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ON RELATIONS BETWEEN WEAK AND STRONG TYPE INEQUALITIES FOR MAXIMAL OPERATORS ON NON-DOUBLING METRIC MEASURE SPACES

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Abstract: In this article we characterize all possible cases that may occur in the relations between the sets of p for which weak type (p,p) and strong type (p,p) inequalities for the Hardy–Littlewood maximal operators, both centered and non-centered, hold in the context of general metric measure spaces.

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1. Introduction

Let $\mathbb{X} = (X, \rho, \mu)$ be a metric measure space with a metric ρ and a Borel measure μ such that the measure of each ball is finite and strictly positive. Define the *Hardy–Littlewood maximal operators*, centered M^c and non-centered M, by

$$M^c f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f|\, d\mu, \quad x \in X,$$

and

$$Mf(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f|\, d\mu, \quad x \in X,$$

respectively. Here B refers to any open ball in (X,ρ) and by $B(x,r)$ we denote the open ball centered at $x \in X$ with radius $r > 0$.

Recall that an operator T is said to be of strong type (p,p) for some $p \in [1,\infty]$ if T is bounded on $L^p = L^p(\mathbb{X})$. Similarly, T is of weak type (p,p) if T is bounded from L^p to $L^{p,\infty} = L^{p,\infty}(\mathbb{X})$ (we use the convention $L^{\infty,\infty} = L^\infty$). Obviously, the operators M^c and M are of strong type (∞,∞) in case of any metric measure space. Moreover, by using the Marcinkiewicz interpolation theorem, if M^c (equivalently M) is of weak or strong type (p_0,p_0) for some $p_0 \in [1,\infty)$, then it is of strong (and hence weak) type (p,p) for every $p > p_0$. If the measure is doubling, that is $\mu(B(x,2r)) \lesssim \mu(B(x,r))$ uniformly in $x \in X$ and $r > 0$, then both M^c and M are of weak type $(1,1)$. However, in general, the weak type $(1,1)$ inequalities may not occur. Furthermore, as we will see, it is even possible to construct a space for which the associated operators M^c and M are not of weak (and hence strong) type (p,p) for every $p \in [1,\infty)$.

Finding examples of metric measure spaces with specific properties of associated maximal operators is usually a nontrivial task; see Aldaz [**1**], for example. H.-Q. Li greatly contributed the program of searching spaces which are peculiar from the point of view of mapping properties of maximal operators. In this context, in [**2**], [**3**], and [**4**], he considered a class of the cusp spaces. In [**2**] H.-Q. Li showed that for any fixed $1 < p_0 < \infty$ there exists a space for which the associated operator M^c is of strong type (p,p) if and only if $p > p_0$. Then, in [**3**] examples of spaces were furnished for which M is of strong type (p,p) if and only if $p > p_0$. Moreover, for every $1 < \tau \leq 2$ there are examples of spaces for which M^c is of weak type $(1,1)$, and M is of strong type (p,p) if and only if $p > \tau$. Finally, in [**4**] H.-Q. Li showed that there are spaces with exponential volume growth for which M^c is of weak type $(1,1)$, while M is of strong type (p,p) for every $p > 1$.

The aim of this article is to complement and strengthen the results obtained by H.-Q. Li. For a fixed metric measure space $\mathbb{X}$ denote by P_s^c and P_s the sets consisting of such $p \in [1,\infty]$ for which the associated operators, M^c and M are of strong type (p,p), respectively. Similarly, let P_w^c and P_w consist of such $p \in [1,\infty]$ for which M^c and M are of weak type (p,p), respectively. Then

(i) each of the four sets is of the form $\{\infty\}$, $[p_0,\infty]$, or $(p_0,\infty]$, for some $p_0 \in [1,\infty)$;

(ii) we have the following inclusions

$$P_s \subset P_s^c, \quad P_w \subset P_w^c, \quad P_s^c \subset P_w^c \subset \overline{P_s^c}, \quad P_s \subset P_w \subset \overline{P_s},$$

where $\overline{E}$ denotes the closure of E in the usual topology of $\mathbb{R} \cup \{\infty\}$.

We will show that the conditions above are the only ones that the sets P_s^c, P_s, P_w^c, and P_w must satisfy. Namely, we will prove the following:

Theorem 1. *Let P_s^c, P_s, P_w^c, and P_w be such that the conditions (i) and (ii) hold. Then there exists a (non-doubling) metric measure space for which the associated Hardy–Littlewood maximal operators, centered M^c and non-centered M, satisfy*

- *M^c is of strong type (p,p) if and only if $p \in P_s^c$,*
- *M is of strong type (p,p) if and only if $p \in P_s$,*
- *M^c is of weak type (p,p) if and only if $p \in P_w^c$,*
- *M is of weak type (p,p) if and only if $p \in P_w$.*

The proof of Theorem 1 is postponed to Section 4.

2. First generation spaces

We begin with a construction of some metric measure spaces called by us the *first generation spaces*. The common property of these spaces is a similarity in the behavior of the associated operators M^c and M, by what we mean the equalities $P_s^c = P_s$ and $P_w^c = P_w$. We begin with an overview of the first generation spaces and then we consider two subtypes separately in detail.

Let $\tau = (\tau_n)_{n\in\mathbb{N}}$ be a fixed sequence of positive integers. Define

$$X_\tau = \{x_n : n \in \mathbb{N}\} \cup \{x_{ni} : i = 1, \dots, \tau_n,\ n \in \mathbb{N}\},$$

where all elements x_n, x_{ni} are pairwise different (and located on the plane, say). We define the metric $\rho = \rho_\tau$ determining the distance between two different elements x and y by the formula

$$\rho(x,y) = \begin{cases} 1 & \text{if } x_n \in \{x,y\} \subset S_n \text{ for some } n \in \mathbb{N}, \\ 2 & \text{in the other case.} \end{cases}$$

By S_n we denote the branch $S_n = \{x_n, x_{n1}, \dots, x_{n\tau_n}\}$ and by S_n' the branch without the root, $S_n' = S_n \setminus \{x_n\}$. Figure 1 shows a model of the space (X_τ, ρ). The solid line between two points indicates that the distance between them equals 1. Otherwise the distance equals 2.

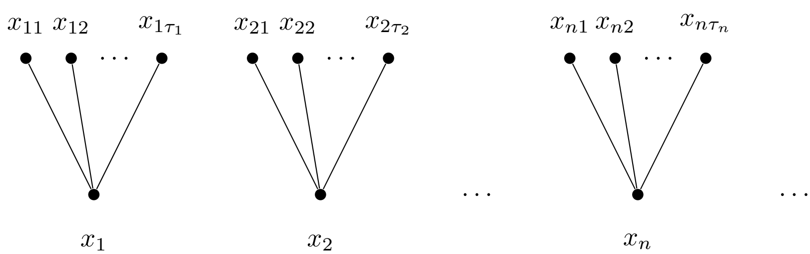

Figure 1.

