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## Some Criteria for Maximal Abstract Monotonicity

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**Abstract.** In this paper, we develop a theory of monotone operators in the framework of abstract convexity. First, we provide a surjectivity result for a broad class of abstract monotone operators. Then, by using an additivity constraint qualification, we prove a generalization of Fenchel's duality theorem in the framework of abstract convexity and give some criteria for maximal abstract monotonicity. Finally, we present necessary and sufficient conditions for maximality of abstract monotone operators.

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## 1 Introduction

Abstract convexity has found many applications in the study of problem of mathematical analysis and optimization. Also, it has found interesting applications to the theory of inequalities. Abstract convexity opens the way for extending some main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets. It is well-known that every convex, proper and lower semicontinuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is taken by an alternative set  $H$  of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of the set  $H$  generate variants of the classical concepts, and have shown important applications in global optimization (see [23, 24, 25, 26]).

Abstract convexity has mainly been used for the study of point-to-point functions. Examples of its use in the analysis of multifunctions can be found in [1, 9, 10, 20]. Several approaches to the theory of monotone multifunctions have established links between maximal monotone multifunctions and convex functions (see [2, 3, 6, 8, 13, 14, 19, 29, 30]). The richness of the theory of monotone operators has given rise to a great number of works and the simplification of proofs and theory that results from the use of convex analysis techniques justifies an interest in these links. Roughly speaking, the study of monotone operators is reduced to the study of the convexification of the coupling function, restricted to the monotone set. However, convexity is sometimes a restrictive assumption, and therefore the problem arises how to generalize the theory of monotone operators via abstract convexity. Recently, a theory of monotone operators has been developed in the framework of abstract convexity (see [5, 15]).

In 1970, Moreau [17] observed that Fenchel conjugation theory and the second conjugate theorem can be established in a very general setting, using two arbitrary sets and arbitrary coupling functions. The second conjugate theorem in this setting, known as Fenchel-Moreau theorem, has given rise to the rich theory of abstract convexity (see [18, 22, 28]). Extensions of Fenchel duality theorem and Fenchel-Rockafellar theorem, which have played key roles in the application of convex analysis, have been presented for abstract convex functions in [7]. The aim of the present paper is to develop a theory of monotone operators in the framework of abstract convexity by using generalized Fenchel duality theorem. In fact, we present criteria for maximal abstract monotonicity and obtain some results on maximal abstract monotonicity by using an additivity constraint qualification.

The structure of the paper is as follows: In section 2, we provide some preliminary definitions and results related to abstract convexity and abstract monotonicity. In section 3, we present a Rockafellar type surjectivity result. In section 4, by using an additivity constraint qualification, we obtain a generalization of Fenchel duality theorem in the framework of abstract convexity and give also criteria for maximal abstract monotonicity. Necessary and sufficient conditions for maximality of abstract monotone operators are given in section 5.

## 2 Preliminaries

Let  $X$  and  $Y$  be two sets. Recall (see [4]) that a set valued mapping (multifunction) from  $X$  to  $Y$  is a mapping  $F : X \longrightarrow 2^Y$ , where  $2^Y$  represents the collection of all subsets of  $Y$ . We define the domain and graph of  $F$  by

$$\text{dom}(F) := \{x \in X : F(x) \neq \emptyset\},$$

and

$$G(F) := \{(x, y) \in X \times Y : y \in F(x)\},$$

respectively. The inverse of  $F$  is the set valued mapping  $F^{-1} : Y \longrightarrow 2^X$  defined by

$$F^{-1}(y) := \{x \in X : y \in F(x)\}.$$

Now, let  $X$  be a set and  $L$  be a set of real valued abstract linear functions  $l : X \rightarrow \mathbb{R}$  defined on  $X$ . For each  $l \in L$  and  $c \in \mathbb{R}$ , consider the shift  $h_{l,c}$  of  $l$  on the constant  $c$  :

$$h_{l,c}(x) := l(x) - c, \quad (x \in X).$$

The function  $h_{l,c}$  is called  $L$ -affine. Recall (see [22]) that the set  $L$  is called a set of abstract linear functions if  $h_{l,c} \notin L$  for all  $l \in L$  and all  $c \in \mathbb{R} \setminus \{0\}$ . The set of all  $L$ -affine functions will be denoted by  $H_L$ . If  $L$  is the set of abstract linear functions, then  $h_{l,c} = h_{l_0,c_0}$  if and only if  $l = l_0$  and  $c = c_0$ .

If  $L$  is a set of abstract linear functions, then the mapping  $(l, c) \rightarrow h_{l,c}$  is a one-to-one correspondence. In this case, we identify  $h_{l,c}$  with  $(l, c)$ , in other words, we consider an element  $(l, c) \in L \times \mathbb{R}$  as a function defined on  $X$  by  $x \rightarrow l(x) - c$  ( $x \in X$ ).

A function  $f : X \rightarrow (-\infty, +\infty]$  is called proper if  $\text{dom } f \neq \emptyset$ , where  $\text{dom } f$  is defined by

$$\text{dom } f := \{x \in X : f(x) < +\infty\}.$$

Let  $\mathcal{F}(X)$  be the set of all functions  $f : X \rightarrow (-\infty, +\infty]$  and the function  $-\infty$ .

Recall (see [22]) that a function  $f \in \mathcal{F}(X)$  is called  $H$ -convex ( $H = L$ , or  $H = H_L$ ) if

$$f(x) = \sup\{h(x) : h \in \text{supp}(f, H)\}, \quad \forall x \in X,$$

where

$$\text{supp}(f, H) := \{h \in H : h \leq f\}$$

is called the support set of the function  $f$ , and  $h \leq f$  if and only if  $h(x) \leq f(x)$  for all  $x \in X$ .

Let  $\mathcal{P}(H)$  be the set of all  $H$ -convex functions  $f : X \rightarrow (-\infty, +\infty]$ . We say that (see [7]) the set valued mapping  $\text{supp}(\cdot, H) : \mathcal{P}(H) \rightarrow 2^H$  is additive in  $f$  and  $g$  if

$$\text{supp}(f + g, H) = \text{supp}(f, H) + \text{supp}(g, H).$$

Note that if  $X$  is a locally convex Hausdorff topological vector space and  $L$  is the set of all real valued continuous linear functionals defined on  $X$ , then  $f : X \rightarrow (-\infty, +\infty]$  is an  $L$ -convex function if and only if  $f$  is lower semi-continuous and sublinear. Also,  $f$  is an  $H_L$ -convex function if and only if  $f$  is lower semi-continuous and convex.

Now, we consider the coupling function  $\langle \cdot, \cdot \rangle : X \times L \rightarrow \mathbb{R}$  is defined by  $\langle x, l \rangle := l(x)$  for all  $x \in X$  and all  $l \in L$ . For a function  $f \in \mathcal{F}(X)$ , define the Fenchel-Moreau  $L$ -conjugate  $f_L^*$  of  $f$  (see [22]) by

$$f_L^*(l) := \sup_{x \in X} (l(x) - f(x)), \quad l \in L.$$

Similarly, the Fenchel-Moreau  $X$ -conjugate  $g_X^*$  of an extended real valued function  $g$  defined on  $L$  is given by

$$g_X^*(x) := \sup_{l \in L} (l(x) - g(l)), \quad x \in X.$$

The function  $f_{L,X}^{**} := (f_L^*)^*$  is called the second conjugate (or biconjugate) of  $f$ , and by definition we have

$$f_{L,X}^{**}(x) := \sup_{l \in L} (l(x) - f^*(l)), \quad x \in X.$$

The following properties of the conjugate function follow directly from the definition.

(i) Fenchel-Young's inequality: if  $f \in \mathcal{F}(X)$ , then

$$f(x) + f_L^*(l) \geq l(x), \quad \forall x \in X; \forall l \in L.$$

(ii) For  $f_1$  and  $f_2 \in \mathcal{F}(X)$ , we have

$$f_1 \leq f_2 \implies f_2^* \leq f_1^*.$$

A set  $C \subset \mathcal{F}(X)$  is called additive if for  $f_1, f_2 \in C$ , then  $f_1 + f_2 \in C$ .

If  $X$  is a set on which an addition  $+$  is defined, then we say that a function  $f \in \mathcal{F}(X)$  is additive if

$$f(x+y) = f(x) + f(y), \quad \forall x, y \in X.$$

Let  $f : X \rightarrow (-\infty, +\infty]$  be a function and  $x_0 \in \text{dom} f$ . Recall (see [22]) that an element  $l \in L$  is called an  $L$ -subgradient of  $f$  at  $x_0$  if

$$f(x) \geq f(x_0) + l(x) - l(x_0), \quad \forall x \in X.$$

The set  $\partial_L f(x_0)$  of all  $L$ -subgradients of  $f$  at  $x_0$  is called  $L$ -subdifferential of  $f$  at  $x_0$ . The subdifferential  $\partial_L f(x_0)$  (see [22, Proposition 1.2]) is non-empty if and only if  $x_0 \in \text{dom} f$  and

$$f(x_0) = \max\{h(x_0) : h \in \text{supp}(f, H_L)\}.$$

Recall (see [7]) that for proper functions  $f, g \in \mathcal{F}(X)$ , the infimal convolution of  $f$  with  $g$  is denoted by  $f \oplus g : X \rightarrow (-\infty, +\infty]$  and is defined by

$$(f \oplus g)(x) := \inf_{x_1+x_2=x} [f(x_1) + g(x_2)], \quad \forall x \in X.$$

The infimal convolution of  $f$  with  $g$  is said to be exact provided the above infimum is achieved for every  $x \in X$  (see [7]).

Now, assume that  $X$  is a set and  $L$  is a set of real valued abstract linear functions  $l : X \rightarrow \mathbb{R}$  defined on  $X$ , with the coupling function  $\langle \cdot, \cdot \rangle : X \times L \rightarrow \mathbb{R}$  defined by  $\langle x, l \rangle := l(x)$  for all  $x \in X$  and all  $l \in L$ . In the following, we present some definitions and properties of abstract monotone operators (see [5, 10, 15, 20]).

(i) A set valued mapping  $T : X \rightarrow 2^L$  is called  $L$ -monotone operator (or, abstract monotone operator) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0 \tag{2.1}$$

for all  $l \in Tx$ ,  $l' \in Tx'$  and all  $x, x' \in X$ .

It is worth noting that if  $X$  is a Banach space with dual space  $X^*$  and  $L := X^*$ , then  $T$  is a monotone operator in the classical sense.

(ii) A set valued mapping  $T : X \rightarrow 2^L$  is called maximal  $L$ -monotone (or maximal abstract monotone) if  $T$  is  $L$ -monotone and  $T = T'$  for any  $L$ -monotone operator  $T' : X \rightarrow 2^L$  such that  $G(T) \subseteq G(T')$ .

(iii) A subset  $S$  of  $X \times L$  is called  $L$ -monotone (or, abstract monotone) if

$$l(x) - l(x') - l'(x) + l'(x') \geq 0, \quad \forall (x, l), (x', l') \in S.$$

(iv) A subset  $S$  of  $X \times L$  is called maximal  $L$ -monotone (or, maximal abstract monotone) if  $S$  is  $L$ -monotone and  $S = S'$  for any  $L$ -monotone set  $S'$  such that  $S \subseteq S'$ .

(v) Let  $T : X \rightarrow 2^L$  be a set valued mapping. Corresponding to the mapping  $T$  define the  $L$ -Fitzpatrick function (or, abstract Fitzpatrick function)  $\varphi_T : X \times L \rightarrow \overline{\mathbb{R}}$  by

$$\varphi_T(x, l) := \sup_{l' \in Tx', x' \in X} [l(x') + l'(x) - l'(x')] \quad (2.2)$$

for all  $x \in X$  and all  $l \in L$ .

There exist examples of abstract convex functions such that their  $L$ -subdifferentials are maximal  $L$ -monotone operators (for more details see [15, 16]).

In the following, we gather some results which will be used later.

