STABILITY OF GENERALIZED LINEAR WEINGARTEN HYPERSURFACES IMMERSED IN THE EUCLIDEAN SPACE

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Abstract: Given a positive function F defined on the unit Euclidean sphere and satisfying a suitable convexity condition, we consider, for hypersurfaces M^n immersed in the Euclidean space $\mathbb{R}^{n+1}$, the so-called k-th anisotropic mean curvatures H_k^F, $0 \leq k \leq n$. For fixed $0 \leq r \leq s \leq n$, a hypersurface M^n of $\mathbb{R}^{n+1}$ is said to be (r, s, F)-linear Weingarten when its k-th anisotropic mean curvatures H_k^F, $r \leq k \leq s$, are linearly related. In this setting, we establish the concept of stability concerning closed (r, s, F)-linear Weingarten hypersurfaces immersed in $\mathbb{R}^{n+1}$ and, afterwards, we prove that such a hypersurface is stable if, and only if, up to translations and homotheties, it is the Wulff shape of F. For $r = s$ and $F \equiv 1$, our results amount to the standard stability studied, for instance, by Alencar–do Carmo–Rosenberg [**1**].

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1. Introduction and statement of the main result

In recent years, following the seminal ideas established by Reilly in [**24, 25**], several authors obtained geometric properties of a hypersurface through the study of the Euler–Lagrange equation associated to certain variational problems (see, for instance, [**1, 4, 5, 6, 14, 11, 29**]). Proceeding into this branch, in this paper we deal with a suitable class of closed hypersurfaces immersed in the Euclidean space $\mathbb{R}^{n+1}$, which are critical points for the variational problem of minimizing a linear combination of certain area functions preserving the volume enclosed. A precise description of our object of study will be given after some preliminaries.

Let $F \colon \mathbb{S}^n \to \mathbb{R}^+$ be a positive smooth function which satisfies the following convexity condition:

$$(1.1) \qquad (D^2 F + FI)_x > 0, \quad \forall\, x \in \mathbb{S}^n,$$

where D^2F denotes the intrinsic Hessian of F on the n-sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$, I denotes the identity on $T_x\mathbb{S}^n$, and > 0 means that the matrix is positive definite. We consider the map

$$(1.2) \qquad \begin{aligned} \phi\colon \mathbb{S}^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto F(x)x + (\nabla_{\mathbb{S}^n} F)_x, \end{aligned}$$

whose image $\mathcal{W}_F = \phi(\mathbb{S}^n)$ is a smooth, convex hypersurfaces in $\mathbb{R}^{n+1}$ called the *Wulff shape* of F (for more details concerning the properties of the Wulff shape see, for instance, [**9, 14, 16, 17, 18, 19, 27**]). We note that, when $F \equiv 1$ we have that the Wulff shape of F is just the n-dimensional Euclidean sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.

Throughout this paper, $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ will stand for a smooth immersion of a closed oriented hypersurface and $N\colon M^n \to \mathbb{S}^n$ will denote its corresponding Gauss map. In this setting, let

$$(1.3) \qquad A_F := D^2F + FI$$

and

$$N_F := \phi \circ N\colon M^n \longrightarrow \mathcal{W}_F,$$

which is called the *generalized Gauss map* into the Wulff shape. Then

$$S_F := -dN_F = -A_F \circ dN$$

is defined as being the *F-Weingarten operator*.

According to the previous definitions, we note that although A_F and dN be symmetric operators, S_F is symmetric if and only if A_F and dN commute, which does not occur in general. We point out that all the roots of the characteristic polynomial of S_F are real. The eigenvalues of S_F are called the *anisotropic principal curvatures* of x and are denoted by $\lambda_1, \ldots, \lambda_n$. Moreover, if the principal curvatures of x are positive, so are the anisotropic principal curvatures λ_i (for a proof, see Lemma 2 in [**10**]).

At each $p \in M^n$, S_F restricts to a linear map $S_F(p)\colon T_pM \to T_pM$. For $1 \leq k \leq n$, let $S_k(p)$ denote the k-th elementary symmetric function on the eigenvalues of $S_F(p)$; thus one gets n smooth functions $S_k\colon M \to \mathbb{R}$, such that

$$(1.4) \qquad \det(tI - S_F) = \sum_{k=0}^{n}(-1)^k S_k t^{n-k},$$

where $S_0 \equiv 1$ by construction. If $p \in M^n$ and $\{\lambda_k\}$ are the eigenvalues with respect to the operator $S_F(p)$, one immediately sees that

$$S_k = \sigma_k(\lambda_1, \ldots, \lambda_n) := \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

where $\sigma_k \in \mathbb{R}[X_1, \ldots, X_n]$ is the *k-th elementary symmetric polynomial* on the indeterminates $X_1, \ldots, X_n$.

