Games and Boolean models

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Giovanni Rossi

Department of Computer Science and Engineering DISI

University of Bologna, email: roxyjean@gmail.com

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1 Introduction

- Game theory firstly appears in 1944 with the book *Games and Economic Behavior* by von-Neumann and Morgestern [30], almost entirely devoted to strategic or non-cooperative games, but still also including a model of cooperative games. Shapley 1953 paper [26] next initiates a systematic study of cooperative coalitional games, known in discrete mathematics as pseudo-Boolean (set) functions [8], namely real-valued functions defined on the Boolean lattice of subsets of a finite set [1]. The first and second halfs of the course shall be respectively devoted to non-cooperative and cooperative games. Key concepts associated with these games are the equilibrium and its generalizations for the former, and the value or solution for the latter. Strategic equilibria basically are situations where everyone is playing a best response to his/her opponents, while a value or solution of coalitional games is, roughly speaking, a worth-sharing criterion specifying how to reward players with the fruits of their cooperation.

- For a n-set $N = \{1, \ldots, n\}$ of players, a non-cooperative game shall consist of a finite product space $\mathbb{S}_1 \times \cdots \times \mathbb{S}_n$ of strategies, and n utilities or payoff functions $u_i : \mathbb{S}_1 \times \cdots \times \mathbb{S}_n \to \mathbb{R}$ measuring how each player $i \in N$ evaluates strategy profiles $s = (s_1, \ldots, s_n) \in \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$. This is the branch of game theory where the notorius prisoner's dilemma and Nash equilibrium apply [19], while a cooperative coalitional game is a set function $v : 2^N \to \mathbb{R}$, where $2^N = \{A : A \subseteq N\}$ is the 2^n-set of coalitions A or subsets of N, and $v(A)$ is thought of as the worth of cooperation among all (and only) players $i \in A$ (or coalition members) [25].

- Cooperative game theory leads to deal with Boolean models (whence the name of the course) because coalitional games v are in fact pseudo-Boolean functions $f^v : \{0,1\}^n \to \mathbb{R}$, while (strictly) Boolean functions have form $f : \{0,1\}^n \to \{0,1\}$. Indeed, power set 2^N bijectively corresponds to the 2^n-set $\{0,1\}^n$ of vertices of the n-dimensional unit hypercube $[0,1]^n$. In particular, *simple* coalitional games $v : 2^N \to \{0,1\}$ (or Boolean functions $f^v : \{0,1\}^n \to \{0,1\}$) are defined to satisfy both *monotonicity*, namely $A \supseteq A \Rightarrow v(A) \geq v(B)$ for all $A, B \in 2^N$, and $v(\emptyset) = 0 = 1 - v(N)$.

- Although cooperative games thus naturally lead to deal with Boolean models, which in turn appear in a wide variety of both theoretical and applicative scenarios [8, 11, 12], still Boolean settings also characterize important strategic environments such as minority games, that formalize interaction between financial agents in a "buy-or-sell world" and also attract considerable attention from the statistical mechanics community [10]. After some definitions and notations contained in the following section, these lecture notes begin with the first issue traditionally addressed in non-cooperative game (and microeconomic [19]) theory, namely how to represent players' preferences over strategy profiles (these latter initially regarded as generic alternatives $x_1, \ldots, x_m \in X$), and under what conditions i's preferences ($i \in N$) over the finite set or product space $\mathbb{S}_1 \times \cdots \times \mathbb{S}_n$ of strategy profiles are representable through a utility function $u_i : \mathbb{S}_1 \times \cdots \times \mathbb{S}_n \to \mathbb{R}$. The whole course material is then organized as follows:

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2 Preliminaries

- Some sets of numbers [1, pp. 1-8]:
  - $\mathbb{N} = \{1, 2, \ldots\}$ natural numbers (countably infinite),
  - $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integer numbers (countably infinite),
  - $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ positive integer numbers (countably infinite),
  - $\mathbb{R}$ real numbers; $\mathbb{R}_+$ positive real numbers (uncountable).
- For $1 < n \in \mathbb{N}$ and $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}}$, let $r \in \mathbb{R}, \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ denote respectively a real number and a n-vector of real numbers, where $\in$ reads "belongs to" or "is an element of".

2.1 Sets

- $\emptyset$ is the empty set, while $|X|$ denotes the cardinality (or number of elements) of a set X (hence $|\emptyset| = 0$).
- For any two sets X, Y the following definitions apply:
  - intersection $X \cap Y = \{x : x \in X \text{ and } x \in Y\}$,
  - union $X \cup Y = \{x : x \in X \text{ and/or } x \in Y\}$,
  - difference $X \backslash Y = \{x : x \in X, x \notin Y\}$,
  - inclusion $X \subseteq Y$ means that every $x \in X$ also satisfies $x \in Y$ (clearly any set X satisfies $X \supseteq \emptyset$),
  - symmetric difference $X \Delta Y = (X \backslash Y) \cup (Y \backslash X) = (X \cup Y) \backslash (X \cap Y)$,
  - proper inclusion $X \subset Y \Leftrightarrow X \subseteq Y, X \neq Y$.
- Clearly $Y \supseteq X \underset{entails}{\Rightarrow} |Y| \geq |X|$ as well as $Y \supset X \Rightarrow |Y| > |X|$.
- For a finite m-set X (i.e. $|X| = m$), power set $2^X = \{Y : Y \subseteq X\}$ contains the 2^m subsets of X.
- For the above sets of numbers, $\mathbb{R} \supset \mathbb{Z} \supset \mathbb{N} \subset \mathbb{Z}_+ \subset \mathbb{R}_+$, but $\mathbb{R} \not\subseteq \mathbb{R}^n$.
- Intervals: for any two real numbers $a, b \in \mathbb{R}$, with $a < b$, define
  - closed interval $[a, b] = \{r \in \mathbb{R} : a \leq r \leq b\}$,
  - open interval $(a, b) = \{r \in \mathbb{R} : a < r < b\} = [a, b] \backslash \{a, b\}$,
  - half-open interval $(a, b] = \{r \in \mathbb{R} : a < r \leq b\} = [a, b] \backslash \{a\}$ as well as $[a, b) = \{r \in \mathbb{R} : a \leq r < b\} = [a, b] \backslash \{b\}$.

  (The closed unit interval $[0, 1]$, as any interval of reals, is uncountable.)
- In the sequel, parenthesis $(\cdot, \cdot)$ and $\{\cdot, \cdot\}$ shall also denote respectively ordered and unordered pairs, hence for generic indices i, j (and mostly for $i, j \in N$), there are two ordered pairs $(i, j) \neq (j, i)$ and a single unordered one $\{i, j\} = \{j, i\}$.

2.2 Mappings

- Consider two finite sets $X = \{x_1, \dots x_m\}$ and Y. A mapping $f : X \to Y$ maps each element $x \in X$ into an element $f(x) = y \in Y$.
- The image of f is $im(f) = \bigcup_{x \in X} f(x) \subseteq Y$.
- The kernel of f is $ker(f) = \bigcup_{y \in im(f)} f^{-1}(y)$, the union involving pair-wise disjoint subsets of X, i.e. any two of which have empty intersection. Hence $ker(f) = P = \{A_1, \dots, A_k\}$ is a partition of X, namely an unordered collection of non-empty and pair-wise disjoint subsets of X, called "blocks", whose union is X. That is,
  - $\emptyset \neq A_l \in 2^X$ for $1 \leq l \leq k$,
  - $A_l \cap A_{l'} = \emptyset$ for $1 \leq l < l' \leq k$,
  - $A_1 \cup \dots \cup A_k = X$.

  In particular, $ker(f) = P = \{A_1, \dots, A_k\}$ means that:

  (i) $k = |im(f)|$ or $im(f) = \{y_1, \dots, y_k\}$, and

  (ii) for each block $A_l, 1 \leq l \leq k$ of P there is a (distinct) $y_l \in im(f)$ such that $A_l = \{x : x \in X, f(x) = y_l\}$.
- A mapping $f : X \to Y$ is
  - surjective if $im(f) = Y$,
  - injective if its kernel $ker(f) = P_\perp = \{\{x_1\}, \dots, \{x_m\}\}$ is the *finest or bottom partition* $P_\perp$ (of X), consisting of m singletons.
  - bijective if it is both injective and surjective, entailing $|X| = |Y|$.
- As each $x \in X$ can be mapped into $|Y|$ distinct $y \in Y$, there are $|Y|^{|X|}$ mappings $f : X \to Y$, and the m^m mappings $f : X \to X$ can be grouped according to their kernel $ker(f)$. Specifically, those m mappings f defined by $f(x) = x_k$ for all $x \in X$, hence obtained by varying $k = 1, \dots, m$, all have the same kernel $ker(f) = P^\top = \{X\}$ given by the *coarsest or top partition* $P^\top$ (of X), namely consisting of a single (whole) block. At the opposite extreme, there are $m!$ bijections $f : X \to X$, i.e. such that $ker(f) = P_\perp$; they are permutations $\pi : \{1, \dots, m\} \to \{1, \dots, m\}$ or elements of the symmetric group $\mathcal{S}_m$ described below.

2.3 Posets: subsets and partitions

- As subsets and partitions of a finite set shall appear rather often throughout the course, it is best to immediately introduce the following basics concepts, definitions and notations [1, 13]. Let $M = \{1, \dots, m\}$, where the first m natural numbers are the elements of M and more generally also represent a labeling of the m elements of a generic set $X = \{x_1, \dots, x_m\}$.
- $(2^M, \supseteq)$ is a fundamental poset (partially ordered set), and $(2^M, \cap, \cup)$ is the Boolean lattice of subsets of M. Apart from proper inclusion $\supset$ already defined, another order relation induced by $\supseteq$ is the *covering relation* $\supset^*$ defined by $A \supset^* B \Leftrightarrow A \supset B, |A| = |B| + 1$ $(A, B \in 2^M)$.

- Although intersection $\cap$ and union $\cup$ are commonly defined as in Section 2.1, still in terms of lattices they are respectively the meet and join operators. Hence $A \cap B$ is the largest subset (of M) included in both A, B, and similarly $A \cup B$ is the smallest subset including both A, B.

- There are $\sum_{0\leq k\leq m}\binom{m}{k} = 2^m = |2^M|$ subsets of M, where the number of k-subsets is $\binom{m}{k} = \frac{m!}{k!(m-k)!} = \binom{m}{m-k}$, i.e. equal to the number of $m-k$-subsets, while factorial $m!$ is defined by $0! := 1$ and $m! = m(m-1)!$.

- The set $\mathcal{P}^M$ of partitions $P = \{A_1, \ldots, A_k\}$ of M also is a fundamental poset $(\mathcal{P}^M, \geqslant)$, where coarsening relation $\geqslant$ (which differs from greater-or-equal $\geq$ between real numbers) is definded as follows: for $P, Q \in \mathcal{P}^M$, the former is coarser than the latter (or the latter is finer than the former), denoted by $P \geqslant Q$, if for each $B \in Q$ there is $A \in P$ such that $A \supseteq B$. Proper coarsening $P > Q$ thus means $P \geqslant Q, P \neq Q$ (i.e. there are at least two blocks $B, B' \in Q$ and a block $A \in P$ such that $A \supseteq (B \cup B')$), while the covering relation $>^*$ is $P >^* Q \Leftrightarrow P > Q, |P| = |Q| - 1$ (i.e. P obtains by merging exactly two blocks of Q).

- $(\mathcal{P}^M, \wedge, \vee)$ is perhaps the main example of geometric lattice [1]. The meet $\wedge$ stands for "coarsest-finer-than" and similarly the join $\vee$ for "finest-coarser-than". Precisely, for $P, Q \in \mathcal{P}^M$ and any $A \in P, B \in Q$ such that $\emptyset \neq A \cap B$,

  - $A \cap B$ is a block of $P \wedge Q$,

  - $A \cup B$ is included in a block of $P \vee Q$.

  The number of partitions of a m-set is the Bell number $\mathcal{B}_m$ obeying recursion $\mathcal{B}_m = \sum_{0\leq k<m}\binom{m-1}{k}\mathcal{B}_k$, where $\mathcal{B}_0 := 1$.

2.4 Formalizing preferences: three ways

- In Section 3, a (rational) preference $\succsim$ over a generic m-set X of alternatives shall be looked at in three different ways:

  - as a *binary relation* or subset $R^{\succsim} \subseteq X \times X$ (i.e. $R^{\succsim} \in 2^{X\times X}$) of ordered pairs of alternatives satisfying certain conditions, namely reflexivity, transitivity and completeness;

  - as an *ordered partition* $\mathfrak{P} = (A_1, \ldots, A_{|P|})$ of X, where the notation $(\cdot,\cdot)$ and $\{\cdot,\cdot\}$ for ordered and unordered pairs is extended to partitions, hence every partition $P = \{A_1, \ldots, A_{|P|}\}$ of X as above corresponds to $|P|!$ distinct ordered partitions $\mathfrak{P} = (A_1, \ldots, A_{|P|})$, each given by a distinct ordering of the $|P|$ blocks of P.

  - as a *subgroup of permutations* π, which are the $m!$ bijective mappings $\pi : M \to M$, i.e. whose kernel is the finest partition of M (see above). The $m!$-set $\mathcal{S}_m = \mathcal{S}(M)$ of permutations is a main example of algebraic group [17], with identity element id defined by $id(i) = i$ for all $i \in M$ and with product "$\circ$" defined by $(\hat{\pi}\circ\tilde{\pi})(i) = \hat{\pi}(\tilde{\pi}(i))$ for all $\hat{\pi}, \tilde{\pi} \in \mathcal{S}_m$. It may be checked that $(\mathcal{S}_m, id, \circ)$ is a group:

  - $(\hat{\pi}\circ\tilde{\pi}) \in \mathcal{S}_m$ for all $\hat{\pi}, \tilde{\pi} \in \mathcal{S}_m$,

  - $\pi\circ(\hat{\pi}\circ\tilde{\pi}) = (\pi\circ\hat{\pi})\circ\tilde{\pi}$ for all $\pi, \hat{\pi}, \tilde{\pi} \in \mathcal{S}_m$,

- $id \circ \pi = \pi \circ id = \pi$ for all $\pi \in \mathcal{S}_m$,
- for every $\pi \in \mathcal{S}_m$ there is $\pi^{-1} \in \mathcal{S}_m$ such that $\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = id$.

3 Preferences

- In the non-cooperative games to be dealt with, the product space of strategy profiles over which players $i \in N$ have preferences in form of a utility function $u_i : \mathbb{S}_1 \times \cdots \times \mathbb{S}_n \to \mathbb{R}$ is finite, i.e. $1 < |\mathbb{S}_i| < \infty$ (see above). Accordingly, these preferences are firstly considered for the general case of a decision maker DM who ranks alternatives $x \in X = \{x_1, \ldots, x_m\}$.

3.1 Binary relations

- A ranking of the m alternatives $x_1, \ldots, x_m \in X$ shall be formalized as *rational preference binary relation* $R^{\succsim}$, or more simply $\succsim$, where $x_i \succsim x_j$ reads "x_i is weakly preferred to x_j" (or "x_i is at least as good as x_j").
- A *binary relation* R on X is any subset $R \subseteq X \times X$ (or $R \in 2^{X \times X}$) of *ordered pairs* of alternatives, hence $(x_i, x_j) \neq (x_j, x_i)$ (see above). For rational preference binary relations $R^{\succsim}$ or $\succsim$ defined below, $(x_i, x_j) \in R^{\succsim}$ shall mean $x_i \succsim x_j$. For $1 \leq i, j \leq m$, product $X \times X$ contains m^2 ordered pairs, out of which m have form $(x_i, x_i), 1 \leq i \leq m$ while $2\binom{m}{2} = m(m-1)$ are (proper) ordered pairs $(x_i, x_j), i \neq j$.
- A binary relation R on a set X is:
  - $+$ reflexive if $(x, x) \in R$ for all $x \in X$,
  - $-$ symmetric if $(x', x) \in R \Rightarrow (x, x') \in R$ for all $x, x' \in X$,
  - $+$ transitive if $(x'', x') \in R \ni (x', x) \Rightarrow (x'', x) \in R$ for all $x, x', x'' \in X$,
  - $+$ complete if $(x, x') \in R$ or $(x', x) \in R$ or both for all $x, x' \in X$,
  - $-$ antisymmetric if $(x', x) \in R$ and $(x, x') \in R \Rightarrow x = x'$ for all $x, x' \in X$,
  - $-$ asymmetric if $(x', x) \in R \Rightarrow (x, x') \in R^c$ for all $x, x' \in X, x \neq x'$,
  - $-$ irreflexive if $(x, x) \in R^c$ for all $x \in X$.
- Reflexive, symmetric and transitive binary relations are known as *equivalence relations* $\mathcal{E} \subset X \times X$; they correspond bijectively to partitions $P = \{A_1, \ldots, A_{|P|}\}$ of X, as each block $A \in P$ is an *equivalence class*, namely a maximal (in terms of inclusion $\supseteq$) subset $A \in 2^X$ satisfying $(x_i, x_j), (x_j, x_i) \in \mathcal{E}$ for all $x_i, x_j \in A$. In computer science, an *apartness relation* is a symmetric and irreflexive binary relation with the additional condition that if two elements are apart, then any other element is apart from at least one of them. Apartness relations are the *complement* $\mathcal{E}^c = (X \times X) \backslash \mathcal{E}$ of equivalence relations $\mathcal{E}$.
- Another main example of a binary relations comes from posets $(2^M, \supseteq)$ and $(\mathcal{P}^M, \geqslant)$ above. Specifically, binary relation $R^{\supseteq} \subset 2^M \times 2^M$ (on 2^M) defined by $R^{\supseteq} = \{(A, B) : A, B \in 2^M, A \supseteq B\}$ is reflexive, transitive,

antisymmetric and asymmetric. The same applies to binary relation $R^{\geqslant} \subset \mathcal{P}^M \times \mathcal{P}^M$ (on $\mathcal{P}^M$) defined by $R^{\geqslant} = \{(P, Q) : P, Q \in \mathcal{P}^M, P \geqslant Q\}$.

- A binary relation R on a m-set X can be represented as a *Boolean matrix* $M^R \in \{0,1\}^{m \times m}$ whose entries are:

$$M^R_{i,j} = \begin{cases} 1 \text{ if } (x_i, x_j) \in R \\ 0 \text{ if } (x_i, x_j) \notin R \end{cases} \text{ for } 1 \leq i, j \leq m,$$

  where i is the row and j is the column. Evidently, if $R = \mathcal{E}$ is an equivalence relation, then the associated Boolean matrix $M^{\mathcal{E}}$ is symmetric.

- **Exercise 1:** For $N = \{1, 2, 3\}$, determine the $\{0,1\}^{8 \times 8}$ matrix representing binary relation $R^{\supseteq} \subset 2^N \times 2^N$ described above, indexing rows/columns $1 \leq i, j \leq 8$ by subsets $A, B \in 2^N$ in a way such that if $A \supset B$ then $i < j$.

3.2 Rational preferences

- In game (and microeconomic [19]) theory, the concern is with *rational preferences* $\succsim$ (on a m-set X of alternatives), namely *reflexive, transitive and complete* binary relations $R^{\succsim} \subseteq X \times X$, with $x_i \succsim x_j \Leftrightarrow (x_i, x_j) \in R^{\succsim}$.

- It is useful to split a rational preference $\succsim$ in its *strong preference* $\succ$ *and indifference* $\sim$ parts. That is to say, for all $x_i, x_j \in X$ such that $x_i \succsim x_j$,
  - if $x_j \not\succsim x_i$, then there is strong preference: $x_i \succ x_j$,
  - if $x_j \succsim x_i$, then there is indifference $x_i \sim x_j$.

- For any rational preference $\succsim$, alternatives $x \in X$ can be labeled with naturals $1, \ldots, m \in \mathbb{N}$ in some (i.e. at least one) way such that $x_i \succsim x_{i+1}$ for $1 \leq i < m$. Then, the Boolean matrix $M^{R^{\succsim}} \in \{0,1\}^{m \times m}$ representing $R^{\succsim}$ has all 1s on and above (or to the right of) the main diagonal, while all remaining 1s (if any) identify squares along the main diagonal. Formally, after suitably labeling alternatives with the first m naturals as above, the generic rational preference may be listed as follows:

$$x_1 \sim \cdots \sim x_{n_1} \succ x_{n_1+1} \sim \cdots \sim x_{n_2} \succ \cdots \sim \cdots \succ x_{m-k} \sim \cdots \sim x_m,$$

  where $m - k = n_{k-1} + 1$. In other terms, $\Delta^n_l := n_l - n_{l-1}$ for $1 < l \leq k$, and $n_0 := 0$ as well as $n_k := n = \sum_{1 \leq l \leq k} \Delta^n_l$.

- Rational preferences $\succsim$ thus correspond bijectively to ordered partitions $\mathfrak{P}^{\succsim} = (A_1, \ldots, A_k)$, with $A_1 = \{x : x \gtrsim x' \text{ for all } x' \in X\}$, i.e. the first block contains all Δ^n_1 optimal alternatives. Then in general the l-th block $A_l, 1 < l < k$ contains all Δ^n_l alternatives x such that $x' \succ x \succ x''$ for all $x' \in A_{l'}, l' < l$ and all $x'' \in A_{l''}, l'' > l$, while the last block $A_k = \{x : x' \succsim x \text{ for all } x' \in X\}$ contains all Δ^n_k worst alternatives.