**Lemma 2.1.** [15]. *Let  $T : X \rightarrow 2^L$  be a maximal  $L$ -monotone operator. Then*

$$\varphi_T(x, l) \geq l(x), \quad \forall x \in X; \forall l \in L, \quad (2.3)$$

*with equality holding if and only if  $l \in Tx$ .*

**Lemma 2.2.** [22, Theorem 7.1]. *Let  $f \in \mathcal{F}(X)$ . Then,  $f = f_{L, X}^{**}$  if and only if  $f$  is an  $H_L$ -convex function.*

**Lemma 2.3.** [7, Theorem 7.1]. *Let  $L$  be an additive set of abstract linear functions and  $f, g : X \rightarrow (-\infty, +\infty]$  be  $H_L$ -convex functions such that  $\text{dom}(f) \cap \text{dom}(g) \neq \emptyset$ . Then the following assertions are equivalent:*

- (i) *The mapping  $\text{supp}(\cdot, H_L)$  is additive in  $f$  and  $g$ .*
- (ii)  *$(f + g)_L^* = f_L^* \oplus g_L^*$  with exact infimal convolution.*

### 3 A surjectivity result

Let  $U$  be an arbitrary set and  $L$  be an additive group of abstract linear functions on  $U$ . We define the coupling between  $U \times L$  and  $L \times U$  as

$$\langle (u, l), (m, v) \rangle = m(u) + l(v),$$

for all  $(u, l) \in U \times L$  and  $(m, v) \in L \times U$ . Let  $X \subseteq U$ . We will say that  $A : X \rightarrow 2^L$  is  $L$ -monotone if so is its extension to  $U$  obtained by assigning empty images to the elements in  $U \setminus X$ . Similarly, a function  $h : X \times L \rightarrow (-\infty, +\infty]$  will be called  $H_{L \times U}$ -convex if it is the restriction of an  $H_{L \times U}$ -convex function on  $U \times L$ .

Given an  $L$ -monotone operator  $A : X \rightarrow 2^L$ , consider the Fitzpatrick family of abstract convex representations of  $A$

$$\begin{aligned} \mathcal{H}_A &= \{h : X \times L \rightarrow (-\infty, +\infty] : h \text{ is } H_{L \times U}\text{-convex,} \\ &\quad h(x, l) \geq l(x) \ \forall (x, l) \in X \times L, \quad h(x, l) = l(x) \ \forall (x, l) \in G(A)\}. \end{aligned}$$

Moreover, for all  $l_0 \in L$ , denote by  $A_{l_0} : X \rightarrow L$  the multifunction such that  $A_{l_0}(x) = A(x) - l_0$ , for all  $x \in X$ . It is easy to check that, for any  $h \in \mathcal{H}_A$ , the function  $h_{l_0} : X \times L \rightarrow (-\infty, +\infty]$ , defined by

$$h_{l_0}(x, l) := h(x, l + l_0) - l_0(x), \ \forall (x, l) \in X \times L,$$

belongs to  $\mathcal{H}_{A_{l_0}}$ . Notice that, for any  $(m, u) \in L \times U$ ,

$$\begin{aligned} (h_{l_0})_{L \times U}^*(m, u) &= \sup_{(x, l) \in X \times L} \{m(x) + l(u) - h(x, l + l_0) + l_0(x)\} \\ &= \sup_{(x, l) \in X \times L} \{(m + l_0)(x) + (l - l_0)(u) - h(x, l)\} \\ &= \sup_{(x, l) \in X \times L} \{(m + l_0)(x) + l(u) - h(x, l)\} - l_0(u) \\ &= h_{L \times U}^*(m + l_0, u) - l_0(u). \end{aligned}$$

If  $A : X \rightarrow 2^L$  is an  $L$ -monotone operator and  $h \in \mathcal{H}_A$ , denote by  $\tilde{A}_h : U \rightarrow 2^L$  the operator defined by

$$G(\tilde{A}_h) = \{(u, l) \in U \times L : h_{L \times U}^*(l, u) = l(u)\}.$$

In particular, when  $h = \varphi_A$ , we will simply write  $\tilde{A}$ , instead of  $\tilde{A}_{\varphi_A}$ , for ease of notation. According to the following proposition,  $\tilde{A}$  is an extension of  $A$ , i.e.  $G(A) \subseteq G(\tilde{A})$ .

**Proposition 3.1.** *Let  $X \subseteq U$  and  $A : X \rightarrow 2^L$  be an  $L$ -monotone operator. Then  $\tilde{A}$  is an extension of  $A$ .*

*Proof:* Notice first that, for any  $(m, x) \in L \times X$ , one has  $(\varphi_A)_{L \times U}^*(m, x) \geq \varphi_A(x, m)$ . Indeed, since  $\varphi_A(y, l) = l(y)$  for all  $(y, l) \in G(A)$ ,

$$\begin{aligned} (\varphi_A)_{L \times U}^*(m, x) &= \sup_{(y, l) \in X \times L} \{m(y) + l(x) - \varphi_A(y, l)\} \\ &\geq \sup_{(y, l) \in G(A)} \{m(y) + l(x) - \varphi_A(y, l)\} \\ &= \sup_{(y, l) \in G(A)} \{m(y) + l(x) - l(y)\} \\ &= \varphi_A(x, m). \end{aligned}$$

Moreover, for all  $(x, m) \in G(A)$ , one has  $(\varphi_A)_{L \times U}^*(m, x) \leq m(x)$ , since

$$\begin{aligned} (\varphi_A)_{L \times U}^*(m, x) &= \sup_{(y, l) \in X \times L} \{m(y) + l(x) - \varphi_A(y, l)\} \\ &= \sup_{(y, l) \in X \times L} \{m(y) + l(x) - m(x) + m(x) - \varphi_A(y, l)\} \\ &\leq m(x) + \sup_{(y, l) \in X \times L} \left\{ \sup_{(z, n) \in G(A)} \{l(z) + n(y) - n(z)\} - \varphi_A(y, l) \right\} \\ &= m(x) + \sup_{(y, l) \in X \times L} \{\varphi_A(y, l) - \varphi_A(y, l)\} \\ &= m(x). \end{aligned}$$

Therefore, for all  $(x, m) \in G(A)$ , one obtains

$$m(x) = \varphi_A(x, m) \leq (\varphi_A)_{L \times U}^*(m, x) \leq m(x),$$

i.e.  $(x, m) \in G(\tilde{A})$ . Thus,  $\tilde{A}$  is an extension of  $A$ . ■

**Definition 3.1.** Let  $f, g : X \times L \rightarrow (-\infty, +\infty]$  be  $H_{L \times U}$ -convex functions. We call an abstract skewed Fenchel functional for  $f$  and  $g$  any  $(m, u) \in L \times U$  such that

$$f_{L \times U}^*(m, u) + g_{L \times U}^*(-m, u) \leq 0.$$

**Remark 3.1.** If  $U$  is an additive set and the elements of  $L$  are odd functions, then, defining the function  $\varrho_2 : X \times L \rightarrow X \times L$  by  $\varrho_2(x, l) = (x, -l)$  for all  $(x, l) \in X \times L$ , the existence of an abstract skewed Fenchel functional for  $f$  and  $g$  is equivalent to the existence of an abstract Fenchel functional for  $f$  and  $g \circ \varrho_2$ , i.e. an element  $(m, u) \in L \times U$  such that

$$f_{L \times U}^*(m, u) + (g \circ \varrho_2)_{L \times U}^*(-m, -u) \leq 0.$$



The proof of this fact is immediate, given that, for all  $(m, u) \in L \times U$ ,

$$\begin{aligned}
(k \circ \varrho_2)_{L \times U}^* (-m, -u) &= \sup_{(x,l) \in X \times L} \{-m(x) + l(-u) - (k \circ \varrho_2)(x, l)\} \\
&= \sup_{(x,l) \in X \times L} \{-m(x) + l(-u) - k(x, -l)\} \\
&= \sup_{(x,l) \in X \times L} \{-m(x) - l(-u) - k(x, l)\} \\
&= \sup_{(x,l) \in X \times L} \{-m(x) + l(u) - k(x, l)\} \\
&= k_{L \times U}^* (-m, u).
\end{aligned}$$

**Theorem 3.1.** *Let  $X \subseteq U$  and  $A, B : X \rightarrow 2^L$  be  $L$ -monotone operators. If there exist  $h \in \mathcal{H}_A$  and  $k \in \mathcal{H}_B$  such that  $h_{L \times U}^*(m, u) \geq m(u)$  and  $k_{L \times U}^*(m, u) \geq m(u)$ , for all  $(m, u) \in L \times U$ , and such that, for any  $l_0 \in L$ , the functions  $h_{l_0}$  and  $k$  admit an abstract skewed Fenchel functional, then  $\mathcal{R}(\tilde{A}_h + \tilde{B}_k) = L$ .*

*Proof:* By hypothesis, there exists an abstract skewed Fenchel functional  $(\bar{m}, \bar{u})$  for  $h_{l_0}$  and  $k$ , i.e.

$$(h_{l_0})_{L \times U}^*(\bar{m}, \bar{u}) + k_{L \times U}^*(-\bar{m}, \bar{u}) \leq 0.$$

Moreover, since  $h_{L \times U}^*(m, u) \geq m(u)$  and  $k_{L \times U}^*(m, u) \geq m(u)$  for all  $(m, u) \in L \times U$ , by hypothesis

$$\begin{aligned}
(h_{l_0})_{L \times U}^*(m, u) + k_{L \times U}^*(-m, u) &= h_{L \times U}^*(m + l_0, u) - l_0(u) + k_{L \times U}^*(-m, u) \\
&\geq (m + l_0)(u) - l_0(u) - m(u) \\
&= m(u) + l_0(u) - l_0(u) - m(u) \\
&= 0.
\end{aligned}$$

Then one concludes

$$(h_{l_0})_{L \times U}^*(\bar{m}, \bar{u}) + k_{L \times U}^*(-\bar{m}, \bar{u}) = 0,$$

from which

$$h_{L \times U}^*(\bar{m} + l_0, \bar{u}) = (\bar{m} + l_0)(\bar{u}) \quad \text{and} \quad k_{L \times U}^*(-\bar{m}, \bar{u}) = -\bar{m}(\bar{u}),$$

so that

$$(\bar{u}, \bar{m} + l_0) \in G(\tilde{A}_h) \quad \text{and} \quad (\bar{u}, -\bar{m}) \in G(\tilde{B}_k).$$

Thus,

$$l_0 = l_0 + \bar{m} - \bar{m} \in \tilde{A}_h(\bar{u}) + \tilde{B}_k(\bar{u}),$$

i.e., as a consequence of the arbitrariness of  $l_0 \in L$ ,

$$\mathcal{R}(\tilde{A}_h + \tilde{B}_k) = L. \quad (3.1)$$

■

**Remark 3.2.** (a) The hypotheses of the previous theorem hold whenever  $A$  and  $B$  are maximal monotone operators of type (D) defined on a Banach space  $X$  and there exist  $h \in \mathcal{H}_A$  and  $k \in \mathcal{H}_B$  such that

$$\text{dom } h - \rho_2(\text{dom } k) = F \times X^*,$$

where  $\bigcup_{\lambda > 0} \lambda F$  is a closed subspace of  $X$ . Indeed, in this case [21, Corollary 4.3] guarantees the existence of a Fenchel functional for  $h_{w^*}$  and  $k \circ \varrho_2$ , for all  $w^* \in X^*$ . Then, identifying  $X$  with its image through the canonical inclusion in  $X^{**}$ , setting  $L := X^*$ ,  $U := X^{**}$  and taking Remark 3.1 into account, the previous theorem applies.

(b) Let  $X, Y$  be reflexive Banach spaces and  $t : X \rightarrow Y$  be an injective and continuous function. Define

$$L := \{f : X \rightarrow \mathbb{R} : \exists y^* \in Y^*, f = y^* \circ t\}$$

and, for all  $l \in L$ , set

$$\|l\|_L := \sup \left\{ \left| \frac{l(x)}{\|t(x)\|_Y} \right| : x \in X, t(x) \neq 0_Y \right\}.$$

It is easy to check that the definition of  $\|\cdot\|_L$  does not depend on the choice of  $y^*$  and that  $(L, \|\cdot\|_L)$  is a normed space. Setting  $U := L^*$ , then  $(t, \text{Id}) : X \times L \rightarrow Y \times L$  is a continuous and injective function,  $L \times L^*$  can be taken as a set of abstract linear functions on  $X \times L$  and the  $H_{L \times L^*}$ -convex functions will be called hidden convex functions [27]. Moreover, one can prove that the function  $\zeta : X \rightarrow L^*$  defined by

$$\zeta(x)(l) = l(x), \quad \forall l \in L,$$

for any  $x \in X$ , is injective. It does indeed take values in  $L^*$ , given that  $\zeta(x)$  is linear and

$$|\zeta(x)(l)| = |l(x)| \leq \|l\|_L \|t(x)\|_Y$$

for all  $x \in X$  and  $l \in L$ , and its injectivity is a direct consequence of that of  $t$ .

