ANALYTICAL STUDY AND GENERALISATION OF SELECTED STOCK OPTION VALUATION MODELS

By

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Matriculation Number: 13PCD00571

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A THESIS SUBMITTED TO THE DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY, COVENANT UNIVERSITY, OTA, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF Ph.D DEGREE IN INDUSTRIAL MATHEMATICS

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ACCEPTANCE

This is to attest that this thesis is accepted in partial fulfillment for the award of the degree of Doctor of Philosophy in Industrial Mathematics, College of Science and Technology, Covenant University, Ota.

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DECLARATION

I, **EDEKI Sunday Onos** (13PCD00571) declare that this research was carried out by me under the supervision of Professor (Mrs) Olabisi O. Ugbebor of the Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Nigeria and Dr. Enahoro A. Owoloko of the Department of Mathematics, Covenant University, Ota, Ogun State, Nigeria. I attest that this thesis has not been presented either wholly or partially for the award of any degree anywhere else. All the sources of data, scholarly publications and information used in this thesis are cited and acknowledged accordingly.

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CERTIFICATION

We certify that this thesis titled "Analytical Study and Generalisation of Selected Stock Option Valuation Models" is an original work carried out by Sunday Onos EDEKI (13PCD00571) in the Department of Mathematics, College of Science and Technology, Covenant University, Ota, Ogun State, Nigeria, under the supervision of Professor Olabisi O. Ugbebor and Dr. Enahoro A. Owoloko. We have examined and found the work acceptable as part of the requirements for the award of Doctor of Philosophy (Ph.D) degree in Industrial Mathematics.

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DEDICATION

To the Almighty God for wisdom, knowledge, provision and protection.

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LIST OF SYMBOLS

Symbol	**Description**
S_t	Stock price process at time t
W_t	Standard Brownian Motion
σ	Stock price volatility
μ	Drift parameter (mean rate of return)
$V(S,t)$	Value of an option in stock at time t
r	Risk-free interest rate
$C^{1,2}(\mathrm{R}^+ \times [0,T])$	The set of all continuous functions which are once differentiable w.r.t. the first variable and twice differentiable w.r.t. the second variable
$p_f(S,t)$	Payoff function on a stock, S, at time t
t	Time parameter
E	Expiration price
ξ	Rate of elasticity
ρ	The constant measuring the liquidity in an illiquid market
$\widehat{\sigma}$	Volatility function in Bakstein and Howison model
Ω	A non-void set
$\breve{\mathrm{B}}$	A $\sigma -$ algebra of subsets of Ω
$\left(\Omega, \breve{\mathrm{B}}\right)$	A measurable space
$\left(\Omega, \breve{\mathrm{B}}, \mho\right)$	A measure space
$\left(\Omega, \breve{\mathrm{B}}, \dddot{P}\right)$	A probability space
$\dddot{P}$	The real world probability measure
$\left(\Omega, \breve{\mathrm{B}}, \dddot{P}, \bar{F}\left(\breve{\mathrm{B}}\right)\right)$	A filtered probability space
$\bar{F}\left(\breve{\mathrm{B}}\right)$	A filtration
$\{X_{*t}(\omega)\}$	A collection of random variables defined on the same probability *space* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}\right)$

$\mathrm{E}(X_*)$	Mathematical expectation of the random variable X_*
$Var(X_*)$	The variance of the random variable X_*
$w(\cdot)$	A differentiable function
$W(k)$	The differential transform of $w(\cdot)$
$\Theta(t)$	A delta-hedge-portfolio
$\Lambda(S,t)$	Contingent claim value (CCV)
$\Lambda(S_t,k_t)$	An investment output
k_t	The total investment (assumed constant), all over a short period of time, t
p^{r*}	Production rate
c^{r*}	Consumption rate
$\Xi(S,t)$	Option value at no dividend yield
Λ^o_{BSM}	The compared volatility of the Black-Scholes model
Λ^o_{CEVM}	The compared volatility of the CEV model
Λ^r_{BSM}	The compared variance of the Black-Scholes model
Λ^r_{CEVM}	The compared variance of the CEV model

LIST OF ABBREVIATIONS

Abbreviations	**Full Meaning**
ADM	Adomian Decomposition Method
ANNM	Artificial Neural Networks Models
Approx	Approximate
ARIMA	Autoregressive Integrated Moving Average
BICGSTABM	Bi-Conjugate Gradient Stabilized Method
BSM	Black-Scholes Model
CCV	Contingent Claim Value
CEV	Constant Elasticity of Variance
CEVM	Constant Elasticity of Variance Model
CEV-BSM	Constant Elasticity of Variance Black-Scholes Model
COPs	Constrained Optimization Problems
DTM	Differential Transformation Method
EMH	Efficient Market Hypothesis
EMM	Equivalent Martingale Measure
ESN	Echo State Networks
ESO	Employee Stock Option
FBM	Fractional Brownian Motion
FDE	Fractional Differential Equation
FTBSE	Fractional Type Black-Scholes Equation
FTBSM	Fractional Type Black-Scholes Model
FVIM	Fractional Variation Iterative Method
G-SIM	Gauss-Seidel Iterative Method
GBM	Geometric Brownian Motion
GMRESM	Generalized Minimal Residual Method
HAM	Homotopy Analysis Method
HPM	Homotopy Perturbation Method

HPT	He's Polynomial Technique
HPSTM	Homotopy Perturbation Sumudu Transform
JIM	Jacobi Iterative Method
LADM	Laplace Adomian Decomposition Method
LCPs	Linear Complementarity Problems
LLWHM	Laplace Legendre Wavelet Hybrid Method
MADM	Modified Adomian Decomposition Method
MDTM	Modified Differential Transformation Method
MHD	Magneto-Hydrodynamic
MsDTM	Multi-step DTM
MT	Mellin Transformation
MVIM	Modified Variation Iterative Method
NGSE	Nigerian Stock Exchange
NGSEINDX	Nigerian Stock Exchange All Share Index
NLFDE	Nonlinear Fractional Differential Equation
OTC	Over-the-Counter
PCGM	Preconditioned Conjugate Gradient Method
PDE	Partial Differential Equation
PDT	Projected Differential Transform
PDTM	Projected Differential Transformation Method
PPM	Power Penalty Method
Ref.	Reference
Rel. error	Relative error
RHPM	Revised Homotopy Perturbation Method
R-FIR	Risk-free Interest Rate
R-IIR	Risk-include Interest Rate
SDE	Stochastic Differential Equation
SMI	Stock Market Index
STM	Sumudu Transform Method

SOR	Successive Over Relaxation
TFBSM	Time-fractional Black-Scholes Model
VIM	Variation Iterative Method
w.r.t.	With respect to

ABSTRACT

In this work, the classical Black-Scholes model for stock option valuation on the basis of some stochastic dynamics was considered. As a result, a stock option valuation model with a non-fixed constant drift coefficient was derived. The classical Black-Scholes model was generalised via the application of the Constant Elasticity of Variance Model (CEVM) with regard to two cases: case *one* was without a dividend yield parameter while case *two* was with a dividend yield parameter. In both cases, the volatility of the stock price was shown to be a non-constant power function of the underlying stock price and the elasticity parameter unlike the constant volatility assumption of the classical Black-Scholes model. The Itô's theorem was applied to the associated Stochastic Differential Equations (SDEs) for conversion to Partial Differential Equations (PDEs), while two approximate-analytical methods: the Modified Differential Transformation Method (MDTM) and the He's Polynomials Technique (HPT) were applied to the Black-Scholes model for stock option valuation; in both cases the integer and time-fractional orders were considered, and the results obtained proved the latter as an extension of the former. In addition, a nonlinear option pricing model was obtained when the constant volatility assumption of the classical linear Black-Scholes option pricing model was relaxed through the inclusion of transaction cost (Bakstein and Howison model). Thereafter, this nonlinear option pricing model was extended to a time-fractional ordered form, and its approximate-analytical solutions were obtained via the proposed solution technique. For efficiency and reliability of the method, two cases with five examples were considered: Case 1 with two examples for time-integer order, and Case 2 with three examples for time-fractional order, and the results obtained show that the time-fractional order form generalises the time-integer order form. Thus, the Black-Scholes and the Bakstein and Howison models for stock option valuation were generalised and extended to time-fractional order, and analytical solutions of these generalised models were provided.

Keywords: Stock options, Stochastic differential equations, Option valuation, Analytical solutions , Fractional calculus, Approximate-analytical methods.

CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

In contemporary financial settings, the role of options in pricing theory is of immense importance as they can be used for risk control and asset hedging. Stock as a basic term refers to a company's assets held by an individual or group in the form of shares. The accountants view stock from two perspectives: as goods on hand to be sold to customers (inventory), and as ownership shares of a corporation. A stock certificate is provided as an evidence of the corporation's common stock (ordinary shares) or preferred stock owner thereby making the stockholders partial owners of the company. Stocks are usually quoted and traded on stock-exchange market. This transaction entails a financial contract known as derivative security (contingent claim) whose value at expiration date is derived from the price process of one or some of the underlying assets (stocks) (Nelson, 1904).

Buying or selling of stocks (shares) in this direction is optional. Hence, an option is defined as a derivative security furnishing its holder(s) the right (but not an obligation), to make a transaction at a specified period for a specified price. Different types of options exist (see Sprenkle, 1964; Fama, 1965; Merton, 1973) and these can be classified in various ways according to the option rights, option styles, underlying assets and so on. Different types of options are described below.

It is a call (or put) option if associated with a buyer (or seller). The timing for the exercise of an option defines its style. It is a *European* option if it can be exercised only at maturity date while it is an *American* option if the exercise can be done on or before the maturity date, else such option expires worthless and its existence as financial instrument ceases. Most exchange traded options follow the American-style of option.

Other kinds of options include the *Asian* option whose final asset price (payoff) is taken as the average of the underlying asset over a predetermined period of time; Asian option is similar to the European option but differs in terms of the final underlying values. *Barrier* option is an option with a general feature indicating that the price of the underlying security must cross a certain level (barrier) before an exercise can be done. *Look back* options are options whose payoffs depend on the maximum or the minimum of the underlying asset price during some predetermined period. As the name implies, a look back option allows its holder to 'look back' over a specified period to determine the payoff.

A *Swap* option or Swaption is a type of option involving two investors with an undertaking to exchange, at a known date in the future, various financial assets according to a prearranged formula that depends on the value of one or more underlying assets. This includes currency swaps and interest rate swaps. In swaption, both the buyer and the seller agree on the price (premium) of the swaption, and the length of the period.

A *Binary* option, also known as 'all or nothing' option, allows the payment of the full amount of payoff if the underlying security meets the specified condition upon expiration, otherwise its value is worthless. An *Exchange Traded* option also referred to as listed option is a type of option traded or listed on a public or regulated trading exchange. This option can be bought or sold by anybody via the services of an appropriate broker. It is a standardized contract such that quantity, underlying asset, date of expiration and strike price are known in advance.

On the other hand, *Over-the-Counter* (OTC) options are options traded in a kind of market called OTC. The concerned investors who invariably do not meet are linked by telephones or electronic connectivity. Options on OTC are less accessible to the general public since they are not traded on exchanges and the terms are more customized and complicated than most exchange traded contracts.

Stock option is therefore defined as the right but not an obligation either to buy or sell stock at a specified price within a stated period of time. Stock option is an example of option whose underlying assets are shares or stocks. It can also be viewed as benefit granted to an employee by the employer or company in the form of an option to purchase the company's stock at a discounted or fixed price. This is commonly called Employee Stock Option (ESO); it can serve as a financial assistance at a time of need.

A Stock Index or Stock Market Index (SMI) is a measurement of the value of a section of stock market. It is computed from the values of selected stocks, usually as a weighted average. SMI is a tool used by investors to describe the market, and to compare returns on specific investment.

An option is beneficial because it protects stock holdings from a decline in market price, helps to increase income against current stock holdings, prepares investors to buy stock at lower prices, helps investors to position themselves for a big market move; even without knowing the way prices will move, and helps investors to benefit from a stock price's rise or fall without incurring the cost of buying or selling the stock outright.

In modern finance, the importance of options in pricing theory cannot be overemphasized as they can be used for asset hedging, and to control risk. This calls for the attention of financial engineers when dealing with finance, actuarial sciences, and other related areas of applied sciences (Habib, 2011).

Investors purchase stocks with the hope that the stocks will appreciate in values and in return, yield income from dividends. Companies or individuals can therefore make a lot of money via stock trading if the market is well understood. Similarly, it can also be a huge risk to investors if proper decision is not taken.

One of the highly volatile variables in stock exchange is the stock price. Its unstable property calls for concern on the investors' part, since sudden change in share prices happens frequently and randomly. Researchers are therefore challenged to consider in their studies the behaviour of this unstable stock parameter in order to render valuable advice to stock investors and owners of corporation, hence, the importance of the study of stock options.

The determinants of the value of stock are the forces of supply and demand. Investors are more concerned with companies' stocks expected to yield significant profits in the future. Thus, investment into stocks requires the minimization or control of risk caused by decrease in stock values.

An efficient way to handle this is the adoption of mathematical model(s) that can give clear suggestions about the future behaviour of stock prices. Although, there are market laws such as Efficient Market Hypothesis (EMH) for market transparency, (Fama, 1970). This EMH gives the same information regarding certain stock to everyone. Based on EMH, the basic information in relation to the stocks is their current value since the past price values say nothing about the future behaviour of the future values. This implies that stock price modelling is geared towards modelling new information about the concerned stocks.

In financial economy, uncertain movements of stock values over time reflects the dynamics of the stock prices. The EMH is one of the reasons for the random movement of stock prices. The EMH states that the present prices of the stock fully reflect the past history of its prices, and that the market is easily affected by any new information about the stock. EMH based its assumption on the premise that stock price changes are Markovian in nature; indicating that the expected future value of stocks depends only on its current value. Thus, predictions can only be expressed as probability distribution because of their uncertainty. That is, prediction of the future price of stock can be done to a certain level of precision if new information about the stock can be anticipated.

The random nature of the stock price process exhibits the same behaviour as a stochastic process known as Brownian motion. This means that some properties of the stock price process are traceable to those of Brownian motion process leading to stochastic modelling of stock prices; since stochastic models are built on random walks, and are often used in theoretical studies because of their simplicity as only the volatility parameter is required (Reiss, 1975). In financial mathematics, trading in an illiquid market in which stock option is an example of illiquid asset has become a topic of great concern to risk managers and hedgers, since assets in such a market cannot be exchanged or traded easily for cash without at least a minimal loss of value.

1.2 Statement of the Problem

Models with regard to drift coefficient and volatility parameters have been noted in literature for option valuation and pricing. The simplest among these with a bearing from the classical Black-Scholes model for option valuation assumes constant (fixed) mean rate of return and volatility. However, it is obvious that the constant nature of these parameters cannot fully explain observed market prices for option valuation unless when modified (Cen and Le, 2011).

The classical Black-Scholes model for stock price valuation is linear, based on the assumptions that both drift and volatility parameters are constants. In most cases, relaxing these assumptions results to nonlinear models (Bakstein and Howison, 2003). The nonlinear transaction-cost model has not been considered in terms of fractional order but for integer ordered form. In most cases, exact solutions of nonlinear models do not exist; even when they do, obtaining them via direct solution or conventional methods seems to possess some setbacks such as linearization, perturbation, computational time consumption and so on. Therefore, because of the nonlinearity there is need for reliable, effective and efficient approximate-analytical method(s).

This research is therefore, motivated to address these gaps by developing a stock option model whose mean rate of return is a non-fixed parameter, generalise the Black-Scholes model for the inclusion of non-constant volatility, and provide analytical solutions to the linear and nonlinear stock option models resulting from the generalisation(s) based on the associated stochastic dynamics.

The application of fast convergent approximate-analytical methods: Modified Differential Transformation Method (MDTM) and the Revised Homotopy Perturbation Method (RHPM) for solving any form of the above models resulting from Stochastic Differential Equations (SDEs) have not been reported in literature to the best of our knowledge.

1.3 Aim and Objectives

The aim of this research work is to study and generalise the classical Black-Scholes, and the Bakstein and Howison pricing models for stock option valuation, and to obtain approximate-analytical solutions of the models resulting from their associated SDEs. The objectives of this research work are to:

(i) derive stock option valuation model that will incorporate the drift coefficient (rate of return) as a non-fixed constant without excluding the other parameters (the risk-free interest rate, and the volatility term;

(ii) obtain a solution for the proposed model in (i) using approximate-analytical methods;

(iii) generalise the Black-Scholes option pricing model via a constant elasticity of variance in order to address the assumption of the constant volatility rate in the Black-Scholes Option Pricing Model;

(iv) determine analytical solutions of European option pricing models via the MDTM and the RHPM;

(v) generalise the nonlinear transaction cost model of Bakstein and Howison (2003) for stocks valuation to a time-fractional order form; and

(vi) obtain an analytical solution of the time-fractional generalised Bakstein and Howison model for stock option valuation in (v).

Note: Throughout this thesis, the term 'fixed constant' will be used to denote a constant that is being suppressed.

1.4 Justification of the Study

The uncertainty in the movement of stock values over time simply reflects the dynamic nature of the stock prices. This follows a stochastic process known as diffusion process. The classical Black-Scholes model for option pricing was based on some assumptions such as constant mean rate of return, and constant volatility. These assumptions could be relaxed by using the Constant Elasticity of Variance (CEV) stochastic dynamics. In addition, there is need for the generalisation of the Baskstein and Howison model induced by CEV to a time-fractional order. This is to permit a smooth running of a timeshare system in fractional ownership style of option pricing.

1.5 Significance of the Study

The findings of this study will be of immense benefit to stock option valuers, and practitioners considering the effect of fractional ownership style of option valuation on effective time management. The unstable nature of stock price movements justifies the need for more efficient valuation models. Thus, stock option Employer-Employee system that applies the recommended generalised models from this study will encourage investment, and employees' alignment with the company's norms and values thereby leading to the growth of both the company (employer) and the employees. The study will also enlarge the scope of operation of the classical Black-Scholes model since the volatility functions in the generalised models are non-constants but functions of the stock price processes. For brokers and hedgers, the study will help in risk management mainly in times of market volatility and uncertainty.

1.6 The Scope of the Study

Assets in a non-liquid market cannot be traded for cash easily without a noticeable loss in its value (no matter how minimal). This is unlike the liquid market which is mainly characterised by the presence of many buyers and sellers who are ever ready and willing to invest. The study therefore covers derivatives in an illiquid market with preference to options whose underlying assets are stocks.

1.7 Limitation of the Study

The derived models were based on Risk-free Interest Rate (R-FIR) meaning that interest rate is constant instead of Risk-include Interest Rate (R-IIR) where interest rate is not constant. Therefore the volatility parameter is modelled as a non-constant function due to the complexity nature in deriving a proper mathematical equation for the calculation of R-IIR. The modelling process avoids assuming both the R-IIR and the volatility parameter to be constants at the same time since the nature of stock market prices is unstable. In addition, some parameters such as the constant measuring the liquidity of the market were chosen hypothetically (or arbitrarily).

1.8 Structure of the Thesis

The remaining parts of the thesis is structured as follows: In Chapter Two, literature review on stock, stock options, stochastic models for stock prices, and approximate-analytical methods are presented. Chapter Three deals with the methodology: the basic definitions, descriptions of theorems on stochastic analysis, and approximate-analytical methods. In Chapter Four, the results are presented in detailed forms in line with discussion and summary of the research findings, while Chapter Five presents the conclusion, contributions to knowledge, recommendations, and open problems for further research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The search for better and efficient predictive models for stock prices is imperative for the valuation of stock prices. A lot of such predictive models have been reported in literature. In this chapter, a review of some key and fundamental results in relation to existing models for stock, stock options, stochastic models for stock prices, and the basic approximate-analytical methods carried out.

2.2 Non-Stochastic Models for Stock Prices

Hanna (1976) proposed a stock price predictive model based on changes in ratio of short interest to trading volume and showed that short ratio produced no evidence that the success of the ratio as a stock market predictor can be attributed to either of its components singly. It was therefore concluded based on the hypothesis of their study that speculative expectations tend to be extremely one-sided at the existence of high probability in relation to stock prices leading to over-discount by investors. Schoneburg (1990) considered the possibility of stock price prediction on a short-term basis using neural network applied to German stocks chosen at random. Though the results were encouraging regarding stock price prediction they were faced with complex problems in some cases with regard to the choice of suitable neural networks.

Fornari and Mele (1997) presented sign and volatility-switching models for valuation of stock prices. They further showed that weak convergence in probability implies convergence in distribution for both models with regard to the diffusion processes.

They recommended for further research, that the response function of the volatility-switching model is linear and hence, needed to be modified.

Rapach and Wohar (2005) employed price-dividend and price-earning ratios to re-examine the predictability of real stock prices. In their work, they used the annual data from Campbell and Shiller (1988), and found out that the price-dividend and price-earnings ratios could be used to predict real stock price growth at long but not in short horizons.

Shane and Stock (2006) investigated the range of how security analysts' earning forecasts and stock prices show temporary income effects of tax-motivated income shifting. They gave regard to the consideration of how market participants anticipated and correctly interpreted temporary income effects of firm's earnings-managerial issues.

Lin *et al.* (2009) investigated the effectiveness of Echo State Networks (ESN) to predict future stock prices in a short-term. Their experimental results indicated that including principle of component analysis (PCA) to filter noise in data pretreatment and choosing appropriate parameters prevent coarse prediction outcome effectively. They compared their results with those from other traditional neural network and pointed out that the application of ESN to long term-stock data mining is yet to be considered. Wu and Hu (2011) proposed a nonlinear price-dividend ratios model for stock price prediction while rejecting the non-predictability hypothesis of stock prices statistically based on in-and out-of-sample tests and as regards the criteria of expected real return per unit of risk.

Adebiyi *et al.* (2014) examined the performance of Autoregressive Integrated Moving Average (ARIMA) and Artificial Neural Networks Models (ANNM) for stock price prediction. They found out that both methods were effective for stock price forecasting but noted that the stock price predictive models with the ANNM showed superior performance over those of ARIMA.

2.3 Stochastic Models for Stock Prices

The first stochastic model for stock price dynamic was proposed by Bachelier (1900) as cited in Akyildirim and Soner (2014). Bachelier's model is driven by a Brownian motion without a drift parameter, that is, for a stock price S_t and a standard Brownian motion W_t, the model follows Stochastic Differential Equation (SDE):

$$dS_t = S_t \sigma dW_t \tag{2.1}$$

where σ is the stock price volatility. In (2.1), the hypothesis of the absolute Brownian motion results to a negative stock prices.

Osborne (1964) refined the Bachelier's model by modelling stock price using stochastic exponent of the Brownian motion. Shortly, Samuelson (1969) modified Osborne's model by introducing the Geometric Brownian Motion (GBM).

Considering the history of option valuation, Black and Scholes (1973) made a major breakthrough based on their option pricing model referred to as Black-Scholes Model (BSM). The most vital point in the BSM is the involvement of a Brownian motion with a drift in the dynamics of the stock price as shown below:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{2.2}$$

where μ, σ and W_t are drift parameter, volatility rate and standard Brownian motion respectively. The Black-Scholes model was based on the following assumptions:

(i) the asset price S_t follows a geometric Brownian motion;

(ii) the drift term μ, and the volatility parameter σ are constants;

(iii) arbitrage-free opportunities (i.e. no risk-free profit); and

(iv) competitive, and frictionless markets, and many others.

Considering $V = V(S,t)$ as the value of an option in stock $S(t) = S$ at time t, then, the partial differential equation (PDE) describing the BSM is:

$$\frac{\partial V}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{2.3}$$

where r is a risk-free interest rate, $V \in C^{2,1}(\mathrm{R}^+ \times [0,T])$, $t \in T$, with a payoff function $P_f(S,t)$ and expiration price E such that:

$$P_f(S,t) = \begin{cases} \max(S-E,0), & \text{for European call option} \\ \max(E-S,0), & \text{for European put option.} \end{cases} \tag{2.4}$$

In a frictionless market, transaction costs are are not permitted, no tax and trade restrictions are not allowed, but in a competitive market, a trader is allowed to sell or buy any quantity of a security without changing the prices.

In practice, some of the Black-Scholes assumptions are not realistic. For instance, the constant volatility assumption was included to preserve the model's linearity for easy solution in terms of analytical solutions. Hence, relaxing this leads to complexity; this aspect will be included in this research work.

Cox and Ross (1976) considered the constant elasticity of variance (CEV) diffusion process governed by the SDE:

$$dS_t = \mu S_t dt + \sigma S_t^{\frac{\xi}{2}} dW_t \tag{2.5}$$

whose solution is S_t, and ξ represents an elasticity rate, while μ, σ, and W_t are as defined earlier.

Merton *et al.* (1977) employed a finite difference method for pricing American style of option for the Black-Scholes model. Beckners (1980) considered the CEV models and their implications for option pricing based on empirical studies and drew a conclusion that the so-called CEV class of models could describe the pattern and behaviour of the actual stock price better than the traditionally applied lognormal model. Hull and White (1987) in their work, examined the problem of pricing a European call on an asset whose volatility is stochastic in nature. They obtained the option price in series form via numerical technique. They did not assume volatility as a traded asset but permitted a constant relationship between instantaneous rate of change of the aggregate consumption and that of volatility. Finally, they noted that stock prices and their volatilities were stochastic processes affected by different sources of risks.

Peters (1989), in modelling stock prices, stressed the need for Fractional Brownian Motion (FBM), saying that large number of natural phenomena possess features traceable to those of random processes or FBM, where the biased random process indicates long term dependency (or memory in between the periods of observations). He applied Hurst Rescaled Range Analysis (RRA: an analysis to investigate fluctuations over time) to bond returns, stock returns, and relative bond returns. Data from SP 500 were analyzed for Hurst exponents, and the research result revealed that each series exhibited a biased process characteristic of FBM. Onah and Ugbebor (1999), in considering a two-dimensional stochastic investment problem, extended the work of Kobila (1993) from a one dimensional stochastic differential equation to a two dimensional form. They solved the resulting PDE using finite difference method and obtained optimal results for investment decision.

Duncan *et al.* (2000) employed stochastic integration with FBM to develop fractional

Black-Scholes formula. They gave two applications of Itô's formula for FBM, namely: the homogeneous choas and the Itô-type stochastic integral. In addition, they introduced multiple Itô, and Stratonovich integrals for FBM, and established link between the two multiple integrals. Ugbebor *et al.* (2001) considered an empirical stochastic model of price-changes at the floor of a stock market where they determined the market growth rate of shares. Delbaen and Shirakawa (2002), in their study of arbitrage free option pricing problem for CEV model, showed that the CEV model allows arbitrage opportunities when the stock price is on strictly positive conditions. Their research was directed to the connection between CEV model, and squared Bessel processes. In addition, they established the existence of a unique equivalent martingale measure (EMM).

Shepp (2002) in an invited paper, presented a model for stock price fluctuations on the concept of information-based. This model is a modification of Black-Scholes model; it incorporates the existence of a stochastic process representing information state in the investor's community.

Carr *et al.* (2002) investigated the effect of diffusion and jumps in a new model for asset returns, and revealed through empirical investigation of time series that index dynamics were devoid of diffusion components but could be present in the dynamics of individual stocks. Chernov *et al.* (2003) considered alternative models for stock price dynamics by evaluating the roles of some volatility specifications such as stochastic volatility factors and jump components.

Bakstein and Howison (2003) in their working paper: a non-arbitrage liquidity model with observable parameters for derivative; emphasizing on the non-constant volatility of market prices for option valuation, derived a nonlinear transaction cost model that leads to market illiquidity. They still maintained that the stock price process follows the SDE in (2.2) but with a volatility function:

$$\sigma = \widehat{\sigma}\left(\tau, S, \frac{\partial V}{\partial S}, \frac{\partial^2 V}{\partial S^2}\right)\left(1 - \rho S \lambda(S) \frac{\partial^2 V}{\partial S^2}\right) \tag{2.6}$$

where $\rho \geq 0$ is a constant measuring the liquidity of the market, $\lambda(S)$ is the risky rate parameter on S, $\tau \in t$ and $\widehat{\sigma}$ represents a new volatility function. Below is the resulting model of Bakstein and Howison:

$$\left.\begin{aligned} \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\left(1 + 2\rho S\frac{\partial^2 V}{\partial S^2}\right)\frac{\partial^2 V}{\partial S^2} - rV = 0, \\ \lambda(S) = 1,\ V(S,0) = f(S). \end{aligned}\right\} \tag{2.7}$$

In an illiquid market, selling or buying of assets easily without a noticeable loss in the asset's value (no matter how minimal it may be) is not possible. The reason for this can be attributed to uncertainty factors which may be transaction cost, shortage of interested trader or buyers and so on (Keynes, 1971). Thus, relaxing the constant volatility assumption of the popular linear Black-Scholes option pricing and valuation model by including transaction cost yielded a nonlinear option pricing model. Bakstein and Howison (2003) saw liquidity as the process of classifying transaction cost of individual trader in connection with the impact of price slippage.

The term 'liquidity' in a professional view, explains the rate at which an underlying asset can be traded or exercised with ease; this means selling or buying in the market without the price of the asset being affected. This portrays that asset's liquidity denotes the ease, and flexibility of that asset as regards quick sales with less concern to the reduction of the asset's price (Amihud and Mendelson, 1986; Acharya and Pedersen, 2005). Examples of liquid assets include cash or money since such can be traded for items like services and goods (immediately) without (or with minimal) loss of value. A liquid market is basically characterized by ever ready and willing investors. *Stock option* is an example of an illiquid asset.

Necula (2008) used the Fourier transform to obtain an explicit fractional Black-Scholes formula for the price of an option whose underlying asset followed a fractional Brownian motion. Their main result was based on the proof of quasi-conditional expectation using Girsanov transform. Thereafter, their results were compared with those obtained via the classical Brownian motion, and concluded that using the fractional Brownian motion, the option price does not depend on the time range between maturity and present. Jumarie (2008) proposed the application of non-random exponential growth process driven by a fractional Brownian motion in modelling stock exchange dynamics. The approach eased the modelling process because of the complex mathematical tasks involved in obtaining solutions with regard to FBM based models.

Wang (2010) used a mean self-financing delta hedging argument to obtain a European call option pricing formula based on multi-fractional Black-Scholes model with transaction cost and showed that opton pricing is significantly affected by long range dependency and scaling. Wang followed the usual assumptions of the Black-Scholes with the following exceptions: the asset price at time t satisfies a multifractional Black-Scholes model, expected return of a hedged portfolio equates that of the option, and traders are rational; hence, maximize utility. Moreso, it was showed that time scaling and Hurst exponent have vital role in the theory of option pricing. Esekon (2013) considered a nonlinear option pricing model: a partial differential equation, having the corresponding nonlinear term as a feedback from price slippage; Esekon's solution was based on the assumption of a travelling wave framework where the nonlinear second order partial differential equation was reduced to first order ordinary differential model. Owoloko and Okeke (2014) applied conditions of normality to reaffirm that the BSM normality assumption does not hold completely. Chen and Wang (2014) proposed a power penalty method (PPM) for a parabolic variational inequality involving a fractional order partial derivative for the valuation of American style option, and proved the convergence of the solution in Sobolev norm at an exponential rate. They employed penalty solution methods to solve the resulting conventional constrained optimization problems (COPs), and extended same method to fractional order differential linear complementarity problems (LCPs) for American options based on Levy processes. González-Gaxiola *et al.* (2015) considered a hypothetical nonlinear option pricing model by means of Laplace Adomian Decomposition Method (LADM) for approximate solution. Their method combined the Laplace transform technique with the usual Adomian Decomposition Method in order to increase efficiency. The approximate solution they obtained were successfully compared with those obtained by Esekon. Though, the work of Esekon (2013) and those of González-Gaxiola *et al.* (2015) were based on integer orders but not on time-fractional orders.

2.4 Approximate-analytical Methods of Solutions

Many approximate-analytical methods such as the Adomian Decomposition Method, Sumudu Transform Method (STM), Homotopy Analysis Method (HAM), Homotopy Perturbation Method (HPM) and even their various modified forms have been introduced and applied by many researchers when dealing with some models arising from pure and applied sciences (Sen, 1988). Most of these methods cannot effectively handle nonlinear cases; even if a few do, the cases needed to be perturbed, linearized, or discretized and therefore, increased the computational work and the error rates (Ravi and Aruna, 2008).

In an attempt to avoid the problems associated with these techniques above, differential transformation method (DTM), modified DTM, and revised Homotopy Perturbation Method (He's Polynonials) are proposed for this study so as to proffer solutions for the stock option valuation models. The choice of this approximate-analytical techniques is for their simplicity and high level of accuracy (Rashidi, 2009, and the related references therein). The He's polynomials was introduced by Ghorbani and Nadjfi (2007); and Ghorbani (2009) where the nonlinear terms were split into a series of polynomials which are calculated using Homotopy Perturbation Method (HPM). The HPM as a approximate-analytical method does not require any small parameter in its model equation. It uses the framework of homotopy from topology to handle the nonlinear systems for convergent solutions in series forms. It is remarked that He's polynomials are compatible with Adomian's polynomials, yet it is shown that the He's polynomials are easier to compute, and are very much user friendly (He, 2003; Mohyud-Din, 2011).

2.5 Differential Transformation Method (DTM) and its Modification

The classical Taylor series method is an analytical method for solving differential equations. However, this method requires a lot of symbolic calculations for derivatives of functions, and as such takes a lot of time to compute higher order derivatives. Hence, the introduction of a approximate-analytical method called differential transformation method (DTM) by Zhou (1986) when solving problem on linear and nonlinear initial value problems of electrical circuits.

Although, the DTM is based on the Taylor series expansion method; it converts the differential equation to an algebraic-recursive equation for easy determination of the Taylor series coefficients, and provides analytical and or approximate solutions in polynomial forms, within a shorter time and with less computation.

Ayaz (2003) studied the two-dimensional DTM for solutions of initial value problem for partial differential equations, where analytical solutions of two diffusion problems were obtained. His work included new theorems to enhance the classical DTM, and the results were compared with those obtained by means of decomposition method. Chen and Ju (2004) combined differential transform method with finite difference method as a hybrid simulation technique to solve transient advective-dispersive transport problems. In their approach, the model parameters were technically varied, while various kinds of inputs data were engaged in order to contest the suitability of the method with respect to the simulation problem. The results emphasized the usefulness of the hybrid method in the prediction of solution of such problems.

Arikoglu and Ozkol (2005) extended the DTM to solve integro-differential equations. They also introduced new theorems with detailed proofs for integral transformations. These new theorems help in transforming the integrals with ease. Based on this, linear and nonlinear integro-differential equations were tested as illustrative examples while the results obtained using their method were more accurate when compared to others existing methods in literature reviewed in this study.They did the same for differential-difference equations but with the presentation of those new theorems in a more general form in order to accommodate a wider range of applications such as differential-difference equations, delayed differential equations, an so on (Arikoglu and Ozkol, 2006).

Momani *et al.* (2007) proposed a generalisation of two dimensional DTM and applied it to a diffusion-wave equation with space and time-fractional derivatives. They based the generalisation on Taylor's formula and Caputo non-integer derivatives which helped them in introducing new theorems. The analytical solutions they obtained via the generalised method were expressed in terms of Mittag-Leffler functions; though, the dependent variable terms of their solved problems were still in the field of the classical DTM which could follow the projected form of the DTM. Ravi and Aruna (2008) applied the DTM as an exact series solution method to solve singular two-point boundary value ordinary differential equations. Based on their illustrative examples, they noted that DTM gives exact solution if it exists, inspite of the method's straight forwardness in application. Rashidi (2009) developed a modified version of the DTM referred to as DTM-Pade and applied it to magneto-hydrodynamic (MHD) boundary-layer equations. He showed that DTM solutions are only valid for small values of the independent term (variable) for MHD. Thus, nullifying its application to MHD boundary-layer models since the independent variable in MHD tends to infinity. This was his motivation for the modification of the DTM. Qiu and Lorenz (2009) studied a modification of the Black-Scholes equation with regard to existence and uniqueness of solution to the Cauchy problem. They based their assumptions on smooth positive function, and allowed the initial function to be 1-periodic. Though, they recommended a more general boundary condition other than the 1-periodic type to be considered in future work.

Dura and Mosneagu (2010) applied numerical methods based on finite differences for solving Black-Scholes equation. Their intention was to create a general numerical scheme for different types of options. As such, their research scope was a complete financial market for European, and exotic options. They considered an option whose payoff depends significantly on two assets with solution domain on the real line. The explicit finite difference method adopted for solving the PDE posed severe constraints on the time-step sizes. They therefore, recommended the implicit finite difference schemes as an approach to overcome such problem.

Jang (2010) proposed a modified version of the DTM (projected DTM) for linear and nonlinear initial value problems. The PDTM was shown to provide approximate as well as exact solutions of linear and nonlinear models. The results computed were compared with those already in literature, say Variation Iterative Method (VIM), and ADM. Tari and Shahmorad (2011) developed DTM and applied it to a system of two dimensional linear and nonlinear Voltera-integro-differential equations of second kind using DTM. They pointed out some of the key merits of the developed method to include high level of accuracy, permissive nature of recursive relation, extension of the technique to linear and nonlinear two-dimensional-type of Volterra integro-differential equation without repeated terms, and so on.

