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Numerical Algorithm of Block Method for General Second Order ODEs using Variable Step Size

(Algoritma Berangka Kaedah Blok bagi ODE Umum Peringkat Kedua Menggunakan Pemboleh Ubah Saiz Langkah)

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ABSTRACT

This paper outlines an alternative algorithm for solving general second order ordinary differential equations (ODEs). Normally, the numerical method was designed for solving higher order ODEs by converting it into an n-dimensional first order equations with implementation of constant step length. Nevertheless, this involved a lot of computational complexity which led to consumption a lot of time. Consequently, a direct block multistep method with utilization of variable step size strategy is proposed. This method was developed for computing the solution at four points simultaneously and the derivation based on numerical integration as well as using interpolation approach. The convergence of the proposed method is justified under suitable conditions of stability and consistency. Five numerical examples are considered and some comparisons are made with the existing methods for demonstrating the validity and reliability of the proposed algorithm.

Keywords: Block method; general second order ordinary differential equations; variable step size

ABSTRAK

Kertas ini menggariskan satu algoritma alternatif untuk menyelesaikan persamaan pembezaan biasa (ODE) umum peringkat kedua. Kebiasaannya, kaedah berangka untuk menyelesaikan ODE peringkat tinggi direka dengan menukarkan ia ke dalam n-dimensi persamaan peringkat pertama dengan perlaksanaan panjang langkah kekal. Walau bagaimanapun, ini melibatkan kerumitan pengiraan yang membawa kepada penggunaan masa yang banyak. Oleh yang demikian, satu kaedah langsung pelbagai langkah blok dengan penggunaan strategi saiz langkah berubah dicadangkan. Kaedah ini dibangunkan bagi menghitung penyelesaian pada empat titik secara serentak dan terbitannya berdasarkan integrasi berangka serta menggunakan pendekatan interpolasi. Penumpuan kaedah yang dicadangkan dijustifikasi mengikut syarat kestabilan dan tekal yang sesuai. Terdapat lima contoh berangka dipertimbangkan dan beberapa perbandingan telah dibuat dengan kaedah yang sedia ada untuk menunjukkan kesahan dan kebolehpercayaan algoritma yang dicadangkan.

Kata kunci: Kaedah blok; persamaan pembezaan biasa umum peringkat kedua; saiz langkah berubah

INTRODUCTION

In modern work of engineering, physics, applied maths and science, second order equations arise very frequently. To date, these equations have been extensively studied and books have been composed along the mathematical methods available for such equations. Generally, an ODE is classified into initial value problems (IVPs) and boundary value problems (BVPs). Hence, second order non-stiff IVPs of the form as (1) will be considered and solved directly.

$$y'' = f\left(x, y, y'\right), \quad y\left(a\right) = y_0, \quad y'\left(a\right) = y_1, \quad x \in \left[a, b\right]. \quad (1)$$

In literature, a various type of numerical methods have been developed in treating such problems as (1). Usually, the problems were solved by reducing the higher order ODE into a system of first order ODEs and solved them using the numerous methods available. However, a number of researchers have attempted the solution of (1) directly without reduction to systems of first order ODEs (Kayode 2008; Majid & Mohamed 2006; Waeleh et al. 2011a). This is an alternative way to save some of the computational time.

In this study, a numerical method based on concurrent computation was developed. The method was designed for generating a set of solutions concurrently, which was referred to the 'block' term (Rosser 1967). The concept of block method was first proposed by Milne (1953), who used block method only as a tool for calculating the starting values for predictor corrector algorithm. Cash (1983) had studied block method based upon Runge-Kutta method for the numerical solution of non-stiff IVPs whereas Fatunla (1991) developed block method for solving special second order ODEs. The block method based on Adams type formula for solving higher order ODEs directly was developed by Omar and Suleiman (2005). However, the authors designed the explicit method for computing solutions only at two points simultaneously using constant step size.

