{dx}{dt}$, $y' = \frac{dy}{dt}$, then the slope of tangent line to the deltoid (2.2) at the point with parameter t is given by

$$\frac{y'}{x'} = \frac{2\,\cos t - 2\,\cos 2t}{-2\,\sin t - 2\,\sin 2t} = -\tan\frac{t}{2}.$$

Equation of the tangent line can be written as

$$x\,\sin\frac{t}{2} + y\,\cos\frac{t}{2} - \sin\frac{3t}{2} = 0.$$

If we set $u = \tan\frac{t}{2}$, then the equation of this tangent line becomes

$$(u^2+1)(ux+y) + u(u^2-3) = 0. \tag{2.4}$$

Let u_1, u_2, and u_3 be the values of parameter u associated with the points A, B, and C, i.e., let $u_1 = \tan\frac{a}{2}$, $u_2 = \tan\frac{b}{2}$, and $u_3 = \tan\frac{c}{2}$. Then

$$\tan\frac{a+b+c}{2} = \frac{s_1 - s_3}{1 - s_2}, \tag{2.5}$$

where s_1, s_2, and s_3 are the elementary symmetric functions of u_1, u_2, and u_3. Tangent lines at A, B, and C have a common point (p, q) if and only if the parameters u_1, u_2, and u_3 are the solutions to the equation

$$(p+1)u^3 + qu^2 + (p-3)u + q = 0,$$

which follows from (2.4) with $x = p$, $y = q$. This is a cubic equation and therefore from each point of the plane there are three tangent lines to the deltoid Γ. The coefficients at u^2 and u^0 are equal, so we get $s_1 = s_3$ and (2.5) implies $\tan\frac{a+b+c}{2} = 0$, which proves (2.3).

Conversely, let (2.3) hold and let the normal lines to the deltoid Γ at A and B meet at the point P. If C' is the foot of the third normal line from the point P to the deltoid Γ and if c' is the value of parameter of the point C' on this deltoid, then, according to the previous proof, we also get that

$\tan\frac{a+b+c'}{2} = 0$. These two facts imply $c' = c$, i.e. $C' = C$, which means that the normal lines at A, B, and C have a common point. ∎

LEMMA 2.7. *Let Q be the set of points of the deltoid Γ with parametric equations (2.2). Normal lines to the curve Γ at points $A, B, C \in Q$ have a common point if and only if*

$$a + b + c \equiv \pi \pmod{2\pi}, \tag{2.6}$$

where a, b, and c are the values of parameter t associated with the points A, B, and C.

PROOF. Based on the initial part of the proof of Lemma 2.6, the slope of the normal line to the deltoid (2.2) at the point with parameter t equals $\cot\frac{t}{2}$. Therefore, the equation of this normal line can be written as

$$x\cos\frac{t}{2} - y\sin\frac{t}{2} - 3\cos\frac{3t}{2} = 0.$$

If we put $u = \tan\frac{t}{2}$ again, then the equation of this normal line becomes $(u^2+1)(x-uy) + 3(3u^2-1) = 0$. If the normal lines at the points A, B, and C to the curve Γ have a common point (p, q), then the values u_1, u_2, and u_3 of the parameter u associated with these points are the solutions to the equation $qu^3 - (p+9)u^2 + qu - (p-3) = 0$, and in addition, the equality (2.5) also holds. The coefficients at u^3 and u are equal, and therefore $s_2 = 1$, and from (2.5) we get

$$\frac{a+b+c}{2} \equiv \frac{\pi}{2} \pmod{\pi},$$

i.e., the statement (2.6).

The converse statement is proved analogously as in Lemma 2.6. ∎

EXAMPLE 2.8. Let Q be the set of points of the rectangular hyperbola Γ with the equation $xy = 1$ (each rectangular hyperbola can be presented in this way) and define a ternary relation $[\,]$ on Q such that $[A, B, C]$ holds if and only if ABC is a triangle with orthocenter D, where $D \in Q$ is a fixed point. Based upon Lemma 2.4 and Theorem 1.2, we get that $(Q, [\,])$ is a cubic structure. It is necessary to define what is meant to be the orthocenter of a triangle when some of the vertices coincide, but we will leave this to the reader.

EXAMPLE 2.9. Let Q be the set of points of the parabola Γ with parametric equations (2.1) and let the points $A, B, C \in Q$ satisfy $[A, B, C]$ if and only if the normal lines to the parabola Γ at A, B, and C have a common point. Then by Lemma 2.5 and Theorem 1.2, the ordered pair $(Q, [\,])$ is a cubic structure.