Cen and Le (2011) considered a numerical method based on central difference spatial discretization on a piecewise uniform mesh, and an implicit time stepping technique for solving the Black-Scholes equation. The stability of their developed numerical scheme permits arbitrary volatility parameter, and arbitrary interest rate term. Emphasizing on their numerical results; singularities of the non-smooth payoff function were handled. In addition, the scheme appeared to be second-order convergent with respect to its concerned spatial variable. They noted that difficulty using the scheme for constructing numerical solutions would be encountered if the the Black-Scholes model is described in an infinite domain. Thus, a preferred truncated domain can be used to overcome the difficulty.

Ravi and Aruna (2012) compared the DTM with PDTM for solving time-dependent Emden-Fowler type equations. Copious examples on linear non-homogeneous, non-homogeneous singular wave-like, and nonlinear time-dependent equations were used to ascertain the effectiveness and efficiency of the proposed methods. They noted that both methods gave the exact solutions, and rated the DTM as an effective method for the solution of both linear and nonlinear models; however, it is faced with some difficulties when constructing recursive relation for nonlinear models, and it also demands a lot of time for the computation using the algebraic recursive relation. This is unlike the projected version which solves the recursive relation with ease and less time.

Merdan (2013) proposed a multi-step DTM (MsDTM) for approximate and analytical solutions of a fractional order Vallis systems with regard to analysis of the stability of equilibrium. In addition, they carefully applied the multi-step DTM as a dependable modification of the classical DTM that develops the series solution convergence. The complex nature of the Vallis systems' dynamics were examined with the change of fractional order while validity of the proposed technique was ascertained by considering the Vallis systems at finite domain for the continuity of chaotic motion but the numerical solutions exhibited periodic motion in other interval range. The technique was used in a direct way without resorting to perturbation, linearization or restrictive assumption. Also, the solutions were provided in terms of convergent series with easily computable components with remarkable performance in terms of results.

Uddin *et al.* (2013) considered solution methods for the Black Scholes model with European options, by studying a weighted average method using different weights numerical approximations, and as such approximate the model using finite difference scheme. They discussed extensively the solutions of the Black-Scholes equation by means of Fourier transformation method for European-type of options. In their approach, the B-S equation was transformed to heat equation in order to obtain numerical solutions of the model. Thereafter, a finite difference scheme was applied to the transformed problem for approximate solutions; and the backward switch of the coordinate transformation to obtain the solution of the original partial differential equation (B-S equation) was carried out. The basic difficulty encountered using the approach is that the scheme required a very little step-size for its convergence; thus, the scheme is very slow in nature. The generated system of linear equations by discretizing of the B-S equation could be handled by a lot of contemporary methods, but for large scaled linear systems. Researchers barely employ direct methods because they are computationally not cheap. So, they were motivated to solve the discretized system of equations via other iterative techniques. Next, they investigated which linear solver converges quickly. To this point, they selected Jacobi Iterative Method (JIM), Gauss-Seidel Iterative Method (G-SIM), Generalized Minimal Residual Method (GMRESM), Preconditioned Conjugate Gradient Method (PCGM), Bi-Conjugate Gradient Stabilized Method (BICGSTABM) and successive over relaxation method. Their study considered only the one dimensional version of the linear Black-Scholes model, and they remarked that a research on a non-linear Black-Scholes equation with higher order accurate schemes likewise multi-dimensional version of the model seem more interesting but challenging; hence, they are left as future research interests.

Agliardi *et al.* (2013) considered the solution of the Black-Scholes equation by means of Mellin Transformation (MT) approach. For the solutions of linear and nonlinear Black–Scholes option pricing models, other methods such as the Adomian Decomposition Method (ADM), Modified ADM (MADM), Modified Variational Iteration Method (MVIM), Homotopy Perturbation Method (HPM), Modified HAM (MHAM), Homotopy Analysis Method (HAM) have been considered for application (Allahviranloo and Behzadi, 2013; Bohner *et al.* 2014).

In recent years, priority has been given to the study of Fractional Differential Equations (FDEs) with their applications (Podlubny, 1999). This is traceable to its wider and important applications in fields not limited to sciences, engineering, management and finance. Fractional calculus appears to be a generalisation of the classical calculus. The greatest advantage in using FDEs lies in their nonlocal property since integer order differential operators are local operators while fractional order differential operators are nonlocal; meaning that a system next state depends not only on its current state but also on all of its historical states (Kilbas *et al.* 2006). Nazari and Shahmorad (2010) considered the solutions of fractional integro-differential equations via nonlocal boundary conditions via fractional DTM. They applied the method to linear fractional integro-differential equation with constant and variable coefficients subject to the given initial conditions.

It is observed that most FDEs do not have exact analytical solutions; and even if they exist, corresponding direct methods seem not available or appear complex in applications. Hence, the involvement of analytical, numerical and approximate-analytical methods for approximate and exact solutions (Ibrahim and Jalab, 2015). In considering solutions of Fractional Type Black-Scholes Equations (FTBSEs) in option pricing settings, Kumar *et al.* (2012) coupled the Homotopy perturbation method with Laplace transform to obtain an accurate and quick solution to the fractional Black-Scholes equation with boundary condition for a European option pricing problem. Based on this coupled method, the solutions: exact and analytical were obtained without any discretization, restrictive suppositions, or identification of Lagrange multiplier. In addition, the method is free from round-off errors, thereby reducing the numerical computations to a reasonable extent.

Elbeleze *et al.* (2013) coupled three powerful approximate-analytical methods viz: Homotopy Perturbation Method (HPM), Sumudu Transform (ST), and He's Polynomials (HP) to obtain the solution of fractional Black-Scholes equation. The fractional derivative was defined in Caputo sense. As a way of ensuring efficiency and reliability of the coupled method, they solved the same equation by Homotopy Laplace Transform Perturbation Method (HPTPM). The results obtained using the two methods agree. The approximate-analytical solutions of the Black-Scholes were presented in power series form with easily computed components.

Ahmad *et al.* (2013) employed Fractional Variation Iterative Method (FVIM) for analytical solutions of linear fractional Black-Scholes equations. The basic aim of their research therein, is to provide an analytical solution of fractional Black-Scholes equation by Variational Iterational Method (VIM) with the modified Riemann-Liouville derivative approach to determine the simplicity and the efficiency of the proposed method. The method was used in a direct way without linearization, perturbation or restrictive assumption and only a few steps lead to highly accurate solutions which are valid for the whole solution domain. It can be concluded that VIM is a very powerful and reliable technique in finding the exact and approximate solutions to the fractional differential equation.

Hariharan *et al.* (2013) employed the Laplace Legendre Wavelet Hybrid Method (LLWHM) for numerical solutions of the linear fractional Black-Scholes European-option pricing model subject to some boundary conditions. Their main point in using the wavelet method was to convert the time-fractional Black-Scholes equation to a set of algebraic functions or equations involving finite number of variables. The employed LLWHM schemes are capable of overcoming the problem of integral values calculation involved in nonlinear PDEs. The LLWHM when compared to the traditional Legendre Wavelet Technique (LWT) for the solutions of differential equations with fractional orders, show higher level of efficiency. They further remarked that LLWHM has less implementation time compared to VIM, HPM, and HAM.

Ghandehari and Ranjbar (2014) considered an extension of the decomposition method via expansion series for analytical solutions of the fractional Black-Scholes option pricing model, and pointed out the main merit of the method to be easy handling of weaknesses resulting from unsatisfied conditions associated with the initial problem. They also proved the convergence of the decomposition power series for the fractional Black-Scholes equation following the pattern of proof of the Mittag-Leffler function of convergence within real and positive domain. Kumar *et al.* (2014) implemented the HPM and HAM to solve the Time-Fractional Black-Scholes Equation (TFBSE) with boundary conditions. The HPM and HAM are two different approximate-analytical methods for solving differential equations; they are related as they are both built on homotopy theory. Kumar *et al.* (2014) described the fractional derivatives in their work in the sense of Caputo. They concluded despite the similarities shared by both methods that HAM solutions are more general compared to HPM solutions.

Recently, Phaochoo *et al.* (2016) applied Meshless Local Petrov–Galerkin Method (MLPGM) for solving the Black–Scholes equation of non-fractional order, and employed Moving Kriging Shape (MCS) functions with the properties of Kronecker delta. They chose their time-based discretization by the Crank–Nicolson method. In their scheme, they noted the relationship between the eigenvalue of the system augmented matrix, the mesh spacing parameter, the shape parameter, the volatility term, and the risk-free interest rate. The MLPGM presented the option value both in regular and irregular nodal points. They submitted that the link between the shape parameters and errors varies by risk-free interest rate, and volatility.

Akrami and Erjaee (2016) implemented a numerical finite method to obtain the solution of the Black-Scholes equation for European and American option types, both in time and asset fractional orders. The early exercise nature of the American option type denied the application of the traditional finite difference method. Thus, finding early boundary before the discretization of the underlying asset becomes essential with respect to each time step. This seemed difficult with the implicit scheme they adopted. Hence, they resorted to the use of an iterative method known as Successive Over Relaxation (SOR) method. They noted that the boundary condition for European options and American options are the same since their payoff are the same at expiration. Numerical solutions of the Black-Scholes American option were emphasized since exact solutions for fractional Black-Scholes American options do not exist.

In this study, a modified version of the DTM referred to as Modified Differential Transform Method (MDTM) and the Revised Homotopy Perturbation Method (RHPM) are hereby adopted and presented for the first time in literature, for solving the classical Black- Scholes equation in option pricing and valuation. Part of our intentions is to test the effectiveness and reliability of the proposed approximate-analytic methods (MDTM), as the RHPM and MDTM would be applied in the later part of the work for solving the resulting nonlinear models from stochastic differential equations, with their generalised forms. The SDEs are based on Itô calculus.

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this section, some fundamental definitions, description of theorems in relation to stochastic analysis: Brownian motion and Itô calculus for the study of stock options are presented. Key properties and theorems of the modified approximate-analytical methods for analytical or approximate solutions are also presented. In addition, the Transformed Black-Scholes Model is derived. The methods and approaches adopted for the accomplishment of the stated objectives in this research include:

(i) Itô approach to Stochastic Differential Equations (SDEs);

(ii) approximate-analytical method: the Modified DTM (MDTM) and the He's polynomials; and

(iii) Maple 18, and Excel 2013 software for calculation and computation.

3.2 Basic Definitions

The definitions below are given according to Henderson and Plaschko (2006), Nkeki (2011) and Owoloko (2014) .

Definition 3.2.1: *Let* Ω *be a set and* $\breve{\mathrm{B}}$ *be a* $\sigma-$algebra *of subsets of* Ω*. Then the pair* $\left(\Omega, \breve{\mathrm{B}}\right)$ *is called a measurable space, while any member of* $\breve{\mathrm{B}}$ *is called a measurable set.*

Definition 3.2.2: *Given that* $\left(\Omega, \breve{\mathrm{B}}\right)$ *is a measurable space, then a map:*

$$\ulcorner\cup\urcorner : \breve{\mathrm{B}} \rightarrow \mathrm{R}^{+}$$

is called a measure on $\left(\Omega, \breve{\mathrm{B}}\right)$ *and the triplet* $\left(\Omega, \breve{\mathrm{B}}, \ulcorner\cup\urcorner\right)$ *is called a measure space provided that:*

$$\ulcorner\cup\urcorner\,(\emptyset) = 0 \text{ and } \ulcorner\cup\urcorner\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \ulcorner\cup\urcorner\,(E_i)$$

where $E_i \in \breve{\mathrm{B}},\ E_i \cap E_j = \emptyset,\ i \neq j.$

Definition 3.2.3: *The measure* $\ulcorner\cup\urcorner = \dddot{P}$ *is called a probability measure if* $\dddot{P}\,(\Omega) = 1$, *while* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}\right)$ *is thus referred to as a probability space.*

Definition 3.2.4: *Let* Ω_* *be the sample space of a random experiment. Then, a real valued function* X_* *defined on* Ω_* *such that* $X_* : \Omega_* \rightarrow \mathrm{R}$ *is called a random variable which may be discrete or continuous.*

Definition 3.2.5: *For* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}\right)$, *the collection of* $\sigma - \text{sub} - \text{algebras}$*:* $\bar{F}\left(\breve{\mathrm{B}}\right) = \left\{\breve{\mathrm{B}}_t : t \in [0, \infty)\right\}$ *is called a fitration if* $\breve{\mathrm{B}}_0$ *contains all the null members of* $\breve{\mathrm{B}}$ *and* $\breve{\mathrm{B}}_0 \subseteq \breve{\mathrm{B}}_s \subseteq \breve{\mathrm{B}}_t \subseteq \breve{\mathrm{B}}$ *whenever* $0 \leq s \leq t$. *Thus,* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}, \bar{F}\left(\breve{\mathrm{B}}\right)\right)$ *is called a filtered probability space or a stochastic basis. A fitration is interpreted as the flow of information.*

Definition 3.2.6: *A Stochastic process* $X_{p*} = \{X_{*t}\,(\omega)\,,\ \omega \in \Omega,\ t \in \tau\}$ *is a collection of random variables defined on the same probability space* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}\right)$.

Definition 3.2.7: *A Stochastic process* $X_{p*} = \{X_{*t}(\omega),\ \omega \in \Omega,\ t \in \tau\}$ *is said to be adapted to a filtration* $\bar{F}\left(\breve{\mathrm{B}}\right)$ *if* X_{*t} *is* $\breve{\mathrm{B}}_t-$*measurable for each* $t \geq 0$.

Definition 3.2.8: *Mathematical Expectation/Variance of* X_* *Let* $X_* \in L^1\left(\Omega, \breve{\mathrm{B}}, \ddot{P}\right)$, *then the mean value or mathematical expectation of the random variable* X_* *is defined and denoted as:*

$$\mathrm{E}(X_*) = \int_{\Omega} X_*(\omega)\ \ddot{P}\ d\omega. \tag{3.1}$$

If $X_* \in L^2\left(\Omega, \breve{\mathrm{B}}, \ddot{P}\right)$ *then we define and denote the variance of* X_* *as:*

$$Var(X_*) = \mathrm{E}(X_* - \mathrm{E}(X_*))^2. \tag{3.2}$$

Definition 3.2.9: *Let* $X_{p*} = \{X_{*t}(\omega),\ \omega \in \Omega,\ t \in \tau\}$ *defined on* $\left(\Omega, \breve{\mathrm{B}}, \ddot{P}\right)$ *be an adapted process to a stochastic base* $\bar{F}\left(\breve{\mathrm{B}}\right)$. *Then,* X_{p*} *is called a martingale if for* $X_{p*} \in L^1\left(\left(\Omega, \breve{\mathrm{B}}, \ddot{P}\right)\right)$, *we have* $\mathrm{E}\left(X_{*t} \middle| \breve{\mathrm{B}}_s\right) = X_{*s}$, *all* $0 \leq s \leq t \leq T$.
Note: X_{p*} *is called supermartingale (resp. submartingale) if:*
$\mathrm{E}\left(X_{*t} \middle| \breve{\mathrm{B}}_s\right) \leq X_{*s}$ $\left(\textit{resp. } \mathrm{E}\left(X_{*t} \middle| \breve{\mathrm{B}}_s\right) \geq X_{*s}\right)$.

Definition 3.2.10: *A continuous real-valued stochastic process* $W(t) = \{W_t : t \in \mathrm{R}^+\}$ *defined on a probability space* $\left(\Omega, \breve{\mathrm{B}}, \ddot{P}\right)$ *is called a Brownian motion provided that the process has stochastically independent increments such that* $W_0 = 0$; *for* $0 \leq s \leq t$, *the increment:* $W_t - W_s$ *is a Gaussian random variable with zero mean and variance* $(t-s)$, *i.e.* $W_t - W_s \sim N(0, t-s)$, *and both* $W_t - W_s$ *and* W_{t-s} *has the same distribution such that* $E[W_t - W_s] = E[W_{t-s}]$.

Definition 3.2.11: *Let $W(t) = \{W_t : t \in \mathrm{R}^+\}$ be a Brownian motion, then a stochastic process $X = \{X(t) : t \in \mathrm{R}^+\}$ satisfying the SDE is given as:*

$$dX(t) = X(t)(\lambda dt + \sigma dW(t)) \tag{3.3}$$

for $X(t_0) = x_0$, $\lambda \in \mathrm{R}$ and $\sigma > 0$ is called a Geometric Brownian Motion (GBM).

Note: In this work, we assume throughout that our underlying assets follow the standard GBM, hence the principles and theories of standard GBM will be adopted.

Figure 3.1 represents a Geometric Brownian Motion (GBM) obtained via the simulation of stock index with the parameters: initial share index ($S_0 = 5306.99$), volatility ($\sigma = 0.07$), drift ($\mu = 0.1$) with a time step ($\Delta t = 0.001$). This shows the pattern and behaviour of the stock market.

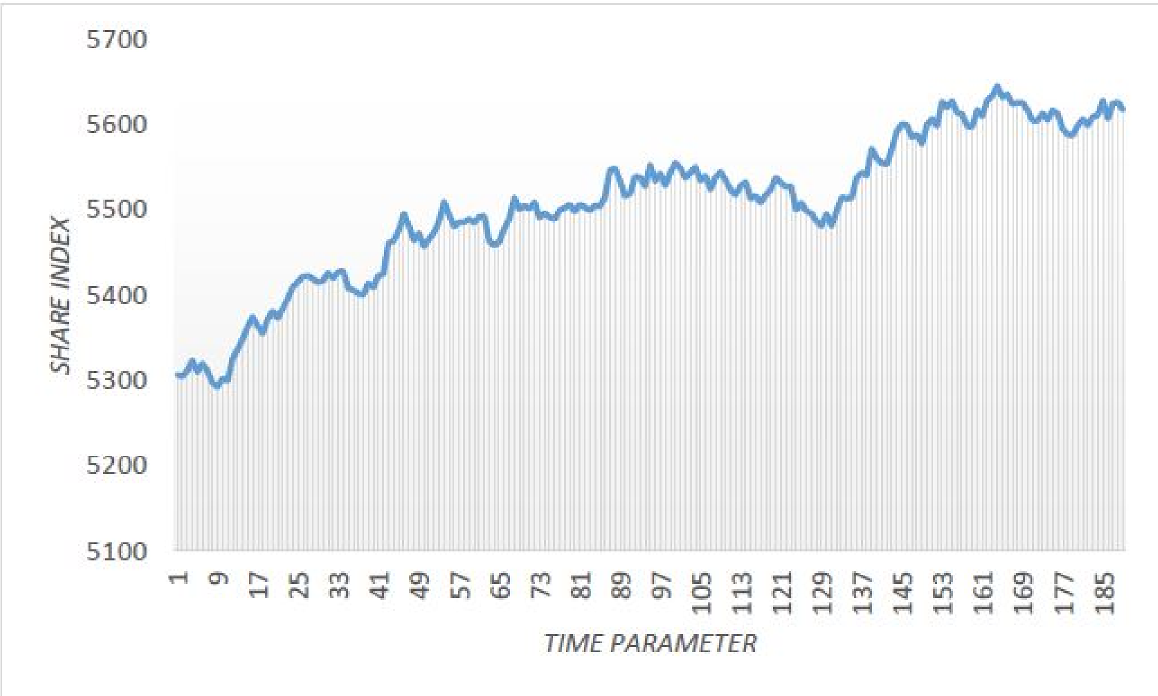

Figure 3.1: GBM obtained via stock data simulation

Definition 3.2.12: *Let* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}, F\left(\breve{\mathrm{B}}\right)\right)$ *be a filtered probability space, and* W *a Brownian motion relative to this space. Then a stochastic process* $(M(t), 0 \leq t \leq T)$ *where each:*

$$M(t) = M_0 + \int_0^t f(s)\, dW(s) + \int_0^t g(s)\, ds, \ t \in \mathrm{R}_+ \tag{3.4}$$

is called an $It\widehat{o}$ *process , where* M_0 *is* $\breve{\mathrm{B}}_0-$*measurable,* $W(f,t) = \int_0^t f(s)\, dW(s)$ *is a stochastic integral and* $\int_0^t g(s)\, ds = I$ *is referred to as a random Lebesgue integral.* For the stochastic integral to be well-defined, the integrand process must be adapted to the associated stochastic base. In differential form, (3.4) can be expressed as a stochastic differential equation (SDE) of the form:

$$dM(t) = g(t)\, dt + f(t)\, dW(t). \tag{3.5}$$

Simply put, an SDE is a differential equation having one or more of its terms as a stochastic process, yielding a solution that is also a stochastic process.

Notation: We denote the set of all continuous functions which are once differentiable with respect to (w.r.t.) the first variable and twice differentiable w.r.t. the second variable by $C^{1,2}(T \times \mathrm{R})$.

Definition 3.2.13: *Quadratic variation: Let* X_{*p} *be a stochastic process on a stochastic base* $\left(\Omega, \breve{\mathrm{B}}, \dddot{P}, F\left(\breve{\mathrm{B}}\right)\right)$, *and* $p : 0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ *be a sequence of partitions of a given interval* $[0, t]$. *Then, the quadratic variation of* X_* *on* $[0, t]$ *is a stochastic process defined and denoted as:* $\langle X_* \rangle (t) = \lim\limits_{|p| \to 0} \sum_{j=0}^{n} |X_*(t_{j+1}) - X_*(t_j)|^2$, where $|p| = \max\limits_{0 \leq j \leq n-1} |t_{j+1} - t_j|$.

3.3 Itô Approach to Stochastic Differential Equations (SDEs)

The importance of SDEs in physical systems and finance cannot be overstated. In describing physical systems, SDEs consider some randomness associated with the systems. One of the ways of solving these SDEs is by applying the Itô's formula as stated below:

Theorem 3.3.1: [Itô, (1946; 1951)]

Suppose $X(t)$, an adapted stochastic process defined on a filtered probability space $F_{PS} = \left(\Omega, \breve{\mathrm{B}}, \ddot{P}, F\left(\breve{\mathrm{B}}\right)\right)$, is a solution of the SDE. Then,

$$dX(t) = H_1 dt + H_2 dW(t) \tag{3.6}$$

where $H_1 = H_1(t, X(t))$ represents the drift coefficient, $H_2 = H_2(t, X(t))$ is the diffusion coefficient, $W(t)$ is a standard Brownian motion representing the intrinsic noise (white noise) in the dynamical system; and $u \in \mathrm{C}^{1,2}(\mathrm{R}^+ \times [0, T])$, such that $u = u(t, x) : (0, \infty) \times \mathrm{R} \to \mathrm{R}$ is a twice continuously differentiable function with time, t. Then $M(t) = u(t, X(t))$ is a stochastic process for which:

$$du(t, X(t)) = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dX(t) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(dX(t))^2, \tag{3.7}$$

whence

$$du(t, X(t)) = \left(\frac{\partial u}{\partial t} + H_1 \frac{\partial u}{\partial x} + \frac{1}{2} H_2^2 \frac{\partial^2 u}{\partial x^2}\right) dt + H_2 \frac{\partial u}{\partial x} dW(t). \tag{3.8}$$

Equation (3.8) is called an Itô's formula.

Note: If $X(t) = W(t)$ (Ref. Definition 3.2.11), then:

$$dX(t) = dW(t). \tag{3.9}$$

Comparing (3.9) with (3.6) shows that $H_1 \equiv 0$ and $H_2 \equiv 1$, therefore, (3.9) becomes:

$$du\left(t, X(t)\right) = \left(\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\right) dt + \frac{\partial u}{\partial x} dW(t). \tag{3.10}$$

Equation (3.10) is called an Itô lemma.

Definition 3.3.1: *Let* $F_{PS} = \left(\Omega, \breve{\mathrm{B}}, \dddot{P}, F\left(\breve{\mathrm{B}}\right)\right)$ *be a filtered probability space, and* W_t *a Brownian motion defined on* F_{PS} , $t \in \mathrm{R}_+$ *then:*

$$dW_t \otimes \Delta_* = \Delta_* \otimes dW_t = \begin{cases} dt\ , \text{ for } \Delta_* = dW_t \\ 0\ \ ,\ \text{otherwise.} \end{cases} \tag{3.11}$$

Equation (3.11) is referred to Itô multiplication table. The corresponding SDEs will be solved using Itô's formula.

3.4 Analysis of the Basic Methods

This subsection introduces the basic concepts and theorems of DTM needed for applications in the remaining sections (Zhou, 1986).

Definition 3.4.1: *Let* $w(x)$ *be a given function of one variabe defined at a point* $x = x_0$*, then the one-dimensional* k^{th} *differential transform of* $w(x)$ *defined as* $W(k)$ *is:*

$$W(k) = \frac{1}{k!}\left(\frac{d^k w(x)}{dx^k}\right)\bigg|_{x=x_0}. \tag{3.12}$$

Definition 3.4.2: *The inverse differential transform of $W(k)$ is a Taylor series expansion of the function $w(x)$ about $x = x_0 = 0$, defined as :*

$$w(x) = \sum_{k=0}^{\infty} W(k)x^k. \tag{3.13}$$

Combining (3.12) and (3.13) yields:

$$w(x) = \sum_{k=0}^{\infty} \left(\frac{d^k w(x)}{dx^k} \right) \frac{x^k}{k!}. \tag{3.14}$$

3.4.1 Some Basic Properties of the Differential Transform Method

The following properties of the DTM are stated below for the purpose of applications while their proofs and further properties can be found in standard numerical texts and referred journals such as Zhou (1986) and the related references therein.

Let $w_1(x)$, $w_2(x)$ and $w_*(x)$ be differentiable functions with differential transforms $W_1(k)$, $W_2(k)$ and $W_*(k)$ respectively, with $n \geq 0$, $\alpha_i \in$ R and δ a kronecker delta, then the following properties ($P1 - P4$) hold.

P1: If $y = \alpha_1 w_1(x) \pm \alpha_2 w_2(x)$ then $Y(k) = \alpha_1 W_1(k) \pm \alpha_2 W_2(k)$.

P2: If $y = x^n$ then $Y(k) = \delta(k - n)$ such that:

$$Y(k) = \delta(k-n) = \begin{cases} 1\ ,\ k = n \\ \\ 0\ ,\ \textit{otherwise.} \end{cases}$$

P3: If $y = w_1(x)w_2(x)$, then $Y(k) = \sum_{\tau=0}^{k} W_1(\tau)W_2(k-\tau)$.

P4: If $y = \frac{d^n}{dx^n}[w_*(x)]$, then $Y(k) = \frac{(k+n)!}{n!} W_*(k+n)$.

In particular, we have:

* If $y = \frac{d}{dx}\left[w_*(x)\right]$, then $Y(k) = (k+1)W_*(k+1)$.

3.4.2 The Overview of the Modified DTM (MDTM)

Suppose $\omega(\varsigma,\tau)$ is analytic in a region D_* at (ς,τ) then by considering the Taylor series expansion of the function: $\omega(\varsigma,\tau)$, preference is given to some variables, say $s_\omega = \tau$ instead of considering all the variables as practiced in the classical Zhou method (DTM). Therefore, the projected DTM of $\omega(\varsigma,\tau)$ with respect to τ at τ_* is defined and expressed as:

$$\Omega\left(\varsigma,\overline{h}\right) = \frac{1}{\overline{h}!}\left[\frac{\partial^{\overline{h}}\omega(\varsigma,\tau)}{\partial\tau^{\overline{h}}}\right]_{\tau=\tau_*} \tag{3.15}$$

and as such:

$$\omega(\varsigma,\tau) = \sum_{\overline{h}=0}^{\infty}\Omega\left(\varsigma,\overline{h}\right)(\tau-\tau_*)^{\overline{h}} \tag{3.16}$$

where (3.15) is referred to as the projected differential inverse transform (PDIT) of $\Omega\left(\varsigma,\overline{h}\right)$ with respect to τ.

3.4.3 Some Fundamental Properties and Features of the MDTM

These fundamental properties $(P5 - P9)$ are as follows:

P5: If $\omega(\varsigma,\tau) = \omega_a(\varsigma,\tau) \pm \beta\omega_b(\varsigma,\tau)$, then $\Omega\left(\varsigma,\overline{h}\right) = \alpha\Omega_a\left(\varsigma,\overline{h}\right) \pm \beta\Omega_b\left(\varsigma,\overline{h}\right)$.

P6: If $\omega(\varsigma,\tau) = \alpha\frac{\partial^n\omega_*(\varsigma,\tau)}{\partial\tau^n}$, then $\Omega\left(\varsigma,\overline{h}\right) = \alpha\frac{\left(\overline{h}+n\right)!}{\overline{h}!}\Omega_*\left(\varsigma,\overline{h}+n\right)$.

P7: If $\omega(\varsigma,\tau) = \alpha\frac{\partial\omega_*(\varsigma,\tau)}{\partial\tau}$, then $\Omega\left(\varsigma,\overline{h}\right) = \alpha\frac{\left(\overline{h}+1\right)!\Omega_*\left(\varsigma,\overline{h}+1\right)}{\overline{h}!} = \alpha\left(\overline{h}+1\right)\Omega_*\left(\varsigma,\overline{h}+1\right)$.

P8: If $\omega(\varsigma,\tau) = c(\varsigma)\frac{\partial^n\omega_*(\varsigma,\tau)}{\partial\varsigma^n}$, then $\Omega\left(\varsigma,\overline{h}\right) = c(\varsigma)\frac{\partial^n\Omega_*\left(\varsigma,\overline{h}\right)}{\partial\varsigma^n}$.

P9: If $\omega(\varsigma,\tau) = c(\varsigma)\,\omega_*^2(\varsigma,\tau)\,,$ then $\Omega\left(\varsigma,\overline{h}\right) = c(\varsigma)\sum_{r=0}^{\overline{h}}\Omega_*(\varsigma,r)\,\Omega_*\left(\varsigma,\overline{h}-r\right).$

3.4.4 Overview of the He's Polynomial

In general form, we consider the equation:

$$\vartheta(\varpi) = 0 \tag{3.17}$$

where ϑ is a differential or an integral operator. Let $H(\varpi,p)$ be a convex homotopy defined by:

$$H(\varpi,p) = p\vartheta(\varpi) + (1-p)\,G(\varpi) \tag{3.18}$$

where $G(\varpi)$ is a functional operator with ϖ_0 as a known solution. Thus, we have:

$$H(\varpi,p) = \begin{cases} G(\varpi)\,, & p = 0 \\ \vartheta(\varpi)\,, & p = 1 \end{cases} \tag{3.19}$$

whenever $H(\varpi,p) = 0$ is satisfied, and $p \in (0,1]$ is an embedded parameter. In Ghorbani and Nadjfi (2007), and Ghorbani (2009), p is used as an expanding parameter to obtain:

$$\varpi = \sum_{j=o}^{\infty} p^j \varpi_j = \varpi_0 + p\varpi_1 + p^2\varpi_2 + \cdots\,. \tag{3.20}$$

From (3.20), the solution is obtained as $p \to 1$.

The method considers $N(\varpi)$ (the nonlinear term) as:

$$N(\varpi) = \sum_{j=o}^{\infty} p^j H_j = H_0 + pH_1 + p^2H_2 + \cdots \tag{3.21}$$

where $H_k's$ are the so-called He's polynomials, which can be computed using:

$$H_k\left(\varpi_0, \varpi_1, \varpi_2, \cdots\right) = \frac{1}{k!}\frac{\partial^k}{\partial p^k}\left(N\left(\sum_{j=0}^{k} p^j \varpi_j\right)\right)_{p=0}. \tag{3.22}$$

3.4.5 Fractional Calculus: The Preliminary

Here, we will give a brief and concise introduction to fractional calculus, some key definitions and theorems that will be used later in the work (Podlubny, 1999; Kilbas *et al.* 2006; Ahmed, 2014). In fractional calculus, the power of the differential operator is considered as a real or complex number. Hence, the following:

Definition 3.4.5.1: Let $D(\cdot)$ and $J(\cdot)$ be differential and integral operators respectively, then for $\gamma(x) = x^k$ (a monomial, of degree k, not necessarily a fraction), we have:

$$D\gamma(x){=}kx^{k-1},\ \ D^2\gamma(x){=}k(k-1)x^{k-2} = \frac{k!}{(k-2)!}x^{k-2}. \tag{3.23}$$

In general,

$$D^m\gamma(x){=}\frac{k!}{(k-m)!}x^{k-m}. \tag{3.24}$$

But in terms of gamma notation, (3.24) is expressed as:

$$D^\alpha\gamma(x){=}\frac{\Gamma\left(k+1\right)}{\Gamma\left(k-\alpha+1\right)}x^{k-\alpha}. \tag{3.25}$$

Equation (3.25) is referred to as fractional derivative of $\gamma(x)$, of order α, if $\alpha \in$ R

Definition 3.4.5.2: Suppose $\gamma(x)$ is defined for $x > 0$, then:

$$(J\gamma)(x) = \int_0^x \gamma(s)ds. \tag{3.26}$$

An arbitrary extension of (3.26) (i.e. Cauchy formula for repeated integration) yields:

$$(J^m\gamma)(x) = \frac{1}{(m-1)!}\int_0^x (x-s)^{m-1}\gamma(s)ds. \tag{3.27}$$

Thus, in gamma sense, (3.27) is expressed as:

$$(J^\alpha\gamma)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-u)^{\alpha-1}\gamma(u)du\ ,\ \alpha > 0,\ t > 0. \tag{3.28}$$

Equation (3.28) is referred to as the Riemann-Liouville fractional integration of order α.

Definition 3.4.5.3: We define Riemann-Liouville fractional derivative (R-LFD) and Caputo fractional derivative (CFD) of $\gamma(x)$ respectively, for $\alpha \in (\hbar - 1, \hbar)$, $\hbar \in$ N, as follows:

$$D^\alpha\gamma(x) = \begin{cases} \frac{d^h\left(J^{h-\alpha}\gamma(x)\right)}{dx^h},\ \mathrm{R-LFD} \\ \\ \frac{J^{\hbar-\alpha}\left(d^h\gamma(x)\right)}{dx^{\hbar}},\ C\mathrm{FD}. \end{cases} \tag{3.29}$$

Note: In (3.29), Riemann-Liouville compute first, the fractional integral of the function and thereafter, an ordinary derivative of the obtained result but the reverse is the case in Caputo sense of fractional derivatives; this allows the inclusion of the traditional initial and boundary conditions in the formulation of the problem.

***Remark* 3.4.5:** {Ref. Lemma 4 in (Song *et al.* 2013)}: The link between the Riemann-Liouville operator and the Caputo fractional differential operator for $\alpha \in (n-1, n)\,, n \in \mathrm{N}$ is:

$$\left(J^{\alpha} D_t^{\alpha}\right) \gamma(t) = \left(D_t^{-\alpha} D_t^{\alpha}\right) \gamma(t) = \gamma(t) - \sum_{k=0}^{n-1} \gamma^{k}(0) \frac{t^k}{k!}. \tag{3.30}$$

$$\therefore \quad \gamma(t) = \left(J^{\alpha} D_t^{\alpha}\right) \gamma(t) + \sum_{k=0}^{n-1} \gamma^{k}(0) \frac{t^k}{k!}. \tag{3.31}$$

Definition 3.4.5.4: The Mittag-Leffler function, $E_{\alpha}(z)$ valid in the whole complex plane C is defined and denoted by the series representation as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad \alpha \geq 0,\ z \in \mathrm{C}. \tag{3.32}$$

***Note*:** $E_{\alpha}(z)$ in (3.32) becomes $E_{\alpha=1}(z) = e^z$ for $\alpha = 1$.

3.4.6 Modified Differential Transform (MDT) of a Function with Fractional Derivative

Let $g(x,t)$ be an analytic function whose PDT is $G(x,k)$.

Thus, if $g(x,t) = D_t^{\alpha} p(x,t)$, then:

$$G(x,k) = \left(\Gamma\left(1 + \frac{k}{q}\right)\right)^{-1} \Gamma\left(1 + \alpha + \frac{k}{q}\right) P(x, k + \alpha q).$$

Consequently, we have:

$$\Gamma\left(1+\alpha+\frac{k}{q}\right)P\left(x,k+\alpha q\right)=\Gamma\left(1+\frac{k}{q}\right)G\left(x,k\right). \tag{3.33}$$

Setting $\alpha q = 1$ in (3.33) gives:

$$P\left(x,k+1\right)=\frac{\Gamma\left(1+\alpha k\right)}{\Gamma\left(1+\alpha\left(1+k\right)\right)}G\left(x,k\right). \tag{3.34}$$

Therefore, $g\left(x,t\right)$ is defined in series form as:

$$g\left(x,t\right)=\sum_{h=0}^{\infty}G\left(x,h\right)t^{\frac{h}{q}}=\sum_{h=0}^{\infty}G\left(x,h\right)t^{\alpha h}. \tag{3.35}$$

3.4.7 Overview of a Local Fractional PDTM

Consider the Nonlinear Fractional Differential Equation (NLFDE):

$$D_t^{\beta}u\left(x,t\right)+L_{[x]}u\left(x,t\right)+N_{[x]}u\left(x,t\right)=q\left(x,t\right),u\left(x,0\right)=g\left(x\right),t>0 \tag{3.36}$$

where $D_t^{\beta}=\frac{\partial^{\beta}}{\partial t^{\beta}}$ is the fractional Caputo derivative of $u=u\left(x,t\right)$; whose projected differential transform is $U(x,h)$, and $L_{[\cdot]}$ and $N_{[\cdot]}$ are linear and nonlinear differential operators with respect to x respectively, while $q=q\left(x.t\right)$ is the source term.

