Modelling argumentation on Axiom of Choice in ASPIC-END – Technical report

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Abstract

In this technical report, we present an application of the structured argumentation methodology to a debate in the foundations of mathematics. We work with ASPIC-END, a recently proposed adaptation of the structured argumentation framework ASPIC+ which can incorporate debates about logical principles, natural deduction style arguments and explanations. We apply this framework to build a preliminary formal model of parts of the debate that mathematicians had about the Axiom of Choice in the early 20th century. Furthermore, we briefly discuss the insight into the strengths and drawbacks of the modeling capacities of ASPIC-END that we have gained from producing this model.

1 Introduction

Structured argumentation theory allows for a fine-grained model of argumentation and argumentative reasoning based on a formal language in which arguments and counterarguments can be constructed. In this technical report, we present preliminary results on the applicability of structured argumentation theory to argumentation about the foundations of mathematics.

The work presented in this technical report is based on the ASPIC+ framework for structured argumentation [see Modgil and Prakken, 2014]. In a recent wokshop paper [see Dauphin and Cramer, 2017] we have proposed some adaptations to ASPIC+ that were mainly motivated by the goal to model arguments and explanations presented in the literature on philosophical logic. These adaptations gave rise to the modified structured argumentation framework ASPIC-END.

In this technical report, we present a model that formalizes parts of the debate that mathematicians had about the Axiom of Choice in the early 20th century [see Moore, 1982]. In 1904, the German mathematician Ernst Zermelo published a proof of the Well-Ordering Theorem, in which he explicitly refered to a set-theoretic principle that came to be known as the Axiom of Choice Zermelo [1904]. In the first years after its publication, this proof received a lot of critique, a significant part of which questioned the general validity of the Axiom of Choice (see Moore [1982]). In the long run, however, the proof got accepted, as the Axiom of Choice got accepted as a valid part of the de-facto standard foundational theory for mathematics, *Zermelo-Fraenkel set theory with the Axiom of Choice* (*ZFC*).

The two critiques of Zermelo's Axiom of Choice that we consider in this paper are those of Peano [1906] and Lebesgue [see Hadamard et al., 1905]. Furthermore, we consider the counterarguments to these cirtiques put forward by Zermelo [1907] and by Hadamard [see Hadamard et al., 1905]. Given that the model still leaves out many contributions to that debate and additionally simplifies some of the contributions that it does take into account, we consider it to only be a preliminary model that we plan to extend in the future. However, we hope that this more extensive model gives some insight into the strengths and drawbacks of the modeling capacities of ASPIC-END, as well as inspiration for further research into this direction.

In Section 9, we discuss how the model could be extended in order to provide a more complete picture of the debate and to link it to debates on related topics within the foundations of mathematics and logic.

2 Some general remarks of the model

We are modeling some sophisticated argumentation that often involves a lot of implicit reasoning steps that are not made explicit. We attempt our formalization of the arguments to be as faithful as possible to the original intention of the authors of the arguments in question, but we cannot avoid making choices about the implicit reasoning steps that could potentially be made differently.

Generally, the purely mathematical and purely logical demonstrations and reasoning is formalized using intuitively strict rules, while the philosophical and metamathematical argumentation and reasoning is formalized using defeasible rules. Most of the attacks between arguments attack defeasible arguments, i.e. philosophical or metamathematical arguments. But given that some of the mathematical and logical principles that were applied in the mathematical and logical reasoning that we model, e.g. the Axiom of Choice, the non-constructivist parts of classical logic and the (inconsistent) set comprehension principle, are attacked by some philosophical or metamathematical arguments, there are also some arguments using only intuitively strict rules that get attacked. Of course, by the design of ASPIC-END, all such attacks have to be undercuts.

All arguments have to start from some assumptions, which are not explicitly backed up by further arguments. Such an assumption is formalized in our model as a premise, i.e. as a rule with no antecedent. Depending on whether this premise is of a purely mathematico-logical nature or has philosophical/metamathematical aspects, it gets modelled either as an intuitively strict rule without antecedent, also called an *axiom*, or as a defeasible rule without antecedent, also called a *defeasible premise*.

Instead of presenting the language and the set of rules of our model at once, we introduce them step by step as we show how to formalize various arguments put forward during the debate. We explicitly mention all the rules needed in out model. The language of our model is a standard first-order language over a vocabulary of predicate symbols, function symbols and constants. This vocabulary is not explicitly listed, but is evident from the list of rules that we put forward and from the explanations we provide about the formalization. We have chosen the names of all predicate symbols, function symbols and constants in such a way that they resemble either the actual words used in the debate[1]

[1]All the arguments from the debate cited in this paper were originally presented in lan-

or the words of some reformulation of the cited arguments that we use when explaining the debate and our formalization of it.

The name of a rule, i.e. the formula that expresses the acceptability of a rule and whose negation can be used to undercut an argumetn using the rule, is denoted with $\text{accept}(\rho)$, where ρ is a constant symbol that refers to the rule as a syntactic object. For some rules we explicitly specify the constant ρ that refers to it by writing (ρ) (for some constant symbol ρ) in front of the rule. However, for other rules we do not specify such a constant symbol to refer to it, as it is mostly not needed.

The inference rules of intuitionistic logic (which are also included in classical logic) are not called into question by any mathematician involved in the debate that we model, so these rules never get undercut. Here are the schemes of intuitively strict rules[2] that are required to model intuitionistic logic in ASPIC-END:

$$\begin{aligned}
(\neg\text{-Elim})\ & \varphi, \neg\varphi \rightsquigarrow \bot;\\
(MP)\ & \varphi, (\varphi \supset \psi) \rightsquigarrow \psi;\\
(\wedge\text{-Intro})\ & \varphi, \psi \rightsquigarrow (\varphi \wedge \psi);\\
(\wedge\text{-Elim}_\text{L})\ & (\varphi \wedge \psi) \rightsquigarrow \varphi;\\
(\wedge\text{-Elim}_\text{R})\ & (\varphi \wedge \psi) \rightsquigarrow \psi;\\
(\vee\text{-Intro}_\text{L})\ & \varphi \rightsquigarrow (\varphi \vee \psi);\\
(\vee\text{-Intro}_\text{R})\ & \psi \rightsquigarrow (\varphi \vee \psi);\\
(=\text{-Intro})\ & \rightsquigarrow t = t;\\
(=\text{-Elim})\ & \varphi, t_1 = t_2 \rightsquigarrow \varphi[t_1/t_2]\\
(\forall\text{-Elim})\ & \forall x\ \varphi \rightsquigarrow \varphi[t/x];\\
(\exists\text{-Intro})\ & \varphi[t/x] \rightsquigarrow \exists x\ \varphi;\\
(\exists\text{-Elim})\ & \exists x\ \varphi, \forall x\ (\varphi \supset \psi) \rightsquigarrow \psi
\end{aligned}$$

For the final rule scheme in this list, i.e. ($\exists$-Elim), we only include in our model those instances for which ψ does not contain x as a free variable.

