ON THE PERIODIC SOLUTIONS OF THE 5–DIMENSIONAL LORENZ EQUATION MODELING COUPLED ROSBY WAVES AND GRAVITY WAVES

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ABSTRACT. Lorenz studied the coupled Rosby waves and gravity waves using the differential system

$$\dot{U} = -VW + b\,VZ,$$
$$\dot{V} = UW - bUZ,$$
$$\dot{W} = -UV,$$
$$\dot{X} = -Z,$$
$$\dot{Z} = b\,UV + X.$$

This system has the two first integrals

$$H_1 = U^2 + V^2,$$
$$H_2 = V^2 + W^2 + X^2 + Z^2.$$

Our main result shows in each invariant set $\{H_1 = h_1 > 0\} \cap \{H_2 = h_2 > 0\}$ there are at least 4 (respectively 2) periodic solutions of the differential system with $b \neq 0$ and $h_2 > h_1$ (respectively $h_2 < h_1$).

1. INTRODUCTION

In [3] E.N. Lorenz studied slow manifolds in coupled Rosby waves and gravity waves. In order to model the behavior he used the following 5-dimensional differential system

(1)
$$\dot{U} = -VW + b\,VZ,$$
$$\dot{V} = UW - bUZ,$$
$$\dot{W} = -UV,$$
$$\dot{X} = -Z,$$
$$\dot{Z} = b\,UV + X,$$

where $b \in \mathbb{R}$ is a parameter. In [4] it was studied the integrability of this differential system and, mainly its analytic integrability. In the present paper we shall use the first integrals of system (1) and the theory of averaging of first order for studying the periodic solutions of

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system (1), for more details on the averaging theory see [2, 5, 6] and the appendix.

In [3] it is proved that

(2)
$$H_1(U, V, W, X, Z) = U^2 + V^2,$$
$$H_2(U, V, W, X, Z) = V^2 + W^2 + X^2 + Z^2,$$

are first integrals of the differential system (1) when $b \neq 0$, and that

(3)
$$H_1(U, V, W, X, Z) = U^2 + V^2,$$
$$H_2(U, V, W, X, Z) = V^2 + W^2,$$
$$H_3(U, V, W, X, Z) = X^2 + Z^2,$$

are first integrals of (1) when $b = 0$.

Our main results on the periodic solutions of the differential system (1) are the following.

Theorem 1. *For every pair of positive real numbers h_1 and h_2 in the invariant set $\{H_1 = h_1\} \cap \{H_2 = h_2\}$, where H_1 and H_2 are given in (2), there are at least 4 (respectively 2) periodic solutions of the differential system (1) with $b \neq 0$ and $h_2 > h_1$ (respectively $h_2 < h_1$). We give explicit approximations of these periodic solutions and of their linear stability.*

Theorem 1 is proved in section 2.

Proposition 2. *For every triple of positive real numbers h_1, h_2 and h_3 the general solution of differential system (1) with $b = 0$ and $h_2 - h_1 > 0$ in the invariant set $\{H_1 = h_1\} \cap \{H_2 = h_2\} \cap \{H_3 = h_3\}$ is*

(4)
$$u(t) = \pm\sqrt{h_1}\, sn\left(\sqrt{h_2 - h_1}\, t \,\middle|\, \frac{h_1}{h_1 - h_2}\right),$$
$$v(t) = \pm\sqrt{h_1 - u(t)^2},$$
$$w(t) = \pm\sqrt{h_2 + u(t)^2},$$
$$x(t) = x_0 \cos t - z_0 \sin t,$$
$$z(t) = z_0 \cos t + x_0 \sin t,$$

with $x_0^2 + z_0^2 = h_3$, and where H_1, H_2 and H_3 are given in (3).

In the statement of Proposition 2 $sn(u|m)$ denotes the sinus elliptic Jacobi function whose period is $4F(\pi/2|m)$, where F is the incomplete elliptic integral of the first kind, for more details see [1].

Proposition 2 is proved in section 3, where we also provide the periodic solutions of the differential system (1) with $b = 0$.