We rewrite (3.36) for $n-1<\beta<n$, $n\in \mathrm{N}$ as:

$$D_t^{\beta}u\left(x,t\right)=-L_{[x]}u\left(x,t\right)-N_{[x]}u\left(x,t\right)+q\left(x,t\right),\ u\left(x,0\right)=g(x). \tag{3.37}$$

Applying the inverse fractional Caputo derivative, $D_t^{-\beta}$ to both sides of (3.37) gives:

$$u\left(x,t\right)=g(x)+D_t^{-\alpha}\left[-L_{[x]}u\left(x,t\right)-N_{[x]}u\left(x,t\right)+q\left(x,t\right)\right],\ u\left(x,0\right)=g(x). \tag{3.38}$$

Thus, the inverse projected differential transform of $U(x,h)$ following (3.35) is given as follows:

$$u\left(x,t\right)=\sum_{h=0}^{\infty}U\left(x,h\right)t^{\beta h}=u\left(x,0\right)+\sum_{h=1}^{\infty}U\left(x,h\right)t^{\beta h},\ u\left(x,0\right)=g(x). \tag{3.39}$$

3.5 Sources of Data: NGSE (2016) and NGSEINDX (2016)

For data simulation in this work, we used historical data between 2000-2015 (15 years) from the Nigerian Stock Exchange with the following overview. Historically, in 1960, the Nigerian Stock Exchange (NGSE) was established as the Lagos Stock Exchange (LSE) but this was changed in 1977 from LSE to the NGSE. As at May 31, 2016, NGSE has about 179 listed companies from different sectors of the economy with *Nestle Nigeria* (NNG) having the highest share unit price of NGN 784.66 and *Amino International* having the least share unit price of NGN 0.25. The Nigerian Stock Exchange is the third largest stock exchange in Africa (NGSE, 2016).

In January 1984, The *Nigerian Stock Exchange All Share Index* (NSE-ASI) was formulated. In the computation of the index, only ordinary shares are included. The index is value-relative and is computed daily. The All-Share Index tracks the general market movement of all listed equities on the Exchange, including those listed on the Alternative Securities Market (ASeM), regardless of the capitalisation (NGSEINDX: IND, 2016).

3.6 The Transformed Black-Scholes Pricing Model

Here, we consider the process of deriving a stock option valuation model that incorporates the drift coefficient (rate of return) as a non-fixed constant without excluding the other parameters: the volatility rate, and the risk-free interest rate.

3.6.1 The Model Assumptions

We assume the following in formulating the transformed Black-Scholes Model:

A_1 : shares are traded over a short period of time interval,

A_2 : stock price movements are stochastic,

A_3 : total investment over the short time interval is constant, and

A_4 : price changes are small almost surely.

3.6.2 Derivation of the Transformed Black-Scholes Model

Suppose that the price of an underlying asset (typically a stock), S, follows a geometric Brownian motion $W(t)$ and satisfies the stochastic differential equation (SDE) in (2.2). Let $\Lambda(S,t)$ be the value of the contingent claim such that $\Lambda \in C^{2,1}\left[R \times [0,T]\right]$, with a payoff function $p_\Lambda\left(S,t\right)$, and expiration price, E such that:

$$p_\Lambda\left(S,t\right) = \begin{cases} \max(S-E,0), \text{ for European call option} \\ \max(E-S,0), \text{ for European put option} \end{cases} \tag{3.40}$$

where $\max\left(S_*,0\right)$ indicates the large value between S_* and 0. Then applying Itô lemma (Itô, 1951) gives:

$$\begin{aligned} d\Lambda &= \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial S^2}(dS)^2 \\ &= \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial S}\left(\alpha S dt + \sigma S dW(t)\right) + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial S^2}(\sigma^2 S^2 dt) \\ &= \left(\frac{\partial \Lambda}{\partial t} + \alpha S\frac{\partial \Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Lambda}{\partial S^2}\right)dt + \sigma S \frac{\partial \Lambda}{\partial S}dW(t) \end{aligned} \tag{3.41}$$

where the Itô multiplicative table (Definition 3.3.1) is used in computing the term: $(dS)^2$ with $(dt)^2 = dt.dW = 0$, and $dW.dW = dt$. Equation (3.41) therefore becomes:

$$\frac{d\Lambda}{dt} = \left(\frac{\partial\Lambda}{\partial t} + \alpha S\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2}\right) + \sigma S\frac{\partial\Lambda}{\partial S}\frac{dW(t)}{dt}. \tag{3.42}$$

Suppose a delta-hedged-portfolio, $\Theta(t)$, is considered by longing a contingent claim, and shorting a delta unit of the underlying asset, then the following dynamics are obtained:

$$\frac{d\Theta(t)}{dt} = \frac{d\Lambda(s,t)}{dt} - \Delta\frac{dS}{dt},\ \Delta = \frac{\partial\Lambda}{\partial S}\ . \tag{3.43}$$

So (3.41), and (3.42) are substituted in (3.43) as follows:

$$\begin{aligned}\frac{d\Theta(t)}{dt} &= \frac{d\Lambda(s,t)}{dt} - \frac{\partial\Lambda}{\partial S}\frac{dS}{dt}\\ &= \left(\frac{\partial\Lambda}{\partial t} + \alpha S\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2}\right) + \sigma S\frac{\partial\Lambda}{\partial S}\frac{dW(t)}{dt}\\ &- \frac{\partial\Lambda}{\partial S}\left(\alpha S + \sigma S\frac{dW(t)}{dt}\right)\\ &= \frac{\partial\Lambda}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\Lambda}{\partial S^2}.\end{aligned} \tag{3.44}$$

Now, suppose a non-unity portfolio, $\Theta(t) = \Theta_*(t)$ that allows the assumption of risk-neutrality is constructed in order to hedge away all randomness. Thereby considering $\Theta_*(t)$ by longing r unit value of the contingent claim and shorting α delta unit of the underlying asset as follows:

$$\Theta_*(t) = r\Lambda(S,t) - \alpha\Delta S. \tag{3.45}$$

Therefore, from (3.44) the following is obtained:

$$
\begin{aligned}
\frac{d\Theta_*(t)}{dt} &= \frac{\partial\Lambda}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\Lambda}{\partial S^2} \\
&= r\Lambda(S,t) - \alpha\Delta S.
\end{aligned}
\tag{3.46}
$$

Thus, simplifying (3.46) gives (3.47) as follows:

$$
\frac{\partial\Lambda}{\partial t} + \alpha S\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\Lambda}{\partial S^2} - r\Lambda = 0. \tag{3.47}
$$

Equation (3.47) has the basic parameters (α, σ, r) as earlier defined in section 2.3 with $\alpha = \mu$. Suppose $\Lambda = \Lambda(S_t, k_t)$ is taken to be an investment output, S_t the price of a risky asset (say stock), and k_t the total investment (assumed constant), all over a short period of time t, with p^{r*} and c^{r*} as production and consumption rates respectively and write $\frac{\partial\Lambda}{\partial t}$ in (3.47) as:

$$
\frac{\partial\Lambda(S_t, k_t)}{\partial t} = p^{r*}(S_t, k_t) - c^{r*}(S_t, k_t) \tag{3.48}
$$

with the assumption that:

$$
\lim_{t\to 0} c^{r*}(S_t, k_t) = 0 \text{ and } \lim_{t\to 0} p^{r*}(S_t, k_t) = S \tag{3.49}
$$

since p^{r*} is proportional to S as $t \to 0$.

Therefore, the limit as $t \to 0$ of both sides of (3.48) using (3.49) is taken as follows:

$$
\lim_{t\to 0}\left(\frac{\partial\Lambda(S_t, k_t)}{\partial t}\right) = \lim_{t\to 0} p^{r*}(S_t, k_t) - \lim_{t\to 0} c^{r*}(S_t, k_t)
$$

That implies that:

$$\begin{aligned} \frac{\partial \Lambda}{\partial t} &= \lim_{t \to 0} p^{r*}\left(S_t, k_t\right) - \lim_{t \to 0} c^{r*}\left(S_t, k_t\right) \\ &= S - 0 \\ &= S. \end{aligned} \tag{3.50}$$

Thus, (3.47) becomes:

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Lambda\left(S_t, k_t\right)}{\partial S^2} + \alpha S \frac{\partial \Lambda\left(S_t, k_t\right)}{\partial S} - r\Lambda\left(S_t, k_t\right) = -S. \tag{3.51}$$

Considering the assumptions: A_1 and A_4, where there is no declaration of new development; as such there is no purchase of new assets. Then, k_t , the total investment over a short trading period could be assumed constant. The implication of this is that $\Lambda\left(S_t, k_t\right)$ becomes $\Lambda\left(S\right)$ which is now a function of one variable. So the associated derivatives will be with respect to S only (giving rise to ODE not PDE as in (4.12)). Therefore, for $\Lambda\left(S\right) = \Lambda = \Lambda_N$, (3.51) becomes:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2 \Lambda_N}{dS^2} + \alpha S \frac{d\Lambda_N}{dS} - r\Lambda_N = -S. \tag{3.52}$$

Equation (3.52), henceforth, referred to as the Transformed Black-Scholes Model (TBSM), is a nonhomogeneous second order Cauchy-Euler differential equation.

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Introduction

In this section, the results are presented and discussed. In addition, the approximate-analytical methods (MDTM and RHTM) introduced earlier in Chapter Three are applied. To aid the discussion and understanding of the results, graphical representation of the results and tables of values are also presented.

4.2 The Theoretical Solution of the Transformed Black-Scholes Model

Here, the solution of the derived TBSM is proposed and numerical calculations are made for comparison.

Proposition 4.2.1:

A twice continuously differentiable function, $\overline{M}_* = \overline{M}_*(S)$ satisfies the transformed Black-Scholes equation:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2\overline{M}}{dS^2} + \alpha S \frac{d\overline{M}}{dS} - r\overline{M} = -S, \ \sigma \neq 0, S \neq 0 \tag{4.1}$$

such that:

$$\overline{M}_*(S) = \left(\frac{2\phi_2 - 1}{\phi_1 - \phi_2}\right) S^{\phi_1} + \left(\frac{1 - 2\phi_1}{\phi_1 - \phi_2}\right) S^{\phi_2} + \frac{S}{r - \alpha}, \ \phi_1 \neq \phi_2, \ r \neq \alpha$$

subject to

$$\overline{M}_*(1) = 0 \ , \ \overline{M'}_*(1) = 1$$

where

$$\phi_1 = \frac{1}{2}\left\{-\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)+\sqrt{\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)^2+\frac{8r}{\sigma^2}}\right\},$$

and

$$\phi_2 = \frac{1}{2}\left\{-\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)-\sqrt{\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)^2+\frac{8r}{\sigma^2}}\right\}.$$

Proof:

Suppose $\overline{M}_*$ is a solution of (4.1), then:

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2\overline{M}_*}{dS^2} + \alpha S \frac{d\overline{M}_*}{dS} - r\overline{M}_* = -S,\ \sigma \neq 0, S \neq 0 \tag{4.2}$$

Therefore, using the Cauchy-Euler approach with $S = e^z$, we have:

$$\frac{dz}{ds} = \frac{1}{S},$$

so

$$\begin{aligned}\frac{d\overline{M}_*}{dS} &= \frac{d\overline{M}_*}{dz}\frac{dz}{dS} \\ &= \frac{1}{S}\frac{d\overline{M}_*}{dz}\end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
\frac{d^2\overline{M}_*}{dS^2} &= \frac{d}{dS}\left(\frac{d\overline{M}_*}{dS}\right) \\
&= \frac{d}{dS}\left(\frac{1}{S}\frac{d\overline{M}_*}{dz}\right) \\
&= -\frac{1}{S^2}\frac{d\overline{M}_*}{dz} + \frac{1}{S}\frac{d}{dS}\left(\frac{d\overline{M}_*}{dz}\right) \\
&= -\frac{1}{S^2}\frac{d\overline{M}_*}{dz} + \frac{1}{S}\frac{d}{dz}\left(\frac{d\overline{M}_*}{dz}\right)\left(\frac{dz}{dS}\right) \\
&= -\frac{1}{S^2}\frac{d\overline{M}_*}{dz} + \frac{1}{S^2}\frac{d^2\overline{M}_*}{dz^2} \\
&= \frac{1}{S^2}\left(\frac{d^2\overline{M}_*}{dz^2} - \frac{d\overline{M}_*}{dz}\right).
\end{aligned} \tag{4.4}
$$

Therefore, (4.3) and (4.4) are substituted in (4.2) as follows:

$$
\frac{1}{2}\sigma^2 S^2\left(\frac{1}{S^2}\left(\frac{d^2\overline{M}_*}{dz^2} - \frac{1}{S^2}\frac{d\overline{M}_*}{dz}\right)\right) + \alpha S\left(\frac{1}{S}\frac{d\overline{M}_*}{dz}\right) - r\overline{M}_* = -e^z,\ \sigma \neq 0, S \neq 0. \tag{4.5}
$$

That implies:

$$
\frac{\sigma^2}{2}\frac{d^2\overline{M}_*}{dz^2} + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{d\overline{M}_*}{dz} - r\overline{M}_* = -e^z,\ \sigma \neq 0. \tag{4.6}
$$

Equation (4.6) is a nonhomogeneous second order linear ODE with constant coefficients. Thus, for the complementary solution, $\overline{M}_{*C}$, the characteristic equation (associated to the homogeneous part of (4.6)) is given as:

$$
\frac{\sigma^2}{2}\phi^2 + \left(\alpha - \frac{\sigma^2}{2}\right)\phi - r = 0 \tag{4.7}
$$

where ϕ is any function that satisfies (4.7), $a_* = \frac{\sigma^2}{2}$, $b_* = \left(\alpha - \frac{\sigma^2}{2}\right)$, and $c_* = -r$, such that:

$$\phi = \frac{-b_* \pm \sqrt{b_*^2 - 4a_*c_*}}{2a_*} \tag{4.8}$$

Therefore, the solution of (4.7) is expressed as:

$$\begin{aligned}
\phi_1 &= \frac{1}{2}\left\{-\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)+\sqrt{\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)^2+\frac{8r}{\sigma^2}}\right\}, \\
\phi_2 &= \frac{1}{2}\left\{-\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)-\sqrt{\left(\frac{2\alpha-\sigma^2}{\sigma^2}\right)^2+\frac{8r}{\sigma^2}}\right\}.
\end{aligned} \tag{4.9}$$

So,

$$\overline{M}_{*C}(z) = Ae^{\phi_1 z} + Be^{\phi_2 z}. \tag{4.10}$$

The nonhomogeneous term in (4.6) is $h(z) = -e^z$, hence, the assumed candidate for the particular solution is $\overline{M}_{*p}(z) = Ce^z$

$$\therefore\ \overline{M'}_{*p}(z) = Ce^z = \overline{M''}_{*p}(z). \tag{4.11}$$

So (4.8) is substituted in (4.3) as follows:

$$\frac{\sigma^2}{2}\frac{d^2\overline{M}_{*p}}{dz^2} + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{d\overline{M}_{*p}}{dz} - r\overline{M}_{*p} = -e^z. \tag{4.12}$$

That is:

$$\frac{\sigma^2}{2}\left(ce^z\right) + \left(\alpha - \frac{\sigma^2}{2}\right)\left(ce^z\right) - r\left(ce^z\right) = -e^z. \tag{4.13}$$

Showing that:

$$((\alpha - r)\,c + 1)\,e^z = 0. \tag{4.14}$$

So,

$$c = \frac{1}{r - \alpha}. \tag{4.15}$$

Hence,

$$\begin{aligned} \overline{M}_{*p} &= ce^z \\ &= \frac{e^z}{r - \alpha}. \end{aligned} \tag{4.16}$$

Thus,

$$\overline{M_*}(z) = Ae^{\phi_1 z} + Be^{\phi_2 z} + \frac{e^z}{r - \alpha}. \tag{4.17}$$

So, replacing e^z with S in (4.17) yields (4.18) below:

$$\overline{M}(S) = AS^{\phi_1} + BS^{\phi_2} + \frac{S}{r - \alpha} \tag{4.18}$$

for

$$\overline{M}\,(1) = 0, \quad \overline{M'}\,(1) = 1, \quad and \quad r - \alpha = \frac{1}{2} \tag{4.19}$$

where

$$A = \frac{2\phi_2 - 1}{\phi_1 - \phi_2}, \;\; \phi_1 \neq \phi_2 \tag{4.20}$$

and

$$B = \frac{1 - 2\phi_1}{\phi_1 - \phi_2}. \tag{4.21}$$

Q.E.D.

Note: $\phi_1 = \frac{1}{2}$ in (4.18) gives same result in (Ugbebor *et al.* 2001).

4.2.1 The DTM Applied to the Transformed Black-Scholes Model (TBSM)

Let the differential transform of $\overline{M_*}(z)$ be $M_*(k)$ (that is, $DT\left(\overline{M_*}(z)\right) = M_*(k)$), then taking the differential transform of (4.6) gives:

$$M_*\left(k+2\right) = \frac{1}{a_*\left(k+1\right)\left(k+2\right)}\left(-\frac{1}{k!} - b_*\left(k+1\right)M_*\left(k+1\right) + c_*M_*\left(k\right)\right) \tag{4.22}$$

with $k \geq 0$, $M_*(0) = 0$, $M_*(1) = 1$ where $a_* = \frac{\sigma^2}{2}$, $b_* = \alpha - \frac{\sigma^2}{2}$ and $r = c_*$.

Equation (4.22) is a recurrence relation for the coefficient terms of the solution $M_*(z)$, defined as:

$$\overline{M_*}(z) = \sum_{n=0}^{\infty} M_*(n)z^n. \tag{4.23}$$

4.2.2 Numerical Calculations and the Transformed Black-Scholes Model

For numerical computation, efficiency and reliability of the model, we use the following values, and represent the solutions graphically in Figure 4.1 and Figure 4.2.

If $\sigma = 2$, $\left(a_* = \frac{\sigma^2}{2} = 2\right)$, $\alpha = 0.4$, $\left(\alpha - \frac{\sigma^2}{2} = -1.6 = b_*\right)$, $r = 0.9 = c_*$, $r - \alpha = \frac{1}{2}$, then:

Case I: (The proposed theoretical solution)

From (4.9) and (4.18) we have that:

$$\phi_1 = 1.181024968 \ , \ \phi_2 = -0.3810249676, A = -1.128036879 \ , \ B = -0.8719631201$$

and

$$\overline{M}_*^P(z) = -1.128036879e^{1.181024968z} - 0.8719631201e^{-0.3810249676z} + 2z. \qquad (4.24)$$

Case II: (The Exact solution)

$$\overline{M}_*^{EXACT}(z) = 2e^z + \left(\frac{-\sqrt{61}}{61} - 1\right) e^{\frac{1}{10}\left(4+\sqrt{61}\right)z} + \left(-1 + \frac{\sqrt{61}}{61}\right) e^{\frac{-1}{10}\left(-4+\sqrt{61}\right)z}. \qquad (4.25)$$

Case III: (The DTM solution with reference to (4.23))

$$\overline{M}_*^{DTM}(z) = z + \frac{3z^2}{20} + \frac{19z^3}{600} - \frac{71z^4}{8000} - \frac{5849z^5}{1200000} - \frac{32107z^6}{12000000} - \frac{2304341z^7}{5040000000} + \cdots \ . \qquad (4.26)$$

Remark 4.2.2: Equations (4.24), (4.25), and (4.26) represent the proposed, exact, and DTM solutions respectively.

Note: In Figure 4.1, the solid red, long dash green, and dot yellow lines are for the proposed, DTM, and exact solutions respectively. From graph above, the relationship between the solutions show that the proposed method is very effective and efficient when compared with the exact, and the DTM method solutions.

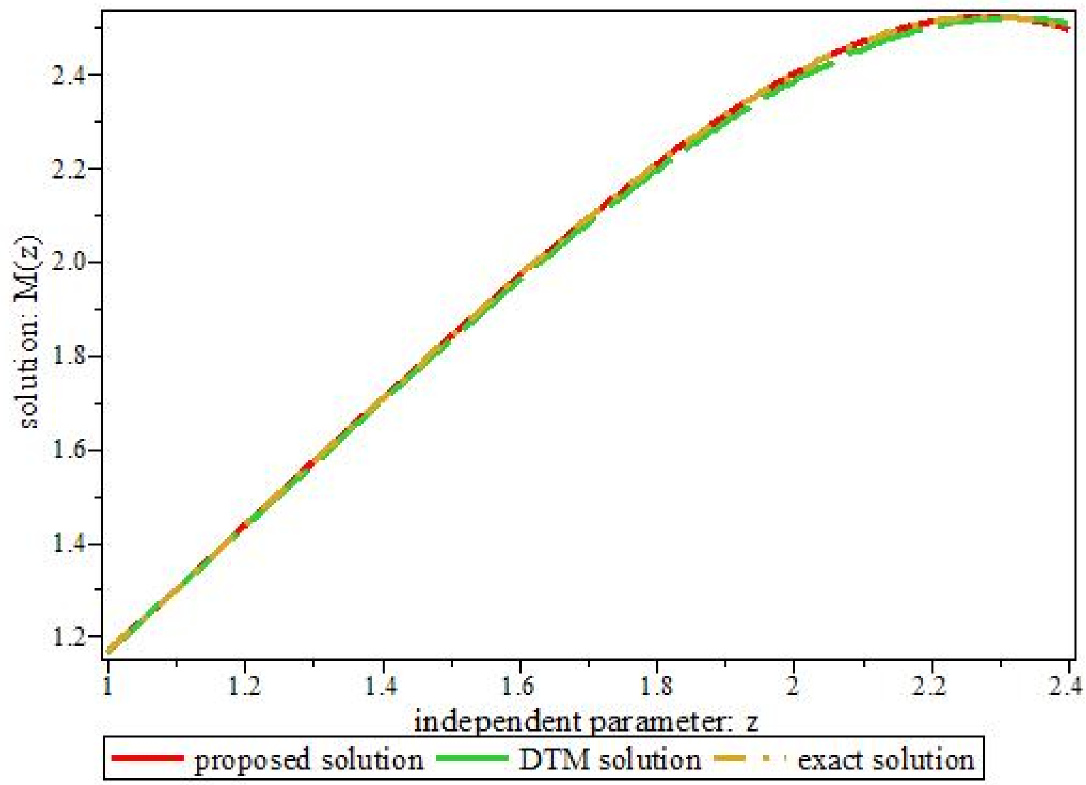

Figure 4.1: The TBSM solution in 2-D

4.3 The Generalised Black-Scholes Model via the CEV Stochastic Dynamics

In this section, the classical Black-Scholes option pricing model is re-visited. A generalised version of the Black-Scholes model via the application of the constant elasticity of variance model (CEVM) with or without dividend yield parameter is presented.

4.3.1 Constant Elasticity of Variance Option Pricing Model: A Case of no Dividend Yield

In what follows, the Black-Scholes model is generalised using the SDE associated with the CEV model in (2.5).

Proposition 4.3.1:

Associated to the CEV stochastic dynamics in (2.5) is a generalised version of the Black-Scholes model:

$$\frac{\partial \Xi}{\partial t} + rS\frac{\partial \Xi}{\partial S} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2 \Xi}{\partial S^2} - r\Xi = 0$$

where $\Xi = \Xi(S,t)$ is the option value at no dividend yield, $S = S_t$ is a corresponding stock price process at time t.

Proof:

Suppose the stock price, S_t at time t, satisfies the SDE in (2.5), with all parameters as earlier defined, and that the value of the contingent claim $\Lambda = \Lambda(S,t)$ is such that

$\Lambda \in C^{2,1}\left(R \times [0,T]\right)$, therefore by Itô lemma, we have:

$$\begin{aligned} d\Lambda &= \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial S^2}\left\langle dS \right\rangle \\ &= \frac{\partial \Lambda}{\partial t}dt + \frac{\partial \Lambda}{\partial S}\left(S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t\right) \\ &+ \frac{1}{2}\frac{\partial^2 \Lambda}{\partial S^2}\left\langle S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t \right\rangle . \end{aligned} \tag{4.27}$$

But for:

$$\begin{aligned} \left\langle S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t \right\rangle &= \left(S_t \mu dt + \sigma S_t^{\frac{\xi}{2}} dW_t\right)^2 \\ &= \left(S_t \mu dt\right)^2 + 2\left(S_t \mu dt\right)\left(\sigma S_t^{\frac{\xi}{2}} d\left(W_t\right)\right) + \left(\sigma S_t^{\frac{\xi}{2}} d\left(W_t\right)\right)^2 \\ &= \left(S_t \mu\right)^2 \left(dt\right)^2 + 2\left(S_t \mu \sigma S_t^{\frac{\xi}{2}}\right)\left(dt\right)\left(d\left(W_t\right)\right) + \sigma^2 S_t^{\xi}\left(d\left(W_t\right)\right)^2 \\ &= 0 + 0 + \sigma^2 S_t^{\xi} dt \\ &= \sigma^2 S_t^{\xi} dt. \end{aligned} \tag{4.28}$$

Since $(dt)^2 = (dt)(dW_t) = 0$, and $(dW_t)^2 = dt$, using the Itô multiplicative table in Definition 3.3.1. Therefore, (4.27) is expressed as:

$$d\Lambda = \left(\frac{\partial \Lambda}{\partial t} + \mu S \frac{\partial \Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2 \Lambda}{\partial S^2}\right)dt + \sigma S^{\frac{\xi}{2}}\frac{\partial \Lambda}{\partial S}dW. \tag{4.29}$$

Let $\Phi(t)$ be a delta-hedged-portfolio by longing a contingent, and shorting a delta unit of the underlying asset such that:

$$\Phi(t) = \Lambda(S,t) - \Delta S,\ d\Phi(t) = d\Lambda(s,t) - \Delta dS,\ \Delta = \frac{\partial \Lambda}{\partial S}\ \mathit{and}\ d\Phi(t) = r\Phi(t)dt \tag{4.30}$$

in order to make the value of the portfolio riskless, where r is a riskless rate, say bank account.

Therefore,

$$d\Phi(t) = \left(\frac{\partial\Lambda}{\partial t} + \mu S\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2\Lambda}{\partial S^2}\right)dt + \sigma S^{\frac{\xi}{2}}\frac{\partial\Lambda}{\partial S}dW - \Delta\left(\mu S dt + \sigma S^{\frac{\xi}{2}}dW\right). \quad (4.31)$$

That implies that:

$$\frac{d\Phi}{dt} = \left(\frac{\partial\Lambda}{\partial t} + \mu S\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2\Lambda}{\partial S^2}\right) + \sigma S^{\frac{\xi}{2}}\frac{\partial\Lambda}{\partial S}\frac{dW}{dt} - \frac{\partial\Lambda}{\partial S}\left(\mu S + \sigma S^{\frac{\xi}{2}}\frac{dW}{dt}\right).$$

Showing that:

$$\frac{d\Phi}{dt} = \frac{\partial\Lambda}{\partial t} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2\Lambda}{\partial S^2} \equiv r\Phi. \quad (4.32)$$

Thus, using (4.30) in (4.32) gives:

$$\frac{\partial\Lambda}{\partial t} + rS\frac{\partial\Lambda}{\partial S} + \frac{1}{2}\sigma^2 S^{\xi}\frac{\partial^2\Lambda}{\partial S^2} - r\Lambda = 0. \quad (4.33)$$

Thus, the proof is complete for $\Lambda(S,t) = \Xi(S,t)$ Q.E.D.

Equation (4.33) is referred to as the generalised Black-Scholes model via CEV (CEV-BSM) on a no-dividend yield basis.

4.3.1.1 Comparison of the Models: The Black-Scholes Model and the CEV-Black-Scholes Model

In this subsection, the basic features of the two models presented in (2.3) and (4.33) are compared using their respective SDEs.

Corollary 4.3.1

Suppose Λ^o_{BSM} denotes the volatility of the BSM, Λ^o_{CEVM} the volatility of the CEV model, Λ^r_{BSM} the variance of the BSM, and Λ^r_{CEVM} the variance of the CEV model. Then:

$$\Lambda^o_{BSM} = \sigma, \ \Lambda^o_{CEVM} = \sigma S^{\frac{\xi}{2}-1}, \ \Lambda^r_{BSM} = \sigma^2, \ \& \ \Lambda^r_{CEVM} = \sigma^2 S^{\xi-2}. \tag{4.34}$$

Proof: For $S_t \neq 0$, the SDEs associated to the BSM and CEVM as contained in (2.2) and (2.5) respectively can be expressed as:

$$\frac{dS_t}{S_t} = \mu + \sigma dW_t,$$

and

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{\frac{\xi}{2}-1} dW_t.$$

Recall from Itô theorem that the volatility of a one-dimensional SDE of the form:

$$\frac{dX}{X} = \varphi_1 d\tau + \varphi_2 dB\left(\tau\right), \ X = X\left(\tau\right) \neq 0$$

is defined and denoted as $V^0_X = \varphi_2$, where $B\left(\tau\right)$ is a standard Brownian motion.

Therefore,

$$\begin{cases} \Lambda^o_{BSM} = \sigma, \\ \\ \Lambda^o_{CEVM} = \sigma S^{\frac{\xi}{2}-1}. \end{cases}$$

But variance by definition, implies the square of the volatility, hence:

$$\begin{cases} \Lambda^r_{BSM} = \sigma^2, \\ \\ \Lambda^r_{CEVM} = \sigma^2 S^{\xi-2}. \end{cases}$$

This completes the proof.

Remark 4.3.1

It is obvious that $\Lambda^o_{CEVM} = h(\sigma, S_t)$ is not a constant function but a function of the underlying asset price S_t .

4.3.1.2 A Note on Elasticity and Elasticity Parameter

For the computation of elasticity, the following relationship is used (Hsu *et al.* 2008):

$$e = \left(\frac{d\Lambda^r_{CEVM}}{dS}\right) \div \left(\frac{\Lambda^r_{CEVM}}{S}\right). \tag{4.35}$$

Therefore, from (4.34), the following is obtained:

$$\frac{d\Lambda^r_{CEVM}}{dS} = (\xi - 2)\,\sigma^2 S^{\xi-3} \text{ and } \frac{\Lambda^r_{CEVM}}{S} = \sigma^2 S^{\xi-3}, \tag{4.36}$$

Therefore,

$$e = \xi - 2. \tag{4.37}$$

Note: It is obvious from (4.36) with little algebra that for any positive constant c such that $c\sigma^2 = 1$, the following is obtained:

$$S_t^e = c\Lambda^r_{CEVM}. \tag{4.38}$$

It has been shown empirically that the relationship between the stock price and the volatility of its return is negative (Hsu *et al.* 2008). This is guaranteed when $\xi < 2$ is considered.

The above result shows that the stock price volatility is a function of the underlying asset price, not a constant. The next case to be considered is that of dividend yield inclusion.

4.3.2 The CEV - Black-Scholes Model on a Basis of Dividend Yield

In this subsection, we consider a pricing model for stock option valuation via the application of the constant elasticity of variance (CEV) model with dividend yield. This incorporates into the Black-Scholes model a non-constant volatility power function of the underlying stock price, and a dividend yield parameter.

Thus, the same approach as in subsection 4.3.1 followed but with the inclusion of dividend yield parameter in the SDE in (2.3). Recall that Cox and Ross (1976) considered the CEV diffusion process governed by the SDE in (4.39) below when $q = 0$:

$$dS_t = S_t\left(\mu - q\right)dt + \sigma S_t^{\frac{\xi}{2}} d\left(W_t\right) \tag{4.39}$$

where $S_t = S_*$ is the solution of (4.39) at time t, ξ represents the rate of elasticity, q denoting a dividend yield parameter, μ, σ, and $W_t = W$ remain as defined in the earlier section.

In what follows, the Black-Scholes model will be generalised using the SDE associated with the CEV model in (4.39).

4.3.2.1 The Generalised CEV-Black-Scholes Model on the Basis of Dividend Yield Parameter

Let S_* be the stock price at time t, satisfying (4.52), such that the value of the contingent claim is $\Lambda_d = \Lambda_d(S,t)$ with $\Lambda_d \in C^{2,1}(R \times [0,T])$, so by applying Itô's lemma, we have:

$$\begin{aligned} d\Lambda_d &= \frac{\partial \Lambda_d}{\partial t}dt + \frac{\partial \Lambda_d}{\partial S_*}dS_* + \frac{1}{2}\frac{\partial^2 \Lambda_d}{\partial S_*^2}\langle dS_* \rangle \\ &= \frac{\partial \Lambda_d}{\partial t}dt + \frac{\partial \Lambda_d}{\partial S_*}\left(S_*(\mu - q)\,dt + \sigma S_*^{\frac{\xi}{2}} d(W)\right) \\ &+ \frac{1}{2}\frac{\partial^2 \Lambda_d}{\partial S_*^2}\left\langle S_*(\mu - q)\,dt + \sigma S_*^{\frac{\xi}{2}} d(W)\right\rangle. \end{aligned} \tag{4.40}$$

But:

$$\begin{aligned} \left\langle S_*(\mu - q)\,dt + \sigma S_*^{\frac{\xi}{2}} dW_t \right\rangle &= \left(S_*(\mu - q)\,dt + \sigma S_*^{\frac{\xi}{2}} dW_t\right)^2 \\ &= (S_*(\mu - q)\,dt)^2 + 2(S_*(\mu - q)\,dt)\left(\sigma S_*^{\frac{\xi}{2}} dW_t\right) \\ &+ \left(\sigma S_*^{\frac{\xi}{2}} dW_t\right)^2 \\ &= (S_*(\mu - q))^2 (dt)^2 + 2\left(S_*(\mu - q)\,\sigma S_*^{\frac{\xi}{2}}\right)(dt)(dW_t) \\ &+ \sigma^2 S_*^{\xi}(dW_t)^2 \\ &= 0 + 0 + \sigma^2 S_*^{\xi} dt \\ &= \sigma^2 S_*^{\xi} dt \end{aligned} \tag{4.41}$$

where

$$(\Theta)(dW_t) = \begin{cases} dt, \text{ for } \Theta = dW_t \\ 0, \qquad \text{otherwise.} \end{cases} \tag{4.42}$$

Hence, (4.40) becomes:

$$d\Lambda_d = \frac{\partial \Lambda_d}{\partial t}dt + \frac{\partial \Lambda_d}{\partial S_*}\left(S_*\left(\mu - q\right)dt + \sigma S_*^{\frac{\xi}{2}} d\left(W\right)\right) + \frac{1}{2}\frac{\partial^2 \Lambda_d}{\partial S_*^2}\left(\sigma^2 S_*^{\xi} dt\right). \tag{4.43}$$

Therefore,

$$d\Lambda_d = \left(\frac{\partial \Lambda_d}{\partial t} + \left(\mu - q\right)S_*\frac{\partial \Lambda_d}{\partial S_*} + \frac{1}{2}\sigma^2 S_*^{\xi}\frac{\partial^2 \Lambda_d}{\partial S_*^2}\right)dt + \sigma S_*^{\frac{\xi}{2}}\frac{\partial \Lambda_d}{\partial S_*}dW. \tag{4.44}$$

Suppose $\Xi(t)$ is a delta-hedge-portfolio constructed by longing a contingent claim, and shorting a delta unit of the concerned underlying asset such that:

$$\begin{aligned} \Xi(t) = -\Delta S_* + \Lambda_d(S_*, t), d\Xi(t) = d\Lambda_d(S_*, t) - \Delta dS_*, \\ \Delta = \frac{\partial \Lambda_d}{\partial S_*}\text{ , and } d\Xi(t) = r\Xi(t)dt. \end{aligned} \tag{4.45}$$

In a bid to making the portfolio value riskless (say bank account) where r is a riskless rate, the following is obtained:

$$d\Xi(t) = d\Lambda_d(S_*, t) - \Delta dS_*.$$

This implies that:

$$\begin{aligned} d\Xi(t) = &\left(\frac{\partial \Lambda_d}{\partial t} + \left(\mu - q\right)S_*\frac{\partial \Lambda_d}{\partial S_*} + \frac{1}{2}\sigma^2 S_*^{\xi}\frac{\partial^2 \Lambda_d}{\partial S_*^2}\right)dt \\ &+ \sigma S_*^{\frac{\xi}{2}}\frac{\partial \Lambda_d}{\partial S_*}dW - \Delta\left(\left(\mu - q\right)S_* dt + \sigma S_*^{\frac{\xi}{2}} dW\right). \end{aligned} \tag{4.46}$$

Therefore:

$$\begin{aligned}\frac{d\Xi}{dt} &= \left(\frac{\partial \Lambda_d}{\partial t} + (\mu - q)\, S_* \frac{\partial \Lambda_d}{\partial S_*} + \frac{1}{2}\sigma^2 S_*^{\xi} \frac{\partial^2 \Lambda_d}{\partial S_*^2}\right) \\ &\quad + \sigma S_*^{\frac{\xi}{2}} \frac{\partial \Lambda_d}{\partial S_*} \frac{dW}{dt} - \frac{\partial \Lambda_d}{\partial S_*} \left((\mu - q)\, S_* dt + \sigma S_*^{\frac{\xi}{2}} d\frac{dW}{dt}\right).\end{aligned} \tag{4.47}$$

Thus:

$$\frac{d\Xi}{dt} = \frac{\partial \Lambda_d}{\partial t} + \frac{1}{2}\sigma^2 S_*^{\xi} \frac{\partial^2 \Lambda_d}{\partial S_*^2} \equiv r\Xi \tag{4.48}$$

So, combining (4.46) and (4.48) gives:

$$\frac{\partial \Lambda_d}{\partial t} + rS_* \frac{\partial \Lambda_d}{\partial S_*} + \frac{1}{2}\sigma^2 S_*^{\xi} \frac{\partial^2 \Lambda_d}{\partial S_*^2} - r\Lambda_d = 0. \tag{4.49}$$

Remark 4.3.2: Equation (4.49) is referred to as the generalised Black-Scholes model via CEV SDE with dividend yield. When $\xi = 2$, (4.49) becomes the classical Black-Scholes model on a dividend yield basis.