An implementation of fixed step size is the most commonly used in deriving numerical method (Anakira et al. 2013; Jator 2012; Jikantoro et al. 2015; Pandey 2014). However, the utilization of variable step size strategy in the numerical method had been adopted by several researchers such as Cash and Girdlestone (2006), Majid and Suleiman (2006), Majid et al. (2012) and Waeleh et al. (2011b). Majid and Suleiman (2006) introduced implicit method for solving higher order ODEs directly. The researchers presented the method in the simple form of the Adams Moulton method with the implementation variable step size. The application of the block method with the implementation of varying the step size has been developed by Waeleh et al. (2011b) and the method proposed was designed for the numerical solutions of ODEs up to order six. Taken together, this research is an extension of the work in Waeleh et al. (2011b) in which the solution is computed at three points concurrently.

METHODS

In the 4-point multistep block method, the closed finite interval $[a, b]$ is partitioned into a series of blocks and each block contains four equal subintervals. According to Figure 1, four equally distant points of the numerical solution will be generated simultaneously.

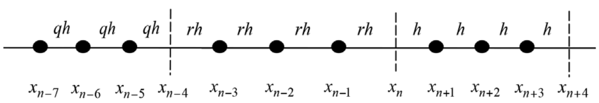

FIGURE 1. 4-point multistep block method

The numerical solution at the point $x_{n+1}, x_{n+2}, x_{n+3}$ and x_{n+4} in the computed block will be obtained by consuming the solutions at the previous block which involved the points x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1} and x_n. After completing the calculation of the numerical solutions in the current block, the process will proceed for the subsequent *i-th* block where the points x_n, x_{n+1} x_{n+2}, x_{n+3} and x_{n+4} in the $(i-1)$*-th* block will be assigned as $x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}$ and x_n, respectively. This 4-point multistep block method will take the form as Adams method and will approximate the numerical solution using variable step size mode. Noting that h is the step size in the computed block whereas rh and qh are the step size in the previous blocks where r and q are the step size ratio.

Numerical integration and the Lagrange interpolation polynomial will be utilized as a step in deriving the formulae of 4-point multistep block method. This suggested method will be performed by simultaneously generating four approximate solutions $y_{n+1}, y_{n+2}, y_{n+3}$ and y_{n+4} with step size h at the points x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4}, respectively. These four solutions will be computed simultaneously using the same back values with step size rh and qh. The formulae of 4-point multistep block method are derived by integrating twice as follows:

Let $x_{n+v} = xn + vh$, where $v = 1,2,3$ and 4.

$$\int_{x_n}^{x_{n+v}} \int_{x_n}^{x} y''\left(x,y,y'\right) dxdx = \int_{x_n}^{x_{n+v}} \int_{x_n}^{x} f\left(x,y,y'\right) dxdx. \tag{2}$$

The derivation proceeds by replacing the function $f(x, y, y')$ in (2) by the interpolating function which is generated from Lagrange polynomial. Furthermore, the set of interpolation points involved in deriving the corrector formulae are $\{(x_{n-4}, f_{n-4}), \ldots, (x_{n+4}, f_{n+4})\}$ and for predictor are $\{(x_{n-7}, f_{n-7}), \ldots, (x_n, f_n)\}$ thus the order of predictor is one order less than corrector. Equation (2) will be integrated over the corresponding interval and consequently produces the corrector and predictor formulae.

The step size will be set to be constant, doubled and halved with the value of step size ratios are $((r = 1, q = 1), (r = 1, q = 2), (r = 1, q = 0.5))$, $(r = 0.5, q = 0.5)$ and $(r = 2, q = 2)$, respectively. This strategy can reduce the number of formulae to be stored in the code and as a result, it will save the amount of storage. The coefficients of the corrector formulae for $r = 1$ will then be tabulated in Tables 1 to 4. To simplify, the general formulae for the 4-point multistep block method can be written in a compact form as follows:

$$y_{n+v}^{d-g} = \sum_{k=0}^{g-1} \frac{(vh)^k}{k!} y_n^{(d-g+k)} + \frac{h^g}{(g-1)!} \sum_{j=s}^{t} \gamma_{v,j}^{g} f_{n+j}, \tag{3}$$

where $\gamma_{v,j}^{g}$ stands for the coefficients of the formulae; d is the order of problem; g is the number of times which (1) is integrated; and k is the number of term when the equation is integrated. For the corrector formulae of 4-point multistep block method, the value of $s = 4$, $t = 4$ and for predictor $s = 7$, $t = 0$.

