# A stability version for a theorem of Erdős on nonhamiltonian graphs 

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Let $n, d$ be integers with $1 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, and set $h(n, d):=\binom{n-d}{2}+d^{2}$ and $e(n, d):=$ $\max \left\{h(n, d), h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right\}$. Because $h(n, d)$ is quadratic in $d$, there exists a $d_{0}(n)=(n / 6)+O(1)$ such that

$$
e(n, 1)>e(n, 2)>\cdots>e\left(n, d_{0}\right)=e\left(n, d_{0}+1\right)=\cdots=e\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)
$$

A theorem by Erdős states that for $d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, any $n$-vertex nonhamiltonian graph $G$ with minimum degree $\delta(G) \geq d$ has at most $e(n, d)$ edges, and for $d \geq d_{0}(n)$ the unique sharpness example is simply the graph $K_{n}-E\left(K_{\lceil(n+1) / 2\rceil}\right)$. Erdős also presented a sharpness example $H_{n, d}$ for each $1 \leq d \leq d_{0}(n)$.

We show that if $d<d_{0}(n)$ and a 2 -connected, nonhamiltonian $n$-vertex graph $G$ with $\delta(G) \geq$ $d$ has more than $e(n, d+1)$ edges, then $G$ is a subgraph of $H_{n, d}$. Note that $e(n, d)-e(n, d+1)=$ $n-3 d-2 \geq n / 2$ whenever $d<d_{0}(n)-1$.
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Dedicated to the memory of Professor H. Sachs.

## 1 Introduction

We use standard notation. In particular, $V(G)$ denotes the vertex set of a graph $G, E(G)$ denotes the edge set of $G$, and $e(G)=|E(G)|$. Also, if $v \in V(G)$, then $N(v)$ denotes the neighborhood of $v$ and $d(v)=|N(v)|$. Ore [4] proved the following Turán-type result:

Theorem 1 (Ore [4]). If $G$ is a nonhamiltonian graph on $n$ vertices, then $e(G) \leq\binom{ n-1}{2}+1$.
This bound is achieved only for the $n$-vertex graph obtained from the complete graph $K_{n-1}$ by adding a vertex of degree 1. Erdős [2] refined the bound in terms of the minimum degree of the graph:

[^0]Theorem 2 (Erdős [2]). Let $n, d$ be integers with $1 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, and set $h(n, d):=\binom{n-d}{2}+d^{2}$. If $G$ is a nonhamiltonian graph on $n$ vertices with minimum degree $\delta(G) \geq d$, then

$$
e(G) \leq \max \left\{h(n, d), h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right\}=: e(n, d) .
$$

This bound is sharp for all $1 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$.
To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, consider the graph $H_{n, d}$ obtained from a copy of $K_{n-d}$, say with vertex set $A$, by adding $d$ vertices of degree $d$ each of which is adjacent to the same $d$ vertices in $A$. An example of $H_{11,3}$ is below.


Figure 1: $H_{11,3}$
By construction, $H_{n, d}$ has minimum degree $d$, is nonhamiltonian, and $e\left(H_{n, d}\right)=\binom{n-d}{2}+d^{2}=$ $h(n, d)$. Elementary calculation shows that $h(n, d)>h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)$ in the range $1 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ if and only if $d<(n+1) / 6$ and $n$ is odd or $d<(n+4) / 6$ and $n$ is even. Hence there exists a $d_{0}:=d_{0}(n)$ such that

$$
e(n, 1)>e(n, 2)>\cdots>e\left(n, d_{0}\right)=e\left(n, d_{0}+1\right)=\cdots=e\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right),
$$

where $d_{0}(n):=\left\lceil\frac{n+1}{6}\right\rceil$ if $n$ is odd, and $d_{0}(n):=\left\lceil\frac{n+4}{6}\right\rceil$ if $n$ is even. Let $H_{n, d}^{\prime}$ denote the graph that is an edge-disjoint union of two complete graphs $K_{n-d}$ and $K_{d+1}$ sharing one vertex.
The result of this note is the following refinement of Theorem 2.
Theorem 3. Let $n \geq 3$ and $d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. Suppose that $G$ is an $n$-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that

$$
\begin{equation*}
e(G)>e(n, d+1)=\max \left\{h(n, d+1), h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right\} . \tag{1}
\end{equation*}
$$