Note that the conditions that ASPIC-END imposes on variable declarations ensure that ($\exists$-Elim) can only be applied when ψ does not have x as a free variable.

In order to simplify the exposition of our model, we sometimes omit implicit reasoning steps that involve only these inference rules of intuitionistic logic. We use the notation $(A_1, \dots, A_n \vdash \psi)$ for an argument that uses multiple of these rules to get from the conclusions of arguments A_1, ..., A_n to the conclusion ψ. Since these rules can never be undercut, this omission does not lead to any attacks on such arguments being overlooked.

guages other than English. We generally work with their translations to English provided by Moore [1982] and Kennedy [1973]. In very few cases we have made minor modifications to the translation that Moore made of German texts in order to make the translation more faithful to the German original. By "actual words used in the debate" we here mean the English translation of these words.

[2]In this techincal report, we use the word *rule* in the way in which it is usually used in the structured argumentation literature. There is one important difference between this usage of *rule* and the way the word is usually used in the logical literature outside of structured argumentation theory: A *rule*, as the word is used in structured argumentation theory, is what would normally be called an instance of a rule. For this reason, it makes sense to speak of a *rule scheme*, which is what would normally be just called a rule.

Furthermore, we will in one place refer to the rule scheme of double negation elimination, which, when added to the above rule schemes, gives a formalization of classical first-order logic in ASPIC-END:

$$(\neg\neg\text{-Elim}_{\varphi})\ \neg\neg\varphi \rightsquigarrow \varphi$$

3 Zermelo's explicitation of the Axiom of Choice

Since 1871, various mathematicians had produced proofs which relied on making infinitely many arbitrary choices, i.e. relied on what came to be known as the Axiom of Choice (see Moore [1982], Chapter 1). However, before 1904, these mathematicians were not aware of the fact that these proofs require a novel mathematical principle (*ibid.*). The first mathematician who explicitly talked about the problem of making infinitely many arbitrary choices was Giuseppe Peano Peano [1890] (page 280), but Peano talked of it as something that cannot be done, i.e. he rejected the kind of inferences that the Axiom of Choice allows. This particular detail of his work did not influence any mathematicians other than some of his Italian colleagues (see Moore [1982], page 76). The first time that a mathematician made explicit reference to the problem of making infinitely arbitrary choices while considering this a valid form of inference was the paper of Zermelo that presented a proof of the Well-Ordering Theorem Zermelo [1904].

"even for an infinite totality of [non-empty] sets there always exist mappings by which each set corresponds to one of its elements"

$$(AC) \rightsquigarrow \forall M\ (\forall m \in M\ \text{non-empty\_set}(m) \supset \exists f\ (\text{domain}(f) = M \wedge \forall m \in M\ f(m) \in m)))$$

"this logical principle cannot be reduced to a still simpler one, but is used everywhere in mathematical deduction without hesitation. So for example the general validity of the theorem that the number of subsets into which a set is partitioned is less than or equal to the number of its elements, cannot be demonstrated otherwise than by assigning to each subset one of its elements."

Zermelo does not explicitly explain what he means by reducing a principle to a simpler principle, so we just take this to be a primitive notion of his meta-mathematical reasoning, rendered in our model by the unary predicate simple. We assume that there is a defeasible premise that asserts that (AC) is simple in this sense. This defeasible premise is justified by the fact that Zermelo presumably put some considerable mathematical effort into attempting to reduce (AC) to a simple principle before claiming in print that this cannot be done.[3]

$$\Rightarrow \text{simple}(AC)$$

[3] The proof of Cohen [1963] that if ZF is consistent, then it does not prove the Axiom of Choice, can be viewed as a confirmation of Zermelo's claim that (AC) cannot be reduced to a simpler principle. However, to get to this conclusion, one would still require some judgement about the simplicity of (AC) in comparison to some other principles that have turned out to imply it, and unlike Cohen's proof, these judgements are not of a purely mathematico-logical nature, and should thus be formalized using defeasible rather than intuitively strict rules. So while progress has been made since Zermelo's claim in 1904 that would allow us to put forward stronger and more elaborate arguments in favour of this claim, these arguments would still be of a defeasible nature, just like Zermelo's original claim.

We formalize Zermelo's claim that his principle "is used everywhere in mathematical deduction without hesitation" as a conjunction of two claims, one asserting that (AC) is widely used in mathematical practice (formalized using the unary predicate widely_used), and one asserting that no mathematician has called this usage to doubt (formalized using the unary predicate calls_to_doubt and the function symbol usage). The first claim is backed up by the example that he produces in the second sentence of the quotation (see the formalization below), while the second claim is made without argument, and therefore gets formalized as a defeasible premise:

$$(\rho_2^{Z^{04}}) \Rightarrow \neg\exists x \text{ calls\_to\_doubt}(x, \text{usage}(AC))$$

The theorem mentioned in the second sentence of the quotation is nowadays usually called the Partition Principle. The content of this theorem is not relevant for the argumentative force of the argument that Zermelo puts forward here. All that is relevant is that he puts forward an example of a theorem from the mathematical literature for which a proof has been published and accepted by the community, and that this proof makes makes implicit use of the principle AC that Zermelo is defending here. As the content of the theorem is not relevant, we simplify the explosition of the model by replacing it by a constant symbol PP ("Partition Principle"). Zermelo makes the claim that this theorem "cannot be demonstrated otherwise than by assigning to each subset one of its elements", which we formalize as a conjunction of two claims: that there is a proof that *demonstrates* PP, and that any proof that demonstrates PP *uses* (AC). We do not assign any specific formal meaning to the words *demonstrate* and *uses*, but consider them primitive concepts of Zermelo's metamathematical reasoning.[4] These two claims are not supported by a further argument, so they get modeled as defeasible premises:[5] [6]

$$\Rightarrow \exists p \text{ demonstrates}(p, PP);$$
$$\Rightarrow \forall p\ (\text{demonstrates}(p, PP) \supset \text{uses}(p, AC))$$