The formulae of 4-point multistep block method in the form of discrete methods for first point corrector formulae when $r = 1$, are shown as (4) and (5).

Integrate once:

$$y'_{n+1} = y'_n + \frac{h}{3628800}\Big[2497f_{n-4} - 25706f_{n-3} + 126286f_{n-2} - 425762f_{n-1} + 2224480f_n + 1909858f_{n+1} - 216014f_{n+2} + 36394f_{n+3} - 3233f_{n+4}\Big]. \tag{4}$$

Integrate twice:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{7257600}\Big[2497f_{n-4} - 25864f_{n-3} + 128472f_{n-2} - 444560f_{n-1} + 2875850f_n + 1258488f_{n+1} - 197216f_{n+2} + 34208f_{n+3} - 3075f_{n+4}\Big]. \tag{5}$$

TABLE 1. Coefficients of first point corrector formulae for 4-point multistep block method when $r = 1$

$r = 1$	$\gamma^g_{1,-4}$	$\gamma^g_{1,-3}$	$\gamma^g_{1,-2}$	$\gamma^g_{1,-1}$	$\gamma^g_{1,0}$	$\gamma^g_{1,1}$	$\gamma^g_{1,2}$	$\gamma^g_{1,3}$	$\gamma^g_{1,4}$
$g = 1$	$\frac{2497}{3628800}$	$-\frac{25706}{3628800}$	$\frac{126286}{3628800}$	$-\frac{425762}{3628800}$	$\frac{2224480}{3628800}$	$\frac{1909858}{3628800}$	$-\frac{216014}{3628800}$	$\frac{36394}{3628800}$	$-\frac{3233}{3628800}$
$g = 2$	$\frac{2497}{7257600}$	$-\frac{25864}{7257600}$	$\frac{128472}{7257600}$	$-\frac{444560}{7257600}$	$\frac{2875850}{7257600}$	$\frac{1258488}{7257600}$	$-\frac{197216}{7257600}$	$\frac{34208}{7257600}$	$-\frac{3075}{7257600}$

TABLE 2. Coefficients of second point corrector formulae for 4-point multistep block method when $r = 1$

$r = 1$	$\gamma^g_{2,-4}$	$\gamma^g_{2,-3}$	$\gamma^g_{2,-2}$	$\gamma^g_{2,-1}$	$\gamma^g_{2,0}$	$\gamma^g_{2,1}$	$\gamma^g_{2,2}$	$\gamma^g_{2,3}$	$\gamma^g_{2,4}$
$g = 1$	$-\frac{23}{113400}$	$\frac{184}{113400}$	$-\frac{494}{113400}$	$-\frac{872}{113400}$	$\frac{43480}{113400}$	$\frac{141928}{113400}$	$\frac{44446}{113400}$	$-\frac{1976}{113400}$	$\frac{127}{113400}$
$g = 2$	$\frac{69}{113400}$	$-\frac{736}{113400}$	$\frac{3820}{113400}$	$-\frac{14208}{113400}$	$\frac{101450}{113400}$	$\frac{130336}{113400}$	$\frac{5796}{113400}$	$\frac{320}{113400}$	$-\frac{47}{113400}$

TABLE 3. Coefficients of third point corrector formulae for 4-point multistep block method when $r = 1$

