Scaling Limits of Critical Systems in Random Geometry

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Preface This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text. The work in Chapter 5 was completed in collaboration with H.Wu, and the work in Chapter 4 with J.Aru and A.Sepúlveda. In all these collaborations, the contribution of each collaborator was equal.

This dissertation is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.

I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Contents

Abstract

This thesis focusses on the properties of, and relationships between, several fundamental objects arising from critical physical models. In particular, we consider Schramm–Loewner evolutions, the Gaussian free field, Liouville quantum gravity and the Brownian continuum random tree.

We begin by considering branching diffusions in a bounded domain $D \subset \mathbb{R}^d$, in which particles are killed upon hitting the boundary ∂D. It is known that such a system displays a phase transition in the branching rate: if it exceeds a critical value, the population will no longer become extinct almost surely. We prove that at criticality, under mild assumptions on the branching mechanism and diffusion, the genealogical tree associated with the process will converge to the Brownian CRT.

Next, we move on to study Gaussian multiplicative chaos. This is the rigorous framework that allows one to make sense of random measures built from rough Gaussian fields, and again there is a parameter associated with the model in which a phase transition occurs. We prove a uniqueness and convergence result for approximations to these measures at criticality.

From this point onwards we restrict our attention to two-dimensional models. First, we give an alternative, “non-Gaussian” construction of Liouville quantum gravity (a special case of Gaussian multiplicative chaos associated with the 2-dimensional Gaussian free field), that is motivated by the theory of multiplicative cascades. We prove that the Liouville (GMC) measures associated with the Gaussian free field can be approximated using certain sequences of “local sets” of the field. This is a particularly natural construction as it is both local and conformally invariant. It includes the case of nested CLE_4, when it is coupled with the GFF as its set of “level lines”.

Finally, we consider this level line coupling more closely, now when it is between SLE_4 and the GFF. We prove that level lines can be defined for the GFF with a wide range of boundary conditions, and are given by SLE_4-type curves. As a consequence, we extend the definition of $\mathrm{SLE}_4(\rho)$ to the case of a continuum of force points.

1 Introduction

1.1 Background

This thesis will aim to explore, and hopefully gain a little insight into, the behaviour of a few canonical models in random geometry. These models, that we will describe shortly, are all linked in that they arise naturally from physical models at their "critical" point. That is, at a point (usually a special temperature), where a phase transition occurs.

We will be interested in the macroscopic behaviour of such systems. Take a discrete lattice model (for example, a random model for the spins of particles in a magnet) and observe what happens in the scaling limit, as the number of particles tends to infinity. Perhaps surprisingly, it turns out that in many cases, what one gets in the end doesn't actually depend very much on the initial model at all. This is the notion of universality.

The main objects studied here will all be universal in this sense. Let us start, as an example, with Brownian motion. Take any random walk in space with mean zero increments, and then speed up the walk and rescale the size of the steps accordingly. The following result tells us that, if we do this in the correct way, the walk will converge to a Brownian motion.

Theorem 1.1 (Donsker's invariance principle (1-dimension))**.** *Let* $(X_k)_{k=1}^{\infty}$ *be a sequence of i.i.d. random variables with mean 0 and variance 1. Set* $S_n = \sum_{k=1}^{n} X_k$ *and define a process* $S = (S_t : t \in [0,1])$ *by linear interpolation:* $S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lfloor t \rfloor + 1} - S_{\lfloor t \rfloor})$*. Then the laws of the process*

$$S^N := \left(\frac{1}{\sqrt{N}} S_{Nt} : 0 \leq t \leq 1 \right)$$

converge weakly to that of a Brownian motion on $(C[0,1], \mathcal{B}(C[0,1]))$*.*

In particular, the scaling limit in the above theorem does not actually depend on the details of the random walk we chose in the first place. For this reason, Brownian motion can be thought of as a canonical notion of random curve.

However, there are some things that Brownian motion lacks the complexity to describe. For example, the natural interfaces that appear in many planar lattice models (say, between positive and negative spins in a magnet) become increasingly fractal as the number of particles tends to infinity, but do not look like the trace of a Brownian motion. A great breakthrough was made in our understanding of these interfaces when Oded Schramm, [Sch00], introduced a family of random fractal curves known as Schramm–Loewner evolutions (SLE) as candidates for their scaling limits. Indeed, it has since been proved that in several cases, [CS12, LSW04, SS09, Smi01], such interfaces do in fact converge to SLE.

Another natural question, which has been of great interest to the probability community in the last few years, is whether the notion of a canonical random path (i.e. Brownian motion) can be extended to higher dimensions. For example, in two-dimensions, can we say what a random sphere should look like? This problem, which can be phrased more concretely in terms of putting a "random metric" on the 2-sphere, has attracted much recent discussion and attention. Although seemingly a very difficult question, astounding progress on this has been made in several directions. We will discuss two different approaches to this now.