4.3.2.2 The CEV-Black-Scholes Model with Parameter Estimation

The elasticity rate parameter ξ, which is the central feature of the model, controls the relationship between the volatility and price of the underlying asset. Therefore, the corresponding graph is displayed in Figure 4.2.

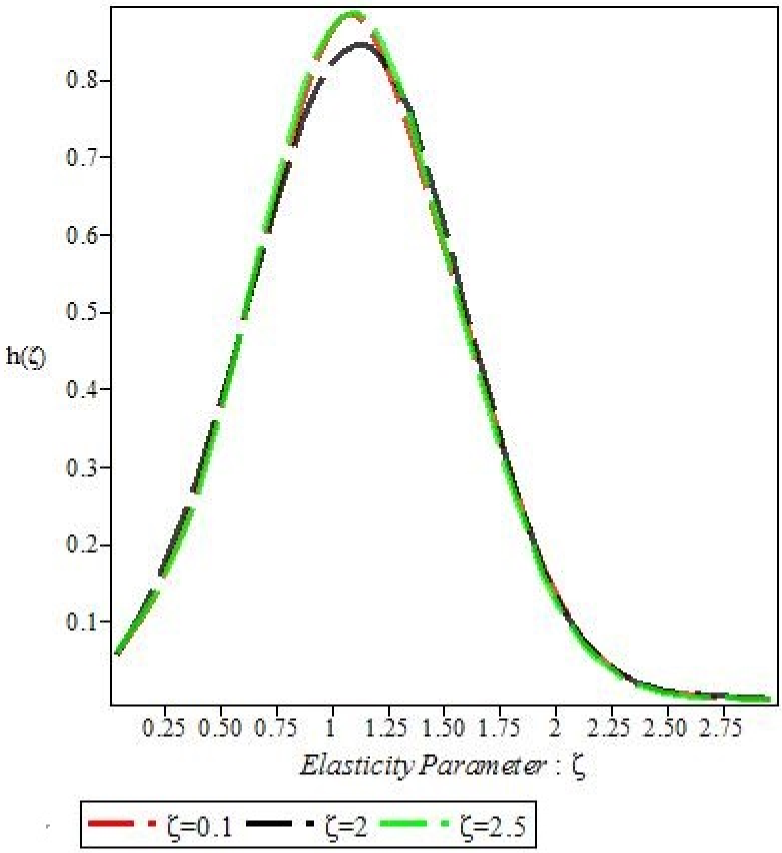

Figure 4.2: Estimates of the CEVM distribution

Figure 4.2 shows the estimates of the CEVM distribution at maturity time, $h(\xi) = (S(T) - 1, 0)^+$ using different values of the elasticity parameter for results interpretation, and to show the economic implications of the elasticity rate. For the purpose of simulation, we consider the following values for the concerned parameters: $T = 2.0$, $\sigma = 0.3$, $\mu = 0.05$ and for $\xi = 0.1$, $\xi = 2$, and $\xi = 2.5$ with red, black, and green dash lines respectively (with reference to section 4.3.1).

Remark 4.3.3

(a) Elasticity is said to be zero if $\xi = 2$; therefore, the stock price is lognormally distributed as indicated in the classical Black-Scholes model.

(b) Elasticity is -1 as proposed by Cox and Ross (1976) if $\xi = 1$.

(c) When $\xi < 2$, the so-called leverage effect observed mainly in equity market, shows that the stock volatility increases as the corresponding price falls.

(d) On the other hand, when $\xi > 2$, the so-called leverage effect observed mainly in commodity market, indicates that the volatility of the commodity increases as the corresponding price increases.

It is remarked here, that in the generalised versions of the Black-Scholes pricing model; a non-constant volatility power function is introduced, and comparison between the CEV models with or without dividend yield is made.

4.4 Analytical Solution of the Black-Scholes Model for European Option Valuation

In this section, analytical solutions of the classical Black-Scholes pricing model for European options are considered via the application of the PDTM, and the Revised

Homotopy Perturbation Method (He's polynomial technique). The cases considered include both the integer and the fractional time-order type.

4.4.1 The MDTM Applied for Analytical Solution of Black-Scholes Pricing Model for European Option Valuation

In this subsection, the MDTM is applied to some examples of the classical Black-Scholes equations (Allahviranloo and Behzadi, 2013; Qiu and Lorenz, 2009; Elbeleze *et al.* 2013) as follows:

***Example 4.4.1.1*:** Consider the following Black-Scholes equation (Allahviranloo and Behzadi, 2013):

$$\frac{\partial \underline{v}}{\partial t} - \frac{\partial^2 \underline{v}}{\partial x^2} + (1-k)\frac{\partial \underline{v}}{\partial x} = k\underline{v} \tag{4.50}$$

subject to:

$$\underline{v}(x,0) = \max(e^x - 1, 0) \tag{4.51}$$

where $\underline{v}$ is the value of the contingent claim, $x = \ln S(t)$, $k \in \mathrm{N}$. For simplicity, the following are re-written as applied in Example 4.4.1:

$$\frac{\partial \underline{v}}{\partial t} = \underline{v}_t,\ \frac{\partial^2 \underline{v}}{\partial x^2} = \underline{v}_{xx},\ \frac{\partial \underline{v}}{\partial x} = \underline{v}_x.$$

Suppose the projected differential transform of $\underline{v}(\circ)$ is $\underline{V}(\circ, h+1)$, then taking the PDT of (4.50) and (4.51) gives:

$$\underline{V}(x,h+1) = \frac{1}{h+1}\left[\underline{V}_{xx}(x,h) + (k-1)\underline{V}_x(x,h) - k\underline{V}(x,h)\right] \tag{4.52}$$

and

$$\underline{V}(x,0) = \max(e^x - 1, 0). \tag{4.53}$$

respectively.

This implies that:

$$\begin{aligned}\underline{V}_x(x,0) &= \underline{V}_{xx}(x,0) \\ &= \max(e^x, 0) \\ &= e^x.\end{aligned} \tag{4.54}$$

Thus, when $h = 0$, the following is obtained:

$$\begin{aligned}\underline{V}(x,1) &= \underline{V}_{xx}(x,0) + (k-1)\underline{V}_x(x,0) - k\underline{V}(x,0) \\ &= \max(e^x, 0) + (k-1)\max(e^x, 0) - K\max(e^x - 1, 0) \\ &= k\{\max(e^x, 0) - \max(e^x - 1, 0)\}\end{aligned} \tag{4.55}$$

whence,

$$\underline{V}_x(x,1) = \underline{V}_{xx}(x,1) = 0. \tag{4.56}$$

So for $h = 1$,

$$\begin{aligned}\underline{V}(x,2) &= \frac{1}{2}\left(\underline{V}_{xx}(x,1) + (k-1)\underline{V}_x(x,1) - k\underline{V}(x,1)\right) \\ &= \frac{-k^2}{2}(\max(e^x, 0) - \max(e^x - 1, 0)).\end{aligned} \tag{4.57}$$

Similarly, from (4.57), the following is obtained:

$$\underline{V}_x(x,2) = 0 = \underline{V}_{xx}(x,2). \tag{4.58}$$

When $h = 2$,

$$\begin{aligned} \underline{V}(x,3) &= \frac{1}{3}\left(\underline{V}_{xx}(x,2) + (k-1)\underline{V}_{x}(x,2)\right) \\ &= \frac{k^3}{6}\left(\max\left(e^x,0\right) - \max\left(e^x - 1,0\right)\right). \end{aligned} \tag{4.59}$$

Hence,

$$\begin{aligned} \underline{v}(x,t) &= \underline{V}(x,0) + \sum_{h=1}^{\infty} \underline{V}(x,h)\, t^h \\ &= \max\left(e^x - 1,0\right) + (kt)\,\mathrm{Z} - \frac{(kt)^2}{2!}\mathrm{Z} + \frac{(kt)^3}{3!}\mathrm{Z}(x) + \cdots \end{aligned} \tag{4.60}$$

where

$$\mathrm{Z}(x) = \left\{\max\left(e^x,0\right) - \max\left(e^x - 1,0\right)\right\}. \tag{4.61}$$

Therefore:

$$\begin{aligned} \underline{v}(x,t) &= \max\left(e^x - 1,0\right) \\ &\quad + \sum_{h=1}^{\infty} (-1)^{h+1} \frac{(kt)^h}{h!} \{\max\left(e^x,0\right) - \max\left(e^x - 1,0\right)\}. \end{aligned} \tag{4.62}$$

Equation (4.62) is the exact solution of (4.50).

Example 4.4.1.2: Consider the following Black-Scholes equation ((Qiu and Lorenz, 2009; Elbeleze *et al.* 2013), Example 7 & Example 2 respectively, for $\alpha = 1$):

$$\frac{\partial \underline{v}}{\partial t} + 0.08\left(2 + Sinx\right)^2 x^2 \frac{\partial^2 \underline{v}}{\partial x^2} + 0.06x\frac{\partial \underline{v}}{\partial x} = 0.06\underline{v} \tag{4.63}$$

subject to:

$$\underline{v}(x,0) - \max\left(x - 25e^{-0.06},0\right) = 0. \tag{4.64}$$

Following the same approach in Example 4.4.1.1 by taking the PDT of (4.63) and (4.64) gives:

$$\begin{aligned}(h+1)\,\underline{V}\,(x,h+1) &= \left(-0.08\,(2+Sinx)^2\,x^2\underline{V}_{xx}\,(x,h)\right)\\ &\quad + \left(-0.06x\underline{V}_x\,(x,h) + 0.06\underline{V}\,(x,h)\right)\end{aligned} \tag{4.65}$$

subject to:

$$\underline{V}\,(x,0) = \max\left(x - 25e^{-0.06}, 0\right). \tag{4.66}$$

This implies that:

$$\underline{V}_x\,(x,0) = 1, \quad \text{and} \quad \underline{V}_{xx}\,(x,0) = 0. \tag{4.67}$$

So, when $h = 0$,

$$\begin{aligned}\underline{V}(x,1) &= \left(-0.06x + 0.06\max(x - 25e^{-0.06}, 0)\right)\\ &= -0.06\left(x - \max(x - 25e^{-0.06}, 0)\right)\end{aligned} \tag{4.68}$$

and

$$\underline{V}_x\,(x,1) = 0 = \underline{V}_{xx}\,(x,1). \tag{4.69}$$

So when $h = 1$,

$$\begin{aligned}\underline{V}(x,2) &= \frac{1}{2}\left(0.06\underline{V}(x,1)\right)\\ &= \frac{-(0.06)^2}{2}\left(x - \max(x - 25e^{-0.06}, 0)\right).\end{aligned} \tag{4.70}$$

As such,

$$\underline{V}_x\,(x,2) = 0 = \underline{V}_{xx}\,(x,2). \tag{4.71}$$

So when $h = 2$,

$$\begin{aligned}\underline{V}(x,3) &= \frac{1}{3}\left(0.06\underline{V}(x,2)\right) \\ &= \frac{-(0.06)^3}{3}\left(x - \max(x - 25e^{-0.06}, 0)\right).\end{aligned} \tag{4.72}$$

Hence,

$$\begin{aligned}\underline{v}\left(x,t\right) &= \sum_{h=0}^{\infty} \underline{V}(x,h)t^h = \underline{V}(x,0) + \underline{V}(x,1)t + \underline{V}(x,2)t^2 + \underline{V}(x,3)t^3 + \cdots \\ &= \max(x - 25e^{-0.06}, 0) + \left(-0.06t - \frac{(0.06t)^2}{2!} - \frac{(0.06t)^3}{3!} + \cdots\right) U\left(x\right)\end{aligned}$$

where

$$U\left(x\right) = \left(x - \max(x - 25e^{-0.06}, 0)\right). \tag{4.73}$$

Therefore:

$$\begin{aligned}\underline{v}\left(x,t\right) &= \max\left(x - 25e^{-0.06}, 0\right) \\ &- \left(0.06t + \frac{(0.06t)^2}{2!} + \frac{(0.06t)^3}{3!} + \cdots\right) U\left(x\right) \\ &= \max\left(x - 25e^{-0.06}, 0\right) - U\left(x\right)\sum_{n=1}^{\infty} \frac{(0.06t)^n}{n!}.\end{aligned} \tag{4.74}$$

So, simplifying (4.74) using (4.73) yields:

$$\begin{aligned}\underline{v}(x,t) &= \max\ \left(x - 25e^{-0.06}, 0\right) + \left(1 - e^{0.06t}\right)\left(x - \max\left(x - 25e^{-0.06}, 0\right)\right) \\ &= x\left(1 - e^{0.06t}\right) + \max\left(x - 25e^{-0.06}, 0\right)e^{0.06t}.\end{aligned} \tag{4.75}$$

Equation (4.75) is the exact solution of (4.63) subject to (4.64).

Figure 4.4 is the 3D plot of the exact solution to the problem in example 4.4.1.2 while Figure 4.3 is the 3D plot of the approximate solution to example 4.4.1.2.

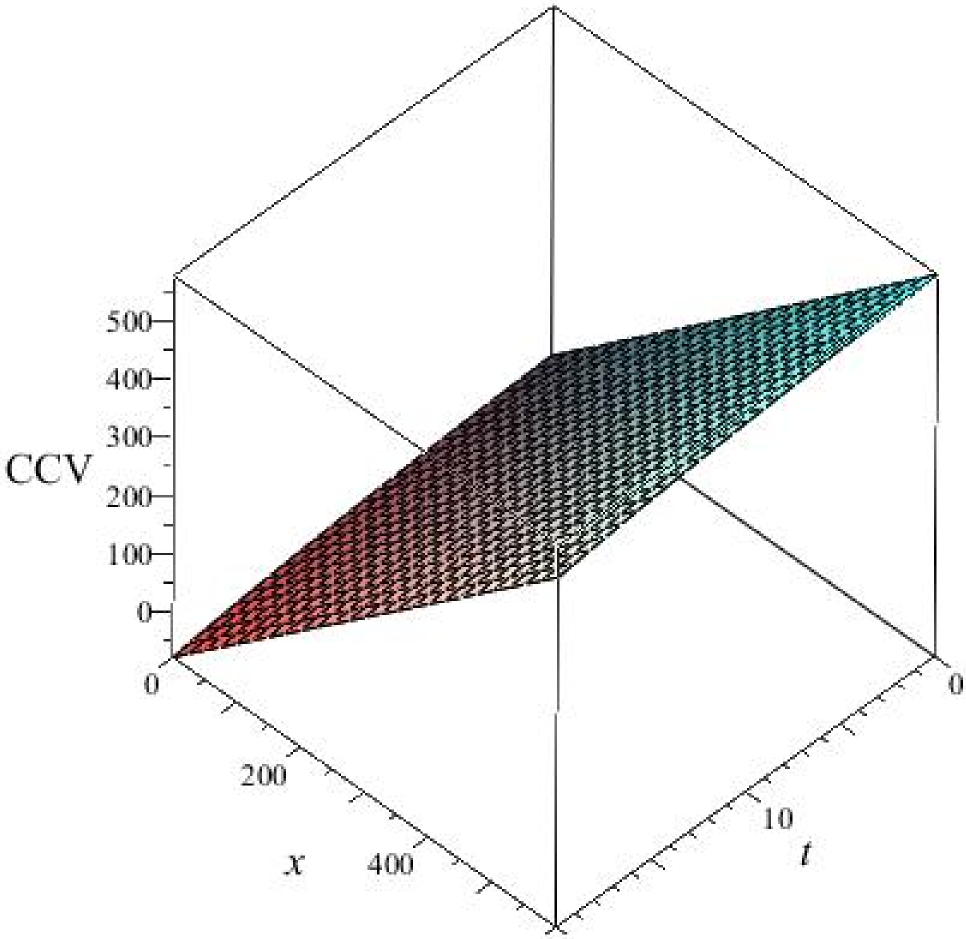

Figure 4.3: The approximate solution for problem example 4.4.1.2

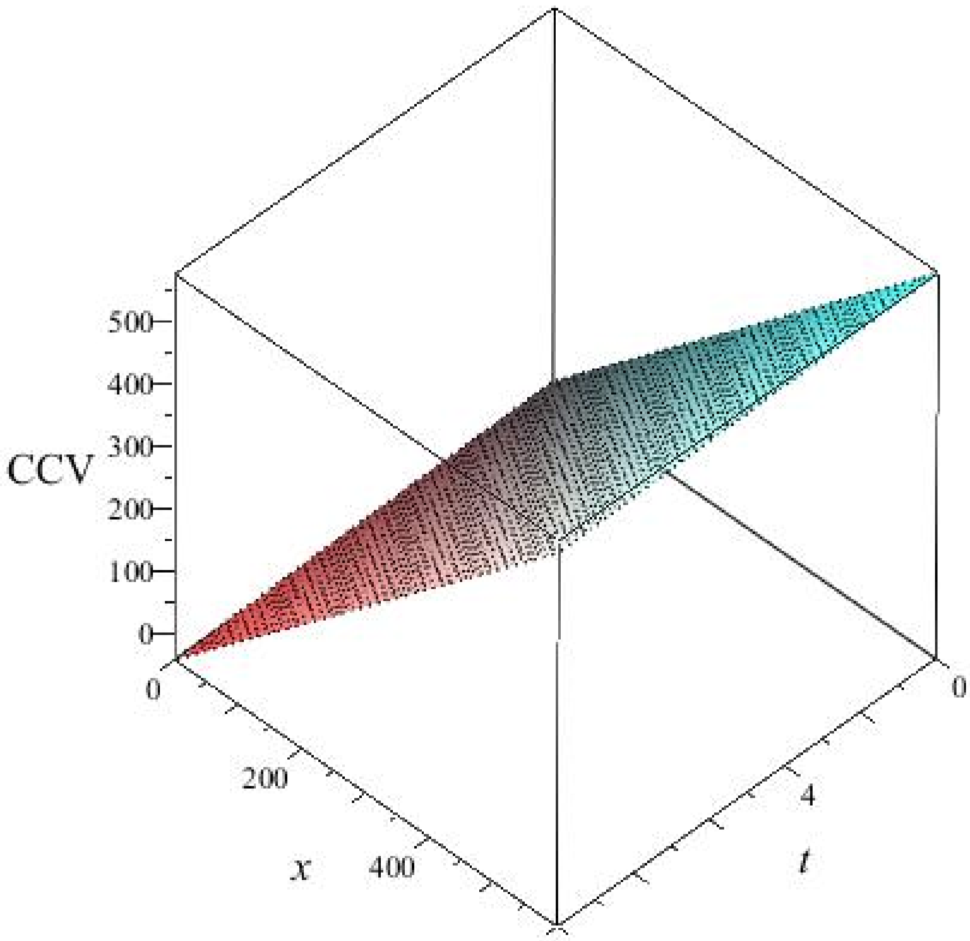

Figure 4.4: The exact solution for problem example 4.4.1.2

Remark 4.4.1.2

The MDTM has been successfully applied to Black-Scholes Equation for European Option Valuation. Some illustrative and numerical examples were solved to test the efficiency of the proposed method. The results obtained converge faster to their exact forms, even with less computation, without linearization or perturbation; showing that the method can also be used easily for approximate solutions in a direct form. These easily computed results represent the analytical values of the associated European call options, the same algorithm can be followed for European put options.

4.4.2 He's Polynomials Applied to the Black-Scholes Pricing Model for Stock Option Valuation

In this subsection, we apply the He's polynomials for solving the classical Black-Scholes pricing model with stock as the underlying asset. Our goal here, is therefore, to provide analytical solutions to the Black–Scholes option pricing model for a particular form of (4.46) using the He's polynomials method as an alternative method, to be used for results comparison.

4.4.2.1 The Pricing Model and the He's Polynomial

Here, the He's Polynomials approach is applied to the model equation as follows, where $\underline{w} = \underline{w}(x,t)$ represents the value of the contingent claim, and $x = \ln S(t)$(Allahviranloo and Behzadi, 2013).

Problem 4.4.2.1.1: Consider the following Black-Scholes equation:

$$\frac{\partial \underline{w}}{\partial t} + x^2 \frac{\partial^2 \underline{w}}{\partial x^2} + \frac{1}{2} x \frac{\partial \underline{w}}{\partial x} = \underline{w} \tag{4.76}$$

subject to:

$$\underline{w}\,(x,0) = \max\left(x^3,0\right) = \begin{cases} x^3, \text{ for } x > 0, \\ 0, \text{ for } x \leq 0. \end{cases} \tag{4.77}$$

Suppose in an integral form, where $I_0^t\,(\cdot) = \int_0^t (\cdot)\,dt$ and (4.76) is re-written as (4.78), then (4.78) is reformulated as (4.79) below:

$$\frac{\partial \underline{w}}{\partial t} = -\left(x^2\frac{\partial^2 \underline{w}}{\partial x^2} + \frac{1}{2}x\frac{\partial \underline{w}}{\partial x} - \underline{w}\right) \tag{4.78}$$

and

$$\underline{w}\,(x,t) = \max\left(x^3,0\right) - pI_0^t\left(x^2\frac{\partial^2 \underline{w}}{\partial x^2} + \frac{1}{2}x\frac{\partial \underline{w}}{\partial x} - \underline{w}\right). \tag{4.79}$$

Applying the convex homotopy method to (4.79) gives:

$$\sum_{n=0}^{\infty} p^n \underline{w}_n = \max\left(x^3,0\right) - pI_0^t\left(x^2\sum_{n=0}^{\infty} p^n\frac{\partial^2 \underline{w}_n}{\partial x^2} + \frac{1}{2}x\sum_{n=0}^{\infty} p^n\frac{\partial \underline{w}_n}{\partial x} - \sum_{n=0}^{\infty} p^n \underline{w}_n\right) \tag{4.80}$$

This implies that:

$$\begin{aligned} \underline{w}_0 + p\underline{w}_1 + p^2\underline{w}_2 + \cdots = {} & \max\left(x^3,0\right) + \\ & - pI_0^t\left\{x^2\left(\frac{\partial^2 \underline{w}_0}{\partial x^2} + p\frac{\partial^2 \underline{w}_1}{\partial x^2} + p^2\frac{\partial^2 \underline{w}_2}{\partial x^2} + \cdots\right)\right. \\ & + \frac{x}{2}\left(\frac{\partial \underline{w}_0}{\partial x} + p\frac{\partial \underline{w}_1}{\partial x} + p^2\frac{\partial \underline{w}_2}{\partial x} + \cdots\right) \\ & - \left(\underline{w}_0 + p\underline{w}_1 + p^2\underline{w}_2 + \cdots\right). \end{aligned} \tag{4.81}$$

So comparing the coefficients of the equal powers of p gives the following:

$$p^{(0)} : \underline{w}_0 = \max\left(x^3,0\right),$$

$$p^{(1)}: \underline{w}_1 = -I_0^t\left(x^2\frac{\partial^2\underline{w}_0}{\partial x^2} + \frac{x}{2}\frac{\partial\underline{w}_0}{\partial x} - \underline{w}_0\right),$$

$$p^{(2)}: \underline{w}_2 = -I_0^t\left(x^2\frac{\partial^2\underline{w}_1}{\partial x^2} + \frac{x}{2}\frac{\partial\underline{w}_1}{\partial x} - \underline{w}_1\right),$$

$$p^{(3)}: \underline{w}_3 = -I_0^t\left(x^2\frac{\partial^2\underline{w}_2}{\partial x^2} + \frac{x}{2}\frac{\partial\underline{w}_2}{\partial x} - \underline{w}_2\right),$$

$$\vdots$$

$$p^{(k)}: \underline{w}_3 = -I_0^t\left(x^2\frac{\partial^2\underline{w}_{k-1}}{\partial x^2} + \frac{x}{2}\frac{\partial\underline{w}_{k-1}}{\partial x} - \underline{w}_{k-1}\right).$$

Thus, simplifying $p^{(1)},\ p^{(2)},\ p^{(3)}$ and so on with $\underline{w}_0 = \max(x^3, 0)$ for $x \geq 0$ gives:

$$\underline{w}_0 = x^3,\ \underline{w}_1 = -\frac{13}{2}x^3t,\ \underline{w}_2 = \frac{169}{8}x^3t^2,\ \underline{w}_3 = \frac{-2197}{48}x^3t^3,$$
$$\underline{w}_4 = \frac{28561}{384}x^3t^4,\ \underline{w}_5 = \frac{-371293}{3840}x^3t^5, \cdots.$$

Therefore:

$$\begin{aligned}
\underline{w}(x,t) &= \underline{w}_1 + \underline{w}_2 + \underline{w}_3 + \underline{w}_4 + \cdots \\
&= x^3 - \frac{13}{2}x^3t + \frac{169}{8}x^3t^2 - \frac{2197}{48}x^3t^3 + \frac{28561}{384}x^3t^4 - \frac{371293}{3840}x^3t^5 + \cdots \\
&= x^3\left(1 + (-6.5t) + \frac{(-6.5t)^2}{2!} + \frac{(-6.5t)^3}{3!} + \frac{(-6.5t)^4}{4!} + \frac{(-6.5t)^5}{5!} + \cdots\right) \\
&\cong x^3e^{-6.5t}.
\end{aligned} \tag{4.82}$$

The exact and the He's polynomial solutions of problem 4.4.2.1.1 are graphically displayed in Figure 4.5 and Figure 4.6 respectively.

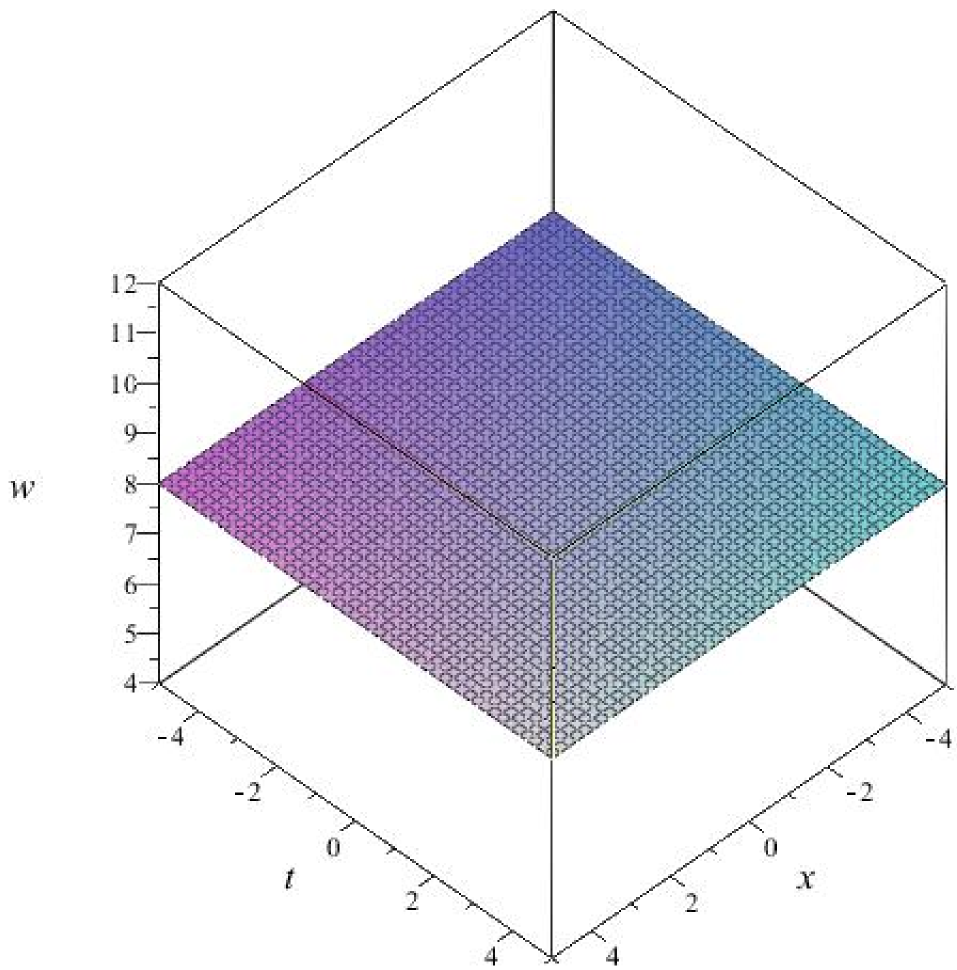

Figure 4.5: The exact solution for problem 4.4.2.1.1

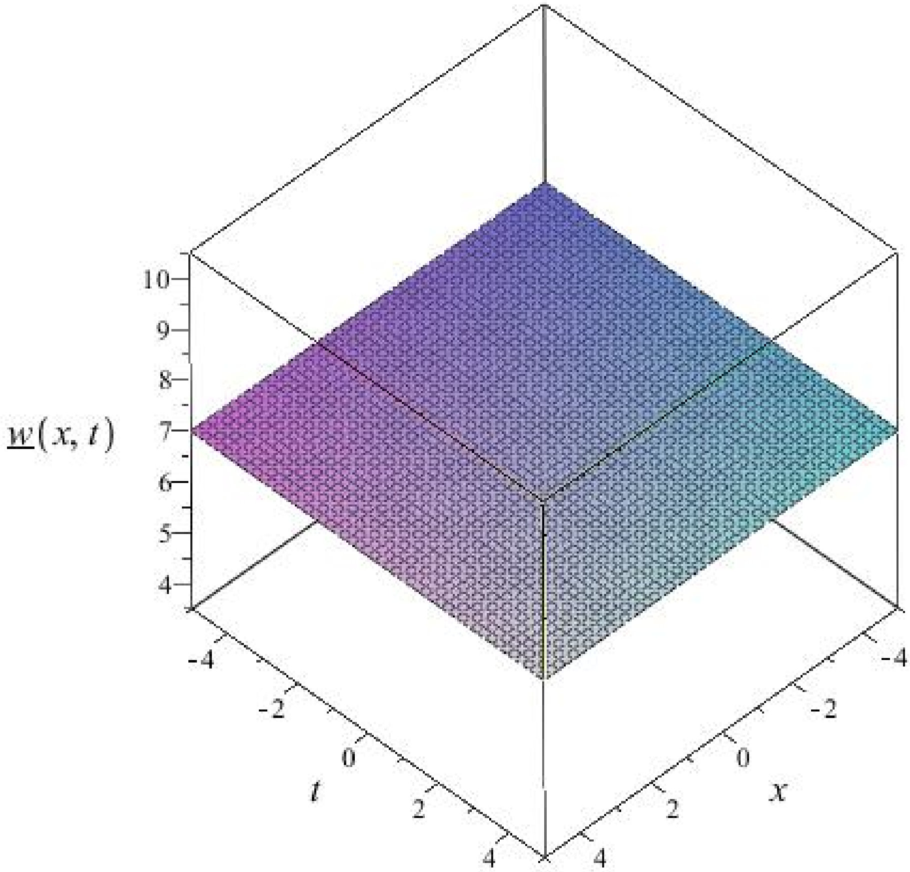

Figure 4.6: The approximate solution for problem 4.4.2.1.1

Note: Figure 4.5 and Figure 4.6 represent the graphs of the exact solution and the He's polynomials solution (including terms up to power 5) respectively, to the classical Black-Scholes model for stock option valuation. These are obtained via the application of He's polynomials as the proposed technique. This technique is very much efficient and reliable as it gives the exact solution of the solved problem in a very simple and quick manner even with less computational work while still maintaining high level of accuracy.

4.5 The Time-Fractional Black-Scholes Option Pricing Model on No-Dividend Paying Equity

To be considered in this section is a generalisation of (2.3) with regard to fractional order, $\beta \in$ R *or* C, not necessarily an integer. This generalisation will be referred to as Time-fractional Black-Scholes Model (TFBSM) of the form:

$$\frac{\partial^{\beta}\Xi}{\partial t^{\beta}} + \zeta_1(S,t)\frac{\partial^2\Xi}{\partial S^2} + \zeta_2(S,t)\frac{\partial\Xi}{\partial S} = r\Xi\ ,\ \beta \in (0,1] \tag{4.83}$$

subject to a given initial or boundary condition(s). We donote $\zeta_i(\cdot,\cdot)\ ,\ i \in$ N, as non-zero functions, while Ξ represents the fair value of the contingent claim associated with the time fractional Black-Scholes model.

4.5.1 Illustrative Examples and Applications

In this section, some examples of time-fractional Black-Scholes equations will be solved with the proposed algorithmic technique: MDTM as follows.

Problem 4.5.1.1: Consider the following time-fractional Black-Scholes equation (problem 4.4.2.1.1 for $\alpha = 1$):

$$\frac{\partial^\alpha \underline{w}}{\partial t^\alpha} + x^2 \frac{\partial^2 \underline{w}}{\partial x^2} + \frac{x}{2}\frac{\partial \underline{w}}{\partial x} - \underline{w} = 0 \ , \ 0 < \alpha \leq 1 \tag{4.84}$$

subject to:

$$\underline{w}\left(x,0\right) = \max\left(x^3,0\right) = \begin{cases} x^3, \text{for } x > 0, \\ 0, \ \text{for } x \leq 0. \end{cases} \tag{4.85}$$

Here, the study will consider $x > 0$ and take $\frac{\partial W(x,\xi)}{\partial x}$ as the projected differential transform (PDT) of $\underline{w} = \underline{w}\left(x,t\right)$ as follows: So taking the PDTM of (4.84) and (4.85) gives:

$$W\left(x,1+\xi\right) = \frac{\Gamma\left(1+\alpha\xi\right)}{\Gamma\left(1+\alpha(1+\xi)\right)}\left(-x^2\frac{\partial^2 W\left(x,\xi\right)}{\partial x^2} - \frac{1}{2}\frac{\partial W\left(x,\xi\right)}{\partial x} + W\left(x,\xi\right)\right) \tag{4.86}$$

subject to:

$$W\left(x,0\right) = x^3. \tag{4.87}$$

That implies:

$$\frac{\partial W\left(x,0\right)}{\partial x} = 3x^2 \ and \ \frac{\partial^2 W\left(x,0\right)}{\partial x^2} = 6x. \tag{4.88}$$

So, when $\xi = 0$,

$$W\left(x,1\right) = \frac{\Gamma\left(1\right)}{\Gamma\left(1+\alpha\right)}\left(-x^2\frac{\partial^2 W\left(x,0\right)}{\partial x^2} - \frac{1}{2}\frac{\partial W\left(x,0\right)}{\partial x} + W\left(x,0\right)\right)$$

showing that:

$$W\left(x,1\right) = \frac{\Gamma\left(1\right)}{\Gamma\left(1+\alpha\right)}\left(-6x^3 - \frac{3}{2}x^3 + x^3\right) = \frac{\left(-6.5x^3\right)}{\Gamma\left(1+\alpha\right)}. \tag{4.89}$$

From (4.89), the following is obtained:

$$\frac{\partial W(x,1)}{\partial x} = \frac{(-19.5x^2)}{\Gamma(1+\alpha)} \; \& \; \frac{\partial^2 W(x,1)}{\partial x^2} = \frac{(-39x)}{\Gamma(1+\alpha)} \tag{4.90}$$

Thus, for $\xi = 1$,

$$\begin{aligned} W(x,2) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(-x^2 \frac{\partial^2 W(x,1)}{\partial x^2} - \frac{1}{2}\frac{\partial W(x,1)}{\partial x} + W(x,1) \right) \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(\frac{39x^3 + 9.75x^3 - 6.5x^3}{\Gamma(1+\alpha)} \right) \\ &= \frac{(6.5)^2 x^3}{\Gamma(1+2\alpha)}. \end{aligned} \tag{4.91}$$

So (4.91) gives:

$$\frac{\partial W(x,2)}{\partial x} = \frac{(126.75x^2)}{\Gamma(1+2\alpha)} \; \& \; \frac{\partial^2 W(x,2)}{\partial x^2} = \frac{(253.5x)}{\Gamma(1+2\alpha)} \tag{4.92}$$

so when $\xi = 2$,

$$\begin{aligned} W(x,3) &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(-x^2 \frac{\partial^2 W(x,2)}{\partial x^2} - \frac{1}{2}\frac{\partial W(x,2)}{\partial x} + W(x,2) \right) \\ &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{-253.5x^3 - 63.375x^3 + 42.25x^3}{\Gamma(1+2\alpha)} \right) \\ &= \frac{(-6.5)^3 x^3}{\Gamma(1+3\alpha)}. \end{aligned} \tag{4.93}$$

Similarly, the recurrence relation below is obtained:

$$W(x,h) = \frac{(-6.5)^h x^3}{\Gamma(1+h\alpha)} \;, h \in \mathrm{Z}^+. \tag{4.94}$$

Hence,

$$
\begin{aligned}
\underline{w}(x,t) &= \sum_{\eta=0}^{\infty} W(x,\eta)\, t^{\alpha\eta} \\
&= W(x,0) + W(x,1)\, t^{\alpha} + W(x,2)\, t^{2\alpha} + W(x,3)\, t^{3\alpha} + \cdots \\
&= x^3 - \frac{(6.5t^{\alpha})x^3}{\Gamma(1+\alpha)} + \frac{(6.5)^2 x^3 t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{(6.5)^3 x^3 t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots \\
&= \left(1 + \frac{(-6.5t^{\alpha})}{\Gamma(1+\alpha)} + \frac{(-6.5t^{\alpha})^2}{\Gamma(1+2\alpha)} + \frac{(-6.5t^{\alpha})^3}{\Gamma(1+3\alpha)} + \cdots\right) x^3 \\
&= x^3 \sum_{n=0}^{\infty} \frac{(-6.5t^{\alpha})^n}{\Gamma(1+n\alpha)} \\
&= x^3 E_{\alpha}(-6.5t^{\alpha}).
\end{aligned}
\tag{4.95}
$$

It is therefore remarked that $\underline{w}(x,t) = x^3 \exp(-6.5t)$ is the exact solution of *problem 4.5.1.1* when $\alpha = 1$ (a special case).