$r = 1$	$\gamma^g_{3,-4}$	$\gamma^g_{3,-3}$	$\gamma^g_{3,-2}$	$\gamma^g_{3,-1}$	$\gamma^g_{3,0}$	$\gamma^g_{3,4}$	$\gamma^g_{3,2}$	$\gamma^g_{3,3}$	$\gamma^g_{3,4}$
$g = 1$	$\frac{81}{44800}$	$-\frac{778}{44800}$	$\frac{3438}{44800}$	$-\frac{9666}{44800}$	$\frac{33440}{44800}$	$\frac{34434}{44800}$	$\frac{57618}{44800}$	$\frac{16202}{44800}$	$-\frac{369}{44800}$
$g = 2$	$\frac{117}{89600}$	$-\frac{1200}{89600}$	$\frac{5904}{89600}$	$-\frac{20376}{89600}$	$\frac{129450}{89600}$	$\frac{194688}{89600}$	$\frac{86760}{89600}$	$\frac{8136}{89600}$	$-\frac{279}{89600}$

TABLE 4. Coefficients of fourth point corrector formulae for 4-point multistep block method when r = 1

$r = 1$	$\gamma^g_{4,-4}$	$\gamma^g_{4,-3}$	$\gamma^g_{4,-2}$	$\gamma^g_{4,-1}$	$\gamma^g_{4,0}$
$g = 1$	$-\frac{107}{14175}$	$\frac{976}{14175}$	$-\frac{3956}{14175}$	$\frac{9232}{14175}$	$-\frac{9080}{14175}$
$g = 2$	$-\frac{10}{14175}$	$\frac{64}{14175}$	$-\frac{48}{14175}$	$-\frac{1216}{14175}$	$\frac{22900}{14175}$

$r = 1$	$\gamma^g_{4,1}$	$\gamma^g_{4,2}$	$\gamma^g_{4,3}$	$\gamma^g_{4,4}$
$g = 1$	$\frac{32752}{14175}$	$\frac{244}{14175}$	$\frac{22576}{14175}$	$\frac{4063}{14175}$
$g = 2$	$\frac{50880}{14175}$	$\frac{24176}{14175}$	$\frac{15808}{14175}$	$\frac{846}{14175}$

CONVERGENCE AND STABILITY OF THE METHOD

The order of the 4-point multistep block method is calculated in a block form with the formulae of the proposed method written in a matrix differentiation equation:

$$\alpha Y_m = h\beta Y'_m + h^2 \lambda F_m, \tag{6}$$

where α, β and λ are the coefficients of the method. Thus, the order and error constant of the method are calculated using the following formulae:

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\
C_2 &= \sum_{j=0}^{k} \left(\frac{j^2}{2!}\alpha_j - j\beta_j - \lambda_j \right) \\
&\vdots \\
C_q &= \sum_{j=0}^{k} \left(\frac{j^q}{q!}\alpha_j - \frac{j^{q-1}}{(q-1)!}\beta_j - \frac{j^{q-2}}{(q-2)!}\lambda_j \right).
\end{aligned} \tag{7}$$

Applying the formulae (7) into (4) and (5) verified that the 4-point multistep block method is of order nine with error constants,

$$C_{11} = \left[\frac{2497}{7257600}, \frac{40321}{239500800}, -\frac{23}{113400}, \frac{481}{1871100}, \frac{113}{89600}, \frac{689}{985600}, -\frac{94}{14175}, -\frac{568}{467775} \right]^T,$$

According to definition in Lambert (1973), the proposed method is consistent since it was of order nine which is greater than one.

In the context of ODEs, an essential, practical criterion for a good functional method is to have a region of absolute stability or simply the stability region. The test equation $y'' = f = \theta y' + \lambda y$ is substituted into the formulae of 4-point multistep block method. Consequently, having written in the matrix form and solving the determinant of $At^2 - (B + Ch + Eh^2)t - (Dh + Fh^2)$ will give the stability polynomials, where A, B, C, D, E, F are the matrices. The stability polynomial obtained was solved for t and it can be seen that the roots were found to be zero and one.

For $r = 1$,

$$\text{Stability polynomial}: t^{16} - 2t^{15} + t^{14} = 0, \tag{8}$$

$t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 0, t = 1, t = 1.$

Hence, the 4-point multistep block method is a zero stable method by Definition 3.2 in Hairer et al. (1987). Since the 4-point multistep block method has established the consistency and zero stability of the method, therefore the method is convergent according to the definition in Lambert (1973).