- Coming to the last representation of rational preferences $\succsim$, consider the *subset* $\mathcal{S}^{\succsim}_m \subseteq \mathcal{S}_m$ *of* $\succsim$*-admissible permutations* whose elements are those $\pi \in \mathcal{S}_m$ (see above) such that if $x_i \succ x_j$, then $\pi(i) < \pi(j)$. Since alternatives are firstly labeled in any way satisfying $x_i \succsim x_{i+1}, 1 \leq i < m$, the identity $id(i) = i$ is an element of this subset, i.e. $id \in \mathcal{S}_m^{\succsim}$. In fact, $(\mathcal{S}_m^{\succsim}, id, \circ)$ is a *subgroup* of the symmetric group $(\mathcal{S}_m, id, \circ)$, meaning that the conditions on pages 6-7 remain valid if $\mathcal{S}_m$ is replaced with $\mathcal{S}_m^{\succsim}$. With the above notation, $|\mathcal{S}_m^{\succsim}| = \prod_{1 \le l \le k} \Delta_n^l!$.

- **Exercise 2:**

  1. Show that for any rational preference $\succsim$ on m-set X, with corresponding binary relation $R^{\succsim} \subseteq X \times X$, the following bounds apply: $\binom{m+1}{2} \le |R^{\succsim}| \le m^2$. What rational preferences attain these bounds?
  2. Discuss the following statement: an equivalence relation with a number of equivalence classes > 1 cannot be a rational preference relation.
  3. For a rational preference $\succsim$ on m-set X corrseponding to ordered partition $\mathfrak{P}^{\succsim} = (A_1, \ldots, A_k)$ (of X), determine both:
     (i) the number $|R^{\succsim}| = \sum_{1 \le i,j \le m} M_{ij}^{R^{\succsim}}$ of 1s in the Boolean matrix $M^{R^{\succsim}} \in \{0,1\}^{m \times m}$ (representing binary relation $R^{\succsim} \subseteq X \times X$),
     (ii) the number $|\mathcal{S}_m^{\succsim}|$ of $\succsim$-admissible permutations.
  4. If the Boolean $m \times m$ matrix representing a rational preference over m alternatives is symmetric, then how many of its entries equal 0?
  5. Can an equivalence relation be complete? Discuss.
  6. For $X = \{x_1, \ldots, x_m\}$, define $f : X \to X$ by $f(x_k) = x_{m-k+1}$ $(1 \le k \le m)$. Identify $ker(f)$. For the bynary relation
  $$R^f = \{(x_k, f(x_k)) : 1 \le k \le m\} \subset X \times X,$$
  count the number of 1s in Boolean matrix $M^{R^f} \in \{0,1\}^{m \times m}$. Is R^f reflexive and/or complete and/or transitive? Identify a ($\supseteq$-)minimal rational preference $R^{\succsim^*}$ satisfying $R^{\succsim^*} \supseteq R$. How many 1s are in Boolean matrix $M^{R^{\succsim^*}}$? (Assume m is even.)

3.3 Preference aggregation

- How to *aggregate n rational preferences* $\succsim_i, i \in N = \{1, \ldots, n\}$ on a m-set $X = \{x_1, \ldots, x_m\}$ of alternatives is an issue with a long history in social choice theory, and more recently also attracting attention from the artificial intelligence AI community. When $X = \times_{i \in N} \mathbb{S}_i$ is the product space of strategy profiles (hence $m = \prod_{i \in N} |\mathbb{S}_i|$), *common interest games* are those where there is a strategy profile $x^* \in X$ such that $x^* \succsim_i x$ for all $x \in X$ and all $i \in N$. Additionally, in *pure common interest games* there is a permutation $\pi^* \in \mathcal{S}_m$ such that for all $i \in N$ and all $x_l, x_k \in X$, if $x_l \succ_i x_k$, then $\pi^*(l) < \pi^*(k)$ [9].

- These (possibly pure) common interest games clearly constitute only a small class of games, where strategic interaction basically has to deal (only) with *coordination*. On the other hand, most non-cooperative games shall be characterized by some degree of *conflict* (and this is especially true for constant-sum games, see below). Then, a well-known criterion for selecting a subset of "socially optimal" strategy profiles/alternatives $x \in X$

is *Pareto optimality.* For any $x, x' \in X$, define x to *Pareto-dominate* x', denoted by $x \succ_* x'$, as follows:

$$x \succ_* x' \Leftrightarrow \begin{cases} x \succsim_i x' \text{ for all } i \in N \text{ (weak preference)}, \\ x \succ_j x' \text{ for at least one } j \in N \text{ (strong preference)}. \end{cases}$$

The non-empty subset $\emptyset \neq X^*_{PO} \subseteq X$ of *Pareto-optimal/efficient* strategy profiles/alternatives consists of all Pareto-undominated ones [19], namely

$$X^*_{PO} = \{x : x' \not\succ_* x \text{ for all } x' \in X\}$$

(where $x' \not\succ_* x$ means x' does not Pareto-dominate x.)

- **Exercise 3:** For player set N and product space of strategy profiles $\times_{i \in N} \mathbb{S}_i = X = \{x_1, \ldots, x_m\}$ as above, let each $i \in N$ have rational preferences $\succsim_i$ on X corresponding to ordered partition $\mathfrak{P}^{\succsim_i} = (A^i_1, \ldots, A^i_{k_i})$ of X. What is the necessary and sufficient condition that $(\mathfrak{P}^{\succsim_i})_{i \in N}$ must satisfy in order for this to be a common interest game?

3.4 Utility representation

- Function $u : X \to \mathbb{R}$ is said to represent rational preference $\succsim$ and to be a *utility function* if $u(x) \geq u(x') \Leftrightarrow x \succsim x'$ for all $x, x' \in X$.
- If u represents $\succsim$, then $\succsim$ may also be represented by any *monotone* transformation u' of u, i.e. $u'(x) = f(u(x))$ for all $x \in X$, with $f : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha, \beta \in \mathbb{R}, \alpha < \beta \Rightarrow f(\alpha) < f(\beta)$.
- Properties of utility functions that are invariant under such monotone transformations are called ordinal, while cardinal ones are not preserved under the same transformations.
- **Exercise 4:** Show that any rational preference $\succsim$ over a finite set of alternatives can be represented by a utility function.

4 Discrete probability

- In non-cooperative game theory players are generally conceived to choose *random (or "mixed") strategies*, meaning that every $i \in N$ plays according to a *discrete probability distribution* σ_i over $\mathbb{S}_i = \{s^1_i, \ldots, s^{|\mathbb{S}_i|}_i\}$. That is, $\sigma_i : \mathbb{S}_i \to [0, 1]$ with $\sum_{s_i \in \mathbb{S}_i} \sigma_i(s_i) = 1$.
- In other terms, when choosing a random strategy σ_i each player $i \in N$ selects a point $\sigma_i \in \Delta_{\mathbb{S}_i}$ in the $|\mathbb{S}_i| - 1$-*dimensional unit simplex* $\Delta_{\mathbb{S}_i} =$

$$= \left\{ \left(\sigma_i(s^1_i), \ldots, \sigma_i(s^{|\mathbb{S}_i|}_i) \right) : \sigma_i(s^k_i) \geq 0 \text{ for } 1 \leq k \leq |\mathbb{S}_i|, \sum_{s_i \in \mathbb{S}_i} \sigma_i(s_i) = 1 \right\}$$

  or *convex set* of probability distributions over $\mathbb{S}_i$, meaning that $(\alpha\sigma_i + (1 - \alpha)\sigma'_i) \in \Delta_{\mathbb{S}_i}$ for all $\alpha \in [0, 1]$ and all $\sigma_i, \sigma'_i \in \Delta_{\mathbb{S}_i}$.

- In this view, non-random (or "pure") strategies are simply random ones where the whole (unit) probability mass is concentrated on a unique *extreme point* $\epsilon \in ex(\Delta_{\mathbb{S}_i})$ of the simplex. There are $|\mathbb{S}_i|$ extreme points $\epsilon_1, \ldots, \epsilon_{|\mathbb{S}_i|} \in \{0,1\}^{|\mathbb{S}_i|}$ of $\Delta_{\mathbb{S}_i}$, each being *a Boolean* $|\mathbb{S}_i|$*-vector* with a unique 1 and $|\mathbb{S}_i|-1$ entries equal to 0. Thus ϵ_k is the degenerate probability distribution $\epsilon_k = \bar{\sigma}_i^k$ defined by $\bar{\sigma}_i^k(s_i^l) = \begin{cases} 0 \text{ if } k \neq l \\ 1 \text{ if } k = l \end{cases}$ $(1 \leq k \leq |\mathbb{S}_i|)$.

4.1 Discrete random variables: lotteries

- A discrete random variable (with finite support) consists of a set of real numbers, i.e. $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}$, and a probability distribution over X, i.e. $p = (p_1, \ldots, p_m) \in \Delta_X$. Here X contains m atomic mutually exclusive events, and $\Delta_X \subset \mathbb{R}_+^m$ is the $m-1$-dimensional unit simplex whose extreme points are indexed by the elements in X; in other terms, $p_k = p(x_k) \geq 0$ is the probability that real quantity or atomic event x_k realizes, with $\sum_{1 \leq k \leq m} p_k = 1$.

- If real numbers $x_1, \ldots, x_m \in X \in \mathbb{R}^m$ are interpreted as money values that the DM may receive, then probability distributions $p \in \Delta_X$ are commonly referred to as *"lotteries"*. The theory of *decision under uncertainty* initiates with the problem of rakning lotteries. For example, any $p, q \in \Delta_X$ may be ranked simply according to their expected value $Ex(p), Ex(q)$, i.e.

$$p \succsim q \Leftrightarrow Ex(p) = \sum_{1 \leq k \leq m} p_k x_k \geq \sum_{1 \leq k \leq m} q_k x_k = Ex(q).$$

Note that the resulting preference (binary relation) $\succsim$ (on Δ_X) is rational.

4.2 Probabilities as set functions

- Conceptually, probabilities are associated with *events or subsets* $A \in 2^X$ of atomic mutually exclusive events $x \in X$. In fact, a probability distribution is a *set function* $p : 2^X \to [0,1]$ satisfying $p(A)+p(B) = p(A \cap B)+p(A \cup B)$ for all $A, B \in 2^X$, as well as $p(\emptyset) = 0 = 1 - p(X)$. Also, $p(A), A \in 2^X$ is thought of as the probability that the realized atomic event x shall satisfy $x \in A$. Then, a main theorem [1, p. 190] on *valuations of distributive lattices* (such as Boolean lattice $(2^X, \cap, \cup)$) entails $p(A) = \sum_{x \in A} p(\{x\})$ for all events $A \in 2^X$.

- Both in decision theory and in cooperative game theory, a central issue is how to map generic monotone set functions $v : 2^X \to \mathbb{R}_+$, namely such that $A \supseteq B \Rightarrow v(A) \geq v(B)$ for all $A, B \in 2^X$, into set functions $\phi(v) : 2^X \to \mathbb{R}_+$ satisfying $\phi(v)(A) = \sum_{x \in A} \phi(v)(\{x\})$ for all $A \in 2^X$. In particular, in decision theory v is a *(discrete) fuzzy measure*, meaning $v(\emptyset) = 0 = 1 - v(N)$ as above, and the concern is with the possibly empty convex set $\mathfrak{C}(v)$ containing those probabilities $\phi(v) = p \in \Delta_X$ such that $p(A) \geq v(A)$ for all $A \in 2^X$ (with equality for $A = X$). Analogously (for $X = N$), in cooperative game theory (see above) $\mathfrak{C}(v)$ is the *core* of coalitional game v, where this latter is monotone and satisfies $v(\emptyset) = 0$ but may take any value $v(N)$ on the grand coalition N. The core $\mathfrak{C}(v)$ is a *set-valued solution/value* concept for coalitional games v, while *point-valued* ones associate with every v a *single* $\phi(v)$ (such that for all $A \in 2^X$ $\phi(v)(A) = \sum_{x\in A} \phi(v)(\{x\})$.

- These topics will be addressed in the sequel, when dealing with the *discrete Choquet integral* with respect to fuzzy measures (in decision theory), and with solutions/values of coalitional games (in the second half of the course). For now, consider at glance the game where every player $i \in N = \{1, \ldots, n\}$ votes on a bill to pass or not, and if the number of those who vote it to pass is greater or equal to $\lfloor \frac{n}{2} \rfloor + 1$, then the bill passes, while if that number is strictly smaller than $\lfloor \frac{n}{2} \rfloor + 1$, then the bill does not pass. In other terms, all coalitions of $\frac{n}{2} + 1$ (for n even) or $\frac{n+1}{2}$ (for n odd) are *minimal winning* ones, while any of their proper subcoalitions is *loosing*. This is the *voting majority game*, a well-known member of the family of *simple games*, which are those monotone $v : 2^N \to \{0, 1\}$ such that $v(\emptyset) = 0 = 1 - v(N)$.

- The voting majority game is evidently *symmetric*, in that players/voters all have equal unit weight. *Voting quota games* $v : 2^N \to \{0, 1\}$ are simple games where, more generally, there are $n + 1$ weights, denoted by $\omega_0, \omega_1, \ldots, \omega_n \in \mathbb{R}_{++}$, which identify as winning those coalitions $A \in 2^N$ such that $\sum_{i\in A} \omega_i \geq \omega_0$, i.e. $v(A) = 1$, and as loosing ones those $A \in 2^N$ such that $\sum_{i\in A} \omega_i < \omega_0$, i.e. $v(A) = 0$. Hence in the voting majority case $\omega_i = 1$ for all $i \in N$, while $\omega_0 = \lfloor \frac{n}{2} \rfloor + 1$.

- A *swing* for a player $i \in N$ in a simple game v is a winning coalition $A \cup i$ such that A is loosing. In other terms, if $v(A\cup i) - v(A) = 1$, then $A \cup i$ is a swing for i. The *Banzhaf value* [5, 25] $\phi^{Ba}(v) = (\phi_1^{Ba}(v), \ldots, \phi_n^{Ba}(v))$ of a simple (voting quota) game v obtains by assigning to each $i = 1, \ldots, n$ the ratio of the total number of i's swings to the maximum possible number 2^{n-1} of such swings, that is

$$\phi_i^{Ba}(v) = \sum_{A \subseteq N\setminus i} \frac{v(A \cup i) - v(A)}{2^{n-1}}.$$

  When v is the majority voting game, $\phi_i^{Ba}(v) = \frac{\binom{n-1}{\frac{n}{2}}}{2^{n-1}}$ if n is even, while $\phi_i^{Ba}(v) = \frac{\binom{n-1}{\frac{n-1}{2}}}{2^{n-1}}$ if n is odd.

- *Chow parameters problem*: given power indices $\phi_1^*, \ldots, \phi_n^* > 0$, determine $n+1$ weights $\omega_0, \omega_1, \ldots, \omega_n$ such that the corresponding voting quota game v has Banzhaf value $\phi^{Ba}(v)$ as close as possible to ϕ^* [11, 12].

- **Exercise 5:**

  1. Consider voting quota game $v : 2^N \to \{0, 1\}$ with weights $\omega_0 = 0.4$ as well as $\omega_i = 0.i$ for $i \in N$, and player set $N = \{1, 2, 3, 4\}$.
     - Compute the Banzhaf value $\phi^{Ba}(v) = (\phi_1^{Ba}(v), \ldots, \phi_4^{Ba}(v))$.
     - Identify the set of minimal winning coalitions.
     - Identify the set of maximal loosing coalitions.

2. Consider a voting quota game $v : 2^N \to \{0,1\}$ with weights $\omega_0 = 90$ as well as $\omega_i = 1$ for $i \in N$, and player set $N = \{1, \ldots, 100\}$.
   - Compute the Banzhaf value $\phi^{Ba}(v) = (\phi_1^{Ba}(v), \ldots, \phi_{100}^{Ba}(v))$.
   - Identify the set of minimal winning coalitions.
   - Identify the set of maximal loosing coalitions.
3. Consider simple game $v : 2^N \to \{0,1\}$ with player set $N = \{1, \ldots, 4\}$ and minimal winning coalitions $\{1,4\}, \{2,3\}, \{2,4\}$ and $\{3,4\}$.
   - Identify the set of maximal loosing coalitions.
   - Compute the Banzhaf value $\phi^{Ba}(v) = (\phi_1^{Ba}(v), \ldots, \phi_4^{Ba}(v))$.
   - Identify weights $\omega_0, \omega_1, \ldots, \omega_4$ such that the resulting voting quota game has Banzhaf value equal to $\phi^{Ba}(v)$.

4.3 Expected utility

- The expected utility theory provides a main result for the representation of preferences $\succsim$ over lotteries $p, q \in \Delta_X$ as described in Section 4.1. That is, $X = \{x_1, \ldots, x_m\} \in \mathbb{R}^m$ is a set of money values [19, p. 171].
- Preference $\succsim$ (over Δ_X) is *continuous* if for any $p, p', q \in \Delta_m$, both the following sets are closed:

$$\{\alpha \in [0,1] : \alpha p + (1-\alpha)p' \succsim q\} \text{ and } \{\alpha \in [0,1] : q \succsim \alpha p + (1-\alpha)p'\}.$$

- Preference $\succsim$ *satisfies the independence axiom* if for any $p, p', q \in \Delta_m$ and for all $\alpha \in [0,1]$,

$$p \succsim p' \Leftrightarrow \alpha p + (1-\alpha)q \succsim \alpha p' + (1-\alpha)q.$$

- Theorem (von Neumann and Morgestern 1944 [30]): if pference $\succsim$ is continuous and satisfies the independence axiom, then there exists a *utility over money values* $u : X \to \mathbb{R}$ such that $\succsim$ is represented by a $Eu : \Delta_X \to \mathbb{R}$ with the following *expected utility* form:

$$Eu(p) = \sum_{1 \le k \le m} p_k u(x_k) \text{ for all } p \in \Delta_X.$$

- Corollary: $Eu : \Delta_X \to \mathbb{R}$ has the expected utility form if and only if it is *linear*, meaning

$$Eu(\alpha_1 p^1 + \ldots + \alpha_k p^k) = \sum_{1 \le l \le k} \alpha_l Eu(p^l)$$

  for all *convex combinations* of any $p^1, \ldots, p^k \in \Delta_X$ (i.e. $\alpha_1, \ldots, \alpha_k \in \mathbb{R}_+$ satisfy $\sum_{1 \le l \le k} \alpha_l = 1$, and Δ_X is convex precisely because any convex combination of probabilities is a probability).

- If $Eu : \Delta_X \to \mathbb{R}$ has the expected utility form and represents preference $\succsim$, then any further $Eu' : \Delta_X \to \mathbb{R}$ representing $\succsim$ has form $Eu'(p) = \beta Eu(p) + \gamma$ (for all $p \in \Delta_X$) for some $\beta \in \mathbb{R}_{++}, \gamma \in \mathbb{R}$.

- In choice experiments, the independence axiom is violated, two main examples being Allais [19, p. 179] and Ellsberg paradoxes (see below).

- Finally, if $u : \mathbb{R} \to \mathbb{R}$ is *concave/linear/convex*, then the DM is said to be *risk-averse/neutral/lover* [19].

- **Exercise 6:** let $X = \mathbb{N}_{10} = \{1, 2, \ldots, 9, 10\}$ be a set of money values, with utility function $u(n) = \ln n, 1 \leq n \leq 10$. Consider two lotteries $p, q \in \Delta_X$ defined as follows: $p(n) = \frac{8-n}{28}$ if $1 \leq n \leq 7$ and $p(n) = 0$ if $7 < n \leq 10$, while $q(n) = \frac{7-n}{21}$ if $n \leq 6$ and $q(n) = 0$ if $6 < n \leq 10$. Compute the vN-M expected utility of the two lotteries, i.e. $Eu(p)$ and $Eu(q)$.

4.4 Ellsberg paradox

- Ellsberg paradox is designed to show that the independence axiom is violated. The DM does not rank lotteries but actions, defined as follows.

- Consider a set $\Omega = \{\omega_1, \ldots, \omega_k\}$ $(k > 1)$ of states of nature and a set $\mathbb{A} = \{a_1, \ldots, a_m\}, m > 1$ of available actions, where the utility function has form $u : \Omega \times \mathbb{A} \to \mathbb{R}_+$.

- In the vN-M expected utility model, given some *subjective belief or probability* $p : 2^\Omega \to [0, 1]$ over states $\omega \in \Omega = \{\omega_1, \ldots, \omega_n\}$ (that is to say, $p(X) = \sum_{\omega \in X} p(\{\omega\})$ for all $X \in 2^\Omega$, see above), actions $a, a' \in \mathbb{A}$ shall be ranked according to their scored expected utility:

$$Eu_a(p) = \sum_{1 \leq l \leq k} p(\{\omega_l\}) u(\omega_l, a) \text{ as well as } a \gtrsim a' \Leftrightarrow Eu_a(p) \geq Eu_{a'}(p).$$

- Ellsberg paradox (1961): a ball is drawn at random from an urn containing 90 balls, 30 red R and each other ball either black B or yellow Y, while there are the following four actions/alternatives $a_1 - a_4$:

$$\begin{aligned} a_1 &: \text{receive 100 if the ball is R,} \\ a_2 &: \text{receive 100 if the ball is B,} \\ a_3 &: \text{receive 100 if the ball is R or Y,} \\ a_4 &: \text{receive 100 if the ball is B or Y.} \end{aligned}$$

  In experiments, $a_1 \succ a_2$ and $a_4 \succ a_3$ (strong preferences).