For $1 \leq k \leq n$, one defines the k-th anisotropic mean curvature H_k^F of x by

(1.5) $$\binom{n}{k} H_k^F = \sigma_k(\lambda_1, \ldots, \lambda_n).$$

There exist many works concerning the properties of the k-th anisotropic mean curvatures of hypersurfaces in $\mathbb{R}^{n+1}$ space. We refer the readers to [**10, 13, 14, 15, 16, 23, 30**].

At this point, we are in a position to define our geometrical object of study: an immersion $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ is said a (r, s, F)-*linear Weingarten hypersurface* if, for some integers r and s satisfying the inequality $0 \leq r \leq s \leq n - 1$, holds the following linear relation

$$a_r b_r H_{r+1}^F + \cdots + a_s b_s H_{s+1}^F = \text{constant},$$

for some nonnegative real numbers a_k, $k \in \{r, \ldots, s\}$, with at least one non zero, where $b_k = (k+1)\binom{n}{k+1}$ and $F \colon \mathbb{S}^n \to \mathbb{R}^+$ is a positive smooth function which satisfies the convexity condition (1.1).

We observe that, when $r = 0$, $s = 1$, and $F = 1$, these hypersurfaces are classically called linear Weingarten hypersurfaces and, in the last years, a vast literature has been produced in the direction to obtain characterization results of them (see, for instance, [**2, 3, 7, 8, 20, 26**]). In this paper we extend this study to the anisotropic case. It is said that a phenomenon has anisotropic behavior when its effects vary with the direction; as opposed to isotropic behavior (homogeneous in all directions). This is observed in many phenomena of nature. Indeed, it appears that anisotropic properties are present in the study of many phenomena. For example, crystals whose propagation of light depends on the direction exhibit optical anisotropy [**28**]. The (s, s, F)-linear Weingarten hypersurfaces are exactly the hypersurfaces with H_{s+1}^F constant. On the other hand, taking into account that all the anisotropic principal curvatures of the Wulff shape $\mathcal{W}_F$ are constant (see, for example, [**14**]), we have that $\mathcal{W}_F$ constitutes a natural example of closed (r, s, F)-linear Weingarten hypersurface immersed in $\mathbb{R}^{n+1}$, for any $0 \leq r \leq s \leq n - 1$.

Motivated by the previous discussion, here we will establish the notion of stability concerning closed (r, s, F)-linear Weingarten hypersurfaces in $\mathbb{R}^{n+1}$. Such concept arises considering the variational problem of minimizing a suitable linear combination of certain functionals (k, F)-th areas for volume-preserving variations (cf. Section 4). Now, we are in position to state our main result.

Theorem 1. *Let r and s be integers satisfying $0 \leq r \leq s \leq n-2$, $n \geq 3$, and let $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be a closed (r,s,F)-linear Weingarten hypersurface with H_{s+1}^F positive. Then, $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ is stable if, and only if, up to translations and homotheties, $x(M)$ is the Wulff shape of F.*

We observed that taking $r = s$ in Theorem 1 we reobtain Theorem 1.3 of [**14**]. The proof of Theorem 1 is given in Section 5.

Remark 1. Related to the isotropic case, Micallef and Moore [**22**] proved that any minimal 2-sphere in a manifold with positive isotropic curvature is unstable. More recently, Li [**21**] showed the nonexistence of stable immersed minimal surfaces uniformly conformally equivalent to $\mathbb{C}$ in any complete orientable 4-dimensional Riemannian manifold with uniformly positive isotropic curvature.

2. The operators P_k, T_k, and L_k

In order to give a description of our variational problem, we will need to define some suitable operators associated to a hypersurface $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ as in the previous section. The first ones are the operators $P_k\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, $0 \leq k \leq n$, which can be defined inductively from the F-Weingarten operator S_F and k-th anisotropic mean curvatures H_k^F by

$$(2.1) \qquad P_0 = I \quad \text{and} \quad P_k = \binom{n}{k} H_k^F I - P_{k-1} \circ S_F,$$

where I denotes the identity in $\mathfrak{X}(M)$. Equivalently,

$$(2.2) \qquad P_k = \sum_{j=0}^{k} (-1)^j \binom{n}{k-j} H_{k-j}^F S_F^j.$$

Note that each $P_k(p)$ is also a linear operator on each tangent space $T_p M$ which commutes with $S_F(p)$.