In summary, the stability regions for the 4-point multistep block method are displayed in Figures 2, 3 and 4. The region lies within the boundary of the dotted lines which is the shaded area. Notice that the stability region is largest when the step size is half ($r = 2$) and the smallest stability region determined by a double step size ($r = 0.5$). Looking at these stability regions obtained, the size of the stability regions increases as the step size becomes smaller hence indicates that the smaller step size will give better accuracy in the numerical approximations.

IMPLEMENTATION

The step of computing the numerical solution begins by calculating the initial step size and finds four initial values

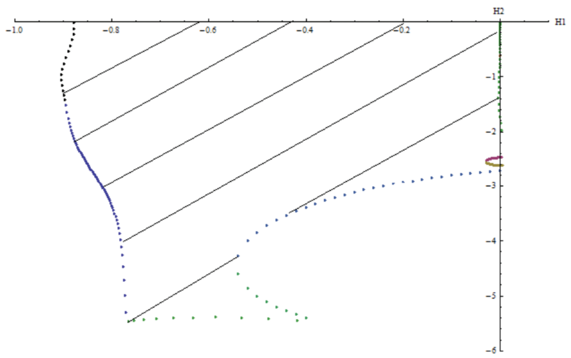

FIGURE 2. Stability region when $r = 1$

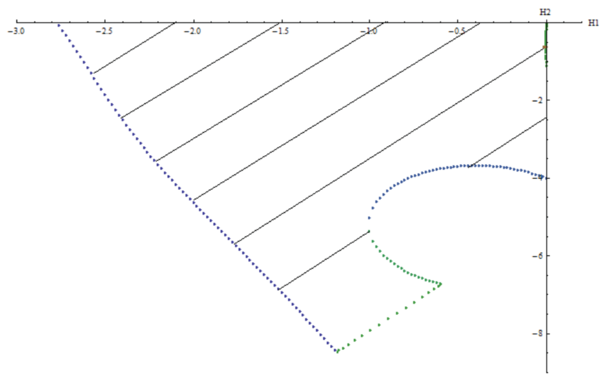

FIGURE 3. Stability region when $r = 2$

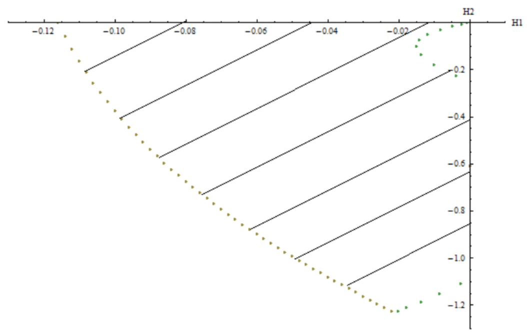

FIGURE 4. Stability region when $r = 1$

in the starting block using Euler method. Subsequently, the 4-point multistep block method can be employed until the end of the interval. In order to gain an efficient and reliable numerical approximation, predictor corrector mode was applied as $P_{k-1}E\left(C_kE\right)^m$. P_{k-1} and C_k stands for an application of the predictor and corrector with different order k, as well as E indicates the evaluation of function f with m iteration.

For validating the current step, local truncation error (LTE) will be calculated by comparing the corrector formulae at the current iteration step with previous iteration. If the LTE is less than the specified tolerance, then the successful step achieved and the next step size will be calculated using the formula as follows:

$$h_{new} = \delta \times h_{old} \times \left(\frac{TOL}{2 \times LTE}\right)^{\frac{1}{k}}, \tag{9}$$

where δ is a safety factor and the value of δ is fixed at 0.5. The step size for the current and previous block are denoted as h_{new} and h_{old}, respectively, while k is the order of the corrector formulae. As has been remarked before, this method was designed with the implementation of varying the step size. For successful step, the next step size will be set to be repeated or doubled by the step size controller. However, for failure step, the step size is set to be halved and the end of the interval for each successive step will be checked using the test provided.