As a consequence of [7, Corollary 5.4], if  $A, B : X \rightarrow 2^L$  are maximal  $L$ -monotone operators and the abstract Fitzpatrick function of  $B$ ,  $\varphi_B : X \times L \rightarrow (-\infty, +\infty]$ , is continuous on  $X \times L$ , then, for all  $l_0 \in L$ , there exists a Fenchel functional  $(\bar{m}, \bar{m}^*) \in L \times L^*$  for  $(\varphi_A)_{l_0}$  and  $\varphi_B \circ \rho_2$ . Therefore, if the functions in  $L$  are odd, identifying  $X$  with  $\zeta(X)$  and taking Remark 3.1 into account, then the surjectivity condition (3.1) holds for the extensions  $\tilde{A}$  and  $\tilde{B}$ .

## 4 Some Results on Abstract Monotonicity

In this section, we present a generalization of Fenchel duality theorem in the framework of abstract convexity, and by using this theorem, we give criteria for maximal abstract monotonicity and obtain some other related results.

Let  $X$  be a set with an operation  $+$  having the following properties:

(A<sub>1</sub>)  $x + y \in X, \quad \forall x, y \in X.$

(A<sub>2</sub>) *There exists a unique element  $0 \in X$  such that  $0 + x = x + 0 = x, \quad \forall x \in X.$*

(A<sub>3</sub>) *For each  $x \in X$  there exists a unique element  $-x \in X$  such that  $x + (-x) = (-x) + x = 0.$*

Let  $L$  be a set of real valued additive abstract linear functions defined on  $X$ . Assume that  $L$  is equipped with the point-wise operation  $+$  of functions such that  $(L, +)$  satisfies the properties (A<sub>1</sub>), (A<sub>2</sub>) and (A<sub>3</sub>), where for each  $l \in L$ , define  $(-l)(x) := -l(x)$  for all  $x \in X$ , and define the function  $0 \in L$  by  $0(x) := 0$  for all  $x \in X$ . We consider the coupling function  $\langle \cdot, \cdot \rangle : X \times L \rightarrow \mathbb{R}$  defined by  $\langle x, l \rangle := l(x)$  for all  $x \in X$  and all  $l \in L$ .

**Remark 4.1.** *Note that for each  $l \in L$ , we have  $l(0) = 0$ . Moreover,  $l(-x) = -l(x)$  for all  $x \in X$  and all  $l \in L$ . Indeed, assume that  $l \in L$  and  $x \in X$  are arbitrary. Then*

$$0 = l(0) = l(x + (-x)) = l(x) + l(-x),$$

and hence  $l(-x) = -l(x)$  for all  $x \in X$  and all  $l \in L$ .

Let  $K \subseteq X \times L$  be any non-empty set such that  $K$  satisfies the properties (A<sub>1</sub>), (A<sub>2</sub>) and (A<sub>3</sub>), where  $-(x, l) := (-x, -l)$  and  $0 := (0, 0) \in K$ . Define  $L^* := \{(l, x) \in L \times X : (x, l) \in K\} \subseteq L \times X$ . It is clear that  $L^*$  satisfies the properties (A<sub>1</sub>), (A<sub>2</sub>) and (A<sub>3</sub>). Define the coupling function  $\langle \cdot, \cdot \rangle_* : K \times L^* \rightarrow \mathbb{R}$  by

$$\langle (x', l'), (l, x) \rangle_* := l(x') + l'(x), \quad \forall (x', l') \in K; \forall (l, x) \in L^*. \quad (4.1)$$

We can consider an element  $(l, x) \in L^*$  as the function defined on  $K$  by :

$$(l, x)(x', l') := \langle (x', l'), (l, x) \rangle_*, \quad \forall (x', l') \in K,$$

and an element  $(x, l) \in K$  as a function is defined on  $L^*$  by :

$$(x, l)(l', x') := \langle (x, l), (l', x') \rangle_*, \quad \forall (l', x') \in L^*.$$

Note that the coupling function  $\langle \cdot, \cdot \rangle_*$  is symmetric, that is

$$\langle (x', l'), (l, x) \rangle_* = \langle (x, l), (l', x') \rangle_*, \quad \text{for all } (x', l') \in K, \text{ and all } (l, x) \in L^*.$$

It is easy to check that  $L^*$  and  $K$  are sets of real valued abstract linear functions. Indeed, if there exist  $(l_0, x_0) \in L^*$  and  $c_0 \in \mathbb{R} \setminus \{0\}$  such that  $h_{(l_0, x_0), c_0} \in L^*$ , where  $h_{(l_0, x_0), c_0} := (l_0, x_0) - c_0$ , then  $h_{(l_0, x_0), c_0} = (l, x)$  for some  $(l, x) \in L^*$ . It follows that

$$l_0(x') + l'(x_0) - c_0 = l(x') + l'(x), \quad \forall (x', l') \in K. \quad (4.2)$$

Since  $(0,0) \in K$ , put  $x' = 0$  and  $l' = 0$  in (4.2). Thus, we have  $c_0 = 0$ . This is a contradiction, because  $c_0 \neq 0$ . Hence,  $h_{(l,x),c} \notin L^*$  for all  $(l,x) \in L^*$  and all  $c \in \mathbb{R} \setminus \{0\}$ . Therefore,  $L^*$  is a set of abstract linear functions. By a similar argument,  $K$  is also a set of abstract linear functions.

**Example 4.1.** Let  $X := \mathbb{Z}$  be the set of all integer numbers endowed with the ordinary addition. Then,  $X$  satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Now, for each  $x \in X$ , define the function  $l_x : X \rightarrow \mathbb{R}$  by  $l_x(y) := xy$  for all  $y \in X$ . Let  $L := \{l_x : x \in X\}$ . It is easy to check that  $L$  is a set of real valued additive abstract linear functions. Hence,  $L$  satisfies Remark 4.1; moreover, it satisfies properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ .

**Example 4.2.** Let  $X$  and  $L$  be as in Example 4.1. Define the function  $T : X \rightarrow L$  by  $T(x) := l_x$  for all  $x \in X$ . Then

$$G(T) = \{(x, l_x) \in X \times L : x \in X\},$$

and  $T$  is a maximal  $L$ -monotone operator (for more details see [5]).

Denote by  $\mathcal{P}(H_{L^*}) := \{h : K \rightarrow (-\infty, +\infty] : h \text{ is a proper } H_{L^*}\text{-convex function}\}$  the set of all proper  $H_{L^*}$ -convex functions defined on  $K$ . Define the transpose operator  $t : K \rightarrow L^*$  by  $t(x, l) := (l, x)$  for all  $(x, l) \in K$ .

**Remark 4.2.** Let  $S$  be any non-empty subset of  $K$ . Then the restriction to  $K$  of the  $L$ -Fitzpatrick function  $\varphi_S : X \times L \rightarrow \overline{\mathbb{R}}$  associated with  $S$ , defined by

$$\varphi_S(x, l) := \sup_{(x', l') \in S} [l(x') + l'(x) - l'(x')], \quad (4.3)$$

is an  $H_{L^*}$ -convex function. Indeed, by definition, we have

$$\begin{aligned} \varphi_S(x, l) &= \sup_{(x', l') \in S} [l(x') + l'(x) - l'(x')] \\ &= \sup_{(x', l') \in S} [\langle (x, l), (l', x') \rangle_* - l'(x')] \\ &= \sup\{\langle (x, l), (l', x') \rangle_* - c : ((l', x'), c) \in \text{supp}(\varphi_S, H_{L^*})\} \end{aligned}$$

for all  $(x, l) \in K$ , and hence the result follows.

**Remark 4.3.** Let  $h : K \rightarrow (-\infty, +\infty]$  be a function. Then,  $h_{L^*}^* \circ t$  is an  $H_{L^*}$ -convex function on  $K$ . To this end, let  $(x, l) \in K$  be arbitrary. Then

$$\begin{aligned} [h_{L^*}^* \circ t](x, l) &= h_{L^*}^*(l, x) \\ &= \sup_{(x', l') \in K} [\langle (x', l'), (l, x) \rangle_* - h(x', l')] \\ &= \sup_{(x', l') \in \text{dom } h} [\langle (x', l'), (l, x) \rangle_* - h(x', l')] \\ &= \sup_{(x', l') \in \text{dom } h} [\langle (x, l), (l', x') \rangle_* - h(x', l')] \\ &= \sup[\langle (x, l), (l', x') \rangle_* - c : ((l', x'), c) \in \text{supp}(h_{L^*}^* \circ t, H_{L^*})], \end{aligned}$$

and hence  $h_{L^*}^* \circ t$  is an  $H_{L^*}$ -convex function on  $K$ .

**Remark 4.4.** Note that for a maximal  $L$ -monotone subset  $S$  of  $K$ , it follows from Lemma 2.1 that

$$\varphi_S(x, l) = l(x) \iff (x, l) \in S. \quad (4.4)$$

Then, for arbitrary  $(x, l) \in K$ , we have

$$\begin{aligned} \varphi_S(x, l) &= \sup_{(x', l') \in S} [l(x') + l'(x) - l'(x')] \\ &= \sup_{(x', l') \in S} [l(x') + l'(x) - \varphi_S(x', l')] \\ &\leq \sup_{(x', l') \in K} [l(x') + l'(x) - \varphi_S(x', l')] \\ &= \sup_{(x', l') \in K} [\langle (x', l'), (l, x) \rangle_* - \varphi_S(x', l')] \\ &= [(\varphi_S)_{L^*}^* \circ t](x, l), \end{aligned}$$

that is

$$(\varphi_S)_{L^*}^* \circ t \geq \varphi_S. \quad (4.5)$$

In the sequel, we will use the following assumption.

**Assumption (D):** Assume that there exists a function  $\gamma \in \mathcal{P}(H_{L^*})$  such that

- (i)  $0 \leq \gamma < +\infty$  on  $K$ ,
- (ii)  $\gamma_{L^*}^* \circ t = \gamma$  on  $K$ ,
- (iii)  $\langle \cdot, \cdot \rangle + \gamma \geq 0$  on  $K$ ,
- (iv) If  $\gamma(x, l) = 0$ , then  $(x, l) = (0, 0)$ .
- (v)  $\gamma(-x, l) = \gamma(x, l)$  for all  $(x, l) \in K$ .
- (vi)  $\gamma(0, 0) = 0$ .

In view of Remark 4.3, the condition  $\gamma \in \mathcal{P}(H_{L^*})$  is automatically satisfied by any function  $\gamma : K \rightarrow (-\infty, +\infty]$  satisfying Assumption (D)(ii).

Notice that, in the case when  $X$  is a Banach space with the dual space  $X^*$  and  $L := X^*$ , the function  $\gamma$  defined by  $\gamma(x, x^*) := \frac{1}{2} (\|x\|^2 + \|x^*\|^2)$  satisfies the Assumption (D).

Define the function  $\delta : K \rightarrow \mathbb{R}$  by

$$\delta(x, l) := l(x) + \gamma(x, l), \quad \forall (x, l) \in K.$$

By Assumption (D)(iii), we have  $\delta(x, l) \geq 0$  for all  $(x, l) \in K$ .

The following two results have been proved in [5].

**Lemma 4.1.** [5, Lemma 4.2]. Let  $(x_0, l_0) \in K$  be arbitrary. Suppose that the Assumption (D) holds. Define the function  $k_{(x_0, l_0)} : K \rightarrow \mathbb{R}$  by

$$k_{(x_0, l_0)}(x, l) := -l(x) + \delta(x_0 - x, l_0 - l), \quad \forall (x, l) \in K.$$

Then,  $k_{(x_0, l_0)}$  is an  $H_{L^*}$ -convex function and

$$[(k_{(x_0, l_0)})_{L^*}^* \circ t](-x, -l) = k_{(x_0, l_0)}(x, l), \quad \forall (x, l) \in K.$$

**Lemma 4.2.** [5, Theorem 4.1]. Let  $f$  and  $g \in \mathcal{P}(H_{L^*})$  be such that  $\text{dom}f \cap \text{dom}g \neq \emptyset$ , and  $\text{supp}(\cdot, H_{L^*})$  be additive in  $f$  and  $g$ . Assume that  $f + g \geq \lambda$  on  $K$  ( $\lambda \in \mathbb{R}$ ). Then there exists  $(l, x) \in L^*$  such that

$$f_{L^*}^*(l, x) + g_{L^*}^*(-(l, x)) \leq \lambda.$$

In the following, we give some examples of a function which satisfies the Assumption (D).