Just as Brownian motion is a limit of random walks, it seems natural to search for a "random sphere" by considering sensible discrete approximations. In this case, the analogous things to look at are random planar maps; essentially, large collections of polygons glued together to give a metric space with the topology of a sphere. A major breakthrough in this direction came when Le Gall [LG13] and Miermont [Mie13] proved, independently, that for certain natural choices of random planar maps these metric spaces have a scaling limit, known as the Brownian map.

Theorem 1.2 (Convergence to the Brownian map)**.** *Suppose that $p = 3$ or an even integer greater than or equal to 4. Let $\mathbf{m}_n$ be a uniformly chosen rooted p-angulation of the sphere (that is, a finite connected graph embedded in $\mathbb{S}^2$, viewed up to orientation preserving homeomorphisms of $\mathbb{S}^2$, with a distinguished oriented root edge and with n faces of degree p.) Let d_{gr} be the graph distance on $\mathbf{m}_n$. Then there exists a positive constant λ_p such that*

$$(\mathbf{m}_n, \frac{\lambda_p}{n^{1/4}} d_{gr}) \xrightarrow{(d)} (\mathbf{m}_\infty, d^*)$$

as $n \to \infty$, with respect to the Gromov–Hausdorff topology on compact metric spaces. The limit $(\mathbf{m}_\infty, d^*)$, known as ***the Brownian map****, is a random compact metric space that is almost surely homeomorphic to $\mathbb{S}^2$.*

This limit is clearly a good candidate for our "uniform" random surface. The only drawback is that it does not come with a canonical conformal embedding into the sphere. Thus, it does not give us immediately the canonical conformal structure that is desirable from the perspective of Liouville quantum gravity.

Another approach to this question has been to try and define the limiting metric space directly in the continuum, using a further canonical object known as the Gaussian free field (GFF). This is the natural analogue of Brownian motion in two dimensional time: it is a planar, conformally invariant Gaussian field, that also arises from numerous physical models. The GFF describes the fluctuations of an electric potential, the scaling limit of various discrete "height-function" models, such as the dimer model, and is also a centrally important object in quantum field theory.

Motivated by non-rigorous constructions in the physics literature, one would like to define a random "Liouville quantum gravity" surface from a Gaussian free field h, roughly, by taking it to have Riemannian metric tensor equal to $e^{\gamma h}\, dz$. In fact, the Gaussian free field is too rough to be defined pointwise, so to define these things rigorously is a non-trivial task. Nevertheless, remarkable progress has been made recently by Miller and Sheffield, who are able to define a metric on the sphere using the GFF, and also connect it to the Brownian map [MS15a, MS15b, MS16e].

The final object that will be considered in this thesis is Aldous' continuum random tree [Ald91]. This is a random metric space that again arises as the scaling limit of various critical models. As an example, take the Galton–Watson tree: a simple and classical model associated with many natural biological processes. It has been proved [Ald93, LGD02, Mie08], that if one takes a critical (multitype) Galton–Watson tree and conditions it to be large in some sense, then it will converge after rescaling to the Brownian CRT.

Theorem 1.3 (Convergence of critical (single-type) Galton–Watson trees)**.** *Suppose that $\mathcal{L}$ is an offspring distribution with $\mathbb{E}[\mathcal{L}]=1$ and $\mathbb{E}[\mathcal{L}^2]=\sigma^2$. Let T_n be a sample of the associated Galton–Watson tree, viewed as a random metric space with distance d_n, and conditioned to reach height n. Then*

$$(T_n, \frac{1}{n}d_n) \xrightarrow{(d)} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$$

with respect to the Gromov–Hausdorff topology as $n\to\infty$, where $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the Brownian continuum random tree: the real tree encoded by a Brownian excursion $\mathbf{e}$ *conditioned to reach height 1.*

In this thesis we will see another example, where the CRT appears as a scaling limit for the genealogy of critical branching diffusions.

To conclude, we remark that one of the most fascinating things about this collection of canonical objects, and indeed many others that have not been mentioned here, is the web of connections between them. For example, it turns out that SLE curves actually appear as level lines and flow lines of the Gaussian free field [Dub09, SS09, MS16a], and in another direction, [DMS12] proves that Liouville quantum gravity can be constructed as a "mating" of two continuum random trees. We will be particularly interested in exploring such connections here. Specifically, we will look more closely at the relationship between SLE and the GFF in Chapter 5, and at the relationship between branching processes and Liouville quantum gravity in Chapter 4.

1.2 Results and Outline

We begin, in Section 1.3, by giving a brief overview of the main objects appearing in this thesis. This will include relevant definitions, results, and a little historical context. We then conclude the introduction by mentioning a few open problems, and discussing possible directions for future research.

In Chapter 2, we move on to our first main topic: invariance principles for branching diffusions in bounded domains. We consider branching diffusions in a bounded domain D of $\mathbb{R}^d$ in which particles are killed upon hitting the boundary ∂D. It is known [Sev58, Wat65] that such a process undergoes a phase transition when the branching rate exceeds a critical value. We investigate the system at criticality, and prove an asymptotic for the probability of survival up to large times. We show further that the genealogical tree associated with such a critical process converges to Aldous' continuum random tree, under appropriate rescaling. This result holds under only a mild assumption on the domain, and is valid for all branching mechanisms with finite variance, and a general class of diffusions.