Zermelo uses this example of the Partition Principle to substantiate his claim that his principle AC is widely used in mathematical deduction. Of course, an argument that concludes that a principle is widely used based on evidence for one single usage of the principle is a comparatively weak argument. But it is still stronger than making the same claim based on no evidence at all, which is what would have been the case if Zermelo had not given this single example. In order to formally account for this inference from a single example of a usage

[4]Even though he did not explicitly use a word that translates to the English word *uses*, his usage of "otherwise than by" suggests that he had in mind some notion of a principle being involved in a demonstration, which we decide to render with the word *uses*.

[5]The defeasible premise that there is a proof that demonstates PP could be replaced by an intuitively strict argument, if we chose to extend the model by the actual mathematical proof from the literature that Zermelo is referring to here. But this would require also extending the model with intuitively strict rules that formalize our semi-formal reasoning about syntactical entities like proofs and their conclusions.

[6]Interestingly, a certain reasonable formalization of the terms *demonstrates* and *uses* in terms of the (historically later) formal systems ZF and ZFC gives the second defeasible premise a reading which is still an open problem in set theory to this day: It is still unknown whether the full force of the Axiom of Choice is needed to prove the Partition Principle, or whether a weaker choice principle is sufficient (though it is known that the Particion Principle cannot be proved in ZF alone).

of AC to the conclusion that it is widely used, we add the following scheme[7] of defeasible rules to our model:

$$\exists p, t \; (\text{demonstrates}(p, t) \wedge \text{uses}(p, \rho)) \Rightarrow \text{widely\_used}(\rho)$$

It is clear that Zermelo puts forward his claims about the simplicity of, the wide use of and the lack of doubt about his principle AC in order to corroborate the acceptability of AC as a basic mathematical principle that does not require mathematical proof. In other words, he implicitly makes use of a metamathematical principle, according to which the simplicity of, wide use of and lack of doubt about a mathematical principle allow one to defeasibly infer the acceptibility of said mathematical principle. This is formalized through the folowing scheme of defeasible rules:

$$\text{simple}(\rho), \text{widely\_used}(\rho), \neg\exists x \; \text{calls\_to\_doubt}(x, \text{usage}(\rho)) \Rightarrow \text{accept}(\rho)$$

Now that we have presented all the rules require to formalize this argument of Zermelo in favour of the acceptibility of AC, we describe the arguments that Zermelo constructs from these rules:

$$\begin{aligned}
Z_1^{04} &= (\Rightarrow \text{simple}(AC)) \\
Z_2^{04} &= (\Rightarrow \neg\exists x \; \text{calls_to_doubt}(x, \text{usage}(AC))) \\
Z_3^{04} &= (\Rightarrow \exists p \; \text{demonstrates}(p, PP)) \\
Z_4^{04} &= (\Rightarrow \forall p \; (\text{demonstrates}(p, PP) \supset \text{uses}(p, AC))) \\
Z_5^{04} &= (Z_3^{04}, Z_4^{04} \vdash \exists p, t \; (\text{demonstrates}(p, t) \wedge \text{uses}(p, AC))) \\
Z_6^{04} &= (Z_5^{06} \Rightarrow \text{widely_used}(AC)) \\
Z_7^{04} &= (Z_1^{06}, Z_6^{06}, Z_2^{06} \Rightarrow \text{accept}(AC))
\end{aligned}$$

4 Peano's response to Zermelo's proof

In 1906, the Italian mathematician Giuseppe Peano published a note in the Rendiconti del Circolo matematico di Palermo in which he responded to Zermelo's proof [Peano, 1906] by criticizing his principle AC. First of all, he points out that he had previously considered and rejected this inference pattern:

> "This assumption, which occurs in several books, was already considered by me in the year 1890, in Math. Ann., 37, p. 210: 'one may not apply an infinite number of times an arbitrary law according to which to a class a is made to correspond an individual of that class ...' " [Peano, 1906, p. 208]

Note that this can be viewed as a counterargument against Zermelo's claim that the principle has been applied in mathematics "without hesitation", based on evidence that Peano himself has previously called this principle to doubt. We formalize this counterargument as follows:

$$(\rho_1^{P^{06}}) \Rightarrow \text{calls\_to\_doubt}(\text{Peano}, \text{usage}(AC))$$

[7]It is a scheme, as ρ may be substituted by an arbitrary term of our language. The particular instance of the scheme that we will make use of is the one where ρ is AC.

Peano then explains how a single arbitrary choice from a non-empty class can be formalized in his *Formulario mathematico*, a semi-formal notational system for mathematical propositions and proofs that he had devised:

> "the form of argument 'if I arbitrarily choose an element x of class a, then proposition p (which does not contain x) follows' is reducible to the form
>
> $$\begin{array}{ll} \exists\, a & (1) \\ x \,\epsilon\, a. \supset .p & (2) \\ (1).(2). \supset .p & \end{array}$$
>
> 'If there exists an a, and if from $x \epsilon a$ follows proposition p, then proposition p may be affirmed.'
> This is the form of argument called 'elimination of x' in Formulario, V, p. 12, Prop. 3.1." [Peano, 1906, p. 208]

Note that –apart from irrelevant notational differences – his elimination of x is the same as our rule scheme ∃-Elim.

The point that Peano is making here is that for any informal argument that makes one arbitrary choice, there is a formalization of this argument in his Formulario system that makes use of elimination of x once. We formalize this by the following defeasible premise:[8]

$$\Rightarrow \forall a, b\ (\text{arb\_choices}(a, 1) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1))$$

Here $\text{arb\_choices}(a, n)$ means that argument a is an informal argument that contains an inference step in which n arbitrary choices are made. $\text{uses}(b, \rho, n)$ means that b is a derivation that makes use of rule ρ n times.