Let  $X := \mathbb{Q}^n \subset \mathbb{R}^n$ , where  $\mathbb{Q}$  is the set of all rational numbers endowed with the ordinary addition. It is clear that  $X$  satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Let  $p : X \rightarrow \mathbb{R}$  be an additive function. Let  $a \in \mathbb{R}$  and  $y \in X$  be arbitrary. Define the function  $l_{y,a} : X \rightarrow \mathbb{R}$  by

$$l_{y,a}(x) := ap(x) + [y, x], \quad \forall x \in X,$$

where  $[\cdot, \cdot]$  is an inner product on  $\mathbb{R}^n$ . The function  $l_{y,a}$  ( $a \in \mathbb{R}$ ,  $y \in X$ ) has the following properties.

- (1)  $l_{y,a}$  is an additive function for each  $a \in \mathbb{R}$  and each  $y \in X$ .
- (2)  $l_{y_1, a_1} + l_{y_2, a_2} = l_{y_1+y_2, a_1+a_2}$  for each  $a_1, a_2 \in \mathbb{R}$  and each  $y_1, y_2 \in X$ .
- (3)  $\alpha l_{y,a} = l_{\alpha y, \alpha a}$  for each  $a \in \mathbb{R}$ ,  $\alpha \in \mathbb{Q}$  and each  $y \in X$ .

Now, let

$$L := \{l_{y,a} : a \in \mathbb{R}, y \in X\}.$$

In view of (1), (2) and (3), it is easy to check that  $L$  is a set of real valued additive abstract linear functions defined on  $X$  which satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define

$$K := \{(x, l_{y,p(y)}) \in X \times L : x, y \in X\} \subset X \times L,$$

and

$$L^* := \{(l_{y,p(y)}, x) \in L \times X : x, y \in X\} \subset L \times X.$$

Since  $p$  is additive and  $p(-x) = -p(x)$  for all  $x \in X$ , it is easy to see that  $K$  and  $L^*$  satisfy the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define the coupling function  $\langle \cdot, \cdot \rangle_*$  on  $K \times L^*$  as in (4.1). Now, we present the following example.

**Example 4.3.** Define the function  $\gamma : K \rightarrow (-\infty, +\infty]$  by

$$\gamma(x, l_{y,p(y)}) := \frac{1}{2}[p(x)^2 + p(y)^2 + \|x\|^2 + \|y\|^2], \quad \forall x, y \in X,$$

where  $\|\cdot\|^2 := [\cdot, \cdot]$ . Then,  $\gamma$  satisfies the Assumption (D). Indeed, it is clear that  $\gamma$  satisfies the Assumption (D)(i), (iii), (iv), (v) and (vi). We show that  $\gamma$  satisfies the Assumption

(D)(ii). Let  $(x, l_{y,p(y)}) \in K$  be arbitrary. Then, one has

$$\begin{aligned}
[\gamma^*_{L^*} \circ t](x, l_{y,p(y)}) &= \gamma^*_{L^*}(l_{y,p(y)}, x) = \sup_{(s, l_{t,p(t)}) \in K} \{ \langle (s, l_{t,p(t)}), (l_{y,p(y)}, x) \rangle_* - \gamma(s, l_{t,p(t)}) \} \\
&= \sup_{(s, l_{t,p(t)}) \in K} \{ l_{t,p(t)}(x) + l_{y,p(y)}(s) - \frac{1}{2}[p(s)^2 + p(t)^2 + \|s\|^2 + \|t\|^2] \} \\
&= \sup_{s, t \in X} \{ p(t)p(x) + [t, x] + p(s)p(y) + [s, y] \\
&= -\frac{1}{2}[p(s)^2 + p(t)^2 + \|s\|^2 + \|t\|^2] \} \\
&= \sup_{s, t \in X} \{ \frac{1}{2}[p(x)^2 + p(y)^2 + \|x\|^2 + \|y\|^2] \\
&\quad - \frac{1}{2}[(p(t) - p(x))^2 + (p(s) - p(y))^2 + \|t - x\|^2 + \|s - y\|^2] \} \quad (4.6) \\
&= \frac{1}{2}[p(x)^2 + p(y)^2 + \|x\|^2 + \|y\|^2] = \gamma(x, l_{y,p(y)}). \quad (4.7)
\end{aligned}$$

Moreover (4.7), together with Remark 4.3, implies that  $\gamma \in \mathcal{P}(H_{L^*})$ , and therefore  $\gamma$  satisfies the Assumption (D).

The following example is a special case of Example 4.3.

**Example 4.4.** Let  $X := \mathbb{Q}$  be the set of all rational numbers endowed with the ordinary addition. Then,  $X$  satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Now, for each  $x \in X$ , define the function  $l_x : X \rightarrow \mathbb{R}$  by  $l_x(y) := xy$  for all  $y \in X$ . Let  $L := \{l_x : x \in X\}$ . It is easy to check that  $L$  is a set of real valued additive abstract linear functions. Hence,  $L$  satisfies Remark 4.1; moreover, it also satisfies properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Also, we have  $\lambda l_x \in L$  for all  $l_x \in L$  and all  $\lambda \in \mathbb{Q}$ , since  $\lambda l_x = l_{\lambda x}$ . Note that  $-l_x = l_{-x}$  for all  $x \in X$ , and  $l_0 = 0$ . Let  $K := X \times L$  and  $L^* := L \times X$ . It is clear that  $K$  and  $L^*$  satisfy the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define the coupling function  $\langle \cdot, \cdot \rangle_*$  on  $K \times L^*$  as in (4.1), and the function  $\gamma : K \rightarrow (-\infty, +\infty]$  by

$$\gamma(x, l_y) := \frac{1}{2}(x^2 + y^2), \quad \forall x, y \in X.$$

Therefore, by a similar argument as in Example 4.3, one can show that  $\gamma$  satisfies the Assumption (D).

**Example 4.5.** Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathbb{R}^2$ . Define the function  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  by

$$T(x, y) := (x, y), \quad \forall (x, y) \in \mathbb{R}^2.$$

It is clear that  $T$  is a continuous linear operator and

$$\langle (x, y), T(x, y) \rangle = \langle (x, y), (x, y) \rangle \geq 0, \quad \forall (x, y) \in \mathbb{R}^2.$$

Also,  $G(T)$  is a linear subspace of  $\mathbb{R}^2 \times \mathbb{R}^2$ .

Let  $K := G(T)$  and  $L^* := G(T)$ . Define the coupling function  $\langle \cdot, \cdot \rangle_* : K \times L^* \rightarrow \mathbb{R}$  by

$$\langle ((x_1, y_1), (x_2, y_2)), ((x_3, y_3), (x_4, y_4)) \rangle_* := \langle (x_1, y_1), (x_3, y_3) \rangle + \langle (x_2, y_2), (x_4, y_4) \rangle$$

for all  $((x_1, y_1), (x_2, y_2)) \in K$  and all  $((x_3, y_3), (x_4, y_4)) \in L^*$ .

Define  $\gamma : K \rightarrow (-\infty, +\infty]$  by  $\gamma((x, y), (x, y)) := \langle (x, y), (x, y) \rangle$  for all  $((x, y), (x, y)) \in K$ . Therefore,  $\gamma$  satisfies the Assumption (D) (for more details see [5]).

**Example 4.6.** Let  $X$  be a reflexive real Banach space with dual space  $X^*$  and duality product  $\langle \cdot, \cdot \rangle : X \times X^* \rightarrow \mathbb{R}$ , defined by

$$\langle x, x^* \rangle := x^*(x), \quad \forall x \in X; \forall x^* \in X^*.$$

We norm  $X \times X^*$  by  $\|(x, x^*)\| := (\|x\|^2 + \|x^*\|^2)^{\frac{1}{2}}$ . Then the dual space of  $X \times X^*$  is  $X^* \times X$ , under the pairing

$$\langle (x, x^*), (y^*, y) \rangle_* := \langle x, y^* \rangle + \langle y, x^* \rangle,$$

for all  $(x, x^*) \in X \times X^*$  and all  $(y^*, y) \in X^* \times X$ . Further,  $\|(y^*, y)\| := (\|y\|^2 + \|y^*\|^2)^{\frac{1}{2}}$ .

Let  $K := X \times X^*$  and  $L^* := X^* \times X$ . Define  $\gamma : K \rightarrow (-\infty, +\infty]$  by  $\gamma(x, l) := \frac{1}{2}\|(x, l)\|^2$  for all  $(x, l) \in K$ . Then,  $\gamma$  satisfies the Assumption (D) (for more details see [5]).

**Remark 4.5.** Let  $X, L, K, L^*$  and  $\gamma$  be as in Example 4.4. Then,  $\partial_{L^*}\gamma(x, l_y) \neq \emptyset$  for each  $(x, l_y) \in K = X \times L$ . Indeed, let  $(x, l_y) \in K$  be arbitrary. Then we have

$$\frac{1}{2}(t - x)^2 + \frac{1}{2}(z - y)^2 \geq 0, \quad \forall t, z \in X,$$

and so

$$\frac{1}{2}(t^2 + z^2) + \frac{1}{2}(x^2 + y^2) \geq tx + zy, \quad \forall t, z \in X.$$

Thus we get

$$\frac{1}{2}(t^2 + z^2) - \frac{1}{2}(x^2 + y^2) \geq -x^2 - y^2 + tx + zy, \quad \forall t, z \in X.$$

This is equivalent, by the definition of  $\gamma$ , to the following inequality:

$$\gamma(t, l_z) - \gamma(x, l_y) \geq \langle (t, l_z), (l_x, y) \rangle_* - \langle (x, l_y), (l_x, y) \rangle_*, \quad \forall (t, l_z) \in K,$$

that is,  $(l_x, y) \in \partial_{L^*}\gamma(x, l_y)$ .

**Theorem 4.1.** Suppose that the Assumption (D) holds. Let  $h \in \mathcal{P}(H_{L^*})$  be such that

$$h(x, l) + \gamma(x, l) \geq 0, \quad \forall (x, l) \in K.$$

If  $\text{supp}(\cdot, H_{L^*})$  is additive in  $h$  and  $\gamma$ , then there exists  $(x_0, l_0) \in K$  such that

$$(h^*_{L^*} \circ t)(x_0, l_0) + \gamma(x_0, l_0) \leq 0.$$



*Proof:* This is an immediate consequence of Lemma 4.2 and the Assumption (D)(ii) and (v).  $\blacksquare$

**Remark 4.6.** Note that necessary and sufficient conditions for additivity of the mapping  $\text{supp}(\cdot, H_L)$  have been given in [7]. Also, in [7] it has given some examples of sets of abstract linear functions and functions  $f$  and  $g$  such that the mapping  $\text{supp}(\cdot, H_L)$  is additive in functions  $f$  and  $g$ .

In the following, we give an example such that the mapping  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_S$  and  $\delta$ .

**Example 4.7.** Let  $X, L, K, L^*$  and  $\gamma$  be as in Example 4.4. Now, let  $S := \{(x, l_x) \in K : x \in X\}$ . It is worth noting that it is not difficult to check that  $S$  is a maximal  $L$ -monotone subset of  $K$ . We have  $\varphi_S(x, l_y) = \frac{1}{4}(x + y)^2$  for all  $(x, l_y) \in K$ . Indeed, let  $(x, l_y) \in K$  be arbitrary. Consider

$$\begin{aligned} \varphi_S(x, l_y) &= \sup_{(t, l_t) \in S} [l_y(t) + l_t(x) - l_t(t)] \\ &= \sup_{t \in X} [ty + tx - t^2] \\ &= \sup_{t \in X} \left[ \frac{1}{4}(x + y)^2 - \left(t - \frac{1}{2}(x + y)\right)^2 \right] \\ &= \frac{1}{4}(x + y)^2, \quad \forall (x, l_y) \in K. \end{aligned}$$

Therefore we have

$$\begin{aligned} \varphi_S(x, l_y) &= \frac{1}{4}(x + y)^2 \\ &= \frac{1}{4}(x^2 + y^2) + \frac{1}{2}xy \\ &= \frac{1}{2}[\gamma(x, l_y) + l_y(x)] \\ &= \frac{1}{2}\delta(x, l_y), \quad \forall (x, l_y) \in K, \end{aligned}$$

and hence

$$\delta(x, l_y) = 2\varphi_S(x, l_y) = \frac{1}{2}(x + y)^2, \quad \forall (x, l_y) \in K.$$

Note that  $\delta = 2\varphi_S$  on  $K$ .