In order to simplify the description, the algorithm of 4-point multistep block method is developed. The approach proceeds following the steps outlined below.

Step 1: Set tolerance and calculate initial step size

Step 2: Compute the initial value in the starting block using Euler method in $PE(CE)^m$ mode

Step 3: Predict the values of y'_p, y_p and f_p using 4-point multistep block method, where $p = 8, 9, 10$ and 11

Step 4: Correct the values of y'_p, y_p and f_p in Step 3 and iterate until it is converge using $\left|y_{11}^{(m)} - y_{11}^{(m-1)}\right| < 0.1 \times \mathrm{TOL}$

Step 5: Calculate LTE, if $LTE < TOL$ then the step success. Else halving the step size with $h_{new} = 0.5 \times h_{old}$ and continue Step 3

Step 6: Calculate error of the computed solution, E_p for $p = 8, 9, 10$ and 11

Step 7: Compute maximum error of the computed block, $\mathrm{MAXE} = \max\limits_{8 \le p \le 11}\left(E_p\right)$

Step 8: Calculate h_{new} using the step size increment formula as (9)

Step 9: If $h_{new} > 2 \times h_{old}$ then $h_{new} = 2 \times h_{old}$. Else $h_{new} = h_{old}$. Continue Step 3

Step 10: If $x_{11} + (4 \times h_{new}) > b$ then $h_{last} = \dfrac{b - x_{11}}{4}$

Step 11: Reset the values of seven back points with h_{last}

Step 12: Do Step 3 and Step 4 for the last block

Step 13: Exit the program and execute the results

NUMERICAL RESULTS

In order to assess the proposed method, five test problems were tested and run at a difference value of tolerance with the results obtained are summarized in Tables 5 to 9. A system as well as single equation of second order ODE are considered with the purpose of showing the competency of the developed method. The following numerical results have shown the comparison between 4-point multistep block method with the method in Abdelrahim and Omar (2016) and Yap and Ismail (2016). In Problems 2 to 5, the numerical results will be examined in terms of total steps taken as well as the value of maximum error. While for

TABLE 5. Numerical results for solving Problem 1

Abdelrahim and Omar (2016)	4-point multistep block method		
MAXE	TOL	TS	MAXE
1.6085(-5)	10^{-2}	18	5.9286(-12)
	10^{-4}	25	6.0546(-15)
	10^{-6}	32	3.1096(-15)
	10^{-8}	38	1.6084(-15)
	10^{-10}	45	9.3702(-16)

TABLE 6. Numerical results for solving Problem 2

Abdelrahim and Omar (2016)		4-point multistep block method		
TS	MAXE	TOL	TS	MAXE
100	5.4296(-6)	10^{-2}	18	7.5088(-12)
		10^{-4}	25	2.8555(-12)
		10^{-6}	32	1.1597(-11)
		10^{-8}	38	5.2300(-12)
		10^{-10}	45	6.5093(-13)

TABLE 7. Numerical results for solving Problem 3

Abdelrahim and Omar (2016)		4-point multistep block method		
TS	MAXE	TOL	TS	MAXE
100	1.8627(-9)	10^{-2}	18	5.6512(-12)
		10^{-4}	25	6.2099(-15)
		10^{-6}	32	4.6563(-14)
		10^{-8}	38	1.4920(-14)
		10^{-10}	45	8.6494(-15)

TABLE 8. Numerical results for solving Problem 4

Yap and Ismail (2016)		4-point multistep block method		
TS	MAXE	TOL	TS	MAXE
120	1.1848(-2)	10^{-3}	73	1.5796(-3)
240	2.5053(-5)	10^{-6}	128	3.6920(-6)
480	4.9288(-8)	10^{-9}	230	5.5238(-9)
960	9.6155(-11)	10^{-12}	423	1.1025(-11)
		10^{-15}	797	9.1186(-13)