2. Proof of Theorem 1

Since in this section we shall prove Theorem 1 we consider $b \neq 0$.

Analyzing the differential system (1) we obtain that the axes U, V and W are filled of equilibria. The Jacobian matrix M of system (1) is

$$M = \begin{pmatrix} 0 & -W + bZ & -V & 0 & bV \\ W - bZ & 0 & U & 0 & -bU \\ -V & -U & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ bV & bU & 0 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalues at the equilibrium point $(U, V, W, X, Z) = (0, 0, 0, 0, 0)$ are $0, 0, 0, i, -i$.

In order to simplify the next computations we shall write the linear part of system (1) at the origin of coordinates in its real Jordan normal form

$$J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For this we do the linear change of variables

$$\begin{pmatrix} \overline{U} \\ \overline{V} \\ \overline{W} \\ \overline{X} \\ \overline{Z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \\ X \\ Z \end{pmatrix}.$$

In the new variables $(\overline{U}, \overline{V}, \overline{W}, \overline{X}, \overline{Z})$ the differential system (1) becomes

$$(5) \qquad \begin{aligned} \dot{\overline{U}} &= -\overline{V}, \\ \dot{\overline{V}} &= \overline{U} - b\,\overline{W}\overline{X}, \\ \dot{\overline{W}} &= b\overline{V}\overline{X} - \overline{X}\overline{Z}, \\ \dot{\overline{X}} &= -b\overline{V}\overline{W} + \overline{W}\overline{Z}, \\ \dot{\overline{Z}} &= \overline{W}\overline{X}, \end{aligned}$$

whose Jacobian matrix at the origin is J.

Since the main tool for proving our results is the averaging theory, we need to transform the differential system (5) into the normal form (10) for applying the averaging theory, see the appendix. Thus, first we introduce in such a system a small parameter. For this we re-scale

our variables as follows $(\overline{U}, \overline{V}, \overline{W}, \overline{X}, \overline{Z}) \to (\varepsilon u, \varepsilon v, \varepsilon w, \varepsilon x, \varepsilon z)$. Then system (5) becomes

(6)
$$\begin{aligned}
\dot{u} &= -v, \\
\dot{v} &= u - \varepsilon b\, wx, \\
\dot{w} &= \varepsilon bvx - \varepsilon xz, \\
\dot{x} &= -\varepsilon bvw + \varepsilon wz, \\
\dot{z} &= \varepsilon wx.
\end{aligned}$$

Now we shall consider the following generalized polar coordinates $(r, \theta, R, \alpha, Z)$ defined by

$$u = r\cos\theta\,, \ v = r\sin\theta\,, \ w = R\cos\alpha\,, \ x = R\sin\alpha \text{ and } z = z.$$

Therefore the differential system (6) in these generalized polar coordinates writes

(7)
$$\begin{aligned}
\dot{r} &= -\varepsilon \frac{1}{2} bR^2 \sin\theta \sin(2\alpha), \\
\dot{\theta} &= 1 - \varepsilon \frac{1}{2r} bR^2 \cos\theta \sin(2\alpha), \\
\dot{R} &= 0, \\
\dot{\alpha} &= \varepsilon\big(z - br\sin\theta\big), \\
\dot{z} &= \varepsilon \frac{1}{2} R^2 \sin(2\alpha).
\end{aligned}$$

As consequence $R = R_0$, with $R_0 \in \mathbb{R}$. As it will be clear later on, we consider $R_0 > 0$.

Now we take as new independent variable in the differential system (7) the variable θ and this system on the invariant set $\{H_1 = \varepsilon^2 R_0^2\}$ can be written as

(8)
$$\begin{aligned}
r' &= -\varepsilon \frac{1}{2} bR_0^2 \sin\theta \sin(2\alpha) + O(\varepsilon^2), \\
\alpha' &= \varepsilon\big(z - br\sin\theta\big) + O(\varepsilon^2), \\
z' &= \varepsilon \frac{1}{2} R_0^2 \sin(2\alpha) + O(\varepsilon^2).
\end{aligned}$$

Here the prime denotes derivative with respect to the variable θ.