Next, in Chapter 3, we turn our attention to the theory of Gaussian multiplicative chaos. This is the framework, developed by Kahane in the 1980s, that allows one to rigorously define random measures of the form "$e^{\gamma h(z)-\frac{\gamma^2}{2}\mathrm{var}(h(z))}\,dz$", when $\gamma > 0$ is a parameter and h is a rough Gaussian field. It turns out that at a critical value $\gamma = \sqrt{2d}$, where d is the dimension of the underlying space, the usual construction of these measures yields something degenerate. Our main result in this chapter is that, if we approximate the field h by convolving with a mollifier, and use a "derivative" renormalisation at criticality (formally, taking the derivative of the subcritical measures in γ and evaluating at $\gamma = 2$), we can obtain a non-trivial limit measure.

One of the fields covered by this result is the 2-dimensional Gaussian free field. As discussed in the Section 1.1, the chaos measure in this case (known as the Liouville measure) is of particular interest. In Chapter 4, we give an alternative construction of this measure, that is inspired by the theory of branching random walks and multiplicative cascades. We prove that the Liouville measure can be approximated, including at criticality, using certain sequence of "local sets" for the Gaussian free field. This is in an extremely simple and natural construction, as it is defined entirely using the local geometry of the field (in particular, not using any Gaussian process machinery). It also resolves a conjecture of Aïdekon, [Aid15].

Finally, in Chapter 5 we study in more detail the "level line" coupling [SS09] between SLE_4 and the Gaussian free field. We generalise the existing theory by considering a free field with general boundary data F, rather than the specific examples of boundary data that had been considered before. We show that the level lines of this field exist as continuous curves, under only the assumption that F is regulated (i.e., admits finite left and right limits at every point), and satisfies certain inequalities. Moreover, we show that these level lines are almost surely determined by the field. This allows us to define and study a generalization of the $\mathrm{SLE}_4(\underline{\rho})$ process, now with a continuum of force points. A crucial ingredient is a monotonicity property in terms of the boundary data which strengthens a result of Miller and Sheffield [MS16a] and is also of independent interest.

Chapter 5 is joint work with Hao Wu from the University of Geneva and Chapter 4 was completed in collaboration with Juhan Aru and Avelio Sepúlveda from ETH Zürich. The work in Chapter 5 is based on the accepted paper [PW17], and the work in Chapters 2, 3 and 4 are based on the preprints [Pow17b],[Pow17a] and [APS17]. All of this work was supported by the UK Engineering and Physical Sciences Research Council (EPSRC) grant EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis.

1.3 Preliminaries

We will now discuss in a little more detail some of the main objects that we will be concerned with.

1.3.1 Brownian Motion

As discussed in the introduction, Brownian motion plays a central role in our understanding of physical systems. It is a canonical object to study for probabilists, since it arises from the most simple model of motion: the random walk. Indeed, Donsker's invariance principle (see for example [MP10]), tells us that any unbiased random walk will converge after rescaling to a Brownian motion.

We will briefly recall here some basic properties of Brownian motion, but this is by no means a complete survey. For a more detailed introduction see, for example, [MP10].

Brownian motion can be characterised in many different ways, but we will use the following definition:

Definition 1.4 (Brownian Motion)**.** *Let $(B_t)_{t\geq 0}$ be a continuous random process in $\mathbb{R}^d$ starting from 0. If $(B_t)_{t\geq 0}$ is a zero-mean Gaussian process with*

$$\mathbb{E}[B_s B_t] = s \wedge t$$

for all $s, t \geq 0$, then we say that $(B_t)_{t\geq 0}$ is a Brownian motion.

Figure 2: A one-dimensional Brownian motion.

Remark 1.5. *To start a Brownian motion at a position x other than 0, as we will often want to do, we simply consider the process $x+B_t$, where B_t is as in Definition 1.4.*

One of the most important properties of Brownian motion is the following Markov property. It says that if one stops a Brownian motion at any given time t, its evolution from time t onwards will simply be that of another Brownian motion, starting from B_t, but otherwise completely independent of what happened before. We write $(\mathcal{F}_s^B)_{s\geq 0}$ for the filtration generated by $(B_s)_{s\geq 0}$.

Lemma 1.6 (Weak Markov Property)**.** *For any $s \geq 0$, the process $(B_{t+s} - B_s)_{t\geq 0}$ is a standard Brownian motion, independent of $\mathcal{F}_s^B$.*

In fact, this property does not only hold at deterministic times. To describe this phenomena we need the notion of *stopping times.*

Definition 1.7 (Stopping times)**.** *A random variable T defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as a Brownian motion $(B_t)_{t\geq 0}$ is a stopping time for the Brownian motion if, for all $t \geq 0$,*

$$\{T \leq t\} \in \mathcal{F}_t^B.$$

These times are precisely those at which we have a stronger version of Lemma 1.6. For T and B as above, we define the *stopped σ-algebra* by setting

$$\mathcal{F}_T^B = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t^B \ \ \forall t \geq 0.\}$$

Lemma 1.8 (Strong Markov Property)**.** *Let B be a standard Brownian motion and T be a stopping time for B which is almost surely finite. Then the process*

$$(B_{t+T} - B_T)_{t\geq 0}$$

is a standard Brownian motion, independent of F_T^B.