TABLE 9. Numerical results for solving Problem 5

Abdelrahim and Omar (2016)		Yap and Ismail (2016)		4-point multistep block method		
TS	MAXE	TS	MAXE	TOL	TS	MAXE
30	1.2927(-12)	10	4.4482(-10)	10^{-3}	22	1.4733(-8)
		20	1.9583(-12)	10^{-6}	32	6.7300(-10)
		40	6.1062(-15)	10^{-9}	43	2.2749(-11)
		80	5.2181(-15)	10^{-12}	52	1.3804(-12)
				10^{-15}	66	6.0099(-15)

problem 1, only the accuracy will be considered as the number of steps taken is not given in Abdelrahim & Omar (2016). The notations used are as follows: TOL: Tolerance; TS: Total steps taken; MAXE: Magnitude of maximum error of the computed solution.

Problem 1

$$y_1'' = 1 - \cos x + \sin(y_2') + \cos(y_2'),\ y_1(0) = 1,\ y_1'(0) = 0,$$

$$x \in [0,1],\quad y_2'' = \frac{1}{4+y_1^2} - \frac{1}{5-\sin(x)^2},\quad y_2(0) = 0,\ y_2'(0) = \pi.$$

Theoretical solution: $y_1 = \cos x$, $y_2 = \pi x$.

Problem 2

$$y_1'' = -e^{-x}y_2,\quad y_1(0) = 1,\quad y_1'(0) = 0,\quad x \in [0,1],$$

$$y_2'' = 2e^x y_1',\quad y_2(0) = 1,\quad y_2'(0) = 1.$$

Theoretical solution: $y_1 = \cos x$, $y_2 = e^x \cos x$.

Problem 3

$$y_1'' = -\frac{y_2}{\sqrt{y_1^2 + y_2^2}},\quad y_1(0) = 1,\quad y_1'(0) = 0,\quad x \in [0,1],$$

$$y_2'' = -\frac{y_2}{\sqrt{y_1^2 + y_2^2}},\quad y_2(0) = 0,\quad y_2'(0) = 1.$$

Theoretical solution: $y_1 = \cos x$, $y_2 = \sin x$.

Problem 4

$$y'' + y = 0.001e^{ix},\quad y(0) = 1,\quad y'(0) = 0.9995i,$$

$$x \in [0,40\pi],$$

which is equivalent to

$$u'' + u = 0.001 \cos x,\quad u(0) = 1,\quad u'(0) = 0,$$

$$v'' + v = 0.001 \sin x,\quad v(0) = 0,\quad v'(0) = 0.9995.$$

Theoretical solution: $y = u(x) + iv(x)$

$$u(x) = \cos x + 0.0005x \sin x$$

$$v(x) = \sin x - 0.0005x \cos x.$$

Problem 5

$$y'' = x(y')^2,\ y(0) = 1,\ y'(0) = \tfrac{1}{2},\ x \in [0,1].$$

Theoretical solution: $y = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$.

Discussion and Conclusion

This study set out with the aim of assessing the accuracy and efficiency of the developed method with other direct block methods whose employed a constant step size. The data in Tables 5 to 7 show that the 4-point multistep block method is significantly outperformed Abdelrahim and Omar (2016) in terms of accuracy as well as steps taken for all tolerance. A similar pattern of performance is observed for Problem 4 with 4-point multistep block method manages to reduce the total step taken, approximately by half.

As can be seen from Table 9, numerical result for this proposed method has less two decimal accuracy compared to Abdelrahim and Omar (2016) with equivalent number of steps. Nevertheless, the proposed method reports comparable outcome with the method in Yap and Ismail (2016) and give a good result for all tolerances. The authors believe the efficiency of the proposed method could be clearly shown if the range of x increases. Taken together, these results indicate that the proposed algorithm manages to solve system and single equation of second order ODEs directly.

To sum up, the accuracy obtained by this developed method is better and comparable with the method proposed in Abdelrahim and Omar (2016) and Yap and Ismail (2016). An implication of implementing variable step size is highlighted in the reduction of steps taken, therefore the efficiency of the method is marked. Thus, these findings offer crucial evidence that implementing variable steps size in the block methods could improve its efficiency as well as preserve the accuracy.

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