- a DM choosing in line with the expected utility theory has some subjective probability $p = (p_0, p_1, p_2, \ldots p_{60})$ where p_k is the probability that the number of black balls is k. Then, each $a \in \{a_1, a_2, a_3, a_4\}$ yields utility $u_a(R), u_a(B), u_a(Y)$ depending on whether the ball, drawn randomly (with uniform distribution), is R or B or Y.

- For any such a subjective probability p, choosing $a_1 \succ a_2$ as well as $a_4 \succ a_3$ is inconsistent with the expected utility model in that:

  (1) $Eu_{a_1}(p) = u(100)\frac{1}{3} + u(0)\frac{2}{3}$,

  (2) $Eu_{a_2}(p) = u(100)\left(\sum_{0 \leq k \leq 60} p_k \frac{k}{90}\right) + u(0)\left(\frac{1}{3} + \sum_{0 \leq k \leq 60} p_k \frac{60-k}{90}\right)$;

(1&2) $a_1 \succ a_2$ or $Eu_{a_1}(p) > Eu_{a_2}(p) \Rightarrow$

$$\Rightarrow \frac{u(100)-u(0)}{3} > \frac{u(100)-u(0)}{90} \sum_{0\leq k\leq 60} p_k k$$

(subtract $u(0)$ from both sides), thus $30 > \sum_{0\leq k\leq 60} p_k k$, i.e. the expected number of B balls in the urn (according to subjective p) is stricly smaller than 30;

(3) $Eu_{a_3}(p) = u(100)\left(\frac{1}{3} + \sum_{0\leq k\leq 60} p_k \frac{60-k}{90}\right) + u(0)\left(\sum_{0\leq k\leq 60} p_k \frac{k}{90}\right)$,

(4) $Eu_{a_4}(p) = u(100)\frac{2}{3} + u(0)\frac{1}{3}$;

(3&4) $a_a \succ a_3$ or $Eu_{a_4}(p) > Eu_{a_3}(p) \Rightarrow$

$$\Rightarrow \frac{u(100)-u(0)}{3} < \frac{u(100)-u(0)}{90} \sum_{0\leq k\leq 60} p_k k$$

(subtract $u(100)$ from both sides), thus $30 < \sum_{0\leq k\leq 60} p_k k$, i.e. the expected number of B balls in the urn (according to the same p) is stricly greater than 30.

- Of course, there is no such a probability p (i.e. satisfying both the above strict inequalities), entailing that the expected utility theory cannot explain these (empirically observed) choices.

5 Strategies

- In *simultaneous-move games* all players move only once, simultaneously, hence choosing a strategy is the same as choosing a move. This is no longer true in *multistage games*, where choosing a strategy means choosing a *sequence of (conditional) moves*. Although the non-cooperative games to be dealt with shall be in simultaneous-move form, still multistage games are briefly described below in order to formally define strategies in a most general setting, namely where players have either perfect or else incomplete information, this latter being commonly modeled by means of partitions.

5.1 Information in multistage games

- As the name clearly suggests, multistage games are played in discrete time $t = 0, 1, \ldots, T$, as $t = 0$ is the starting point or *root of the game tree* (defined hereafter), where some (at last one, and possibly all) players move; next, depending on previous moves, at each $t \geq 1$ a *node* is reached, corresponding either to a moment where at least one player has to move, or else to an end of the game or *leaf*. The concern here is only with games where $T < \infty$ (for any leaf).

- Multistage games are thus commonly represented by a *rooted and directed (game) tree* $\mathfrak{T} = (\mathbb{V}, E), \mathbb{V} = \{v_0, v_1, \ldots, v_{|\mathbb{V}|-1}\}, E \subseteq \mathbb{V}\times\mathbb{V}$, namely a cycle-free graph whose edges $(v, v') \in E$ are ordered pairs of vertices or nodes (i.e. $(v, v') \neq (v', v)$; also, $(v, v) \notin E$ for all $v \in \mathbb{V}$). Game tree $\mathfrak{T}$ is *rooted* at v_0 (denoting the start) and at each node either some player moves or else the game ends (i.e. in a leaf). Visually, the root may be placed on top so that a game course is a descending path to some leaf. In this view, existence of an edge $(v, v') \in E$, with v' immediately below v, means that node v' is reachable from node v through precisely one move choice by those players who move at v (again, the number of nodes is finite, and in particular $5 \leq |\mathbb{V}| < \infty$, as there must be at least two players, each with minimally two actions).

- Let $\mathbb{V}^L$ be the vertex subset containing all leaves (hence in simultaneous-move games $\mathbb{V} \backslash \mathbb{V}^L = \{v_0\}$). Denote by $\mathbb{V}_i \subseteq (\mathbb{V} \backslash \mathbb{V}^L)$ the subset of nodes where each player $i \in N$ moves. Also, for every $v \in \mathbb{V}_i$, let $\mathbb{A}_i^v$ be the set of moves available to i at v, with $\mathbb{A}_i = \bigcup_{v \in \mathbb{V}_i} \mathbb{A}_i^v$ containing all moves available to i (i.e. independently from nodes $v \in \mathbb{V}_i$).

- Under *perfect information*, this notation enables to *formally define a strategy* s_i, for a player $i \in N$, as any mapping $s_i : \mathbb{V}_i \rightarrow \mathbb{A}_i$ satisfying $s_i(v) \in \mathbb{A}_i^v$ for every $v \in \mathbb{V}_i$. In words, a strategy s_i specifies an admissible move for i at each node (that may be reached) where i has to move. Denote by $\mathbb{S}_i$ the set of all such strategies s_i available to $i \in N$.

- As already mentioned, information is modeled by means of partitions. Specically, for every $i \in N$, denote by $\mathbb{P}_i \subset \mathcal{P}^{\mathbb{V}_i}$ the set of partitions $P = \{B_1, \ldots, B_k\}$ of $\mathbb{V}_i$ satisfying, for any two nodes $v, v' \in \mathbb{V}_i$,

$$\text{if } \mathbb{A}_i^v \neq \mathbb{A}_i^{v'}, \text{ then } \{v, v'\} \not\subseteq B \text{ for all } B \in P.$$

  In words, if at nodes $v, v' \in \mathbb{V}_i$ the sets $\mathbb{A}_i^v, \mathbb{A}_i^{v'}$ of moves available to i are different, then v and v' must be apart in all $P \in \mathbb{P}_i$. The finest partition $P_\perp \in \mathbb{P}_i$ (consisting of $|\mathbb{V}_i|$ singletons, see above) clearly satisfies this condition, and in fact corresponds precisely to the case where i has perfect information.

- A player i endowed with *incomplete information* cannot distinguish between certain nodes $v, v' \in \mathbb{V}_i$ with same available moves $\mathbb{A}_i^v = \mathbb{A}_i^{v'}$ (as otherwise i could of course distinguish between v and v'). This is formalized by endowing i with a partition $P = \{B_1, \ldots, B_k\} \in \mathbb{P}_i$ such that $P > P_\perp$, i.e. strictly coarser (see Section 2.3) than the finest one. Blocks $B_1, \ldots, B_k$ of P are *information sets.* Then, a strategy s_i with incomplete information P must be constant on each information set, i.e. $s_i : P \rightarrow \mathbb{A}_i$. That is, in addition to the definition with perfect information, s_i must also satisfy: $P \ni B \supseteq \{v, v'\} \Rightarrow s_i(v) = s_i(v')$.

- Players' preferences are defined over the set $\mathbb{V}^L$ of leaves, where these latter correspond bijectively to strategy profiles $s = (s_1, \ldots, s_n) \in \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$. Hence $u_i : \times_{j \in N} \mathbb{S}_j \rightarrow \mathbb{R}$ for every $i \in N$ and non-cooperative games Γ are traditionally denoted by triples $\Gamma = (N, \mathbb{S}, u)$, with $u : \mathbb{S} \rightarrow \mathbb{R}^n$.

5.2 Dominated and dominant strategies

- A central issue in non-cooperative game theory is how to figure what strategy profiles are more likely to prevail in a given strategic interaction. This leads to investigate not only the *equilibrium conditions* (identifying those profiles from which no player has an incentive to unilaterally deviate, see below), but also whether certain strategies (and thus certain profiles) have a chance to be rationally played or not.

- In game $\Gamma = (N, \mathbb{S}, u)$, for each $i \in N$ consider the set

$$\mathbb{S}_{-i} = \underset{j \in N \backslash i}{\times} \mathbb{S}_j = \mathbb{S}_1 \times \cdots \times \mathbb{S}_{i-1} \times \mathbb{S}_{i+1} \times \cdots \times \mathbb{S}_n$$

of $n-1$-tuples of strategies for players $j \in N \backslash i$, with generic element $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in \mathbb{S}_{-i}$.

- Strategy $s_i \in \mathbb{S}_i$ is *weakly dominated* if there is a $\hat{s}_i \in \mathbb{S}_i \backslash s_i$ such that

$$u_i(\hat{s}_i, s_{-i}) - u_i(s_i, s_{-i}) \geq 0 \text{ for all } s_{-i} \in \mathbb{S}_{-i},$$

with strict inequality for at least one $s_{-i} \in \mathbb{S}_{-i}$. Then, $\hat{s}_i$ weakly dominates s_i. Here $(\hat{s}_i, s_{-i}), (s_i, s_{-i}) \in \mathbb{S}_i \times \mathbb{S}_{-i}$ are the two strategy profiles where all players $j \in N \backslash i$ choose $n-1$-profile $s_{-i} \in \mathbb{S}_{-i}$ in both, while $i \in N$ chooses respectively strategies $\hat{s}_i, s_i \in \mathbb{S}_i$.

- Similarly, $\hat{s}_i$ is said to *strongly dominate* s_i if the inequality is strict for all $s_{-i} \in \mathbb{S}_{-i}$. A strategy strongly dominating all others is said to be *dominant*. Clearly if there is a dominant strategy it is unique.

5.3 Deletion of dominated strategies

- Dominated strategies are unlikely to be chosen by rational players, enabling to conceive an iterated process where at each step some strongly dominated strategy, for some player, is deleted, until residual strongly dominated strategies no longer exist, for no player. When a strategy $\hat{s}_i \in \mathbb{S}_i$ of a player $i \in N$ is deleted in any given game $\Gamma = (N, \times_{j \in N} \mathbb{S}_j, u)$, the number of correspondigly deleted strategy profiles from the whole set $\times_{j \in N} \mathbb{S}_j$ clearly is $|\mathbb{S}_{-i}| = \prod_{j \in N \backslash i} |\mathbb{S}_j|$. The resulting sequence of games is $\Gamma^t = (N, \times_{j \in N} \mathbb{S}^t_j, u), t = 0, 1, \ldots$ with $\Gamma = \Gamma^0 = (N, \times_{j \in N} \mathbb{S}^0_j, u)$.

- The removal of all strategies, i.e. across all players, that are strongly dominated in the original game Γ^0 relies only on rationality, as nobody rational chooses a strongly dominated strategy, independently from other players' rationality (in turn) and payoffs. However, any further deletion of strongly dominated strategies requires both that each player has complete knowledge of the game, and that this individual complete knowledge also is common knowledge (i.e. everyone knows that everyone knows) [19]. After deleting any strongly dominated strategy further strategies may become strongly dominated (and thus deleted) simply because the smaller the number $\left|\mathbb{S}^t_{-i}\right|$ of $n-1$-tuples of strategies for other players $j \in N \backslash i$, the more likely that any (residual) strategy $s_i \in \mathbb{S}^t_i$ becomes strongly dominated, in that it remains a viable response to only those $n-1$-tuples $s_{-i} \in \mathbb{S}^t_{-i}$.

- The order in which strongly dominated strategies are deleted is irrelevant. Conversely, if weakly (rather than strongly) dominated strategies are deleted, then the order of deletion does affect the final outcome, as

Table 1: *Deleting weakly dominated strategies*

$(u_i(a_i, a_j), u_j(a_i, a_j))$	$a_j = L$	$a_j = R$
$a_i = U$	$(5, 1)$	$(4, 0)$
$a_i = M$	$(6, 0)$	$(3, 1)$
$a_i = D$	$(6, 4)$	$(4, 4)$

shown in Table 1. For player i, both U and M are weakly dominated by D; if U is deleted first, then L has to be deleted, and finally M as well, so that (D, R) remains the only surviving outcome; conversely, if M is deleted first, then R has to be deleted, and finally U as well, so that (D, L) is the only residual outcome.

5.4 Equilibrium

- Given any game $\Gamma = (N, \times_{j \in N} \mathbb{S}_j, u)$, for the case of non-random (or pure) strategies an equilibrium is a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in \times_{j \in N} \mathbb{S}_j$ from which no player has an incentive to unilaterally deviate, hence where

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*) \text{ for all } s_i \in \mathbb{S}_i \text{ and all } i \in N.$$

- Another way to define equilibria is in terms of best responses as follows. For every $i \in N$, define the best response mapping $BR_i : \mathbb{S}_{-i} \to 2^{\mathbb{S}_i}$ by

$$BR_i(\hat{s}_{-i}) = \{\hat{s}_i : u_i(\hat{s}_i, \hat{s}_{-i}) \geq u_i(s_i, \hat{s}_{-i}) \text{ for all } s_i \in \mathbb{S}_i\}.$$

  As $\{u_i(s_i, \hat{s}_{-i}) : s_i \in \mathbb{S}_i\} \in \mathbb{R}^{|\mathbb{S}_i|}$ is a finite set of real numbers, its maximum exists for any $\hat{s}_{-i} \in \mathbb{S}_{-i}$, hence $BR_i(\hat{s}_{-i}) \neq \emptyset$. Then, strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in \times_{j \in N} \mathbb{S}_j$ is an equilibrium if at s^* every i is playing a best response, i.e. $s_i^* \in BR_i(s_{-i}^*)$ for all $i \in N$.

- The set of these equilibria s^* for a generic game Γ may be empty but may also consist of several strategy profiles, and such a (possible) multiplicity leads to investigate alternative equilibrium refinement criteria [19].

- The popular prisoner's dilemma is a simple non-cooperative game with only two players, each with the same two strategies, and still where there exists a unique equilibrium which, in particular, is Pareto-dominated (see above). For both players i and j, strategy C (confess) strongly dominates strategy NC (non-confess), as shown in Table 2. After deleting strongly

Table 2: *Prisoner's dilemma payoff matrix (-1 = 1 year in jail)*

$(u_i(a_i, a_j), u_j(a_i, a_j))$	$a_j = NC$	$a_j = C$
$a_i = NC$	$(-2, -2)$	$(-10, -1)$
$a_i = C$	$(-1, -10)$	$(-5, -5)$

dominated strategies, the only surviving outcome is C, C, which is thus an equilibrium in dominant strategies. However, C, C also is *strongly Pareto-dominated*, as with NC, NC both players receive a *strictly greater payoff*:

$$u_i(NC, NC) = u_j(NC, NC) = -2 > -5 = u_i(C, C) = u_j(C, C).$$

6 Random strategies

- As already mentioned, the expected utility theory in Section 4.3 was conceived to deal with non-cooperative games where players i may choose each a probability distribution over strategy set $\mathbb{S}_i$. Traditionally, these probability distributions are called *mixed* strategies, while *pure* ones are those $s_i \in \mathbb{S}_i$ just considered. Here, every s_i shall be regarded as the probability distribution fully concentrated on a single extreme of the $|\mathbb{S}_i| - 1$-dimensional unit simplex $\Delta_{\mathbb{S}_i}$ defined in Section 4, i.e. $\Delta_{\mathbb{S}_i} =$

$$= \left\{ \left(\sigma_i(s_i^1), \ldots, \sigma_i(s_i^{|\mathbb{S}_i|}) \right) : \sigma_i(s_i^k) \geq 0 \text{ for } 1 \leq k \leq |\mathbb{S}_i|, \sum_{s_i \in \mathbb{S}_i} \sigma_i(s_i) = 1 \right\}.$$

  There are $|\mathbb{S}_i|$ extreme points $\epsilon_1, \ldots, \epsilon_{|\mathbb{S}_i|} \in \{0,1\}^{|\mathbb{S}_i|}$ of $\Delta_{\mathbb{S}_i}$, each being a Boolean $|\mathbb{S}_i|$-vector with a unique 1 and $|\mathbb{S}_i| - 1$ entries equal to 0. Thus ϵ_k is the probability distribution $\epsilon_k = \bar{\sigma}_i^k$ defined by $\bar{\sigma}_i^k(s_i^l) = \begin{cases} 0 \text{ if } k \neq l \\ 1 \text{ if } k = l \end{cases}$ $(1 \leq k \leq |\mathbb{S}_i|)$. Equivalently, $\{\epsilon_1, \ldots, \epsilon_{|\mathbb{S}_i|}\}$ is the canonical basis of $\mathbb{R}^{|\mathbb{S}_i|}$ (with axes indexed by strategies $s_i \in \mathbb{S}_i$), and $\Delta_{\mathbb{S}_i} = co.hu(\{\epsilon_1, \ldots, \epsilon_{|\mathbb{S}_i|}\})$ is the convex hull of these extreme points.

- A random strategy $\sigma_i \in \Delta_{\mathbb{S}_i}$ is a point in this simplex, and $\sigma_i^k = \sigma_i(s_i^k)$ is the probability (or frequency in repeated games) by which $i \in N$ plays according to (non-random) strategy $s_i^k \in \mathbb{S}_i, 1 \leq k \leq |\mathbb{S}_i|$ (when adopting random strategy σ_i). On the other hand, probability distributions over the product space $\times_{j \in N} \mathbb{S}_j$ of strategy profiles are points in the $|\times_{j \in N} \mathbb{S}_j| - 1$-dimensional unit simplex, denoted by $\Delta_{\mathbb{S}}^{\times}$, whose elements $p \in \Delta_{\mathbb{S}}^{\times}$ are those functions $p : \times_{j \in N} \mathbb{S}_j \to [0,1]$ satisfying

$$\sum_{s \in \times_{j \in N} \mathbb{S}_j} p(s) = 1, \text{ where } s = (s_1, \ldots, s_n).$$

- Note that $\Delta_{\mathbb{S}}^{\times} \neq \times_{j \in N} \Delta_{\mathbb{S}_j}$ and players choose their random strategies *independently*. Thus any profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \times_{j \in N} \Delta_{\mathbb{S}_j}$ of random strategies chosen by the n players induces probability distribution $p_\sigma \in \Delta_{\mathbb{S}}^{\times}$ over non-random strategy profiles $s = (s_1, \ldots, s_n) \in \times_{j \in N} \mathbb{S}_j$ defined by

$$p_\sigma(s) = p_\sigma(s_1, \ldots, s_n) = \prod_{j \in N} \sigma_j(s_j).$$

- For given $p_\sigma \in \Delta_{\mathbb{S}}^{\times}$, each player $i \in N$ may be seen as facing a lottery or random variable taking each utility value $u_i(s), s = (s_1, \ldots, s_n) \in \times_{j \in N} \mathbb{S}_j$

with probability $p_\sigma(s)$. Hence at any profile $\sigma = (\sigma_1, \ldots, \sigma_n) \in \times_{j \in N} \Delta_{\mathbb{S}_j}$ of random strategies i's expected utility is

$$Eu_i(\sigma) = \sum_{s \in \times_{j \in N} \mathbb{S}_j} p_\sigma(s) u_i(s) = \sum_{s \in \times_{j \in N} \mathbb{S}_j} \left(\prod_{j \in N} \sigma_j(s_j) \right) u_i(s).$$

Strategy profiles $s, s' \in \times_{j \in N} \mathbb{S}_j$ are ranked accoring to $u_i(s) \lesseqqgtr u_i(s')$, while random strategy profiles $\sigma, \sigma' \in \times_{j \in N} \Delta_{\mathbb{S}_j}$ are ranked according to $Eu_i(\sigma) \lesseqqgtr Eu_i(\sigma')$ [19, pp. 167-182, 232].