Example 2.10. Each deltoid Γ could be (up to similarity) presented by the parametric equations (2.2). Let Q be the set of points of this deltoid and let $[A, B, C]$ hold if and only if the tangent lines to the deltoid Γ at A, B, and C have a common point. Then, according to Lemma 2.6 and Theorem 1.2, $(Q, [\ ])$ is a cubic structure.

Example 2.11. Let Q be the set of points of the deltoid Γ with parametric equations (2.2) and let $[A, B, C]$ hold if and only if the normal lines to the deltoid Γ at A, B, and C have a common point. Then $(Q, [\ ])$ is a cubic structure due to Lemma 2.7 and Theorem 1.2.

Because of the previous examples, each theorem which could be proved in an abstract cubic structure has a number of geometric consequences. Some of them are well-known, but there are some new ones as well. [1] presents a case of a model given in Example 2.1.

3. Simple properties of abstract cubic structures

Let $(Q, [\ ])$ be any cubic structure. Elements of the set Q will be called *points*, and the triples of the form $[a, b, c]$, for $a, b, c \in Q$, will be called *lines*, although in concrete examples 'points' can be lines and 'lines' can be points or circles, or something else. The same geometrical names, i.e., triangle, quadrangle, complete quadrangle, etc. will be used in the same way.

According to Theorem 1.1, the proofs of various theorems can be done directly in the cubic structure itself or in the associated TSM-quasigroup $(Q, \cdot)$. We will illustrate this by proving three simple theorems. The reader can easily conclude which geometrical consequences follow from these theorems in cases of specific models given in the aforementioned examples.

Theorem 3.1. *Let $\{a_1, a_2, a_3, a_4\}$ be a complete quadrangle and let for each $i, j \in \{1, 2, 3\}$, $i < j$, the point b_{ij} be such that $[a_i, a_j, b_{ij}]$. Then there is a point c such that $[b_{12}, b_{34}, c]$, $[b_{13}, b_{24}, c]$, and $[b_{14}, b_{23}, c]$.*

Proof. According to C1, there is a point c such that $[b_{12}, b_{34}, c]$, and the remaining two statements follow from C3 based on the following tables:

$$\begin{array}{cc|c|} \cline{3-3} a_1 & a_3 & b_{13} \\ a_2 & a_4 & b_{24} \\ b_{12} & b_{34} & c \\ \cline{3-3} \end{array} \qquad\qquad \begin{array}{cc|c|} \cline{3-3} a_1 & a_4 & b_{14} \\ a_2 & a_3 & b_{23} \\ b_{12} & b_{34} & c \\ \cline{3-3} \end{array}$$

□

Theorem 3.2. *If $[a, c, e]$ and $[b, d, f]$, then $[a, x_1, x_2]$, $[b, x_2, x_3]$, $[c, x_3, x_4]$, $[d, x_4, x_5]$, and $[e, x_5, x_6]$ imply $[f, x_6, x_1]$.*

Proof. According to C1, there is a point y such that $[x_3, x_6, y]$. Now, from the tables

$$\begin{array}{cc|c|} \cline{3-3} x_4 & x_5 & d \\ x_3 & x_6 & y \\ c & e & a \\ \cline{3-3} \end{array} \qquad\qquad \begin{array}{cc|c|} \cline{3-3} b & d & f \\ x_3 & y & x_6 \\ x_2 & a & x_1 \\ \cline{3-3} \end{array}$$

it follows that $[d, y, a]$, and then $[f, x_6, x_1]$. In addition to property C3, in these tables we also used several times the property C2. ∎

THEOREM 3.3. *Let* $[a_1, a_2, a_3]$ *and* $[b_1, b_2, b_3]$, *and for each* $i, j \in \{1, 2, 3\}$ *let* $[a_i, b_j, c_{ij}]$. *Then* $[c_{11}, c_{22}, c_{33}]$, *and also for each* $i \in \{1, 2, 3\}$, $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j < k$, *one has* $[c_{ii}, c_{jk}, c_{kj}]$ *also holds.*

PROOF. Because of Theorem 1.1, we get the assumptions $a_1a_2 = a_3$, $b_1b_2 = b_3$ and $a_ib_j = c_{ij}$ for each $i, j \in \{1, 2, 3\}$. Therefore, we get

$$c_{11}c_{22} = a_1b_1 \cdot a_2b_2 \overset{(1.1)}{=} a_1a_2 \cdot b_1b_2 = a_3b_3 = c_{33},$$
$$c_{ii}c_{jk} = a_ib_i \cdot a_jb_k \overset{(1.1)}{=} a_ia_j \cdot b_ib_k = a_kb_j = c_{kj}.$$ ∎

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