(So we have $d<d_{0}(n)$.) Then $G$ is a subgraph of either $H_{n, d}$ or $H_{n, d}^{\prime}$.
This is a stability result in the sense that for $d<n / 6$, each 2-connected, nonhamilitonian $n$-vertex graph with minimum degree at least $d$ and "close" to $h(n, d)$ edges is a subgraph of the extremal graph $H_{n, d}$. Note that $h(n, d)-h(n, d+1)=n-3 d-2$ is at least $n / 2$ for $d<d_{0}-1$. Note also that $e\left(H_{n, d}^{\prime}\right)>e(n, d+1)$ only when $d=O(\sqrt{n})$.
We will use the following well-known theorems of Pósa.
Theorem 4 (Pósa [5). Let $n \geq 3$. If $G$ is a nonhamiltonian $n$-vertex graph, then there exists $1 \leq k \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ such that $G$ has a set of $k$ vertices with degree at most $k$.

Theorem 5 (Pósa [6]). Let $n \geq 3,1 \leq \ell<n$ and let $G$ be an $n$-vertex graph such that $d(u)+d(v) \geq n+\ell$ for every non-edge $u v$ in $G$. Then for every linear forest $F$ with $\ell$ edges contained in $G$, the graph $G$ has a hamiltonian cycle containing all edges of $F$.

## 2 Proof of Theorem 3

Call a graph $G$ saturated if $G$ is nonhamiltonian but for each $u v \notin E(G), G+u v$ has a hamiltonian cycle. Ore's proof [4] of Dirac's Theorem [1] yields that

$$
\begin{equation*}
\text { for every n-vertex saturated graph } G \text { and for each uv } \notin E(G), d(u)+d(v) \leq n-1 \text {. } \tag{2}
\end{equation*}
$$

First we show two facts on saturated graphs with many edges.
Lemma 6. Let $G$ be a saturated $n$-vertex graph with $e(G)>h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)$. Then for some $1 \leq k \leq$ $\left\lfloor\frac{n-1}{2}\right\rfloor, V(G)$ contains a subset $D$ of $k$ vertices of degree at most $k$ such that $G-D$ is a complete graph.

Proof. Since $G$ is nonhamiltonian, by Theorem 4 , there exists some $1 \leq k \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ such that $G$ has $k$ vertices with degree at most $k$. Pick the maximum such $k$, and let $D$ be the set of the vertices with degree at most $k$. Since $e(G)>h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right), k<\left\lfloor\frac{n-1}{2}\right\rfloor$. So, by the maximality of $k,|D|=k$. Suppose there exist $x, y \in V(G)-D$ such that $x y \notin E(G)$. Among all such pairs, choose $x$ and $y$ with the maximum $d(x)$. Since $y \notin D, d(y)>k$. Let $D^{\prime}:=V(G)-N(x)-\{x\}$ and $k^{\prime}:=\left|D^{\prime}\right|=n-1-d(x)$. By (2),

$$
\begin{equation*}
d(z) \leq n-1-d(x)=k^{\prime} \text { for all } z \in D^{\prime} . \tag{3}
\end{equation*}
$$

So $D^{\prime}$ is a set of $k^{\prime}$ vertices of degree at most $k^{\prime}$. Since $y \in D^{\prime}, k^{\prime} \geq d(y)>k$. Thus by the maximality of $k$, we get $k^{\prime}=n-1-d(x)>\left\lfloor\frac{n-1}{2}\right\rfloor$. Equivalently, $d(x)<\left\lceil\frac{n-1}{2}\right\rceil$. For all $z \in D^{\prime}+\{x\}$, either $z \in D$ where $d(z) \leq k \leq\left\lfloor\frac{n-1}{2}\right\rfloor$, or $z \in V(G)-D$, and so $d(z) \leq d(x) \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. It follows that $e(G) \leq h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)$, a contradiction.

