The minimum number of triangular edges and a symmetrization method for multiple graphs

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Abstract

We give an asymptotic formula for the minimum number of edges contained in triangles in a graph having n vertices and e edges. Our main tool is a generalization of Zykov's symmetrization method that can be applied for several graphs simultaneously.

1 Graphs with few triangular edges

Erdős, Faudree, and Rousseau [3] showed that a graph on n vertices and at least $\lfloor n^2/4\rfloor + 1$ edges has at least $2\lfloor n/2\rfloor + 1$ edges in triangles. To see that this result is sharp, consider the graph obtained by adding one edge to the larger side of the complete bipartite graph $K_{\lceil n/2\rceil,\lfloor n/2\rfloor}$. We consider a more general problem, where the number of edges may be larger than $\lfloor n^2/4\rfloor + 1$. Given a graph G, denote by $\mathrm{Tr}(G)$ the number of edges of G contained in triangles, and let $\mathrm{Tr}(n, e) := \min\{\mathrm{Tr}(G) : |V(G)| = n,\, e(G) = e\}$. With this notation the above result of Erdős, Faudree, and Rousseau can be reformulated as

$$\mathrm{Tr}(n, \lfloor n^2/4\rfloor + 1) = 2\lfloor n/2\rfloor + 1. \tag{1}$$

Note that $\mathrm{Tr}(n, e) = 0$ whenever $e \le n^2/4$, because in that case there exist triangle-free (even bipartite) graphs with n vertices and e edges. To avoid trivialities, we usually implicitly assume that $e > n^2/4$.

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Given integers a, b and c, $(a \geq 2)$, we define a family of graphs $\mathcal{G}(a,b,c)$ as follows, see Figure 1 below. The vertex set V of a graph G in this class has a partition $V = A \cup B \cup C$ where $|A| = a$, $|B| = b$, and $|C| = c$, such that B and C are independent sets, $B \cup C$ induces a complete bipartite graph $K_{b,c}$, the vertices of C have neighbors only in B, and $G[A]$ and $G[A,B]$ are 'almost complete graphs', namely, they span more than $\binom{|A|-1}{2} + |A||B|$ edges. The edges of $G[B,C]$ are the non-triangular edges.

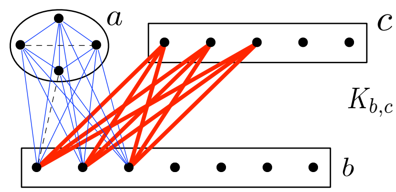

Figure 1: A graph from $\mathcal{G}(a,b,c)$.

Given integers $n \geq 3$ and $n^2/4 < e \leq \binom{n}{2}$, we define a class of graphs, $\mathcal{G}(n,e)$, with many non-triangular edges as follows. Put a graph $G \in \mathcal{G}(a,b,c)$ into the class $\mathcal{G}(n,e)$ if it has n vertices and e edges. Define $g(n,e)$ as $\min\{\mathrm{Tr}(G) : G \in \mathcal{G}(n,e)\}$. We have

$$\mathrm{Tr}(n,e) \leq g(n,e) = \min\{e - bc : a+b+c = n,\ a,b,c \in \mathbb{N} \cup \{0\},\ \binom{a}{2} + ab + bc \geq e\}. \tag{2}$$

We believe that one can extend the Erdős, Faudree, Rousseau theorem [3] as follows.

Conjecture 1. *Suppose that G is an n-vertex graph with e edges, such that $e > n^2/4$ and it has the minimum number of triangular edges, i.e.,* $\mathrm{Tr}(G) = \mathrm{Tr}(n,e)$. *Then $G \in \mathcal{G}(n,e)$.*

In particular, we conjecture that $\mathrm{Tr}(n,e) = g(n,e)$. We prove a slightly weaker result.

Theorem 2. *For $e > n^2/4$ we have $g(n,e) - (3/2)n \leq \mathrm{Tr}(n,e) \leq g(n,e)$.*

Our main tool, presented in Section 2, is a new symmetrization method, a generalization of previous results by Zykov and Motzkin and Straus such that it can be applied to more than one graph simultaneously.

In Section 3, we use the new symmetrization method to prove a lemma about triangular edges of a given graph. In Section 4, using the lemma of Section 3 we complete the proof of Theorem 2. In Section 5 we introduce more problems for future research, our method can be used to solve some of them (see [5]).