6.1 Dominance

- Random strategy $\sigma_i \in \Delta_{\mathbb{S}_i}$ is (strongly) dominated by $\hat{\sigma}_i \in \Delta_{\mathbb{S}_i} \backslash \sigma_i$ if $Eu_i(\hat{\sigma}_i, \sigma_{-i}) - Eu_i(\sigma_i, \sigma_{-i}) > 0$ for all $n-1$-tuples σ_{-i} of random strategies for players $j \in N \backslash i$, i.e. $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n) \in \times_{j \in N \backslash i} \Delta_{\mathbb{S}_j}$. This means that $Eu_i(\hat{\sigma}_i, \sigma_{-i}) - Eu_i(\sigma_i, \sigma_{-i}) =$

$$= \sum_{(s_1, \ldots, s_n) \in \times_{j \in N} \mathbb{S}_j} \left[\left(\prod_{j \in N \backslash i} \sigma_j(s_j) \right) \Big(\hat{\sigma}_i(s_i) - \sigma_i(s_i) \Big) \right] u_i(s_1, \ldots, s_n) > 0$$

is strictly positive for all $\sigma_{-i} \in \underset{j \in N \backslash i}{\times} \Delta_{\mathbb{S}_j}$, which is the case if and only if

$$Eu_i(\hat{\sigma}_i, s_{-i}) - Eu_i(\sigma_i, s_{-i}) = \sum_{s_i \in \mathbb{S}_i} \Big(\hat{\sigma}_i(s_i) - \sigma_i(s_i) \Big) u_i(s_i, s_{-i}) > 0$$

for all $s_{-i} \in \mathbb{S}_{-i}$. The reason is that every $\hat{s}_{-i} \in \mathbb{S}_{-i}$ has associated the $n-1$-tuple of extreme points $\hat{\sigma}_{-i} \in \times_{j \in N \backslash i} ex(\Delta_{\mathbb{S}_j})$ defined by $\hat{\sigma}_j(\hat{s}_j) = 1$ for all $j \in N \backslash i$ (where $ex(\Delta)$ is the set of extreme points of Δ). In other terms, $Eu_i(\sigma_i, s_{-i})$ is i's expected utility when all $j \in N \backslash i$ do not randomize according to s_{-i}, while i's random strategy is σ_i.

- If $\tilde{s}_i \in \mathbb{S}_i$ is a strongly dominated (non-random) strategy, then any random strategy $\sigma_i \in \Delta_{\mathbb{S}_i}$ placing strictly positive probability $\sigma_i(\tilde{s}_i) > 0$ on $\tilde{s}_i$ also is strongly dominated. To see this, let $\hat{s}_i \in \mathbb{S}_i \backslash \tilde{s}_i$ be any strategy strongly dominating $\tilde{s}_i$, and denote by $\sigma_i \in \Delta_{\mathbb{S}_i}$ any random strategy such that $\sigma_i(s_i) > 0$. Define $\sigma'_i \in \Delta_{\mathbb{S}_i} \backslash \sigma_i$ by

$$\begin{aligned} \sigma'_i(s_i) &= \sigma_i(s_i) \text{ for all } s_i \in \mathbb{S}_i \backslash \{\tilde{s}_i, \hat{s}_i\}, \\ \sigma'_i(\tilde{s}_i) &= 0, \\ \sigma'_i(\hat{s}_i) &= \sigma_i(\tilde{s}_i) + \sigma_i(\hat{s}_i). \end{aligned}$$

Then, $Eu_i(\sigma'_i, s_{-i}) - Eu_i(\sigma_i, s_{-i}) =$

$$\begin{aligned} &= \Big[\sigma_i(\tilde{s}_i) + \sigma_i(\hat{s}_i) \Big] u_i(\hat{s}_i, s_{-i}) - \sigma_i(\tilde{s}_i) u_i(\tilde{s}_i, s_{-i}) - \sigma_i(\hat{s}_i) u_i(\tilde{s}_i, s_{-i}) = \\ &= \Big[\sigma_i(\tilde{s}_i) + \sigma_i(\hat{s}_i) \Big] \Big[u_i(\hat{s}_i, s_{-i}) - u_i(\tilde{s}_i, s_{-i}) \Big] > 0. \end{aligned}$$

6.2 Best responses

- Random strategy $\sigma_i \in \Delta_{\mathbb{S}_i}$ is a best response to $\sigma_{-i} \in \times_{j\in N\setminus i}\Delta_{\mathbb{S}_j}$ if

$$Eu_i(\sigma_i, \sigma_{-i}) \geq Eu_i(\sigma'_i, \sigma_{-i}) \text{ for all } \sigma'_i \in \Delta_{\mathbb{S}_i}.$$

- Let $\mathbb{P}(\Delta_{\mathbb{S}_i}) = \{Y : Y \subseteq \Delta_{\mathbb{S}_i}\}$ contain all subsets of simplex $\Delta_{\mathbb{S}_i}$. Then, $BR_i : \times_{j\in N\setminus i}\Delta_{\mathbb{S}_j} \to \mathbb{P}(\Delta_{\mathbb{S}_i})$ is the (random strategy) best response mapping (for $i \in N$), defined for all $\sigma_{-i} \in \times_{j\in N\setminus i}\Delta_{\mathbb{S}_j}$ by

$$BR_i(\sigma_{-i}) = \{\sigma_i : \sigma_i \in \Delta_{\mathbb{S}_i}, Eu_i(\sigma_i, \sigma_{-i}) \geq Eu_i(\sigma'_i, \sigma_{-i}) \text{ for all } \sigma'_i \in \Delta_{\mathbb{S}_i}\}.$$

- To see the form of BR_i, which in turn entails $\emptyset \neq BR_i(\sigma_{-i}) \in \mathbb{P}(\Delta_{\mathbb{S}_i})$ for all $\sigma_{-i} \in \times_{j\in N\setminus i}\Delta_{\mathbb{S}_j}$, note that $Eu_i(\sigma_i, \sigma_{-i}) =$

$$\begin{aligned}
&= \sum_{(s_1,\ldots,s_n)\in\times_{j\in N}\mathbb{S}_j} \left[\sigma_i(s_i)\left(\prod_{j\in N\setminus i}\sigma_j(s_j)\right)\right] u_i(s_1,\ldots,s_n) = \\
&= \sum_{s_i\in\mathbb{S}_i}\sigma_i(s_i)\left[\sum_{s_{-i}\in\mathbb{S}_{-i}}\left(\prod_{j\in N\setminus i}\sigma_j(s_j)\right)u_i(s_i,s_{-i})\right] = \\
&= \sum_{s_i\in\mathbb{S}_i}\sigma_i(s_i)Eu_i(s_i,\sigma_{-i}),
\end{aligned}$$

where $Eu_i(s_i, \sigma_{-i})$ is i's expected utility when all $j \in N\setminus i$ randomize according to σ_{-i} while i plays s_i with probability 1. Thus geometrically

$$Eu_i(\sigma_i, \sigma_{-i}) = \sum_{s_i\in\mathbb{S}_i}\sigma_i(s_i)Eu_i(s_i, \sigma_{-i}) = \langle\sigma_i(\cdot), Eu_i(\cdot, \sigma_{-i})\rangle$$

is the scalar product of $\sigma_i(\cdot) \in \Delta_{\mathbb{S}_i} \subset \mathbb{R}_+^{|\mathbb{S}_i|}$ and $Eu_i(\cdot, \sigma_{-i}) \in \mathbb{R}^{|\mathbb{S}_i|}$. This means that the set of best responses to any σ_i is the convex hull of a non-empty, possibly singleton subset of $ex(\Delta_{\mathbb{S}_i})$, i.e. $BR_i(\sigma_{-i}) = \Delta_{\mathbb{S}^*_i}$,

$$\mathbb{S}^*_i = \mathbb{S}^*_i(\sigma_{-i}) = \{s_i : s_i \in \mathbb{S}_i, Eu_i(s_i, \sigma_{-i}) \geq Eu_i(s'_i, \sigma_{-i}) \text{ for all } s'_i \in \mathbb{S}_i\}$$

being the non-empty, possibly singleton subset of strategies $s_i \in \mathbb{S}_i$ where $Eu_i(s_i, \sigma_{-i})$ attains its maximum (over its $|\mathbb{S}_i|$, at most, distinct values), while $\Delta_{\mathbb{S}^*_i}$ is the $|\mathbb{S}^*_i| - 1$-dimensional unit simplex whose extreme points are indexed by strategies $s_i \in \mathbb{S}^*_i$.

6.3 Equilibrium

- Random strategy profile $\sigma^* = (\sigma^*_1, \ldots, \sigma^*_n) \in \times_{j\in N}\Delta_{\mathbb{S}_j}$ is a (Nash) equilibrium if, again, no player has an incentive to unilaterally deviate, i.e.

$$Eu_i(\sigma^*) = Eu_i(\sigma^*_i, \sigma^*_{-i}) \geq Eu_i(\sigma_i, \sigma^*_{-i}) \text{ for all } \sigma_i \in \Delta_{\mathbb{S}_i} \text{ and all } i \in N.$$

- In terms of best responses, random strategies $\sigma^*_1, \ldots, \sigma^*_n$ must satisfy

$$\sigma^*_i \in BR_i(\sigma^*_{-i}) \text{ for all } i \in N.$$

Accordingly, consider the (whole) best response correspondence
$\mathbb{B}R : \times_{i\in N}\Delta_{\mathbb{S}_i} \to \times_{i\in N}\mathbb{P}(\Delta_{\mathbb{S}_i})$ defined by

$$\mathbb{B}R(\sigma_1,\ldots,\sigma_n) = \times_{i\in N}BR_i(\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_n)$$

for all $\sigma = (\sigma_1,\ldots,\sigma_n) \in \times_{i\in N}\Delta_{\mathbb{S}_i}$.

- In view of Kakutani fixed point theorem for upper hemicontinuous correspondences [19, pp. 950,953], [6, pp. 88-90], best response correspondence $\mathbb{B}R$ has a (i.e. at least one) fixed point σ^*, namely such that $\sigma^* \in \mathbb{B}R(\sigma^*)$, which is precisely the above condition $\sigma_i^* \in BR_i(\sigma_{-i}^*), i \in N$ identifying σ^* as an equilibrium.
- Hence the set of random strategy equilibria of any game is non-empty, and clearly includes the set of non-random strategy equilibria (if any).

6.4 Exercises

- **Exercise 7:** For player set $N = \{1,\ldots,100\}$ with binary strategy sets $\mathbb{S}_i = \{0,1\}$ for all $i \in N$, every strategy profile $(s_1,\ldots,s_n) = s$ $(n = 100)$ is an element of $\{0,1\}^n$, i.e. a vertex of the n-dimensional unit hypercube $[0,1]^n$. For all players $i \in N$, define utilities

$$u_i : \{0,1\}^n \to \left\{\frac{1}{2n}, \frac{1}{2(n-1)},\ldots,\frac{1}{2[n-(n-2)]},\frac{1}{2}\right\},$$

at any strategy profile $s = (s_i, s_{-i}) \in \{0,1\}^n$, by:

$$\begin{aligned} u_i(s_i,s_{-i}) &= \frac{1}{2\sum_{j\in N}s_j} \text{ if } s_i = 1, \text{ while} \\ u_i(s_i,s_{-i}) &= \frac{1}{2(n-\sum_{j\in N}s_j)} \text{ if } s_i = 0. \end{aligned}$$

1. Are there dominated/dominant strategies? If yes, then show how $s_i = 1$ dominates $s_i = 0$ or the opposite. If no, then show that neither $s_i = 1$ nor $s_i = 0$ are dominated. Are there Pareto-dominated strategy profiles? If yes, then provide one Pareto-dominated strategy profile. If no, then show that any strategy profile cannot be Pareto-dominated.
2. Are there pure-strategy equilibria $s = (s_1,....,s_n) \in \{0,1\}^n$? If yes, then provide one equilibrium. If no, then show that at any strategy profile some player may profitably deviate.
3. Verify whether the profile $(\sigma_1^*,\ldots,\sigma_n^*) \in \times_{i\in N}\Delta_{\mathbb{S}_i}$ of random strategies where each player $i \in N$ plays both $s_i = 1$ and $s_i = 0$ with equal probability $\sigma_i^*(0) = \frac{1}{2} = \sigma_i^*(1)$ is an equilibrium or not.
4. Is this a (possibly pure) common interest game? Is it a constant-sum game (i.e. $\sum_{i\in N} u_i(s) = const$)?

- **Exercise 8:** For the two-player constant-sum game with $N = \{1,2\}$ and

$\mathbb{S}_1 = \{1, 2, 3, 4, \ldots, 50\}$ as well as $\mathbb{S}_2 = \{51, 52, 53, 54, \ldots 100\}$, while utilities are

$$\begin{aligned} u_1(s_1, s_2) &= 1 \text{ if } s_1 + s_2 \text{ is odd,} \\ u_1(s_1, s_2) &= 0 \text{ if } s_1 + s_2 \text{ is even,} \\ u_2(s_1, s_2) &= 1 - u_1(s_1, s_2). \end{aligned}$$

(a) Identify the set of pure strategy equilibria.
(b) Identify the set of mixed strategy equilibria.

- **Exercise 9:** Consider the following 3×3 two-player constant-sum game where players are A and B while strategy sets are $\mathbb{S}_A = \mathbb{S}_B = \{0, 1, 2\}$, with payoffs

Table 3: *Payoff matrix* 3×3 *game*

u_A, u_B	0	1	2
0	$1/2, 1/2$	$1, 0$	$0, 1$
1	$1, 0$	$0, 1$	$1, 0$
2	$0, 1$	$1, 0$	$0, 1$

1. What strategies are dominated (either weakly or strongly)?
2. Determine the two best response correspondences in pure strategies.
3. Determine the set of pure-strategy equilibria.
4. Determine the set of mixed-strategy equilibria.

- **Exercise 10:** Consider the 2x2 game, i.e. with two players i, j each with two strategies $\mathbb{S}_i = \{0, 1\} = \mathbb{S}_j$, where payoffs are as follows.

Table 4: *Payoff matrix* 2×2 *game*

u_i, u_j	0	1
0	$0, 0$	$7, 2$
1	$2, 7$	$6, 6$

Determine all equilibria, both with random and non-random strategies, and check whether they are pair-wise comparable in terms of Pareto-dominance (using the expected utility criterion for random strategy equilibria, if any).

- **Exercise 11:** As in Exercise 7, let $N = \{1, \ldots, 100\}$ and $\mathbb{S}_i = \{0, 1\}$ for all $i \in N$. For every $s \in \{0, 1\}^{100}$, define payoffs by

$$\text{if } \sum_{i \in N} s_i \in \{2k : 0 \leq k \leq 50\}, \text{ then } u_i(s) = \begin{cases} 1 \text{ if } i \in \{2k : 1 \leq k \leq 50\}, \\ 0 \text{ if } i \notin \{2k : 1 \leq k \leq 50\}; \end{cases}$$

if $\sum_{i\in N} s_i \notin \{2k : 0 \leq k \leq 50\}$, then $u_i(s) = \begin{cases} 0 \text{ if } i \in \{2k : 1 \leq k \leq 50\}, \\ 1 \text{ if } i \notin \{2k : 1 \leq k \leq 50\}. \end{cases}$

Is this a constant-sum game?

Is this a (possibly pure) common interest game?

Are there dominated strategies?

Are there Pareto-dominated strategy profiles?

Is there any equilibrium with non-random strategies?

Can you find an equilibrium with random strategies?

Are there equilibria where some players randomize while some other do not?

7 Strong equilibrium

- In non-cooperative game theory, much attention has been devoted to methods for strengthening the above standard equilibrium conditions. As already mentioned, when the idea is to select a sufficiently smaller proper subset of a whole large set of multiple equilibria, then the concern is with *equilibrium refinements*. One way to strengthen the equlibrium conditions is by requiring that not only single players but also coalitions have no incentive to (unilaterally but coalitionally) deviate. This approach leads to define strong equilibria as follows. Firstly focus on the case of non-random strategies, which shall be considered again when dealing with potential and congestion games in the sequel.
- Denoting by $2^N = \{A : A \subseteq N\}$ the (power) set of all 2^n coalitions, for every $A \in 2^N$ let $\mathbb{S}_A = \underset{i\in A}{\times} \mathbb{S}_i$ and $\mathbb{S}_{A^c} = \underset{j\in A^c}{\times} \mathbb{S}_j$, where $A^c = N \backslash A$ is the complement of $A \neq \emptyset$ (in 2^N).
- A strong equilibrium [2] is any strategy profile $s \in \times_{j\in N}\mathbb{S}_j$ from which no coalition has an incentive to deviate, meaning that for all $A \in 2^N$ there is no coalitional deviation $\hat{s}_A \in \mathbb{S}_A \backslash s_A$ such that

  $$u_i(\hat{s}_A, s_{A^c}) > u_i(s_A, s_{A^c}) = u_i(s) \text{ for all coalition members } i \in A.$$

- Hence $s \in \times_{j\in N}\mathbb{S}_j$ is a strong equilibrium if for all $A \in 2^N$ and all $\hat{s}_A \in \mathbb{S}_A$ inequality $u_i(\hat{s}_A, s_{A^c}) \leq u_i(s_A, s_{A^c}) = u_i(s)$ holds for some $i \in A$.
- In words, $s \in \mathbb{S}$ is a strong equilibrium if for no coalition $\emptyset \neq A \in 2^N$ is there a choice $\hat{s}_A \in \mathbb{S}_A \backslash s_A$ such that at $(\hat{s}_A, s_{A^c}) \in \mathbb{S}$ all coalition members get a utility strictly greater than at $s = (s_A, s_{A^c})$.
- Clearly the set of (non-random strategy) strong equilibria is a subset of the set of (non-random strategy) equilibria, and thus may well be empty. In fact, even the set of random strategy strong equilibria can be empty.
- For $\emptyset \neq A \in 2^N$, let $\Delta_{\mathbb{S}_A} = \times_{i\in A}\Delta_{\mathbb{S}_i}$ and $\Delta_{\mathbb{S}_{A^c}} = \times_{j\in A^c}\Delta_{\mathbb{S}_j}$. Then, $\sigma \in \times_{j\in N}\Delta_{\mathbb{S}_j}$ is a random strategy strong equilibria if for for no coalition A there exists a deviation $\sigma'_A \neq \sigma_A$ such that $Eu_i(\sigma'_A, \sigma_{A^c}) > Eu_i(\sigma_A, \sigma_{A^c})$ for all coalition members $i \in A$, where $(\sigma_A, \sigma_{A^c}) = \sigma$.

- **Exercise 12:** Is it true that any common interest game has a strong equilibrium (with non-random strategies)? Discuss. And may a (non-random) strategy profile be both: (i) a strong equilibrium, and (ii) Pareto-dominated? Discuss.

8 Potential games

- For the remaining part of the course devoted to non-cooperative games, strategies shall only be non-random ones, as attention turns on a class of games with non-empty set of equilibria defined in terms of potential functions [20] as follows.

- $\Gamma = (N, \times_{j\in N}\mathbb{S}_j, u)$ is a potential game if it admits a potential function, namely a $\mathbf{P} : \times_{j\in N}\mathbb{S}_j \to \mathbb{R}$ such that for each player $i \in N$, for all pairs $s_i, s'_i \in \mathbb{S}_i$ of strategies for i, and for all $n-1$-tuples $s_{-i} \in \mathbb{S}_{-i}$ of strategies for players $j \in N\backslash i$, if $u_i(s_i, s_{-i}) \neq u_i(s'_i, s_{-i})$, then

$$[\mathbf{P}(s_i, s_{-i}) - \mathbf{P}(s'_i, s_{-i})][u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i})] > 0.$$

- In words, for any strategy profile and unilateral deviation from it, the potential varies in the same way (i.e. positive or negative) as the deviator's utility. When it exists, $\mathbf{P}$ is said to be an *ordinal potential* for Γ.

- Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n_{++}$ be a vector of strictly positive weights associated with players. A *w-potential* is an ordinal potential $\mathbf{P}$ satisfying: for each player $i \in N$, for all pairs $s_i, s'_i \in \mathbb{S}_i$ of strategies for i, and for all $n-1$-tuples $s_{-i} \in \mathbb{S}_{-i}$ of strategies for players $j \in N\backslash i$,

$$w_i\big[\mathbf{P}(s_i, s_{-i}) - \mathbf{P}(s'_i, s_{-i})\big] = u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}).$$

  A w-potential with $w_i = 1$ for all $i \in N$ is an *exact potential*.

- A potential game $\Gamma = (N, \times_{j\in N}\mathbb{S}_j, u)$ with potential $\mathbf{P}$ has the same non-empty set of equilibria as game $\Gamma_{\mathbf{P}} = (N, \times_{j\in N}\mathbb{S}_j, u^{\mathbf{P}})$, where utilities are defined by $u^{\mathbf{P}}_i(s) = \mathbf{P}(s)$ for all $i \in N$ and all $s \in \times_{j\in N}\mathbb{S}_j$. In fact, equilibria s^* of Γ are (by definition) *local maximizers* of potential $\mathbf{P}$, where "locality" is in terms of the following notion of neighborhood

$$\mathcal{N}(s) = \bigcup_{i\in N}\{\hat{s} : \hat{s} = (\hat{s}_i, s_{-i}), \hat{s}_i \in \mathbb{S}_i\} \subset \times_{j\in N}\mathbb{S}_j$$

  of strategy profiles $s \in \times_{j\in N}\mathbb{S}_j$. Hence the set of equilibria is

$$\{s^* : s^* \in \times_{j\in N}\mathbb{S}_j, \mathbf{P}(s^*) \geq \mathbf{P}(s) \text{ for all } s \in \mathcal{N}(s^*)\}.$$

- In particular, $\mathbf{P}$ surely has at least one global maximizer strategy profile s^*, i.e. such that $\mathbf{P}(s^*) \geq \mathbf{P}(s)$ for all $s \in \times_{j\in N}\mathbb{S}_j$, and thus potential maximization provides an equilibrium refinement criterion (see above).