Peano continues:

> "The assumption of two successive arbitrary elements has the form:
>
> $$\begin{array}{ll} \exists\, a & (1) \\ x \,\epsilon\, a. \supset .\exists\, b & (2) \\ x \,\epsilon\, a.y \,\epsilon\, b. \supset .p & (3) \\ (1).(2).(3). \supset .p\ \text{''} & \end{array}$$
>
> [Peano, 1906, p. 208]

In this case, four propositions are involved (three hypotheses and a conclusion). Note that in his semi-formal system the intermediate step resulting from just one application of elimination of x does not need to be written down, whereas in our fully formal system such an omission is not allowed, so that there are actually five propositions involved in two consecutive applications of ∃-Elim).

The point that Peano is making here is that for any informal argument that makes two arbitrary choices, there is a formalization of this argument in his

[8] Given that it is a statement about the connection between something informal and something formal, it is not purely mathematico-logical, but has a metamathematical character that justifies the choice of a defeasible premise instead of an (intuitively strict) axiom.

Formulario system that makes use of elimination of x twice. We formalize this by the following defeasible premise:

$$\Rightarrow \forall a, b\ (\text{arb\_choices}(a, 2) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2))$$

At this point, Peano is ready to make his main argument against Zermelo's principle AC:

> "The assumption of two arbitrary elements x and y leads to an argument with three hypotheses (1), (2), (3), and a thesis (4). In general the assumption of n successive arbitrary elements leads to an argument which consists of $n+2$ propositions. Therefore we may not suppose $n = \infty$, that is, we cannot construct an argument with an infinite number of propositions." [Peano, 1906, p. 209]

To formalize this argument, we first need some rules that allow him to conclude the claims of the first two sentences of this citation from the above mentioned defeasible premises. Note that he makes a generalization about n arbitrary choices from evidence about what happens in the case of one or two arbitrary choice. This kind of generalization is of course not a mathematical demonstration, but a common form of argumentation in the informal exposition of mathematical ideas. So we formalize it as a defeasible rule:

$$\begin{aligned}
&\forall a, b\ (\text{arb_choices}(a, 1) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1)),\\
&\quad \forall a, b\ (\text{arb_choices}(a, 2) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2))\\
&\quad \Rightarrow \forall n, a, b\ (\text{arb_choices}(a, n) \wedge \text{formalizes}(b, a) \wedge \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, n));
\end{aligned}$$

The property of his system that n applications of elimination of x lead to an argument that involves $n+2$ propositions, on the other hand, can be considered a mathematical statement about his system, so we formalize it using an intuitively strict rule (which, given the lack of proof of this statement, is here just treated as an axiom, i.e. an intuitively strict rule without antecedent):

$$\rightsquigarrow \forall n\ (\text{uses}(a, x\text{-elim}, n) \supset \text{involves\_propositions}(a, n+2))$$

For concluding the claim in the final sentence, we need two axioms that formalize his mathematical assumptions that $\infty + 2 = \infty$ and that there is no Formulario argument that involves infinitely many propositions:

$$\begin{aligned}
&\rightsquigarrow \infty + 2 = \infty;\\
&\rightsquigarrow \neg\exists a\ (\text{Formulario}(a) \wedge \text{involves_propositions}(a, \infty))
\end{aligned}$$

These rules allow us to conclude that no informal argument that makes infinitely many choices can be formalized in the Formulario. Peano considers this an attack on the usability of Zermelo's principle AC. Here Peano is making the implicit assumption that an informal mathematical argument is acceptable if and only if it can be formalized in the Formulario. To get an attack on AC, we additionally need the premise that the acceptability of the Axiom of Choice implies the acceptability of some informal argument that makes infinitely many arbitrary choices:

$$(\rho_4^{P^{06}}) \Rightarrow \forall a\ (\text{accept}(a) \equiv \exists b(\text{formalizes}(b,a) \land \text{Formulario}(b)));$$
$$\rightsquigarrow (\text{accept}(AC) \supset \exists a\ (\text{accept}(a) \land \text{arb\_choices}(a,\infty)))$$

Now that we presented all the rules we require to model Peano's argument against Zermelo's principle AC, let us describe the arguments that Peano constructs from these rules:

$P_1^{06} = (\Rightarrow \text{calls\_to\_doubt}(\text{Peano}, \text{usage}(AC)))$
$P_2^{06} = (P_1^{06} \rightsquigarrow \exists x\ \text{calls\_to\_doubt}(x, \text{usage}(AC)))$
$P_3^{06} = (\Rightarrow \forall a, b\ (\text{arb\_choices}(a,1) \land \text{formalizes}(b,a) \land \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 1)))$
$P_4^{06} = (\Rightarrow \forall a, b\ (\text{arb\_choices}(a,2) \land \text{formalizes}(b,a) \land \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, 2)))$
$P_5^{06} = (P_3^{06}, P_4^{06} \Rightarrow \forall n, a, b\ (\text{arb\_choices}(a,n) \land \text{formalizes}(b,a) \land \text{Formulario}(b) \supset \text{uses}(b, x\text{-elim}, n)))$
$P_6^{06} = (\rightsquigarrow \forall n\ (\text{uses}(a, x\text{-elim}, n) \supset \text{involves\_propositions}(a, n+2)))$
$P_7^{06} = (\rightsquigarrow \infty + 2 = \infty)$
$P_8^{06} = (\rightsquigarrow \neg\exists a\ (\text{Formulario}(a) \land \text{involves\_propositions}(a,\infty)))$
$P_9^{06} = (\Rightarrow \forall a(\text{accept}(a) \equiv \exists b\ (\text{formalizes}(b,a) \land \text{Formulario}(b))))$
$P_{10}^{06} = (P_5^{06}, P_6^{06}, P_7^{06}, P_8^{06}, P_9^{06} \vdash \neg\exists a\ (\text{accept}(a) \land \text{arb\_choices}(a,\infty)))$
$P_{11}^{06} = (\rightsquigarrow (\text{accept}(AC) \supset \exists a\ (\text{accept}(a) \land \text{arb\_choices}(a,\infty))))$
$P_{12}^{06} = (\mathsf{Assume}_\neg(\text{accept}(AC)))$
$P_{13}^{06} = (P_{10}^{06}, P_{11}^{06}, P_{12}^{06} \vdash \bot)$
$P_{14}^{06} = (\mathsf{ProofbyContrad}(P_{13}^{06}, \neg\text{accept}(AC)))$