A variant of Brownian motion that will be relevant to what comes next is the *Brownian Bridge*. This is a Brownian motion on some interval of time, let's say $[0,1]$, that is conditioned to start and end at 0. Although this is an event of probability zero, one can make sense of this conditioning, and obtain a process which is characterised as follows.

Definition 1.9 (Brownian Bridge)**.** *Let $(W_t)_{t\in[0,1]}$ be a continuous Gaussian process with*

- $W_0 = W_1 = 0$ *almost surely.*
- $\mathbb{E}[W_t] = 0\ \forall t$ *and*
- $\mathrm{cov}(W_s, W_t) = s \wedge t - st$ *for all* $s, t \geq 0$.

Then we say that W is a Brownian bridge on $[0,1]$.

We conclude this section by mentioning a very special property of Brownian motion in two dimensions. This is the property of *conformal invariance.* We will see it appearing again and again in the following sections, and it will play a central role in this thesis. To define it, we need the following fundamental theorem:

Theorem 1.10 (Riemann Mapping Theorem)**.** *Let $\mathbb{D}$ be the open unit disc and $D \subset \mathbb{C}$ be any proper, simply connected domain. Then there exists a conformal isomorphism $\Phi : D \to \mathbb{D}$. That is a bijection Φ such that Φ and Φ^{-1} are holomorphic with non-vanishing derivative.*

Lemma 1.11 (Conformal Invariance of Brownian Motion)**.** *Let D and D' be simply connected complex domains and let B, B' be Brownian motions in D and D' respectively, starting at z, z' and stopped upon leaving D, D'. Let $\Phi : D \to D'$ be a conformal isomorphism. Then $\Phi(B)$ and B' have the same law, up to a time change.*

1.3.2 Gaussian free field

The planar Gaussian free field (GFF) is another canonical, conformally invariant object in two dimensions. In fact, we will see that it is the natural generalisation of Brownian motion, or more specifically, the Brownian bridge, to two-dimensional time. This section will mainly follow the approach taken in [Ber15b]. We refer the reader to these excellent notes for further details and proofs, as well as to [She07] and [Wer].

We will begin by briefly discussing the discrete counterpart of the GFF, the *Discrete Gaussian free field*, in order to get some intuition. Take a finite graph $G = (V, E)$ with a special set of boundary edges ∂, and let X_n be a random walk on G. Write $\mathbb{P}_x$ for the law of X_n, when it is started from position $x \in V$.

Definition 1.12 (Discrete Green Function)**.** *The discrete Green function $G(x, y)$ is defined, for $x, y \in V$ by*

$$G(x,y) := \frac{1}{d(y)} \mathbb{E}_x \left(\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = y; \tau > n\}} \right)$$

where τ is the first time that X_n hits ∂.

We then define the Discrete GFF on G in a very simple way. Let

$$V = \{x_1, \cdots, x_n\}.$$

Definition 1.13 (Discrete GFF)**.** *The Discrete GFF is the centered Gaussian vector $(h(x_i))_{1 \le i \le n}$ on V, with covariances given by the discrete Green function.*

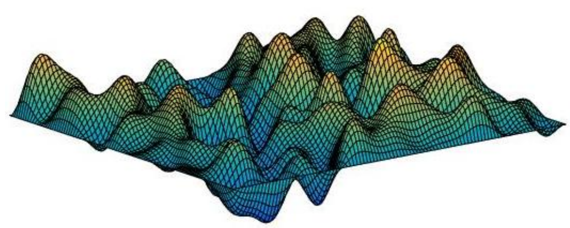


Figure 3: A simulation of the discrete GFF (extended as a continuous function).

From this definition, one can in fact prove that the density of h with respect to $\prod dx_i$ takes a very special form. It is proportional to

$$\exp\left(-\frac{1}{4} \sum_{x,y \in V : x \sim y} (h(x) - h(y))^2 \right)$$

where the term $\sum_{x,y \in V: x \sim y} (h(x) - h(y))^2$ is what is known as the *Dirichlet energy* of h: an energy that is minimised by harmonic functions. Thus, the DGFF can be considered a natural pertubation of harmonic functions, just as the random walk, or Brownian motion, can be considered a perturbation of linear ones.

We would like the continuum Gaussian free field to also be such a perturbation. Unfortunately, it turns out that in this case, as we will discuss shortly, it cannot even be defined as a function! With this in mind, perhaps the simplest way to think of the GFF is as a limit of discrete Gaussian fields, in the same way that Brownian motion is often thought of as the limit of random walks. Indeed, one can define it in this way, and this is useful to bear in mind. However, we will approach things more directly.

To do this, we first need an analogue of the discrete Green function. Let $D \subset \mathbb{C}$ be a bounded domain, and let $p^D(t,x,y)$ be the transition density [1] for a Brownian motion killed when leaving D.

