Ramsey theory on Steiner triples

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Abstract

We call a partial Steiner triple system C (configuration) t-Ramsey if for large enough n (in terms of C, t), in every t-coloring of the blocks of any Steiner triple system STS(n) there is a monochromatic copy of C.

We prove that configuration C is t-Ramsey for every t in three cases:

- C is acyclic
- every block of C has a point of degree one
- C has a triangle with blocks $123, 345, 561$ with some further blocks attached at points 1 and 4

This implies that we can decide for all but one configurations with at most four blocks whether they are t-Ramsey. The one in doubt is the *sail* with blocks $123, 345, 561, 147$.

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1 Introduction

A *Steiner triple system of order* n, STS(n), is an n-element set V, called *points* and a set $\mathcal{B}$ of 3-element subsets of V called *blocks*, such that each pair of elements of V appears in exactly one block of $\mathcal{B}$. A *partial Steiner triple system of order* n, PTS(n) is defined by requiring only that each pair of elements in V is in at most one block. Sometimes a PTS(n) is referred to as a *configuration* on n points. Also, in hypergraph theory, a PTS(n) is called a 3-*uniform linear hypergraph on* n *vertices*. As is customary, we assume that every point of a PTS(n) is in at least one block. The number of blocks containing $v \in V$ is called the *degree* of v. A set $S \subset V$ in an STS(n) is *independent* if $|S \cap B| \leq 2$ for all $B \in \mathcal{B}$.

A configuration C is *unavoidable* if there is an $n_0 = n_0(C)$ such that every STS(n) with $n \geq n_0$ must contain C. It is known ([1, 4]) that all but two configurations with at most four blocks are unavoidable. The two exceptions are the Pasch configuration with blocks $123, 345, 561, 246$ and the one with blocks $123, 345, 561, 267$. To decide whether a configuration is unavoidable can be very difficult. The most spectacular example is the following conjecture of Erdős.

A configuration is called r-*sparse* for some $r \geq 4$ if it does not contain any configuration with $i + 2$ points and i blocks for all $2 \leq i \leq r$. Erdős conjectured [5] that for every $r \geq 4$ there exists r-sparse STS(n) for every large enough admissible ($\equiv 1, 3 \pmod 6$) n. Thus, supposing that this conjecture is true, unavoidable configurations have at most as many blocks as the number of points minus three.

We call a configuration C t-*Ramsey* if there exists a constant $n_0(C, t)$ such that for all admissible $n \geq n_0(C, t)$ there is a monochromatic copy of C in every t-coloring of the blocks of any STS(n). If C is t-Ramsey then the smallest possible value of $n_0(C, t)$ is denoted by $R(C, t)$. Clearly, a configuration is 1-Ramsey if and only if it is unavoidable.

Notice that the nature of $R(C, t)$ differs slightly from classical Ramsey numbers. For example, proving that in any 2-coloring of the edges of K_6 there is a monochromatic triangle, establishes that the 2-color Ramsey number of a triangle is at most 6. However, although in every 2-coloring of the blocks of STS(7) (the Fano plane) there is a monochromatic triangle (a *triangle* is the configuration with blocks $123, 345, 561$), the blocks of STS(9) can be 2-colored without having monochromatic triangles.

Assume that the blocks of an STS(n) are colored with t colors. This coloring defines a natural *induced coloring* on the complete graph with vertex set V by coloring every pair of V with the color of the block containing the pair. A natural tool to establish bounds on $R(C, t)$, one can use results of Ramsey theory on graphs. This is illustrated with the next result.

Proposition 1. *Let C be the triangle. Then $R(C, 2) = 13$.*

Proof. Suppose that $n \geq 13$ and consider a 2-coloring of the blocks of any STS(n). We apply the result $R(K_4 - e, K_4 - e) = 10$ ([3],[7] Table III.) for the complete 2-colored K_n in the induced coloring ($K_4 - e$ denotes the graph obtained from K_4 by deleting an edge). We obtain a monochromatic, say red $K_4 - e$ in the induced coloring, say with vertex set $W = \{1, 2, 3, 4\}$ and with edges as the pairs of W except $(3, 4)$. If there is a block $B \in \mathcal{B}$ covering three edges of $K_4 - e$, without loss of generality $B = \{1, 2, 3\}$, then B together with the blocks through the pairs $(1, 4), (2, 4)$ form a red triangle. If there is no block covering three edges of $K_4 - e$, then the three blocks through the pairs $(1, 2), (1, 3), (2, 3)$ define a red triangle. This proves that $R(C, 2) \leq 13$.