The operator $T_k\colon \mathfrak{X}(M) \to \mathfrak{X}(M)$, $0 \leq k \leq n$, is defined by

$$(2.3) \qquad T_k = P_k \circ A_F.$$

Note that, since A_F and dN are symmetric, from (2.2) we have that the transformations T_k are all self-adjoint operators. So, likewise $S_F \circ AF$, $dN \circ S_F$, and $dN \circ P_k$ are symmetric. Moreover, taking into account that

$$T_{k-1} \circ dN = P_{k-1} \circ A_F \circ dN = -P_{k-1} \circ S_F,$$

we get

$$(2.4) \qquad P_k = \binom{n}{k} H_k^F I - P_{k-1} \circ S_F = \binom{n}{k} H_k^F I + T_{k-1} \circ dN.$$

Finally, we define the operator $L_k \colon \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ by

$$(2.5) \qquad L_k(f) = \mathrm{div}(T_k(\nabla f)).$$

Equivalently,

$$(2.6) \qquad L_k(f) = \sum_{i,j} \Big[(T_k)_{ij} f_j \Big]_i,$$

where we denote the coefficients of covariant differential of f and T_k with respect to a (local) orthonormal frame $\{e_1, \ldots, e_n\}$ on M^n by f_i and $(T_k)_{ij}$, respectively. In particular, we the F-Laplacian is defined by

$$(2.7) \qquad \Delta_F(f) := L_0(f) = \mathrm{div}(A_F(\nabla f)).$$

3. Key lemmas

This section is devoted to other auxiliary results which will be also necessary to prove Theorem 1. Initially, we recall suitable inequalities concerning the k-th anisotropic mean curvatures, which will be very useful to show our main result.

Lemma 1. *Let* $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ *be a closed hypersurface and* $F \colon \mathbb{S}^n \to \mathbb{R}^+$ *satisfying the condition* (1.1). *If* H_{s+1}^F *is positive on* M^n *then for* $1 \le k \le s$, *we have:*
  (i) *each k-th anisotropic mean curvature H_k^F is positive;*
  (ii) $H_1^F H_{k+1}^F - H_{k+2}^F \ge 0.$
Moreover equality holds for some k in (ii) if, and only if, the anisotropic principal curvatures are equal.

Proof: Item (i) corresponds to Lemma 10 of [**10**]. For (ii), it is known that the following generalization of the Cauchy–Schwarz type inequality holds true (see, for instance, [**12**, Theorem 51, p. 52, and Theorem 144, p. 104]) for any $1 \le k \le n - 1$:

$$(3.1) \qquad (H_k^F)^2 - H_{k-1}^F H_{k+1}^F \ge 0,$$

the equality occurring for some k if, and only if, at this point the anisotropic principal curvatures are equal. As $H_k^F > 0$ for any $0 \le k \le s + 1$, we can write the inequality (3.1) as follows:

$$\frac{H_k^F}{H_{k-1}^F} \ge \frac{H_{k+1}^F}{H_k^F},$$

for any $1 \leq k \leq s+1$, and equality holds for some k if, and only if, at this point the anisotropic principal curvatures are equal. Hence, we have the inequalities

$$(3.2) \qquad H_1^F \geq \frac{H_2^F}{H_1^F} \geq \frac{H_3^F}{H_2^F} \geq \cdots \geq \frac{H_{k+2}^F}{H_{k+1}^F},$$

for all $0 \leq k \leq s$, and equality holds in (3.2) if, and only if, the anisotropic principal curvatures are equal. Thus, from (3.2) we obtain the results, finishing the proof. $\qquad \square$

Now, we will quote well known result which will be used later (for a proof, see He and Li [**13, 15**] or Palmer [**23**]).

Lemma 2. *Let $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be an isometric immersion of a closed orientable Riemannian manifold M^n and $F \colon \mathbb{S}^n \to \mathbb{R}^+$ satisfying the condition (1.1). If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = \text{constant} \neq 0$, then up to translations and homotheties, $x(M)$ is the Wulff shape of F.*

In the next lemma, we recall the so-called Minkowski formulas (see He and Li [**13, 15**]).