- A *path* in $\times_{j\in N}\mathbb{S}_j$ is a (finite) sequence $s^0, s^1, \ldots, s^t, \ldots, s^T \in \times_{j\in N}\mathbb{S}_j$ of *distinct* strategy profiles such that $s^t \in \mathcal{N}(s^{t-1})$ for all $0 < t \leq T$. (Recall that this is the traditional definition of path in graph theory, namely for the simple graph $G = (\times_{j\in N}\mathbb{S}_j, E)$ on strategy profiles as vertices, and with edge set $E = \{\{s, s'\} : s' \in \mathcal{N}(s)\}$.)

- Hence a path is a sequence of unilateral deviations by single players, i.e. for each $0 < t \leq T$ there is some $i \in N$ such that $s^t = (s_i^t, s_{-i}^{t-1})$, although clearly any fixed player $i \in N$ may well be the (unique) deviator at several nodes $s^{t_1}, s^{t_2}, \ldots, s^{T_i}$.

- For $0 < t \leq T$, let $i_t \in N$ be the deviator at t. An *improvement path* is a path where $u_{i_t}(s^t) > u_{i_t}(s^{t-1})$ for all $0 < t \leq T$. In particular, along a *best-response improvement path*

$$s_{i_t}^t \in BR_i(s_{-i_t}^{t-1}) \text{ for all } 0 < t \leq T.$$

- A game is said to have the *finite improvement property* if every improvement path is finite, and potential games do have this property.

- **Exercise 13:** Consider the game with two players i and j whose strategy sets are $\mathbb{S}_i = \{a, m, b\}$ and $\mathbb{S}_j = \{s, c, d\}$. Payoffs are as follows.

Table 5: *Payoff matrix*

$u_i(\cdot,\cdot), u_j(\cdot,\cdot)$	s	c	d
a	$4, 6$	$8, 2$	$4, 1$
m	$8, 4$	$2, 4$	$1, 2$
b	$7, 4$	$1, 1$	$0, 2$

1. What strategies are dominated (either strongly or weakly)?
2. Determine the two best response correspondences.
3. Is (m, s) a (pure-strategy) equilibrium?
4. Determine the set of mixed-strategy equilibria.
5. Is there a finite improvement path? Is there an infinite one?
6. What strategy profiles are pareto-efficient/optimal?

9 Congestion games

- Congestion games are characterized by a set $M = \{a_1, \ldots, a_m\}$ of facilities and by strategy sets $\mathbb{S}_i \subset 2^M$ (or $\mathbb{S}_i \in 2^{2^M}$) for players $i \in N$ consisting of families of subsets of facilities (where $2^M = \{A : A \subseteq M\}$ is the power set of all subsets of facilities, see above). The name of these games comes from thinking of M as the edge set of a given graph $G = (V, E), E = M$, which in turn represents a transportation network. If each player $i \in N$ has to go from some origin vertex $v_i \in V$ to some destination one $v_i' \in V$, then the set $\mathbb{S}_i$ of strategies consists of all (edge sets of) existing paths in G connecting v_i to v_i'. Finally, for each n-tuble of chosen paths, each i's payoff depends on the congestion encountered along i's (chosen) path.

- A congestion game form $F = (N, M, \times_{j \in N} \mathbb{S}_j)$ identifies a whole class of congestion games, each obtained by specifying players' payoffs. Following [18], these payoffs are denoted by $\pi^i : \times_{j \in N} \mathbb{S}_j \to \mathbb{R}$ $(i \in N)$.

- Every strategy profile $s = \{s_1, \dots, s_n\} \in \times_{j\in N}\mathbb{S}_j$ identifies *congestion vector* $c(s) = (c_{a_1}(s), \dots, c_{a_m}(s)) \in \mathbb{Z}_+^m$ defined by

$$c_{a_k}(s) = |\{i \in N : a_k \in s_i\}| \text{ for } 1 \leq k \leq m.$$

- The game is said to be *monotone* when each facility $a_k \in M$ has an associated utility function $u_{a_k} : \mathbb{Z}_+ \to \mathbb{R}$ satisfying $u_{a_k}(l) < u_{a_k}(l')$ whenever $l > l'$ (formalizing that utility decreases as congestion increases) and each $i \in N$ gets a payoff

$$\pi^i(s) = \sum_{a_k \in s_i} u_{a_k}(c_{a_k}(s))$$

given by the sum over chosen facilities $a_k \in s_i$ of the corresponding utility.

- A congestion game form (and thus any game derived from it) is *symmetric* when the strategy set is the same across players: $\mathbb{S}_1 = \dots = \mathbb{S}_n$ [18]. When strategies are paths in a transportation network, symmetry corresponds to the case where all players share the same origin and destination.

- An *exact potential* $\mathbf{P} : \mathbb{S} \to \mathbb{R}$ for these games is

$$\mathbf{P}(s) = \sum_{a_k \in M} \sum_{l=1}^{c_{a_k}(s)} u_{a_k}(l), \tag{1}$$

as for all players $i \in N$ and strategy profiles $s = (s_i, s_{-i}) \in \mathbb{S}_i \times \mathbb{S}_{-i}$, any unilateral deviation $s'_i \in \mathbb{S}_i \backslash s_i$ results in variation $\mathbf{P}(s_i, s_{-i}) - \mathbf{P}(s'_i, s_{-i}) =$

$$= \sum_{a \in s_i \backslash s'_i} u_a(c_a(s_i, s_{-i})) - \sum_{a' \in s'_i \backslash s_i} u_{a'}(c_{a'}(s'_i, s_{-i})) =$$

$= \pi^i(s_i, s_{-i}) - \pi^i(s'_i, s_{-i})$.

- Congestion games Γ being potential games, the set $NE(\Gamma) \neq \emptyset$ of their equilibria is non-empty. In fact, under quite mild conditions the subset $SE(\Gamma) \subseteq NE(\Gamma)$ of strong equilibria of these games is non-empty as well.

- Theorem [18, Section 2]: if $|s_i| = 1$ for all $s_i \in \mathbb{S}_i$ and all $i \in N$ (i.e. if all players only have singleton strategies), then $SE(\Gamma) = NE(\Gamma)$ (this obtains by showing that $NE(\Gamma) \subseteq SE(\Gamma)$).

- For the general case of non-singleton strategies, a fundamental condition is the (non)-existence of bad configurations: union $\cup_{j\in N}\mathbb{S}_j$ of all n strategy sets displays a *bad configuration* if there are two facilities $a, a' \in M$ and three strategies $s, s', s'' \in \cup_{j\in N}\mathbb{S}_j$ such that $a \in s \not\ni a'$ and $a \notin s' \ni a'$ while $a, a' \in s''$ (this indeed may be regarded as an acyclity condition for deviations of non-singleton coalitions, to be compared with the finite improvement property of all potential games).

- [18, Section 4, Theorem 4.1] For any *symmetric congestion game form* $F = (N, M, \times_{j\in N}\mathbb{S}_j)$, if union $\cup_{j\in N}\mathbb{S}_j$ displays no bad configuration, then $SE(\Gamma) = NE(\Gamma)$ for all monotone congestion games Γ derived from F.

- **Exercise 14:** Consider game $\Gamma = (N, \times_{j\in N}\mathbb{S}_j, u)$ where the strategy set of every player $i \in N = \{1, \ldots, n\}$ is the 2^{n-1}-set of coalitions where i is included, i.e. $\mathbb{S}_i = 2_i^N := \{A : A \subseteq N, A \ni i\}$. Utilities $u_i : \times_{j\in N}\mathbb{S}_j \to \mathbb{R}$ are:
$$u_i(s) = \frac{|A_i|}{|\{j : A_j = A_i\}|}$$
for all $s = (s_1, \ldots, s_n) = (A_1, \ldots, A_n) \in 2_1^N \times \cdots \times 2_n^N$ and all $i \in N$. This is the ratio of the number $|A_i|$ of players in the coalition $A_i = s_i$ chosen by i, divided by the number $|\{j : A_j = A_i\}| \in \{1, 2, \ldots, |A_i| - 1, |A_i|\}$ of players who choose the same coalition A_i, including i.

  1. Are there weakly dominated strategies? Are there dominant strategies? Is this a common interest game?
  2. Is this a potential and, in particular, a monotone and/or symmetric congestion game with facilities corresponding to non-empty coalitions $\emptyset \neq A \in 2^N$?
  3. Starting from strategy profile $\overline{s}$ defined by $\overline{s}_i = N$ for all $i \in N$, complete a *best-response* improvement path (see above). Does the path end at a strong equilibrium?
  4. Compute the difference between the values of the exact potential at the end of the path and at $\overline{s}$.
  5. Consider profile s^* where $s_i^* = \begin{cases} \{1, \ldots, i, i+2, \ldots, n\} \text{ if } i < n, \\ \{2, 3, \ldots, n\} \text{ if } i = n. \end{cases}$ Is s^* an equilibrium? Is it different from the end of the best-response improvement path determined above?
  6. Evaluate $\max_{s\in\mathbb{S}} \sum_{i\in N} u_i(s)$. At what profiles $s \in \times_{j\in N}\mathbb{S}_j$ is this maximum attained?
  7. Consider profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_n^*) \in \times_{j\in N}\Delta_{\mathbb{S}_j}$ of random strategies where the probability $\sigma_j^*(A) = p_a$ that every player $j \in N$ chooses coalition/strategy $A \in 2_j^N$ depends only on $|A| = a \in \{1, \ldots, n\}$ and is the same for all players j. Hence $\sum_{1\leq a\leq n} \binom{n-1}{a-1} p_a = 1$. Verify whether
  $$\frac{1 - (1 - p_a)^a}{p_a} = \frac{1 - (1 - p_{a+1})^{a+1}}{p_{a+1}}$$
  satisfies the equilibrium condition or not. Can you determine a random strategy equilibrium for $n = 2$?

- **Exercise 15:** Consider game $\Gamma = (N, \times_{j\in N}\mathbb{S}_j, u)$ where the strategy set of every player $i \in N = \{1, \ldots, n\}$ consists of the $\mathcal{B}_n$ partitions of N, i.e. $\mathbb{S}_i = \mathcal{P}^N$ (see above). Let $s_i = \{A_1, \ldots, A_{|s_i|}\} \in \mathcal{P}^N$ denote the generic strategy of any player i. Utilities $u_i : \times_{j\in N}\mathbb{S}_j \to \mathbb{R}$ are:
$$u_i(s) = \sum_{A\in s_i} \frac{|A|}{|\{j : A \in s_j\}|},$$
namely the sum over the blocks $A \in s_i$ of the chosen partition of the ratio of their size $|A|$ to the number of players (including i) who have chosen a partition one of whose block is A.

1. Are there weakly dominated strategies? Are there dominant strategies? Is this a common interest game?
2. Is this a potential and, in particular, a monotone and/or symmetric congestion game?
3. Starting from strategy profile $\overline{s}$ defined by $\overline{s}_i = \{N\} = P^\top$ for all $i \in N$, complete a *best-response* improvement path. Does the path end at a strong equilibrium?
4. Compute the difference between the values of the exact potential at the end of the path and at $\overline{s}$.
5. Consider profile s^* where every i chooses the partition $s_i^* = \{N \backslash i, \}$ where i is a singleton and all other $n-1$ players are in a unique block. Is s^* an equilibrium? Is it different from the end of the best-response improvement path determined above?
6. Evaluate $\max_{s \in \mathbb{S}} \sum_{i \in N} u_i(s)$. At what profiles $s \in \times_{j \in N} \mathbb{S}_j$ is this maximum attained?

- **Exercise 16:** Consider game $\Gamma = (N, \times_{j \in N} \mathbb{S}_j, u)$ where the strategy set $\mathbb{S}_i = \mathcal{S}_n$ of every player $i \in N = \{1, \ldots, n\}$ is the symmetric group of the $n!$ permutations $s_i : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Denote by $s_i(j)$ the position where $j \in N$ is mapped by the permutation/strategy s_i of i. Utilities $u_i : \times_{j \in N} \mathbb{S}_j \to \mathbb{R}$ are:

$$u_i(s) = \sum_{1 \le k \le n} \frac{1}{|\{j : s_j(k) = s_i(k)\}|},$$

i.e. the sum over all players k of 1 divided by the number of those who choose a permutation mapping k into the $s_i(k)$-th position, including i.

1. Are there weakly dominated strategies? Are there dominant strategies? Is this a common interest game?
2. Is this a potential and, in particular, a monotone and/or symmetric congestion game with facilities corresponding to the n^2 ordered pairs $(i, j), 1 \le i, j \le n$?
3. Starting from strategy profile $\overline{s}$ defined by $\overline{s}_i = id$ for all $i \in N$, where $id(k) = k$, complete a *best-response* improvement path. Does the path end at a strong equilibrium?
4. Compute the difference between the values of the exact potential at the end of the path and at $\overline{s}$.
5. Consider profile s^* where every i chooses the permutation

$$s_i^*(k) = \begin{cases} k - i + 1 \text{ if } k \ge i, \\ n + k - i + 1 \text{ if } k < i. \end{cases}$$

   Is s^* an equilibrium? Is it different from the end of the best-response improvement path determined above?
6. Evaluate $\max_{s \in \mathbb{S}} \sum_{i \in N} u_i(s)$. At what profiles $s \in \times_{j \in N} \mathbb{S}_j$ is this maximum attained?

10 Choquet expected utility theory

- With the Choquet expected utility theory the focus turns on set functions, namely taking real values on the Boolean lattice $(2^\Omega, \cap, \cup)$ of subsets of a finite set $\Omega = \{\omega_1, \ldots, \omega_m\}$, here consisting of states of nature. In fact, while cooperative coalitional games are real-valued functions defined on coalitions or subsets of a finite player set, in decision under uncertainty set functions are (discrete) *fuzzy measures/probabilities* taking [0-1]-values on events or subsets $A = \{\omega_{i_1}, \ldots, \omega_{i_{|A|}}\} \in 2^\Omega$ of atomic and mutually exclusive events $\omega \in \Omega$. Thus a fuzzy probability is any $\eta : 2^\Omega \rightarrow [0,1]$ satisfying the general requirements $\eta(\emptyset) = 1 - \eta(\Omega)$ and monotonicity: $A \supseteq B \Rightarrow \eta(A) \geq \eta(B)$ for all events $A, B \in 2^\Omega$.

- Recall that a (traditional) probability $p : 2^\Omega \rightarrow [0,1]$ is defined to satisfy $p(\emptyset) = 0 = 1 - p(\Omega)$ and

$$p(A) + p(B) = p(A \cup B) + p(A \cap B) \text{ for all } A, B \in 2^\Omega.$$

Then, a general result concerning valuations and atoms of distributive lattices (such as $(2^\Omega, \cup, \cap)$) detailed in the sequel entails $p(A) = \sum_{i \in A} p(\{i\})$ for all $A \subseteq \Omega$. Hence geometrically $p \in \Delta_m \subset \mathbb{R}^m_+$. On the other hand, a fuzzy probability η geometrically may be seen as $\eta \in [0,1]^{2^m}$.

- A decision maker with utility $u : \Omega \times \mathbb{A} \rightarrow \mathbb{R}_+$, where $\mathbb{A}$ is a set of available actions, in the vN-M expected utility model has subjective belief or (traditional) probability p, and thus ranks actions $a, a' \in \mathbb{A}$ according to their scored expected utility, i.e. $a \succsim a'$ whenever

$$E_p[u(\cdot, a)] = \sum_{\omega \in \Omega} p(\{\omega\}) u(\omega, a) \geq \sum_{\omega \in \Omega} p(\{\omega\}) u(\omega, a') = E_p[u(\cdot, a')]$$

(see Ellsberg paradox above).

- The issue thus is how to rank actions when in subjective beliefs traditional probabilities $p \in \Delta_m$ are replaced with fuzzy ones $\eta \in [0,1]^{2^m}$. This is an aggregation problem: for every action $a \in \mathbb{A}$, the aim is to aggregate the m values $u(\omega_1, a), \ldots, u(\omega_m, a) \in \mathbb{R}_+$ taken by random variable $u(\cdot, a)$ into a unique one $E_\eta[u(\cdot, a)]$. In this view, aggregation $E_p[u(\cdot, a)]$ through traditional probabilities p corresponds to *weighted averaging.*

- Choquet (discrete) integration E^C_η works as follows: for every action $a \in \mathbb{A}$ relabel states according to $(\cdot) : \{1, \ldots, m\} \rightarrow \{1, \ldots, m\}$ in non-decreasing order, meaning $u(\omega_{(1)}, a) \leq \cdots \leq u(\omega_{(m)}, a)$, and slso set $u(\omega_{(0)}, a) := 0$ and/or $\eta(\{\omega_{(m+1)}, \ldots, \omega_{(m)}\}) = \eta(\{\emptyset\}) = 0$; then,

$$E^C_\eta[u(\cdot, a)] = \sum_{1 \leq i \leq m} \Big[u(\omega_{(i)}, a) - u(\omega_{(i-1)}, a) \Big] \eta(\{\omega_{(i)}, \omega_{(i+1)}, \ldots, \omega_{(m)}\}) =$$

$$= \sum_{1 \leq i \leq m} u(\omega_{(i)}, a) \Big[\eta(\{\omega_{(i)}, \omega_{(i+1)}, \ldots, \omega_{(m)}\}) - \eta(\{\omega_{(i+1)}, \ldots, \omega_{(m)}\}) \Big].$$

- The discrete Choquet integral E_η^C is also sometimes regarded as an extension of η from the set $\{0,1\}^m$ of vertices of the m-dimensional unit hypercube $[0,1]^m$ to the whole m-cube, as

$$E_\eta^C[\chi_A] = \eta(A) \text{ for every}$$
$$\chi_A = (\chi_A(\omega_1), \ldots, \chi_A(\omega_m)) \in \{0,1\}^m,$$
$$\chi_A(\omega_i) = \begin{cases} 1 \text{ if } \omega_i \in A, \\ 0 \text{ if } \omega_i \in A^c = \Omega \backslash A, \end{cases}$$

and thus $E_\eta^C[x]$ for $x = (x_1, \ldots, x_m) \in [0,1]^m$ is the extension of η from $\{0,1\}^m$ to $[0,1]^m$.

- **Exercise 17:** For $\mathbb{A} = \{a, a'\}$ and $\Omega = \{\omega_1, \ldots, \omega_4\}$, with

$$\eta(A) = \frac{\left(\sum_{\omega_i \in A} i\right)^2}{100} \text{ for all } A \in 2^\Omega,$$

and $u(\omega_i, a) = i$ as well as $u(\omega_i, a') = 5 - i$ for $i = 1, \ldots, 4$, determine $E_\eta^C[u(\cdot, a)]$ and $E_\eta^C[u(\cdot, a')]$.

11 Cooperative games

- In most general terms, cooperative games may be defined to be functions taking real values on a poset (partially ordered set), which in turn is grounded on a player set. Although in the 70s attention has also been placed on cooperative games with a continuum of players [4], still the concern commonly is with a finite player set $N = \{1, \ldots, n\}$.

- Cooperative games are mostly intended to be *coalitional games*, namely set functions $v : 2^N \to \mathbb{R}$, where subsets $A \in 2^N$ are *coalitions* [25, 26].

- In 1963, a further type of cooperative games was introduced, involving partitions $P = \{A_1, \ldots, A_{|P|}\}$ of N or *coalition structures*. Specifically, *games in partition function form* are functions taking real values on pairs (A, P) such that $A \in 2^N$ and $P \in \mathcal{P}^N$ is a partition of N (see above) such that $A \in P$. These pairs (A, P) are sometimes referred to as *embedded coalitions or embedded subsets* [15, 16].

- In 1990, a third type of cooperative games was introduced and named *global games* [14]. These are simply real-valued partition functions.

- The idea behind coalitional games v is that the worth $v(A)$ of coalitions $A \in 2^N$ quantifies the worth of cooperation among all anly coalition members $i \in A$. However, such a worth cooperation within coalitions might also depend on the response of non-members $j \in A^c = N \backslash A$. In this view, games in partition function form formalize the idea that alternative such responses correspond to alternative partitions of A^c, hence the worth of cooperation within A may take up to $\mathcal{B}_{|A^c|}$ distinct values, where $\mathcal{B}_n$ is the number of partitions of a n-set. Finally, global games are meant to model cooperation over global issues such as global warming, where every partition P of players has an associated worth to be interpreted as the utility level common to all players attained when the whole cooperation takes the form of P, with blocks corresponding to international ageements.

11.1 Posets

- Any set $X \ni x, y, z, \ldots$ endowed with a binary order relation $\geqslant$ satisfying

  1. $x \geqslant x$ for all $x \in X$ (reflexivity),
  2. $x \geqslant y$ and $y \geqslant z$ entail $x \geqslant z$ for all $x, y, z \in X$ (transitivity),
  3. $x \geqslant y$ and $y \geqslant x$ entail $x = y$ (antisymmetry),

  is *partially ordered*, i.e. a poset. Here $\geqslant$ denotes the generic order relation. In the sequel, the concern shall be mainly with inclusion $\supseteq$ as the order relation, while maintaing $\geqslant$ as the coarsening order relation between partitions $P, Q \in \mathcal{P}^N$ introduced in Section 2.3. (see also [13, Chapters 1, 2], [1, Chapter 2], [29, Chapter 1]).

- Grounded on $N = \{1, \ldots, n\}$, the posets $(X, \geqslant)$ to be dealt with are finite, i.e. $|X| < \infty$, and also have a bottom element $x_\perp \in X$, i.e. $x \geqslant x_\perp$ for all $x \in X$, as well as a top element $x^\top \in X$, i.e. $x^\top \geqslant x$ for all $x \in X$.