Note that we can explicitly describe any ball: for $n \in \mathbb{N}$,

$$B(x_n, r) = \begin{cases} \{x_n\} & \text{for } 0 < r \le 1, \\ S_n & \text{for } 1 < r \le 2, \\ X_\tau & \text{for } 2 < r, \end{cases}$$

and for $i \in \{1, \dots, \tau_n\}$, $n \in \mathbb{N}$,

$$B(x_{ni}, r) = \begin{cases} \{x_{ni}\} & \text{for } 0 < r \le 1, \\ \{x_n, x_{ni}\} & \text{for } 1 < r \le 2, \\ X_\tau & \text{for } 2 < r. \end{cases}$$

We define the measure $\mu = \mu_{\tau,F}$ on X_τ by letting $\mu(\{x_n\}) = d_n$ and $\mu(\{x_{ni}\}) = d_n F(n,i)$, where $F > 0$ is a given function and $d = (d_n)_{n \in \mathbb{N}}$ is an appropriate sequence of strictly positive numbers with $d_1 = 1$ and d_n chosen (uniquely!) in such a way that $\mu(S_n) = \mu(S_{n-1})/2$, $n \ge 2$. Note that this implies $\mu(X_\tau) < \infty$. Moreover, observe that μ is non-doubling. From now on we shall use the sign $|E|$ instead of $\mu(E)$ for $E \subset X_\tau$. It will be clear from the context when the symbol $|\cdot|$ refers to the measure and when it denotes the absolute value sign.

For a function f on X_τ (which is in fact a 'sequence' of numbers) the Hardy–Littlewood maximal operators, centered M^c and non-centered M, are given by

$$M^c f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \sum_{y \in B(x,r)} |f(y)| \cdot |\{y\}|, \quad x \in X_\tau,$$

and

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \sum_{y \in B} |f(y)| \cdot |\{y\}|, \quad x \in X_\tau,$$

respectively. In this setting M is of weak type (p,p) for some $1 \le p < \infty$ if $\|Mf\|_{p,\infty} \lesssim \|f\|_p$ uniformly in $f \in \ell^p(X_\tau, \mu)$, where $\|g\|_p = \big(\sum_{x \in X_\tau} |g(x)|^p |\{x\}|\big)^{1/p}$ and $\|g\|_{p,\infty} = \sup_{\lambda > 0} \lambda |E_\lambda(g)|^{1/p}$ with $E_\lambda(g) = \{x \in X_\tau : |g(x)| > \lambda\}$. Similarly, M is of strong type (p,p) for some $1 \le p \le \infty$ if $\|Mf\|_p \lesssim \|f\|_p$ uniformly in $f \in \ell^p(X_\tau, \mu)$, where $\|g\|_\infty = \sup_{x \in X_\tau} |g(x)|$. Here the notation $A \lesssim B$ is used to indicate that $A \le CB$ with a positive constant C independent of significant quantities. Moreover, for given a function $f \ge 0$ and a set $E \subset X_\tau$ we denote the average value of f on E by

$$A_E(f) = \frac{1}{|E|} \sum_{x \in E} f(x) |\{x\}|.$$

Analogous definitions and comments apply to M^c instead of M and then to both M and M^c in the context of the space (Y_τ, μ) in Section 3.

We are ready to describe two subtypes of the first generation spaces.

2.1. We first construct and investigate first generation spaces for which the equalities $P_s^c = P_s$ and $P_w^c = P_w$ hold and, in addition, there is no significant difference between the incidence of the weak and strong type inequalities, by what we mean that $P_s^c = P_w^c$ and $P_s = P_w$. Of course, combining all these equalities, we obtain that for such spaces all four sets take the same form. In the first step, for any fixed $p_0 \in [1, \infty]$ we construct a space denoted by $\hat{\mathbb{X}}_{p_0}$ for which $P_s^c = P_s = P_w^c = P_w = [p_0, \infty]$ (by $[\infty, \infty]$ we mean $\{\infty\}$). Then, after slight modifications, for any fixed $p_0 \in [1, \infty)$ we get a space $\hat{\mathbb{X}}'_{p_0}$ for which $P_s^c = P_s = P_w^c = P_w = (p_0, \infty]$.

Fix $p_0 \in [1, \infty]$ and let $\hat{\mathbb{X}}_{p_0} = (X_\tau, \rho, \mu)$ be the first generation space with $\tau_n = \lfloor \frac{(n+1)^{p_0}}{n} \rfloor$ in the case $p_0 \in [1, \infty)$, or $\tau_n = 2^n$ in the case $p_0 = \infty$, and $F(n, i) = n$, $i = 1, \dots, \tau_n$, $n \in \mathbb{N}$. The key point for considerations that follow is that we have: for $p_0 \neq 1$,

$$\lim_{n\to\infty} \frac{n\tau_n}{(n+1)^p} = \infty, \quad 1 \le p < p_0,$$

and for $p_0 \neq \infty$,

$$\frac{n\tau_n}{(n+1)^{p_0}} \le 1, \quad n \in \mathbb{N}.$$

Proposition 2. *Fix $p_0 \in [1, \infty]$ and let $\hat{\mathbb{X}}_{p_0}$ be the metric measure space defined above. Then the associated maximal operators, centered M^c and non-centered M, are not of weak type (p, p) for $1 \le p < p_0$, but are of strong type (p, p) for $p \ge p_0$.*

Proof: It suffices to prove that M^c fails to be of weak type (p, p) for $1 \le p < p_0$ and M is of strong type (p_0, p_0). First we show that M^c is not of weak type (p, p) for $1 \le p < p_0$. Consider $p_0 \in (1, \infty]$ and fix $p \in [1, p_0)$. Let $f_n = \delta_{x_n}$, $n \ge 1$. Then $\|f_n\|_p^p = d_n$ and $M^c f_n(x_{ni}) \ge \frac{1}{n+1}$, $i = 1, \dots, \tau_n$. This implies that $|E_{1/(2(n+1))}(M^c f_n)| \ge n\tau_n d_n$ and hence

$$\limsup_{n\to\infty} \frac{\|M^c f_n\|_{p,\infty}^p}{\|f_n\|_p^p} \ge \lim_{n\to\infty} \frac{n\tau_n d_n}{(2(n+1))^p d_n} = \infty.$$

In the next step we show that M is of strong type (p_0, p_0). Consider $p_0 \in [1, \infty)$, since the case $p_0 = \infty$ is trivial. Let $f \in \ell^{p_0}(\hat{\mathbb{X}}_{p_0})$. Without any loss of generality we assume that $f \ge 0$. Denote $\mathcal{D} = \{\{x_n, x_{ni}\} :$

$n \in \mathbb{N}$, $i = 1, \dots, \tau_n\}$. We use the estimate

$$\|Mf\|_{p_0}^{p_0} \le \sum_{B \subset X_\tau} \sum_{x \in B} A_B(f)^{p_0} |\{x\}| = \sum_{B \subset X_\tau} A_B(f)^{p_0} |B|.$$