Lemma 3. *Let $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be a closed hypersurface and $F \colon \mathbb{S}^n \to \mathbb{R}^+$ satisfying the condition (1.1). For each $0 \leq j \leq n-1$, the following Minkowski-type formulas hold*

$$\int_M (F H_k^F + H_{k+1}^F \langle x, N \rangle) \, dM = 0.$$

At this point, we will fix some notation. Given $f \in \mathcal{C}^\infty(M)$ smooth function, we define:

$$(3.3) \qquad I_k[f] := L_k f + \langle T_k dN, dN \rangle f;$$

$$(3.4) \qquad R_{r,s}[f] := \sum_{k=r}^{s} (k+1) a_k I_k[f].$$

We will also need of the following result due to He and Li [**13**].

Lemma 4. *For each $0 \leq k \leq n-1$, we have:*

$$(3.5) \qquad \begin{aligned} I_k[F(N)] &= -\binom{n}{k+1} \langle \nabla H_{k+1}^F, \nabla_{S^n} F \rangle + n \binom{n}{k+1} H_1^F H_{k+1}^F \\ &\quad - (k+2) \binom{n}{k+2} H_{k+2}^F; \end{aligned}$$

$$(3.6) \qquad I_k[\langle x, N \rangle] = -\binom{n}{k+1} (\langle \nabla H_{k+1}^F, x^\top \rangle - (k+1) H_{k+1}^F).$$

Finally, we also need of the following symmetry result, which will be crucial to establish the proof of Theorem 1:

Lemma 5. *For $f, g \in \mathcal{C}^\infty(M)$ and $0 \le r \le s \le n - 1$, we have:*

$$\int_M g R_{r,s}[f] \, dM = \int_M f R_{r,s}[g] \, dM.$$

Proof: By (3.4) we have

$$(3.7) \qquad \int_M g R_{r,s}[f] \, dM = \sum_{k=r}^{s} (k+1) a_k \int_M g I_k[f] \, dM.$$

By Stokes theorem and from the symmetry of T_k, we have

$$
\begin{aligned}
(3.8) \qquad \int_M g L_k(f) \, dM &= \int_M g \operatorname{div} T_k \nabla f \, dM \\
&= -\int_M \langle T_k \nabla f, \nabla g \rangle \, dM \\
&= -\int_M \langle \nabla f, T_k \nabla g \rangle \, dM \\
&= \int_M f \operatorname{div} T_k \nabla g \, dM = \int_M f L_k(g) \, dM.
\end{aligned}
$$

Thus, by (3.3) and (3.8)

$$
\begin{aligned}
(3.9) \qquad \int_M g I_k[f] \, dM &= \int_M g(L_k(f) + \langle T_k dN, dN \rangle f) \, dM \\
&= \int_M g L_k(f) \, dM + \int_M f g \langle T_k dN, dN \rangle \, dM \\
&= \int_M f L_k(g) \, dM + \int_M f g \langle T_k dN, dN \rangle \, dM \\
&= \int_M f(L_k(g) + \langle T_k dN, dN \rangle g) dM = \int_M f I_k[g] \, dM.
\end{aligned}
$$

Therefore, by (3.7), (3.9), and (3.4), we have

$$\int_M g R_{r,s}[f] \, dM = \sum_{k=r}^{s} (k+1) a_k \int_M f I_k[g] \, dM = \int_M f R_{r,s}[g] \, dM,$$

finishing the proof.   □

4. Description of the variational problem

A *variation* of a closed oriented hypersurface $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ is a smooth map
$$X \colon M^n \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^{n+1}$$
such that, for all $t \in (-\epsilon, \epsilon)$, the map $X_t \colon M^n \to \mathbb{R}^{n+1}$ given by $X_t(p) = X(t, p)$ is an immersion such that $X_0 = x$. In what follows, dM_t will denote the volume element of the metric induced on M^n by X_t and N_t will stand for the unit normal vector field along X_t.

The *variational field* associated to the variation X is the vector field $\left. \dfrac{\partial X}{\partial t} \right|_{t=0}$. In this setting, denoting by $(\cdot)^{\top}$ the tangential component on M^n, we have that

$$\frac{\partial X}{\partial t} = f N_t + \left(\frac{\partial X}{\partial t} \right)^{\top},$$

where

(4.1)
$$f = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle.$$

The *volume* of the variation X is the functional
$$\mathcal{V} \colon (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$$
$$t \longmapsto \mathcal{V}(t) = \frac{1}{n+1} \int_M \langle X_t, N_t \rangle \, dM_t,$$
and we say X is *volume-preserving* if $\mathcal{V}(t) = \mathcal{V}(0)$, for all $t \in (-\epsilon, \epsilon)$.