First note that the differential system (8) has the additional first integral

$$H_2(r, \alpha, z) = \varepsilon^2\big(r^2 + z^2 + R_0^2 \cos^2\alpha\big),$$

obtained from (2) after the successive change of variables listened above.

Now isolating r from the expression of $H_2 = \varepsilon^2 \overline{h}_2 > 0$ we obtain that

$$r = \sqrt{\overline{h}_2 - z^2 - R_0^2 \cos^2 \alpha}.$$

So substituting r into system (8) this reduces to the system

(9)
$$\alpha' = \varepsilon(z - b\sqrt{\overline{h}_2 - z^2 - R_0^2 \cos^2 \alpha} \ \sin \theta) + O(\varepsilon^2),$$

$$z' = \varepsilon \frac{1}{2} R_0^2 \sin(\alpha) \cos(\alpha) + O(\varepsilon^2),$$

on the invariant set $\{H_1 = \varepsilon^2 R_0^2 = h_1\} \cap \{H_2 = \varepsilon^2 \overline{h}_2 = h_2\}$.

Note that the differential system (9) is written into the normal form (10) for applying the averaging theory, see for more details the appendix. Using the notation of the appendix we have

$$x = (\alpha, z), \quad t = \theta, \quad F_1(t, x) = F_1(\theta, \alpha, z),$$

where

$$F_1(\theta, \alpha, z) = \begin{pmatrix} F_{11}(\theta, \alpha, z) \\ F_{12}(\theta, \alpha, z) \end{pmatrix} = \begin{pmatrix} z - b\sqrt{\overline{h}_2 - z^2 - R_0^2 \cos^2 \alpha} \ \sin \theta \\ \frac{1}{2} R_0^2 \sin(\alpha) \cos(\alpha) \end{pmatrix}.$$

We note that our differential system (9) satisfies all the assumptions of Theorem 3 of the appendix, so we can apply it.

Now we consider the averaging function (12) of the appendix

$$f(\alpha, z) = \begin{pmatrix} f_1(\alpha, z) \\ f_2(\alpha, z) \end{pmatrix} = \begin{pmatrix} \int_0^{2\pi} F_{11}(\theta, \alpha, z) d\theta \\ \int_0^{2\pi} F_{12}(\theta, \alpha, z) d\theta \end{pmatrix} = \begin{pmatrix} z \\ \frac{1}{2} R_0^2 \sin(\alpha) \cos(\alpha) \end{pmatrix}.$$

The zeros of the averaged function $f(\alpha, z)$ in the interval $[0, 2\pi)$ are $z_1 = (0, 0)$, $z_2 = (0, \pi/2)$, $z_3 = (0, \pi)$ and $z_4 = (0, 3\pi/2)$.

The Jacobian Jf of the function f at these zeros are

$$Jf(z_1) = Jf(z_3) = -R_0^2 \neq 0 \text{ and } Jf(z_2) = Jf(z_4) = R_0^2 \neq 0.$$

Clearly we need that $R_0 > 0$, otherwise the Jacobian is zero and the averaging theory does not provide information about the periodic solutions.

On the other hand, the eigenvalues associated to the zeros z_1 and z_3 are $-R_0$ and R_0, while the ones to the zeros z_2 and z_4 are $-iR_0$ and iR_0. Therefore according to statement (b) of Theorem 3 the two periodic solutions of system (9) associated to the zeros z_1 and z_3 are unstable, having a stable and an unstable invariant manifolds formed

each one by two cylinders. While the two periodic solutions of system (9) associated to the zeros z_1 and z_3 are linear stable.

Now we go back through the changes of variables in order to estimate in the initial variables (U, V, W, X, Z) how are the four periodic orbits that we found in the invariant set $\{H_1 = \varepsilon^2 R_0^2 = h_1\} \cap \{H_2 = \varepsilon^2 \overline{h}_2 = h_2\}$.