Note that each $x \in X_\tau$ belongs to at most three different balls which are not elements of $\mathcal{D}$. Combining this with Hölder's inequality, we obtain

$$\sum_{B \notin \mathcal{D}} A_B(f)^{p_0} |B| \le \sum_{B \notin \mathcal{D}} \sum_{x \in B} f(x)^{p_0} |\{x\}| \le 3\|f\|_{p_0}^{p_0}.$$

Therefore

$$(1) \qquad \|Mf\|_{p_0}^{p_0} \le 3\|f\|_{p_0}^{p_0} + \sum_{n \in \mathbb{N}} \sum_{i=1}^{\tau_n} \left(\frac{f(x_n) + nf(x_{ni})}{n+1} \right)^{p_0} |\{x_n, x_{ni}\}|.$$

Finally, we use the inequalities $(f(x_n) + nf(x_{ni}))^{p_0} \le (2f(x_n))^{p_0} + (2nf(x_{ni}))^{p_0}$ and $|\{x_n, x_{ni}\}| \le 2|\{x_{ni}\}| = 2n|\{x_n\}|$ to estimate the double sum in (1) by

$$2^{p_0+1} \left(\sum_{n \in \mathbb{N}} \frac{n\tau_n}{(n+1)^{p_0}} f(x_n)^{p_0} |\{x_n\}| + \sum_{n \in \mathbb{N}} \sum_{i=1}^{\tau_n} \left(\frac{nf(x_{ni})}{n+1} \right)^{p_0} |\{x_{ni}\}| \right) \le 2^{p_0+1} \|f\|_{p_0}^{p_0}. \quad \square$$

A modification of arguments from the proof of Proposition 2 shows that, for a fixed $p_0 \in [1, \infty)$, replacing the former τ_n by $\tau_n' = \lfloor (\log(n) + 1) \frac{(n+1)^{p_0}}{n} \rfloor$ leads to the space $\hat{\mathbb{X}}_{p_0}'$ for which $P_s^c = P_s = P_w^c = P_w = (p_0, \infty]$. Moreover, it may be noted that only the properties $\lim_{n \to \infty} \frac{n\tau_n'}{(n+1)^p} = \infty$, $1 \le p \le p_0$, and $\sup_{n \in \mathbb{N}} \frac{n\tau_n'}{(n+1)^p} < \infty$, $p > p_0$, are essential.

2.2. In contrast to the former case, for the spaces we now construct and study, the equalities $P_s^c = P_s$ and $P_w^c = P_w$ still hold, but there is a difference between the incidence of the weak and strong type inequalities. For any fixed $p_0 \in [1, \infty)$ we construct a space denoted by $\widetilde{\mathbb{X}}_{p_0}$ for which $P_s^c = P_s = (p_0, \infty]$ and $P_w^c = P_w = [p_0, \infty]$. We begin with the case $p_0 = 1$, which is discussed separately because it is relatively simple and may be helpful to outline the core idea behind the more difficult case $p_0 \in (1, \infty)$.

By $\widetilde{\mathbb{X}}_1$ we denote the first generation space (X_τ, ρ, μ) with construction based on $\tau_n = n$ and $F(n, i) = 2^i$. Recall that μ is non-doubling.

Proposition 3. *Let $\widetilde{\mathbb{X}}_1$ be the metric measure space defined above. Then the associated maximal operators, centered M^c and non-centered M, are not of strong type* $(1,1)$*, but are of weak type* $(1,1)$*.*

Proof: First we note that M^c fails to be of strong type $(1,1)$. Indeed, let $f_n = \delta_{x_n}$, $n \geq 1$. Then $\|f_n\|_1 = d_n$ and for $i = 1, \dots, n$ we have $M^c f_n(x_{ni}) \geq (1+2^i)^{-1} > 1/2^{i+1}$ and hence $\|M^c f_n\|_1 \geq \sum_{i=1}^{n} 2^i d_n / 2^{i+1} = n\|f_n\|_1/2$.

In the next step we show that M is of weak type $(1,1)$. Let $f \in \ell^1(\widetilde{\mathbb{X}}_1)$, $f \geq 0$, and consider $\lambda > 0$ such that $E_\lambda(Mf) \neq \emptyset$. If $\lambda < A_{X_\tau}(f)$, then $\lambda |E_\lambda(Mf)|/\|f\|_1 \leq 1$ follows. Therefore, from now on assume that $\lambda \geq A_{X_\tau}(f)$. With this assumption, if for some $x \in S_n$ we have $Mf(x) > \lambda$, then any ball B containing x and realizing $A_B(f) > \lambda$ must be a subset of S_n. Take any $n \in \mathbb{N}$ such that $E_\lambda(Mf) \cap S_n \neq \emptyset$. If $\lambda < A_{S_n}(f)$, then

$$\frac{\lambda |E_\lambda(Mf) \cap S_n|}{\sum_{x \in S_n} f(x)|\{x\}|} \leq 1. \tag{2}$$

Assume that $\lambda \geq A_{S_n}(f)$ and take $x \in E_\lambda(Mf) \cap S_n$. Now, any ball B containing x and realizing $A_B(f) > \lambda$ must be a proper subset of S_n. If $E_\lambda(Mf) \cap S'_n = \emptyset$, then $x = x_n$ so we obtain $f(x_n) > \lambda$ and hence (2) again follows. In the opposite case, if $E_\lambda(Mf) \cap S'_n \neq \emptyset$, denote $j = \max\{i \in \{1, \dots, n\} : Mf(x_{ni}) > \lambda\}$. Then $f(x_{nj}) > \lambda$ or $\frac{f(x_n)|\{x_n\}| + f(x_{nj})|\{x_{nj}\}|}{|\{x_n\}| + |\{x_{nj}\}|} > \lambda$. Therefore, $f(x_n)|\{x_n\}| + f(x_{nj})|\{x_{nj}\}| > \lambda |\{x_{nj}\}|$ and combining this with the estimate $|E_\lambda(Mf) \cap S_n| \leq 2|\{x_{nj}\}|$, we obtain

$$\frac{\lambda |E_\lambda(Mf) \cap S_n|}{\sum_{x \in S_n} f(x)|\{x\}|} \leq \frac{2\lambda |\{x_{nj}\}|}{f(x_n)|\{x_n\}| + f(x_{nj})|\{x_{nj}|\}} \leq 2.$$

Since $\frac{\lambda |E_\lambda(Mf) \cap S_n|}{\sum_{x \in S_n} f(x)|\{x\}|} \leq 2$ for any branch S_n such that $E_\lambda(Mf) \cap S_n \neq \emptyset$, we have

$$\frac{\lambda |E_\lambda(Mf)|}{\|f\|_1} \leq 2,$$

and, consequently, the weak type $(1,1)$ estimate $\|Mf\|_{1,\infty} \leq 2\|f\|_1$ follows. □