To prove that $R(C, 2) \geq 13$ we exhibit a 2-coloring of STS(9) (the affine plane of order 3): color the blocks in two parallel classes red and the blocks in the other two parallel classes blue, there is no monochromatic triangle. $\square$

We can prove that a configuration is t-Ramsey in two basic cases. The first case is when C is *acyclic* defined recursively as follows. A configuration $C = (V, \mathcal{B})$ is *acyclic* if either $|\mathcal{B}| = 1$, or it can be obtained from an acyclic configuration C' by adding a new block which intersects $V(C')$ in at most one point.

Theorem 1. *Acyclic configurations are t-Ramsey for every t. In fact, $R(C, t) \leq 6t|V(C)|$.*

Our other result is for *graphlike* configurations, where every block contains a point of degree one. To a graphlike configuration C we associate a graph G_C, obtained from the blocks of C by removing a point of degree one from each block.

Theorem 2. *Every graphlike configuration is t-Ramsey for every t with $R(C, t) = O((R_t(G_C))^3)$, where $R_t(G_C)$ denotes the t-color Ramsey number of the graph G_C.*

The two theorems above show that all but four of the (24) configurations with at most four blocks are t-Ramsey for every t. The two natural exceptions are the avoidable ones mentioned before. There are two further small configurations that are unavoidable but neither acyclic nor graphlike. One of them has blocks $123, 345, 561, 478$. We shall apply the induced matching lemma of Ruzsa and Szemerédi [9] to show that it is also t-Ramsey. In fact, we prove this for a more general family, $D_{p,q}$, obtained from the triangle with blocks $123, 345, 561$ by attaching p blocks at point 1 and q blocks at point 4. For details on the 24 small configurations and the two avoidable ones among them see [4, 6].

Theorem 3. *For fixed non-negative integers p, q, the configuration $D_{p,q}$ is t-Ramsey for every t.*

Theorems 1, 2, 3 leave only one (unavoidable) small configuration for which we could not decide wether it is even 2-Ramsey: the *sail* with blocks $123, 345, 561, 147$.

Corollary 1. *The unavoidable configurations with at most four blocks, except possibly the sail, are t-Ramsey for any $t \geq 1$.*

2 Acyclic configurations

The *density* of a configuration C, denoted $\epsilon(C)$, is defined to be the number of blocks divided by the number of points. For the proof of Theorem 1 we need two lemmas.

Lemma 1. *Let $C = (V, \mathcal{B})$ be any configuration.*

(a) There exists a point $v \in V$ such that $\deg(v) > \epsilon(C)$.

(b) There exists a subconfiguration $(V', \mathcal{B}')$ such that $\deg(v) > \epsilon(C)$ for every $v \in V'$.

Proof.

(a) Suppose to the contrary that $\deg(v) \leq \epsilon(C)$ for every $v \in V$. Then

$$3|\mathcal{B}| = \sum_{v \in V} \deg(v) \leq |V|\epsilon(C) = |\mathcal{B}|,$$

which is absurd, since $|\mathcal{B}| > 0$. Hence, there must be some point v with $\deg(v) > \epsilon(C)$.

(b) Proceeding by induction on $|\mathcal{B}|$, clearly the claim holds for the configuration consisting of exactly one block. Suppose it also holds for all configurations with fewer than n blocks, and let $C = (V, \mathcal{B})$ be a configuration with n blocks.

If $\deg(v) > \epsilon(C)$ for every $v \in V$ then we are done, so suppose that for some $v_0 \in V$ we have $\deg(v_0) \leq \epsilon(C)$.

Removing v_0 from C yields a subconfiguration $D = (V', \mathcal{B}')$. Then $|V'| = |V| - 1$ and $|\mathcal{B}'| = |\mathcal{B}| - \deg(v_0)$, since we must remove each block containing v_0 when removing v_0. Yet D must have at least one block, since it follows from (a) and $\epsilon(C) \geq \deg(v_0)$ that some point has degree larger than $\deg(v_0)$. It remains to show that $\epsilon(D) \geq \epsilon(C)$. Then it would follow by induction that there exists a subconfiguration D' of D such that the degree of each point exceeds $\epsilon(C)$.