Lemma 7. Under the conditions of Lemma 6 , if $k=\delta(G)$, then $G=H_{n, \delta(G)}$ or $G=H_{n, \delta(G)}^{\prime}$.
Proof. Set $d:=\delta(G)$, and let $D$ be a set of $d$ vertices with degree at most $d$. Let $u \in D$. Since $\delta(G) \geq|D|=d$, $u$ has a neighbor $w \in V(G)-D$. Consider any $v \in D-\{u\}$. By Lemma 6, $w$ is adjacent to all of $V(G)-D-\{w\}$. It also is adjacent to $u$, therefore its degree is at least $n-d$. We obtain

$$
d(w)+d(v) \geq(n-d)+d=n
$$

Then by (2), $w$ is adjacent to $v$, and hence $w$ is adjacent to all vertices of $D$.
Let $W$ be the set of vertices in $V(G)-D$ having a neighbor in $D$. We have obtained that $W \neq \emptyset$ and

$$
\begin{equation*}
N(u) \cap(V(G)-D)=W \text { for all } u \in D \tag{4}
\end{equation*}
$$

Let $G^{\prime}=G[D \cup W]$. If $|W|=1$, then $G=H_{n, d}^{\prime}$. If $\left|V\left(G^{\prime}\right)\right|=2 d$, then by (4), each vertex $u \in D$ has the same $d$ neighbors in $V(G)-D$. Because $d(u)=d, D$ is an independent set. Thus $G=H_{n, d}$. Otherwise, $d+2 \leq\left|V\left(G^{\prime}\right)\right| \leq 2 d-1,|D| \geq 2$.

Fix a pair of vertices $w_{1}, w_{2} \in W$. For any $x, y \in V\left(G^{\prime}\right)$,

$$
d(x)+d(y) \geq d+d \geq\left|V\left(G^{\prime}\right)\right|+1
$$

Therefore by Theorem 5, $G^{\prime}$ has a hamiltonian cycle $C$ that uses the edge $w_{1} w_{2}$. Since $G^{\prime \prime}:=$ $G-\left(V\left(G^{\prime}\right)-\left\{w_{1}, w_{2}\right\}\right)$ is a complete graph, it contains a hamiltonian $w_{1}, w_{2}$-path $P$. Then $P \cup\left(C-w_{1} w_{2}\right)$ is a hamiltonian cycle of $G$, a contradiction.

Proof of Theorem 3. Suppose that an $n$-vertex, nonhamiltonian graph $G$ satisfies the constraints of Theorem 3 for some $1 \leq d \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. We may assume $G$ is saturated, since if a graph containing $G$ is a subgraph of $H_{n, d}$ or $H_{n, d}^{\prime}$, then $G$ is as well.

By Lemma 6, $G$ has a set $D$ of $k \leq\left\lfloor\frac{n-1}{2}\right\rfloor$ vertices with degree at most $k$ such that $G-D$ is a complete graph. Therefore $e(G) \leq\binom{ n-k}{2}+k^{2}=h(n, k)$. If $k>d$, then $e(G) \leq \max \{h(n, d+$ $\left.1), h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right\}=e(n, d+1)$, a contradiction. Thus $k \leq d$. Furthermore, $k \geq \delta(G) \geq d$, and hence $k=d$. Also, since $\left.e(G)>h\left(n,\left\lfloor\frac{n-1}{2}\right\rfloor\right)\right)$, we have $d+1 \leq d_{0}(n) \leq(n+8) / 6$. Applying Lemma 7 completes the proof.

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