- For all $x, y \in X$, the corresponding interval or segment [24] is the subset $[x, y] = \{z : x \leqslant z \leqslant y\} \subseteq X$ (hence if $x \leqslant y$, then $[x, y] \neq \emptyset$ while $[y, x] = \emptyset$).

- A chain is a subset $\mathcal{K} \subset X$ any two of whose elements are comparable, i.e. for all $x, y \in \mathcal{K}$, either $[x, y] \neq \emptyset$ or else $[y, x] \neq \emptyset$ hold.

- Dually, an antichain is a subset $\mathcal{AK} \subset X$ any two of its elements are uncomparable, i.e. for all $x, y \in \mathcal{AK}$, both $[x, y] = \emptyset$ and $[y, x] = \emptyset$ hold.

- The length of a chain $\mathcal{K} = \{x_0, x_1, \ldots, x_k\}$ is $|\mathcal{K}| - 1 = k$.

- The covering relation, denoted by $>^*$, is defined as follows:

  $x >^* y \Leftrightarrow [y, x] = \{x, y\}$ for any $x, y \in X$.

- For $z \geqslant y$, a zy-chain $\mathcal{K}_*^{zy} = \{y = x_0, x_1, \ldots, x_k = z\}$ is said to be maximal if $x_l >^* x_{l-1}$ for all $0 < l \leq k$.

- If for any $y, z \in X$ all maximal $z - y$-chains have the same length, then poset $(X, \geqslant)$ is said to satisfy the Jordan-Dedekind JD condition, in which case for every element $x \in X$ the length of any maximal $x - x_\perp$-chain is the *rank* of x. Formally, for any poset $(X, \geqslant)$ with bottom element $x_\perp$ and satisfying the JD condition, the rank function $r : X \to \mathbb{Z}_+$ is defined recursively by

  1. $r(x_\perp) = 0$,
  2. $x >^* y \Rightarrow r(x) = r(y) + 1$.

  Thus the rank measures the height of elements (in the Hasse diagram [1]).

11.2 Maximal chains of subsets and permutations

- For maximal chains $\mathcal{K}_*^{N\emptyset} = \{\emptyset = A_0, A_1, \ldots, A_n = N\}$ of subsets, the covering relation clearly is

$$A_k \supset^* A_{k-1} \Leftrightarrow A_k \supset A_{k-1}, |A_k| - 1 = |A_{k-1}| \ (0 < k \leq n).$$

- The cardinality of subsets is precisely their rank: $|A_k| = k = r(A_k)$, $0 \leq k \leq n$. Hence the poset $(2^N, \supseteq)$ of coalitions has $n+1$ levels, and

$$|\{A : r(A) = k\}| = \binom{n}{k} = \binom{n}{n-k} = |\{A : r(A) = n-k\}| \text{ for } 0 \leq k \leq n.$$

- Maximal $N\emptyset$-chains $\mathcal{K}_*^{N\emptyset}$ of subsets correspond bijectively to permutations $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, with $\pi(i)$ denoting the position where $i \in N$ is mapped by $\pi \in \mathcal{S}_n$ (see Section 2.4). The bijection $\pi \leftrightarrow \mathcal{K}_{*\pi}^{N\emptyset}$ between permutations $\pi \in \mathcal{S}_n$ and maximal chains $\mathcal{K}_{*\pi}^{N\emptyset} = \{A_0^\pi, A_1^\pi, \ldots, A_n^\pi\}$ is

$$A_k^\pi = \{i : \pi(i) \leq k\} \text{ for } 0 \leq k \leq n.$$

- For every subset $A \in 2^N$, there are $|A|!(n-|A|)! = n!/\binom{n}{|A|}$ maximal chains $\mathcal{K}_*^{N\emptyset}$ "meeting" A, i.e. such that $A \in \mathcal{K}_*^{N\emptyset}$. Similarly for every $A, A' \in 2^N$ where $A' \supset^* A$, there are $|A|!(n-|A|-1)!$ maximal chains $\mathcal{K}_*^{N\emptyset}$ meeting both A, A', i.e. such that $A, A' \in \mathcal{K}_*^{N\emptyset}$.

11.3 Boolean and geometric lattices

- Given a (finite) poset $(X, \geqslant)$, if for every subset $\{x_1, \ldots, x_k\} = Y \subseteq X$ there exist:

  1. the greatest lower bound or infimum *inf* $Y = \bigwedge Y = \underline{y} \in X$, namely
     (a) $\underline{y} \leqslant x_1, \ldots, x_k$, and
     (b) $\underline{y} \geqslant y'$ for all $y' \in X$ satisfying $y' \leqslant x_1, \ldots, x_k$;
  2. the least upper bound or supremum *sup* $Y = \bigvee Y = \overline{y} \in X$, namely
     (a) $\overline{y} \geqslant x_1, \ldots, x_k$, and
     (b) $\overline{y} \leqslant y''$ for all $y'' \in X$ satisfying $y'' \geqslant x_1, \ldots, x_k$;

  then $\bigwedge Y$ and $\bigvee Y$ are unique for every $Y \subseteq X$ and $(X = L, \wedge, \vee)$ is a complete lattice, where $\wedge$ and $\vee$ are the *meet* and *join* (operators), respectively. The bottom and top elements $x_\perp$ and $x^\top$ clearly are
  $x_\perp = \bigwedge X = inf\ X$ and $x^\top = \bigvee X = sup\ X$.

- A (complete) lattice $(L, \wedge, \vee)$ is distributive if

$$\begin{aligned} x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \\ x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \end{aligned}$$

  for all $x, y, z \in L$.

- The set $L_{\mathcal{A}} \subset L$ of *atoms* of a lattice $(L, \wedge, \vee)$ with rank function

$$r : L \to \mathbb{Z}_+ \text{ is } L_{\mathcal{A}} = \{x : x >^* x_\perp\} = \{x : r(x) = 1\},$$

  namely the set of elements covering the bottom.

- The set $L_{\mathcal{J}}$ of *join-irreducible elements* of a lattice $(L, \wedge, \vee)$ is

$$L_{\mathcal{J}} = \{x : x = y \vee z \Rightarrow x = y \text{ or } x = z \text{ or both}\}.$$

- A lattice $(L, \wedge, \vee)$ is *atomic* if every element is a join of atoms, i.e. for all $x \in L$ there are $a_1, \ldots, a_k \in L_{\mathcal{A}}$ such that $x = a_1 \vee \cdots \vee a_k$. If L is atomic, then $L_{\mathcal{J}} \backslash L_{\mathcal{A}} = \{x_\perp\}$.

- For $x \in L$ and $y_1, \ldots, y_k \in L_{\mathcal{J}}$, *join-decomposition* $x = y_1 \vee \cdots \vee y_k$ is *irredundant* if

$$x > y_1 \vee \cdots \vee y_{l-1} \vee y_{l+1} \vee \cdots \vee y_k \text{ for all } 1 \leq l \leq k,$$

  and redundant otherwise.

- A lattice $(L, \wedge, \vee)$ is complemented if for every element $x \in L$ there is some (i.e. at least one) element $x' \in X$ such that $x \wedge x' = x_\perp$ and $x \vee x' = x^\top$.

- Coalitional games $v : 2^N \to \mathbb{R}$ thus are *lattice functions*, as $(2^N, \cap, \cup)$ is the Boolean lattice of subsets of N, which is distributive, atomic and complemented. The set of atoms $\{\{1\}, \ldots, \{n\}\}$ consists of the n singletons $\{i\}, i \in N$ and every $A \in 2^N$ has the unique join-decomposition $A = \cup_{i \in A}\{i\}$, i.e. $A = \{i_1, \ldots, i_{|A|}\} = \{i_1\} \cup \cdots \cup \{i_{|A|}\}$, which is clearly irredundant. Every $A \in 2^N$ also has the unique complement $A^c = N \backslash A$, as $A \cap A^c = \emptyset$ and $A \cup A^c = N$.

- Global games $h : \mathcal{P}^N \to \mathbb{R}$ are lattice functions defined on the *geometric lattice* $(\mathcal{P}^N, \wedge, \vee)$ of partitions of N. A (finite) lattice $(L, \wedge, \vee)$ is geometric [1, p. 52] if

  (a) it is atomic (or a *point lattice*), and

  (b) for any two elements $x, y \in L$ such that $x >^* y$,

  there is an atom (or point) $a \in L_{\mathcal{A}}$ such that $y \ngeqslant a$

  satisfying $x = y \vee a$.

  The atoms are those partition with $n-1$ blocks, namely $n-2$ singletons and one pair. Let $N_2 = \{\{i, j\} : 1 \leq i < j \leq n\}$ be the $\binom{n}{2}$-set of (onordered) pairs of players, and denote by $[ij]$ the atom whose non-singleton block is $\{i, j\} \in N_2$. Thus $\mathcal{P}^N$ has $\binom{n}{2}$ atoms. As for join-decompositions of any partition $P = \{A_1, \ldots, A_{|P|}\}$, the unique maximal and generally redundant one evidently is

$$P = \bigvee_{[ij] \leqslant P} [ij] = \bigvee_{A \in P} \bigvee_{\{i,j\} \subseteq A} [ij],$$

  while any minimal or irredundant one requires $|A| - 1$ (distinct) atoms for each bloch $A \in P$, and

$$\sum_{A \in P} (|A| - 1) = n - |P| = r(P) \text{ is the rank of partitions.}$$

The number of maximal chains of partitions is

$$\underbrace{\binom{n}{2}\binom{n-1}{2}\cdots\binom{3}{2}\binom{2}{2}}_{n-1 \text{ times}} = \frac{n!(n-1)!}{2^{n-1}}.$$

Partitions $P \neq P_\perp, P^\top$ have several complements which also are generally comparable in terms of coarsening $\geqslant$. To see this, consider a co-atom (or dual atom) $P = \{A, A^c\}, \emptyset \subset A \subset N$. Any of the $|A|(n - |A|)$ atoms $[ij]$ such that $i \in A, j \in A^c$ is a complement of P, but complements also are those partitions with $n - k$ blocks, $k \leq \min\{|A|, n - |A|\}$, of the form $[i_1 j_1] \vee \cdots \vee [i_k, j_k]$ where $i_1, \ldots, i_k \in A$ and $j_1, \ldots, j_k \in A^c$ are all distinct.

And of course $[i_1 j_1] \vee \cdots \vee [i_k, j_k] > [i_1 j_1] \vee \cdots \vee [i_{k-1}, j_{k-1}]$.

Hence there are $(\geqslant)$-comparable complements for all co-atoms $\{A, A^c\}$ such that $\min\{|A|, n - |A|\} > 1$. On the other hand, every atom $[ij]$ has 2^{n-2} distinct complements, but any two of them are uncomparable. In fact, *all complemets of a partition have the same rank (and thus are pairwise uncomparable) if and only if the partition is a modular element of $\mathcal{P}^N$, namely with a number of non-singleton blocks* ≤ 1. Thus there are $2^n - n$ modular partitions: $P_\perp$ and $P^\top$ together with all those

$$P_\perp^A := \{A, \{i_1\}, \ldots, \{i_{n-|A|}\}\} \text{ where } 1 < |A| < n, \{i_1, \ldots, i_{n-|A|}\} = A^c.$$

11.4 Möbius function

- The *incidence algebra* [1, p. 138], [24, p. 344], [29, p. 149]) of a (locally) finite poset $(X, \geqslant)$ consists of those functions $f : X \times X \to \mathbb{R}$ defined over *ordered* pairs of poset elements satisfying $x \not\leqslant y \Rightarrow f(x, y) = 0$ and equipped with

  1. usual multiplication by scalars $\alpha \in \mathbb{R}$ and sum $(f + f')(x, y) = f(x, y) + f'(x, y)$,
  2. product $h = f \cdot f'$ given by convolution:

  $$h(x, y) = \sum_{x \leqslant z \leqslant y} f(x, z) f'(z, y),$$

  3. the identity element or Kronecker delta

  $$\delta(x, y) = \begin{cases} 1 \text{ if } x = y, \\ 0 \text{ otherwise.} \end{cases}$$

- Two fundamental elements of the incidence algebra are the zeta function ζ, defined by $\zeta(x, y) = 1$ for all $y \geqslant x$, and its inverse, namely the Möbius function μ. That is, $\delta = \zeta \cdot \mu = \mu \cdot \zeta$ or $\mu = \zeta^{-1}$, where μ is defined recursively by

$$\mu(x, y) = \begin{cases} 1 \text{ if } x = y, \\ -\sum_{x \leqslant z < y} \mu(x, z) \text{ if } x < y. \end{cases}$$

- In terms of matrices, any element of the incidence algebra corresponds to a $|X| \times |X|$ matrix, with rows giving the first element of the ordered pair and columns giving the second one. Lattice elements can always be numbered in such a way that the zeta matrix is upper triangular (i.e. all elements below the main diagonal are zeros), and thus the Möbius matrix is the inverse of the zeta matrix.
- Example: for the poset consisting of a chain $\mathcal{K} = \{x_0, x_1, \ldots, x_k\}$ the Möbius function is
  $\mu(x_0, x_0) = 1$,
  $\mu(x_0, x_1) = -1$ and
  $\mu(x_0, x_l) = 0$ for $1 < l \leq k$.
- The Möbius function of $(2^N, \supseteq)$ is $\mu(A, B) = (-1)^{|B \backslash A|}$.

11.5 Möbius inversion

- Any poset function $f : X \to \mathbb{R}$ has Möbius inversion $\mu^f : X \to \mathbb{R}$ given by
  $$\mu^f(x) = \sum_{x_\perp \leqslant y \leqslant x} \mu(y, x) f(y),$$
  with f and μ^f linked by the following combinatorial *"analog of the fundamental theorem of the calculus"* [24]
  $$f(x) = \sum_{x_\perp \leqslant y \leqslant x} \mu^f(x),$$
  and thus recursively
  $$\mu^f(x) = f(x) - \sum_{x_\perp \leqslant y < x} \mu^f(y).$$
- For coalitional games or set functions $v : 2^N \to \mathbb{R}$,
  $$\begin{aligned} \mu^v(A) &= \sum_{B \subseteq A} (-1)^{|A \backslash B|} v(B), \\ v(A) &= \sum_{B \subseteq A} \mu^v(B), \\ \mu^v(A) &= v(A) - \sum_{B \subset A} \mu^v(B). \end{aligned}$$
- Sometimes $\mu^v(A)$ is called the Harsanyi (1959) dividend, and interpreted as the net (possibly negative) added worth of cooperation within coalition $A \in 2^N$ with respect to all proper subcoalitions $B \subset A$.
- **Exercise 18:** Regarding $N = \{1, \ldots, n\}$ as a set of atomic and mutually exclusive events or states of nature, let $\eta : 2^N \to [0, 1]$ be a fuzzy measure and for any $f : N \to \mathbb{R}_+$ denote by $E^C_\eta[f]$ the (discrete) Choquet integral of f with respect to η (over N, see Section 10). Show that
  $$E^C_\eta[f] = \sum_{A \in 2^N} \mu^\eta(A) \min\{f(i) : i \in A\}.$$
  [Firstly check that $\eta(A) - \eta(A \backslash i) = \sum_{B \in 2^A \backslash 2^{A \backslash i}} \mu^\eta(B)$ for all $i \in A \in 2^N$.]

11.6 Vector spaces and bases

- For a finite poset $(X \geqslant)$, any function $f : X \to \mathbb{R}$ is a point in a vector space $f \in \mathbb{R}^{|X|}$.
- For the element δ of the incidence algebra (of X, see above), and for all $x \in X$, let $\delta(x, \cdot) : X \to \{0, 1\}$. That is, $\delta(x, y) = 1$ if $y = x$ and $\delta(x, y) = 0$ if $y \neq x$ for all $y \in X$. Then, the $|X|$-collection $\{\delta(x, \cdot) : x \in X\}$ of poset functions is the so-called canonical basis of $\mathbb{R}^{|X|}$. In particular, any poset function $f : X \to \mathbb{R}$ or $f \in \mathbb{R}^{|X|}$ can be expressed as a linear combination of basis elements. The coefficients of the linear combination are precisely the $|X|$ values taken by f. That is to say, $f = \sum_{x \in X} f(x)\delta(x, \cdot)$.
- Alternatively, consider the element of the incidence algebra of X given by the zeta function ζ, and for all $x \in X$ let $\zeta(x, \cdot) : X \to \{0, 1\}$. That is, $\zeta(x, y) = 1$ if $y \geqslant x$ and $\delta(x, y) = 0$ if $y \not\geqslant x$ for all $y \in X$. Then, the $|X|$-collection $\{\zeta(x, \cdot) : x \in X\}$ of poset functions is again a linear basis of $\mathbb{R}^{|X|}$. In particular, any poset function $f : X \to \mathbb{R}$ or $f \in \mathbb{R}^{|X|}$ can be expressed as a linear combination of basis elements with coefficients given by the Möbius inversion μ^f of f. That is to say, $f = \sum_{x \in X} \mu^f(x)\zeta(x, \cdot)$.
- In fact, from above, for every (real) value $f(y), y \in X$ taken by f, it holds

$$f(y) = \sum_{x \in X} \mu^f(x)\zeta(x, y) = \sum_{x_\perp \leqslant x \leqslant y} \mu^f(x)\zeta(x, y).$$

- In terms of coalitional games $v : 2^N \to \mathbb{R}$, for every $A \in 2^N$ the game $\zeta(A, \cdot) : 2^N \to \{0, 1\}$ is the so-called unanimity game (see chapters 2 and 7, photocopies), denoted by $u_A(\cdot) = \zeta(A, \cdot)$, hence $u_A(B) = 1$ if $A \subseteq B$ and $u_A(B) = 0$ if $A \not\subseteq B$ (for all $A, B \in 2^N$). Thus, every coalitional game v may be expressed as linear combination $v(\cdot) = \sum_{A \in 2^N} u_A(\cdot)\mu^v(A)$, in that every (real) value $v(B), B \in 2^N$ taken by v is given by

$$v(B) = \sum_{A \in 2^N} u_A(B)\mu^v(A) = \sum_{A \subseteq B} \mu^v(A).$$

  [Notation: $\sum_{A \subseteq B} \mu^v(A) = \sum_{A \in 2^B} \mu^v(A)$.]

- Finally note that for any poset $(X, \geqslant)$ and all elements $x \in X$ it holds

$$\mu^{\zeta(x,\cdot)}(y) = \delta(x, y) = \begin{cases} 1 \text{ if } x = y, \\ 0 \text{ otherwise,} \end{cases} \quad \text{as } \mu \cdot \zeta = \delta,$$

  and thus for $(2^N, \supseteq)$ and generic unanimity game $u_A, A \in 2^N$ it holds

$$\mu^{u_A}(B) = \begin{cases} 1 \text{ if } B = A, \\ 0 \text{ otherwise.} \end{cases}$$

11.7 Lattice functions

- Real-valued functions $f : L \to \mathbb{R}$ defined on a lattice $(L, \wedge, \vee)$ are:
  1. bottom-normalized if $f(x_\perp) = 0$,

2. monotone or order-preserving if
   $f(x) \geq f(y)$ for all $x, y \in L$ such that $x \geqslant y$,
3. antitone or order-reversing if
   $f(x) \geq f(y)$ for all $x, y \in L$ such that $x \leqslant y$,
4. supermodular if $f(x \wedge y) + f(x \vee y) \geq f(x) + f(y)$ for all $x, y \in L$,
5. submodular if $f(x \wedge y) + f(x \vee y) \leq f(x) + f(y)$ for all $x, y \in L$,
6. modular or valuations (of X) if they are both supermodular and submodular, i.e. $f(x \wedge y) + f(x \vee y) = f(x) + f(y)$ for all $x, y \in L$,
7. totally positive if their Möbius inversion takes only non-negative values: $\mu^f(x) \geq 0$ for all $x \in L$.

11.8 Set functions and Boolean or Pseudo-Boolean functions

- The 2^n-set $\{0,1\}^n$ of vertices of the n-dimensional unit hypercube $[0,1]^n$ corresponds bijectively to power set 2^N, in that characteristic functions $\chi_A : N \to \{0,1\}$, $A \in 2^N$ are defined by $\chi_A(i) = 1$ if $i \in A$ and $\chi_A(i) = 0$ if $i \in A^c$.

- For this reason, set functions $v : 2^N \to \mathbb{R}$ are commonly dealt with also in terms of pseudo-Boolean functions $f^v : \{0,1\}^n \to \mathbb{R}$.

- In this view, coalitional games $v : 2^N \to \mathbb{R}_+$ are set functions or, equivalently, pseudo-Boolean functions. In this context, bottom-normalization $v(\emptyset) = 0 = f^v(\chi_\emptyset)$ is by large and far a universal assumption (quantifying the idea that the cooperation of no player is worth nothing). Another standard assumption is monotonicity, i.e. $A \in 2^B \subseteq 2^N \Rightarrow v(A) \leq v(B)$, quantifying the idea that the cooperation within larger coalitions can be no smaller than the cooperation within larger coalitions.

- Boolean functions $f : \{0,1\}^n \to \{0,1\}$ correspond to so-called *simple (coalitional) games* $w : 2^N \to \{0,1\}$, and generally model voting situations. Unanimity games $u_A, \emptyset \neq A \in 2^N$ provide perhaps the most basic example of simple coalitional games.