Now fix $p_0 \in (1, \infty)$ and consider $\widetilde{\mathbb{X}}_{p_0} = (X_\tau, \rho, \mu)$, with construction based on $\tau_n = \tau_n(p_0)$ and $F(n,i) = F_{p_0}(n,i)$, defined as follows. Let $c_n = \lfloor \frac{(n+1)^{p_0}}{n} \rfloor$ and

$$e_n = \max\left\{ k \in \mathbb{N} : 2^{k-1} \leq c_n \text{ and } 2^{1-k-p_0} \geq \left(\frac{1}{1+n}\right)^{p_0} \right\}, \quad n \in \mathbb{N}.$$

Note that $\lim_{n\to\infty} e_n = \infty$. Then, for $j \in \{1,\dots,e_n\}$, $n \in \mathbb{N}$, define m_{nj} by the equality

$$2^{1-j}\left(\frac{1}{1+m_{nj}}\right)^{p_0} = \left(\frac{1}{1+n}\right)^{p_0},$$

and s_{nj} by

$$s_{nj} = \min\{k \in \mathbb{N} : k m_{nj} \geq 2^{1-j} n c_n\}.$$

Observe that for $j \in \{1,\dots,e_n\}$, $n \in \mathbb{N}$,

$$1 \leq m_{nj} \leq n, \quad 2^{1-j} n c_n \leq s_{nj} m_{nj} \leq 2^{2-j} n c_n.$$

Finally, denote $\tau_n = \sum_{j=1}^{e_n} s_{nj}$, $n \in \mathbb{N}$, and $F(n,i) = m_{nj(n,i)}$, $i = 1,\dots,\tau_n$, $n \in \mathbb{N}$, where

$$j(n,i) = \min\left\{k \in \{1,\dots,e_n\} : \sum_{j=1}^{k} s_{nj} \geq i\right\}.$$

Proposition 4. *Let $\widetilde{\mathbb{X}}_{p_0}$ be the metric measure space defined above. Then the associated maximal operators, centered M^c and non-centered M, are not of strong type (p_0,p_0), but are of weak type (p_0,p_0).*

Proof: First we note that M^c is not of strong type (p_0,p_0). Indeed, let $f_n = \delta_{x_n}$, $n \geq 1$. Then $\|f_n\|_{p_0}^{p_0} = d_n$ and for $i = 1,\dots,\tau_n$ we have $M^c f_n(x_{ni}) \geq (1+m_{nj(n,i)})^{-1}$ and hence

$$\|M^c f_n\|_{p_0}^{p_0} \geq \sum_{j=1}^{e_n}\sum_{k=1}^{s_{nj}} \left(\frac{1}{1+m_{nj}}\right)^{p_0} d_n m_{nj} = d_n \sum_{j=1}^{e_n} \frac{s_{nj} m_{nj}}{(1+m_{nj})^{p_0}}$$

$$\geq d_n \sum_{j=1}^{e_n} \frac{2^{1-j} n c_n}{(1+m_{nj})^{p_0}} = d_n \sum_{j=1}^{e_n} \frac{n c_n}{(1+n)^{p_0}} = e_n \frac{n c_n}{(1+n)^{p_0}} \|f_n\|_{p_0}^{p_0}.$$

Since $\lim_{n\to\infty} e_n = \infty$ and $\lim_{n\to\infty} \frac{n c_n}{(1+n)^{p_0}} = 1$, we are done.

In the next step we show that M is of weak type (p_0,p_0). Let $f \in \ell^{p_0}(\widetilde{\mathbb{X}}_{p_0})$, $f \geq 0$, and consider $\lambda > 0$ such that $E_\lambda(Mf) \neq \emptyset$. If $\lambda < A_{X_\tau}(f)$, then using the inequality $\|f\|_1 \leq \|f\|_{p_0} |X_\tau|^{1/q_0}$, where q_0 is the exponent conjugate to p_0, we obtain $\lambda^{p_0}|E_\lambda(Mf)|/\|f\|_{p_0}^{p_0} < 1$. Therefore, from now on assume that $\lambda \geq A_{X_\tau}(f)$. Take any $n \in \mathbb{N}$ such that $E_\lambda(Mf) \cap S_n \neq \emptyset$. If $\lambda < A_{S_n}(f)$, then using similar argument as above we have

$$\frac{\lambda^{p_0}|E_\lambda(Mf) \cap S_n|}{\sum_{x \in S_n} f(x)^{p_0} |\{x\}|} \leq 1. \tag{3}$$

Assume that $\lambda \geq A_{S_n}(f)$. If $E_\lambda(Mf)\cap S'_n = \emptyset$, then $f(x_n) > \lambda$ and hence (3) again follows. In the opposite case, we have $|E_\lambda(Mf)\cap S_n| \leq 2|E_\lambda(Mf)\cap S'_n|$. Assume that $f(x_n) < (1+m_{ne_n})\lambda/2$. If $x \in E_\lambda(Mf)\cap S'_n$, then $f(x) \geq \lambda/2$ and hence

$$\frac{\lambda^{p_0}|E_\lambda(Mf)\cap S_n|}{\sum_{x\in S_n} f(x)^{p_0}|\{x\}|} \leq \frac{2\lambda^{p_0}|E_\lambda(Mf)\cap S'_n|}{\sum_{x\in S_n} f(x)^{p_0}|\{x\}|} \leq 2^{p_0+1}.$$

Otherwise, if $f(x_n) \geq (1+m_{ne_n})\lambda/2$, denote $r = \min\{j \in \{1,\ldots,e_n\} : f(x_n) \geq (1+m_{nj})\lambda/2\}$. Let $S_n^{(r)} = \{x_{ni} : i \in \{1,\ldots,\sum_{j=1}^{r-1} s_{nj}\}\}$. Note that the case $S_n^{(r)} = \emptyset$ is possible. Assume that $S_n^{(r)} \neq \emptyset$. If $x \in E_\lambda(Mf)\cap S_n^{(r)}$, then $f(x) > \lambda/2$ and hence

$$\frac{\lambda^{p_0}|E_\lambda(Mf)\cap S_n^{(r)}|}{\sum_{x\in S_n^{(r)}} f(x)^{p_0}|\{x\}|} \leq 2^{p_0+1}.$$

Moreover, we have

$$\begin{aligned}
\frac{\lambda^{p_0}|E_\lambda(Mf)\cap (S_n\setminus S_n^{(r)})|}{f(x_n)^{p_0}|\{x_n\}|} &\leq \left(\frac{2}{1+m_{nr}}\right)^{p_0}\frac{|S_n\setminus S_n^{(r)}|}{|\{x_n\}|}\\
&\leq \left(\frac{2}{1+m_{nr}}\right)^{p_0}\frac{2|(S_n\setminus S_n^{(r)})\cap S'_n|}{|\{x_n\}|}\\
&\leq \left(\frac{2}{1+m_{nr}}\right)^{p_0} 2\sum_{j=r}^{e_n} nc_n 2^{2-j}\\
&< 2^{p_0+4-r} nc_n \left(\frac{1}{1+m_{nr}}\right)^{p_0}\\
&= 2^{p_0+3}\frac{nc_n}{(1+n)^{p_0}} \leq 2^{p_0+3}.
\end{aligned}$$