Finally, We show that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_S$  and  $\delta$ . It is easy to check that

$$\text{supp}(\varphi_S, H_{L^*}) + \text{supp}(\delta, H_{L^*}) \subseteq \text{supp}(\varphi_S + \delta, H_{L^*}).$$

For the converse inclusion, suppose that  $f \in H_{L^*}$  and  $f \leq \varphi_S + \delta$ . Let  $f_1 := \frac{1}{3}f$  on  $K$ , and  $f_2 := \frac{2}{3}f$  on  $K$ . Since  $f \in H_{L^*}$  and  $\lambda(l_y, x) \in L^* = L \times X$  for all  $(l_y, x) \in L^*$  and all  $\lambda \in \mathbb{Q}$ , it is easy to check that  $f_1, f_2 \in H_{L^*}$ ,  $f_1 \leq \varphi_S$  and  $f_2 \leq \delta$ . Thus, we have  $f_1 \in \text{supp}(\varphi_S, H_{L^*})$  and  $f_2 \in \text{supp}(\delta, H_{L^*})$ . Also, we have  $f = f_1 + f_2$ , and hence

$$\text{supp}(\varphi_S, H_{L^*}) + \text{supp}(\delta, H_{L^*}) = \text{supp}(\varphi_S + \delta, H_{L^*}).$$

Now, let  $S$  be a non-empty subset of  $K$ . Define the function  $\psi_S : K \rightarrow (-\infty, +\infty]$  by

$$\psi_S(x, l) := \sup_{(x', l') \in S} [l(x') + l'(x) - l'(x') - l(x)], \quad \forall (x, l) \in K.$$

It is clear that

$$\varphi_S(x, l) = \psi_S(x, l) + l(x), \quad \forall (x, l) \in K. \quad (4.8)$$

**Theorem 4.2.** *Suppose that the Assumption (D) holds. Let  $S$  be a non-empty subset of  $K$ , and  $\text{supp}(\cdot, H_{L^*})$  be additive in  $\varphi_S$  and  $\gamma$ . If  $S$  is maximal  $L$ -monotone, then there exists  $(x_0, l_0) \in S$  such that  $\delta(x_0, l_0) = 0$ .*

*Proof:* By maximality of  $S$  we conclude that  $\psi_S(x, l) \geq 0$  for all  $(x, l) \in K$ . Therefore, by Assumption (D) and (4.8) we deduce that

$$\varphi_S(x, l) + \gamma(x, l) = \psi_S(x, l) + l(x) + \gamma(x, l) = \psi_S(x, l) + \delta(x, l) \geq 0, \quad \forall (x, l) \in K.$$

Since by Remark 4.2  $\varphi_S$  is an  $H_{L^*}$ -convex function, it follows from Theorem 4.1 that there exists  $(x_0, l_0) \in K$  such that

$$[(\varphi_S)^*_{L^*} \circ t](x_0, l_0) + \gamma(x_0, l_0) \leq 0. \quad (4.9)$$

But, by Remark 4.4 we have  $(\varphi_S)^*_{L^*} \circ t \geq \varphi_S$  on  $K$ , and hence it follows from (4.9) that

$$\varphi_S(x_0, l_0) + \gamma(x_0, l_0) \leq 0.$$

This implies that

$$\psi_S(x_0, l_0) + \delta(x_0, l_0) = \varphi_S(x_0, l_0) + \gamma(x_0, l_0) \leq 0.$$

This, together with the definition of  $\psi_S$ , implies that

$$l_0(x) + l(x_0) - l(x) - l_0(x_0) + \delta(x_0, l_0) \leq 0, \quad \forall (x, l) \in S.$$

Thus, we have

$$l(x) - l_0(x) - l(x_0) + l_0(x_0) \geq \delta(x_0, l_0), \quad \forall (x, l) \in S, \quad (4.10)$$

and so

$$l(x) - l_0(x) - l(x_0) + l_0(x_0) \geq 0, \quad \forall (x, l) \in S, \quad (4.11)$$

because  $\delta(x_0, l_0) \geq 0$ . Since  $S$  is a maximal  $L$ -monotone set, it follows from (4.11) that  $(x_0, l_0) \in S$ . Therefore, in view of (4.10) we conclude that  $\delta(x_0, l_0) \leq 0$ , and hence  $\delta(x_0, l_0) = 0$ , which completes the proof.  $\blacksquare$

In the following, we give an example of a non-empty set  $S$  such that the mapping  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_S$  and  $\gamma$ .

**Example 4.8.** Let  $X$ ,  $L$ ,  $K$  and  $L^*$  be as in Example 4.4. Let

$$S := \{(1, l_1)\},$$

and let  $g \in \mathcal{P}(H_{L^*})$  be arbitrary. We show that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_S$  and  $g$ . Hence, in particular,  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_S$  and  $\gamma$ . We have

$$\varphi_S(x, l_y) = l_y(1) + l_1(x) - l_1(1) = x + y - 1, \quad \forall (x, l_y) \in K.$$

Therefore, one has

$$\varphi_S(x, l_y) = \langle (x, l_y), (l_1, 1) \rangle_* - 1, \quad \forall (x, l_y) \in K.$$

It is easy to see that

$$\text{supp}(\varphi_S, H_{L^*}) + \text{supp}(g, H_{L^*}) \subseteq \text{supp}(\varphi_S + g, H_{L^*}).$$

For the converse inclusion, let  $f \in H_{L^*}$  and  $f \leq \varphi_S + g$ . Let  $f_1 := \varphi_S$ . Then,  $f_1 \in H_{L^*}$ , and so  $f_1 \in \text{supp}(\varphi_S, H_{L^*})$ . Define

$$f_2(x, l_y) := f(x, l_y) + \langle (x, l_y), (l_{-1}, -1) \rangle_* + 1, \quad \forall (x, l_y) \in K.$$

By using the properties of  $L^*$ , and since  $f \in H_{L^*}$  and  $f \leq \varphi_S + g$ , it is easy to show that  $f_2 \in H_{L^*}$  and  $f_2 \leq g$ . Thus,  $f_2 \in \text{supp}(g, H_{L^*})$ . Also, we have  $f = f_1 + f_2$ , and hence the proof is complete.

Let  $S$  be a subset of  $K$  and  $(x_0, l_0) \in K$ . Define the translation of  $S$  by

$$S^t := S - (x_0, l_0) := \{(x, l) - (x_0, l_0) : (x, l) \in S\}.$$

**Lemma 4.3.** If  $S$  is a maximal  $L$ -monotone subset of  $K$  and  $(x, l) \in K$ , then  $S^t := S - (x, l)$  is a maximal  $L$ -monotone subset of  $K$ .

*Proof:* Since  $S$  is  $L$ -monotone, it is easy to check that  $S^t$  is an  $L$ -monotone set. Now, we show that  $S^t$  is maximal. To this end, let  $(x_0, l_0) \in K$  be arbitrary and

$$l_0(x_0) - l_0(x') - l'(x_0) + l'(x') \geq 0, \quad \forall (x', l') \in S^t. \quad (4.12)$$

Let  $(x_1, l_1) \in S$  be arbitrary and let  $x' = x_1 - x$  and  $l' = l_1 - l$ . Then we have  $(x', l') = (x_1, l_1) - (x, l) \in S^t$ , and therefore it follows from (4.12) that

$$l_0(x_0) - l_0(x_1 - x) - (l_1 - l)(x_0) + (l_1 - l)(x_1 - x) \geq 0, \quad \forall (x_1, l_1) \in S,$$

and so

$$l_1(x_1) - l_1(x + x_0) - (l + l_0)(x_1) + (l + l_0)(x + x_0) \geq 0, \quad \forall (x_1, l_1) \in S. \quad (4.13)$$

Since  $S$  is maximal  $L$ -monotone, in view of (4.13) we conclude that  $(x + x_0, l + l_0) \in S$ , and hence  $(x_0, l_0) \in S^t$ , which completes the proof.  $\blacksquare$

The following theorem gives criteria for maximal abstract monotonicity, which is a generalization of [29, Theorem 10.3].

**Theorem 4.3.** *Suppose that the Assumption (D) holds. Let  $S$  be an  $L$ -monotone subset of  $K$ . Then the following assertions are true:*

(1) *Let  $\text{supp}(\cdot, H_{L^*})$  be additive in  $\varphi_{S^t}$  and  $\gamma$ , where  $S^t$  is a translation of  $S$ . If  $S$  is a maximal  $L$ -monotone set, then for each  $(x, l) \in K$  there exists  $(x_0, l_0) \in S$  such that  $\delta(x_0 - x, l_0 - l) = 0$ .*

(2) *If for each  $(x, l) \in K$  there exists  $(x_0, l_0) \in S$  such that  $\delta(x_0 - x, l_0 - l) = 0$ , then  $S$  is a maximal  $L$ -monotone set.*

*Proof:* (1). Assume that  $S$  is a maximal  $L$ -monotone set and  $(x, l) \in K$  is given. Then, by Lemma 4.3,  $S^t := S - (x, l)$  is a maximal  $L$ -monotone set. Therefore, since by the hypothesis  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{S^t}$  and  $\gamma$ , in view of Theorem 4.2 there exists  $(x_1, l_1) \in S^t$  such that

$$\delta(x_1, l_1) = 0. \quad (4.14)$$

Since  $(x_1, l_1) \in S^t$ , it follows that there exists  $(x_0, l_0) \in S$  such that  $x_1 = x_0 - x$  and  $l_1 = l_0 - l$ . Thus, the result follows from (4.14).

(2). We have by the hypothesis  $S$  is an  $L$ -monotone set. Now, we show that  $S$  is a maximal  $L$ -monotone subset of  $K$ . To this end, let  $(x_0, l_0) \in K$  and

$$l_0(x_0) - l_0(x) - l(x_0) + l(x) \geq 0, \quad \forall (x, l) \in S. \quad (4.15)$$

By the hypothesis for this  $(x_0, l_0) \in K$  there exists  $(x_1, l_1) \in S$  such that  $\delta(x_0 - x_1, l_0 - l_1) = \delta(x_1 - x_0, l_1 - l_0) = 0$ . Therefore, by the definition of  $\delta$  and (4.15) we obtain

$$\begin{aligned} 0 &= \delta(x_0 - x_1, l_0 - l_1) = (l_0 - l_1)(x_0 - x_1) + \gamma(x_0 - x_1, l_0 - l_1) \\ &= l_0(x_0) + l_0(-x_1) - l_1(x_0) - l_1(-x_1) + \gamma(x_0 - x_1, l_0 - l_1) \\ &= l_0(x_0) - l_0(x_1) - l_1(x_0) + l_1(x_1) + \gamma(x_0 - x_1, l_0 - l_1) \geq \gamma(x_0 - x_1, l_0 - l_1) \geq 0. \end{aligned}$$

This implies that  $\gamma(x_0 - x_1, l_0 - l_1) = 0$ , and so by Assumption (D)(iv), we deduce that  $x_0 = x_1$  and  $l_0 = l_1$ . Consequently,  $(x_0, l_0) = (x_1, l_1) \in S$ . Hence,  $S$  is a maximal  $L$ -monotone subset of  $K$ . This completes the proof.  $\blacksquare$

In the following, we give an example of a non-empty set  $S$  such that the mapping  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{S^t}$  and  $\gamma$ , where  $S^t$  is any translation of  $S$ .