2 The symmetrization method

In this section, we describe Zykov's symmetrisation process [10]. It starts with a K_p-free graph G with vertex set $\{v_1, \ldots, v_n\}$ and at each step takes two nonadjacent vertices v_i and v_j such that $\deg(v_i) > \deg(v_j)$ and replaces all edges incident to v_j by new edges incident to v_j and to the neighborhood $N(v_i)$. We do the same if $\deg(v_i) = \deg(v_j)$, $N(v_i) \neq N(v_j)$ and $i < j$. Symmetrization does not increase the size of the largest clique and does not decrease the number of edges. When the process terminates it yields a complete multipartite graph with at most $p-1$ parts.

This way Zykov [10] gave a proof of Turán's theorem which states that the number of edges of a K_p-free graph is at most as large as in a complete $(p-1)$-partite graph with almost equal parts. It seems that this method cannot be used directly to determine $\mathrm{Tr}(n,e)$ because we need to increase simultaneously the number of edges and the number of non-triangular edges. In the rest of the section this method will be generalized to settings involving more than one graph.

Let us recall a continuous version of Zykov's symmetrisation method, due to Motzkin and Straus [8]. Given a graph G with vertex set $\{v_1, \dots, v_n\}$ define a real polynomial

$$f(G, \mathbf{x}) := \sum \{x_i x_j : v_i v_j \in E\}.$$

Define a simplex $S_n := \{\mathbf{x} \in \mathbb{R}^n : \forall x_i \geq 0 \text{ and } \sum x_i = 1\}$. Let $f(G) := \max\{f(G, \mathbf{x}) : \mathbf{x} \in S_n\}$. Motzkin and Straus [8] provided an alternative proof of an asymptotic version of Turán's theorem by observing a remarkable connection between the clique number, $\omega(G)$, and $f(G)$. They proved that $f(G) = (\omega - 1)/(2\omega)$. Their main tool was a continuous version of Zykov's symmetrization as follows.

Theorem 3. (Motzkin and Straus [8]) *Given a graph G on n vertices and a vector $\mathbf{x} \in S_n$, there exists $\mathbf{y} \in S_n$ such that $f(G, \mathbf{x}) \leq f(G, \mathbf{y})$ and* $\mathrm{support}(\mathbf{y})$ *induces a complete subgraph.*

We generalize this result so that it can be applied simultaneously for several graphs.

Theorem 4. *Let G be a graph on n vertices and let $G_1, G_2, \dots, G_d$ be subgraphs of G with the same vertex set. For every $\mathbf{x} \in S_n$ there exists a subset $K \subseteq V(G)$ and a vector $\mathbf{y} \in S_n$ with support K such that $f(G_i, \mathbf{x}) \leq f(G_i, \mathbf{y})$ for every $1 \leq i \leq d$ and $\alpha(G[K]) \leq d$.*

To prove Theorem 4 we need the following lemma.

Lemma 5. *Suppose that $\mathbf{a}_1, \dots, \mathbf{a}_d \in \mathbb{R}^{d+1}$. Then there exists a non-zero vector $\mathbf{z} \in \mathbb{R}^{d+1}$ such that $\mathbf{a}_i^T \mathbf{z} \geq 0$ for every $1 \leq i \leq d$ and the sum of the coordinates is 0, namely $\sum_{1 \leq i \leq d+1} z_i = 0$.*

Proof. Let $\mathbf{j} \in \mathbb{R}^{d+1}$ be the all 1 vector and define the matrix A as $\{\mathbf{a}_1, \dots, \mathbf{a}_d, \mathbf{j}\}$. If $\det(A) = 0$, then there are non-trivial solutions of $A^T \mathbf{z} = \mathbf{0}$. If $\det(A) \neq 0$ define $\mathbf{a} := (1, \dots, 1, 0)^T \in \mathbb{R}^{d+1}$. There is a unique solution $\mathbf{z}$ of $A^T \mathbf{z} = \mathbf{a}$. Clearly, $\mathbf{z} \neq 0$ so we are done. □