Therefore, regardless of the posibilities, $S_n^{(r)} = \emptyset$ or $S_n^{(r)} \neq \emptyset$, we obtain $\frac{\lambda^{p_0}|E_\lambda(Mf)\cap S_n|}{\sum_{x\in S_n} f(x)^{p_0}|\{x\}|} \leq 2^{p_0+3}$. Since $\lambda^{p_0}|E_\lambda(Mf)\cap S_n|/\sum_{x\in S_n} f(x)^{p_0}|\{x\}| \leq 2^{p_0+3}$ for any branch S_n such that $E_\lambda(Mf)\cap S_n \neq \emptyset$, we have $\lambda^{p_0}|E_\lambda(Mf)|/\|f\|_{p_0}^{p_0} \leq 2^{p_0+3}$ and, consequently, $\|Mf\|_{p_0,\infty}^{p_0} \leq 2^{p_0+3}\|f\|_{p_0}^{p_0}$. □

3. Second generation spaces

Now we construct and study metric measure spaces called by us the *second generation spaces*. The common attribute of these spaces is a significant difference in the behavior of the associated operators M^c and M, by what we mean that M^c is of strong type $(1,1)$, which implies the basic property $P_s^c = P_w^c = [1,\infty]$, while P_s (and possibly P_w) is a proper subset of $[1,\infty]$. Let $\tau = (\tau_n)_{n\in\mathbb{N}}$ be a fixed sequence of positive integers. Define

$$Y_\tau = \{y_n : n \in \mathbb{N}\} \cup \{y_{ni}, y'_{ni} : i = 1, \dots, \tau_n,\, n \in \mathbb{N}\},$$

where all elements y_n, y_{ni}, y'_{ni} are pairwise different. We define the metric $\rho = \rho_\tau$ determining the distance between two different elements x and y by the formula

$$\rho(x,y) = \begin{cases} 1 & \text{if } \{x,y\} = T_{ni} \text{ or } y_n \in \{x,y\} \subset T_n \setminus T'_n \\ & \text{for some } n \in \mathbb{N},\, i \in \{1,\dots,\tau_n\}, \\ 2 & \text{in the other case.} \end{cases}$$

By T_n we denote the branch $T_n = \{y_n, y_{n1}, \dots, y_{n\tau_n}, y'_{n1}, \dots, y'_{n\tau_n}\}$. Additionally, we denote $T'_n = \{y'_{n1}, \dots, y'_{n\tau_n}\}$ and $T_{ni} = \{y_{ni}, y'_{ni}\}$. Figure 2 shows a model of the space (Y_τ, ρ).

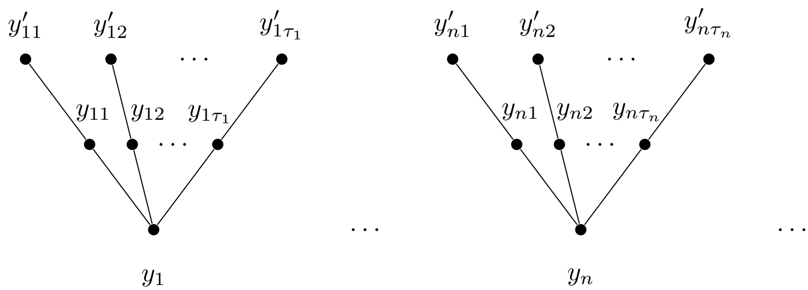

Figure 2.

Note that we can explicitly describe any ball: for $n \in \mathbb{N}$,

$$B(y_n, r) = \begin{cases} \{y_n\} & \text{for } 0 < r \le 1, \\ T_n \setminus T'_n & \text{for } 1 < r \le 2, \\ Y_\tau & \text{for } 2 < r, \end{cases}$$

and for $i \in \{1,\dots,\tau_n\}$, $n \in \mathbb{N}$,

$$B(y_{ni}, r) = \begin{cases} \{y_{ni}\} & \text{for } 0 < r \le 1, \\ \{y_n\} \cup T_{ni} & \text{for } 1 < r \le 2, \\ Y_\tau & \text{for } 2 < r, \end{cases}$$

and

$$B(y'_{ni},r)=\begin{cases}\{y'_{ni}\} & \text{for } 0<r\le 1,\\ T_{ni} & \text{for } 1<r\le 2,\\ Y_\tau & \text{for } 2<r.\end{cases}$$

We define the measure $\mu=\mu_{\tau,F}$ by letting $\mu(\{y_n\})=d_n$, $\mu(\{y_{ni}\})=\frac{d_n}{\tau_n}$, and $\mu(\{y'_{ni}\})=d_nF(n,i)$, where $F>0$ is a given function and $d=(d_n)_{n\in\mathbb{N}}$ is an appropriate sequence of strictly positive numbers with $d_1=1$ and d_n chosen (uniquely!) in such a way that $|T_n|=|T_{n-1}|/2$, $n\ge 2$. Note that this implies $|Y_\tau|<\infty$ and observe that μ is non-doubling.

We are ready to describe two subtypes of the second generation spaces.

3.1. We first construct spaces for which apart from the basic property $P_s^c=P_w^c=[1,\infty]$ we also have $P_s=P_w$. In the first step, for any fixed $p_0\in(1,\infty]$ we construct a space denoted by $\hat{\mathbb{Y}}_{p_0}$ for which $P_s=P_w=[p_0,\infty]$. Then, after a slight modification, for any fixed $p_0\in[1,\infty)$ we get a space $\hat{\mathbb{Y}}'_{p_0}$ for which $P_s=P_w=(p_0,\infty]$.

Fix $p_0\in(1,\infty]$ and let $\hat{\mathbb{Y}}_{p_0}$ be the second generation space with $\tau_n=\lfloor\frac{(n+1)^{p_0}}{n}\rfloor$ in the case $p_0\in(1,\infty)$, or $\tau_n=2^n$ when $p_0=\infty$, and $F(n,i)=n$, $i=1,\dots,\tau_n$, $n\in\mathbb{N}$.

Proposition 5. *Let $\hat{\mathbb{Y}}_{p_0}$ be the metric measure space defined above. Then the associated centered maximal operator M^c is of strong type $(1,1)$, while the non-centered M is not of weak type (p,p) for $1\le p<p_0$, but is of strong type (p,p) for $p\ge p_0$.*

Proof: First we show that M^c is of strong type $(1,1)$. Let $f\in\ell^1(\hat{\mathbb{Y}}_{p_0})$, $f\ge 0$. Denote $\mathcal{G}=\{\{y_n\}\cup T_{ni}: n\in\mathbb{N}, i=1,\dots,\tau_n\}$ and $\mathcal{B}_y=\{B(y,\frac12),B(y,\frac32),B(y,\frac52)\}$, $y\in Y_\tau$. We use the estimate

$$\|M^cf\|_1\le\sum_{y\in Y_\tau}\sum_{B\in\mathcal{B}_y}A_B(f)|\{y\}|.$$

Note that each $y\in Y_\tau$ belongs to at most four different balls which are not elements of $\mathcal{G}$. Thus we obtain

$$\sum_{y\in Y_\tau}\sum_{B\in\mathcal{B}_y\setminus\mathcal{G}}A_B(f)|\{y\}|\le\sum_{B\notin\mathcal{G}}\sum_{y\in B}f(y)|\{y\}|\le 4\|f\|_1.$$