**Example 4.9.** *Let  $X$  and  $L$  be as in Example 4.4. Define the function  $T : X \rightarrow L$  by  $T(x) := l_x$  for all  $x \in X$ . Then*

$$G(T) = \{(x, l_x) \in X \times L : x \in X\},$$

*and  $T$  is a maximal  $L$ -monotone operator. Let  $K := G(T) \subseteq X \times L$ , and*

$$L^* := \{(l_x, x) \in L \times X : x \in X\} \subseteq L \times X.$$

*It is easy to check that  $K$  and  $L^*$  satisfy the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define the coupling function  $\langle \cdot, \cdot \rangle_*$  on  $K \times L^*$  as in (4.1). Assume that  $\gamma$  is defined on  $K$  as in*

*Example 4.4.* Thus, we have  $\gamma(x, l_x) = x^2 = l_x(x)$  for all  $x \in X$ . Moreover,  $\gamma$  satisfies the Assumption (D). Let  $S := G(T)$ . Since  $T$  is a maximal  $L$ -monotone operator, we deduce that  $S$  is a maximal  $L$ -monotone set. Thus, it follows from Lemma 2.1 that

$$\varphi_S(x, l_x) = l_x(x) = x^2 = \gamma(x, l_x), \quad \forall x \in X.$$

Now, let  $(x_0, l_{x_0}) \in K$  be arbitrary and  $S^t := S - (x_0, l_{x_0})$ . Therefore, we have

$$\begin{aligned} \varphi_{S^t}(x, l_x) &= \sup_{(y, l_y) \in S^t} [l_x(y) + l_y(x) - l_y(y)] \\ &= \sup_{(y, l_y) \in S^t} [2xy - y^2] \\ &= \sup_{s \in X} [2x(s - x_0) - (s - x_0)^2] \\ &= \sup_{s \in X} [x^2 - (x - (s - x_0))^2] \\ &= x^2 = \varphi_S(x, l_x), \quad \forall x \in X. \end{aligned}$$

Hence,  $\varphi_{S^t} = \varphi_S = \gamma$  on  $K$ .

Now, we show that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{S^t}$  and  $\gamma$ . It is clear that

$$\text{supp}(\varphi_{S^t}, H_{L^*}) + \text{supp}(\gamma, H_{L^*}) \subseteq \text{supp}(\varphi_{S^t} + \gamma, H_{L^*}).$$

Let  $f \in \text{supp}(\varphi_{S^t} + \gamma, H_{L^*})$  be arbitrary. Then,  $f \in H_{L^*}$  and  $f \leq \varphi_{S^t} + \gamma$ . Let  $f_1 = f_2 := \frac{1}{2}f$ . Therefore, by using the properties of  $L^*$  and since  $f \in H_{L^*}$ , we conclude that  $f_1, f_2 \in H_{L^*}$ ,  $f_1 \leq \varphi_{S^t}$  and  $f_2 \leq \gamma$ , that is,  $f_1 \in \text{supp}(\varphi_{S^t}, H_{L^*})$  and  $f_2 \in \text{supp}(\gamma, H_{L^*})$ . Since  $f = f_1 + f_2$ , it follows that

$$f \in \text{supp}(\varphi_{S^t}, H_{L^*}) + \text{supp}(\gamma, H_{L^*}).$$

## 5 Necessary and Sufficient Conditions for Maximal Abstract Monotonicity

In this section, we give a generalization of [11, Theorem 2.1] and, by using this generalization, we obtain necessary and sufficient conditions for maximality of abstract monotone operators.

Let  $X, L, K$  and  $L^*$  be as in section 4, which satisfy the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define the coupling function  $\langle \cdot, \cdot \rangle_*$  on  $K \times L^*$  as in (4.1). Suppose that the Assumption (D) holds. Let  $(x_0, l_0) \in K$  be arbitrary. Define the function  $k_{(x_0, l_0)} : K \rightarrow \mathbb{R}$  by

$$k_{(x_0, l_0)}(x, l) := -l(x) + \delta(x_0 - x, l_0 - l), \quad \forall (x, l) \in K. \quad (5.1)$$

In view of Lemma 4.1, we have  $k_{(x_0, l_0)} \in \mathcal{P}(H_{L^*})$  and

$$[(k_{(x_0, l_0)})_{L^*}^* \circ t](-x, -l) = k_{(x_0, l_0)}(x, l), \quad \forall (x, l) \in K. \quad (5.2)$$

Also, we define the function  $\alpha : K \rightarrow \mathbb{R}$  by

$$\alpha(x, l) := \gamma(x, -l), \quad \forall (x, l) \in K. \quad (5.3)$$

It is easy to see, by Assumption (D)(i), (ii) and (v), that

$$\alpha \in \mathcal{P}(H_{L^*}), \quad \alpha_{L^*}^* \circ t = \alpha, \quad \text{and} \quad \alpha(-x, -l) = \alpha(x, l), \quad \forall (x, l) \in K. \quad (5.4)$$

In the following, we give an assumption.

**Assumption (C):** Assume that the infimal convolution  $\alpha \oplus k_{(x_0, l_0)}$  of  $\alpha$  with  $k_{(x_0, l_0)}$  is exact and an  $H_{L^*}$ -convex function for each  $(x_0, l_0) \in K$ .

The following example shows that for a function  $\alpha$  and a function  $k_{(x_0, l_0)}$  the infimal convolution  $\alpha \oplus k_{(x_0, l_0)}$  of  $\alpha$  with  $k_{(x_0, l_0)}$  is exact and an  $H_{L^*}$ -convex function for each  $(x_0, l_0) \in K$ .

**Example 5.1.** Let  $X, L, K, L^*$  and  $\gamma$  be as in Example 4.4. Let  $(x_0, l_{y_0}) \in K$  be arbitrary. By (5.3), one has  $\alpha(x, l_y) = \gamma(x, -l_y) = \frac{1}{2}(x^2 + y^2)$  for all  $(x, l_y) \in K$ . Also, in view of (5.1) and Example 4.7, we have

$$k_{(x_0, l_{y_0})}(x, l_y) = -l_y(x) + \delta(x_0 - x, l_{y_0} - l_y) = -xy + \frac{1}{2}[(x_0 - x) + (y_0 - y)]^2, \quad \forall (x, l_y) \in K.$$

Therefore, one has

$$\begin{aligned} (\alpha \oplus k_{(x_0, l_{y_0})})(x, l_y) &= \inf_{(x_1, l_{y_1}) + (x_2, l_{y_2}) = (x, l_y)} [\alpha(x_1, l_{y_1}) + k_{(x_0, l_{y_0})}(x_2, l_{y_2})] \\ &= \inf_{(x_1, l_{y_1}) + (x_2, l_{y_2}) = (x, l_y)} \left\{ \frac{1}{2}(x_1^2 + y_1^2) - x_2 y_2 \right. \\ &\quad \left. + \frac{1}{2}[(x_0 - x_2) + (y_0 - y_2)]^2 \right\} \\ &= \inf_{x_1, y_1 \in X} \left\{ \frac{1}{2}(x_1^2 + y_1^2) - (x - x_1)(y - y_1) \right. \\ &\quad \left. + \frac{1}{2}[x_0 + y_0 - x + x_1 - y + y_1]^2 \right\}. \end{aligned}$$

It is not difficult to see that the above infimum is achieved at  $x_1 = \frac{1}{2}(x - x_0 - y_0)$ ,  $y_1 = \frac{1}{2}(y - x_0 - y_0)$  for each  $(x, l_y) \in K$ . Moreover, we have

$$(\alpha \oplus k_{(x_0, l_{y_0})})(x, l_y) = \frac{1}{4}(x^2 + y^2) - \frac{1}{2}(x + y)(x_0 + y_0), \quad \forall (x, l_y) \in K.$$

That is, the infimal convolution  $\alpha \oplus k_{(x_0, l_{y_0})}$  is exact for each  $(x_0, l_{y_0}) \in K$ . Also, by using the properties of  $L^*$ , one has

$$(\alpha \oplus k_{(x_0, l_{y_0})})(x, l_y) = \frac{1}{2}\gamma(x, l_y) + \langle (x, l_y), (l_{t_0}, t_0) \rangle_*, \quad \forall (x, l_y) \in K,$$

where  $t_0 = -\frac{1}{2}(x_0 + y_0) \in X$ . Therefore the infimal convolution  $\alpha \oplus k_{(x_0, l_{y_0})}$  is an  $H_{L^*}$ -convex function for each  $(x_0, l_{y_0}) \in K$ .

**Theorem 5.1.** *Suppose that the Assumptions (C) and (D) hold. Then the mapping  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\alpha$  and  $k_{(x_0, l_0)}$  for each  $(x_0, l_0) \in K$ .*

*Proof:* It is enough to show that

$$(\alpha + k_{(x_0, l_0)})_{L^*}^* = \alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*$$

and the infimal convolution  $\alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*$  is exact. Then by Lemma 2.3 the result follows. To this end, let  $(x_0, l_0) \in K$  be fixed and arbitrary. Then, by (5.2) and (5.4), we have

$$\begin{aligned} [(\alpha \oplus k_{(x_0, l_0)})_{L^*}^* \circ t](x, l) &= (\alpha \oplus k_{(x_0, l_0)})_{L^*}^*(l, x) \\ &= \sup_{(x', l') \in K} [\langle (x', l'), (l, x) \rangle_* - (\alpha \oplus k_{(x_0, l_0)})(x', l')] \\ &= \sup_{(x', l') \in K} \{ \langle (x', l'), (l, x) \rangle_* \\ &\quad - \inf_{(x_1, l_1) + (x_2, l_2) = (x', l')} [\alpha(x_1, l_1) + k_{(x_0, l_0)}(x_2, l_2)] \} \\ &= \sup_{(x_1, l_1) \in K} [\langle (x_1, l_1), (l, x) \rangle_* - \alpha(x_1, l_1)] \\ &\quad + \sup_{(x_2, l_2) \in K} [\langle (x_2, l_2), (l, x) \rangle_* - k_{(x_0, l_0)}(x_2, l_2)] \\ &= [\alpha_{L^*}^* \circ t](x, l) + [(k_{(x_0, l_0)})_{L^*}^* \circ t](x, l) \\ &= \alpha(x, l) + k_{(x_0, l_0)}(-x, -l) = \alpha(-x, -l) + k_{(x_0, l_0)}(-x, -l) \\ &= (\alpha + k_{(x_0, l_0)})(-x, -l), \quad \forall (x, l) \in K. \end{aligned} \tag{5.5}$$

On the other hand, by (5.2) and (5.4), one has

$$\begin{aligned} [(\alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*) \circ t](x, l) &= (\alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*)(l, x) \\ &= \inf_{(l_1, x_1) + (l_2, x_2) = (l, x)} [\alpha_{L^*}^*(l_1, x_1) + (k_{(x_0, l_0)})_{L^*}^*(l_2, x_2)] \\ &= \inf_{(x_1, l_1) + (x_2, l_2) = (x, l)} [[\alpha_{L^*}^* \circ t](x_1, l_1) + [(k_{(x_0, l_0)})_{L^*}^* \circ t](x_2, l_2)] \\ &= \inf_{(x_1, l_1) + (x_2, l_2) = (x, l)} [\alpha(x_1, l_1) + k_{(x_0, l_0)}(-x_2, -l_2)] \\ &= \inf_{(x_1, l_1) + (x_2, l_2) = (x, l)} [\alpha(-x_1, -l_1) + k_{(x_0, l_0)}(-x_2, -l_2)] \\ &= \inf_{(x_1, l_1) + (x_2, l_2) = (-x, -l)} [\alpha(x_1, l_1) + k_{(x_0, l_0)}(x_2, l_2)] \\ &= (\alpha \oplus k_{(x_0, l_0)})(-x, -l), \quad \forall (x, l) \in K. \end{aligned} \tag{5.6}$$

Since, by Assumption (C), the infimal convolution  $\alpha \oplus k_{(x_0, l_0)}$  is exact, it follows from (5.2), (5.4) and (5.6) that the infimal convolution  $\alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*$  is exact for each  $(x_0, l_0) \in K$ .