Problem 4.5.1.2: Consider the following time-fractional Black-Scholes equation (with reference to example 4.4.1.1, and Hariharan *et al.*, 2013) for $\alpha = 1$.

$$
\frac{\partial^{\alpha}\underline{w}}{\partial t^{\alpha}} = \frac{\partial^2 \underline{w}}{\partial x^2} + (k-1)\frac{\partial \underline{w}}{\partial x} + k\underline{w}, \quad \underline{w} = \underline{w}(x,t), \quad 0 < \alpha \leq 1 \tag{4.96}
$$

subject to:

$$
\underline{w}(x,0) = \max(e^x - 1, 0). \tag{4.97}
$$

By method of solution, taking the MDTM of (4.96) and (4.97) gives:

$$
W(x,\xi+1) = \frac{\Gamma(1+\alpha\xi)}{\Gamma(1+\alpha(1+\xi))}\left(\frac{\partial^2 W(x,\xi)}{\partial x^2} + (k-1)\frac{\partial W(x,\xi)}{\partial x} - kW(x,\xi)\right) \tag{4.98}
$$

and

$$W(0,\xi) = \max(e^x - 1, 0). \tag{4.99}$$

respectively. So,

$$\frac{\partial W(0,\xi)}{\partial x} = \frac{\partial^2 W(0,\xi)}{\partial x^2} = \max(e^x, 0). \tag{4.100}$$

Thus, when $\xi = 0$, we have:

$$\begin{aligned} W(x,1) &= \frac{\Gamma(1)}{\Gamma(1+\alpha)} \left(\frac{\partial^2 W(x,0)}{\partial x^2} + (k-1) \frac{\partial W(x,0)}{\partial x} - kW(x,0) \right) \\ &= \frac{1}{\Gamma(1+\alpha)} \left\{ \max(e^x, 0) + (k-1)\max(e^x, 0) - K\max(e^x - 1, 0) \right\}. \end{aligned}$$

That implies:

$$W(x,1) = \frac{1}{\Gamma(1+\alpha)} \left\{ k\left(\max(e^x, 0) - \max(e^x - 1, 0)\right) \right\}. \tag{4.101}$$

From (4.101),

$$\frac{\partial W(x,1)}{\partial x} = \frac{\partial^2 W(x,1)}{\partial x^2} = 0. \tag{4.102}$$

When $\xi = 1$,

$$\begin{aligned} W(x,2) &= \frac{-k\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left(W(x,1)\right) \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{-k}{\Gamma(1+\alpha)} \left\{ k\left(\max(e^x, 0) - \max(e^x - 1, 0)\right) \right\} \right] \\ &= \frac{-k^2}{\Gamma(1+2\alpha)} \left\{ \max(e^x, 0) - \max(e^x - 1, 0) \right\}. \end{aligned} \tag{4.103}$$

From (4.103),

$$\frac{\partial W(x,2)}{\partial x} = 0 = \frac{\partial^2 W(x,2)}{\partial x^2} \tag{4.104}$$

so, when $\xi = 2$,

$$\begin{aligned} W(x,3) &= \frac{-k\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(W(x,2)\right) \\ &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[\frac{-k}{\Gamma(1+2\alpha)}\left\{(-k^2)\left(\max(e^x,0)-\max(e^x-1,0)\right)\right\}\right] \\ &= \frac{k^3}{\Gamma(1+3\alpha)}\left\{\max(e^x,0)-\max(e^x-1,0)\right\}. \end{aligned} \tag{4.105}$$

From (4.105),

$$\frac{\partial W(x,3)}{\partial x} = 0 = \frac{\partial^2 W(x,3)}{\partial x^2} \tag{4.106}$$

so, when $\xi = 3$,

$$\begin{aligned} W(x,4) &= \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(-kW(x,3)\right) \\ &= \frac{-k^4}{\Gamma(1+4\alpha)}\left(\max(e^x,0)-\max(e^x-1,0)\right). \end{aligned} \tag{4.107}$$

Suppose the following is set:

$$B^*(x) = \left(\max(e^x,0)-\max(e^x-1,0)\right) \tag{4.108}$$

then,

$$
\begin{aligned}
\underline{w}(x,t) &= \sum_{\eta=0}^{\infty} W(x,\eta)\, t^{\alpha\eta} \\
&= W_{x,0} + W_{x,1}t^{\alpha} + W_{x,2}t^{2\alpha} + W_{x,3}t^{3\alpha} + \cdots \\
&= \max(e^x - 1, 0) \\
&+ B^*(x)\left\{\frac{(kt^{\alpha})}{\Gamma(1+\alpha)} - \frac{(kt^{\alpha})^2}{\Gamma(1+2\alpha)} + \frac{(kt^{\alpha})^3}{\Gamma(1+3\alpha)} - \frac{(kt^{\alpha})^4}{\Gamma(1+4\alpha)} + \cdots\right\} \\
&= \max(e^x - 1, 0) + B^*(x)\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(kt^{\alpha})^n}{\Gamma(1+n\alpha)} \\
&= \max(e^x - 1, 0) - B^*(x)\sum_{n=1}^{\infty}\frac{(-kt^{\alpha})^n}{\Gamma(1+n\alpha)} \\
&= \max(e^x - 1, 0) - \{\max(e^x, 0) - \max(e^x - 1, 0)\}\{-1 + E_{\alpha}(-kt^{\alpha})\} \\
&= \max(e^x - 1, 0)\, E_{\alpha}(-kt^{\alpha}) + \max(e^x, 0)\,(1 - E_{\alpha}(-kt^{\alpha}))
\end{aligned}
\tag{4.109}
$$

where $E_{\alpha}(-kt^{\alpha})$ denotes a one-parameter Mittag-Leffler function.

It is remarked that a special case of ***Problem 4.5.1.2*** at $\alpha = 1$ has an exact solution:

$$
\begin{aligned}
\underline{w}(x,t) &= \max(e^x - 1, 0) \\
&+ \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(kt)^n}{n!}\{\max(e^x, 0) - \max(e^x - 1, 0)\} \\
&= \max(e^x - 1, 0)\, e^{-kt} + \max(e^x, 0)\left(1 - e^{-kt}\right).
\end{aligned}
\tag{4.110}
$$

Problem 4.5.1.3: Consider the following fractional Black-Scholes equation (Ref. example 4.4.1.2, and Kumar *et al.* 2014 for $\alpha = 1$):

$$
\frac{\partial^{\alpha}\underline{w}}{\partial t^{\alpha}} + 0.08\,(2 + Sinx)^2\, x^2\frac{\partial^2\underline{w}}{\partial x^2} + 0.06x\frac{\partial\underline{w}}{\partial x} - 0.06\underline{w} = 0 \tag{4.111}
$$

subject to:

$$\underline{w}\left(x,0\right)=\max\left(x-25e^{-0.06},0\right). \tag{4.112}$$

By way of solving Problem 4.5.1.3, we take the MDTM of (4.111) and (4.112) which gives:

$$\begin{aligned}W\left(x,\xi+1\right)=\frac{\Gamma\left(1+\alpha\xi\right)}{\Gamma\left(1+\alpha(1+\xi)\right)}&\left\{-0.08\left(2+Sinx\right)^{2}x^{2}\frac{\partial^{2}W\left(x,\xi\right)}{\partial x^{2}}\right.\\&\left.-0.06x\frac{\partial W\left(x,\xi\right)}{\partial x}+0.06W\left(x,\xi\right)\right\}\end{aligned} \tag{4.113}$$

subject to:

$$W\left(x,0\right)=\max\left(x-25e^{-0.06},0\right). \tag{4.114}$$

Thus, from (4.114),

$$\frac{\partial W\left(x,0\right)}{\partial x}=1,\quad and\quad \frac{\partial^{2}W\left(x,0\right)}{\partial x^{2}}=0. \tag{4.115}$$

So, when $\xi=0$,

$$\begin{aligned}W\left(x,1\right)&=\frac{\Gamma\left(1\right)}{\Gamma\left(1+\alpha\right)}\left\{-0.06x+0.06\max(x-25e^{-0.06},0)\right\}\\&=\frac{-0.06}{\Gamma\left(1+\alpha\right)}\left\{x-\max(x-25e^{-0.06},0)\right\}.\end{aligned} \tag{4.116}$$

So, from (4.116),

$$\frac{\partial W\left(x,1\right)}{\partial x}=0=\frac{\partial^{2}W\left(x,1\right)}{\partial x^{2}}, \tag{4.117}$$

so for $\xi = 1$,

$$\begin{aligned} W(x,2) &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}(0.06W(x,1)) \\ &= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[0.06\left\{\frac{-0.06}{\Gamma(1+\alpha)}\left(x-\max(x-25e^{-0.06},0)\right)\right\}\right] \\ &= \frac{-(0.06)^2}{\Gamma(1+2\alpha)}\left\{x-\max(x-25e^{-0.06},0)\right\}. \end{aligned} \tag{4.118}$$

So, from (4.118),

$$\frac{\partial W(x,2)}{\partial x} = 0 = \frac{\partial^2 W(x,2)}{\partial x^2}, \tag{4.119}$$

so when $\xi = 2$,

$$\begin{aligned} W(x,3) &= \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}(0.06W(x,3)) \\ &= -\frac{(0.06)^3}{\Gamma(1+3\alpha)}\left\{x-\max(x-25e^{-0.06},0)\right\}. \end{aligned} \tag{4.120}$$

Similarly, in recurrent form, the following holds:

$$W(x,\eta) = -\frac{(0.06)^\eta}{\Gamma(1+\eta\alpha)}\left\{x-\max(x-25e^{-0.06},0)\right\}, \eta \in \mathrm{N}. \tag{4.121}$$

Set in (4.121) the following:

$$N^*(x) = \left\{x-\max(x-25e^{-0.06},0)\right\}. \tag{4.122}$$

Hence,

$$
\begin{aligned}
\underline{w}(x,t) &= \sum_{h=0}^{\infty} W(x,h)\, t^{\alpha h} \\
&= W(x,0) + W(x,1)\, t^{\alpha} + W(x,2)\, t^{2\alpha} + W(x,3)\, t^{3\alpha} + \cdots \\
&= \max\left(x - 25e^{-0.06}, 0\right) \\
&- N^{*}(x)\left\{\frac{(0.06t^{\alpha})}{\Gamma(1+\alpha)} + \frac{(0.06t^{\alpha})^{2}}{\Gamma(1+2\alpha)} + \frac{(0.06t^{\alpha})^{3}}{\Gamma(1+3\alpha)} + \cdots\right\} \\
&= \max\left(x - 25e^{-0.06}, 0\right) - N^{*}(x)\sum_{n=1}^{\infty}\frac{(0.06t^{\alpha})^{n}}{\Gamma(1+n\alpha)} \\
&= \max\left(x - 25e^{-0.06}, 0\right) - N^{*}(x)\left\{-1 + E_{\alpha}(0.06t^{\alpha})\right\}.
\end{aligned}
\tag{4.123}
$$

Further simplification of (4.123) yields:

$$
\begin{aligned}
\underline{w}(x,t) &= \max\left(x - 25e^{-0.06}, 0\right) \\
&- \left(x - \max(x - 25e^{-0.06}, 0\right)\left(-1 + E_{\alpha}(0.06t^{\alpha})\right) \\
&= x\left\{1 - E_{\alpha}(0.06t^{\alpha})\right\} + \max\left(x - 25e^{-0.06}, 0\right) E_{\alpha}(0.06t^{\alpha}).
\end{aligned}
\tag{4.124}
$$

It is remarked that when $\alpha = 1$, *Problem 4.5.1.3* has a special case whose exact solution is:

$$
\begin{aligned}
\underline{w}(x,t) &= \max\ \left(x - 25e^{-0.06}, 0\right) + (1 - e^{0.06t})\left\{x - \max(x - 25e^{-0.06}, 0)\right\} \\
&= x\left(1 - e^{0.06t}\right) + \max\left(x - 25e^{-0.06}, 0\right) e^{0.06t}.
\end{aligned}
\tag{4.125}
$$

Note: References are made to Figures 4.7-4.12 for the graphical solutions of problems 4.5.1.1, 4.5.1.2, and 4.5.1.3 in terms of their contingent claim values .

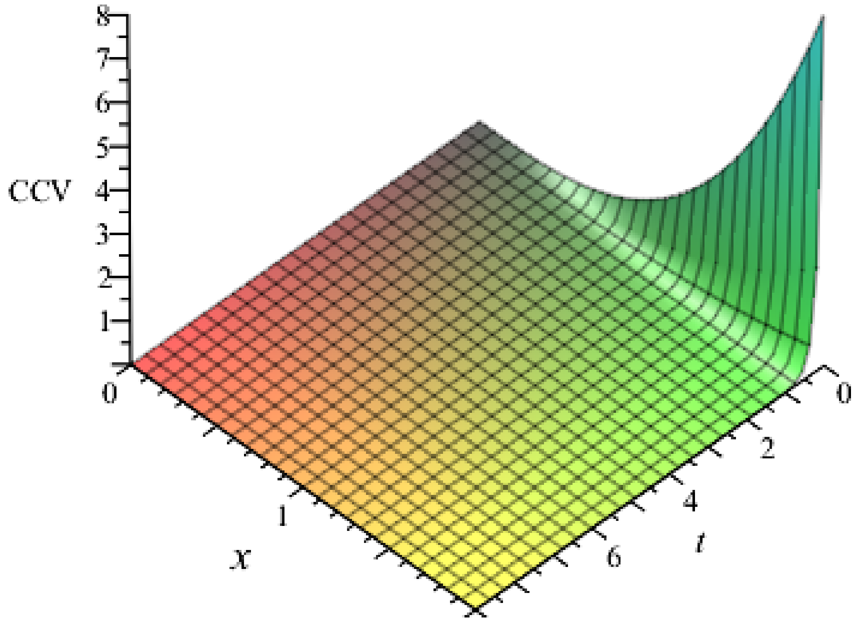

Figure 4.7: Contingent Claim Value (CCV) for $x \in [0, 2]$, $t \in [0, 9]$

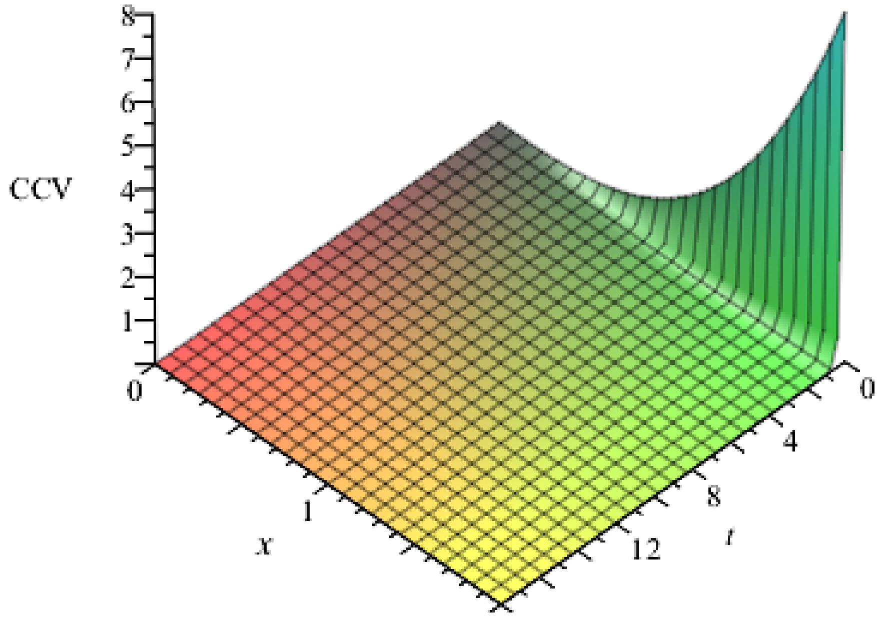

Figure 4.8: Contingent Claim Value (CCV) for $x \in [0, 2]$, $t \in [0, 18]$

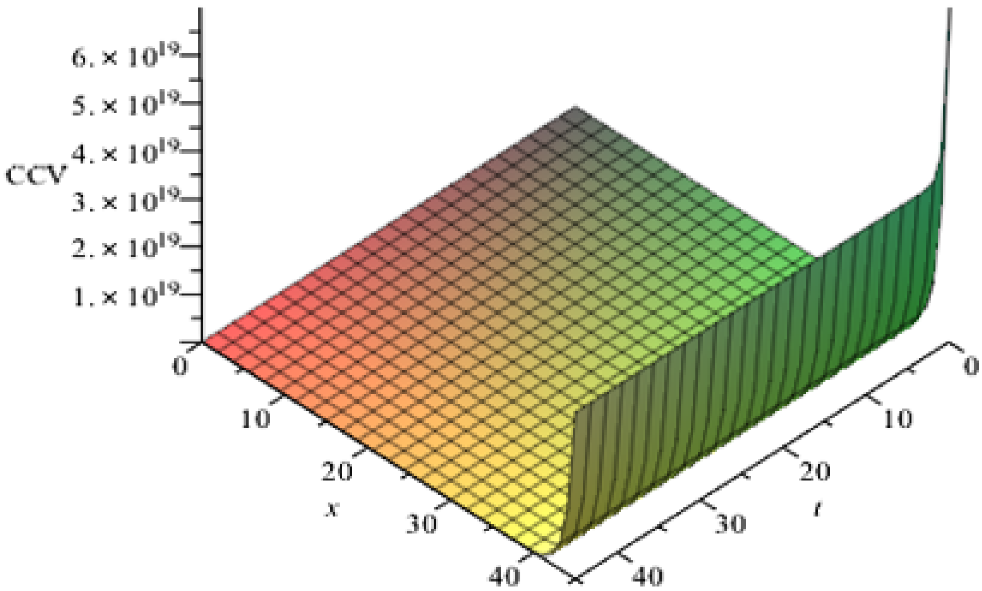

Figure 4.9: Contingent Claim Value (CCV) for $x \in [0, 45]$, $t \in [0, 45]$

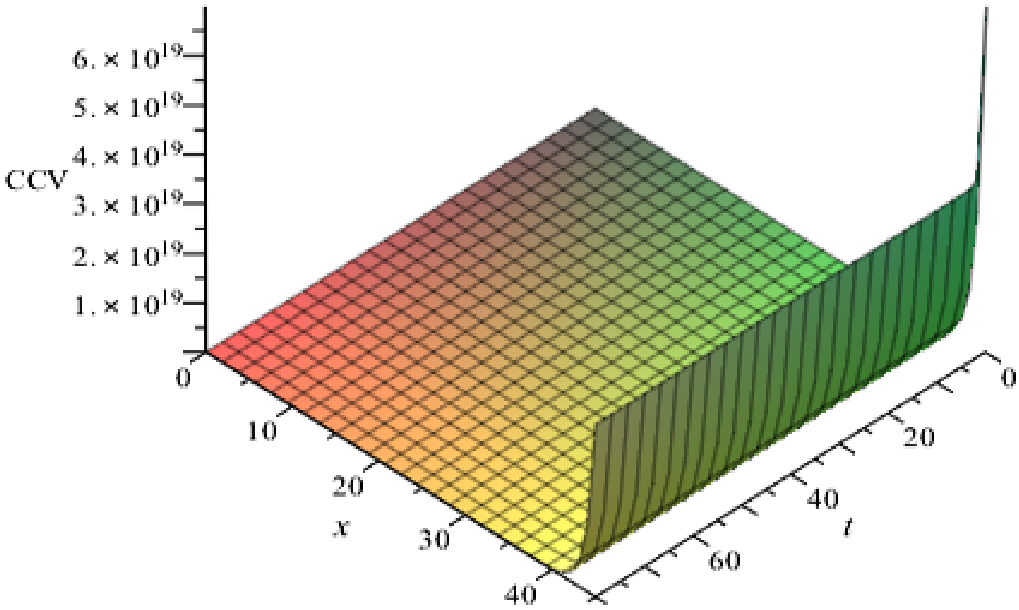

Figure 4.10: Contingent Claim Value (CCV) for $x \in [0, 45]$, $t \in [0, 80]$

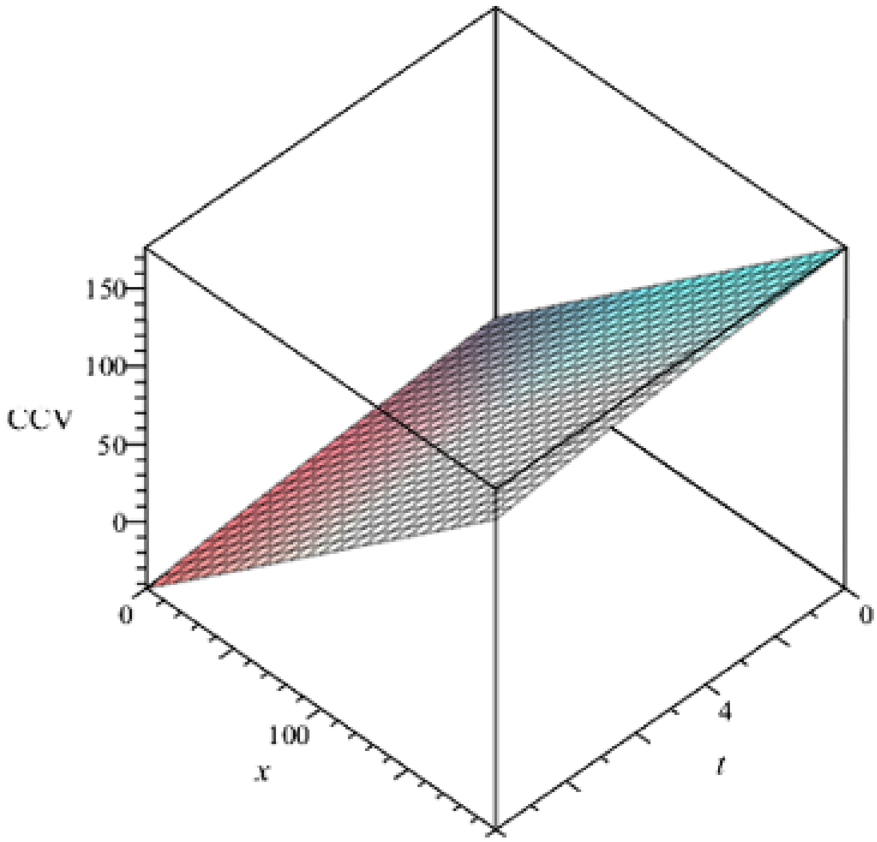

Figure 4.11: Contingent Claim Value (CCV) for $x \in [0, 200]\,,\ t \in [0, 10]$

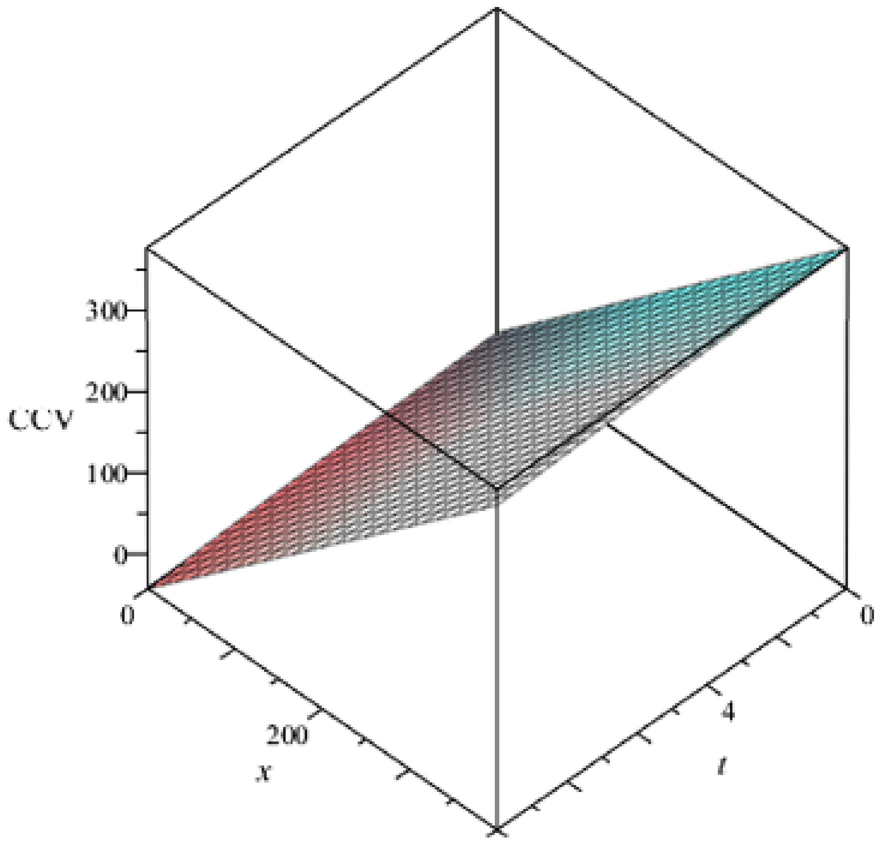

Figure 4.12: Contingent Claim Value (CCV) for $x \in [0, 400]$, $t \in [0, 10]$

Remark 4.5.2

In section 4.5, analytical solutions of the time-fractional Black-Scholes equations for European call option were obtained via the proposed relatively new approximate-analytic technique, PDTM. The graphical representation of the solutions were presented in Figures 4.7-4.12. Figure 4.7 and Figure 4.8 are for *problem 4.5.1.1* , Figure 4.9 and Figure 4.10 are for *problem 4.5.1.2,* while Figure 4.11 and Figure 4.12 are for *problem 4.5.1.3.* For each case, same interval is used for x while different intervals for t.. The application of this method for analytical solutions of time-fractional Black-Scholes model is new to the best of our knowledge. These present results showed that the result in subsection 4.4.1 is a special case of this present work for $\alpha = 1$. Consequently, it is remarked that the time-fractional Black-Scholes equation for European option valuation is a generalisation of the classical Black-Scholes equations for European option valuation at order, $\alpha = 1$.

4.6 The Generalised Bakstein and Howison Model

In this section, the Bakstein and Howison Model is generalised. This is done by using the volatility function associated with the transaction-cost of Baskstein and Howison (2003) in the CEV-BSM in section 4.3. Thereafter, consider the analytical solutions of this generalised Bakstein and Howison model, and its extended version to time-fractional order type are considered.

4.6.1 The Generalisation Procedures

This subsection considers a case where the volatility parameter, σ can be expressed as a function of the following: time, τ, stock price, S, and the differential coefficients of the option price, Φ. In particular, that of non-constant modified volatility function:

$$\sigma = \widehat{\sigma}\left(\tau, S, \frac{\partial \Phi}{\partial S}, \frac{\partial^2 \Phi}{\partial S^2}\right) \tag{4.126}$$

is to be considered. So, the CEV-BSM in (4.46) becomes:

$$\frac{\partial \Phi}{\partial \tau} + rS\frac{\partial \Phi}{\partial S} + \frac{1}{2}S^{\xi}\,\widehat{\sigma}^2\left(\tau, S, \frac{\partial \Phi}{\partial S}, \frac{\partial^2 \Phi}{\partial S^2}\right)\frac{\partial^2 \Phi}{\partial S^2} - r\Phi = 0. \tag{4.127}$$

Note: Equation (4.46) can be improved using (4.139) from the aspect of transaction costs inclusion, large trader and illiquid markets effect. In this regard, the approach of Frey and Patie (2002) and Frey and Stremme (1997) for the effects on the price with the result is followed:

$$\sigma = \widehat{\sigma}\left(\tau, S, \frac{\partial \Phi}{\partial S}, \frac{\partial^2 \Phi}{\partial S^2}\right)\left(1 - \rho S \lambda(S)\frac{\partial^2 \Phi}{\partial S^2}\right) \tag{4.128}$$

where σ is the traditional volatility, ρ is a constant measuring the liquidity of the market, and λ is the price of risk.

Following the assumption that the price of risk is unity, a special case where $\lambda(S) = 1$ and a little algebra with the notion that:

$$1 \approx (1 - f_*)^2\left(1 + 2f_* + \mathrm{O}\left(f_*\right)^3\right)$$

where f_* is a variable function. Thus, (4.128) is written as:

$$\begin{aligned}
\left\{\widehat{\sigma}\left(\tau, S, \frac{\partial\Phi}{\partial S}, \frac{\partial^2\Phi}{\partial S^2}\right)\right\}^2 =& \widehat{\sigma}^2 \\
=& \left\{\sigma\left(1-\rho S\frac{\partial^2\Phi}{\partial S^2}\right)^{-1}\right\}^2 \\
=& \sigma^2\left(1-\rho S\frac{\partial^2\Phi}{\partial S^2}\right)^{-2} \\
\approx& \sigma^2\left(1+2\rho S\frac{\partial^2\Phi}{\partial S^2}\right).
\end{aligned} \tag{4.129}$$

Therefore (4.127) becomes:

$$\frac{\partial\Phi}{\partial\tau} + rS\frac{\partial\Phi}{\partial S} + \frac{1}{2}S^{\xi}\left[\sigma^2\left(1+2\rho S\frac{\partial^2\Phi}{\partial S^2}\right)\right]\frac{\partial^2\Phi}{\partial S^2} - r\Phi = 0 \tag{4.130}$$

such that $\Phi(S,T) = h(S)$, $S \in [0,\infty)$. For the translation, $t+\tau = T$ (T is time at maturity such that $\tau \in [0,T]$) and using $\underline{w}(S,t) = \Phi(S,\tau)$, (4.130) becomes:

$$\frac{\partial\underline{w}}{\partial t} + rS\frac{\partial\underline{w}}{\partial S} + \frac{1}{2}S^{\xi}\sigma^2\left(1+2\rho S\frac{\partial^2\underline{w}}{\partial S^2}\right)\frac{\partial^2\underline{w}}{\partial S^2} - r\underline{w} = 0,\ \underline{w}(S,0) = h(S). \tag{4.131}$$

Note: For $\xi = 2$, equation (4.131) has an exact solution (Esekon, 2013) of the form:

$$\underline{w}(S,t) = S - \rho^{-1}\sqrt{S_0}\left\{\sqrt{S}\exp\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t + \frac{\sqrt{S_0}}{4}\exp\left(r+\frac{\sigma^2}{4}\right)t\right\}. \tag{4.132}$$

For $\sigma,\ S_0,\ S,\ |\rho| > 0$ while $r,\ t \geq 0,\quad S_0$ as an initial stock price, with:

$$\underline{w}(S,0) = \max\left\{S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right),\ 0\right\}. \tag{4.133}$$

Remark 4.6.1.1: It is obvious that (4.131) generalises the Bakstein and Howison (2003) model. González-Gaxiola *et al.* (2015) considered the approximate solution of a particular case of (4.131) via the application of the Adomian Decomposition Method. Thus, our results will be compared with theirs.

4.6.1.1 The MDTM Applied to the Generalised Nonlinear Model

In this subsection, the MDTM approach is applied to the model equation (4.131) as follows. Equation (4.131) is re-expressed as:

$$\frac{\partial \underline{w}}{\partial t} = -rS\frac{\partial \underline{w}}{\partial S} - \frac{1}{2}S^{\xi}\sigma^{2}\left(1 + 2\rho S\frac{\partial^{2}\underline{w}}{\partial S^{2}}\right)\frac{\partial^{2}\underline{w}}{\partial S^{2}} + r\underline{w} \tag{4.134}$$

subject to:

$$\underline{w}\left(S,0\right) = \max\left\{S - \rho^{-1}\left(\sqrt{S_{0}S} + \frac{S_{0}}{4}\right), 0\right\}.$$

For simplicity, (4.134) is re-expressed as follows:

$$\frac{\partial \underline{w}}{\partial t} = -\left\{rS\frac{\partial \underline{w}}{\partial S} + \frac{1}{2}S^{\xi}\sigma^{2}\left\{\frac{\partial^{2}\underline{w}}{\partial S^{2}} + 2\rho S\left(\frac{\partial^{2}\underline{w}}{\partial S^{2}}\right)^{2}\right\} - r\underline{w}\right\}. \tag{4.135}$$

At projection, the transformation of (4.135) using MDTM yields:

$$\begin{aligned}(k+1)\,H_{k+1}\left(S\right) &= -rSH_{k}'\left(S\right) + rH_{k}\left(S\right)\\ &\quad - \left\{\frac{1}{2}S^{\xi}\sigma^{2}\left(H_{k}''\left(S\right) + 2\rho S\sum_{n=0}^{k}H_{n}''\left(S\right)H_{k-n}''\left(S\right)\right)\right\}.\end{aligned} \tag{4.136}$$

Re-writing (4.136) for $H_{k+1} = H_{k+1}(S)$ gives:

$$H_{k+1} = \frac{-1}{k+1}\left\{rSH_{k}' + \frac{1}{2}S^{\xi}\sigma^{2}\left(H_{k}'' + 2\rho S\sum_{n=0}^{k}H_{n}''H_{k-n}''\right) - rH_{k}\right\} \tag{4.137}$$

subject to:

$$H_0 = \max\left\{S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right),\ 0\right\}. \tag{4.138}$$

When $k = 0$,

$$H_1 = -\left(rSH_0' + \frac{1}{2}S^{\xi}\sigma^2\left(H_0'' + 2\rho S H_0'' H_0''\right) - rH_0\right). \tag{4.139}$$

When $k = 1$,

$$\begin{aligned} H_2 &= -\frac{1}{2}\left\{rSH_1' + \frac{1}{2}S^{\xi}\sigma^2\left(H_1'' + 2\rho S\sum_{n=0}^{1} H_1'' H_{1-n}''\right) - rH_1\right\} \\ &= \frac{-1}{2}\left(rSH_1' + \frac{1}{2}S^{\xi}\sigma^2\left(H_1'' + 4\rho S H_0'' H_1''\right) - rH_1\right). \end{aligned} \tag{4.140}$$

When $k = 2$,

$$\begin{aligned} H_3 &= -\frac{1}{3}\left\{rSH_2' + \frac{1}{2}S^{\xi}\sigma^2\left(H_2'' + 2\rho S\sum_{n=0}^{2} H_n'' H_{2-n}''\right) - rH_2\right\} \\ &= \frac{-1}{3}\left(rSH_2' + \frac{1}{2}S^{\xi}\sigma^2\left(H_2'' + 2\rho S\left(2H_0'' H_2'' + H_1'' H_1''\right)\right) - rH_1\right). \end{aligned} \tag{4.141}$$

When $k = 3$,

$$\begin{aligned} H_4 &= -\frac{1}{4}\left\{rSH_3' + \frac{1}{2}S^{\xi}\sigma^2\left(H_3'' + 2\rho S\sum_{n=0}^{3} H_n'' H_{3-n}''\right) - rH_3\right\} \\ &= \frac{-1}{4}\left(rSH_3' + \frac{1}{2}S^{\xi}\sigma^2\left(H_3'' + 4\rho S\left(H_0'' H_3'' + 2H_1'' H_2''\right)\right) - rH_3\right). \end{aligned} \tag{4.142}$$

When $k = 4$,

$$\begin{aligned} H_5 &= -\frac{1}{5}\left\{rSH_4' + \frac{1}{2}S^{\xi}\sigma^2\left(H_4'' + 2\rho S\sum_{n=0}^{4} H_n'' H_{4-n}''\right) - rH_4\right\} \\ &= \frac{-1}{5}\left(rSH_4' + \frac{1}{2}S^{\xi}\sigma^2\left(H_4'' + 2\rho S\left(2H_0'' H_4'' + 2H_1'' H_3'' + H_2'' H_2''\right)\right) - rH_4\right). \end{aligned} \tag{4.143}$$

In general, for an integer k_* such that $k_* \in [1, \infty)$, the following is obtained:

$$H_{k_*} = \frac{-1}{k_*}\left\{rSH'_{k_*-1} + \frac{1}{2}S^2\sigma^2\left(H''_{k_*-1} + 2\rho S\sum_{n=0}^{k_*-1} H''_n H''_{k_*-n-1}\right) - rH_{k_*-1}\right\}. \quad (4.144)$$

4.6.1.2 Numerical Illustration and Applications

Recall (4.132) and (4.133) as follows:

$$\underline{w}(S,t) = \underline{w} = S - \rho^{-1}\sqrt{S_0}\left\{\sqrt{S}\exp\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t + \frac{\sqrt{S_0}}{4}\exp\left(r+\frac{\sigma^2}{4}\right)t\right\} \quad (4.145)$$

and

$$\underline{w}(S,0) = \max\left\{S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right),\ 0\right\}. \quad (4.146)$$

For numerical computation, the following cases will be considered:

Case 4.6.1.2.1 For $r = 0$, $|\rho| = 0.01$, $\xi = 2$, $\sigma = 0.4$, $S_0 = 4$, thus the exact solution and initial condition are:

$$\underline{w}(S,t) = S + 200\left\{\sqrt{S}\exp\left(\frac{t}{50}\right) + \frac{1}{2}\exp\left(\frac{t}{100}\right)\right\} \quad (4.147)$$

and

$$\underline{w}(S,0) = S + 200\sqrt{S} + 100 \quad (4.148)$$

respectively.