The next lemma is a well known result and a proof of it can be found in [**6**].

Lemma 6. *Let $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be a closed oriented hypersurface. If $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ is a variation of x, then*

$$\frac{d\mathcal{V}}{dt} = \int_M f \, dM_t.$$

In particular, X is volume-preserving if, and only if, $\int_M f \, dM_t = 0$ for all $t \in (-\epsilon, \epsilon)$.

We can reason as in the proof of Lemma 7 of [**10**] to get the following:

Lemma 7. *Let $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be a closed oriented hypersurface with Gauss map N. For any $f \in \mathcal{C}^{\infty}(M)$ satisfying*

(4.2)
$$\int_M f \, dM = 0,$$

there exists a volume-preserving variation $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ of x such that its variational vector field is fN.

For each $k \in \{0, \ldots, n\}$, we define the (k, F)-*area functional* $\mathcal{A}_{k,F}$: $(-\epsilon, \epsilon) \to \mathbb{R}$ associated to the variation $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ by

$$(4.3) \qquad \mathcal{A}_{k,F}(t) = \int_M F(N_t) S_k \, dM_t.$$

Note that for $F \equiv 1$ and $k = 0$, it is the classical area functional.

The result below follows from Lemma 3.1 of [**14**].

Lemma 8. *Let* $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ *be a closed oriented hypersurface. If* $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ *is a variation of* x *then*

$$(4.4) \quad \frac{\partial H_{k+1}^F}{\partial t} = \frac{k+1}{b_k} \{ L_k(f) + f \langle T_k \circ dN_t, dN_t \rangle \} + \left\langle \left(\frac{\partial X}{\partial t} \right)^{\top}, \nabla H_{k+1}^F \right\rangle,$$

where f *is defined in* (4.1).

The previous lemma allows us to obtain the first variation of the (k, F)-area functional (cf. Theorem 3.3 of [**14**]).

Lemma 9. *Let* $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ *be a closed oriented hypersurface. If* $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ *is a variation of* x, *then*

$$(4.5) \qquad \mathcal{A}_{k,F}'(t) = -b_k \int_M H_{k+1}^F f \, dM_t,$$

where f *is defined in* (4.1).

Now, motivated by the concept of (r, s, F)-linear Weingarten hypersurface, it is natural to consider the variational problem of minimizing the following functional

$$(4.6) \qquad \begin{aligned} \mathcal{B}_{r,s,F} \colon (-\epsilon, \epsilon) &\longrightarrow \mathbb{R} \\ t &\longmapsto \mathcal{B}_{r,s,F}(t) = a_r \mathcal{A}_{r,F}(t) + \cdots + a_s \mathcal{A}_{s,F}(t) \end{aligned}$$

for all variations $X \colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ of $x \colon M^n \hookrightarrow \mathbb{R}^{n+1}$ which preserve the volume $\mathcal{V}(t)$. The *Jacobi functional* associated to this variational problem is given by

$$(4.7) \qquad \begin{aligned} \mathcal{J}_{r,s,F} \colon (-\epsilon, \epsilon) &\longrightarrow \mathbb{R} \\ t &\longmapsto \mathcal{J}_{r,s,F}(t) = \mathcal{B}_{r,s,F}(t) + \lambda \mathcal{V}(t), \end{aligned}$$

where λ is a constant to be determined.

As an immediate consequence of Lemmas 6 and 9 we get

$$\mathcal{J}_{r,s,F}'(t) = \int_M \left\{ -\sum_{k=r}^{s} a_k b_k H_{k+1}^F + \lambda \right\} f \, dM_t.$$

In order to make an appropriated choice of λ, let

$$\overline{\mathcal{H}} = \frac{1}{\mathcal{A}_{0,1}(0)} \int_M \left\{ \sum_{k=r}^{s} a_k b_k H_{k+1}^F(0) \right\} dM$$

be a mean value of the k-th anisotropic mean curvatures of M^n, $r \leq k \leq s$. We point out to the fact that, in case $\sum_{k=r}^{s} a_k b_k H_{k+1}^F(0)$ is constant, one has

$$(4.8) \qquad \overline{\mathcal{H}} = \sum_{k=r}^{s} a_k b_k H_{k+1}^F(0) = \sum_{k=r}^{s} a_k b_k H_{k+1}^F,$$

and this notation will be used in what follows without further comments. Hence, choosing $\lambda = \overline{\mathcal{H}}$, we arrive at

$$(4.9) \qquad \mathcal{J}'_{r,s,F}(t) = \int_M \left\{ - \sum_{k=r}^{s} a_k b_k H_{k+1}^F + \overline{\mathcal{H}} \right\} f\, dM_t.$$

From (4.9) we observe that the critical points of the variational problem described above are exactly the closed (r, s, F)-linear Weingarten hypersurfaces. This fact allow us to define a closed (r, s, F)-linear Weingarten hypersurface $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ being *stable* when $\mathcal{B}''_{r,s,F}(0) \geq 0$, for all volume-preserving variations $X\colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ of x.