Therefore

$$\|M^cf\|_1\le 4\|f\|_1+\sum_{n\in\mathbb{N}}\sum_{i=1}^{\tau_n}A_{B(y_{ni},\frac32)}(f)|\{y_{ni}\}|.$$

It suffices to see that the last term of the above expression is estimated by

$$\sum_{n\in\mathbb{N}} \tau_n f(y_n)|\{y_{n1}\}| + \sum_{n\in\mathbb{N}}\sum_{i=1}^{\tau_n}\big(f(y_{ni})|\{y_{ni}\}| + f(y'_{ni})|\{y'_{ni}\}|\big) = \|f\|_1.$$

In the next step we show that M is not of weak type (p,p) for $1 \le p < p_0$. Indeed, fix $p < p_0$ and let $f_n = \delta_{y_n}$, $n \ge 1$. Then $\|f_n\|_p^p = d_n$ and $Mf_n(y'_{ni}) \ge \frac{1}{n+1+(1/\tau_n)} \ge \frac{1}{n+2}$, $i = 1,\dots,\tau_n$. This implies that $|E_{1/(2(n+2))}(Mf_n)| \ge n\tau_n d_n$ and hence we obtain

$$\limsup_{n\to\infty} \frac{\|Mf_n\|_{p,\infty}^p}{\|f_n\|_p^p} \ge \lim_{n\to\infty} \frac{n\tau_n d_n}{(2(n+2))^p d_n} = \infty.$$

To complete the proof, it suffices to show that M is of strong type (p_0,p_0) in the case $p_0 \in (1,\infty)$. Let $f \in \ell^{p_0}(\hat{\mathbb{Y}}_{p_0})$, $f \ge 0$. We use the estimate

$$\|Mf\|_{p_0}^{p_0} \le \sum_{B\subset Y_\tau}\sum_{y\in B} A_B(f)^{p_0}|\{y\}| = \sum_{B\subset Y_\tau} A_B(f)^{p_0}|B|.$$

Once again note that each $y \in Y_\tau$ belongs to at most four different balls which are not elements of $\mathcal{G}$. Combining this with Hölder's inequality, we obtain

$$\sum_{B\notin\mathcal{G}} A_B(f)^{p_0}|B| \le \sum_{B\notin\mathcal{G}}\sum_{y\in B} f(y)^{p_0}|\{y\}| \le 4\|f\|_{p_0}^{p_0}.$$

Therefore

$$\begin{aligned} (4)\quad \|Mf\|_{p_0}^{p_0} &\le 4\|f\|_{p_0}^{p_0} \\ &+ \sum_{n\in\mathbb{N}}\sum_{i=1}^{\tau_n}\left(\frac{f(y_n) + 1/\tau_n f(y_{ni}) + nf(y'_{ni})}{1 + 1/\tau_n + n}\right)^{p_0} |\{y_n, y_{ni}, y'_{ni}\}|. \end{aligned}$$

Finally, we use the inequalities

$$\begin{aligned} &(f(y_n) + 1/\tau_n f(y_{ni}) + nf(y'_{ni}))^{p_0} \\ &\qquad \le (3f(y_n))^{p_0} + (3f(y_{ni})/\tau_n)^{p_0} + (3nf(y'_{ni}))^{p_0}, \end{aligned}$$

and $|\{y_n, y_{ni}, y'_{ni}\}| \le 3|\{y'_{ni}\}| = 3n|\{y_n\}|$ to estimate the double sum in (4) by

$$3^{p_0+1}\left(\sum_{n\in\mathbb{N}} \frac{n\tau_n f(y_n)^{p_0}}{(n+1)^{p_0}}|\{y_n\}| + \sum_{n\in\mathbb{N}}\sum_{i=1}^{\tau_n} \frac{(f(y_{ni})/\tau_n)^{p_0} + (nf(y'_{ni}))^{p_0}}{(1+1/\tau_n+n)^{p_0}}|\{y'_{ni}\}|\right) \le 3^{p_0+1}\|f\|_{p_0}^{p_0}. \quad \square$$

Note that in the same way as it was done at the end of Subsection 2.1, replacing the former τ_n by $\tau'_n = \left\lfloor(\log(n)+1)\frac{(n+1)^{p_0}}{n}\right\rfloor$, $p_0 \in [1,\infty)$, results in obtaining the space $\hat{\mathbb{Y}}'_{p_0}$ for which $P_s = P_w = (p_0, \infty]$.

3.2. In contrast to the former case the spaces we now construct, apart from the basic property $P_s^c = P_w^c = [1,\infty]$, satisfy $P_s \subsetneq P_w$. Namely, for any fixed $p_0 \in [1,\infty)$ we construct a space $\widetilde{\mathbb{Y}}_{p_0}$ for which $P_s = (p_0,\infty]$ and $P_w = [p_0,\infty]$. We consider the cases $p_0 = 1$ and $p_0 > 1$ separately, similarly as it was done in Section 2.

By $\widetilde{\mathbb{Y}}_1$ we denote the second generation space (Y_τ, ρ, μ) with construction based on $\tau_n = n$ and $F(n,i) = 2^i$. Recall that μ is non-doubling.

Proposition 6. *Let $\widetilde{\mathbb{Y}}_1$ be the metric measure space defined above. Then the associated centered operator M^c is of strong type $(1,1)$, while the non-centered M is of weak type $(1,1)$, but is not of strong type $(1,1)$.*

Proof: First note that it is easy to verify that M^c is of strong type $(1,1)$, by using exactly the same argument as in the proof of Proposition 5. In the next step we show that M is not of strong type $(1,1)$. Indeed, let $f_n = \delta_{y_n}$, $n \ge 1$. Then $\|f_n\|_1 = d_n$ and for $i = 1,\dots,n$ we have $Mf_n(y'_{ni}) \ge (1+1/n+2^i)^{-1} > 1/2^{i+1}$ and hence we obtain $\|Mf_n\|_1 \ge \sum_{i=1}^n 2^i d_n/2^{i+1} = n\|f_n\|_1/2$.