So, applying the PDTM with the parameters in *Case 4.6.1.2.1* through (4.137)-

(4.1143) gives the following:

$$H\left(S,0\right) = H_0 = S + 200\sqrt{S} + 100, \tag{4.149}$$

$$H(S,1) = H_1 = -\frac{S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right), \tag{4.150}$$

$$\begin{aligned} H(S,2) = &-\frac{S^2}{25}\left(-\frac{8}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right) - \frac{4S}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2\right.\\ &-\frac{4S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} + \frac{300}{S^4}\right)\\ &-\frac{S}{50}\left(\frac{-50\left(\frac{-4}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2 - \frac{4}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} - \frac{300}{S^4}\right)\right)}{S^{3/2}}\right)\\ &-\frac{1}{S^{3/2}}\left(50\left(-\frac{8}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right) - \frac{4S}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2\right.\right.\\ &\left.\left.\left.-\frac{4S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} + \frac{300}{S^4}\right)\right)\right)\right). \end{aligned} \tag{4.151}$$

Whence,

$$
\begin{aligned}
\underline{w}(S,t) &= \sum_{\eta=0}^{\infty} H(S,\eta)\, t^{\eta} \\
&= H(S,0) + H(S,1)\, t + H(S,2)\, t^2 + H(S,3)\, t^3 + \cdots \\
&= \left(S + 200\sqrt{S} + 100\right) - \frac{S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right) t \\
&+ \left(-\frac{S^2}{25}\left(-\frac{8}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right) - \frac{4S}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2\right.\right. \\
&- \frac{4S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} + \frac{300}{S^4}\right) \\
&- \frac{1}{50\sqrt{S}}\left(-50\left(\frac{-4}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2 - \frac{4}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} - \frac{300}{S^4}\right)\right)\right) \\
&- \frac{1}{S^{3/2}}\left(50\left(-\frac{8}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right) - \frac{4S}{25}\left(\frac{75}{S^{5/2}} + \frac{100}{S^3}\right)^2\right.\right. \\
&\left.\left.\left.\left.\left. - \frac{4S}{25}\left(\frac{-50}{S^{3/2}} - \frac{50}{S^2}\right)\left(\frac{-375}{2S^{7/2}} + \frac{300}{S^4}\right)\right)\right)\right)\right)\right) t^2 + \cdots .
\end{aligned}
\tag{4.152}
$$

Figure 4.13 is the graphics for approximate solution for *Case 4.6.1.2.1*, for $S \in [0.1, 10]$ and $t \in [0, 1]$ while Figure 4.14 is the graphics for exact solution for *Case 4.6.1.2.1*, for $S \in [0.1, 10]$ and $t \in [0, 1]$.

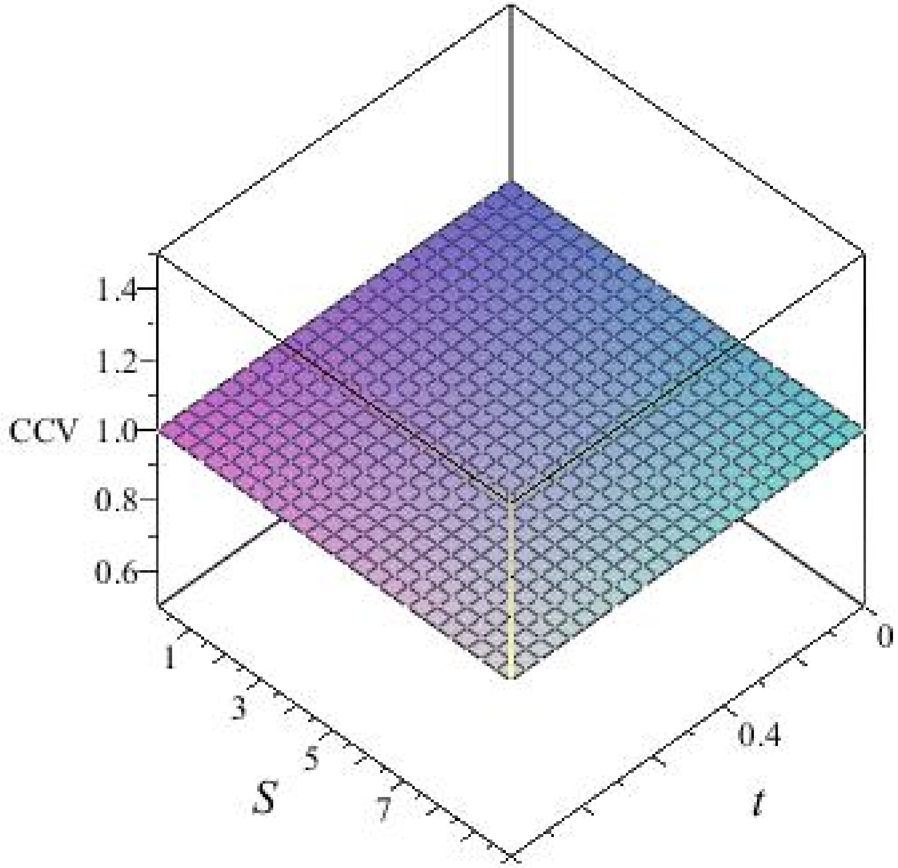

Figure 4.13: Approximate solution for *Case 4.6.1.2.1*

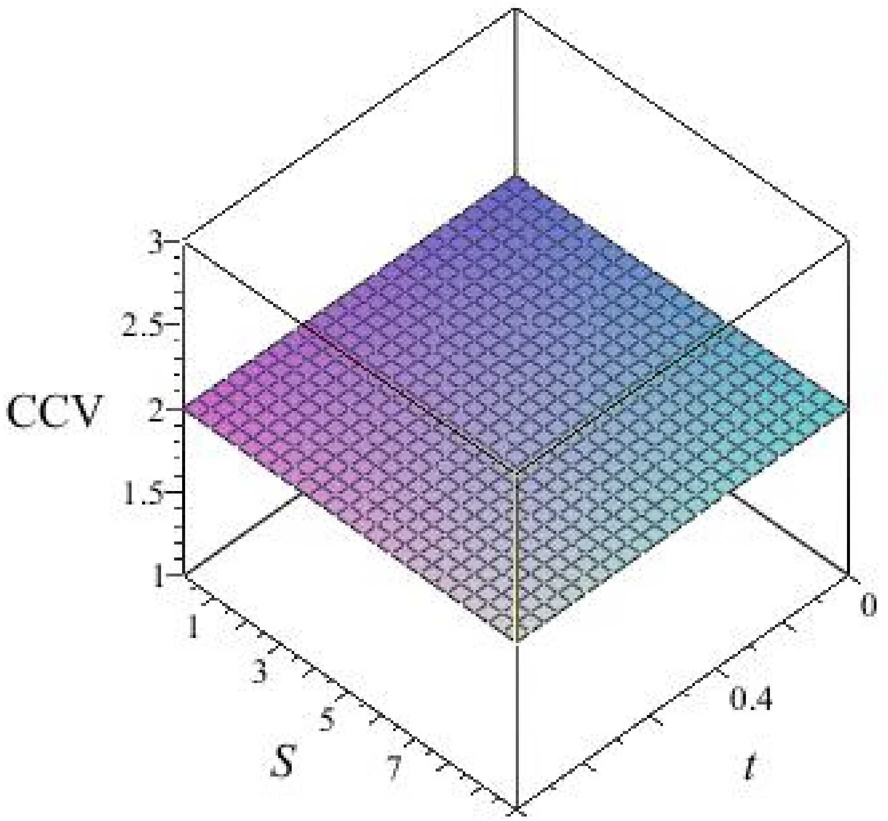

Figure 4.14: Exact solution for *Case 4.6.1.2.1*

Case 4.6.1.2.2 For $r = 0.06,\ |\rho| = 0.01,\ \sigma = 0.4,\ \xi = 2,\ S_0 = 4$, we thus have the exact solution and initial condition as:

$$\underline{w}\,(S,t) = S + 200\left(\sqrt{S}\exp\left(\frac{t}{20}\right) + \frac{1}{2}\exp\left(\frac{t}{10}\right)\right) \tag{4.153}$$

and

$$\underline{w}\,(S,0) = S + 200\sqrt{S} + 100 \tag{4.154}$$

respectively.

Following the same procedure as in *Case 4.6.1.2.1*, by applying the PDTM with the parameters in *Case 4.6.2.2.2* through (4.153)-(4.154) gives the following:

$$H\,(S,0) = H_0 = S + 200\sqrt{S} + 100, \tag{4.155}$$

$$H\,(S,1) = H_1 = \frac{1}{2500}\left(6S^3 + 1200S^{5/2} + 600S^2 - 75S - 2500S^{1/2} + 5000\right), \tag{4.156}$$

$$\begin{aligned} H\,(S,2) = H_2 = {} & \frac{9S^6}{781250} + \frac{36S^{11/2}}{15625} + \frac{18S^5}{15625} - \frac{9S^4}{62500} - \frac{3S^{7/2}}{625} + \frac{501S^3}{62500} \\ & - \frac{3411S^{5/2}}{15625} - \frac{222S^2}{625} - \frac{48S^{3/2}}{625} + \frac{9S}{5000} + \frac{\sqrt{S}}{100} - \frac{1}{25}, \end{aligned} \tag{4.157}$$

$$\vdots$$

Whence,

$$
\begin{aligned}
\underline{w}\,(S,t) &= \sum_{\eta=0}^{\infty} W\,(S,\eta)\, t^{\eta} \\
&= H\,(S,0) + H\,(S,1)\, t + H\,(S,2)\, t^{2} + H\,(S,3)\, t^{3} + \cdots \\
&= \left(S + 200\sqrt{S} + 100\right) \\
&+ \left(\frac{1}{2500}\left(6S^{3} + 1200S^{5/2} + 600S^{2} - 75S - 2500S^{1/2} + 5000\right)\right) t \\
&+ \left(\frac{9S^{6}}{781250} + \frac{36S^{11/2}}{15625} + \frac{18S^{5}}{15625} - \frac{9S^{4}}{62500} - \frac{3S^{7/2}}{625} + \frac{501S^{3}}{62500}\right. \\
&\left. - \frac{3411S^{5/2}}{15625} - \frac{222S^{2}}{625} - \frac{48S^{3/2}}{625} + \frac{9S}{5000} + \frac{\sqrt{S}}{100} - \frac{1}{25}\right) t^{2} + \cdots .
\end{aligned}
\tag{4.158}
$$

Note: In Tables 4.1-4.3, we present in comparison, the exact and the approximate solutions for time $t = 0,\ 0.5\ \textit{and}\ 1$ respectively. Table 4.1 is for the solutions of Case 4.6.1.2.2 when $t = 0$. This table shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0$ and $S \in [0, 5]$. Table 4.2 is for the solutions of Case 4.6.1.2.2 when $t = 0.5$. This table shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0.5$ and $S \in [0, 5]$. Table 4.3 is for the solutions of Case 4.6.1.2.2 when $t = 1$. This table shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 1$ and $S \in [0, 5]$.

Table 4.1: The solutions of Case 4.6.1.2.2 at $t = 0$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.0	100.000	100.000	0.00000
0.5	241.921	241.921	1.2E-16
1.0	301.000	301.000	0.00000
1.5	346.449	346.449	0.00000
2.0	384.843	384.843	0.00000
2.5	418.728	418.728	1.4E-16
3.0	449.410	449.410	0.00000
3.5	477.666	477.666	1.2E-16
4.0	504.000	504.000	0.00000
4.5	528.764	528.764	0.00000
5.0	552.214	552.214	0.00000

Table 4.2: The solutions of Case 4.6.1.2.2 at $t = 0.5$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.0	105.127	101.990	0.0298
0.5	250.629	243.300	0.0292
1.0	311.190	302.528	0.0278
1.5	357.777	348.674	0.0254
2.0	397.130	388.385	0.0220
2.5	431.860	424.306	0.0175
3.0	463.307	457.863	0.0117
3.5	492.265	489.986	0.0046
4.0	519.253	521.378	0.0041
4.5	544.631	552.641	0.0147
5.0	568.662	584.355	0.0276

Table 4.3: The solution of Case 4.6.1.2.2 at $t = 1$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.0	110.517	103.960	0.05933
0.5	259.689	244.590	0.05815
1.0	321.771	303.729	0.05607
1.5	369.525	350.145	0.05244
2.0	409.861	390.570	0.04707
2.5	445.458	427.789	0.03966
3.0	477.688	463.451	0.02980
3.5	507.367	498.813	0.01686
4.0	535.026	535.043	3.3E-05
4.5	561.034	573.390	0.02202
5.0	585.660	615.287	0.05059

4.6.2 The Time-Fractional Generalised Bakstein and Howison Model

In this section, the extension of the nonlinear option pricing model in section 4.6.1 (that is, the generalised Bakstein and Howison Model) to a time-fractional order type is considered. Thereafter, the PDTM is applied to the extended version of the model for analytical solutions. It is remarked with regard to the consulted literature, that this is the first time such nonlinear option pricing model is generalised and extended to time-fractional order type.

In what follows, (4.144) considered with respect to time-fractional order, thus considering the model:

$$\frac{\partial^\alpha \underline{w}}{\partial t^\alpha} = -rS\frac{\partial \underline{w}}{\partial S} - \frac{1}{2}S^\xi \sigma^2 \left(1 + 2\rho S \frac{\partial^2 \underline{w}}{\partial S^2}\right)\frac{\partial^2 \underline{w}}{\partial S^2} + r\underline{w} \tag{4.159}$$

subject to:

$$\underline{w}\,(S,0) = \left(S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right)\right)^+. \tag{4.160}$$

4.6.2.1 The MDTM and the Extended Nonlinear Model

In this subsection, the MDTM approach is applied to the extended nonlinear model (the time-fractional Generalised Bakstein and Howison Model) (4.159). Equation (4.159) re-expressed as:

$$\frac{\partial^\alpha \underline{w}}{\partial t^\alpha} = -\left(rS\frac{\partial \underline{w}}{\partial S} + \frac{1}{2}S^\xi \sigma^2 \left(\frac{\partial^2 \underline{w}}{\partial S^2} + 2\rho S\left(\frac{\partial^2 \underline{w}}{\partial S^2}\right)^2\right) - r\underline{w}\right). \tag{4.161}$$

At projection, the transformation of (4.161) using PDTM yields the following:

$$PDT\left[\frac{\partial^{\alpha}\underline{w}}{\partial t} = -\left(rS\frac{\partial \underline{w}}{\partial S} + \frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}\underline{w}}{\partial S^{2}} + 2\rho S\left(\frac{\partial^{2}\underline{w}}{\partial S^{2}}\right)^{2}\right) - r\underline{w}\right)\right]. \tag{4.162}$$

and

$$PDT\left[\underline{w}\left(S,0\right) = \max\left(S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right), 0\right)\right]. \tag{4.163}$$

Thus, the following is obtained:

$$\begin{aligned}\frac{\Gamma\left(1+\alpha\left(1+k\right)\right)}{\Gamma\left(1+\alpha k\right)}H\left(S,k+1\right) = -&\left(rS\frac{\partial H\left(S,k\right)}{\partial S} - rH\left(S,k\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,k\right)}{\partial S^{2}} + 2\rho S\sum_{n=0}^{k}\frac{\partial^{2}H\left(S,n\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,k-n\right)}{\partial S^{2}}\right)\right).\end{aligned} \tag{4.164}$$

As such,

$$\begin{aligned}H\left(S,k+1\right) = -&\frac{\Gamma\left(1+\alpha k\right)}{\Gamma\left(1+\alpha\left(1+k\right)\right)}\left(rS\frac{\partial H\left(S,k\right)}{\partial S} - rH\left(S,k\right)\right.\\ &\left.+\frac{1}{2}S^{2}\sigma^{2}\left(\frac{\partial^{2}H\left(S,k\right)}{\partial S^{2}} + 2\rho S\sum_{n=0}^{k}\frac{\partial^{2}H\left(S,n\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,k-n\right)}{\partial S^{2}}\right)\right.\end{aligned} \tag{4.165}$$

subject to:

$$H\left(\left(S,0\right)\right) = \max\left(S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right), 0\right). \tag{4.166}$$

For $k = 0$, we have:

$$\begin{aligned}H\left(S,1\right) = -&\frac{1}{\Gamma\left(1+\alpha\right)}\left(rS\frac{\partial H\left(S,0\right)}{\partial S} - rH\left(S,0\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}} + 2\rho S\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}}\right)\right).\end{aligned} \tag{4.167}$$

For $k = 1$, the following is obtained:

$$\begin{aligned} H\left(S,2\right) &= -\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}\left(rS\frac{\partial H\left(S,1\right)}{\partial S} - rH\left(S,1\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}} + 2\rho S\sum_{n=0}^{1}\frac{\partial^{2}H\left(S,n\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,1-n\right)}{\partial S^{2}}\right)\right)\\ &= -\frac{\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}\left(rS\frac{\partial H\left(S,1\right)}{\partial S} - rH\left(S,1\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}} + 4\rho S\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}}\right)\right). \end{aligned} \tag{4.168}$$

For $k = 2$, the following is obtained:

$$\begin{aligned} H\left(S,3\right) &= -\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\left(rS\frac{\partial H\left(S,2\right)}{\partial S} - rH\left(S,2\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,2\right)}{\partial S^{2}} + 2\rho S\sum_{n=0}^{2}\frac{\partial^{2}H\left(S,n\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,2-n\right)}{\partial S^{2}}\right)\right)\\ &= -\frac{\Gamma\left(1+2\alpha\right)}{\Gamma\left(1+3\alpha\right)}\left\{rS\frac{\partial H\left(S,2\right)}{\partial S} - rH\left(S,2\right) + \frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,2\right)}{\partial S^{2}}\right)\right.\\ &\left.+S^{\xi+1}\sigma^{2}\rho\left(2\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,2\right)}{\partial S^{2}} + \frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}}\right)\right\}. \end{aligned} \tag{4.169}$$

For $k = 3$, the following is obtained:

$$\begin{aligned} H\left(S,4\right) &= -\frac{\Gamma\left(1+3\alpha\right)}{\Gamma\left(1+4\alpha\right)}\left(rS\frac{\partial H\left(S,3\right)}{\partial S} - rH\left(S,3\right)\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(\frac{\partial^{2}H\left(S,3\right)}{\partial S^{2}} + 2\rho S\sum_{n=0}^{3}\frac{\partial^{2}H\left(S,n\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,3-n\right)}{\partial S^{2}}\right)\right)\\ &= -\frac{\Gamma\left(1+3\alpha\right)}{\Gamma\left(1+4\alpha\right)}\left(rS\frac{\partial H\left(S,3\right)}{\partial S} - rH\left(S,3\right) + \frac{1}{2}S^{\xi}\sigma^{2}\frac{\partial^{2}H\left(S,3\right)}{\partial S^{2}}\right.\\ &\left.+\frac{1}{2}S^{\xi}\sigma^{2}\left(4\rho S\left(\frac{\partial^{2}H\left(S,0\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,3\right)}{\partial S^{2}} + \frac{\partial^{2}H\left(S,1\right)}{\partial S^{2}}\frac{\partial^{2}H\left(S,2\right)}{\partial S^{2}}\right)\right)\right). \end{aligned} \tag{4.170}$$

For $k=4$, the following is obtained:

$$
\begin{aligned}
H(S,5) &= -\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}\left(rS\frac{\partial H(S,4)}{\partial S} - rH(S,4)\right.\\
&\left.+\frac{1}{2}S^{\xi}\sigma^2\left(\frac{\partial^2 H(S,4)}{\partial S^2} + 2\rho S\sum_{n=0}^{4}\frac{\partial^2 H(S,n)}{\partial S^2}\frac{\partial^2 H(S,4-n)}{\partial S^2}\right)\right)\\
&= -\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}\left(rS\frac{\partial H(S,4)}{\partial S} - rH(S,4) + \frac{1}{2}S^{\xi}\sigma^2\left(\frac{\partial^2 H(S,4)}{\partial S^2}\right.\right.\\
&\left.\left.+4\rho S\left(\frac{\partial^2 H(S,0)}{\partial S^2}\frac{\partial^2 H(S,4)}{\partial S^2} + \frac{\partial^2 H(S,1)}{\partial S^2}\frac{\partial^2 H(S,3)}{\partial S^2} + \frac{1}{2}\left(\frac{\partial^2 H(S,2)}{\partial S^2}\right)^2\right)\right)\right).
\end{aligned} \tag{4.171}
$$

$\vdots$

In general, for $m \geq 1$, $m \in \mathrm{N}$, the following is obtained:

$$
\begin{aligned}
H(S,m) = &-\frac{\Gamma(1+\alpha(m-1))}{\Gamma(1+\alpha m)}\left(rS\frac{\partial H(S,m-1)}{\partial S} - rH(S,m-1)\right.\\
&\left.+\frac{1}{2}S^2\sigma^2\left(\frac{\partial^2 H(S,m-1)}{\partial S^2} + 2\rho S\sum_{n=0}^{m-1}\frac{\partial^2 H(S,n)}{\partial S^2}\frac{\partial^2 H(S,m-n-1)}{\partial S^2}\right)\right).
\end{aligned} \tag{4.172}
$$

4.6.2.2 Numerical Illustration and Applications

In this subsection, two cases are considered. *Case 4.6.3.2.1* has two examples with time-integer order while *Case 4.6.3.2.2* has three examples with time-fractional order. Equation (4.132) and (4.133) recalled as follows:

$$
\underline{w}(S,t) = w = S - \rho^{-1}\sqrt{S_0}\left(\sqrt{S}\exp\left(\frac{r+\frac{\sigma^2}{4}}{2}\right)t + \frac{\sqrt{S_0}}{4}\exp\left(r+\frac{\sigma^2}{4}\right)t\right)
$$

and

$$\underline{w}(S,0) = \max\left(S - \rho^{-1}\left(\sqrt{S_0 S} + \frac{S_0}{4}\right),\ 0\right)$$

respectively. For numerical illustration, some examples for different values of S, t, and α over fixed values for the other parameters are considered. Hence, for $r = 0.06$, $|\rho| = 0.01$, $\sigma = 0.4$, $\xi = 2$ and $S_0 = 4$, the exact solution and initial condition are as follows as:

$$\underline{w}(S,t) = S + 200\left(\sqrt{S}\exp\left(\frac{t}{20}\right) + \frac{1}{2}\exp\left(\frac{t}{10}\right)\right) \tag{4.173}$$

and

$$\underline{w}(S,0) = S + 200\sqrt{S} + 100. \tag{4.174}$$

Thus, by applying the MDTM with the above parameters, the following are obtained:

$$H(S,0) = 100 + 200\sqrt{S} + S, \tag{4.175}$$

$$H(S,1) = \frac{1}{2500\Gamma(1+\alpha)}\left(5000 - 2500S^{1/2} - 75S + 600S^2 + 1200S^{5/2} + 6S^3\right), \tag{4.176}$$

$$H(S,2)=\frac{1}{312500\Gamma(1+2\alpha)}\left(\begin{array}{l}-125000+31250S^{1/2}+5625S-240000S^{3/2}\\-1080000S^{2}-697200S^{5/2}-5400S^{3}\\+3600S^{4}+7200S^{9/2}+36S^{5}\end{array}\right), \tag{4.177}$$

$$\vdots$$

Whence,

$$\begin{aligned}\underline{w}(S,t)&=\sum_{h=0}^{\infty}H(S,h)\,t^{h\alpha}\\&=H(S,0)+H(S,1)\,t^{\alpha}+H(S,2)\,t^{2\alpha}+H(S,3)\,t^{3\alpha}+\cdots\\&=\left(100+200\sqrt{S}+S\right)\\&+\left(\frac{1}{2500\Gamma(1+\alpha)}\left(5000-2500S^{1/2}-75S+600S^{2}+1200S^{5/2}+6S^{3}\right)\right)t^{\alpha}\\&+\left\{\frac{1}{312500\Gamma(1+2\alpha)}\left(-125000+31250S^{1/2}+5625S-240000S^{3/2}\right.\right.\\&\left.\left.-1080000S^{2}-697200S^{5/2}-5400S^{3}+3600S^{4}+7200S^{9/2}+36S^{5}\right)\right\}t^{2\alpha}+\cdots.\end{aligned} \tag{4.178}$$

Based on (4.178), two Cases viz: *Case 4.6.3.2.1* and *Case 4.6.3.2.2* for integer and time-fractional order are considered respectively. Tables 4.4-4.8 are for the comparison of solutions in line with their relative absolute errors (all for cases 4.6.3.2.1 and 4.6.3.2.2) as thus specified as follows: Table 4.4 and Table 4.5 are for *Case 4.6.3.2.1* for an integer power of the time parameter. In a similar way, Tables 4.6-4.8 are for *Case 4.6.3.2.2* for fractional powers of the time parameter. Also, we present in comparison, the exact and the approximate solutions for different values of t and α.

Table 4.4: The solutions of *Case 4.6.3.2.1* at $t = 0$, and $\alpha = 1$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.5	241.92136	241.92136	0.000000000
1.0	301.00000	301.00000	0.000000000
1.5	346.44900	346.44898	5.77286E-08
2.0	384.84280	384.84272	2.07877E-07
2.5	418.72780	418.72777	7.16456E-08
3.0	449.41020	449.41017	6.67542E-08
3.5	477.66580	477.66574	1.25611E-07
4.0	504.00000	504.00000	0.000000000
4.5	528.76410	528.76407	5.67361E-08
5.0	552.21360	552.21360	0.000000000
5.5	574.54160	574.54158	3.48104E-08
6.0	595.89800	595.89795	8.3907E-08
6.5	616.40200	616.40196	6.48927E-08
7.0	636.15030	636.15027	4.71587E-08
7.5	655.22260	655.22256	6.1048E-08

Table 4.5: The solutions of *Case 4.6.3.2.1* at $t = 0.5, \alpha = 1$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.5	250.62857	243.22119	0.029555210
1.0	311.19020	301.95640	0.029672528
1.5	357.77700	347.18204	0.029613307
2.0	397.13010	385.49469	0.029298736
2.5	431.86030	419.48813	0.028648547
3.0	463.30670	450.53158	0.027573786
3.5	492.26490	479.47976	0.025972073
4.0	519.25320	506.93393	0.023724977
4.5	544.63150	533.35939	0.020696765
5.0	568.66200	559.14593	0.016734141
5.5	591.54260	584.64223	0.011665043
6.0	613.42690	610.17683	0.005298219
6.5	634.43730	636.07195	0.002576535
7.0	654.67290	662.65295	0.012189370
7.5	674.21540	690.25510	0.023790172

Table 4.6: The solutions of *Case 4.6.3.2.2* for $t = 0.5$ & $\alpha = 0.5$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.01	125.64342	123.22544	0.019244780
0.02	134.14750	131.45494	0.020071638
0.03	140.67510	137.77139	0.020641251
0.04	146.17980	143.09741	0.021086292
0.05	151.03060	147.79036	0.021454195
0.06	155.41710	152.03352	0.021770963
0.07	159.45170	155.93576	0.022050188
0.08	163.20770	159.56801	0.022300970
0.09	166.73610	162.97956	0.022529854
0.10	170.07380	166.20626	0.022740363
0.11	173.24890	169.27521	0.022936307
0.12	176.28310	172.20744	0.023119970
0.13	179.19370	175.01969	0.023293285
0.14	181.99470	177.72552	0.023457716
0.15	184.69770	180.33610	0.023614804
0.16	187.31240	182.86079	0.023765698
0.17	189.84680	185.30748	0.023910437

Table 4.7: The solutions of *Case 4.6.3.2.2* for $t = 0.5$ & $\alpha = 1.5$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.01	125.64342	121.02823	0.036732445
0.02	134.14750	129.30043	0.036132392
0.03	140.67510	135.65022	0.035719754
0.04	146.17980	141.00492	0.035400787
0.05	151.03060	145.72370	0.035137912
0.06	155.41710	149.99079	0.034914498
0.07	159.45170	153.91560	0.034719605
0.08	163.20770	157.56945	0.034546471
0.09	166.73610	161.00186	0.034391113
0.10	170.07380	164.24889	0.034249308
0.11	173.24890	167.33777	0.034119293
0.12	176.28310	170.28963	0.033999118
0.13	179.19370	173.12131	0.033887296
0.14	181.99470	175.84644	0.033782632
0.15	184.69770	178.47625	0.033684502
0.16	187.31240	181.02014	0.033592330
0.17	189.84680	183.48606	0.033504594

Table 4.8: The solutions of *Case 4.6.3.2.2* for $t = 1$ & $\alpha = 2.5$

S	$\underline{w}$ (exact)	$\underline{w}$ (approx)	Rel. error
0.01	125.64342	120.21220	0.043227254
0.02	134.14750	128.50205	0.042083900
0.03	140.67510	134.86540	0.041298709
0.04	146.17980	140.23152	0.040691532
0.05	151.03060	144.96038	0.040191988
0.06	155.41710	149.23655	0.039767503
0.07	159.45170	153.16972	0.039397385
0.08	163.20770	156.83132	0.039069113
0.09	166.73610	160.27099	0.038774507
0.10	170.07380	163.52487	0.038506401
0.11	173.24890	166.62024	0.038260907
0.12	176.28310	169.57828	0.038034389
0.13	179.19370	172.41585	0.037824153
0.14	181.99470	175.14662	0.037627909
0.15	184.69770	177.78186	0.037444105
0.16	187.31240	180.33096	0.037271638
0.17	189.84680	182.80190	0.037108342

Table 4.4 shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0$, $\alpha = 1$ and $S \in [0, 5]$. Table 4.5 shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0.5$, $\alpha = 1$ and $S \in [0, 5]$. Table 4.6 shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0.5$, $\alpha = 0.5$ and $S \in [0, 5]$. Table 4.7 shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 0.5$, $\alpha = 1.5$ and $S \in [0, 5]$. Table 4.8 shows the exact solution in column 2, the approximate solution in column 3, and the corresponding relative absolute errors in column 4 for $t = 1$, $\alpha = 2.5$ and $S \in [0, 5]$.

Remark 4.6.2.3

In subsection 4.6.2, a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black-Scholes model assumption was considered. The considered case here is an extension of the nonlinear model in subsection 4.6.1. The approximate-analytical solutions obtained via the PDTM showed that the results in subsection 4.6.1 and those of González-Gaxiola *et al.* (2015) are special cases of this present extension.

4.7 Discussion and Summary of Results

Here, a concise discussion and summary of the research results in relation to the associated objectives are presented. For more clarification and illustration, tables and graphs are used at different stages.

4.7.1 The Transformed Black-Scholes Model

For the robustness of the proposed Transformed Black-Scholes Model (TBSM) in section 4.2.1, three (3) sets of obtained solutions viz: the exact solution, the approximate solution, and the proposed theoretical solution were compared. These solutions are presented in Table 4.9, where $\overline{M}_*^{DTM}(z)$ denotes approximate solution via DTM and $\overline{M}_*^{P}(z)$ denotes proposed theoretical solution. The absolute errors in both cases are respectively defined as:

$$\text{Abs Error} : \overline{M}_*^{DTM}(z) = \left|\text{Exact Solution} - \overline{M}_*^{DTM}(z)\right|$$

and

$$\text{Abs Error} : \overline{M}_*^{P}(z) = \left|\text{Exact Solution} - \overline{M}_*^{P}(z)\right|$$

Table 4.9: The solutions of the TBSM and their absolute errors

z	*Exact solution*	$\overline{M}_*^{DTM}(z)$	$\overline{M}_*^{P}(z)$	Abs Error : $\overline{M}_*^{DTM}(z)$	Abs Error : $\overline{M}_*^{P}(z)$
1.0	1.166054	1.164705	1.166054	0.00134876	1.51796E-09
1.1	1.299419	1.297003	1.299419	0.00241637	1.54226E-09
1.2	1.434336	1.430218	1.434336	0.00411742	1.54817E-09
1.3	1.569919	1.563193	1.569919	0.00672601	1.53075E-09
1.4	1.705042	1.694444	1.705042	0.01059818	1.48408E-09
1.5	1.838290	1.822103	1.838290	0.01618656	1.40113E-09
1.6	1.967908	1.943852	1.967908	0.02405591	1.27359E-09
1.7	2.091749	2.056850	2.091749	0.03489897	1.09159E-09
1.8	2.207197	2.157645	2.207197	0.04955222	8.43522E-10
1.9	2.311097	2.242086	2.311097	0.06901105	5.15711E-10
2.0	2.399660	2.305216	2.399660	0.09444344	9.20788E-11
2.1	2.468361	2.341160	2.468361	0.12720134	4.46218E-10
2.2	2.511821	2.342993	2.511821	0.16882864	1.12124E-09
2.3	2.523668	2.302604	2.523668	0.22106420	1.9588E-09
2.4	2.496381	2.210543	2.496381	0.28583858	2.98902E-09
2.5	2.421110	2.055848	2.421110	0.36526218	4.247E-09
2.6	2.287467	1.825864	2.287467	0.46160279	5.77366E-09
2.7	2.083291	1.506041	2.083291	0.57724954	7.61657E-09
2.8	1.794375	1.079715	1.794375	0.71466030	9.83099E-09

Note: Table 5.9 shows the relationship between these solutions as presented in columns 2, 3, and 4, respectively. Similarly, in the last two columns, the absolute errors between the solutions are displayed. It is obvious that the theoretical solution (in column 4) converges faster to the exact solution (in column 2). It is noticed that our results agree with those of Ugbebor *et al.* (2001) at $\phi_1 = \frac{1}{2}$; hence, their model is a particular type of this present model. Therefore, the proposed model is very robust and reliable.

4.7.2 The Generalised Black-Scholes Model

The generalised Black-Scholes Model via the CEV SDEs in section 4.3 for two cases (with and without dividend yield parameters) is discussed here.

4.7.2.1 Comparison of the SDE Models (BSM and CEV-BSM)

This subsection discusses by comparing the fundamental features of the associated models presented in section 4.3 as follows.

Let S_{N*} and S_* be the solutions of the SDEs in (4.39) and (2.2) , indicating no dividend and dividend yields respectively. Suppose further that V_0^{BSM} denotes the volatility of the Black-Scholes model, V_0^{CEVM} is the volatility of the CEV model without dividend yield, V_{0*}^{CEVM} is the volatility of the CEV model with dividend yield, V_{BSM}^{ar} the variance of the Black-Scholes model, V_{CEVM}^{ar} the variance of the CEV model without dividend yield, V_{CEVM}^{ar*} the variance of the CEV model with dividend yield, $\bar{m}_{CEVM}^{N*}$ the drift term of the CEV model without dividend yield, and $\bar{m}_{CEVM}^{*}$ the drift term of the CEV model with dividend yield. Then, the following are deduced easily.