5 Zermelo's response to Peano

In 1908, Zermelo wrote an article responding to multiple critiques of his proof of the Well-Ordering Theorem. The article contains the following response to Peano's arguments:

> "First of all, how does Peano arrive at his own fundamental principles and how does he justify admitting them into the *Formulaire*, since he cannot prove them either? Obviously, through analyzing the rules of inference that have historically been recognized as valid and by referring both to the intuitive evidence for the rules and to their necessity for science – considerations which may be argued just as well for the disputed Principles. This Axiom, without being formulated in a scholastic manner, has been applied successfully, and very frequently, in the most diverse mathematical fields, particularly set theory, by R. Dedekind, G. Cantor, F. Bernstein, A. Schoenflies, and J. König among others. Such extensive usage of a principle can only be explained through its self-evidence, which, naturally, must not be confused with its provability. While this self-evidence may be

> subjective to a certain degree, it is in any case an essential source of mathematical principles, though not a basis for mathematical proofs. Thus Peano's statement, that self-evidence has nothing to do with mathematics, does not do justice to obvious facts. However, what can be objectively decided, the question of necessity for science, I would like now to submit to judgment by presenting a series of elementary and fundamental theorems and problems, which, in my opinion, could not be settled without the Axiom of Choice."

Zermelo continues by listing seven theorems of set theory, which he believed not to be provable without the Axiom of Choice. Some of these theorems were already widely considered as proven among set theorists of his time, e.g. the theorem that a countable union of countable sets is contable. Others had been implicitly assumed by many set theorists without explicit proof, e.g. that every Dedekind finite set is finite. He also repeated in the list the Partition Principle mentioned in his 1904 article.

In the second sentence of this quotation, Zermelo mentions three criteria for admitting fundamental principles: being historically recognized as valid, being intuitively evident, and being necessary for science. In this passage he seems interested in providing strong evidence for admitting a fundamental principle by satisfying all three of these criteria. We formalize this by the following defeasible rule scheme:

$$\text{hist\_rec\_as\_valid}(\rho) \wedge \text{int\_evident}(\rho) \wedge \text{nec\_for\_science}(\rho) \Rightarrow \text{accept}(\rho)$$

Zermelo cites the frequent usage of the Axiom of Choice as evidence both for its being historically recognized as valid, and for it intuitive evidence. This is formlized by the following rules:

$$\begin{aligned}
&\Rightarrow \text{used}(\text{Dedekind}, AC);\\
&\Rightarrow \text{used}(\text{Cantor}, AC);\\
&\Rightarrow \text{used}(\text{Bernstein}, AC);\\
&\Rightarrow \text{used}(\text{Schoenflies}, AC);\\
&\Rightarrow \text{used}(\text{König}, AC);\\
&\text{used}(\text{Dedekind}, \rho) \wedge \text{used}(\text{Cantor}, \rho) \wedge \text{used}(\text{Bernstein}, \rho) \wedge\\
&\quad \text{used}(\text{Schoenflies}, \rho) \wedge \text{used}(\text{König}, \rho) \Rightarrow \text{widely_used}(\rho);\\
&\text{widely_used}(\rho) \Rightarrow \text{hist_rec_as_valid}(\rho);\\
&\text{widely_used}(\rho) \Rightarrow \text{int_evident}(\rho);
\end{aligned}$$

Finally, the seven theorems that Zermelo puts forward as examples for where the Axiom of Choice is needed serve as evidence for the Axiom of Choice being necessary for science. As in the case of the Particion Principle that Zermelo already mentioned in his 1904 article, the precise content of these theorems is not of great importance for the argumentative power of his argument. So we will replace the theorems other than PP, which we have already given a name in Section 3, by the placeholder names $Th2$, ..., $Th7$ corresponding to the numbering used by Zermelo in his paper:

$$\Rightarrow \exists p \; \text{demonstrates}(p, Th2);$$

$\Rightarrow \forall p$ (demonstrates$(p, Th2) \supset$ uses(p, AC));
$\Rightarrow \exists p$ demonstrates$(p, Th3)$;
$\Rightarrow \forall p$ (demonstrates$(p, Th3) \supset$ uses(p, AC));
$\Rightarrow \exists p$ demonstrates$(p, Th4)$;
$\Rightarrow \forall p$ (demonstrates$(p, Th4) \supset$ uses(p, AC));
$\Rightarrow \exists p$ demonstrates$(p, Th5)$;
$\Rightarrow \forall p$ (demonstrates$(p, Th5) \supset$ uses(p, AC));
$\Rightarrow \exists p$ demonstrates$(p, Th6)$;
$\Rightarrow \forall p$ (demonstrates$(p, Th6) \supset$ uses(p, AC));
$\Rightarrow \exists p$ demonstrates$(p, Th7)$;
$\Rightarrow \forall p$ (demonstrates$(p, Th7) \supset$ uses(p, AC));
$\exists p$ (demonstrates$(p, PP) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th2) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th3) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th4) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th5) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th6) \wedge$ uses(p, ρ)), $\exists p$ (demonstrates$(p, Th7) \wedge$ uses(p, ρ)) $\Rightarrow$ nec_for_science(ρ)

Now the rules provided in this subsection can be combined into a new argument in favour of the acceptability of the Axiom of Choice:

$Z_1^{08} = (\Rightarrow \text{used(Dedekind}, AC))$
$Z_2^{08} = (\Rightarrow \text{used(Cantor}, AC))$
$Z_3^{08} = (\Rightarrow \text{used(Bernstein}, AC))$
$Z_4^{08} = (\Rightarrow \text{used(Schoenflies}, AC))$
$Z_5^{08} = (\Rightarrow \text{used(König}, AC))$
$Z_6^{08} = (Z_1^{08}, Z_2^{08}, Z_3^{08}, Z_4^{08}, Z_5^{08} \Rightarrow \text{widely\_used}(AC))$
$Z_7^{08} = (Z_6^{08} \Rightarrow \text{hist\_rec\_as\_valid}(AC))$
$Z_8^{08} = (Z_6^{08} \Rightarrow \text{int\_evident}(AC))$
$Z_9^{08} = (Z_3^{04}, Z_4^{04} \vdash \exists p\ (\text{demonstrates}(p, PP) \wedge \text{uses}(p, AC)))$
$Z_{10}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th2))$
$Z_{11}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th2) \supset \text{uses}(p, AC)))$
$Z_{12}^{08} = (Z_{10}^{08}, Z_{11}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th2) \wedge \text{uses}(p, AC)))$
$Z_{13}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th3))$
$Z_{14}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th3) \supset \text{uses}(p, AC)))$
$Z_{15}^{08} = (Z_{13}^{08}, Z_{14}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th3) \wedge \text{uses}(p, AC)))$
$Z_{16}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th4))$
$Z_{17}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th4) \supset \text{uses}(p, AC)))$
$Z_{18}^{08} = (Z_{16}^{08}, Z_{17}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th4) \wedge \text{uses}(p, AC)))$
$Z_{19}^{08} = (\Rightarrow \exists p\ \text{demonstrates}(p, Th5))$
$Z_{20}^{08} = (\Rightarrow \forall p\ (\text{demonstrates}(p, Th5) \supset \text{uses}(p, AC)))$
$Z_{21}^{08} = (Z_{19}^{08}, Z_{20}^{08} \vdash \exists p\ (\text{demonstrates}(p, Th5) \wedge \text{uses}(p, AC)))$

$$
\begin{aligned}
Z_{22}^{08} &= (\Rightarrow \exists p \text{ demonstrates}(p, Th6)) \\
Z_{23}^{08} &= (\Rightarrow \forall p \text{ (demonstrates}(p, Th6) \supset \text{uses}(p, AC))) \\
Z_{24}^{08} &= (Z_{22}^{08}, Z_{23}^{08} \vdash \exists p \text{ (demonstrates}(p, Th6) \wedge \text{uses}(p, AC))) \\
Z_{25}^{08} &= (\Rightarrow \exists p \text{ demonstrates}(p, Th7)) \\
Z_{26}^{08} &= (\Rightarrow \forall p \text{ (demonstrates}(p, Th7) \supset \text{uses}(p, AC))) \\
Z_{27}^{08} &= (Z_{25}^{08}, Z_{26}^{08} \vdash \exists p \text{ (demonstrates}(p, Th7) \wedge \text{uses}(p, AC))) \\
Z_{28}^{08} &= (Z_{9}^{08}, Z_{12}^{08}, Z_{15}^{08}, Z_{18}^{08}, Z_{21}^{08}, Z_{24}^{08}, Z_{27}^{08} \Rightarrow \text{nec\_for\_science}(AC)) \\
Z_{29}^{08} &= (Z_{7}^{08}, Z_{8}^{08}, Z_{28}^{08} \Rightarrow \text{accept}(AC))
\end{aligned}
$$

Even though the conclusion of Peano's argument P_{14}^{06} is the negation of the conclusion of Zermelo's new argument Z_{29}^{08}, this does not constitute a direct attack from Z_{29}^{08} to P_{14}^{06}. The reason for this is that the conclusion of P_{14}^{06} is attained by a proof by contradiction, and such a proof cannot be rebutted on the top level. However, it is possible to construct from Z_{29}^{08} an argument attack on P_{14}^{06} by making use of some of Peano's subarguments that Zermelo does not intend to attack:

$$
\begin{aligned}
Z_{30}^{08} &= (\mathsf{Assume}_{\neg}(\forall a \text{ (accept}(a) \equiv \exists b(\text{formalizes}(b, a) \wedge \text{Formulario}(b))))); \\
Z_{31}^{08} &= (Z_{30}^{08}, P_{5}^{06}, P_{6}^{06}, P_{7}^{06}, P_{8}^{06}, P_{11}^{06} \vdash \neg\text{accept}(AC)); \\
Z_{32}^{08} &= (Z_{29}^{08}, Z_{31}^{08} \rightsquigarrow \bot); \\
Z_{33}^{08} &= (\mathsf{ProofbyContrad}(Z_{32}^{08}, \neg\forall a \text{ (accept}(a) \equiv \exists b(\text{formalizes}(b, a) \wedge \text{Formulario}(b))))));
\end{aligned}
$$

Now Z_{33}^{08} directly rebuts P_{9}^{06}, which is a subargument of P_{14}^{06}, so Z_{33}^{08} indirectly attacks P_{14}^{06} as well. This is how our model formalizes Zermelo's attack on Peano's argument. Note that without taking into account preferences, there would also be an attack back from Peano's argument P_{14}^{06} onto Zermelo's argument Z_{29}^{08}. We explain in Section 7 how this is avoided through the use of preferences.

6 Lebesgue's and Hadamard's letters

In this subsection, we extend the model with a somewhat simplified formalization of arguments from two more mathematicians – Henri Lebesgue and Jacques Hadamard – who had participated in this debate before Peano's response. The purpose of this addition to the model is mainly to illustrate the possibility of attacks on classical inference rules in the context of such foundational debates. In order to not complicate the exposition of the model much more, we simplify the formalization of these arguments a bit, while acknowledging that this simplification makes the formalization less faithful to the wording used by Lebesgue and Hadamard than it could be.

In 1905, French mathematician Émile Borel, who himself had critiqued Zermelo's proof of the Well-Ordering Theorem, asked his colleague Henri Lebesgue to comment on the proof. Lebesgue responded in a letter to Borel that shortly afterwards got published together with four other letters on the topic in the Bulletin de la Société mathématique de France [Hadamard et al., 1905]. Lebesgue rejected Zermelo's statement that he had proved the Well-Ordering Theorem, and a central statement in his justification for this rejection is the following:

> "I believe that we can only build solidly by granting that it is impossible to demonstrate the existence of an object without defining it."

Lebesgue attributes this principle to the German mathematician Leopold Kronecker, who is now often considered a forerunner of later constructivist and intuitionistic approaches to mathematics. Lebesgue does not make any precise statement about which forms of inference involving existential statements are acceptable and which ones are not. However, it is clear that he intends this to be an attack on the existence claim that the Axiom of Choice makes, namely that there exists a certain choice function. At the same time, it is fair to assume that this statement puts him at odds with any non-constructive proof of an existential statements. So in order to capture the argumentative force of this claim, we assume that it implies (through two intuitively strict rules) both a rejection of the Axiom of Choice (AC) and a rejection of double negation elimination applied to an existential statement ($\neg\neg$-Elim$_{\exists x\ \psi}$). But in order to keep the formalization simple, we do not formalize the internal structure of this claim, but instead formalize it as a propositional variable.