To complete the proof, it suffices to show that M is of weak type $(1,1)$. Let $f \in \ell^1(\widetilde{\mathbb{Y}}_1)$, $f \ge 0$, and consider $\lambda > 0$ such that $E_\lambda(Mf) \ne \emptyset$. If $\lambda < A_{Y_\tau}(f)$, then $\lambda|E_\lambda(Mf)|/\|f\|_1 < 1$ follows. Therefore, from now on assume that $\lambda \ge A_{Y_\tau}(f)$. With this assumption, if for some $y \in T_n$ we have $Mf(y) > \lambda$, then any ball B containing y and realizing $A_B(f) > \lambda$ must be a subset of T_n. Take any $n \in \mathbb{N}$ such that $E_\lambda(Mf) \cap T_n \ne \emptyset$. If $\lambda < A_{T_n}(f)$, then

$$\frac{\lambda|E_\lambda(Mf)\cap T_n|}{\sum_{y\in T_n} f(y)|\{y\}|} \le 2. \tag{5}$$

Assume that $\lambda \geq A_{T_n}(f)$ and take $y \in E_\lambda(Mf) \cap T_n$. Now, any ball B containing y and realizing $A_B(f) > \lambda$ must be a proper subset of T_n. First, consider the case $E_\lambda(Mf) \cap T'_n = \emptyset$. If $y_n \in E_\lambda(Mf) \cap T_n$, then we obtain $\sum_{y \in T_n \setminus T'_n} f(y)|\{y\}| > \lambda |\{y_n\}|$ and since $|E_\lambda(Mf) \cap T_n| \leq 2|\{y_n\}|$, (5) follows. Otherwise, if $y_n \notin E_\lambda(Mf) \cap T_n$, then, necessarily, $f(y) > \lambda$ for every $y \in E_\lambda(Mf) \cap T_n$ and hence (5) again follows. Finally, in the case $E_\lambda(Mf) \cap T'_n \neq \emptyset$, denote $j = \max\{i \in \{1, \ldots, n\} : Mf(y'_{ni}) > \lambda\}$. Therefore, $\sum_{y \in T_n} f(y)|\{y\}| > \lambda |\{y'_{nj}\}|$ and combining this with the estimate $|E_\lambda(Mf) \cap T_n| \leq 2|\{y'_{nj}\}|$, we conclude that (5) follows. Since $\lambda |E_\lambda(Mf) \cap T_n| / \sum_{y \in T_n} f(y)|\{y\}| \leq 2$ for any branch T_n such that $E_\lambda(Mf) \cap T_n \neq \emptyset$, we have $\lambda |E_\lambda(Mf)| / \|f\|_1 \leq 2$ and, consequently, $\|Mf\|_{1,\infty} \leq 2\|f\|_1$. □

Now, fix $p_0 \in (1, \infty)$ and consider $\widetilde{\mathbb{Y}}_{p_0} = (Y_\tau, \rho, \mu)$ with construction based on $\tau_n = \tau_n(p_0)$ and $F(n,i) = F_{p_0}(n,i)$, defined in the same way as in Subsection 2.2, by using the auxiliary sequences c_n, e_n, and m_{nj}, s_{nj}, $j \in \{1, \ldots, e_n\}$, $n \in \mathbb{N}$.

Proposition 7. *Let $\widetilde{\mathbb{Y}}_{p_0}$ be the metric measure space defined above. Then the associated centered maximal operator M^c is of strong type $(1,1)$, while the non-centered M is of weak type (p_0, p_0), but is not of strong type (p_0, p_0).*

Proof: First note once again that it is easy to verify that M^c is of strong type $(1,1)$, by using the same argument as in the proof of Proposition 5. In the next step we show that M is not of strong type (p_0, p_0). Indeed, let $f_n = \delta_{x_n}$, $n \geq 1$. Then $\|f_n\|_{p_0}^{p_0} = d_n$ and for $i = 1, \ldots, \tau_n$ we have $Mf_n(y'_{ni}) \geq (1 + 1/\tau_n + m_{nj(n,i)})^{-1} \geq (2(1 + m_{nj(n,i)}))^{-1}$ and hence

$$\|Mf_n\|_{p_0}^{p_0} \geq \sum_{j=1}^{e_n} \sum_{k=1}^{s_{nj}} \frac{d_n m_{nj}}{(2(1 + m_{nj}))^{p_0}} = d_n \sum_{j=1}^{e_n} \frac{s_{nj} m_{nj}}{(2(1 + m_{nj}))^{p_0}}$$

$$\geq d_n \sum_{j=1}^{e_n} \frac{2^{1-j-p_0} n c_n}{(1 + m_{nj})^{p_0}} = 2^{-p_0} d_n \sum_{j=1}^{e_n} \frac{n c_n}{(1+n)^{p_0}}$$

$$= 2^{-p_0} e_n \frac{n c_n}{(1+n)^{p_0}} \|f_n\|_{p_0}^{p_0}.$$

Since $\lim_{n \to \infty} e_n = \infty$ and $\lim_{n \to \infty} \frac{n c_n}{(1+n)^{p_0}} = 1$, we are done.

To complete the proof, it suffices to show that M is of weak type (p_0, p_0). Let $f \in \ell^{p_0}(\widetilde{\mathbb{Y}}_{p_0})$, $f \geq 0$, and consider $\lambda > 0$ such that $E_\lambda(Mf) \neq \emptyset$. If $\lambda < A_{Y_\tau}(f)$, then using the inequality $\|f\|_1 \leq$

$\|f\|_{p_0}|Y_\tau|^{1/q_0}$, we obtain $\lambda^{p_0}|E_\lambda(Mf)|/\|f\|_{p_0}^{p_0} < 1$. Therefore, from now on assume that $\lambda \geq A_{Y_\tau}(f)$. Take any $n \in \mathbb{N}$ such that $E_\lambda(Mf) \cap T_n \neq \emptyset$. If $\lambda < A_{T_n}(f)$, then using similar argument as above we have

$$\tag{6} \frac{\lambda^{p_0}|E_\lambda(Mf) \cap T_n|}{\sum_{y \in T_n} f(y)^{p_0}|\{y\}|} \leq 1.$$

Consider $\lambda \geq A_{T_n}(f)$. Assume that $E_\lambda(Mf) \cap T'_n = \emptyset$. If $\lambda < A_{T_n \setminus T'_n}(f)$, then (6) again follows. Otherwise, if $\lambda \geq A_{T_n \setminus T'_n}(f)$, then we consider two cases. If $y_n \in E_\lambda(Mf)$, then we obtain $f(y_n) \geq \lambda$ and hence

$$\frac{\lambda^{p_0}|E_\lambda(Mf) \cap T_n|}{\sum_{y \in T_n} f(y)^{p_0}|\{y\}|} \leq \frac{2\lambda^{p_0}|\{y_n\}|}{\sum_{y \in T_n} f(y)^{p_0}|\{y\}|} \leq 2.$$

In the other case, if $y_n \notin E_\lambda(Mf)$, then $f(y) > \lambda$ holds for every $y \in E_\lambda(Mf) \cap T_n$ and hence (6) follows one more time. Now assume that $E_\lambda(Mf) \cap T'_n \neq \emptyset$. See that $|E_\lambda(Mf) \cap T_n| \leq 3|E_\lambda(Mf) \cap T'_n|$. Consider the case $f(y_n) < (1 + 1/\tau_n + m_{ne_n})\lambda/3$. If $y'_{ni} \in E_\lambda(Mf) \cap T'_n$ for some $i \in \{1, \dots, \tau_n\}$, then $f(y'_{ni}) \geq \lambda/3$ or $f(y_{ni})|\{y_{ni}\}| \geq |\{y'_{ni}\}|\lambda/3$ and hence $f(y'_{ni})^{p_0}|\{y'_{ni}\}| + f(y_{ni})^{p_0}|\{y_{ni}\}| \geq |\{y'_{ni}\}|(\lambda/3)^{p_0}$, which implies

$$\frac{\lambda^{p_0}|E_\lambda(Mf) \cap T_n|}{\sum_{y \in T_n} f(y)^{p_0}|\{y\}|} \leq \frac{3\lambda^{p_0}|E_\lambda(Mf) \cap T'_n|}{\sum_{y \in T_n} f(y)^{p_0}|\{y\}|} \leq 3^{p_0+1}.$$