Definition 1.14 (Continuum Green Function)**.** *We define the Green function on D by*

$$G_D(x,y) := \int_0^\infty p^D(t,x,y)\, dt$$

Similarly to Definition 1.13, we would like to define the continuum GFF to be the centered Gaussian function with covariances given by the Green function. However, there is now a problem, as the Green function is not actually well defined. In fact, it can be characterised as the only function in D that blows up logarithmically on the diagonal and is harmonic away from it. [2]

Although this singularity prevents us from defining the GFF pointwise, we can still define it in other ways. One approach, using the Kolmogorov extension theorem, is to define it as functional acting on the set

$$\mathcal{M} := \{\rho = \rho^+ - \rho^- \text{ s.t. } \int_{D^2} \rho^\pm(x)\rho^\pm(y) G_D(x,y)\, dxdy < \infty\}$$

of signed measures with finite Green energy. We can think of this action as "integrating" the GFF against these measures (which will make more sense in light of the following discussion, see Lemma 1.17.)

Definition 1.15 (Gaussian free field zero boundary conditions [Ber15a])**.** *There exists a unique stochastic process $(h_\rho)_{\rho\in\mathcal{M}}$ indexed by $\mathcal{M}$ such that for every $\rho_1, \cdots, \rho_n \in \mathcal{M}$*

$$(h_{\rho_1}, \cdots h_{\rho_n})$$

is a centered Gaussian vector with

$$\operatorname{cov}(h_{\rho_i}, h_{\rho_j}) = \int_{D^2} G_D(x,y)\rho_i(x)\rho_j(y)\, dxdy.$$

Remark 1.16. *Recall Definition 1.9 of the Brownian bridge: it is the unique centered Gaussian process on $[0,1]$ with covariances given by the function $G(s,t) = s \wedge t - st$. In fact, this function G is exactly the Green function on $[0,1]$.*

[1] That is, $p^D(t,x,y) = p(t,x,y)\mathbb{P}^t_{x,y}(W_s \in D \; \forall s \in [0,t])$, where $p(t,x,y) = (2\pi t)^{-1}\exp(-|x-y|^2/2t)$ is the transition density for Brownian motion in $\mathbb{R}^d$ and $\mathbb{P}^t_{x,y}$ is the law of a Brownian bridge from x to y on $[0,t]$.

[2] More precisely, it is the only function such that $G_D(x,y) = -\log(|x-y|) + O(1)$ as $y \to x$, $\forall x \in D$ and $\Delta G_D(x,\cdot) = -2\pi\delta_x(\cdot)$ as a distribution.

We can intepret the above as saying that the GFF has "covariances given by the Greens' function". But how does the definition fit in with our intuition in terms of Dirichlet energy? This can be made clearer with an alternative approach.

It turns out that one can also view the GFF as a *random distribution.* That is, as a continuous linear map $\mathcal{D}(D) \to \mathbb{R}$, where $\mathcal{D}(D)$ is the set of smooth, compactly supported functions in D. [3] We denote the set of distributions by $\mathcal{D}'(D)$, and endow it with the weak-* topology.

Define the Sobolev space $H_0^1(D)$ to be the Hilbert space completion of $\mathcal{D}(D)$ with respect to the *Dirichlet inner product*:

$$(f,g)_\nabla := \int_D \nabla f \cdot \nabla g.$$

Let $(f_n)_{n\geq 1}$ be an orthonormal basis of $H_0^1(D)$ and let $(\alpha_n)_{n\geq 1}$ be a collection of i.i.d $N(0,1)$ random variables. Then we can also construct the GFF using the following approximation:

Lemma 1.17. *Set*

$$h_n := \sum_{i=1}^n \alpha_i f_i.$$

This sum converges almost surely in the space of distributions (in fact, in the Sobolev space $H_0^{-\varepsilon}(D)$ for any $\varepsilon > 0$). Moreover, the limit agrees in distribution with the process h from Definition 1.15, when it is restricted to $\mathcal{D}(D)$.

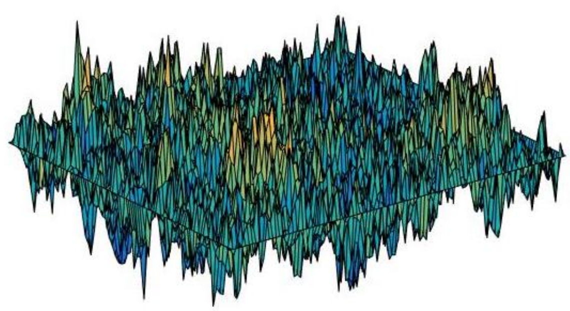


Figure 4: A simulation of the discrete GFF using a lot of terms.

Remark 1.18. *In fact, a density argument can be used to show that the above definition extends directly to give a process satisfying Definition 1.15 (see [Ber15b, Theorem 1.16]). With this in mind, we write (h,ρ) for h_ρ in Definition 1.15, as is standard for distributions.*

[3] We use the topology on $\mathcal{D}(D)$ which is defined by saying that a sequence $f_n \to 0$ if there exists a compact $K \subset D$ supporting all the f_n, and such that f_n and f_n' converge uniformly to 0 on D.