- Exercise: let $\{A_1, \dots, A_k\} = \mathcal{AK} \subset 2^N$ be an antichain of coalitions (or subsets) and consider the following two simple games:

  $w^+, w^{\max} : 2^N \to \{0,1\}$ defined by

  $w^+(B) = \sum_{1 \leq l \leq k} u_{A_l}(B)$ and $w^{\max}(B) = \max\limits_{1 \leq l \leq k} u_{A_l}(B)$ for all $B \in 2^N$.

  Determine their Möbius inversions $\mu^{w^+}, \mu^{w^{\max}} : 2^N \to \mathbb{Z}$. Does anything change in your answer/computation if $\{A_1, \dots, A_k\} = \mathcal{K} \subset 2^N$ is, instead, a chain?

- For generic (i.e. non-simple) coalitional games $v : 2^N \to \mathbb{R}_+$, apart from bottom-normalization and monotonicity, further assumptions sometimes used are:

  supermodularity (traditionally called convexity):

  $v(A \cap B) + v(A \cup B) \geq v(A) + v(B)$ for all $A, B \in 2^N$, and superadditivity: for all $A, B \in 2^N$, if $A \cap B = \emptyset$, then

$v(A \cup B) \geq v(A) + v(B)$.

Clearly, the former entails the latter (that is, the latter is a weaker condition than the former). Also note that total positivity (i.e. $\mu^v(A) \geq 0$ for all $A \in 2^N$) entails both supermodularity and monotonicity.

- Exercise: Show that if $v : 2^N \to \mathbb{R}_+$ is supermodular, then for all (players) $i \in N$ and for all $A \subseteq B \subseteq N \backslash i$ (i.e. $i \notin B$) it holds

  $v(A \cup i) - v(A) \leq v(B \cup i) - v(B)$.

- Exercise: let $N = \{1, 2, 3, 4, 5, 6, 7\}$ and consider the symmetric set function $v : 2^N :\to \mathbb{Z}_+$ defined by $v(A) = |A|^{|A|}$ for all $A \in 2^N$. Compute its Möbius inversion $\mu^v : 2^N \to \mathbb{Z}_+$. Recall that $0^0 = 1$ (see https://www.math.hmc.edu/funfacts/ffiles/10005.3-5.shtml). Additionally, let , $w : 2^N \to \mathbb{Z}_+, w(\emptyset) = 0$. Determine both $w, \mu^w : 2^N :\to \mathbb{Z}_+$ such that $v(A) = w(A)$ for all $A \in 2^N, A \neq \emptyset$.

11.9 Polynomial multilinear extension of set functions

- From exercise 1. p. 34 above, for given fuzzy measure (or normalized and monotone set function) η, the discrete Choquet integral, regarded as an operator over the set of all integrands $g : N \to [0, 1]$ or equivalently $g \in [0, 1]^n$, may well be regarded as an extension of η from the set $\{0, 1\}^n$ of vertices (for integrands $\chi_A \in \{0, 1\}^n, A \in 2^N$) to the whole n-dimensional hypercube $[0, 1]^n$ (for generic integrands $g \in [0, 1]^n$).

- By the way, nobody ever asked how to perform Choquet (discrete) integration when the integrand also takes negative values.... Such a (discrete) non-additive integration, i.e. with respect to a fuzzy measure, may be symmetric (Šipoš integral) or else asymmetric (Choquet integral), see

  https://arxiv.org/pdf/0804.1760v1.pdf page 4, where the notation is very much different from the one adopted here, and where the dual γ^* of a fuzzy measure γ mentioned in class is alternatively called "conjugate", and denoted by $\overline{\gamma}$.

- A far more important such an extension (i.e. from $\{0, 1\}^n$ to $[0, 1]^n$) of set functions $v : 2^N \to \mathbb{R}$, or equivalently of pseudo-Boolean functions $f^v : \{0, 1\}^n \to \mathbb{R}$, is the polynomial multilinear one, or MLE for short, denoted by $\hat{f}^v : [0, 1]^n \to \mathbb{R}$ and formalized hereafter.

- The MLE $\hat{f}^v : [0, 1]^n \to \mathbb{R}$ of v (or of f^v) takes values

$$\hat{f}^v(\chi_B) = \sum_{A \in 2^N} \left(\prod_{i \in A} \chi_B(i) \right) \mu^v(A) = \sum_{A \subseteq B} \mu^v(A) = v(B)$$

  on vertices $\chi_B \in \{0, 1\}^n, B \in 2^N$, and

$$\hat{f}^v(q) = \sum_{A \in 2^N} \left(\prod_{i \in A} q_i \right) \mu^v(A)$$

on any point $q = (q_1, \ldots, q_n) \in [0,1]^n$ of the n-hypercube. Conventionally, $\prod_{i \in \emptyset} q_i := 1$ [8, p. 157], hence $\hat{f}^v(\chi_\emptyset) = v(\emptyset)$. Therefore, $\hat{f}^v : [0,1]^n \to \mathbb{R}$ is a polynomial multilinear extension of set function v in that:

1. it is a polynomial with n variables $q_1, \ldots, q_n$,
2. it is linear in each variable (i.e. no variable q_i ever appears in form $q_i^k, k > 1$),
3. it is an extension of f^v from the 2^n-set $\{0,1\}^n$ of vertices of the n-dimensional unit hypercube $[0,1]^n$ to the whole of it;
4. in particular, the (non-zero) values $\mu^v(A), A \in 2^N$ taken by Möbius inversion μ^v are the coefficients of the polynomial, while its degree is $\max\{|A| : \mu^v(A) \neq 0\}$.

- **First derivatives:** for the MLE $\hat{f}^v$ of set functions v, the first (order) i-th derivative [8, p. 157] of $\hat{f}^v$ at q is $\frac{\partial \hat{f}^v}{\partial q_i}(q) = \hat{f}^v_i(q) =$

$$= \hat{f}^v(q_1, \ldots, q_{i-1}, 1, q_{i+1}, \ldots, q_n) - \hat{f}^v(q_1, \ldots, q_{i-1}, 0, q_{i+1}, \ldots, q_n) =$$

$$= \sum_{A \ni i} \left(\prod_{j \in A \backslash i} q_j \right) \mu^v(A).$$

- Although it is not used in the sequel, for the sake of completeness it may be mentioned that the second order ij-th derivative of $\hat{f}^v$ at q is

  $\hat{f}^v_{ij}(q) = \frac{\partial \hat{f}^v_i}{\partial q_j}(q) = \frac{\partial \hat{f}^v_j}{\partial q_i}(q) = \frac{\partial \hat{f}^v}{\partial q_i \partial q_j}(q)$, see [8, p. 207].

- Note that $\hat{f}^v_i(q)$ factually is a function of only $n-1$ variables, namely $q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n$. However, it is still to be regarded as a function of the n variables $q_1, \ldots, q_n$.

- At vertices $\chi_B, B \in 2^N$ of $[0,1]^n$, this $\hat{f}^v_i$ yields

$$\hat{f}^v_i(\chi_B) = \sum_{A \ni i} \left(\prod_{j \in A \backslash i} \chi_B(j) \right) \mu^v(A).$$

  Therefore, $\hat{f}^v_i(\chi_B) = v(B) - v(B \backslash i)$ if $i \in B$
  and $\hat{f}^v_i(\chi_B) = v(B \cup i) - v(B)$ if $i \notin B$.
  In fact, $\prod_{j \in A \backslash i} \chi_B(j) = 1$ if $A \backslash i \subseteq B$
  and $\prod_{j \in A \backslash i} \chi_B(j) = 0$ if $A \backslash i \not\subseteq B$.

- For $i \in N$ and $B \subseteq N \backslash i$, difference $v(B) - v(B \backslash i)$ is fundamental in cooperative (i.e. coalitional) game theory; it is the so-called "marginal contribution" of player $i \in B$ to coalition $B \in 2^N$ [25]. In particular, such a marginal contribution equals $\sum_{A \subseteq B : A \ni i} \mu^v(A)$.
  The same quantity, of course, is $\sum_{A \subseteq B} \mu^v(A \cup i)$ for $i \notin B$.

- Exercise: let $\mathbf{\frac{1}{2}} = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) \in [0,1]^n$ be the center of the n-dimensional unit hypercube.

1. Compute
$$\left(\hat{f}_1^v\left(\mathbf{\frac{1}{2}}\right),\ldots,\hat{f}_n^v\left(\mathbf{\frac{1}{2}}\right)\right)\in\mathbb{R}_+^n$$
for generic coalitional game $v\in\mathbb{R}^{2^n}$. You should (quite easily) obtain
$$\begin{aligned}\hat{f}_i^v\left(\mathbf{\frac{1}{2}}\right) &= \sum_{A\subseteq N\backslash i}\frac{\mu^v(A\cup i)}{2^{|A|}} = \\ &= \sum_{A\subseteq N\backslash i}\frac{v(A\cup i)-v(A)}{2^{n-1}} =: \phi_i^{Ba}(v) \text{ for } 1\leq i\leq n,\end{aligned}$$
but do display every single step of your computations and carefully comment them. This is in fact the (i-th component of the) so-called Banzhaf value of Banzhaf power index [5, 25] of coalitional games, mostly applying to the case where v is a simple game, that is to say $v:2^N\rightarrow\{0,1\}$ with $v(\emptyset)=1-v(N)=0$ and
$A\subseteq B\in 2^N\Rightarrow v(A)\leq v(B)$ for all $A,B\in 2^N$ (monotonicity).
2. Let $N=\{1,2,3,4\}$ and consider two games v and w defined as follows:
   (a) v is not simple and in particular $v(\emptyset)=0$ while
   $$v(A)=\prod_{i\in A} i \text{ for all } \emptyset\neq A\in 2^N,$$
   (b) w is simple and in particular
   $$w(A)=\begin{cases}1 \text{ if } \sum_{i\in A} i\geq 3,\\ 0 \text{ if } \sum_{i\in A} i<3\end{cases}.$$
   Compute both $\phi^{Ba}(v)=(\phi_1^{Ba}(v),\phi_2^{Ba}(v),\phi_3^{Ba}(v),\phi_4^{Ba}(v))\in\mathbb{R}^4$ and $\phi^{Ba}(w)=(\phi_1^{Ba}(w),\phi_2^{Ba}(w),\phi_3^{Ba}(w),\phi_4^{Ba}(w))\in\mathbb{R}^4$

12 Solutions of coalitional games

- Conceptually, a solution of cooperative games should quantify the a priori worth (or value), for each player, of cooperating (i.e. joining the grand coalition N).
- Formally, the solution or value of a (monotone) coalitional game v assigns to each player $i\in N$ the expectation

  $E_{p_i}[v(A\cup i)-v(A):A\subseteq N\backslash i]$

  of a random variable taking 2^{n-1} values given by player i's marginal contributions to coalitions $A\subseteq N\backslash i$, where $p_i:2^{N\backslash i}\rightarrow[0,1]$ is a probability distribution.

  In this view, the Banzhaf value above obtains by means of the simplest such a probability p_i, namely the uniform one:

  $p_i(A)=2^{1-n}=\frac{1}{2^{n-1}}$ for all $A\subseteq N\backslash i$ and for all $i\in N$.

- Hence a solution ϕ is a mapping $v \xrightarrow{\phi} \phi(v)$ associating with every game v a modular game $\phi(v)$, i.e. a valuation of Boolean lattice $(2^N, \cap, \cup)$. That is to say, $\phi(v) : 2^N \to \mathbb{R}_+$ is a further coalitional game satisfying

$$\phi(v)(A \cap B) + \phi(v)(A \cup B) = \phi(v)(A) + \phi(v)(B)$$

  for all $A, B \in 2^N$.

- In view of a fundamental theorem (by Davis-Rota) concerning valuations of distributive lattices (see [1, Theorem 4.63, p. 190], solutions or values $\phi(v)$ of coalitional games v have Möbius inversion living only on atoms of $(2^N, \cap, \cup)$, that is to say

$$\mu^{\phi(v)}(A) = \begin{cases} \phi(v)(\{i\}) = \phi_i(v) \text{ if } |A| = 1, \text{ i.e. } A = \{i\} \text{ for some } i \in N, \\ 0 \text{ otherwise.} \end{cases}$$

- Thus geometrically $\phi : \mathbb{R}^{2^n} \to \mathbb{R}^n$ and $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$ as well as $\phi(v)(A) = \sum_{i \in N} \phi_i(v)$. That is, solutions ϕ map games $v \in \mathbb{R}^{2^n}$ into games $\phi(v) \in \mathbb{R}^n$ corresponding to points in a vector subspace $\mathbb{R}^n$ of $\mathbb{R}^{2^n}$.

12.1 Axiomatic characterization of solutions

- Solutions $\phi(v) = (\phi_1(v), \ldots, \phi_n(v)) \in \mathbb{R}^n$ of coalitional games $v \in \mathbb{R}^{2^n}$ are usually classified in terms of the following axioms:

  - linearity (L) for all scalars $\alpha \in \mathbb{R}_{++}$ and for any two games v, v' $\phi(\alpha v) = \alpha\phi(v)$ and $\phi(v + v') = \phi(v) + \phi(v')$,
  - dummy (D) for any player $i \in N$, if $v(A \cup i) = v(A) + v(\{i\})$ for all $A \subseteq N \backslash i$, then i is said to ba a dummy in game v and $\phi_i(v) = v(\{i\})$,
  - symmetry (S) for any two players $i, j \in N$, if $v(A \cup i) = v(A \cup j)$ for all $A \subseteq N \backslash \{i, j\}$, then $\phi_i(v) = \phi_j(v)$,
    (in photocopies, symmetry is most often stated in terms of permutations $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$),
  - efficiency (E) $\sum_{i \in N} \phi_i(v) = v(N)$ for all games v.
  - Weber (1988) in [25, Chapter 7] (see photocopies), also considers the monotonicity (or positivity) axiom: $\phi_i(v) \geq 0$ for all monotone games v ($v(\emptyset) = 0$) and for all players $i \in N$.

- Observations:

  1. given that the collection $\{u_A : A \in 2^N\}$ is a linear basis of the vector space $\mathbb{R}^{2^n}$ of coalitional games, any solution $\phi : \mathbb{R}^{2^n} \to \mathbb{R}^n$ satisfying linearity has form

$$\begin{aligned} \phi(v) &= \sum_{A \in 2^N} \mu^v(A)\phi(u_A), \text{ i.e.} \\ \phi_i(v) &= \sum_{A \in 2^N} \mu^v(A)\phi_i(u_A) \text{ for all } i \in N. \end{aligned}$$

2. any set function v is a valuation of Boolean lattice $(2^N, \cap, \cup)$ (i.e. $v(A \cap B) + v(A \cup B) = v(A) + v(B)$ for all $A, B \in 2^N$) if and only if $v(A \cup i) = v(A) + v(\{i\})$ for all $A \subseteq N \backslash i$ and all $i \in N$ (see above); hence valuations are (all and only) the fixed points of solutions ϕ satisfying dummy; that is, ϕ satisfies dummy $\Leftrightarrow \phi_i(v) = v(\{i\})$ for all $i \in N$.

12.2 The Shapley value: existence and uniqueness

- Theorem (Shapley (1953) ch. 2 in photocopies): there exists a unique solution ϕ^{Sh} satisfying axioms L, D, S and E.

- Proof (sketch): firslty (from observation 1 above), since ϕ^{Sh} satisfies L, then $\phi_i^{Sh}(v) = \sum_{A \in 2^N} \mu^v(A) \phi_i^{Sh}(u_A)$; hence, it remains to show that for any element u_A of the basis the remaining axioms D, S and E provide existence and uniqueness; in fact, they yield $\phi_i^{Sh}(u_A) = \frac{1}{|A|}$ if $i \in A$ and $\phi_j^{Sh}(u_A) = 0$ if $j \in A^c$ for all $u_A, \emptyset \neq A \in 2^N$ and all $i, j \in N$, as:

  - D entails $\phi_j(u_A) = 0$ for all $j \in A^c$,
  - S entails $\phi_i(u_A) = \phi_{i'}(u_A)$ for all $i, i' \in A$,
  - E entails $\sum_{i \in N} \phi_i^{Sh}(u_A) = u_A(N) = 1 = \sum_{i \in A} \phi_i^{Sh}(u_A) =$
    $= |A| \phi_i^{Sh}(u_A)$ for all $i \in A \Rightarrow$
    $\Rightarrow \phi_i^{Sh}(u_A) = \frac{1}{|A|}$ for all $i \in A$ and $\phi_j^{Sh}(u_A) = 0$ for all $j \in A^c$.

- Therefore,

$$\begin{aligned}\phi_i(v) &= \sum_{A \subseteq N \backslash i} \frac{\mu^v(A \cup i)}{|A| + 1} \\ &= \sum_{A \subseteq N \backslash i} \frac{|A|!(n - |A| - 1)!}{n!} [v(A \cup i) - v(A)]\end{aligned}$$

  for all games v and all players $i \in N$, where $n!$ is the total number of maximal chains from $\emptyset$ to N, while $|A|!(n - |A| - 1)!$ is the number of such maximal chains meeting both A and $A \cup i$ (see above).

12.3 Integrating MLE first derivatives

- Consider computing

$$\int_{(0,\ldots,0)=\chi_\emptyset}^{(1,\ldots,1)=\chi_N} \hat{f}_i^v(q_1, \ldots, q_n) dq_1 \cdots dq_n \text{ for every } 1 \leq i \leq n$$

that is, the integral of the i-th derivative $\hat{f}_i^v$ of the MLE $\hat{f}^v$ of v along the main diagonal of the n-dimensional unit hypercube.

$$\begin{aligned}\int_{\chi_\emptyset}^{\chi_N} \hat{f}_i^v(q_1,\ldots,q_n)dq_1\cdots dq_n &= \sum_{A\in 2^N\backslash 2^{N\backslash i}}\left(\int_0^1 t^{|A|-1}dt\right)\mu^v(A)\\ &= \sum_{A\in 2^N\backslash 2^{N\backslash i}}\left[\frac{t^{|A|}}{|A|}\right]_0^1\mu^v(A)\\ &= \sum_{A\in 2^N\backslash 2^{N\backslash i}}\frac{\mu^v(A)}{|A|}=\phi_i^{Sh}(v).\end{aligned}$$

12.4 Random-order and probabilistic solutions

- For every player $i \in N$, consider the set $2^{N\backslash i}$ of coalitions that do not contain i; then, a probabilistic solution ϕ is defined by a n-tuple of probability distributions $p_i : 2^{N\backslash i} \to [0,1], i \in N$ as follows (see above)

$$\phi_i^{p_i}(v) = \sum_{A\in 2^{N\backslash i}} p_i(A)[v(A\cup i) - v(A)] \text{ for every } i \in N$$

- Exercise: show that any probabilistic solution satisfies axioms (see above) L and D; also determine under what conditions (on distributions $p_i, i \in N$) it also satisfies axiom S.
- Consider the set $\mathcal{S}(N)$ of all permutations of players, i.e. the symmetric group of order $n!$ (see [17]); then, a random-order solution ϕ is defined by a probability distribution $p : \mathcal{S}(N) \to [0,1]$ as follows:

$$\phi_i^p(v) = \sum_{\pi\in\mathcal{S}(N)} p(\pi)[v(\{j\in N : \pi(j)\leq\pi(i)\}) - v(\{j\in N : \pi(j)<\pi(i)\})]$$

  for every $i \in N$.
- Exercise: show that any random-order solution satisfies axioms L, D and E; also determine under what conditions it also satisfies axiom S.
- Exercise: show that the random-order form of the Shapley value $\phi^{Sh}(v)$ of coalitional games v obtains by means of the uniform distribution over permutations of players, i.e. $p_U : \mathcal{S}(N) \to [0,1]$ defined by $p_U(\pi) = (n!)^{-1}$ for all $\pi \in \mathcal{S}(N)$.

12.5 Weights and Shapley values

- Consider n strictly positive weights $\omega_1,\ldots,\omega_n \in \mathbb{R}_{++}$. These weights define the weighted Shapley value $\phi^{\omega-Sh}(v) \in \mathbb{R}^n$ of any game v as follows:

$$\phi_i^{\omega-Sh}(v) = \sum_{A\in 2^{N\backslash i}}\left(\frac{\omega_i}{\omega_i+\sum_{j\in A}\omega_j}\right)\mu^v(A\cup i).$$

- That is, for any unanimity game $u_A, \emptyset \neq A \in 2^N$ and for all players $i \in N$

$$\phi_i^{\omega-Sh}(u_A) = \begin{cases} \omega_i \Big/ \sum_{j\in A} \omega_j \text{ if } i \in A, \\ 0 \text{ if } i \in A^c. \end{cases}$$

- In random-order form, weighted Shapley values are given by $\phi_i^{\omega-Sh}(v) =$

$$= \sum_{\pi\in\mathcal{S}(N)} p_\omega(\pi)[v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})],$$

  where probability distribution $p_\omega : \mathcal{S}(N) \to [0,1]$ is

$$p_\omega(\pi) = \prod_{1\leq i\leq n} \left(\omega_{\pi^{-1}(i)} \Big/ \sum_{1\leq j\leq i} \omega_{\pi^{-1}(j)} \right)$$

  for all permutations $\pi \in \mathcal{S}(N)$, with $\pi^{-1}(i) = i'$ denoting the player $i' \in N$ mapped into position $i \in N$ by permutation π.