Proof of Theorem 4. Let $\mathbf{y} \in S_n$ be a vector whose support has minimum size among vectors $\mathbf{y}' \in S_n$ satisfying $f(G_i, \mathbf{x}) \leq f(G_i, \mathbf{y}')$ for every $1 \leq i \leq d$. If $\{v_1, v_2, \dots, v_{d+1}\} \subseteq \mathrm{support}(\mathbf{y})$ is an independent set, then for any $\mathbf{z} = (z_1, \dots, z_{d+1}, 0, 0, \dots)^T \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $1 \leq i \leq d$ we have $f(G_i, \mathbf{y} + t\mathbf{z}) = f(G_i, \mathbf{y}) + t(\mathbf{a}_i^T \mathbf{z})$ for some $\mathbf{a}_i \in \mathbb{R}^{d+1}$. Here $\mathbf{a}_i$ depends only on G_i and $\mathbf{y}$, not on $\mathbf{z}$ or t. Apply Lemma 5 to obtain a non-zero vector $\mathbf{z} = (z_1, \dots, z_{d+1}, 0, 0, \dots)^T$ with $\sum_{1 \leq i \leq d+1} z_i = 0$ and $\mathbf{a}_i^T \mathbf{z} \geq 0$ for $1 \leq i \leq d$. Choosing an appropriate $t > 0$ we have $\mathbf{y} + t\mathbf{z} \in S_n$ and $\mathrm{support}(\mathbf{y} + t\mathbf{z}) \subseteq \mathrm{support}(\mathbf{y}) - \{v_j\}$ for some $1 \leq j \leq d+1$. This is a contradiction, so $\mathbf{y}$ has the desired property. □

3 Maximizing the weight of non-triangular edges in a weighted graph

Lemma 6. *Let G_1 be a graph on n vertices $\{v_1, \dots, v_n\}$ and let G_2 be a subgraph of G_1 whose edges are some of the non triangular edges of G_1, $E(G_2) \neq \emptyset$. For every $\mathbf{x} \in S_n$ there exists a subset $K \subseteq V$ and a vector $\mathbf{y} \in S_n$ with support K such that $f(G_1, \mathbf{x}) \leq f(G_1, \mathbf{y})$ and $f(G_2, \mathbf{x}) \leq f(G_2, \mathbf{y})$. Furthermore, the graph $H := G_1[K]$ contains exactly one edge e of G_2 and $H \setminus V(e)$ is a complete graph.*

Proof. By Theorem 4, we know that there is a $\mathbf{y} \in S_n$ such that $f(G_1, \mathbf{x}) \leq f(G_1, \mathbf{y})$, $f(G_2, \mathbf{x}) \leq f(G_2, \mathbf{y})$ and $\alpha(H) \leq 2$. Let $\mathbf{y} = (y_1, \dots, y_n)$ be such a vector whose support has minimal size. We claim that $K := \text{support}(\mathbf{y})$ satisfies the required properties. First we show that the structure of $G_2[K]$ is rather simple, then we show that by finding an appropriate $\mathbf{y}'$ one can further reduce K if $G_2[K]$ has two or more edges.

Recall that $\frac{\partial}{\partial z_k} f$ stands for the partial derivative of the function $f(z_1, z_2, ..., z_n)$ with respect to the variable z_k. Suppose that v_k and $v_h \in K$ are nonadjacent vertices such that

$$\frac{\partial}{\partial y_k} f(G_1, \mathbf{y}) \geq \frac{\partial}{\partial y_h} f(G_1, \mathbf{y}) \quad \text{and} \quad \frac{\partial}{\partial y_k} f(G_2, \mathbf{y}) \geq \frac{\partial}{\partial y_h} f(G_2, \mathbf{y}). \tag{3}$$

In other words, $\sum\{y_\ell : v_k v_\ell \in E(G_i[K])\} \geq \sum\{y_\ell : v_h v_\ell \in E(G_i[K])\}$ for $i = 1, 2$. Define the vector $\mathbf{y}' \in S_n$ by

$$y'_\ell = \begin{cases} y_k + y_h & \ell = k \\ 0 & \ell = h \\ y_\ell & \text{otherwise.} \end{cases}$$

We have $f(G_i, \mathbf{y}) \leq f(G_i, \mathbf{y}')$ for $i \in \{1, 2\}$ and $\text{support}(\mathbf{y}') = K \setminus \{v_h\}$, a contradiction. We conclude that condition (3) does not hold.

Without loss of generality, we may suppose that $v_1 v_2$ is a G_2-edge of H. From now on, in this section if we talk about 'edges', 'degrees' etc., then we always mean H-edges, degree in H, etc., except if it is otherwise stated.

If $f(G_1, \mathbf{y}) \leq 1/4$ then define $\mathbf{y}' = (1/2, 1/2, 0, \dots, 0)$. We obtain $f(G_2, \mathbf{y}) \leq f(G_1, \mathbf{y}) \leq 1/4 = f(G_1, \mathbf{y}') = f(G_2, \mathbf{y}')$. This implies $K = \{1, 2\}$ and we are done. So from now on, we suppose that $f(G_1, \mathbf{y}) > 1/4$. Then the Motzkin-Straus theorem implies that the graph H is not triangle-free.