Lebesgue does not put forward any argument to support this belief other than attributing the idea to Kronecker. For this reason, we have decided to model it as a defeasible premise:

$$
\begin{aligned}
(\rho_1^{L^{05}})\ &\Rightarrow \text{existence\_proof\_requires\_definition};\\
&\text{existence\_proof\_requires\_definition} \rightsquigarrow \neg\text{accept}(AC);\\
&\text{existence\_proof\_requires\_definition} \rightsquigarrow \neg\text{accept}(\neg\neg\text{-Elim}_{\exists x\ \psi})
\end{aligned}
$$

Borel sent a copy of Lebesgue's response to Jacques Hadamard, who reacted to it in another letter, which was published together with Lebesgue's letter in the Bulletin de la Société mathématique de France. In this letter, Hadamard defends Zermelo's proof against Lebesgue's critique. In this letter, he calls the following argument the "essence of the debate":

> "From the invention of the infinitesimal calculus to the present, it seems to me, the essential progress in mathematics has resulted from successively annexing notions which, for the Greeks or the Renaissance geometers or the predecessors of Riemann, were "outside mathematics" because it was impossible to describe them."

In order to keep the exposition of the model simple – just as for Lebesgue's argument – we will not analyse the internal structure of this claim, but just formalize it as a propositional variable and defeasible premise, to which we assign the argumentative force that it was intended to have by including rule $\rho_2^{H^{05}}$ that allows it to be used to attack Lebesgue's argument:

$$
\begin{aligned}
&\Rightarrow \text{progress\_by\_accepting\_existence\_of\_undescribables};\\
(\rho_2^{H^{05}})\ &\text{progress\_by\_accepting\_existence\_of\_undescribables} \Rightarrow \neg\text{existence\_proof\_requires\_definition}
\end{aligned}
$$

We can describe the arguments that Lebesgue and Hadamard construct:

$$
\begin{aligned}
L_1^{05} &= (\Rightarrow \text{existence\_proof\_requires\_definition}) \\
L_2^{05} &= (L_1^{05} \rightsquigarrow \neg\text{accept}(AC)) \\
L_3^{05} &= (L_1^{05} \rightsquigarrow \neg\text{accept}(\neg\neg\text{-Elim}_{\exists x\ \psi})) \\
H_1^{05} &= (\Rightarrow \text{progress\_by\_accepting\_existence\_of\_undescribables}) \\
H_2^{05} &= (H_1^{05} \Rightarrow \neg\text{existence\_proof\_requires\_definition})
\end{aligned}
$$

Now while H_2^{05} rebuts L_1^{05} directly and thus rebuts L_2^{05} and L_3^{05} indirectly, L_1^{05} also rebuts H_2^{05}. The preference ordering we consider in the next section does not give a preference to one of these arguments over the other, so these rebuts are successful in both directions. This looks like we have no way to make up our mind between accepting Hadamard's and Lebesgue's arguments in the current model. But assuming, as we will in the next section, that all rules in Section 5 are prefered to $\rho_1^{L^{05}}$, we can construct the following successful attack on Lebesgue's argument:

$$
\begin{aligned}
I_1 &= (\mathsf{Assume}_{\neg}(\text{existence\_proof\_requires\_definition})) \\
I_2 &= (I_1 \rightsquigarrow \neg\text{accept}(AC)) \\
I_3 &= (Z_{29}^{08}, I_2 \rightsquigarrow \bot) \\
I_4 &= (\mathsf{ProofbyContrad}(I_3, \neg\text{existence\_proof\_requires\_definition}))
\end{aligned}
$$

Now I_4 rebuts L_1^{05}. This is an example of an implicit argument that is not explicitly stated in the debate that we model, but that can be derived from other rules in the model. One of the strengths of our methodological approach is precisely that it allows to identify such implicit arguments that no one has put forward, but that could be put forward and that could have a relevance influence on the outcome of the debate.

7 Preferences in our model

Without imposing preferences on the set of rules, all attacks in our model would become *practically* bidirectional. By this we mean that even though there can be a unidirection attack from some argument A to some argument B, in such a case there will always be an attack back onto A from some argument B' that is closely related to B and accepted in the same circumstances as B. If we had included in our model a formalization of the proof of the Well-Ordering Theorem, then some attacks would have been unidirectional (not just formally, but also in the practical sense just alluded) even without preferences, as they would be undercuts of the use of the Axiom of Choice in the proof of the Well-Ordering Theorem, but as the model stands currently, it does not involve such undercuts.

In order to make the model more interesting and more realistic, it is therefore a good idea to include in it some preference order on the rules, which gives rise to a preference order on the arguments. One drawback of our methodology is that it gives no methodological guidance on how to select a preference order on the rules. So for now, we just have to follow our common sense of the relative strength of different rules and different argument. We will just specify some instances of rules being prefered to other rules, leaving most pairs of rules uncomparable on the preference order, as comparison is only needed for some pairs of rules.

The defeasible ($\rho_2^{Z^{04}}$) of Zermelo's 1904 argument, which claims that no one has called the Axiom of Choice into doubt, is clearly weaker than Peano's defeasible rule ($\rho_1^{P^{06}}$) that claims that Peano has called the Axiom of Choice into doubt, as Peano can know better than Zermelo what he has called into doubt, and can even provide a reference to a publication, where he has called this inference pattern into doubt in print. So we assume $\rho_2^{Z^{04}} < \rho_1^{P^{06}}$.