Finally, by (5.5) we have

$$\begin{aligned}
[(\alpha + k_{(x_0, l_0)})_{L^*}^* \circ t](x, l) &= (\alpha + k_{(x_0, l_0)})_{L^*}^*(l, x) \\
&= \sup_{(x', l') \in K} [\langle (x', l'), (l, x) \rangle_* - (\alpha + k_{(x_0, l_0)})(x', l')] \\
&= \sup_{(l', x') \in L^*} [\langle (-x, -l), (-l', -x') \rangle_* - (\alpha \oplus k_{(x_0, l_0)})_{L^*}^*(-l', -x')] \\
&= (\alpha \oplus k_{(x_0, l_0)})_{L^*, K}^{**}(-x, -l), \quad \forall (x, l) \in K. \tag{5.7}
\end{aligned}$$

Since, by Assumption (C), the infimal convolution  $\alpha \oplus k_{(x_0, l_0)}$  is an  $H_{L^*}$ -convex function, it follows from Lemma 2.2 that  $(\alpha \oplus k_{(x_0, l_0)})_{L^*, K}^{**} = \alpha \oplus k_{(x_0, l_0)}$ . This, together with (5.6) and (5.7) implies that

$$(\alpha + k_{(x_0, l_0)})_{L^*}^* \circ t = [\alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*] \circ t,$$

that is,  $(\alpha + k_{(x_0, l_0)})_{L^*}^* = \alpha_{L^*}^* \oplus (k_{(x_0, l_0)})_{L^*}^*$ , which completes the proof.  $\blacksquare$

In the sequel, let  $X' \subseteq X$  be any non-empty set such that  $X'$  satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Also, let  $L' \subseteq L$  be any non-empty set such that  $L'$  satisfies the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Define  $K := X' \times L'$  and  $L^* := L' \times X'$ . It is clear that  $K$  and  $L^*$  satisfy the properties  $(A_1)$ ,  $(A_2)$  and  $(A_3)$ . Now, suppose that the Assumption (D) holds and define  $\mathcal{D}$  as follows:

$$\mathcal{D} := \{(x, l) \in K : \delta(x, -l) = 0\}. \tag{5.8}$$

It is worth noting that under the hypotheses of Theorem 4.2, for each maximal  $L'$ -monotone subset  $S$  of  $K$  we have  $\mathcal{D} \neq \emptyset$ .

Define the set valued mapping  $d : X' \rightarrow 2^{L'}$  by

$$d(x) := \{l \in L' : \delta(x, -l) = 0\}, \quad \forall x \in X'. \tag{5.9}$$

It is clear that  $G(d) = \mathcal{D}$ . Also, define the set valued mapping  $-d : X' \rightarrow 2^{L'}$  by  $(-d)(x) := -d(x)$  for each  $x \in X'$ . Therefore, we have

$$(-d)(x) = \{l \in L' : \delta(x, l) = 0\}, \quad \forall x \in X'. \tag{5.10}$$

It is easy to check that  $d(-x) = -d(x)$  for each  $x \in X'$ .

In the following, we give an example of a function  $d$  such that  $\text{dom}(d) = X$ . Moreover,  $d$  is single-valued.

**Example 5.2.** Let  $X, L, K, L^*$  and  $\gamma$  be as in Example 4.4. Now,  $d : X \rightarrow 2^L$  is defined by

$$d(x) := \{l_y \in L : \delta(x, -l_y) = 0\}, \quad \forall x \in X.$$

(Note that  $-l_y = l_{-y}$ .) Therefore, for each  $x \in X$ , we have

$$\begin{aligned}
d(x) &= \{l_y \in L : \delta(x, -l_y) = 0\} = \{l_y \in L : \gamma(x, -l_y) = l_y(x)\} \\
&= \{l_y \in L : \frac{1}{2}(x^2 + y^2) = xy\} = \{l_y \in L : (x - y)^2 = 0\} = \{l_y \in L : x = y\} = \{l_x\}.
\end{aligned}$$



This implies that  $\text{dom}(d) = X$ , and also,  $d$  is single-valued. Moreover, one has  $d^{-1}(l_y) = \{y\}$  for each  $y \in X$ , that is,  $\text{dom}(d^{-1}) = L$  and  $d^{-1}$  is single-valued.

**Proposition 5.1.** *Suppose that the Assumption (D) holds. Then the set valued mapping  $d$  is an  $L'$ -monotone operator.*

*Proof:* Assume that  $(x, l)$ , and  $(x_0, l_0) \in G(d)$  are arbitrary. Then, by the definition of  $d$ , we have  $\delta(x, -l) = 0$  and  $\delta(x_0, -l_0) = 0$ . This implies that

$$\gamma(x, -l) = l(x) \quad \text{and} \quad \gamma(x_0, -l_0) = l_0(x_0). \quad (5.11)$$

Note that by Assumption (D)(v) we get

$$\gamma(-x, l) = \gamma(x, -l), \quad \forall (x, l) \in K. \quad (5.12)$$

In view of Fenchel-Young inequality and the Assumption (D)(ii) we conclude that

$$\gamma(x_1, l_1) + \gamma(x_2, l_2) \geq l_1(x_2) + l_2(x_1), \quad \forall (x_1, l_1), (x_2, l_2) \in K. \quad (5.13)$$

By putting  $(x_1, l_1) = (-x, l)$  and  $(x_2, l_2) = (x_0, -l_0)$  in (5.13), we obtain

$$\gamma(-x, l) + \gamma(x_0, -l_0) \geq l(x_0) + l_0(x).$$

This, together with (5.11) and (5.12) implies that

$$l(x) - l(x_0) - l_0(x) + l_0(x_0) \geq 0,$$

which completes the proof. ■

**Proposition 5.2.** *Suppose that the Assumptions (C) and (D) hold. Then the set valued mapping  $d$  is a maximal  $L'$ -monotone operator.*

*Proof:* By Proposition 5.1,  $d$  is an  $L'$ -monotone operator. For maximality of  $d$ , suppose that  $(x_0, l_0) \in K$  and

$$l'(x') - l'(x_0) - l_0(x') + l_0(x_0) \geq 0, \quad \forall (x', l') \in G(d). \quad (5.14)$$

Define the function  $k_{(x_0, l_0)}$  on  $K$  as in (5.1), and the function  $\alpha$  on  $K$  as in (5.3). Since  $\delta(x', l') \geq 0$  for all  $(x', l') \in K$ , we have

$$\begin{aligned} \alpha(x, l) + k_{(x_0, l_0)}(x, l) &= \gamma(x, -l) - l(x) + \delta(x_0 - x, l_0 - l) = \delta(x, -l) + \delta(x_0 - x, l_0 - l) \\ &\geq 0, \quad \forall (x, l) \in K. \end{aligned}$$

Therefore, it follows from Lemma 4.2 and Theorem 5.1 that there exists  $(x_1, l_1) \in K$  such that

$$[\alpha^*_{L^*} \circ t](x_1, l_1) + [(k_{(x_0, l_0)})^*_{L^*} \circ t](-x_1, -l_1) \leq 0. \quad (5.15)$$

By using the Assumption (D)(ii) and (v) we have  $[\alpha^*_{L^*} \circ t](x_1, l_1) = \gamma(x_1, -l_1)$ . Since, by (5.2), one has

$$[(k_{(x_0, l_0)})^*_{L^*} \circ t](-x_1, -l_1) = k_{(x_0, l_0)}(x_1, l_1),$$

we conclude from (5.15) that

$$\gamma(x_1, -l_1) - l_1(x_1) + \delta(x_0 - x_1, l_0 - l_1) \leq 0. \quad (5.16)$$

This implies that

$$0 \leq \delta(x_1, -l_1) = \gamma(x_1, -l_1) - l_1(x_1) \leq \gamma(x_1, -l_1) - l_1(x_1) + \delta(x_0 - x_1, l_0 - l_1) \leq 0,$$

and hence  $\delta(x_1, -l_1) = 0$ . That is,  $(x_1, l_1) \in G(d)$ . Therefore, it follows from (5.14) that

$$l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0) \geq 0. \quad (5.17)$$

On the other hand, since  $\delta(x_1, -l_1) = 0$ , we conclude from (5.16) that  $\delta(x_0 - x_1, l_0 - l_1) = 0$ . Thus, by using (5.17) and the definition of  $\delta$  we have

$$\begin{aligned} 0 &= \delta(x_0 - x_1, l_0 - l_1) \\ &= (l_0 - l_1)(x_0 - x_1) + \gamma(x_0 - x_1, l_0 - l_1) \\ &= l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0) + \gamma(x_0 - x_1, l_0 - l_1) \\ &\geq \gamma(x_0 - x_1, l_0 - l_1) \\ &\geq 0, \end{aligned}$$

and so  $\gamma(x_0 - x_1, l_0 - l_1) = 0$ . This, together with the Assumption (D)(iv) implies that  $x_0 = x_1$  and  $l_0 = l_1$ . Consequently,  $(x_0, l_0) = (x_1, l_1) \in G(d)$ , and hence the proof is complete.  $\blacksquare$

In the sequel, for any non-empty subset  $S$  of  $K$ , define

$$h_S(x, l) := \varphi_S(-x, l), \quad \forall (x, l) \in K := X' \times L'. \quad (5.18)$$

The following theorem is a generalization of [11, Theorem 2.1].

**Theorem 5.2.** *Suppose that the Assumptions (C) and (D) hold. Let  $A : X' \rightarrow 2^{L'}$  be any  $L'$ -monotone operator. Assume that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_S$ , where  $A'$  is any operator whose graph is a translation of  $G(A)$ ,  $h_S$  is defined by (5.18) and  $S$  is any non-empty subset of  $K$ . Then the following assertions are equivalent:*

- (1)  $A$  is a maximal  $L'$ -monotone operator.
- (2)  $G(A) + G(-B) = K := X' \times L'$  for every maximal  $L'$ -monotone operator  $B : X' \rightarrow 2^{L'}$  such that  $\varphi_B$  is finite-valued.
- (3) There exist an operator  $B : X' \rightarrow 2^{L'}$  such that  $G(A) + G(-B) = K$  and a point  $(\tilde{x}, \tilde{l}) \in G(B)$  such that

$$\tilde{l}(\tilde{x}) - \tilde{l}(x) - l(\tilde{x}) + l(x) > 0, \quad \forall (x, l) \in G(B) \setminus \{(\tilde{x}, \tilde{l})\}.$$

*Proof:* (1)  $\implies$  (2). Assume that (1) holds and  $B : X' \longrightarrow 2^{L'}$  is any maximal  $L'$ -monotone operator such that  $\varphi_B$  is finite-valued. Let  $(x_0, l_0) \in K$  be arbitrary. Consider the set valued mapping  $A' : X' \longrightarrow 2^{L'}$  such that  $G(A') := G(A) - (x_0, l_0)$ . Since  $A$  is a maximal  $L'$ -monotone operator, in view of Lemma 4.3 we conclude that  $A'$  is also a maximal  $L'$ -monotone operator. Let  $S := G(B)$ . Then,  $\varphi_S$  is finite-valued and by Remark 4.2 we have  $h_S \in \mathcal{P}(H_{L^*})$ . Since  $A'$  and  $B$  are maximal  $L'$ -monotone operators, it follows from Lemma 2.1 that

$$\begin{aligned} \varphi_{A'}(x, l) + h_S(x, l) &\geq l(x) + l(-x) \\ &= l(x + (-x)) \\ &= l(0) = 0, \quad \forall (x, l) \in K. \end{aligned}$$

Since by the hypothesis  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_S$ , it follows from Lemma 4.2 that there exists  $(\bar{l}, \bar{x}) \in L^* := L' \times X'$  such that

$$(\varphi_{A'})^*_{L^*}(\bar{l}, \bar{x}) + (h_S)^*_{L^*}(-\bar{l}, -\bar{x}) \leq 0. \quad (5.19)$$

But we have

$$\begin{aligned} (h_S)^*_{L^*}(-\bar{l}, -\bar{x}) &= \sup_{(x, l) \in K} [\langle (x, l), (-\bar{l}, -\bar{x}) \rangle_* - h_S(x, l)] \\ &= \sup_{(x, l) \in K} [\langle (x, l), (-\bar{l}, -\bar{x}) \rangle_* - \varphi_S(-x, l)] \\ &= \sup_{(x, l) \in K} [\langle (-x, l), (\bar{l}, -\bar{x}) \rangle_* - \varphi_S(-x, l)] = (\varphi_S)^*_{L^*}(\bar{l}, -\bar{x}). \end{aligned}$$

Therefore, it follows from (5.19) that

$$(\varphi_{A'})^*_{L^*}(\bar{l}, \bar{x}) + (\varphi_S)^*_{L^*}(\bar{l}, -\bar{x}) \leq 0.$$

This, together with Remark 4.4 implies that

$$\varphi_{A'}(\bar{x}, \bar{l}) + \varphi_S(-\bar{x}, \bar{l}) \leq 0. \quad (5.20)$$

On the other hand, by maximality of  $A'$  and  $B$  we have

$$\varphi_{A'}(\bar{x}, \bar{l}) + \varphi_S(-\bar{x}, \bar{l}) \geq \bar{l}(\bar{x}) + \bar{l}(-\bar{x}) = \bar{l}(\bar{x} + (-\bar{x})) = \bar{l}(0) = 0. \quad (5.21)$$

Thus, we deduce from (5.20), (5.21) and the maximality of  $A'$  and  $B$  that

$$\varphi_{A'}(\bar{x}, \bar{l}) = \bar{l}(\bar{x}) \quad \text{and} \quad \varphi_S(-\bar{x}, \bar{l}) = \bar{l}(-\bar{x}),$$

and hence

$$(\bar{x}, \bar{l}) \in G(A') \quad \text{and} \quad (-\bar{x}, \bar{l}) \in S = G(B).$$

Then, we have

$$\begin{aligned} (x_0, l_0) &= (x_0, l_0) + (\bar{x}, \bar{l}) + (-\bar{x}, -\bar{l}) \\ &\in (x_0, l_0) + G(A') + G(-B) \\ &= G(A) + G(-B). \end{aligned}$$

This completes the proof of the implication (1)  $\implies$  (2).