Furthermore, we can reason as in [6] to obtain the following stability criterion: *a closed (r, s, F)-linear Weingarten hypersurface $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ is stable if, and only if, $\mathcal{J}''_{r,s,F}(0) \geq 0$, for all $f \in \mathcal{C}^\infty(M)$ such that $\int_M f\, dM = 0$.*

The sought formula for the second variation of $\mathcal{J}_{r,s,F}$ is a straightforward consequence of Lemmas 8 and 9.

Proposition 1. *Let $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ be closed (r, s, F)-linear Weingarten hypersurface. If $X\colon M^n \times (-\epsilon, \epsilon) \to \mathbb{R}^{n+1}$ is a variation of x, then $\mathcal{J}''_{r,s,F}(0)$ is given by*

$$(4.10) \quad \mathcal{J}''_{r,s,F}(0)(f) = - \sum_{k=r}^{s} (k+1) a_k \int_M \{ L_k(f) + \langle T_k \circ dN, dN \rangle f \} f\, dM,$$

for $f \in \mathcal{C}^\infty(M)$.

Proof: From (4.9) we obtain

$$\mathcal{J}''_{r,s,F}(0) = \frac{\partial}{\partial t}\left(\int_M \left\{-\sum_{k=r}^{s} a_k b_k H_{k+1}^F + \overline{\mathcal{H}}\right\} f \, dM_t\right)\bigg|_{t=0}$$

$$= -\int_M \sum_{k=r}^{s} a_k b_k \left(\frac{\partial H_{k+1}^F}{\partial t}\bigg|_{=0}\right) f \, dM$$

$$+ \int_M \left(-\sum_{k=r}^{s} a_k b_k H_{k+1}^F + \overline{\mathcal{H}}\right)\frac{\partial}{\partial t}(f \, dM_t)\bigg|_{t=0}.$$

Consequently, taking into account (4.4) and (4.8), we get

$$\mathcal{J}''_{r,s,F}(0) = -\sum_{j=r}^{s}(k+1)a_k \int_M \{L_k(f) + \langle T_j \circ dN, dN\rangle f\}f \, dM$$

$$- \int_M \left\langle \left(\frac{\partial X}{\partial t}\right)^{\top}, \nabla\left(\underbrace{\sum_{k=r}^{s} a_k b_k H_{k+1}^F}_{0}\right)\right\rangle f \, dM$$

$$= -\sum_{k=r}^{s}(k+1)a_k \int_M \{L_k(f) + \langle T_k \circ dN, dN\rangle f\}f \, dM.$$

To finish the proof, we observe that the above expression depends only on the hypersurface $x\colon M^n \hookrightarrow \mathbb{R}^{n+1}$ and on the function $f \in \mathcal{C}^\infty(M)$. $\quad\square$

5. Proof of Theorem 1

In what follows, we will consider the set

$$\mathcal{G} = \left\{f \in \mathcal{C}^\infty(M) : \int_M f \, dM = 0\right\}.$$

If $x(M)$ is (up to translations and homotheties) the Wulff shape of F, from the proof of Theorem 1.3 in [**14**] we have that

$$-\int_M \{L_k(f) + \langle T_k \circ dN, dN\rangle f\}f \, dM \geq 0, \quad \forall f \in \mathcal{G} \quad \text{and} \quad 0 \leq k \leq n.$$

So, from (4.10), $\mathcal{J}''_{r,s,F}(f) \geq 0$ for all $f \in \mathcal{G}$. Therefore, x is stable.