Claim 1. *There are no two adjacent edges of $G_2[K]$.*

Proof of Claim 1. Assume, to the contrary, that $v_1 v_2$ and $v_1 v_3 \in E(H)$ are G_2 edges. We claim that

$$v_2 \text{ and } v_3 \text{ are non-adjacent, } \deg(v_1) = 2, \text{ and } \\ H \setminus \{v_1, v_2, v_3\} \text{ is a complete graph.} \tag{4}$$

Indeed, v_2 and v_3 are non-adjacent, otherwise the triangle $v_1v_2v_3$ contains G_2 edges. Suppose, to the contrary, that $|N(v_1)| > 2$, i.e., there exists a vertex $v_4 \neq v_2, v_3$, such that $v_1v_4 \in E(H)$. Since $\alpha(H) \leq 2$ and $v_2v_3 \notin E(H)$, without loss of generality, $v_3v_4 \in E(H)$. Then the triangle $v_1v_3v_4$ contains a G_2 edge (namely v_1v_3), a contradiction, so we must have $N(v_1) = \{v_2, v_3\}$. Finally, the condition $\alpha(H) \leq 2$ implies that $K \setminus (N(v_1) \cup \{v_1\})$ induces a complete graph (cf., Figure 2).

The statement (4) already implies that the structure of G_2 edges is rather simple in H. Using Condition (3) and other techniques, we reach a contradiction, considering three possible cases.

Case $1a$**.** Assume that there is no G_2 edge connecting $\{v_1, v_2, v_3\}$ to $K \setminus \{v_1, v_2, v_3\}$.

Then $\frac{\partial}{\partial y_2} f(G_2, \mathbf{y}) = \frac{\partial}{\partial y_3} f(G_2, \mathbf{y})$ (namely, both are y_1). Since v_2 and v_3 are non-adjacent the conditions of (3) hold, a contradiction.

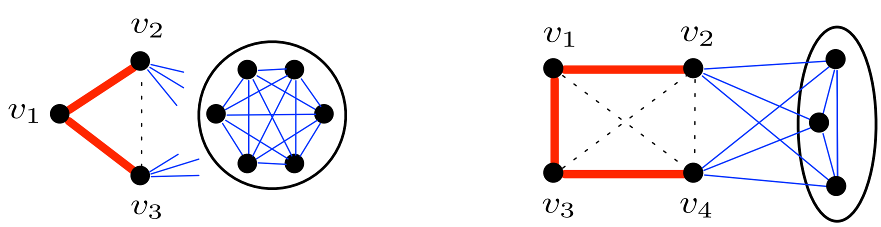

Figure 2: The structure of H in Cases $1a$ and $1b$. The G_2 edges are bold.

Case $1b$**.** Assume that there is a G_2 edge, say v_3v_4, connecting $\{v_1, v_2, v_3\}$ to $K \setminus \{v_1, v_2, v_3\}$ such that $v_2v_4 \notin E(H)$.

According to (4) the set $A := \{v_1, ..., v_4\}$ spans only these three G_2 edges, v_1 and v_3 are degree 2 vertices, and $(K \setminus A) \cup \{v_i\}$ are complete graphs for $i \in \{2, 4\}$. Since H must contain triangles we have $|K \setminus A| \geq 2$ and H does not contain further G_2 edge (see Figure 2). Suppose that $y_1 \geq y_3$. We obtain that

$$\frac{\partial}{\partial y_2} f(G_2, \mathbf{y}) = y_1 \geq \frac{\partial}{\partial y_4} f(G_2, \mathbf{y}) = y_3,$$

and

$$\frac{\partial}{\partial y_2} f(G_1, \mathbf{y}) = y_1 + \sum_{\ell > 4} y_\ell \geq \frac{\partial}{\partial y_4} f(G_1, \mathbf{y}) = y_3 + \sum_{\ell > 4} y_\ell.$$

Since v_2 and v_4 are non-adjacent, this contradicts Condition (3).

Case $1c$**.** Assume that there is a G_2 edge, say v_3v_4, connecting $\{v_1, v_2, v_3\}$ to $K \setminus \{v_1, v_2, v_3\}$ such that $v_2v_4 \in E(H)$.