According to statement (a) of Theorem 3 the periodic solution of system (9) associated to the zero $z_1 = (\alpha, z) = (0, 0)$ is of the form

$$\alpha(\theta, \varepsilon) = O(\varepsilon), \quad z(\theta, \varepsilon) = O(\varepsilon).$$

This periodic solution in the differential system (8) is

$$r(\theta, \varepsilon) = \sqrt{\overline{h}_2 - R_0^2} + O(\varepsilon), \quad \alpha(\theta, \varepsilon) = O(\varepsilon), \quad z(\theta, \varepsilon) = O(\varepsilon),$$

and in the differential system (7) becomes

$$r(t, \varepsilon) = \sqrt{\overline{h}_2 - R_0^2} + O(\varepsilon), \quad \theta(t, \varepsilon) = t + O(\varepsilon), \quad R(t, \varepsilon) = R_0,$$
$$\alpha(t, \varepsilon) = O(\varepsilon), \quad z(t, \varepsilon) = O(\varepsilon).$$

This periodic solution in the differential system (6) writes

$$u(t, \varepsilon) = \sqrt{\overline{h}_2 - R_0^2} \cos t + O(\varepsilon), \quad v(t, \varepsilon) = \sqrt{\overline{h}_2 - R_0^2} \sin t + O(\varepsilon),$$
$$w(t, \varepsilon) = R_0 + O(\varepsilon), \quad x(t, \varepsilon) = O(\varepsilon), \quad z(t, \varepsilon) = O(\varepsilon).$$

Now for the differential system (5) this last periodic solution writes

$$\overline{U}(t, \varepsilon) = \varepsilon\sqrt{\overline{h}_2 - R_0^2} \cos t + O(\varepsilon^2), \quad \overline{V}(t, \varepsilon) = \varepsilon\sqrt{\overline{h}_2 - R_0^2} \sin t + O(\varepsilon^2),$$
$$\overline{W}(t, \varepsilon) = \varepsilon R_0 + O(\varepsilon^2), \quad \overline{X}(t, \varepsilon) = O(\varepsilon^2), \quad \overline{Z}(t, \varepsilon) = O(\varepsilon^2).$$

Finally for our differential system (1) we get the periodic solution

$$U(t, \varepsilon) = O(\varepsilon^2), \quad V(t, \varepsilon) = -\varepsilon R_0 + O(\varepsilon^2), \quad W(t, \varepsilon) = O(\varepsilon^2),$$
$$X(t, \varepsilon) = \varepsilon\sqrt{\overline{h}_2 - R_0^2} \cos t + O(\varepsilon^2), \quad Z(t, \varepsilon) = -\varepsilon\sqrt{\overline{h}_2 - R_0^2} \sin t + O(\varepsilon^2),$$

or equivalently, since $h_1 = \varepsilon^2 R_0^2$ and $h_2 = \varepsilon^2 \overline{h}_2$ we have that the periodic solution corresponding to the zero z_1 for the differential system (1) in the invariant set $\{H_1 = h_1\} \cap \{H_2 = h_2\}$ is

$$U(t, \varepsilon) = O(\varepsilon), \quad V(t, \varepsilon) = -\sqrt{h_1} + O(\varepsilon), \quad W(t, \varepsilon) = O(\varepsilon),$$
$$X(t, \varepsilon) = \sqrt{h_2 - h_1} \cos t + O(\varepsilon), \quad Z(t, \varepsilon) = -\sqrt{h_2 - h_1} \sin t + O(\varepsilon),$$

once we have substitute R_0 and $\overline{h}_2$ by $\sqrt{h_1}/\varepsilon$ and h_2/ε^2, respectively. Note that this last solution satisfies that it is on the invariant set $\{H_1 = h_1\} \cap \{H_2 = h_2\}$.