- Exercise: check the probabilistic form of $\phi^{\omega-Sh} : \mathbb{R}^{2^n} \to \mathbb{R}^n$.

DONE: 22 nov :: TOPIC (2.10) Axiomatic characterization of solutions, TOPIC (2.11) the Shapley value: existence and uniqueness, TOPIC (2.12) integrating MLE first derivatives along the main diagonal of the n-hypercube, TOPIC (2.13) random-order and probabilistic solutions, TOPIC (2.14) weighted Shapley values in terms of Mobius inversion and in random-order form.

12.6 Core of coalitional games

- The core $Core(v)$ of a coalitional game v is the set of all valuations ϕ of Boolean lattice $(2^N, \cap, \cup)$ satisfying $\phi(A) \geq v(A)$ for all $A \in 2^N$ and $\phi(N) = v(N)$.

- Hence $Core(v) \subset \mathbb{R}^n$ is a compact convex polyhedron, possibly empty, of dimension at most $n-1$ [28]. (The dimension of a polyhedron is the maximum number of its affinely independent points minus one; in general, see http://math.mit.edu/∼goemans/18433S07/polyhedral.pdf for examples and other definitions, such as that of facets.)

- The necessary and sufficient conditions (on v) for non-emptiness of the core, i.e. $Core(v) \neq \emptyset$, are known as the Shapley-Bondareva conditions: a map $\kappa : 2^N \to [0,1]$ is said to be balanced if

$$\sum_{A\in 2^N} \kappa(A)\chi_A = \chi_N;$$

  then, $Core(v) \neq \emptyset$ if and only if for all balanced maps κ it holds

$$\sum_{A\in 2^N} \kappa(A)v(A) \leq v(N)$$

  (see [27]).

- Supermodularity (or convexity) of v is a sufficient (but not necessary) condition for $Core(v) \neq \emptyset$. In this case, the set $ex(Core(v))$ of extreme points of the core correspond to those (distinct) valuations

  $\phi^\pi(v) = (\phi^\pi_1(v), \ldots, \phi^\pi_n(v))$ defined each by a permutation $\pi \in \mathcal{S}(N)$ as follows:

  $$\phi^\pi_i(v) = v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})$$

  for all $i \in N$.

- Hence if v is supermodular, then the Shapley value belongs to the core, i.e. $\phi^{Sh}(v) \in Core(v)$, and in particular it obtains as the peculiar convex combination of all extreme points given by their arithmetic mean (i.e., the Shapley value is the center of the core).

- Exercise: denote by $\int_N^C f d\eta$ the discrete Choquet integral of $f : N \to [0,1]$ with respect to a fuzzy measure $\eta : 2^N \to [0,1]$ (see above). Show that if η is supermodular, then

  $$\int_N^C f d\eta = \min_{p \in Core(\eta)} \int_N f dp$$

  for any integrand $f : N \to [0,1]$, where $\int_N f dp$ denotes the traditional integral or expectation of f with respect to a (additive) probability p, i.e.

  $$\int_N f dp = \sum_{i \in N} f(i) p(\{i\}).$$

12.7 Cooperation restrictions

- The general idea behind cooperation restrictions relies upon the assumption that there is a set system $\mathcal{F} \subset 2^N$ of feasible coalitions, so that cooperation can only occur within coalitions $A \in \mathcal{F}$, while unfeasible coalitions $A \in \mathcal{F}^c = 2^N \backslash \mathcal{F}$ cannot form.

- The set system $\mathcal{F}$ was firslty modeled by means of some exogenously given (i.e. generic but fixed) partition $P \in \mathcal{P}^N$ of the player set N. In particular, such a modeling resulted in two alternative forms of cooperation restrictions, which are referred to as "type I" and "type II" in the sequel.

- Secondly, cooperation restrictions were modeled by means of a given graph $G = (N, E)$ with players as vertices, and this provided quite many different approches to the solution concept of graph-restricted games.

- Afterwards, cooperation restrictions were modeled in alternative ways (such as combinatorial geometries and/or so-called augmenting systems) by requiring the set system $\mathcal{F}$ of feasible coalitions to satisfy certain conditions rather than others. A general summary can be found in

  https://arxiv.org/pdf/1304.1075.pdf, published in 2013, where the reference list is exhaustive.

12.7.1 Partition constraints type I

- Let $P = \{B_1, \ldots, B_{|P|}\} \in \mathcal{P}^N$ be a partition of N. Aumann and Dréze (1974) (see [3]) consider the case where the set system of feasible coalitions, denoted by $\mathcal{F}_P \subset 2^N$, is determined by partition P as follows:

$$\mathcal{F}_P = 2^{B_1} \cup \cdots \cup 2^{B_{|P|}},$$

  i.e. only those coalitions $A \subseteq B \in P$ included in a block B of P can form.

- The share assigned to each player $i \in N$ by the Aumann-Dréze solution of any game v with coalition structure P, denoted by $\phi^{AD}(v, P)$, is

$$\phi_i^{AD}(v, P) = \phi_i^{Sh}(v^B)$$

  for all $i \in B$ and all $B \in P$, where $v^B : 2^N \to \mathbb{R}_+$ is the original game v restricted to power set 2^B (or equivalently to block $B \in P$), and defined by $v^B(A) = v(B \cap A)$ for all $A \in 2^N$.

- Exercise: check that

$$\phi_i^{AD}(v, P) = \sum_{B \in P} \phi_i^{Sh}(v^B) \text{ for all } i \in N \text{ and all } (v, P) \in \mathbb{R}^{2^n} \times \mathcal{P}^N.$$

- Exercise: for any coalitional game $v : 2^N \to \mathbb{R}_+$ and partition (i.e. coalition structure) $P = \{B_1, \ldots, B_{|P|}\}$, consider a (further) restricted game $v_{/P} : 2^N \to \mathbb{R}_+$ defined by

$$v_{/P}(A) = \sum_{B \in P} v(A \cap B) \text{ for all } A \in 2^N.$$

  1. Verify whether $v_{/P}$ may be equivalently defined as follows:
     $v_{/P}(A) = v(A)$ for all feasible coalitions $A \in \mathcal{F}_P$, and
     $\mu^{v_{/P}}(A') = 0$ for all unfeasible coalitions $A' \in \mathcal{F}_P^c = 2^N \backslash \mathcal{F}_P$.
     [Hint: use induction on $\sum_{B \in P} |A \cap B| = |A| \geq 2$.]
  2. Also verify whether $\phi_i^{AD}(v, P) = \phi_i^{Sh}(v_{/P})$ for all $i \in N$ and all $(v, P) \in \mathbb{R}^{2^n} \times \mathcal{P}^N$.

  DONE: 22 nov :: TOPIC (2.15) core of coalitional games, supermodularity (or convexity) and extreme points of the polyhedron, the Shapley value as a convex combination of extreme points, TOPIC (2.16) cooperation restrictions in general and partition constraints type I.

12.7.2 Partition constraints type II

- In Owen (1977) (see [22]), a given (i.e. generic but fixed) partition or coalition structure $P = \{B_1, \ldots, B_{|P|}\}$ is intended to model situations where the family of feasible coalitions, denoted by $\mathcal{F}_P^* \subset 2^N$, is

$$\mathcal{F}_P^* = \bigcup_{B \in P} \left(2^B \cup 2^{P \backslash B} \right),$$

that is, the generic feasible coalition $A \in \mathcal{F}_P^*$ obtains as $A = \hat{A} \cup \tilde{A}$, i.e. the union of a subset $\hat{A} \in 2^B, B \in P$ of some block and the merging $\tilde{A} \in 2^{P \backslash B}$ of some (i.e. zero or one or two or more) blocks $B' \in P \backslash B$. In fact, for any block $B \in P$, Boolean lattice $(2^{P \backslash B}, \cap, \cup)$ has $|P| - 1$ atoms given by blocks $B' \in P, B' \neq B$. More precisely, any partition P of N identifies the field of sets 2^P (i.e., closed under union, intersection and complementation, see https://en.wikipedia.org/wiki/Field_of_sets), and the same applies to (field of sets) $2^{P \backslash B}$ for any block B of P.

- In order to consider the associated solution $\phi^{Ow}(v, P)$ in random-order form, define $\mathcal{S}^P(N) \subseteq \mathcal{S}(N)$ to be the set of P-admissible permutations $\pi : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ of players, where $|\mathcal{S}^P(N)| = |P|! \prod_{B \in P} |B|!$ since an admissible permutation $\pi \in \mathcal{S}^P(N)$ works as follows: it firstly selects one of the $|P|!$ available orders for blocks $B_1, \ldots, B_{|P|}$, and next puts all members of each block in consecutive positions; in other terms, these permutations $\pi \in \mathcal{S}^P(N)$ satisfy the following condition: for all blocks $B \in P$ and for all pairs $i, j \in B$, if $\pi(i) < \pi(i') < \pi(j)$, then $i' \in B$. Hence $\mathcal{S}^{P_\perp}(N) = \mathcal{S}(N)$ while proper inclusion $\mathcal{S}^P(N) \subset \mathcal{S}(N)$ holds for all partitions P such that $P^\top > P > P_\perp$, i.e. strictly finer than the top one and stricly coarser than the bottom one.

- The associated solution (in random-order form) thus is $\phi_i^{Ow}(v, P) =$

$$= \sum_{\pi \in \mathcal{S}^P(N)} \frac{v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})}{|P|! \prod_{B \in P} |B|!}$$

  for all $i \in N$ (and all games v).

- Exercise: for $N = \{1, 2, 3, 4\}$ and $v : 2^N \rightarrow \mathbb{R}_+$ defined by $v(A) = (|A|)^{\frac{1}{2}}$ for all $A \in 2^N$, consider coalition structure $P = 13|24$;

  1. compute both $\phi^{AD}(v, P), \phi^{Ow}(v, P) \in \mathbb{R}^4$, i.e. the Aumann-Dréze and Owen values;
  2. for restricted game $v_{\mathcal{F}_P^*} : 2^N \rightarrow \mathbb{R}_+$ defined by $v_{\mathcal{F}_P^*}(A) = v(A)$ for all $A \in \mathcal{F}_P^*$ and $\mu^{v_{\mathcal{F}_P^*}}(A') = 0$ for all $A' \in 2^N \backslash \mathcal{F}_P^*$, determine $v_{\mathcal{F}_P^*}(A')$ for all $A' \in 2^N \backslash \mathcal{F}_P^*$, checking whether the Shapley value $\phi^{Sh}(v_{\mathcal{F}_P^*}) \in \mathbb{R}_+^4$ equals the Owen value $\phi^{Ow}(v, P) \in \mathbb{R}_+^4$ or not, where

$$\phi_i^{Sh}(v_{\mathcal{F}_P^*}) = \sum_{A \in 2^{N \backslash i}} \frac{\mu^{v_{\mathcal{F}_P^*}}(A \cup i)}{|A| + 1} \text{ for all } i \in N;$$

  3. let $\omega_i = i$ for all $i \in N$ and $p_\omega : \mathcal{S}(N) \rightarrow [0, 1]$ be a probability distribution over the $4! = |\mathcal{S}(N)|$ permutations of players such that $p_\omega(\pi') = 0$ for all non-P-admissible permutations $\pi' \in \mathcal{S}(N) \backslash \mathcal{S}^P(N)$, while for any two P-admissible permutations $\pi, \hat{\pi} \in \mathcal{S}^P(P)$ it holds

$$\frac{p_\omega(\pi)}{p_\omega(\hat{\pi})} = \frac{\prod_{1 \leq i \leq 4} \left(\omega_{\pi^{-1}(i)} \middle/ \sum_{1 \leq j \leq i} \omega_{\pi^{-1}(j)} \right)}{\prod_{1 \leq i \leq 4} \left(\omega_{\hat{\pi}^{-1}(i)} \middle/ \sum_{1 \leq j \leq i} \omega_{\hat{\pi}^{-1}(j)} \right)};$$

compute the random-order solution $\phi^{p_\omega}(v) \in \mathbb{R}^4_+$ identified by probability p_ω, i.e. $\phi_i^{p_\omega}(v) =$

$$= \sum_{\pi \in \mathcal{S}^P(N)} p_\omega(\pi)[v(\{j \in N : \pi(j) \leq \pi(i)\}) - v(\{j \in N : \pi(j) < \pi(i)\})]$$

for $1 \leq i \leq 4$;

4. verify whether $\phi^{Sh}(v_{\mathcal{F}^*_P}) \in Core(v_{\mathcal{F}^*_P})$ and/or $\phi^{Ow}(v, P) \in Core(v_{\mathcal{F}^*_P})$ and/or $\phi^{p_\omega}(v) \in Core(v_{\mathcal{F}^*_P})$.

12.7.3 Graph-restricted games

- A (coalitional) game with cooperation structure (sometimes also called a communication situation or conference structure) is a pair (v, G) where v is a coalitional game and $G = (N, E)$ is a graph with players $i \in N$ as vertices, i.e. $E \subseteq N_2 := \{A : A \in 2^N, |A| = 2\}$, hence the edge set E is a subset of the $\binom{n}{2}$-set $N_2 = \{\{i, j\} : 1 \leq i < j \leq n\}$ of all (unordered) pairs $\{i, j\}$ of players.

- The standard interpretation is that any two players $i, j \in N$ can cooperate only if they are connected through graph G, i.e. only if either $\{i, j\} \in E$ or else there is a $i - j$-path $P_{ij} \subseteq G$, where $P_{ij} = (V(P_{ij}), E(P_{ij}))$ is a subgraph (of G) with vertex set $V(P_{ij}) = \{i = i_1, \ldots, i_k = j\}$ and edge set $E(P_{ij}) = \{\{i_l, i_{l+1}\} : 1 \leq l < k\}$.

- Therefore, together with their role in game v (quantified by marginal contributions, see above), players may also be crucial for enabling cooperation between other players within graph G.

- For every coalition $A \in 2^N$, let $G(A) = (A, E(A)) \subseteq G$ be the subgraph spanned by A, i.e. with vertex set A and edge set
  $E(A) = \{\{i, j\} : \{i, j\} \in E, \{i, j\} \subseteq A\}$

- Also let $A/G = \{B_1, \ldots, B_{|A/G|}\}$ denote the partition of A whose blocks are the vertex sets of the components $G(B_1), \ldots G(B_{|A/G|}) \subseteq G(A)$ or maximal connected subgraphs of $G(A)$.

- Define $v_{/G} : 2^N \to \mathbb{R}_+$ by $v_{/G}(A) = \sum_{B \in A/G} v(B)$ for all $A \in 2^N$.

- Exercise: verify that $v_{/G}(A) = v(A)$ for all $A \in 2^N$ such that $|A/G| = 1$, and $\mu^{v_{/G}}(A') = 0$ for all $A' \in 2^N$ such that $|A'/G| > 1$ (in fact, this is true as shown in [23, Theorem 2, p. 212 and Appendix, pp. 218-219]; however, much shorter proof is again by induction on the number and size of the components of $G(A)$, i.e. on $|A/G| \geq 2$ and $\sum_{B \in A/G} |B| \geq 2$; that is, firstly consider a pair $\{i, j\} \subseteq N$ such that $\{i, j\} \notin E$, and then use induction); can you determine $\mu^{v_{/G}}$ when $G = (N, E)$ is a $1 - n$-path, i.e. $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-2, n-1\}, \{n-1, n\}\}$?

- In games with cooperation structure (v, G) the family (or set system) $\mathcal{F}_G$ of feasible coalitions contains all connected ones, that is to say

$$\mathcal{F}_G = \{A \in 2^N : |A/G| = 1\}$$

- In order to characterize the Myerson solution $\phi^{My}(v,G)$ for games v with cooperation structure $G=(N,E)$, let $G\backslash\{i,j\}=(N,E\backslash\{i,j\})$ denote the graph obtained from G by deleting edge $\{i,j\}$, and consider the following two axioms:
  - fairness F: for all edges $\{i,j\}\in E$
  $$\phi_i(v,G)-\phi_i(v,G\backslash\{i,j\})=\phi_j(v,G)-\phi_j(v,G\backslash\{i,j\},$$
  - component efficiency CE: $\sum_{i\in N}\phi_i(v,G)=v_{/G}(N)$.
- Theorem (Myerson (1977), see [21] in photocopies): there exixts a unique solution of games with cooperation structure that satisfies both F and CE; it is $\phi^{My}(v,G)=\phi^{Sh}(v_{/G})$.
- Another solution of games with cooperation structure (v,G) is the so-called position value $\phi^{Po}(v,G)$, which works as follows (see [7]):
  - firstly, game v is turned into "arc game" $w_v:2^E\rightarrow\mathbb{R}_+$ defined by $w_v(E')=v_{/G}\left(\underset{\{i,j\}\in E'}{\cup}\{i,j\}\right)$ for all edge subsets $E'\in 2^E$;
  - secondly, each arc or edge $\{i,j\}\in E$ receives the share $\psi_{\{i,j\}}(w_v)$ given by Shapley solution of arc game w_v, i.e. $\psi_{\{i,j\}}(w_v)=\phi^{Sh}_{\{i,j\}}(w_v)$
  - finally, each player $i\in N$ receives the sum over all edges where i is one endvertex of one half of such edges' shares, that is to say
  $$\phi_i^{Po}(v,G)=\sum_{j\in N\backslash i:\{i,j\}\in E}\frac{\psi_{\{i,j\}}(w_v)}{2}.$$
- There exists no axiomatic characterization for the position value (when graph G is generic, i.e. non-cycle-free, see [7, Theorem 3.2, p. 311]).
- Exercise: let graph $G=(N,E)$ have edge set given by $E=\{\{1,i\}:1<i\leq n\}$, and consider the $\binom{n-1}{2}$ unanimity games $u_{\{i,j\}}$ such that $1<i<j\leq n$; also define coalitional games $v,w:2^N\rightarrow\mathbb{R}$ by
$$\begin{aligned} v(A) &= \sum_{1<i<j\leq n}u_{\{i,j\}}(A) \text{ and}\\ w(A) &= \max_{1<i<j\leq n}u_{\{i,j\}}(A) \text{ for all } A\in 2^N; \end{aligned}$$
compute both $\phi^{My}(v,G),\phi^{Po}(v,G)\in\mathbb{R}^n$.
- Exercise: let $N=\{1,2,3,4\}$ and consider the set system $\mathcal{F}$ of feasible coalitions defined by
$$\mathcal{F}=\{\emptyset,\{2\},\{4\},\{1,2\},\{2,4\},\{1,2,3\},\{1,2,4\},N\}$$
and coalitional game $v:2^N\rightarrow\mathbb{R},v(\emptyset)=0$ defined by
$$v(A)=\prod_{i\in A}i \text{ for all } A\in 2^N, A\neq\emptyset;$$

1. compute the Shapley value $\phi^{Sh}(v_{/\mathcal{F}}) \in \mathbb{R}^4$ of the restricted game $v_{/\mathcal{F}} : 2^N \to \mathbb{R}$ defined by

$$\begin{aligned} v_{/\mathcal{F}}(A) &= v(A) \text{ for all } A \in \mathcal{F}, \\ \mu^{v_{/\mathcal{F}}}(A) &= 0 \text{ for all } A \in \mathcal{F}^c = 2^N \backslash \mathcal{F}; \end{aligned}$$

2. identify the set $\mathcal{S}^{\mathcal{F}}(N) \subset \mathcal{S}(N)$ of $\mathcal{F}$-admissible permutations $\pi : \{1,2,3,4\} \to \{1,2,3,4\}$ defined by

$$\pi \in \mathcal{S}^{\mathcal{F}}(N) \Leftrightarrow \{i : \pi(i) \leq k\} \in \mathcal{F}, 1 \leq k \leq 4,$$

   and compute the random-order solution $\phi^{p_U^{\mathcal{F}}}(v) \in \mathbb{R}^4$ given by

$$\phi_i^{p_U^{\mathcal{F}}}(v) = \sum_{\pi \in \mathcal{S}^{\mathcal{F}}(N)} \frac{v(\{j : \pi(j) \leq \pi(i)\}) - v(\{j : \pi(j) < \pi(i)\})}{|\mathcal{S}^{\mathcal{F}}(N)|}$$

   for all $i \in N$ (check: $\phi^{p_U^{\mathcal{F}}}(v) = \phi^{Sh}(v_{/\mathcal{F}})$ or $\phi^{p_U^{\mathcal{F}}}(v) \neq \phi^{Sh}(v_{/\mathcal{F}})$?);
3. identify (and possibly solve) the system of equations with unknown variables $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{R}_+$ such that the weighted Shapley value $\phi^{\omega-Sh}(v_{/\mathcal{F}}) \in \mathbb{R}^4$ satisfies $\phi^{\omega-Sh}(v_{/\mathcal{F}}) = \phi^{p_U^{\mathcal{F}}}(v)$;
4. noting that for any coalitional game $w : 2^N \to \mathbb{R}$ it holds

$$\mu^w(A) \geq 0 \text{ for all } A \in 2^N \Rightarrow w \text{ is supermodular},$$

   compare the two cores $Core(v)$ and $Core(v_{/\mathcal{F}})$.

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