Now since $\deg(v_0) \le \epsilon(C)$, we have $|\mathcal{B}'| \ge |\mathcal{B}| - \epsilon(\mathcal{C})$. It follows that,

$$\epsilon(D) = \frac{|\mathcal{B}'|}{|V'|} \ge \frac{|\mathcal{B}| - \epsilon(\mathcal{C})}{|V| - 1}.$$

But

$$|\mathcal{B}| - \epsilon(C) = \frac{|\mathcal{B}||V|}{|V|} - \frac{|\mathcal{B}|}{|V|} = \frac{|\mathcal{B}|(|V| - 1)}{|V|}.$$

Thus, $\epsilon(D) \ge |\mathcal{B}|/|V| = \epsilon(C)$, as desired. $\square$

Lemma 2. *Let C be an acyclic configuration, and let S be a PTS(n). If $\deg(v) \ge |V(C)|$ for every $v \in V(S)$, then there exists an injective hypergraph homomorphism $f\colon V(C) \to V(S)$. Hence, some subconfiguration of S is isomorphic to C.*

Proof. We will proceed by induction on $|V(C)|$. Clearly, if $|V(C)| = 3$, then the claim is true.

Now assume that the claim is true for acyclic configurations C such that $|V(C)| < m$ for some integer m. Let C be an acyclic configuration with $|V(C)| = m$, and suppose S is a PTS(n) such that for every point v, $\deg(v) \ge |V(C)|$.

Since C is acyclic, there is an $A = \{p, q, r\} \in E(C)$ such that $\deg(p) = \deg(q) = 1$. Remove A from $E(C)$ to yield another acyclic configuration D. Then for every $v \in V(S)$, $\deg(v) \ge m > |V(D)|$. It follows from the induction hypothesis that there exists an injective homomorphism $f\colon V(D) \to V(S)$.

Suppose $\deg(r) = 1$ in C . Since every point in S has degree at least m, clearly S must have at least $2m + 1 > m$ points. But $|f[V(D)]| = |V(D)| < m$, so $V(S) \setminus f[V(D)]$ must be nonempty. Choose any $x \in V(S) \setminus f[V(D)]$. Since $\deg(x) \ge m$ and $|f[V(D)]| < m$, there must be a block $B \in E(S)$ such that $B = \{x, y, z\}$ with $y, z \notin f[V(D)]$. Then the function $\tilde{f}\colon V(C) \to V(S)$ defined by $\tilde{f}(p) = x$, $\tilde{f}(q) = y$, $\tilde{f}(r) = z$, and $\tilde{f}(v) = f(v)$ for $v \in V(D)$, is clearly an injective homomorphism as desired.

Finally, suppose $\deg(r) > 1$ in C, then $r \in V(D)$. Proceeding as before, since $\deg(f(r)) \ge m$ and $|f[V(D)]| < m$, there must be a block $B \in E(S)$ such that $B = \{f(r), y, z\}$ with $y, z \notin f[V(D)]$, and define an injective homomorphism $\tilde{f}\colon V(C) \to V(S)$ by $\tilde{f}(q) = y$, $\tilde{f}(p) = z$, and $\tilde{f}(v) = f(v)$ for $v \in V(D)$. $\square$

Proof of Theorem 1. Let C be an acyclic configuration and let $S = (V, \mathcal{B})$ be an STS(n) with $n \ge 6t|V(C)|$. Color $\mathcal{B}$ with t colors. Using the fact that $|\mathcal{B}| = n(n-1)/6$, it follows that there exists a subconfiguration $T = (V', \mathcal{B}')$ of S such that all blocks of T have the same color, and

$$|\mathcal{B}'| \ge \frac{1}{t}|\mathcal{B}| = \frac{n(n-1)}{6t}.$$

Then

$$\epsilon(T) = \frac{|\mathcal{B}'|}{|V'|} \geq \frac{1}{n}|\mathcal{B}'| \geq \frac{n-1}{6t}.$$

Now by Lemma 1(b), there exists a subconfiguration U of T such that for every point v in U,

$$\deg(v) > \epsilon(T) \geq \frac{n-1}{6t}.$$

But $n \geq 6t|V(C)|$, so it follows that,

$$\deg(v) > \frac{6t|V(C)|-1}{6t} \geq |V(C)| - 1.$$

Hence, $\deg(v) \geq |V(C)|$ for every point v in U. It follows from Lemma 2 that U, and therefore T, contains a subconfiguration isomorphic to C.

Thus, for $n \geq 6t|V(C)|$, every t-coloring of an STS(n) results in a monochromatic copy of C. $\square$

To find the exact value of $R(C,t)$ is a difficult problem, even for the configurations with two blocks. Let A be the configuration of two intersecting blocks. To find $R(A,t)$ is equivalent to the problem of finding the *chromatic index* of STSs, the minimum number of colors needed to color the blocks so that blocks of the same color must be disjoint. In fact, $R(A,t)$ is the minimum n such that every STS of order at least n has chromatic index larger than t. It follows from an important result of Pippenger and Spencer [8] that the chromatic index of STS(n) is asymptotic to $n/2$ (see [4] p. 366). This translates into the statement that $R(A,t)$ is asymptotic to $2t$.