In a similar way we can study the periodic solutions of the differential system (1) corresponding to the zeros z_k for $k = 2, 3, 4$, and we obtain

$$U(t, \varepsilon) = -\sqrt{h_1} + O(\varepsilon), \quad V(t, \varepsilon) = O(\varepsilon), \quad W(t, \varepsilon) = O(\varepsilon),$$

$$X(t, \varepsilon) = \sqrt{h_2} \cos t + O(\varepsilon), \quad Z(t, \varepsilon) = -\sqrt{h_2} \sin t + O(\varepsilon),$$

for the zero z_2,

$$U(t, \varepsilon) = O(\varepsilon), \quad V(t, \varepsilon) = \sqrt{h_1} + O(\varepsilon), \quad W(t, \varepsilon) = O(\varepsilon),$$

$$X(t, \varepsilon) = \sqrt{h_2 - h_1} \cos t + O(\varepsilon), \quad Z(t, \varepsilon) = -\sqrt{h_2 - h_1} \sin t + O(\varepsilon),$$

for the zero z_3, and

$$U(t, \varepsilon) = \sqrt{h_1} + O(\varepsilon), \quad V(t, \varepsilon) = O(\varepsilon), \quad W(t, \varepsilon) = O(\varepsilon),$$

$$X(t, \varepsilon) = \sqrt{h_2} \cos t + O(\varepsilon), \quad Z(t, \varepsilon) = -\sqrt{h_2} \sin t + O(\varepsilon),$$

for the zero z_4. This completes the proof of Theorem 1.

3. Proof of Proposition 2

Using an algebraic manipulator as mathematica the general solution (4) of the differential system (1) with $b = 0$ is easily obtained.

We note that system (1) with $b = 0$ has the axes u, v and w filled of equilibria.

Moreover we remark that the 3–dimensional subspace $\{x = 0\} \cap \{z = 0\}$ and the 2–dimensional subspace $\{u = 0\} \cap \{v = 0\} \cap \{w = 0\}$ are filled of periodic solutions if we remove the equilibria which are in these subspaces.

When the quotient $h_1/(h_1 - h_2)$ satisfies $4F(\pi/2|h_1/(h_1 - h_2)) = 2\pi$, the 3–dimensional invariant sets $\{H_1 = h_1\} \cap \{H_2 = h_2\}$ are filled of period solutions of period 2π.

Appendix: The averaging theory of first order

Consider the differential system

$$(10) \qquad \dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x, \varepsilon), \quad x(0) = x_0$$

with $x \in D$, where D is an open subset of $\mathbb{R}^n$, $t \geq 0$. We also assume that the functions $F_1(t, x)$ and $F_2(t, x, \varepsilon)$ are $T-$periodic in t. We define in D the averaged differential system

$$(11) \qquad \dot{y} = \varepsilon f(y), \quad y(0) = x_0,$$

where

$$(12) \qquad f(y) = \frac{1}{T} \int_0^T F_1(t, y) dt.$$

The next result shows that under the convenient hypotheses the equilibria of the averaged system will provide $T-$periodic solutions of system (10).

Theorem 3. *Consider the two initial value problems* (10) *and* (11). *Suppose that*

  (i) *the functions F_1, $\partial F_1/\partial x$, $\partial^2 F_1/\partial x^2$, F_2 and $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0, \infty) \times D$ and $\varepsilon \in (0, \varepsilon_0]$;*
  (ii) *the functions F_1 and F_2 are $T-$periodic in t (T independent of ε).*

Then the following statements hold.

  (a) *If p is an equilibrium point of the averaged system* (11) *satisfying*
  $$\det \left(\frac{\partial f}{\partial y} \right) \bigg|_{y=p} \neq 0,$$
  then there is a $T-$periodic solution $x(t, \varepsilon)$ of system (10) *such that $(0, \varepsilon) \to p$ as $\varepsilon \to 0$.*
  (b) *The kind of stability or instability of the periodic solution $x(t, \varepsilon)$ coincides with the kind of stability or instability of the equilibrium point p of the averaged system* (11). *The equilibrium point p has the kind of stability behavior of the Poincaré map associated to the periodic solution $x(t, \varepsilon)$.*

For a proof of Theorem 3, see sections 6.3, 11.8 of Verhulst [6].

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