(2)  $\implies$  (3). Suppose that (2) holds. Let  $B := d$ , where  $d$  is defined by (5.9). Since the Assumptions (C) and (D) hold, then in view of Proposition 5.2 we conclude that  $B$  is a maximal  $L'$ -monotone operator. Also, we have

$$\begin{aligned} G(B) &= \{(x, l) \in K : \delta(x, -l) = 0\} \\ &= \{(x, l) \in K : \gamma(x, -l) = l(x)\}. \end{aligned}$$

Thus, by Assumption (D)(i) and (ii), one has

$$\begin{aligned} \varphi_B(x, l) &= \sup_{(x', l') \in G(B)} [l(x') + l'(x) - l'(x')] & (5.22) \\ &= \sup_{(x', l') \in G(B)} [\langle (x', -l'), (l, -x) \rangle_* - \gamma(x', -l')] \\ &\leq \sup_{(x', l') \in K} [\langle (x', -l'), (l, -x) \rangle_* - \gamma(x', -l')] \\ &= [\gamma^*_{L^*} \circ t](-x, l) \\ &= \gamma(-x, l) < +\infty, \quad \forall (x, l) \in K. \end{aligned}$$

This implies that  $\varphi_B$  is finite-valued. Therefore, by the hypothesis (2), we have  $G(A) + G(-B) = K$ .

Moreover, since by Assumption (D)(vi) we have  $\gamma(0, 0) = 0$ , we deduce that  $(0, 0) \in G(B)$ . Let  $(\tilde{x}, \tilde{l}) := (0, 0) \in G(B)$ . Thus, for every  $(x, l) \in G(B) \setminus \{(\tilde{x}, \tilde{l})\}$ , one has

$$\tilde{l}(\tilde{x}) - \tilde{l}(x) - l(\tilde{x}) + l(x) = l(x) = \gamma(x, -l) > 0,$$

because if  $\gamma(x, -l) = 0$ , then by Assumption (D)(iv) we have  $(x, l) = (0, 0)$ , which is a contradiction.

(3)  $\implies$  (1). Assume that (3) holds. By the hypothesis,  $A$  is an  $L'$ -monotone operator. For maximality of  $A$ , suppose that  $(x_0, l_0) \in K$  and

$$l(x) - l(x_0) - l_0(x) + l_0(x_0) \geq 0, \quad \forall (x, l) \in G(A). \quad (5.23)$$

Assume that  $(\tilde{x}, \tilde{l}) \in G(B)$  is such that

$$l'(x') - l'(\tilde{x}) - \tilde{l}(x') + \tilde{l}(\tilde{x}) > 0, \quad \forall (x', l') \in G(B) \setminus \{(\tilde{x}, \tilde{l})\}. \quad (5.24)$$

Since

$$(x_0 + \tilde{x}, l_0 - \tilde{l}) \in K = G(A) + G(-B),$$

we have

$$(x_0 + \tilde{x}, l_0 - \tilde{l}) = (x_1, l_1) + (x_2, l_2) \quad (5.25)$$

for some  $(x_1, l_1) \in G(A)$  and some  $(x_2, l_2) \in G(-B)$ . It follows from (5.25) that  $x_0 - x_1 = x_2 - \tilde{x}$  and  $l_0 - l_1 = l_2 + \tilde{l}$ . Hence, since  $(x_2, -l_2), (\tilde{x}, \tilde{l}) \in G(B)$ ,  $(x_1, l_1) \in G(A)$  and (5.23) and (5.24) hold, we conclude that

$$\begin{aligned}
0 &\leq l_1(x_1) - l_1(x_0) - l_0(x_1) + l_0(x_0) \\
&= (l_0 - l_1)(x_0 - x_1) \\
&= (l_2 + \tilde{l})(x_2 - \tilde{x}) \\
&= -[(-l_2)(x_2) - (-l_2)(\tilde{x}) - \tilde{l}(x_2) + \tilde{l}(\tilde{x})] \\
&\leq 0.
\end{aligned}$$

This implies that

$$(-l_2)(x_2) - (-l_2)(\tilde{x}) - \tilde{l}(x_2) + \tilde{l}(\tilde{x}) = 0,$$

and so in view of (5.23) we deduce that  $x_2 = \tilde{x}$  and  $-l_2 = \tilde{l}$ . Therefore, (5.25) yields

$$(x_0 + \tilde{x}, l_0 - \tilde{l}) = (x_1, l_1) + (\tilde{x}, -\tilde{l}),$$

and hence  $(x_0, l_0) = (x_1, l_1) \in G(A)$ , which proves the maximality of  $A$ .  $\blacksquare$

**Remark 5.1.** *It is worth noting that in Theorem 5.2 the additivity of the mapping  $\text{supp}(\cdot, H_{L^*})$  in  $\varphi_{A'}$  and  $h_S$  is only necessary for the implication (1)  $\implies$  (2).*

**Corollary 5.1.** *Suppose that the Assumptions (C) and (D) hold. Let  $A : X' \longrightarrow 2^{L'}$  be an  $L'$ -monotone operator. Assume that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_{G(d)}$ , where  $A'$  is any operator whose graph is a translation of  $G(A)$  and  $h_{G(d)}$  and  $d$  are defined by (5.18) and (5.9), respectively. Then the following assertions are equivalent:*

- (1)  $A$  is a maximal  $L'$ -monotone operator.
- (2)  $G(A) + G(-d) = K := X' \times L'$ .

*Proof:* (1)  $\implies$  (2). Assume that (1) holds. Since the Assumptions (C) and (D) hold, it follows from Proposition 5.2 that  $d$  is a maximal  $L'$ -monotone operator and in view of (5.22)  $\varphi_d$  is finite-valued, thus (2) follows from the proof of the implication (1)  $\implies$  (2) of Theorem 5.2 (indeed, in that proof the equality  $G(A) + G(-B) = K$  requires the additivity of  $\text{supp}(\cdot, H_{L^*})$  only in  $\varphi_{A'}$  and  $h_{G(B)}$ ).

(2)  $\implies$  (1). Suppose that (2) holds. Since by the hypothesis  $A$  is an  $L'$ -monotone operator, we have  $G(A)$  is an  $L'$ -monotone subset of  $K$ . Now, let  $(x, l) \in K$  be arbitrary. Then, by the hypothesis (2), there exists  $(x_0, l_0) \in G(A)$  such that  $(x - x_0, l - l_0) \in G(-d)$ , and so by (5.10) we have  $\delta(x_0 - x, l_0 - l) = 0$ . Therefore, in view of Theorem 4.3(2) we conclude that  $G(A)$  is a maximal  $L'$ -monotone set, and hence  $A$  is a maximal  $L'$ -monotone operator.  $\blacksquare$

**Theorem 5.3.** *Suppose that the Assumptions (C) and (D) hold. Let  $A : X' \longrightarrow 2^{L'}$  be a maximal  $L'$ -monotone operator. Assume that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_{G(d)}$ , where  $A'$  is any operator whose graph is a translation of  $G(A)$  and  $h_{G(d)}$ ,  $d$  are defined by (5.18) and (5.9), respectively. Then,  $\mathcal{R}(A + d) = L'$ .*

*Proof:* It is clear that  $\mathcal{R}(A + d) \subseteq L'$ . Since  $A$  is a maximal  $L'$ -monotone operator, it follows from Corollary 5.1 that  $G(A) + G(-d) = K := X' \times L'$ . Now, let  $l \in L'$  be arbitrary. Then,  $(0, l) \in K$ , and so there exist  $(x_1, l_1) \in G(A)$ ,  $x_2 \in X'$ , and  $l_2 \in d(x_2)$  such that  $(0, l) = (x_1, l_1) + (x_2, -l_2)$ . This implies that  $x_1 = -x_2$  and  $l = l_1 - l_2$ . Since  $d(-x) = -d(x)$  for every  $x \in X'$ , we have

$$\begin{aligned} l &= l_1 - l_2 \in l_1 - d(x_2) \\ &= l_1 + d(-x_2) \\ &\subseteq A(-x_2) + d(-x_2) \\ &= (A + d)(-x_2) \\ &\subseteq \mathcal{R}(A + d), \end{aligned}$$

which completes the proof. ■

In the following, we give an example of an  $L$ -monotone operator  $A : X \rightarrow 2^L$  and a non-empty subset  $S$  of  $K := X \times L$  such that the mapping  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_{G(d)}$ , where  $A'$  is any operator whose graph is a translation of  $G(A)$  and  $h_S$  is defined by (5.18).

**Example 5.3.** Let  $X, L, K$  and  $L^*$  be as in Example 4.4. Consider the set valued mapping  $A : X \rightarrow 2^L$  such that

$$G(A) := \{(1, l_1)\}.$$

Let  $g \in \mathcal{P}(H_{L^*})$  and  $S \subseteq K$  be arbitrary. We show that  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $g$ . Hence, in particular,  $\text{supp}(\cdot, H_{L^*})$  is additive in  $\varphi_{A'}$  and  $h_S$ . It is clear that  $A$  is an  $L$ -monotone operator. Let  $(x_0, l_{y_0}) \in K$  be arbitrary. Then, one has

$$G(A)^t := G(A) - (x_0, l_{y_0}) = \{(1 - x_0, l_{1-y_0})\}.$$

Therefore, we have

$$\begin{aligned} \varphi_{A'}(x, l_y) &= l_y(1 - x_0) + l_{1-y_0}(x) - l_{1-y_0}(1 - x_0) \\ &= -yx_0 - xy_0 + x_0 + y_0 - x_0y_0 + x + y - 1 \\ &= \langle (x, l_y), (l_{1-y_0}, 1 - x_0) \rangle_* - x_0y_0 + x_0 + y_0 - 1, \quad \forall (x, l_y) \in K. \end{aligned}$$

It is easy to see that

$$\text{supp}(\varphi_{G(A)^t}, H_{L^*}) + \text{supp}(g, H_{L^*}) \subseteq \text{supp}(\varphi_{G(A)^t} + g, H_{L^*}).$$

For the converse inclusion, let  $f \in H_{L^*}$  and  $f \leq \varphi_{A'} + g$ . Let  $f_1 := \varphi_{A'}$ . Then,  $f_1 \in H_{L^*}$ , and so  $f_1 \in \text{supp}(\varphi_{A'}, H_{L^*})$ . Define

$$f_2 := f - \varphi_{A'}.$$

By using the properties of  $L^*$ , and since  $f \in H_{L^*}$ ,  $-\varphi_{A'} \in H_{L^*}$  and  $f \leq \varphi_{A'} + g$ , it is easy to check that  $f_2 \in H_{L^*}$  and  $f_2 \leq g$ . Thus,  $f_2 \in \text{supp}(g, H_{L^*})$ . Also, we have  $f = f_1 + f_2$ , and hence the proof is complete.

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