According to (4) the set $A := \{v_1, ..., v_4\}$ spans only these four edges, v_1 and v_3 have degree 2, and $K \setminus \{v_1, v_3\}$ is a complete graph of size at least 3 (see Figure 3). H does not contain other

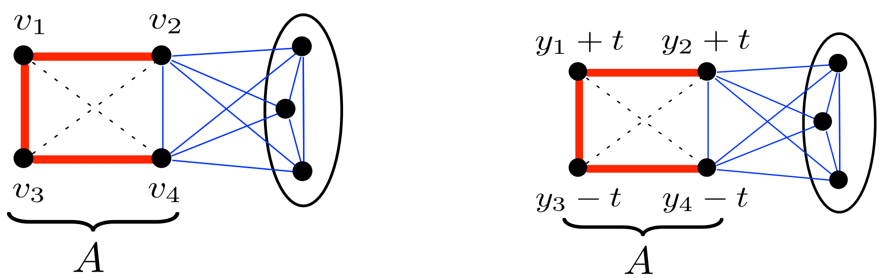

Figure 3: The structure of H in Case 1c, and the change of the weights.

G_2 edges. We have

$$f(G_1,\mathbf{y}) = (y_1+y_4)(y_2+y_3) + (y_2+y_4)\left(\sum_{\ell>4} y_\ell\right) + \sum_{i>j>4}\sum y_i y_j$$

and

$$f(G_2,\mathbf{y}) = y_1y_2 + y_1y_3 + y_3y_4.$$

Substitute $\mathbf{y}' := \mathbf{y}'(t) = \mathbf{y} + t(\mathbf{e}_1+\mathbf{e}_2-\mathbf{e}_3-\mathbf{e}_4)$ into the above equations (Figure 3). Note that $\mathbf{y}' \in S_n$ if $t \in I := [\max\{-y_1,-y_2\}, \min\{y_3,y_4\}]$. We get $f(G_1,\mathbf{y}') = f(G_1,\mathbf{y})$ and

$$f(G_2,\mathbf{y}') - f(G_2,\mathbf{y}) = t^2 + t(y_2-y_4).$$

The right hand side is a convex polynomial of t and it takes its maximum on I in one of the endpoints. Taking this optimal t we obtain that $\max_{t\in I} f(G_2,\mathbf{y}') > f(G_2,\mathbf{y})$ and $|\text{support}(\mathbf{y}')| < |\text{support}(\mathbf{y})|$, a contradiction. This completes the proof of Claim 1 that H has no adjacent G_2 edges. □

Claim 2. *There are no two parallel edges of $G_2[K]$.*

Proof of Claim 2. According to Claim 1, $G_2[K]$ is a matching, $\{v_1v_2,\ v_3v_4, \dots,\ v_{2k-1}v_{2k}\}$. We will show $k=1$. Assume, to the contrary, that v_1v_2 and v_3v_4 are two disjoint G_2 edges of H.

Define $A := \{v_1,\dots,v_4\}$. Since v_1v_2 and v_3v_4 are two non-triangular edges, the set A can contain at most two more edges of H, and those should be disjoint to each other. So without losing generality, we may assume that v_1v_4 and $v_2v_3 \notin E(G_1)$, (cf., Figure 4).

Let $B_i := \{v \in K \setminus A : vv_i \in E(H)\}$ for $1 \le i \le 4$. We claim that $B_1 = B_3$. Indeed, if $v_5 \in B_1$ then $v_2v_5 \notin E(G_1)$, otherwise $\{v_1,v_2,v_5\}$ forms a triangle. Then $v_5v_3 \in E(H)$ otherwise $\{v_2,v_3,v_5\}$ forms an independent set. Hence $v_5 \in B_3$, implying $B_1 \subseteq B_3$. By symmetry $B_3 \subseteq B_1$, we obtain $B_1 = B_3$ and similarly $B_2 = B_4$.

Since v_1v_2 is a G_2 edge we have $B_1 \cap B_2 = \emptyset$ (actually, $\{A, B_1, B_2\}$ is a partition of K). We distinguish two cases.