Furthermore, the rules that Zermelo requires for his 1908 argument are comparatively strong: For example, he makes claims about certain people having used the Axiom of Choice implicitly, which can be verified by reading the proofs produced by the mathematicians in question. The rule that allows him to conclude frequent usage of the Axiom of Choice from five cited instances of such usage is clearly stronger than the similar rule from 1904, by which he made this conclusion based on one instance of such usage. Also the central premise used to conclude the acceptability of the Axiom of Choice based on three criteria seems to be a philosophically strong point of his argument. In contrast, the two unsupported rules that stand in conflict with Zermelo's 1908 argumentation, namely Peano's rule $\rho_4^{P^{06}}$, which claims that all acceptable informal mathematical arguments can be formalized in the Formulario, and Lebesgue's rule $\rho_1^{L^{05}}$ that claims that proving the existence of an object requires defining it, are weaker than those rules from Zermelo's 1908 argumentation. So we assume that each of $\rho_4^{P^{06}}$ and $\rho_1^{L^{05}}$ is weaker than all the rules introduced in Section 5.

8 Depicting the relevant arguments

The specified set of rules of our model allow for infinitely many arguments to be constructed, so that the EAF corresponding to the model will also be infinite. However, only a small finite subset of this infinite EAF contains attacks that are relevant for the overall status of the acceptability of the Axiom of Choice, which was the focus of attention of the debate that we have formally modeled. For example, an implicit argument similar to I_4 could be built based on Z_7^{04} instead of on $Z_2^{08}9$, but as Z_7^{04} will not be accepted in any argumentative core extension of the overall EAF, this implicit argument will also never be accepted. In this subsection, we present only the small subset of relevant arguments, as well as all defeats between them.

The restriction of the single argumentative extension to this set of relevant arguments is $\{P_2^{06}, H_2^{05}, Z_{33}^{08}, Z_{29}^{08}, I4\}$, which means that these argument are accepted in our model, while $\{14^{06}$, Z_2^{04}, Z_7^{04}, L_2^{05} and L_1^{05} are rejected. Note that this set of relevant arguments contains two arguments with conclusion accept(AC), namely Z_7^{04} and Z_{29}^{08}. While the first one gets rejected, the second one gets accepted, so that overall, the claim accept(AC) gets accepted in our model.

Of course, this acceptance of the Axiom of Choice in our formal model of

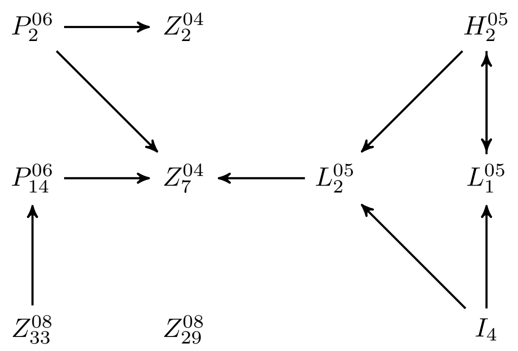

Figure 1: The relevant arguments and attacks from the example

the debate is to a certain extend an artifact of the choice of arguments that we formalized and of the preference order that we imposed. We could have gotten an opposite result if we had chosen to formalize only weak arguments in favour of the Axiom of Choice and strong arguments against it, or if we had just made significantly different judgements about the preference order on the rules involved in our model. So at the current level of development, such a model cannot be seriously defended as a method for deciding which side in a debate is right. What it can do, however, is to help us discover relevant implicit arguments like argument I_4 in our model, to help us get a more precise understanding of what assumptions are made and what is at stake in a given debate, and to point towards weaknesses of the current methodology of structured argumentation theory, like the lack of a methodological guidance for choosing a preference order on the rules.

9 Conclusion and proposed extensions to the model

The parts of the debate presented and formalized in this subsection were, of course, only a small part of the debate that mathematicians had about the Axiom of Choice in the early 20th century, and additionally some of the considered arguments have been formalized in a simplified way. So it is obvious that the model could be expanded to a more extensive formal model of that debate. One obvious extension that has already been alluded to above is to include the proof of the Well-Ordering Theorem, so that attacks on the Axiom of Choice would also be attacks on the Well-Ordering Theorem, as actually intended by the mathematicians involved in the debate.

Some of the points that were raised during the debate touch on other issues from the foundations of mathematics that were discussed at the time. For example, the German mathematician Felix Bernstein criticized Zermelo's proof of the Well-Ordering Theorem not for the usage of the Axiom of Choice, but for its similarity to Burali-Forti's Paradox [see Moore, 1982, p. 110]. Bernstein had somewhat peculiar ideas about how Burali-Forti's Paradox should be resolved, ideas which later turned out not to be tenable, but which at the time led him to think that the resolution of Burali-Forti's Paradox also blocks the possibility of a construction that Zermelo used in his proof of the Well-Ordering Theorem. The face that this idea of Bernstein, unlike rejection of the Axiom of Choice, turned out to not be a viable position, should be explainable by a formal model that incorporates his argumentation.

Zermelo wrote his 1908 response to his critiques [Zermelo, 1907] in conjunction with another paper [Zermelo, 1908], in which he proposed an axiomatization of set theory including his new Axiom of Choice as well as other set-theoretic principles. This axiomatization, which after later modifications by Fraenkel gave rise to ZFC, also had to avoid the two set-theoretic paradoxes that were hotly discussed at the time, namely Russel's Paradox and Burali-Forti's Paradox. An interesting extension of the model from this paper would be one that covers these as well as other competing resolutions to these paradoxes. This will also bring into the picture the notion of explanation of a paradox defined in this paper.

A model of the debate about these paradoxes could also be naturally combinedd with a model of the debate about semantic paradoxes like the Liar Paradox, which we have already looked at superficially in the model in Section 4 of Dauphin and Cramer [2017]. Semantic paradoxes are a topic that many philosophical logicians continue to work on and that has given rise to a number of relatively novel non-classical logics like paraconsistent logic [see Priest, 2006], paracomplete logic [see Field, 2008] and substructural logics [see Beall and Murzi, 2013]. This area of research is characterized by a combination of formal rigour, philosophical depth and debate about the acceptability of various logical principles, which is likely to make it a fruitful field for testing the applicability of structured argumentation theory to debates in the formal sciences. As works like that of Field [2006] show, the topic of semantic paradoxes is also connected to the philosophical interpretation of Gödel's Second Incompleteness Theorem, which has also been studied intensively within the philosophy of mathematics.

An overarching formal model of these foundational debates across multiple formal sciences is certainly still a distant goal. But given the potential insights that it could provide into foundational research in the long run, this distant goal could become a driving force for research on structured argumentation models of debates in the formal sciences.

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