Note that for any distribution f in $\mathcal{D}'(D)$, and function $g \in \mathcal{D}(D)$, one can define the Dirichlet inner product $(f,g)_\nabla$ using the definition of the distributional derivative. This definition is just the natural extension of the Gauss–Green formula (integration by parts), and so we have that $(f,g)_\nabla = -(f,\Delta g)$. Thus, for the Gaussian free field, whenever g is such that $-\Delta g = \rho \in \mathcal{M}$ we set

$$(h,g)_\nabla := (h,\rho).$$

This is then a Gaussian random variable with mean 0, and variance $(g,g)_\nabla$.

We should also point out that, in all of the above, we have been considering a Gaussian free field with *zero boundary conditions.* In fact, we will often want to work in a more general setting, where we allow it to have *non-zero mean.* The way we do this is simply to add on a deterministic harmonic function to the field. This harmonic function will be referred to as the *mean* of the distribution, and its restriction to the boundary will be referred to as its *boundary conditions.*

Now we will review some of the key properties of the Gaussian free field. The first is conformal invariance, and follows directly from the corresponding conformal invariance of the Green function (which in turn follows from the conformal invariance of Brownian motion, Lemma 1.11.)

Lemma 1.19 (Conformal Invariance of the GFF)**.** *Let h be a Gaussian free field in D, and $\phi : D \to D'$ be a conformal map. Then $h \cdot \phi^{-1}$ has the law of a GFF on D'.*

The second is a spatial Markov property. See [Ber15b, Theorem 1.17] for a proof.

Lemma 1.20 (Weak Markov Property for the GFF)**.** *Take a deterministic closed set $A \subset D$ and let h be a GFF with zero boundary conditions on D. Then we can write $h = h_A + h^A$ where*

- *h_A is a random distribution that is a harmonic function when restricted to $D \setminus A$;*
- *h^A is a zero-boundary GFF in $D \setminus A$; and*
- *h_A, h^A are independent.*

Note the analogy between this Lemma, and Lemma 1.6. The latter says that a Brownian motion can be written, from any fixed time t onwards, as a sum of two parts: B_t, plus another independent Brownian motion. The above Lemma says that a Gaussian free field can be written, in any fixed open set, as a sum of two parts: a harmonic function, plus another independent Gaussian free field. It is therefore perhaps not particularly surprising that there is also a strong Markov property for the Gaussian free field. This is described by the notion of a local set.

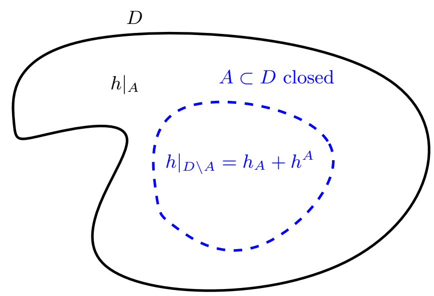


Figure 5: Weak Markov property for the GFF.

Definition 1.21 (Local sets, Strong Markov Property)**.** *Take a (now possibly random) relatively closed set $A \subset D$ and let h be a GFF with zero boundary conditions on D. We say that A is a local set for h if there exists a distribution h_A, that is harmonic when restricted to $D \setminus A$, and is such that, conditionally on A, $h^A = h - h_A$ is a zero-boundary Gaussian free field in $D \setminus A$.*

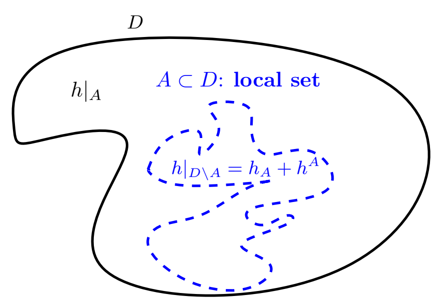


Figure 6: Strong Markov Property for the GFF.

These sets are exactly analogous to stopping times for Brownian motion - they are random "stopping" sets at which a strong Markov property holds. One can even draw stronger analogies between particular types of local sets for the GFF and stopping times for Brownian motion. For examples of this, see [ASW15]. Local sets will play an important role in two of the papers included in this thesis, and we will see many natural, non-deterministic examples later on.

One final thing that we will discuss here is the so-called *circle average* process of the free field. This will be important in the following sections, as it is particularly useful way of approximating the field. Essentially we approximate the value of the Gaussian free field h at a point z by taking its average on smaller and smaller circles around z. That is, we define $h_\varepsilon(z) := (h, \rho_\varepsilon^z)$. where ρ_ε^z is the uniform measure on the circle of radius ε around z, and let $\varepsilon \to 0$. One can prove (see [DS11, Propn 3.1]) that a modification of h exists for which this process is jointly Hölder in z and ε. In fact (for fixed z) it has the same law as a Brownian motion.

Lemma 1.22. *Fix $z \in D$ let $h_\varepsilon(z)$ be defined as above, for $\varepsilon \leq \varepsilon_0 < dist(z, \partial D)$. Then*

$$(h_{e^{-t}}(z))_{t \geq \log(1/\varepsilon_0)}$$

is a Brownian motion started from $h_{e^{-\log(1/\varepsilon_0)}}(z)$.