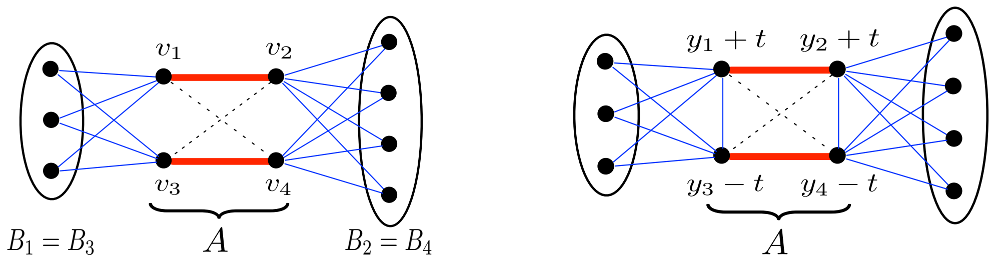

Figure 4: The structure of H in Case 2, and the change of the weights in Case 2b.

Case 2a**.** Assume first that $v_1v_3 \notin E(H)$.

Suppose that $y_2 \geq y_4$. Since no G_2-edge joins A to $K \setminus A$ and $B_1 = B_3$ we obtain that

$$\frac{\partial}{\partial y_1} f(G_2, \mathbf{y}) = y_2 \geq \frac{\partial}{\partial y_3} f(G_2, \mathbf{y}) = y_4,$$

and

$$\frac{\partial}{\partial y_1} f(G_1, \mathbf{y}) = y_2 + \sum_{y_\ell \in B_1} y_\ell \geq \frac{\partial}{\partial y_3} f(G_1, \mathbf{y}) = y_4 + \sum_{y_\ell \in B_1} y_\ell.$$

Since v_1 and v_3 are non-adjacent, this contradicts (3).

So we may assume that A contains the edge v_1v_3. By symmetry, we may assume that A contains the edge v_2v_4, too.

Case 2b**.** Finally, A contains the edges v_1v_3 and v_2v_4 (see Figure 4).

We have

$$\begin{aligned} f(G_1, \mathbf{y}) &= (y_1 + y_4)(y_2 + y_3) \\ &+ (y_1 + y_3)\left(\textstyle\sum_{y_\ell \in B_1} y_\ell\right) + (y_2 + y_4)\left(\textstyle\sum_{y_\ell \in B_2} y_\ell\right) + \sum_{v_i, v_j \notin A,}\ \sum_{v_iv_j \in E(H)} y_iy_j, \end{aligned}$$

and

$$f(G_2, \mathbf{y}) = y_1y_2 + y_3y_4 + \cdots + y_{2k-1}y_{2k}.$$

Substitute $\mathbf{y}' := \mathbf{y}'(t) = \mathbf{y} + t(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)$ into the above equations. Note that $\mathbf{y}' \in S_n$ if $t \in I := [\max\{-y_1, -y_2\}, \min\{y_3, y_4\}]$. We get $f(G_1, \mathbf{y}') = f(G_1, \mathbf{y})$ and

$$f(G_2, \mathbf{y}') - f(G_2, \mathbf{y}) = 2t^2 + t(y_1 + y_2 - y_3 - y_4).$$

The right hand side is convex, it takes its maximum on I in one of the endpoints. Taking this optimal t we obtain that $\max_{t \in I} f(G_2, \mathbf{y}') > f(G_2, \mathbf{y})$ and $|\text{support}(\mathbf{y}')| < |\text{support}(\mathbf{y})|$, a contradiction. This completes the proof of Claim 2. $\square$

The end of the proof of Lemma 6. Claims 1 and 2 imply that H has a unique G_2 edge. We claim that the vertices in H which are not adjacent to any G_2 edge of H induce a clique. To see this, consider two such vertices v_i and v_j. We have $\frac{\partial}{\partial y_i} f(G_2, \mathbf{y}) = 0 = \frac{\partial}{\partial y_j} f(G_2, \mathbf{y})$ so the inequalities of (3) hold. Therefore v_i and v_j must be adjacent to avoid a contradiction. □

4 A continuous lower bound for the number of triangular edges

In this section, by using Lemma 6, we will prove the main result of this paper, i.e., Theorem 2. Recall that $g(n,e) := \min\{e - bc : G \in \mathcal{G}(a,b,c) \text{ with } e(G) \geq e,\ a+b+c=n\}$ (see (2)). We define $t(n,e)$ to be a real valued version of $g(n,e)$ as follows,

$$t(n,e) := \min\{e - bc : a+b+c = n,\ a,b,c \in \mathbb{R}_+,\ \frac{1}{2}a^2 + ab + bc \geq e\}. \tag{5}$$

Obviously, $t(n,e) \leq g(n,e)$ for $n^2/4 \leq e \leq \binom{n}{2}$. Furthermore,

$$g(n,e) - (3/2)n \leq t(n,e). \tag{6}$$

Indeed, suppose that $(a,b,c) \in \mathbb{R}^3_+$ yields the optimal value, $t(n,e) = e - bc$. It is a straightforward calculation to show that the choice of $(a',b',c') := (\lceil a+1 \rceil, \lceil b \rceil, n - a' - b')$ satisfies (2) and the difference between $(e - b'c')$ and $(e - bc)$ is at most $(3/2)n$.