For volatilities:

$$V_0^{BSM} = \sigma, \;\; V_0^{CEVM} = \sigma S^{\frac{\xi}{2}-1}, \;\; \text{and} \;\; V_{0*}^{CEVM} = \sigma_* S_*^{\frac{\xi}{2}-1}.$$

For the variances:

$$V_{BSM}^{ar} = \sigma^2, \; V_{CEVM}^{ar} = \sigma^2 S^{\xi-2}, \;\; \text{and} \;\; V_{CEVM}^{ar*} = \sigma_*^2 S_*^{\xi-2}.$$

For the mean parameters:

$$\bar{m}_{CEVM}^{N*} = \mu, \quad \text{and} \quad \bar{m}_{CEVM}^{*} = \mu - q.$$

Note: In both cases of the governing SDEs; with or without dividend parameter, it is clear that the variances $V_{CEVM}^{ar} = g\left(\sigma, S_*\right)$ and $V_{CEVM}^{ar*} = h\left(\sigma, S_{N*}\right)$ are functions of the underlying asset prices S_{N*} and S_* respectively. This implies that the stock price volatilities in both cases are variable functions. This result is of immense contribution in the generalisation of the classical Black-Scholes pricing model as clearly depicted in section 4.3.

4.7.3 Cases of the Generalised Bakstein and Howison Model - Section 4.6.1

The aspect of the result discussion here is on the solution of the generalised Bakstein and Howison Model. To do this, reference is made to the numerical illustration presented in *Case 4.6.1.2.2* in subsection 4.6.1.1. Thus, the approximate and the exact solutions are displayed in Figure 4.15 and Figure 4.16 respectively.

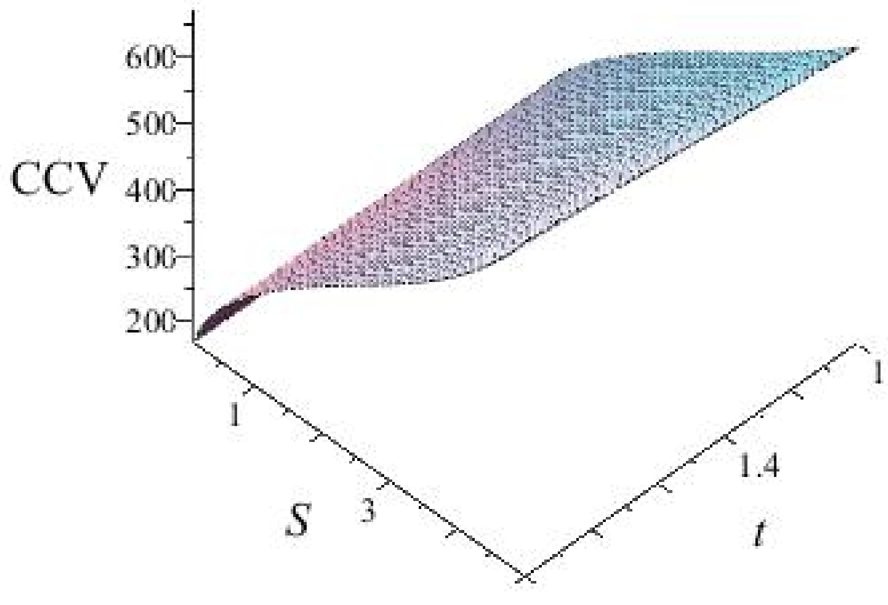

Figure 4.15: Exact solution for *Case 4.6.1.2.2*

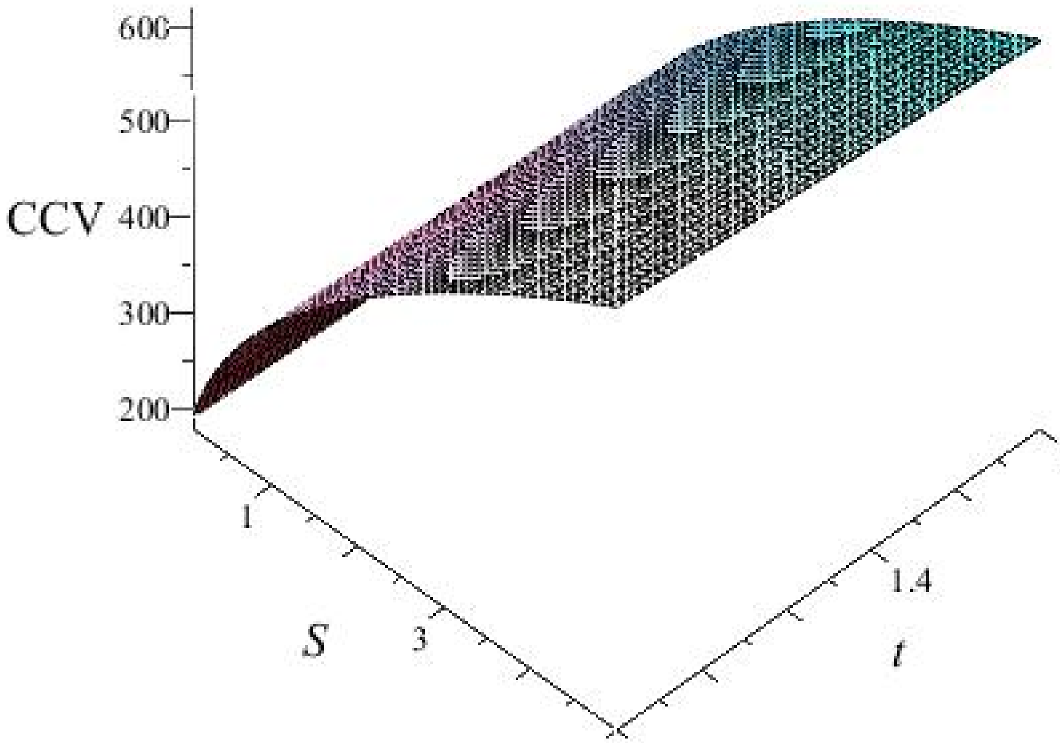

Figure 4.16: Approximate solution for *Case 4.6.1.2.2*

The results are obtained with less computational time while still maintaining high level of accuracy. In addition, these results conform to the associated exact solution obtained by Esekon (2013) and the approximate solutions obtained by González-Gaxiola *et al.* (2015) using the Adomian decomposition method.

4.7.4 The Generalised Bakstein and Howison Model: Time-fractional case - Section 4.6.2

This subsection discusses the solution of the time-fractional generalised Bakstein and Howison Model. To do this, reference is made to the numerical illustration in subsection 4.6.2.1. Figure 4.17 and Figure 4.18 are for *Case 4.6.3.2.1* for an integer power of the time parameter. In a similar way, Figures 5.5-5.7 are for *Case 4.6.3.2.2* for fractional powers of the time parameter. Also, in comparison, the exact and the approximate solutions for different values of t and α are presented.

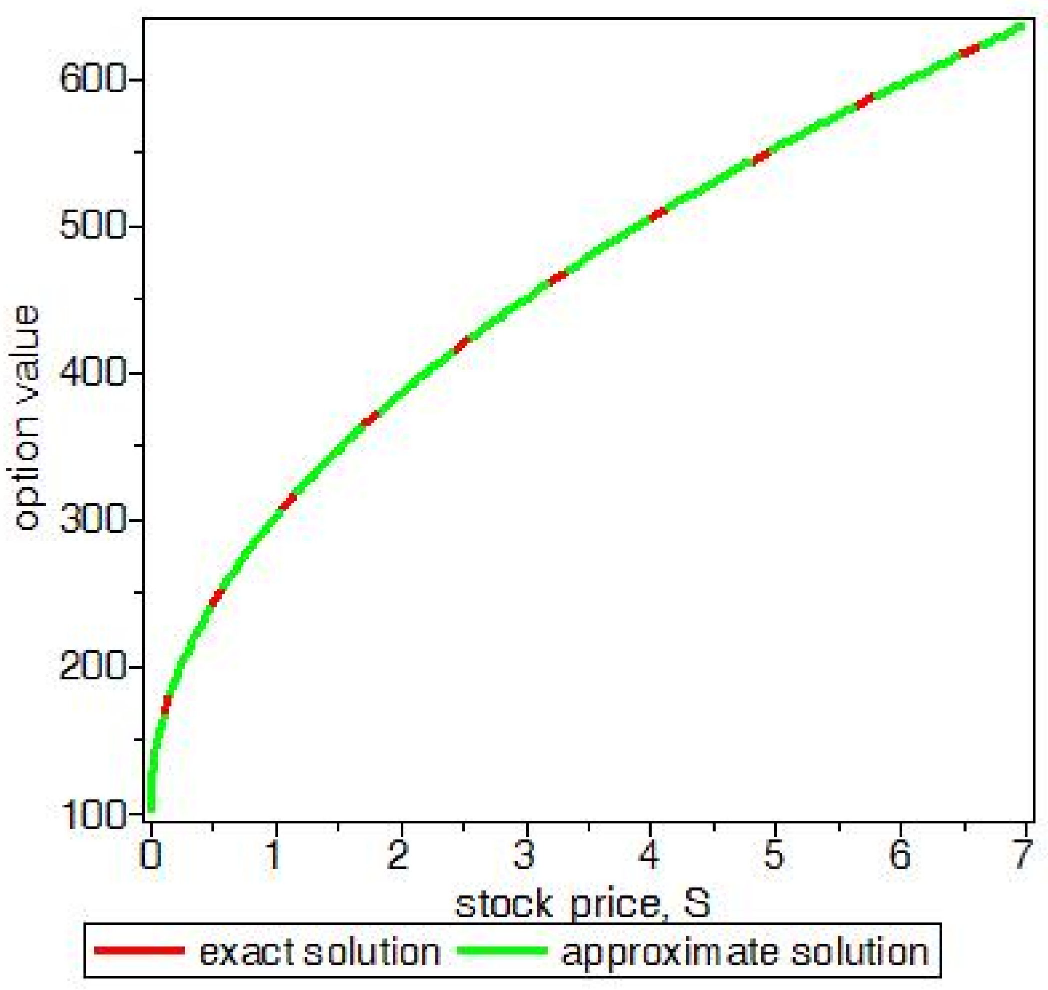

Figure 4.17: Solutions of *Case 4.6.3.2.1* using Table 4.4

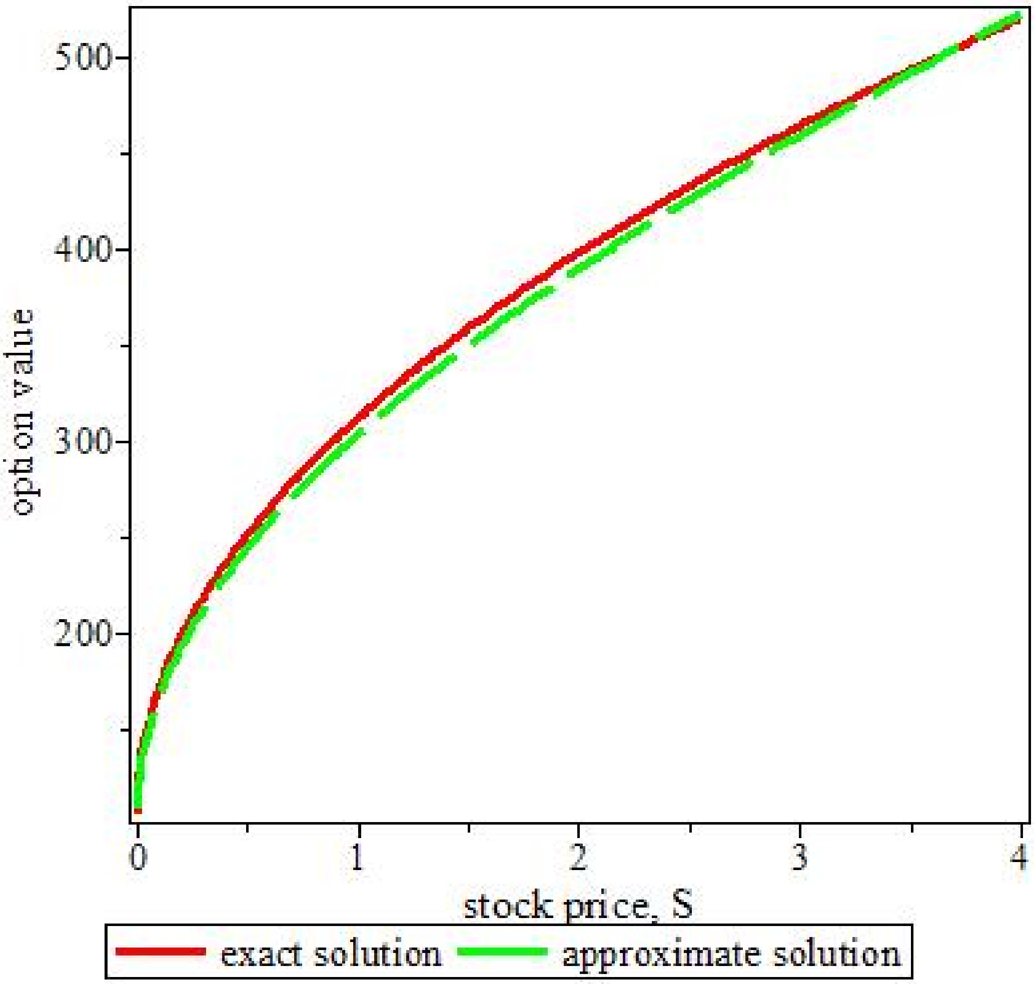

Figure 4.18: Solutions of *Case 4.6.3.2.1* using Table 4.5

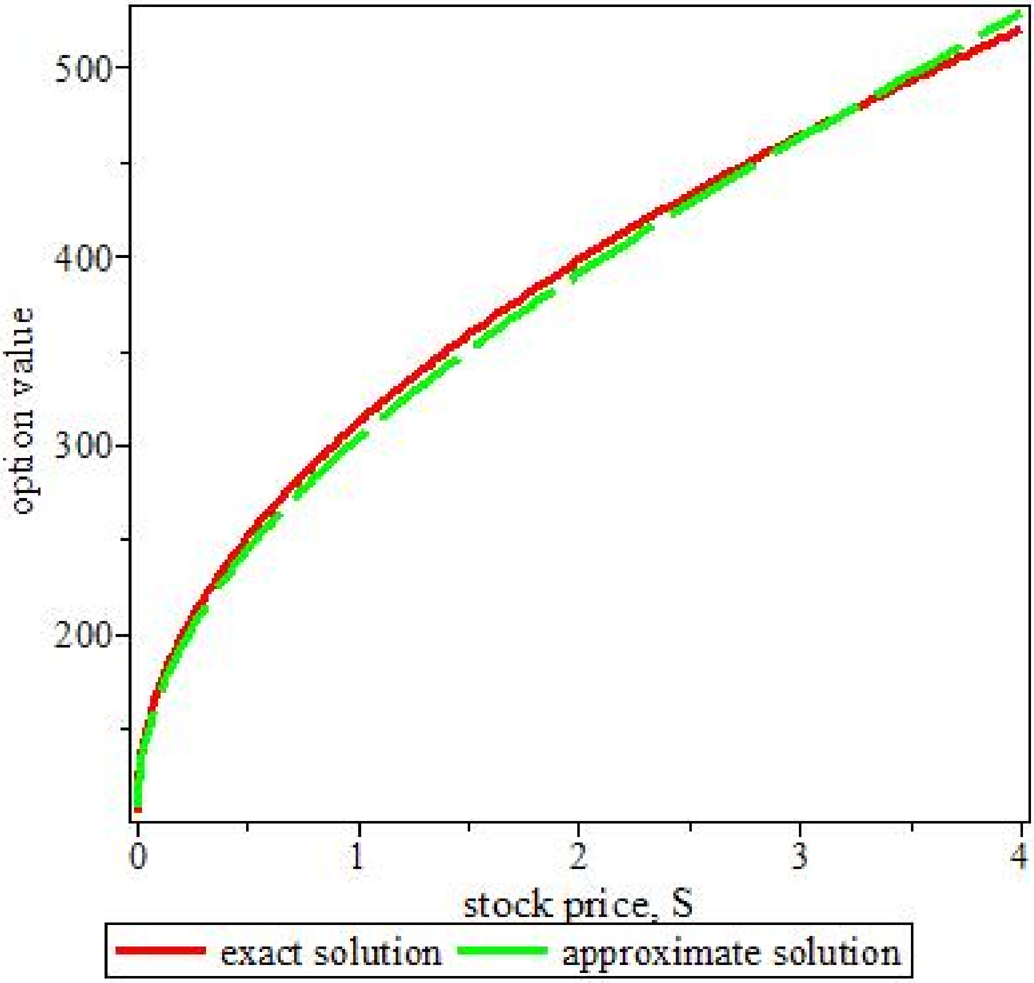

Figure 4.19: Solutions of *Case 4.6.3.2.2* using Table 4.6

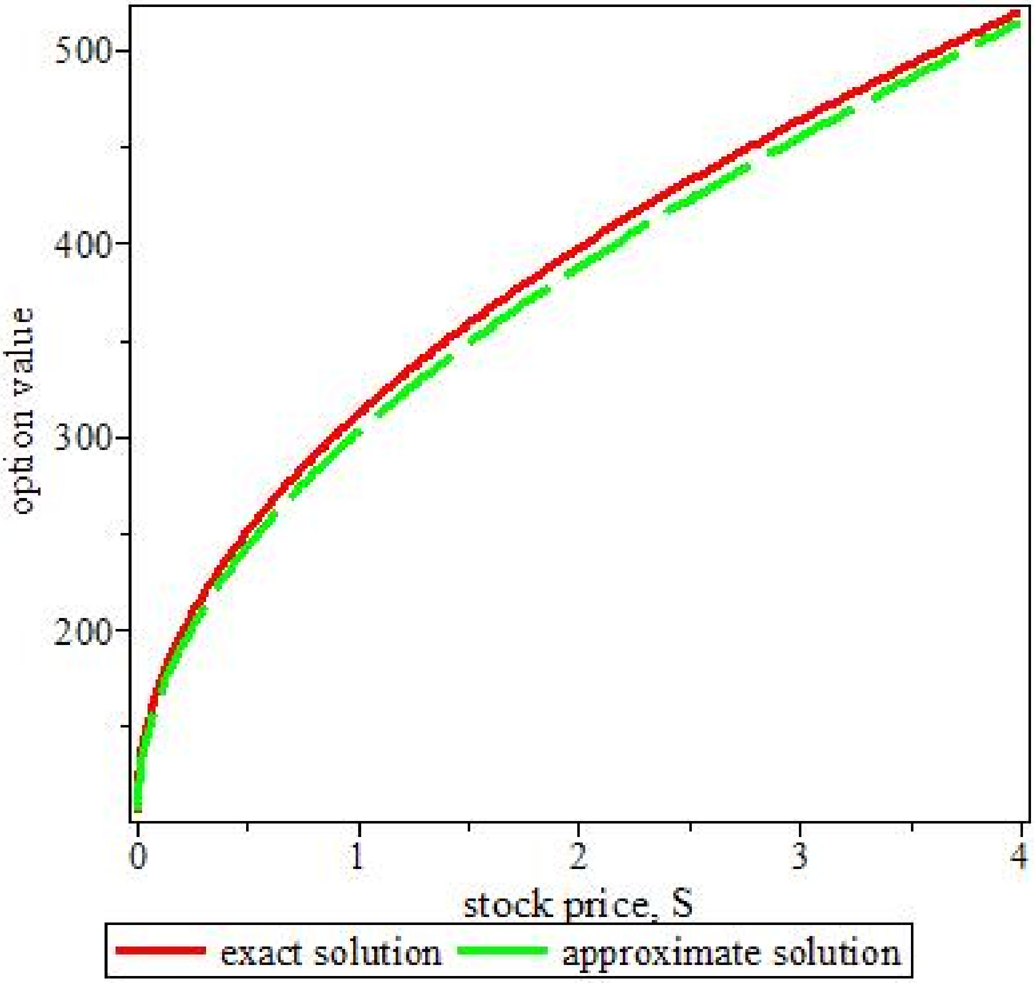

Figure 4.20: Solutions of *Case 4.6.3.2.2* using Table 4.7

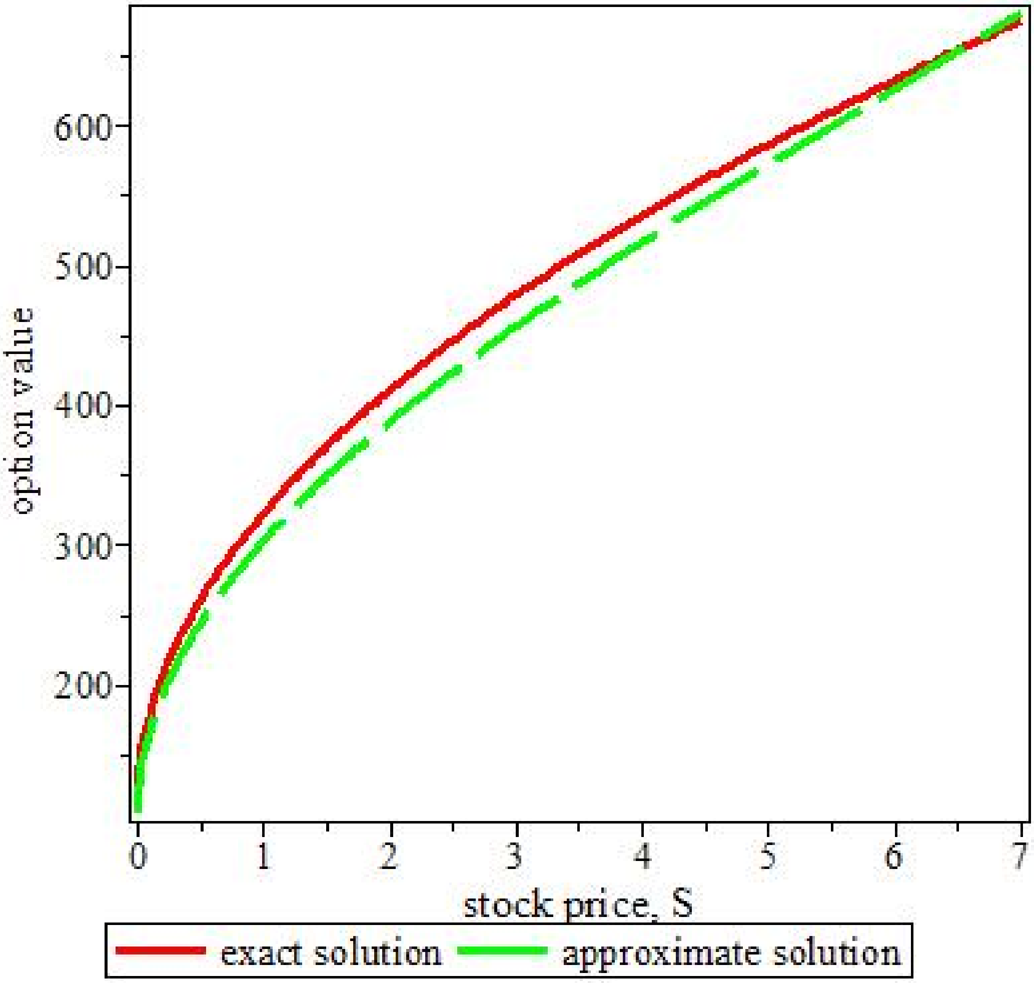

Figure 4.21: Solutions of *Case 4.6.3.2.2* using Table 4.8

In Figure 4.17, the exact and the approximate solutions of the solved problem in *Case 4.6.3.2.1* for $t = 0$ and $\alpha = 1$ are presented. This is with respect to Table 4.4. In Figure 4.18, the exact and the approximate solutions of the solved problem in *Case 4.6.3.2.1* for $t = 0.5$ and $\alpha = 1$ are presented. This is with respect to Table 4.5. In Figure 4.19, the exact and the approximate solutions of the solved problem in *Case 4.6.3.2.2* for $t = 0.5$ and $\alpha = 0.5$ are presented. This is with respect to Table 4.6. In Figure 4.20, the exact and the approximate solutions of the solved problem in *Case 4.6.3.2.2* for $t = 0.5$ and $\alpha = 1.5$ are presented. This is with respect to Table 4.7. In Figure 4.21, the exact and the approximate solutions of the solved problem in *Case 4.6.3.2.2* for $t = 1$ and $\alpha = 2.5$ are presented. This is with respect to Table 4.5.

Remark 4.7.1

In section 4.7.4, the study considered a time-fractional nonlinear transaction-cost model for stock option valuation in an illiquid market setting driven by a relaxed Black-Scholes model assumption. The case considered in this subsection is an extension of the nonlinear model in subsection 4.5.1. The approximate-analytical solutions obtained via the PDTM showed that the results in subsection 4.5.1 and those of González-Gaxiola *et al.* (2015) are particular cases of this present extension. It is also note that the fractional time parameter α, serves as a control parameter since the diffusion of the stock option price relies on the history of the price.

CHAPTER FIVE

CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, the general summary and conclusion of the research work are presented. Recommendations and further areas of research are also made and suggested respectively.

5.2 Conclusion

In this thesis, some established conditions for transforming the classical Black-Scholes model for stock option valuation from its PDE form to an equivalent ODE form have been considered. Consequently, a proposition regarding the theoretical value of stock options was given, and a relatively new approximate-analytical technique was proposed for the solution of the Transformed Black-Scholes Model (TBSM). Test cases showed that obtained results agreed with the exact solution.

The study proposed CEV-Black-Scholes model for stock option valuation. The model was derived in two forms as a generalisation of the classical Black-Scholes pricing model using the CEV stochastic dynamics. In the first form, the concentration was on the case without a dividend yield parameter; thereafter, the case with a dividend yield parameter was incorporated. These models served as alternatives to the traditional lognormal model for stock prices. In CEV model, the price variations of the underlying asset are negatively correlated with variations in the level of volatility; this helps in reducing the known volatility smile effects of the lognormal model. The key merit of these models is that the stock price volatility is a function of the underlying asset price but not a constant as assumed in the classical Black-Scholes pricing model.

In considering the European Option, the computational method: PDTM was successfully applied to the Black-Scholes model for European Option Valuation (to the best knowledge of the researcher, such application has not been considered in literature). The obtained solutions of the solved problems were expressed in simple forms of convergent series as no linearization or perturbation is needed. These computed results denote the analytical values of the associated European call options. The same technique can be applied to European put options. In addition, the study provided an accurate and exact solutions to the classical Black-Scholes model for stock option valuation by the means of He's polynomials. This technique gives the exact solution of the solved problem in a very simple and quick manner even with less computational work while still maintaining high level of accuracy.

With regard to the time-fractional Black-Scholes model for stock option valuation (an extension of the classical Black-Scholes equations in time-fractional form), analytical solutions were obtained via the proposed relatively novel computational Method (PDTM). The results obtained converged faster to their exact form of solutions (even without giving up accuracy). It was remarked, to the best of the researcher's knowledge, that this proposed technique has not been reported in literature for solving time-fractional Black-Scholes equations. This therefore, showed that the result in subsection 4.4.1 is a special case of this present work for $\alpha = 1$. Thus, the time-fractional Black-Scholes equation for stock option valuation is a generalisation of the classical Black-Scholes equations for European option valuation at order, $\alpha = 1$.

In an extensive manner, the Bakstein and Howison model: a nonlinear transaction-cost model for stock prices in an illiquid market, obtained when the constant volatility assumption of the classical linear Black-Scholes option pricing model was relaxed through the inclusion of transaction cost was studied. Thereafter, the study generalised and extended this nonlinear option pricing model to a time-fractional ordered form, and obtained approximate-analytical solutions for both forms. Two cases with five examples: case 1 with two examples for time-integer order, and case 2 with three examples for time-fractional order were considered. The obtained results conformed with the associated exact solutions obtained by Esekon (2013), and showed that the work of González-Gaxiola *et al.* (2015) using the Adomian decomposition method is a particular case of this present work when $\alpha = 1$.

5.3 Contributions to Knowledge

The following contributions are made to the body of knowledge:

(i) a stock option valuation model where the drift coefficient (rate of return) is a non-fixed constant parameter has been derived, and a approximate-analytical solution is provided;

(ii) the classical Black-Scholes option pricing model is generalised via a constant elasticity of variance stochastic dynamics thereby addressing the assumption of the constant volatility in the Black-Scholes Option Pricing Model;

(iii) analytical solutions of the Black–Scholes pricing model for European option valuation are obtained via the proposed approximate-analytical methods;

(iv) the nonlinear transaction cost model of Bakstein and Howison (2003) for stocks valuation is generalised, and analytical solution is provided via a proposed approximate-analytical method;

(v) the generalised Bakstein and Howison model (2003) for stocks valuation has been extended to a time-fractional order form; and

(vi) an analytical solution of the extended generalised model in (v) is obtained using the MDTM.

5.4 Recommendations

'Timeshare' as a term in a business setting refers to an arrangement where joint owners possess the right to a property on the ground of time-sharing agreement among them. Meanwhile, in fractional ownership style of trade, business owners are entitled to purchase a fixed time period share (mainly on monthly basis). It is therefore recommended that:

(i) the time-fractional model be adopted in fractional ownership style of stock option valuation for effective and efficient time management;

(ii) the nonlinear model be adopted in an Employer-Employee stock option system since dividend yield parameter is incorporated; and

(iii) the constant elasticity parameter in the nonlinear model enlarges the scope of operation of the classical linear Black-Scholes model.

5.5 Open Problems

Basically, this study proposed stock option valuation model via the application of the constant elasticity of variance dynamics with or without dividend yield. This model (CEV-BSM) acts as a geralisation and extension of the classical Black-Scholes option valuation model. Hence, for further research,

(a) the CEV-BSM could be extended to fractional order both in time and space as a generalisation, and the approximate-analytical solutions of the resulting

PDEs for option valuation be obtained in a similar way using the proposed approximate-analytical methods in this work.

(b) In addition, the generalised Bakstein and Howison nonlinear stock option model was only with regard to time-fractional order; this could be considered with respect to space-fractional order.

REFERENCES

Acharya, V. and Pedersen, L. H. (2005). Asset pricing with liquidity risk. *Journal of Financial Economics,* **77 (2)**: 375-410.

Adebiyi, A. A., Adewumi, A. O. and Ayo, C. K. (2014). Comparison of ARIMA and artificial neural network models for stock price prediction. *Journal of Applied Mathematics,* **2014**: 1-8. Article ID 614342.

Agliardi, R., Popivanov, P. and Slovova, A. (2013). On nonlinear Black-Scholes equations. *Nonlinear Analysis and Differential Equations.* **2013 (1)**: 75-81.

Ahmad, J., Shakeel, M., UI-Hassan, Q. M. and Mohyud-Din, S. T. (2013). Analytical solution of Black-Scholes model using fractional variational iteration method. *International Journal of Modern Mathematical Sciences,* **5 (3)**: 133-142.

Ahmed, S. (2014). On the generalized fractional integrals of the generalized Mittag-Leffler function. *Springerplus,* **3**: 198-202. DOI:10.1186/2193-1801-3-198.

Akrami, M. H. and Erjaee, G. H. (2016). Numerical solutions for fractional Black-Scholes option pricing equation, *Global Analysis and Discrete Mathematics* **1 (1)**: 9–14.

Akyildirim, E. and Soner, H. M. (2014). A brief history of mathematics in finance. *Borsa Istanbul Review* **14 (1)**: 57–63.

Allahviranloo, T. and Behzadi, Sh. S. (2013). The use of iterative methods for solving Black-Scholes equations. *International Journal of Industrial Mathematics*, **5 (1)**: 1-11.

Amihud, Y. and Mendelson, H. (1986). Asset pricing and the bid-ask spread. *Journal of Financial Economics,* **17**: 223-249.

Arikoglu, A. and Ozkol, I. (2006). Solution of differential-difference equations by using differential transform method. *Applied Mathematical Computation,* **181 (1)**: 153-162.

Arikoglu, A. and Ozkol, I. (2005). Solution of boundary value problems for integro-differential equations by using differential transform method. *Applied Mathematical Computation,* **168**: 1145-1158.

Ayaz, F. (2003). On the two-dimensional differential transform method. *Applied Mathematical Computation,* **143**: 361-374.

Bachelier, L. (1900). Théorie de la speculation, *Annales Scientifiques de L'École Normale Supérieure,* **17**, 21-86. English translation by Boness A. J. (1964). In Cootner, P. H. (ed.): The random character of stock market prices. *Cambridge, MA: MIT Press,* 17-75.

Bakstein, D. and Howison, S. (2003). A non-arbitrage liquidity model with observable parameters for derivatives. *Mathematical Institute, Oxford University.* Available at: http://eprints.maths.ox.ac.uk/53/.

Beckners, S. (1980). The constant elasticity of variance model and its implications for option pricing. *The Journal of Finance*, **35 (3)**: 661-673.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, **81 (3)**: 637-654.

Bohner, M., Sanchez, F. H. M. and Rodriguez, S. (2014). European call option pricing using the Adomian decomposition method. *Advances in Dynamical Systems and Applications*, **9 (1)**: 75-85.

Campbell, J. Y. and Shiller, R. J. (1988). Stock prices, earnings, and expected dividends. *Journal of Finance,* **43**: 661-676.

Carr, P., Geman, H., Madan, D.B. and Yor, M. (2002). The fine structure of asset returns: an empirical investigation. *The Journal of Business,* **75 (2)**: 305-332.

Cen, Z. and Le, A. (2011). A robust and accurate finite difference method for a generalized Black-Scholes equation. *Journal of Computational and Applied Mathematics*, **235 (13)**: 3728-3733.

Chen, C. K. and Ju, S. P. (2004). Application of differential transformation to transient advective-dispersive transport equation. *Applied Mathematical Computation,* **155 (1)**: 25-38.

Chen, W. and Wang, S. (2014). A penalty method for a fractional order parabolic variational inequality governing American put option valuation. *Computers and Mathematics with Applications,* **67 (1)**: 77–90.

Chernov, M., Gallant, A. R., Ghysels, E. and George Tauchen, G. (2003). Alternative models for stock price dynamics. *Journal of Econometrics*, **116 (1-2)**: 225-257.

Cox, J. and Ross, S. (1976). The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, **3 (1-2)**: 145-166.

Delbaen, F. and Shirakawa, H. (2002). A note of option pricing for constant elasticity of variance model. *Asia-Pacific Financial Markets*, **9 (2)**: 85-99.

Dura, D. and Mosneagu, A. M. (2010). Numerical approximation of Black-Scholes equation. Analele Stiitifice Ale Universitatii "Al.i Cuza" Din Iasa (S.N.) *Matematica, Tomul* **56**: 39-64.

Duncan, T. E., Hu, Y. and Pasik-Duncan, B. (2000). Stochastic calculus for fractional Brownian motion. *SIAM Journal* on *Control* and *Optimization,***38**: 582-612.

Elbeleze, A. A., Kilicman, A. and Taib, B. M. (2013). Homotopy perturbation method for fractional Black-Bcholes European option pricing equations using Sumudu transform. *Mathematical Problems in Engineering,* **(2013)**: 1-7.

Esekon, J. E. (2013). Analytic solution of a nonlinear Black-Scholes equation. *International Journal of Pure and Applied Mathematics*, **82 (4)**: 547-555.

Fama, E. F. (1965). The behaviour of stock-market prices. *Journal of Business*, **38 (1)**: 34-105.

Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *Journal of Finance,* **25 (2)**: 383–417.

Fornari, F. and Mele, A. (1997). Sign-and volatility-switching arch models: theory and applications to international stock markets. *Journal of Applied Econometrics,* **12**: 49-65.

Frey, R. and Patie, P. (2002). Risk management for derivatives in illiquid markets: A simulation study. *Advances in Finance and Stochastics*, Springer, Berlin, 137-159. ISBN 9783540434641.

Frey, R. and Stremme, A. (1997). Market volatility and feedback effects from dynamic hedging. *Mathematical Finance,* **4**: 351-374.

Ghandehari, M. A. M. and Ranjbar, M. (2014). European option pricing of fractional version of the Black-Scholes Model: Approach via expansion in series. *International Journal of Nonlinear Science,* **17 (2)**: 105-110.

Ghorbani, A. (2009). Beyond Adomian's polynomials: He polynomials. *Chaos, Solitons and Fractals*, **39 (3)**: 1486–1492.

Ghorbani, A. and Nadjfi, J. S. (2007). He's homotopy perturbation method for calculating Adomian's polynomials. *International Journal of Nonlinear Sciences and Numerical Simulation*, **8 (2)**: 229-332.

González-Gaxiola, O., Ruíz de Chávez, J. and Santiago, J. A. (2015). A nonlinear option pricing model through the Adomian decomposition method. *International Journal of Applied and Computational Mathematics*, **1 (2)**: 1-15.

Habib, A. (2011). The calculus of finance. Universities Press (India) Private Ltd. ISBN 8173717230.