1.3.3 Gaussian multiplicative chaos

Gaussian multiplicative chaos theory was introduced by Kahane [Kah85] in the 1980s in order to rigorously define measures of the form

$$\mu^\gamma(dz) = \mathrm{e}^{\gamma h(z)}\, dz$$

where h is a rough Gaussian field and $\gamma > 0$ is a parameter. Although Kahane's original construction of these measures was restricted to a special class of fields h (having so called σ-positive kernels) it has since been extended (at least for some values of γ) to general log-correlated fields [Ber15a, RV10]. By a log-correlated field we mean a Gaussian field h defined as in Definition 1.15, but with the Green function G_D replaced by another non-negative definite kernel K satisfying

$$K(x, y) = -\log|x - y| + \mathrm{O}(1) \quad \text{as } x \to y.$$

The basic idea to construct the measures μ^γ, is to approximate the field h using a sequence of smooth fields. A natural way to do this is to convolve h with a mollifying measure θ, supported in a compact set and with total mass 1. This yields the approximations

$$h_\varepsilon(z) := h \star \theta_\varepsilon(z) = (\theta_{z,\varepsilon}, h)$$

for $\varepsilon > 0$, where θ_ε is the image of the measure θ under the map $x \mapsto \varepsilon x$, and $\theta_{\varepsilon,z}$ is the image of θ_ε under the map $x \mapsto x + z$. Then one can define the approximate measures

$$\mu_\varepsilon^\gamma := \mathrm{e}^{\gamma h_\varepsilon(z)}\, \mathbb{E}[\mathrm{e}^{\gamma h_\varepsilon(z)}]^{-1}$$

and try to take a limit as $\varepsilon \to 0$. Note that for each z, the process $\mathrm{e}^{\gamma h_\varepsilon(z)}\, \mathbb{E}[\mathrm{e}^{\gamma h_\varepsilon(z)}]^{-1}$ is a martingale. We have the following result.

Theorem 1.23 ([Ber15a, RV10])**.** *Let h be a log-correlated field and θ a measure supported in $B(0,1)$, with total mass 1 and satisfying $\int \log(1/|w - v|)\theta(dw) = \mathrm{O}(1)$ uniformly in $v \in B(0,5)$. Then for $\gamma < \sqrt{2d}$, μ_ε^γ converges weakly in probability to a random measure μ^γ as $\varepsilon \to 0$. Moreover, μ^γ a.s. has no atoms, $\mu^\gamma(A) > 0$ for any $A \subset D$ open, and μ^γ does not depend on the choice of mollifier θ.*

Remark 1.24. *This includes the case when h is the 2d-GFF in a domain D and θ is uniform measure on the unit circle. In Chapter 4, based on [APS17], we will give an alternative construction of this measure using only the local geometry of the field and its local sets.*

Note that this theorem only holds for $\gamma < \sqrt{2d}$. Indeed, it is known [RV10], that the measures μ_ε^γ converge to the 0 measure almost surely if $\gamma \geq \sqrt{2d}$. Therefore, to define something non-trivial in the critical ($\gamma = \sqrt{2d}$) and supercritical ($\gamma > \sqrt{2d}$) cases, it is necessary to give an extra *push* as to the measures $\varepsilon \to 0$. That is, to make an additional renormalisation. These cases turn out to be much more tricky to deal with than the subcritical case. One of the main reasons for this is that the μ_ε^γ mass of a fixed set is no longer uniformly integrable, meaning that existence of a limit is much harder to show.

We first discuss the critical case. In analogy to the corresponding results for Gaussian multiplicative cascades (a discrete counterpart of Gaussian multiplicative chaos related to the branching random walk, see Section 1.3.5) we expect to be able to renormalise at criticality and obtain a non-trivial limit measure in two different ways:

(1) by instead considering $\sqrt{\log(1/\varepsilon)}\mu_\varepsilon^{\sqrt{2d}}$ or

(2) by taking the derivative measures $D_\varepsilon(dz) := \frac{d}{d\gamma}\mu_\varepsilon^\gamma(dz)|_{\gamma=\sqrt{2d}}$.

The first is a deterministic renormalisation, known as the Seneta–Heyde rescaling, and has the advantage of yielding a sequence of positive approximating measures. The downside, however, is that $\sqrt{\log(1/\varepsilon)}\, \mathrm{e}^{\gamma h_\varepsilon(z)}\, \mathbb{E}[\mathrm{e}^{\gamma h_\varepsilon(z)}]^{-1}$ is no longer a martingale for each z. On the other hand, the derivative, random, renormalisation preserves the martingale property. The drawback here is that it gives only a sequence of signed measures.

Both problems are reasonably difficult to get around. However, in [DRSV14a, DRSV14b] the authors considered a special class of fields h, with so-called $\star$-scale invariant kernels, and used a different sequence of "cut-off" approximations to the field to define the critical chaos measures. In this specific set-up, they were able to show that both sequences (1) and (2) yield a non-trivial limiting measure, and in fact, they are the same up to a constant.