We cannot prove Conjecture 1 that $g(n,e) \leq \mathrm{Tr}(n,e)$ (i.e., that they are equal), but as an application of Lemma 6 we will show that $t(n,e)$ is a lower bound for $\mathrm{Tr}(n,e)$.

Theorem 7. *For $e > n^2/4$ we have $t(n,e) \leq \mathrm{Tr}(n,e)$.*

Proof. Suppose that G_1 is a graph with n vertices, e edges and minimum number of edges in triangles, i.e., G_1 has $\mathrm{Tr}(n,e)$ triangle edges. Let G_2 be the subgraph of G_1 consisting of the edges not in any triangle of G_1. Consider the vector $(1/n)\mathbf{j} = (1/n, 1/n, \ldots, 1/n) \in \mathbb{R}^n$. By Lemma 6 there exists a $\mathbf{y} = (y_1, \ldots, y_n) \in S_n$ with support K such that $G_2[K]$ consists of a single edge, say v_1v_2. Moreover

$$\frac{e}{n^2} = f(G_1, (1/n)\mathbf{j}) \leq f(G_1, \mathbf{y}) \tag{7}$$

and

$$\frac{e - \mathrm{Tr}(n,e)}{n^2} = f(G_2, (1/n)\mathbf{j}) \leq f(G_2, \mathbf{y}) = y_1y_2. \tag{8}$$

Assume that $y_1 \geq y_2$ and define $a := \left(\sum_{k \neq 1,2} y_k\right) n$, $b := y_1 n$, $c := y_2 n$. Then (8) yields that $\mathrm{Tr}(n,e) \geq e - bc$. We claim that the reals a, b, and c satisfy the constraints in (5), hence $e - bc \geq t(n,e)$, completing the proof.

Indeed, since v_1v_2 is not in any triangle, $N(v_1) \cap N(v_2) = \emptyset$, we get from (7) that

$$\begin{aligned} \frac{e}{n^2} &\leq f(G_1, \mathbf{y}) \\ &= y_1y_2 + y_1\Big(\sum_{y_k \in N(v_1), k \neq 2} y_k\Big) + y_2\Big(\sum_{y_k \in N(v_2), k \neq 1} y_k\Big) + \sum_{i<j,\ i,j \neq 1,2} \sum y_iy_j \\ &\leq \frac{bc}{n^2} + \frac{b}{n} \times \frac{a}{n} + \frac{1}{2}\Big(\frac{a}{n}\Big)^2. \end{aligned}$$

□

5 Further problems, minimizing C_{2k+1} edges

In addition to the question of minimizing the number of triangular edges, Erdős, Faudree and Rousseau [3] also considered a conjecture of Erdős [2] regarding pentagonal edges asserting that a graph on n vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges has at most $n^2/36 + O(n)$ non-pentagonal edges. This value can be obtained by considering a graph having two components, a complete graph on $[2n/3] + 1$ vertices and a complete bipartite graph on the rest. This conjecture was mentioned in the papers of Erdős [2] and also in the problem book of Fan Chung and Graham [1].

Erdős, Faudree, and Rousseau [3] proved that if G is a graph with n vertices and at least $\lfloor n^2/4 \rfloor + 1$ edges then for any fixed $k \geq 2$ at least $\frac{11}{144}n^2 - O(n)$ edges of G are in cycles of length $2k+1$. So there is a jump of $\Omega(n^2)$ in the number of C_5-edges, while the construction of $\mathcal{G}(n, e)$ shows that for K_3-edges the change is smoother, $\mathrm{Tr}(n, n^2/4 + x) = O(n\sqrt{x})$.

In a forthcoming paper [5] we give an example of graphs with $\lfloor n^2/4 \rfloor + 1$ edges and $n^2/8(2 + \sqrt{2}) + O(n) = n^2/27.31...$ non-pentagonal edges, disproving Erdős' conjecture. Using the weighted symmetrization method we show that this coefficient is asymptotically the best possible for $e > (n^2/4) + o(n^2)$. On the other hand, we asymptotically establish the conjecture of Erdős that for every $k \geq 3$, the maximum number of non-C_{2k+1} edges in a graph of size exceeding $(n^2/4) + o(n^2)$ is at most $n^2/36 + o(n^2)$, as in the graph of two-components described above.