Hanna, M. (1976). A stock predictive model based on changes in ratios of short interest to trading volume. *The Journal of Financial and Quantitative Analysis,* **11 (5)**: 857-872.

Hariharan, G., Padma, S., and Pirabaharan, P. (2013). An efficient wavelet based approximation method to time fractional Black-Scholes European option pricing problem arising in financial market. *Applied Mathematical Sciences*, **7 (69)**: 3445-3456.

He, J. H. (2003). Homotopy perturbation method: A new nonlinear analytical technique. *Applied Mathematics and Computation*, **135 (1)**: 73-79.

Henderson, D. and Plaschko, P. (2006). Stochastic differential equations in science and engineering. World Scientific Publishing Co. Pte. Ltd, London. ISBN: 978-981-4480-53-6.

Hsu, Y. L. Lin, T. I. and Lee, C. F. (2008). Constant elasticity of variance (CEV) option pricing model: Integration and detailed derivation. *Mathematics and Computers in Simulation,* **79 (1)**: 60–71.

Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatility. *The Journal of Finance*, **42 (2)**: 271-301.

Ibrahim, R. W. and Jalab, H. A. (2015). Existence of Ulam stability for iterative fractional differential equations based on fractional entropy. *Entropy,* **17**: 3172-3181.

Itô, K. (1946). On a stochastic integral equation. *Proceedings of the Japan Academy,* **22**: 32-35.

Itô, K. (1951). On stochastic differential equations. *Memoirs American Mathematical Society*, **4**: 1-51. Doi.org/10.1090/memo/0004.

Jang, B. (2010). Solving linear and nonlinear initial value problems by the projected differential transform method. *Computer Physics Communications*, **181 (5)**: 848-854.

Jumarie, G. (2008). Stock exchange fractional dynamics defined as fractional exponential growth driven by (usual) Gaussian white noise: Application to fractional Black-Scholes equations, *Insurance: Mathematics and Economics*, **42 (1)**: 271–287.

Keynes, J. M. (1971). A treatise on money: The pure theory of money. Johnson, E. and Moggridge, D. (eds.) Macmillan, London, ISBN: 978-0-404-15000-6.

Kilbas, A. A., Srivastava, H. M. and Trujilo, J. J. (2006). Theory and applications of fractional differential equations. 1st Edn. North-Holland Mathematical Studies, Elsevier, Amsterdam. ISBN: 9780080462073.

Kobila, T. O. (1993). A class of solvable stochastic investment problems involving singular controls. *Stochastics and Stochastics Reports,* **43:** 29-63.

Kumar, S., Yildirin, A., Khan, Y., Jafari, H., Sayevand, K. and Wei, L. (2012). Analytical solution of fractional Black-Scholes European option pricing equations using Laplace transform. *Journal of Fractional Calculus and Applications,* **2 (8)**: 1-9.

Kumar, S., Kumar, D. and Singh, J. (2014). Numerical computation of fractional Black-Scholes equation arising in financial market. *Egyptian Journal of Basic and Applied Sciences*, **1**: 177-183.

Lin, X., Yang, Z. and Song, Y. (2009). Short-term stock price prediction based on echo state networks. *Expert Systems with Applications,* **36**, 7313–7317.

Merdan, M. (2013). Numerical solution of the fractional-order Vallis systems using multi-step differential transformation method. *Applied Mathematical Modelling,* **37 (8)**: 6025-6036.

Merton, R. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science*, **4:** 141–183.

Merton, R. C., Brennan, M. J. and Schwartz, E. S. (1977). The valuation of American put options. *Journal of Finances,* **32 (2)**: 449-462.

Mohyud-Din, S. T. (2011). Solution of troesch's problem using He's polynomials.

Revista de la Unión Matemática Argentina, **52 (1)**: 143-148.

Momani, S., Odibat, Z., and Eturk, V. S. (2007). Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation. *Physics Letter A.*, **370 (5-6)**: 379-387.

Nazari, D. and S. Shahmorad (2010). Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. *Journal* of Computational and *Applied Mathematics,* **234 (3)**: 883-891.

Necula, C. (2008). Option pricing in a fractional Brownian motion environment. *Finance and Banking, Bucharest University of Economics*, **94**: 109-126.

Nelson, S. A. (1904). The A B C of options and arbitrage. The Wall Street Library, **VI**, New York, 96p.

NGSE-Nigerian Stock Exchange, (2016). https://www.african-markets.com/en/stock-markets/ngse/listed-companies. Retrived: May 31, 2016.

NGSEINDX:IND-Nigerian Stock Exchange All Share Index, (2016) www.nse.com.ng/products/indices. Retrived: May 31, 2016.

Nkeki, C. I. (2011). On optimal portfolio management of the accumulated phase of a defined contributory pension scheme. Unpublished Ph.D Thesis, Department of Mathematics, University of Ibadan, Nigeria.

Onah, S. E. and Ugbebor, O. O. (1999). Solution of a two-dimensional stochastic investment problem. *Applied Mathematics and Computation*, **98:** 75-82.

Osborne, M. F. M. (1964). Brownian motion in the stock market. In: *The random character of stock market prices*, P. H. Cootner (Editor). *Cambridge, MA: MIT Press*, 100-128.

Owoloko, E. A. (2014). On the valuation of banks as growth enterprises. Unpublished Ph.D Thesis, Department of Mathematics, Covenant University, Ota, Nigeria.

Owoloko, E. A. and Okeke, M. C. (2014). Investigating the imperfection of the B-S Model: A case study of an emerging stock market. *British Journal of Applied Science and Technology,* **4 (29)**: 4191-4200.

Peters, E. E. (1989). Fractal structure in the capital markets. *Financial Analysts Journal*, **45 (4)**: 32-37.

Phaochoo, P., Luadsong, A. and Aschariyaphotha, N. (2016). A numerical study of the European option by the MLPG method with moving kriging interpolation. *SpringerPlus*, **5** : 305-318.

Podlubny, I. (1999). Fractional differential equations. *Mathematics in Science and Engineering*, **198**, 1st Edn. ISBN: 9780080531984.

Qiu, Y. and Lorenz, J. (2009). A nonlinear Black-Scholes equation. *International Journal of Business Performance and Supply Chain Modelling,* **1 (1)**: 33-40.

Rapach, D. E. and Wohar, M. E. (2005). Valuation ratios and long-horizon stock price predictability. *Journal of Applied Economics,* **20**: 327-344.

Rashidi, M. M. (2009). The modified differential transform method for solving MHD boundary-layer equations. *Computer Physics Communications*, **180 (11)**: 2210-2217.

Ravi, K. A. S. V. and Aruna, K. (2008). Solution of singular two-point boundary value problems using differential transform method. *Physics Letter A,* **372 (26)**: 4671-4673.

Ravi, K. A. S. V. and Aruna, K. (2012). Comparison of two dimensional DTM and PDTM for solving Time- Dependent Emden-Fowler Type Equations. *International Journal of Nonlinear Science*, **13 (2)**: 228-239.

Reiss, W. (1975). A note on stock market prices in Germany and the random walk hypothesis. *Kyklos,* **28 (4)**: 878-883.

Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics,* **51 (3)**: 239-246.

Schoneburg, E. (1990). Stock price prediction using neural networks: A project report. *Neurocomputing,* **2**: 17-27.

Sen, A. K. (1988). An application of the Adomian decomposition method to the transcient behaviour of a model of biochemical reaction. *Journal of Mathematical Analysis and Applications,* **131**: 232-245.

Shane, P. B. and Stock, T. (2006). Security analyst and stock market efficiency in anticipating tax-motivated income shifting. *The Accounting Review*, **81 (1)**: 227-250.

Shepp, L. (2002). A model for stock price fluctuations based on information. *IEEE Transactions on Information Theory,* **48 (6)**: 1372-1378.

Song, J. Yin, F., Cao. X. and Lu, F. (2013). Fractional variational iteration method versus Adomian's decomposition method in some fractional partial differential equations. *Journal of Applied Mathematics*, **(2013)**: 10 pages . Article ID: 392567.

Sprenkle, C. (1964). Warrant prices as indicators of expectations and preferences in the random character of stock market prices, Cootner, P. (ed.) MIT Press, Cambridge MA, 412-474.

Tari, A. and Shahmorad, S. (2011). Differential transform method for the system of two-dimensional nonlinear Volterra integro-differential equations. *Computers and Mathematics with Applications*, **61 (9)**: 2621-2629.

Uddin, M. K. S., Ahmed, M. and Bhowmik, S.K. (2013). A note on numerical solution of linear Black-Scholes model. *GANIT: Journal of Bangladesh Mathematical Society*, **33:** 103-115.

Ugbebor, O. O., Onah, E. S. and Ojowu, O. (2001). An empirical stochastic model of stock price changes. *Journal of the Nigeria Mathematical Society,* **20**: 95-101.

Wang, X.-T. (2010). Scaling and long-range dependence in option pricing I: Pricing European option with transaction costs under the fractional Black-Scholes model. *Physica A,* **389**: 438-444.

Wu, J-L. and Hu, Y-H, (2011). Price–dividend ratios and stock price predictability. *Journal of forecasting,* (wileyonlinelibrary.com).

Zhou, J. K. (1986). Differential transformation and its application for electrical circuits, *Huazhong University Press,* Wuhan, China.

APPENDIX A: MAPLE CODE FOR FIGURES (A1-A5)

A1: MAPLE CODE FOR FIGURES 4.1 and 4.2

FIGURE 4.1

```
Psoln=z+(3/20)*z∧2+(19/600)*z∧3-(71/8000)*z∧4-(5849/1200000)*z∧5-
(32107/12000000)*z∧6, z = 1 .. 2.5

dtm := z+(3/20)*z∧2+(19/600)*z∧3-(71/8000)*z∧4-(5849/1200000)*z∧5-
(32107/12000000)*z∧6

exact := exp((1/10*(4+sqrt(61)))*z)*(-(1/61)*sqrt(61)-1)+exp(-(1/10*(-
4+sqrt(61)))*z)*(-1+(1/61)*sqrt(61))+2*exp(z)

plot([-1.128036879*exp(1.181024968*z)-.8719631201*exp(-.3810249676*z)+2*exp(z),
z+(3/20)*z∧2+(19/600)*z∧3-(71/8000)*z∧4-(5849/1200000)*z∧5-(32107/12000000)*z∧6,
exp((1/10*(4+sqrt(61)))*z)*(-(1/61)*sqrt(61)-1)+exp(-(1/10*(-4+sqrt(61)))*z)*(-
1+(1/61)*sqrt(61))+2*exp(z)], z = 1 .. 2.4, labels = ["independent pa-
rameter: z", "solution: M(z)"], labeldirections = ["horizontal", "verti-
cal"], labelfont = ["ROMAN", 12], linestyle = [solid, longdash, dashdot],
titlefont = ["ROMAN", 12], thickness = 3, legend = ["proposed solution",
"DTM solution", "exact solution"])
```

FIGURE 4.2

```
S := CEVProcess(s0, mu, sigma, xi);

Diffusion(S(t));

simplify(Drift(S(t)∧(2-xi)));
```

simplify(Diffusion(S(t)∧(2-xi)));

S := SamplePath(S(t), t = 0 .. T, timesteps = 100, replications = 10∧4);

The following set of examples estimates the distribution of

max(0, _S(2)-1);

for different values of the elasticity parameter ExpectedValue(max(S(T)-1, 0), timesteps = 100, replications = 10∧4);

xi := 2.0;

S1 := SampleValues(S(T), timesteps = 100, replications = 10∧4);

xi := 2.5;

S2 := SampleValues(S(T), timesteps = 100, replications = 10∧4);

xi := .1;

S3 := SampleValues(S(T), timesteps = 100, replications = 10∧4);

P1 := Statistics[FrequencyPlot](S1, range = 0 .. 3, bincount = 10, averageshifted = 5,

thickness = 3, color = red);

P2 := Statistics[FrequencyPlot](S2, range = 0 .. 3, bincount = 10, averageshifted = 5,

thickness = 3, color = black);

P3 := Statistics[FrequencyPlot](S3, range = 0 .. 3, bincount = 10, averageshifted = 5,

thickness = 3, color = green);

plots[display](P1,P2,P3,gridlines=true,tickmarks=[,5]);

A2: MAPLE CODE FOR FIGURES 4.3 & 4.4

FIGURE 4.3

V(x,t)=

-1.412646800*t+x-4.742163550*10∧(-63)*t∧28-2.213009657*10∧(-60)*t∧27-9.958543459*10∧(-58)*t∧26-8.064113665*10∧(-102)*t∧42-5.644879565*10∧(-99)*t∧41-3.857334369*10∧(-96)*t∧40-2.571556247*10∧(-93)*t∧39-1.671511560*10∧(-90)*t∧38-1.058623988*10∧(-87)*t∧37-6.528181261*10∧(-85)*t∧36-3.916908757*10∧(-82)*t∧35-2.284863441*10∧(-79)*t∧34-1.294755950*10∧(-76)*t∧33-7.121157724*10∧(-74)*t∧32-3.797950787*10∧(-71)*t∧31-9.811372865*10∧(-66)*t∧29-1.962274573*10∧(-68)*t∧30-2.045863932*10∧(-110)*t∧45-1.534397949*10∧(-107)*t∧44+

-1.125225163*10∧(-104)*t∧43-1.798070347*10∧(-52)*t∧24-4.315368832*10∧(-55)*t∧25-7.192281387*10∧(-50)*t∧23-1.010915106*10∧(-44)*t∧21-2.757041199*10∧(-47)*t∧22-1.179400957*10∧(-39)*t∧19-3.538202871*10∧(-42)*t∧20-1.120430909*10∧(-34)*t∧17-3.734769697*10∧(-37)*t∧18-8.465477980*10∧(-30)*t∧15-3.174554241*10∧(-32)*t∧16-4.938195487*10∧(-25)*t∧13-2.116369495*10∧(-27)*t∧14-6.538536619*10∧(-16)*t∧9-3.923121971*10∧(-18)*t∧10-2.139884712*10∧(-20)*t∧11-1.069942356*10∧(-22)*t∧12-1.525658544*10∧(-7)*t∧5-1.525658544*10∧(-9)*t∧6-1.307707324*10∧(-11)*t∧7-9.807804929*10∧(-14)*t∧8-47.08822668-0.8475880802e-3*t∧3-0.1271382120e-4*t∧4-0.4237940401e-1*t∧2 plot3d(x-25*exp(-0.6e-1)-(x-x+25*exp(-0.6e-1))*(sum((0.6e-1*t)∧i/factorial(i), i = 1..10)), t = 0 .. 10, x = 0 .. 1000, axes = boxed)

FIGURE 4.4 *vexact = x*(1-exp(0.6e-1*t))+(x-exp(-0.6e-1))*exp(0.6e-1*t); plot3d(x*(1-

exp(0.6e-1*t))+(x-exp(-0.6e-1))*exp(0.6e-1*t), t = 0 .. 10, x = 0 .. 1000, axes = boxed);

A3: MAPLE CODE FOR FIGURES 4.7-4.12

FIGURE 4.7

w(x,t) = x∧3*exp(-6.5*t) plot3d(x∧3*exp(-6.5*t), t = 0 .. 9, x = 0 .. 2, axes = boxed)

FIGURE 4.8 plot3d(x∧3*exp(-6.5*t), t = 0 .. 18, x = 0 .. 2, axes = boxed)

FIGURE 4.9

w(x,t_exact) = (-1+exp(x))*exp(-2*t)+(1-exp(-2*x))*exp(x)

plot3d((-1+exp(x))*exp(-2*t)+(1-exp(-2*x))*exp(x), t = 0 .. 45, x = 0 .. 45, axes = boxed)

FIGURE 4.10

plot3d((-1+exp(x))*exp(-2*t)+(1-exp(-2*x))*exp(x), t = 0 .. 80, x = 0 .. 45, axes = boxed)

FIGURE 4.11

w(x,t) = x-25*exp(-0.6e-1)+(1-exp(0.6e-1*t))*(x-x+25*exp(-0.6e-1))

plot3d(x-25*exp(-0.6e-1)+(1-exp(0.6e-1*t))*(x-x+25*exp(-0.6e-1)), t = 0 .. 10, x = 0 .. 200, axes = boxed)

FIGURE 4.12

plot3d(x*(1-exp(0.6e-1*t))+(x-25*exp(-0.6e-1))*exp(0.6e-1*t), t = 0 .. 10, x = 0 .. 400,

axes = boxed)

A4: MAPLE CODE FOR FIGURES 4.13-4.16

FIGURE 4.13

w(s,t) =-(1/25)*(2*s(-(50*(s+sqrt(s)))/s∧(5/2))∧2*s∧(9/2)*t-25*s∧(11/2)-5000*s∧5-2500*s∧(9/2)-1200*s∧(5/2)*t∧2-4700*s∧2*t∧2-5900*s∧(3/2)*t∧2-4800*s*t∧2-6300*t∧2*sqrt(s)-4000*t∧2)/s∧(9/2)

plot3d(((1/25)*(2*s(-(50*(s+sqrt(s)))/s∧(5/2))∧2*s∧(9/2)*t-25*s∧(11/2)-5000*s∧5-2500*s∧(9/2)-1200*s∧(5/2)*t∧2-4700*s∧2*t∧2-5900*s∧(3/2)*t∧2-4800*s*t∧2-6300*t∧2*sqrt(s)-4000*t∧2)/s∧(9/2)

), S = 0.1 .. 10, t = 0 .. 1, axes = boxed);

FIGURE 4.14

w(exact) = s+200*(sqrt(s)*exp((1/50)*t)+(1/2)*exp((1/100)*t))

plot3d(s+200*(sqrt(s)*exp((1/50)*t)+(1/2)*exp((1/100)*t)), S = .1 .. 10,
t = 0 .. 1, axes = boxed)

Figure 4.15

w(s,t_exact) = s+200*(sqrt(s)*exp((1/20)*t)+(1/2)*exp((1/10)*t))

3D Plot s+200*s∧(1/2)*exp(1/20*t)+100*exp(1/10*t)

plot3d(s+200*s∧(1/2)*exp(1/20*t)+100*exp(1/10*t), t = 1 .. 2, s = 0.1 .. 5, axes = boxed);

Figure 4.16

w(s,t) =s+200*sqrt(s)+100+(-(3/50)*s*(1+100/sqrt(s))-(2/25)*s∧2*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6))*t+(-(3/100)*s*(-3/50-3/sqrt(s)-(4/25)*s*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6)-(2/25)*s∧2*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s)))-(1/25)*s∧2*(19/(2*s∧(3/2))+8/s∧2+(6/625)*s+(48/25)*sqrt(s)+24/25-(8/25)*s*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s))-(2/25)*s∧2*(-375/(2*s∧(7/2))-300/s∧4+3/s∧(3/2))+(2*(19/(2*s∧(3/2))+8/s∧2+(6/625)*s+(48/25)*sqrt(s)+24/25-(8/25)*s*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s))-(2/25)*s∧2*(-375/(2*s∧(7/2))-300/s∧4+3/s∧(3/2))))/sqrt(s)-(3/50)*s*(-(3/50)*s*(1+100/sqrt(s))-(2/25)*s∧2*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6))))*t∧2

plot3d(s+200*sqrt(s)+100+(-(3/50)*s*(1+100/sqrt(s))-(2/25)*s∧2*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6))*t+(-(3/100)*s*(-3/50-3/sqrt(s)-(4/25)*s*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6)-(2/25)*s∧2*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s)))-(1/25)*s∧2*(19/(2*s∧(3/2))+8/s∧2+(6/625)*s+(48/25)*sqrt(s)+24/25-(8/25)*s*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s))-(2/25)*s∧2*(-375/(2*s∧(7/2))-300/s∧4+3/s∧(3/2))+(2*(19/(2*s∧(3/2))+8/s∧2+(6/625)*s+(48/25)*sqrt(s)+24/25-(8/25)*s*(75/s∧(5/2)+100/s∧3-3/50-6/sqrt(s))-(2/25)*s∧2*(-375/(2*s∧(7/2))-300/s∧4+3/s∧(3/2))))/sqrt(s)-(3/50)*s*(-(3/50)*s*(1+100/sqrt(s))-(2/25)*s∧2*(-50/s∧(3/2)-50/s∧2-(3/50)*s-12*sqrt(s)-6))))*t∧2, t = 1 .. 2, s = .1 .. 5, axes = boxed)

A5: MAPLE CODE FOR FIGURES 4.17-4.21

e := s+200*(sqrt(s)*exp((1/20)*t)+(1/2)*exp((1/10)*t))

a := s+200*sqrt(s)+100+(1/1250)*(6*s∧(7/2)+1200*s∧3+600*s∧(5/2)-75*s∧(3/2)-2500*s+5000*sqrt(s))*t∧alpha/(sqrt(s)*GAMMA(1+alpha)) +(1/3125000)*(36*s∧5+7200*s∧(9/2)+3600*s∧4-5400*s∧3-697200*s∧(5/2)-1080000*s∧2-240000*s∧(3/2)+5625*s+31250*sqrt(s)-125000)/GAMMA(1+2*alpha)

Figure 4.17

> t := .0; > alpha := 1;

plot([e, a], s = 0. .. 7, legend = [”exact solution”, ”approximate solution”], thickness = 3, color = [red, green], titlefont = [”ROMAN”, 12], labels = [”stock price, S”, ”option value”], labeldirections = [”horizontal”, ”vertical”], labelfont = [”HELVETICA”, 13], linestyle = [solid, longdash], axesfont = [”HELVETICA”, ”ROMAN”, 12], legendstyle = [font = [”HELVETICA”, 12], location = bottom])

Figure 4.18 > t := .5; > alpha := 1;

plot([e, a], s = 0. .. 4, legend = [”exact solution”, ”approximate solution”], thickness = 3, color = [red, green], titlefont = [”ROMAN”, 12], labels = [”stock price, S”, ”option value”], labeldirections = [”horizontal”, ”vertical”], labelfont = [”ROMAN”, 13], linestyle = [solid, longdash], axesfont = [”ROMAN”, ”ROMAN”, 12], legendstyle = [font = [”ROMAN”, 12], location = bottom])

Figure 4.19

> t := 0.5; > alpha :=0 .5;

plot([e, a], s = 0. .. 4, legend = [”exact solution”, ”approximate solution”], thickness = 3, color = [red, green], titlefont = [”ROMAN”, 12], labels = [”stock price, S”,

”option value”], labeldirections = [”horizontal”, ”vertical”], labelfont = [”ROMAN”, 13], linestyle = [solid, longdash], axesfont = [”ROMAN”, ”ROMAN”, 12], legendstyle = [font = [”ROMAN”, 12], location = bottom])

Figure 4.20

> t := 0.5; > alpha := 1.5;

plot([e, a], s = 0. .. 4, legend = [”exact solution”, ”approximate solution”], thickness = 3, color = [red, green], titlefont = [”ROMAN”, 12], labels = [”stock price, S”, ”option value”], labeldirections = [”horizontal”, ”vertical”], labelfont = [”ROMAN”, 13], linestyle = [solid, longdash], axesfont = [”ROMAN”, ”ROMAN”, 12], legendstyle = [font = [”ROMAN”, 12], location = bottom])

Figure 4.21

> t := 1; > alpha := 2.5;

plot([e, a], s = 0. .. 7, legend = [”exact solution”, ”approximate solution”], thickness = 3, color = [red, green], titlefont = [”ROMAN”, 12], labels = [”stock price, S”, ”option value”], labeldirections = [”horizontal”, ”vertical”], labelfont = [”ROMAN”, 13], linestyle = [solid, longdash], axesfont = [”ROMAN”, ”ROMAN”, 12], legendstyle = [font = [”ROMAN”, 12], location = bottom])

APPENDIX B
NGSEINDX Simulated Data (2000 - 2016): Figure 3.1

Time	Drift	Uncertainty	Change	Share Index *(in Naira)*
0.000	---	---	---	5306.900000
0.001	0.010000000	-0.043319058	-0.033319058	5305.131791
0.002	0.009996668	0.115325360	0.125322029	5311.782507
0.003	0.010009200	0.210607455	0.220616655	5323.490410
0.004	0.010031262	-0.258396925	-0.248365663	5310.309896
0.005	0.010006425	0.172786950	0.182793376	5320.010553
0.006	0.010024705	-0.173585465	-0.163560760	5311.330550
0.007	0.010008349	-0.277082869	-0.267074520	5297.157172
0.008	0.009981641	-0.088793263	-0.078811622	5292.974718
0.009	0.009973760	0.167673014	0.177646774	5302.402254
0.010	0.009991525	-0.050491963	-0.040500439	5300.252936
0.011	0.009987475	0.463875026	0.473862500	5325.400347
0.012	0.010034861	0.202619091	0.212653952	5336.685677
0.013	0.010056126	0.208848056	0.218904182	5348.302704
0.014	0.010078017	0.273180729	0.283258746	5363.334965
0.015	0.010106343	0.204240466	0.214346809	5374.710135
0.016	0.010127777	-0.214883391	-0.204755613	5363.843955
0.017	0.010107302	-0.166109764	-0.156002462	5355.565064
0.018	0.010091701	0.306309787	0.316401488	5372.356175
0.019	0.010123342	0.158845746	0.168969088	5381.323196
0.020	0.010140239	-0.160802204	-0.150661965	5373.327714
0.021	0.010125172	0.203355354	0.213480527	5384.656911
0.022	0.010146520	0.213070191	0.223216711	5396.502798
0.023	0.010168842	0.246663520	0.256832362	5410.132637
0.024	0.010194525	0.097555250	0.107749775	5415.850811
0.025	0.010205300	0.110452376	0.120657676	5422.253989
0.026	0.010217366	0.005064239	0.015281605	5423.064974
0.027	0.010218894	-0.080847630	-0.070628736	5419.316774
0.028	0.010211831	-0.091483394	-0.081271562	5415.003777
0.029	0.010203704	0.030288076	0.040491780	5417.152631
0.030	0.010207753	0.165390527	0.175598281	5426.471457
0.031	0.010225313	-0.124382874	-0.114157561	5420.413232
0.032	0.010213897	0.119081154	0.129295052	5427.274789
0.033	0.010226827	0.013802604	0.024029431	5428.550005
0.034	0.010229230	-0.386613519	-0.376384289	5408.575667
0.035	0.010191591	-0.062933167	-0.052741576	5405.776728
0.036	0.010186317	-0.079026004	-0.068839686	5402.123474
0.037	0.010179433	-0.046069622	-0.035890188	5400.218817
0.038	0.010175844	0.247173344	0.257349188	5413.876082

APPENDIX B Cont'd:

0.039	0.010201579	-0.091240427	-0.081038848	5409.575428
0.040	0.010193475	0.247012792	0.257206267	5423.225109
0.041	0.010219196	0.024529112	0.034748308	5425.069167
0.042	0.010222671	0.664834213	0.675056884	5460.893761
0.043	0.010290176	0.048952139	0.059242316	5464.037691
0.044	0.010296101	0.215816947	0.226113048	5476.037287
0.045	0.010318712	0.363046064	0.373364776	5495.851378
0.046	0.010356049	-0.288344298	-0.277988250	5481.098823
0.047	0.010328250	-0.320006678	-0.309678428	5464.664500
0.048	0.010297282	0.139334492	0.149631774	5472.605304
0.049	0.010312245	-0.298471749	-0.288159504	5457.312968
0.050	0.010283429	0.156049316	0.166332745	5466.140083
0.051	0.010300062	0.124577290	0.134877353	5473.297887
0.052	0.010313550	0.252917309	0.263230859	5487.267287
0.053	0.010339873	0.418555437	0.428895310	5510.028332
0.054	0.010382763	-0.305876601	-0.295493838	5494.346771
0.055	0.010353213	-0.254773373	-0.244420160	5481.375636
0.056	0.010328771	0.075802942	0.086131713	5485.946559
0.057	0.010337384	-0.004602368	0.005735016	5486.250910
0.058	0.010337958	0.047383766	0.057721724	5489.314148
0.059	0.010343730	-0.086773440	-0.076429709	5485.258095
0.060	0.010336087	0.116773632	0.127109720	5492.003685
0.061	0.010348798	0.011556986	0.021905785	5493.166199
0.062	0.010350989	-0.567070462	-0.556719473	5463.621657
0.063	0.010295317	-0.103145717	-0.09285040	5458.694179
0.064	0.010286032	0.072573065	0.082859096	5463.091428
0.065	0.010294318	0.279583217	0.289877535	5478.474937
0.066	0.010323305	0.215509748	0.225833054	5490.459674
0.067	0.010345889	0.440596228	0.450942117	5514.390721
0.068	0.010390983	-0.259545306	-0.249154323	5501.168346
0.069	0.010366067	0.056461581	0.066827648	5504.714825
0.070	0.010372750	-0.071504018	-0.061131268	5501.470648
0.071	0.010366637	0.138576286	0.148942923	5509.374904
0.072	0.010381531	-0.348504393	-0.338122862	5491.431060
0.073	0.010347719	0.092918984	0.103266703	5496.911320
0.074	0.010358046	-0.109384736	-0.09902669	5491.656072
0.075	0.010348143	-0.049796525	-0.039448382	5489.562585
0.076	0.010344198	0.183757590	0.194101789	5499.863374
0.077	0.010363608	0.040555608	0.050919217	5502.565605
0.078	0.010368700	0.063643988	0.074012689	5506.493385
0.079	0.010376102	-0.157989103	-0.147613001	5498.659710

APPENDIX B Cont'd:

0.080	0.010361340	0.136616978	0.146978318	5506.459702
0.081	0.010376038	-0.067872629	-0.057496591	5503.408415
0.082	0.010370289	-0.087790849	-0.07742056	5499.299786
0.083	0.010362546	0.107287102	0.117649648	5505.543333
0.084	0.010374311	-0.032207925	-0.021833614	5504.384645
0.085	0.010372128	0.194916196	0.205288324	5515.279090
0.086	0.010392657	0.586509041	0.596901698	5546.956067
0.087	0.010452347	0.039208527	0.049660874	5549.591521
0.088	0.010457313	-0.293561406	-0.283104093	5534.567469
0.089	0.010429003	-0.344214902	-0.333785900	5516.853785
0.090	0.010395624	0.035153231	0.045548855	5519.271020
0.091	0.010400179	0.380959662	0.391359841	5540.040093
0.092	0.010439315	-0.042356478	-0.031917163	5538.346285
0.093	0.010436123	-0.208022334	-0.197586211	5527.860577
0.094	0.010416365	0.465487652	0.475904016	5553.116332
0.095	0.010463955	-0.360214549	-0.349750594	5534.555417
0.096	0.010428980	0.152395676	0.162824656	5543.196356
0.097	0.010445263	-0.277369174	-0.266923912	5529.030972
0.098	0.010418570	0.272164311	0.282582881	5544.027364
0.099	0.010446828	0.203439398	0.213886226	5555.378090
0.100	0.010468217	-0.124403057	-0.11393484	5549.331684
0.101	0.010456824	-0.221925324	-0.211468501	5538.109263
0.102	0.010435677	0.097697668	0.108133345	5543.847789
0.103	0.010446490	0.119720243	0.130166733	5550.755611
0.104	0.010459507	-0.307760592	-0.297301085	5534.978139
0.105	0.010429777	0.086540600	0.096970376	5540.124255
0.106	0.010439474	-0.305221592	-0.294782119	5524.480464
0.107	0.010409995	0.249976990	0.260386985	5538.298942
0.108	0.010436034	0.114971705	0.125407739	5544.954208
0.109	0.010448575	-0.194100604	-0.183652030	5535.207975
0.110	0.010430210	-0.220091682	-0.209661473	5524.081454
0.111	0.010409244	-0.120071900	-0.109662656	5518.261764
0.112	0.010398277	0.185343782	0.195742059	5528.649602
0.113	0.010417851	0.071671833	0.082089685	5533.006015
0.114	0.01042606	-0.366675586	-0.356249525	5514.100210
0.115	0.010390435	0.040924726	0.051315162	5516.823457
0.116	0.010395567	-0.158237396	-0.147841829	5508.977635
0.117	0.010380783	0.155910411	0.166291194	5517.802542
0.118	0.010397412	0.115505239	0.125902651	5524.484073
0.119	0.010410002	0.245502363	0.255912366	5538.065088
0.120	0.010435593	-0.126758742	-0.116323148	5531.891932

APPENDIX B Cont'd:

0.121	0.010423961	-0.093430378	-0.083006417	5527.486865
0.122	0.010415660	0.012299281	0.022714941	5528.692322
0.123	0.010417932	-0.541342018	-0.530924086	5500.516711
0.124	0.010364840	0.136185043	0.146549882	5508.293968
0.125	0.010379495	-0.185064879	-0.174685385	5499.023589
0.126	0.010362026	-0.060628333	-0.050266307	5496.356006
0.127	0.010356999	-0.180046716	-0.169689716	5487.350744
0.128	0.010340030	-0.116366284	-0.106026254	5481.724039
0.129	0.010329428	0.242066685	0.252396113	5495.118448
0.130	0.010354667	-0.259656289	-0.249301622	5481.888261
0.131	0.010329737	0.304464663	0.314794400	5498.594085
0.132	0.010361217	0.297012804	0.307374021	5514.906116
0.133	0.010391954	-0.032692211	-0.022300257	5513.722661
0.134	0.010389724	0.003822956	0.014212680	5514.476915
0.135	0.010391145	0.434986366	0.445377512	5538.112654
0.136	0.010435683	0.095171133	0.105606816	5543.717101
0.137	0.010446244	-0.066220044	-0.055773800	5540.757241
0.138	0.010440666	0.577594514	0.588035181	5571.963681
0.139	0.010499470	-0.217595886	-0.207096416	5560.973282
0.140	0.010478760	-0.114306953	-0.103828193	5555.463224
0.141	0.010468377	-0.043639684	-0.033171307	5553.702856
0.142	0.010465060	0.327134407	0.337599468	5571.618924
0.143	0.010498820	0.382157865	0.392656685	5592.456817
0.144	0.010538086	0.142607988	0.153146074	5600.584127
0.145	0.010553401	-0.022681374	-0.012127974	5599.940506
0.146	0.010552188	-0.279781925	-0.269229737	5585.652755
0.147	0.010525265	0.040232181	0.050757446	5588.346405
0.148	0.010530341	-0.207095456	-0.196565115	5577.914887
0.149	0.010510684	0.403447153	0.413957837	5599.883218
0.150	0.010552080	0.126500080	0.137052160	5607.156436
0.151	0.010565785	-0.166374601	-0.155808816	5598.887819
0.152	0.010550204	0.523114000	0.533664204	5627.208845
0.153	0.010603571	-0.137084113	-0.126480543	5620.496651
0.154	0.010590922	0.131418552	0.142009474	5628.032953
0.155	0.010605123	-0.267755718	-0.257150594	5614.386228
0.156	0.010579408	-0.046325291	-0.035745883	5612.489229
0.157	0.010575834	-0.266899565	-0.256323731	5598.886381
0.158	0.010550201	-0.027918201	-0.017368000	5597.964679
0.159	0.010548465	0.355068172	0.365616636	5617.367591
0.160	0.010585026	-0.14694731	-0.136362284	5610.130980
0.161	0.010571390	0.340978763	0.351550153	5628.787398

APPENDIX B Cont'd:

0.162	0.010606545	0.082217614	0.092824159	5633.713480
0.163	0.010615827	0.215032345	0.225648172	5645.688404
0.164	0.010638392	-0.257046298	-0.246407906	5632.611784
0.165	0.010613751	0.044727055	0.055340806	5635.548664
0.166	0.010619286	-0.223499301	-0.212880015	5624.251336
0.167	0.010597998	0.027390979	0.037988977	5626.267369
0.168	0.010601796	-0.012683807	-0.002082010	5626.156879
0.169	0.010601588	-0.171668085	-0.161066497	5617.609241
0.170	0.010585482	-0.246075249	-0.235489767	5605.112038
0.171	0.010561933	-0.016784758	-0.006222825	5604.781795
0.172	0.010561310	0.156412561	0.166973871	5613.642933
0.173	0.010578008	-0.16116245	-0.150584442	5605.651569
0.174	0.010562949	0.20792926	0.218492209	5617.246732
0.175	0.010584799	-0.091958273	-0.081373474	5612.928322
0.176	0.010576661	-0.34186142	-0.331284759	5595.347374
0.177	0.010543533	-0.133900865	-0.123357332	5588.800920
0.178	0.010531197	-0.026922107	-0.016390910	5587.931071
0.179	0.010529558	0.203111016	0.213640574	5599.268764
0.180	0.010550922	0.124188263	0.134739185	5606.419239
0.181	0.010564396	-0.133540311	-0.122975915	5599.893031
0.182	0.010552098	0.169027055	0.179579153	5609.423114
0.183	0.010570056	0.033137488	0.043707544	5611.742627
0.184	0.010574427	0.303475143	0.314049570	5628.408926
0.185	0.010605832	-0.412689671	-0.402083839	5607.070741
0.186	0.010565624	0.329001727	0.339567350	5625.091238
0.187	0.010599580	0.021477420	0.032077000	5626.793532
0.188	0.010602788	-0.182846317	-0.172243529	5617.652742