Let B be the configuration of two disjoint blocks. Then $R(B,t)$ is the minimum n such that the blocks of any STS of order at least n cannot be decomposed into t parts so that these parts contain pairwise intersecting blocks. It seems that this problem is not investigated yet. It is not difficult to see that $R(B,2) = 9$ because STSs of order at least 9 contain three pairwise disjoint blocks. For larger t we give the following bounds.

Theorem 4. *For $t \geq 3$, $2t-1 \leq R(B,t) \leq 3t+1$.*

Lemma 3. *For $n \geq 9$, the maximum number of pairwise intersecting blocks in any STS(n) is $\frac{n-1}{2}$.*

Proof. In any STS(n), any point is in exactly $\frac{n-1}{2}$ blocks, so equality is possible in the lemma.

Suppose that $\mathcal{A}$ is a set of pairwise intersecting blocks. We may assume that $n \geq 13$ since STS(9) has exactly four parallel classes so we cannot have more than four blocks in $\mathcal{A}$. Let v be a point of maximum degree, say k, in $\mathcal{A}$.

Observe that if $k \geq 4$ then all edges of $\mathcal{A}$ must contain v, proving the lemma. If $k = 1$ ($k = 2$) the $\mathcal{A}$ has at most one (four) blocks and the proof is finished. Thus $k = 3$ and in this case all blocks of $\mathcal{A}$ must be inside the union of the three blocks of $\mathcal{A}$ containing v. Thus $|\mathcal{A}| \leq 7 \leq \frac{n-1}{2}$ except when $n = 13$ and the proof is finished since none of the two STS(13) contain STS(7). $\Box$

Proof of Theorem 4. First we give the proof for $R(B,t) \leq 3t + 1$.

An STS(n) with $n \geq 9$ has $\frac{1}{3}\binom{n}{2}$ blocks, which means a t-colored STS(n) has at least $\frac{1}{3t}\binom{n}{2}$ blocks in one color. Lemma 3 implies that if $\frac{1}{3t}\binom{n}{2} > \frac{n-1}{2}$, i.e. if $n > 3t$, then there exists two disjoint monochromatic blocks. So if $n \geq 3t + 1$, we will have a monochromatic B in a t-colored STS(n). Thus $R(B,t) \leq 3t + 1$.

For the lower bound we need the result of Sauer and Schönheim [10] who proved that for every admissible n, there always exists a STS(n) $= (V, \mathcal{B})$ with a maximum independent set I of size at least $\frac{n-1}{2}$. Then there are at most $n - \frac{n-1}{2} = \frac{n+1}{2}$ vertices in $J = V \setminus I$. Assuming $J = \{v_1, \ldots, v_t\}$, we can partition $\mathcal{B}$ by placing a block $Z \in \mathcal{B}$ in class i if i is the smallest integer for which $v_i \in Z$. Clearly, blocks in the same class intersect. So $\frac{n+1}{2} \geq t$ therefore $2t - 1 \leq R(B,t)$. $\Box$

3 Graphlike Configurations

A set $S \subset V$ in an STS is *scattering* if it is independent and for any two blocks B_1, B_2 such that $|B_1 \cap S| = |B_2 \cap S| = 2$, the points $B_1 \setminus S, B_2 \setminus S$ are different. Note that any subset of a scattering set is a scattering set. The blocks defined by the pairs of a scattering set S with $s = |S|$ determine $\binom{s}{2}$ points in $V \setminus S$. This implies that any scattering set S in STS(n) satisfies $\binom{s+1}{2} \leq n$ and Colbourn, Dinitz and Stinson [2] proved that for all admissible n there is an STS(n) with a scattering set S that gives equality.

We need that *any* STS(n) has a large scattering set.

Proposition 2. *Within any $STS(n)$ there exists a scattering set of size s such that*

$$n \leq \binom{s}{2}(s-1) + s.$$

Proof. Clearly every STS(n) has a non-empty scattering set, so let S be a maximal scattering set, $|S| = s$. Note that two distinct points in S uniquely determine a block with a point outside of S. Let $T \subseteq V$ be the set of points outside S contained in a block with two elements from S. Then $|T| = \binom{s}{2}$. Now consider the set $U = V \setminus (S \cup T)$. Fix $u \in U$ and consider all of the blocks $\{u, s, x\}$ where $s \in S$. Then x cannot be in S, or else u would be in T. Also, x cannot be in U for all such blocks, or else S