Reciprocally, supposing that x is stable, we have $\mathcal{J}''_{r,s,F}(f) \geq 0$, for all $f \in \mathcal{G}$. Moreover, from Lemma 3 we can choose

$$f = \gamma F(N) + \xi\langle x, N\rangle$$

as the test function, where

$$\gamma = \frac{\int_M F \sum_{k=r}^{s} a_k b_k H_k^F \, dM}{\int_M F(N) \, dM} \quad \text{and} \quad \xi = \sum_{k=r}^{s} a_k b_k H_{k+1}^F = \text{constant.}$$

We also note that, from (3.3), (3.4), and Proposition 1 we have that

$$\mathcal{J}''_{r,s,F}(0)(f) = -\int_M f R_{r,s}[f] \, dM \geq 0.$$

On the other hand, from (3.5) and (3.6) we get

$$R_{r,s}[f] = \sum_{k=r}^{s} (k+1) a_k I_k[f] = \sum_{k=r}^{s} (k+1) a_k I_k[\gamma F(N) + \xi \langle x, N \rangle]$$

$$= \sum_{k=r}^{s} (k+1) a_k (\gamma I_k[F(N)] + \xi I_k[\langle x, N \rangle])$$

$$= \sum_{k=r}^{s} (k+1) a_k \left[\gamma \left(-\binom{n}{k+1} \langle \nabla H_{k+1}^F, \nabla_{S^n} F \rangle \right.\right.$$

$$+ n \binom{n}{k+1} H_1^F H_{k+1}^F - (k+2) \binom{n}{k+2} H_{k+2}^F \right)$$

$$\left. + \xi \left(-\binom{n}{k+1} \langle \nabla H_{k+1}^F, x^\top \rangle - (k+1) \binom{n}{k+1} H_{k+1}^F \right) \right].$$

Hence, we get

$$(5.1) \quad R_{r,s}[f] = \sum_{k=r}^{s} a_k b_k [\gamma (n H_1^F H_{k+1}^F - (n-k-1) H_{k+2}^F) - \xi (k+1) H_{k+1}^F].$$

Furthermore, from Lemma 5 we also have that

$$\mathcal{J}''_{r,s,F}(0)(f) = -\int_M f R_{r,s}[f] \, dM$$

$$(5.2) \qquad\qquad = -\int_M (\gamma F + \xi \langle x, N \rangle) R_{r,s}[f] \, dM$$

$$= -\int_M (\gamma F R_{r,s}[f] + \xi f R_{r,s}[\langle x, N \rangle]) \, dM.$$

But, from (5.1) and Lemma 4 we have

$$\gamma F R_{r,s}[f] = \sum_{k=r}^{s} a_k b_k F [\gamma^2 (n H_1^F H_{k+1}^F - (n-k-1) H_{k+2}^F) - \gamma \xi (k+1) H_{k+1}^F].$$

Consequently, we get

$$
\gamma F R_{r,s}[f] = \sum_{k=r}^{s} a_k b_k F \Big[\gamma^2 ((n-k-1) H_1^F H_{k+1}^F + (k+1) H_1^F H_{k+1}^F
$$

$$
- (n-k-1) H_{k+2}^F) - \gamma \xi (k+1) H_{k+1}^F \Big]
$$

(5.3)

$$
= \sum_{k=r}^{s} a_k b_k F \Big\{ \gamma^2 (n-k-1) [H_1^F H_{k+1}^F - H_{k+2}^F]
$$

$$
+ \gamma^2 (k+1) H_1^F H_{k+1}^F - \gamma \xi (k+1) H_{k+1}^F \Big\}.
$$

Moreover,

$$
\xi f R_{r,s}[\langle x, N \rangle] = \xi f \sum_{k=r}^{s} (k+1) a_k I_k[\langle x, N \rangle]
$$

$$
= \xi f \sum_{k=r}^{s} a_k b_k (-\langle \nabla H_{k+1}^F, x^\top \rangle - (k+1) H_{k+1}^F)
$$

(5.4)

$$
= -\xi f \sum_{k=r}^{s} a_k b_k (k+1) H_{k+1}^F
$$

$$
= - \sum_{k=r}^{s} a_k b_k (k+1) \xi (\gamma F + \xi \langle x, N \rangle) H_{k+1}^F.
$$