Finally, in the case $f(y_n) \geq (1 + 1/\tau_n + m_{ne_n})\lambda/3$, denote $r = \min\{j \in \{1, \dots, e_n\} : f(y_n) \geq \frac{(1+1/\tau_n+m_{nj})\lambda}{3}\}$. Let $T_n^{(r)} = \{y'_{ni} : i \in \{1, \dots, \sum_{j=1}^{r-1} s_{nj}\}\}$. Note that the case $T_n^{(r)} = \emptyset$ is possible. Assume that $T_n^{(r)} \neq \emptyset$. If $y'_{ni} \in E_\lambda(Mf) \cap T_n^{(r)}$, then $f(y'_{ni})^{p_0}|\{y'_{ni}\}| + f(y_{ni})^{p_0}|\{y_{ni}\}| \geq |\{y'_{ni}\}|(\lambda/3)^{p_0}$ and hence

$$\frac{\lambda^{p_0}|E_\lambda(Mf) \cap T_n^{(r)}|}{\sum_{i : y'_{ni} \in T_n^{(r)}} (f(y'_{ni})^{p_0}|\{y'_{ni}\}| + f(y_{ni})^{p_0}|\{y_{ni}\}|)} \leq 3^{p_0+1}.$$

Moreover, we have

$$\begin{aligned}
\frac{\lambda^{p_0}|E_\lambda(Mf)\cap(T_n\setminus T_n^{(r)})|}{f(y_n)^{p_0}|\{y_n\}|} &\le \Big(\frac{3}{1+m_{nr}}\Big)^{p_0}\frac{|T_n\setminus T_n^{(r)}|}{|\{y_n\}|}\\
&\le \Big(\frac{3}{1+m_{nr}}\Big)^{p_0}\frac{3|(T_n\setminus T_n^{(r)})\cap T_n')|}{|\{y_n\}|}\\
&\le \Big(\frac{3}{1+m_{nr}}\Big)^{p_0} 3\sum_{j=r}^{e_n} nc_n 2^{2-j}\\
&< 2^{3-r}3^{p_0+1}nc_n\Big(\frac{1}{1+m_{nr}}\Big)^{p_0}\\
&= 4\cdot 3^{p_0+1}\frac{nc_n}{(1+n)^{p_0}} \le 4\cdot 3^{p_0+1}.
\end{aligned}$$

Therefore, regardless of the possibilities, $T_n^{(r)}=\emptyset$ or $T_n^{(r)}\neq\emptyset$, we obtain $\lambda^{p_0}|E_\lambda(Mf)\cap T_n|/\sum_{y\in T_n} f(y)^{p_0}|\{y\}| \le 4\cdot 3^{p_0+1}$. Since $\lambda|E_\lambda(Mf)\cap T_n|/\sum_{y\in T_n} f(y)^{p_0}|\{y\}| \le 4\cdot 3^{p_0+1}$ for any branch T_n such that $E_\lambda(Mf)\cap T_n \neq \emptyset$, we have $\lambda^{p_0}|E_\lambda(Mf)|/\|f\|_{p_0}^{p_0} \le 4\cdot 3^{p_0+1}$ and consequently $\|Mf\|_{p_0,\infty}^{p_0} \le 4\cdot 3^{p_0+1}\|f\|_{p_0}^{p_0}$. □

4. Proof of Theorem 1

All spaces discussed above were constructed in such a way as to avoid any interactions between the different branches in the context of considerations relating to the existence of the weak and strong type inequalities. Therefore we can construct a new space consisting of two types of branches, one borrowed from some first generation space and one from some second generation space, and to ensure that the operators M^c and M inherit a particular property of a particular space. We explain the construction of such a space in detail proving Theorem 1.

Proof of Theorem 1: We consider a few cases. If the equalities $P_s^c = P_s$ and $P_w^c = P_w$ are supposed to hold, then the expected space may be chosen to be a first generation space. If, in turn, we have $P_s^c = P_w^c = [1,\infty]$, but $P_s \neq [1,\infty]$, then the expected space may be chosen to be a second generation space. Finally, in other cases we can find spaces $\mathbb{X} = (X,\rho_X,\mu_X)$ and $\mathbb{Y} = (Y,\rho_Y,\mu_Y)$, of first and second generation, respectively, for which

- $P_s^c(\mathbb{X}) = P_s(\mathbb{X}) = P_s^c$ and $P_w^c(\mathbb{X}) = P_w(\mathbb{X}) = P_w^c$,
- $P_s^c(\mathbb{Y}) = P_w^c(\mathbb{Y}) = [1,\infty]$, $P_s(\mathbb{Y}) = P_s$, and $P_w(\mathbb{Y}) = P_w$.

Using $\mathbb{X}$ and $\mathbb{Y}$ and assuming that $X \cap Y = \emptyset$ we construct the space $\mathbb{Z} = (Z, \rho_Z, \mu_Z)$ as follows. Denote $Z = X \cup Y$. We define the metric ρ_Z on Z by

$$\rho_Z(x,y) = \begin{cases} \rho_X(x,y) & \text{if } \{x,y\} \subset X, \\ \rho_Y(x,y) & \text{if } \{x,y\} \subset Y, \\ 2 & \text{in the other case,} \end{cases}$$

and the measure μ_Z on Z by

$$\mu_Z(E) = \mu_X(E \cap X) + \mu_Y(E \cap Y), \quad E \subset Z.$$

It is not hard to show that $\mathbb{Z}$ has the following properties

- $P_s^c(\mathbb{Z}) = P_s^c(\mathbb{X}) \cap P_s^c(\mathbb{Y}) = P_s^c \cap [1,\infty] = P_s^c$,
- $P_s(\mathbb{Z}) = P_s(\mathbb{X}) \cap P_s(\mathbb{Y}) = P_s^c \cap P_s = P_s$,
- $P_w^c(\mathbb{Z}) = P_w^c(\mathbb{X}) \cap P_w^c(\mathbb{Y}) = P_w^c \cap [1,\infty] = P_w^c$,
- $P_w(\mathbb{Z}) = P_w(\mathbb{X}) \cap P_w(\mathbb{Y}) = P_w^c \cap P_w = P_w$,

and therefore it may be chosen to be the expected space. Finally, it is not hard to see that μ_Z is non-doubling, since it is bounded and there is a ball B in Z with radius $r = 1$ and $|B| < \epsilon$ for any arbitrarily small $\epsilon > 0$. □

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