would not be a maximal scattering set. Thus $x \in T$ for at least one block. This gives an injection from U to $S \times T$. However, we have over-counted the pairs (s, x) where $s \in S$ and $x \in T$ which lie in blocks containing two points of S and one point of T. We counted each of these twice, but none of them may lie in a block with a point from U. Thus $|U| \leq s\binom{s}{2} - 2\binom{s}{2}$. This gives

$$n = |S| + |T| + |U| \leq s + \binom{s}{2} + s\binom{s}{2} - 2\binom{s}{2} = \binom{s}{2}(s-1) + s,$$

as desired. $\square$

Proof of Theorem 2. Let C be a graphlike configuration. By Proposition 2, we can choose $N = O((R_t(G_C)^3))$ such that $n \geq N$ guarantees every STS(n) will have a scattering set of size $s = R_t(G_C)$. Then, in the coloring induced on K_s by the blocks containing the pairs of the scattering set in a t-colored STS(n), there must be a monochromatic copy of G_C. By the definition of a scattering set, the blocks whose coloring induces this copy of G_C all have a point of degree one, and thus constitute a monochromatic copy of C in the STS(n). $\square$

4 The configuration $D_{p,q}$

A *matching* in a graph G is a set of pairwise vertex disjoint edges. An *induced matching* M in G is a matching which is an induced subgraph of G, i.e., within the vertex set of M the only edges of G are the edges of M. We need the following well-known result.

Theorem 5. *(Ruzsa and Szemerédi, [9]) If the edge set of a graph on n vertices is the union of at most cn induced matchings (where c is a fixed constant), then the graph has $o(n^2)$ edges.*

Proof of Theorem 3. Assume we have a t-coloring on the blocks of a STS(n) = $(V, \mathcal{B})$, $V = \{v_1, \ldots, v_n\}$. Let $\mathcal{B}_i \subset \mathcal{B}$ denote the set of blocks containing v_i whose color appears most frequently among the blocks containing v_i. For example if $t = 2$ and v_1 is in more red blocks than blue, than $\mathcal{B}_1$ consists of all red blocks containing v_1. Moreover, if v_2 appears in more blue blocks than in red, then $\mathcal{B}_2$ consists of the blue blocks containing v_2. Note that distinct $\mathcal{B}_i$-s may be of different colors. Also, in case of ties, the color can be selected arbitrarily. Now, at least n/t of the $\mathcal{B}_i$-s consists of blocks of the same color, so without loss of generality, $\mathcal{B}_1, \ldots, \mathcal{B}_m$ are of same color c, where $m \geq n/t$.

We define a graph G on V with edge set $E = E_1 \cup E_2 \cup \ldots \cup E_m$ where

$$E_i = \{(v_j, v_k) : \{v_i, v_j, v_k\} \in \mathcal{B}_i\}.$$

Note that each E_i is a matching, $|E_i| \geq \frac{n-1}{2t}$ and $|E(G)| \geq m\left(\frac{n-1}{2t}\right) \geq \frac{n(n-1)}{2t^2}$. Suppose that n is large enough to satisfy

$$|E_i| \geq \frac{n-1}{2t} > 2p + q + 4. \tag{1}$$

Since we have a quadratic number of edges in G, Theorem 5 implies that for every sufficiently large n, *some* E_i is not an induced matching. Thus there exists $j \neq i$ for which we have a three-edge path e, f, g in G such that $e, g \in E_i, f \in E_j$. Condition (1) implies (applied for i) that $|E_i| \geq p + 3$ thus we can find p edges $e_1, \ldots, e_p \in E_i$ different from e, g and not containing v_j. Now we apply condition (1) for j which gives that $|E_j| > 2p + q + 4$ ensuring q edges $f_1, \ldots, f_q \in E_j$ so that these edges are disjoint from the $2p$ vertices of $e_1 \cup \ldots \cup e_p$ and also disjoint from the edges of the path e, f, g and from v_i (at most three edges of E_j can intersect the path efg since $f \in E_j$). Now the blocks defined by v_i with the pairs $e, g, e_1, \ldots, e_p$ and the blocks defined by v_j with the pairs $f, f_1, \ldots, f_q$ give a $D_{p,q}$ configuration in color c. □

5 Concluding remarks

It seems reasonable to conjecture that unavoidable configurations are t-Ramsey for every t. However, we could not decide whether the sail is t-Ramsey (even for $t = 2$).

It seems that certain properties that are trivial in Ramsey theory become difficult for Steiner systems. For example, we do not see how to prove that if C is 2-Ramsey then two disjoint copies of C is also 2-Ramsey.

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