Therefore, from (5.2), (5.3), (5.4) we have

$$
\mathcal{J}_{r,s,F}''(0)(f) = - \sum_{k=r}^{s} a_k b_k \int_M \Big[F \big\{ \gamma^2 (n-k-1) [H_1^F H_{k+1}^F - H_{k+2}^F]
$$

$$
+ \gamma^2 (k+1) H_1^F H_{k+1}^F - \gamma \xi (k+1) H_{k+1}^F \big\}
$$

$$
- (k+1) \xi [\gamma F + \xi \langle x, N \rangle] H_{k+1}^F \Big] \, dM
$$

$$
= - \sum_{k=r}^{s} a_k b_k \gamma^2 (n-k-1) \int_M F (H_1^F H_{k+1}^F - H_{k+2}^F) \, dM
$$

$$
- \sum_{k=r}^{s} a_k b_k \int_M \Big[F \big\{ \gamma^2 (k+1) H_1^F H_{k+1}^F - \gamma \xi (k+1) H_{k+1}^F \big\}
$$

$$
- (k+1) \xi [\gamma F + \xi \langle x, N \rangle] H_{k+1}^F \Big] \, dM.
$$

Thus, from Lemma 3 we get

$$
\mathcal{J}_{r,s}''(0)(f) = -\sum_{k=r}^{s} a_k b_k \gamma^2 (n-k-1) \int_M F(H_1^F H_{k+1}^F - H_{k+2}^F)\, dM
$$

$$
-\sum_{k=r}^{s} a_k b_k \int_M F\{\gamma^2(k+1)H_1^F H_{k+1}^F - \gamma\xi(k+1)H_{k+1}^F
$$

(5.5)
$$
- (k+1)\xi\gamma H_{k+1}^F + (k+1)\xi^2 H_k^F\}\, dM
$$

$$
= -\sum_{k=r}^{s} a_k b_k \gamma^2(n-k-1) \int_M F(H_1^F H_{k+1}^F - H_{k+2}^F)\, dM
$$

$$
-\sum_{k=r}^{s} a_k b_k (k+1) \int_M F\{H_1^F H_{k+1}^F \gamma^2 - 2\xi H_{k+1}^F \gamma + \xi^2 H_k^F\}\, dM.
$$

Now, for each point $p \in M^n$, we consider the polynomial in z given by

$$
\mathcal{P}_{k,p}(z) = H_1^F H_{k+1}^F z^2 - 2\xi H_{k+1}^F z + \xi^2 H_k^F.
$$

In this case, we have that the discriminant of $\mathcal{P}_{k,p}$ is

$$
\Delta = 4\xi^2[(H_{k+1}^F)^2 - H_1^F H_{k+1}^F H_k^F] = 4\xi^2(H_{k+1}^F)^2\left(1 - \frac{H_1^F H_k^F}{H_{k+1}^F}\right) \le 0,
$$

where the last inequality in the expression above is given by Lemma 1. Thus, we have that $\mathcal{P}_{k,p} \ge 0$, $\forall\, p \in M^n$, and $\forall\, k \in \{r,\ldots,s\}$. In particular,

(5.6)
$$
H_1^F H_{k+1}^F \gamma^2 - 2\xi H_{k+1}^F \gamma + \xi^2 H_k^F = \mathcal{P}_{k,p}(\gamma) \ge 0,
$$

$\forall\, p \in M^n$ and $\forall\, k \in \{r,\ldots,s\}$. So, from (5.5), (5.6), and Lemma 1, we have

(5.7)
$$
\mathcal{J}_{r,s}''(0)(f) = -\sum_{k=r}^{s} a_k b_k \gamma^2(n-k-1) \int_M F(H_1^F H_{k+1}^F - H_{k+2}^F)\, dM
$$

$$
-\sum_{k=r}^{s} a_k b_k (k+1) \int_M F\mathcal{P}_{k,p}(\gamma)\, dM \le 0.
$$

From (5.7), since x is stable and taking into account that each therm of $\mathcal{J}_{r,s}''(0)(f)$ is nonpositive, we easily see that they are, in fact, identically zero. Consequently, using once more (5.7), we get that $H_1^F H_{k+1}^F - H_{k+2}^F = 0$ on M^n, for all $r \le k \le s$. So, from item (ii) of Lemma 1, we have that the anisotropic principal curvatures are equal. Therefore, Lemma 2 assures us that, up to translations and homotheties, $x(M)$ is the Wulff shape of F, finishing the proof of Theorem 1.   □

Remark 2. Concerning our constraint on the sign of H_{s+1}^F in Theorem 1, we observe that it is crucial to the success of our technique, since the Gardin type inequalities of Lemma 1 constitutes our algebraic tool to detect the anisotropic umbilicity of the closed (r, s, F)-linear Weingarten hypersurface. We also note that this phenomenon already occurs in the context of the works of Alencar–do Carmo–Rosenberg [**1**] and Barbosa–Colares [**6**].

Acknowledgements

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