More generally, given a graph F, one can define $h(n, e, F)$ as the minimum number of F-edges among all graphs of n vertices and e edges. In a forthcoming paper [5] we asymptotically determine $h(n, \lambda n^2, F)$ for any fixed λ, when $1/4 < \lambda < 1/2$ and F is 3-chromatic. Many problems, e.g., an F with a higher chromatic number, or natural generalizations for hypergraphs remain open.

A remark on very dense graphs. One can verify Conjecture 1 for $n \leq 8$ and in general for $e \geq \binom{n}{2} - (3n - 13)$. This and (1) yield the exact value of $\mathrm{Tr}(n, e)$ for all pairs with $n \leq 10$ except $\mathrm{Tr}(10, 27)$. More details can be found in the `arXiv` version [4].

A remark on keeping equalities. Taking $\mathbf{a} := \mathbf{e}_\ell^T \in \mathbb{R}^{d+1}$ for some $1 \leq \ell \leq d$ instead of $\mathbf{a} := (1, \ldots, 1, 0)^T \in \mathbb{R}^{d+1}$ in the proof of Lemma 5 one can obtain a sharper version of it. Namely, there exists a non-zero vector $\mathbf{z} \in \mathbb{R}^{d+1}$ such that $\sum_{1 \leq i \leq d+1} z_i = 0$ and $\mathbf{a}_\ell^T \mathbf{z} \geq 0$, but $\mathbf{a}_i^T \mathbf{z} = 0$ for every $1 \leq i \leq d$, $i \neq \ell$.

This sharper version of Lemma 5 yields a sharper version of Theorem 4. Namely, there exists an appropriate vector $\mathbf{y} \in S_n$ such that $f(G_\ell, \mathbf{x}) \leq f(G_\ell, \mathbf{y})$ and $f(G_i, \mathbf{x}) = f(G_i, \mathbf{y})$ for every $1 \leq i \leq d$, $i \neq \ell$.

Then the proof of Lemma 6 can be adjusted so that given $\ell \in \{1, 2\}$ one can find an appropriate vector $\mathbf{y} \in S_n$ such that $f(G_\ell, \mathbf{x}) = f(G_\ell, \mathbf{y})$ and $f(G_{3-\ell}, \mathbf{x}) \leq f(G_{3-\ell}, \mathbf{y})$.

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New developments (as of May 2016). Since the first public presentations of our results (e.g., in the Combinatorics seminar of the Department Mathematics and Computer Science at Emory University, December 6, 2013, in the Oberwolfach Combinatorics Workshop, Jan 5–11, 2014) and posting the present manuscript on `arXiv` [4] on November 4, 2014, there were (at least) two remarkable achievements.

Gruslys and Letzter [6] using a refined version of the symmetrization method proved that there exists an n_0 such that $\mathrm{Tr}(n, e) = g(n, e)$ for all $n > n_0$. The second part of our Conjecture 1, namely that the extremal graph should be from a $\mathcal{G}(a, b, c)$, is still open.

Grzesik, P. Hu, and Volec [7] using Razborov's flag algebra method showed that every n-vertex graph with $\lfloor n^2/4 \rfloor + 1$ edges has at least $(n^2/4) - n^2/8(2 + \sqrt{2}) - \varepsilon n^2$ pentagonal edges for $n > n_0(\varepsilon)$ for every $\varepsilon > 0$. They also proved that those graphs have at most $n^2/36 + \varepsilon n^2$ C_{2k+1}-edges for $n > n_k(\varepsilon)$ for every $\varepsilon > 0$ and $k \geq 3$. In [5] we were able to prove the same results only for graphs with $\lfloor n^2/4 \rfloor + \varepsilon n^2$ edges (for $n > n_0(k, \varepsilon)$, $k \geq 2$). Let's close with a slightly corrected version of Erdős conjecture.

Conjecture 8. *Suppose that G is an n-vertex graph with e edges, such that $e > n^2/4$ and it has the minimum number of C_{2k+1}-edges, $k \geq 3$, $n > n_k$. Then G is connected and has two blocks, one of them is a complete bipartite graph and the other one is almost complete.*

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