Theorem 1.25. *Suppose h has a $\star$-scale invariant kernel $K(x,y) = \int_1^\infty k(u|x-y|)/u\, dx$, and the approximate fields h_ε have kernels given by*

$$K_\varepsilon(x,y) := \int_1^{1/\varepsilon} \frac{k(u|x-y|)}{u}\, du.$$

Then the two sequences of approximating measures (1) and (2) converge to the same limiting measure, up to a constant $\sqrt{2/\pi}$ - the limit of (1) is $\sqrt{2/\pi}$ times the limit of (2).

They also extended the above result to cut-off approximations for the 2d-GFF, which does not quite fit into the $\star$-scale invariant framework.

However, these approximations do not correspond to convolving the field with a mollifier. To complete the picture at criticality therefore, it remains to show that (1) and (2) yield (the same) limiting measure when h is a general log-correlated field and h_ε is a convolution approximation.

This was done for (1) when h has $\star$-scale invariant kernel by Junnila and Saksman [JS17], and also when h is the 2d-GFF [JS17, HRV15]. The corresponding result (2) is the main result of [Pow17a] (and Chapter 3.)

In the supercritical case ($\gamma > \sqrt{2d}$) less is known, but there are still alternative constructions that should lead to non-trivial chaos measures. We refer the reader to the survey [RV14] for a discussion of what is expected from the physics literature, and of recent mathematical progress (see in particular Section 6.2.)

1.3.4 Liouville quantum gravity

Liouville quantum gravity is a canonical model of a random Riemannian surface, first studied by physicists in the 1980's [Pol81a, Pol81b]. It was introduced to extend the notion of Feynman path integrals (closely related to Brownian motion) to Feynman integrals over surfaces. Naïvely, one would like to define a Liouville quantum gravity surface as the Riemannian surface with metric tensor given by

$$\mathrm{e}^{\gamma h(z)}\, dz$$

where $\gamma > 0$ is a real parameter, and h is an instance of the Gaussian free field. This would result in a random surface, parameterised by say, the sphere, in which areas where h is large correspond to those with large area, and areas where h is small correspond to those with small area.

Note the connection with the chaos measures μ^γ described in the previous section. The difference is that here we want to define a random surface (as a metric space, together with a conformal structure) rather than just a volume measure. It turns out that this is a much more difficult problem.

Connection to the Brownian map. As discussed in Section 1.1, another approach to LQG is to consider the scaling limit of certain sequences of *random planar maps.* This yields a random metric space, known as the Brownian map, by the work of

Le Gall [LG13] and Miermont [Mie13]: see Theorem 1.2. This metric space has the topology of the sphere, but comes without any canonical conformal embedding. So, on one hand we have a metric space with no conformal structure, and on the other, we have a conformal structure with no metric. It has long been conjectured that these two objects should be closely related in the case when $\gamma = \sqrt{8/3}$, known as *pure gravity.* Moving forward, however, has proved extremely difficult.

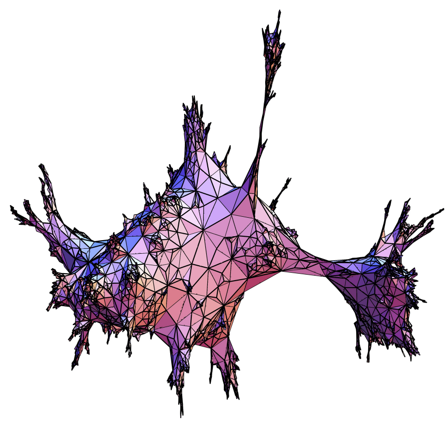


Figure 7: A Random Planar Map (Simulation by Nicolas Curien).

Despite the difficulties presented by this problem, Miller and Sheffield [MS15a, MS15b, MS16e] have recently found a way to connect the two. Using a random growth process called QLE they are able to put a metric on $\sqrt{8/3}$-Liouville quantum gravity which agrees in law with the Brownian map. Conversely, they can show that the Brownian map comes almost surely with a canonical conformal structure, and that the resulting conformal sphere (equipped with a measure) agrees in law with $\sqrt{8/3}$-Liouville quantum gravity. This represents an enormous breakthrough in our understanding of these random surfaces.

1.3.5 Branching Brownian motion and connections with Gaussian multiplicative chaos.

In this section, we will provide a (very) brief introduction to branching Brownian motion. This will not only be the central object of study in Chapter 2, but is also linked to the theory of Gaussian multiplicative chaos. The process in the simplest case (binary branching) is defined as follows:

Definition 1.26 (Branching Brownian Motion)**.** *We start, at time 0, with a single particle at position $x \in \mathbb{R}^d$.*

- *This particle moves according the law of a Brownian motion in $\mathbb{R}^d$.*
- *After an $Exp(\beta)$-distributed waiting time, independent of the motion, the particle splits into two.*
- *Each offspring particle repeats stochastically the behaviour of the initial particle, starting from the point of fission, and independently of one another.*

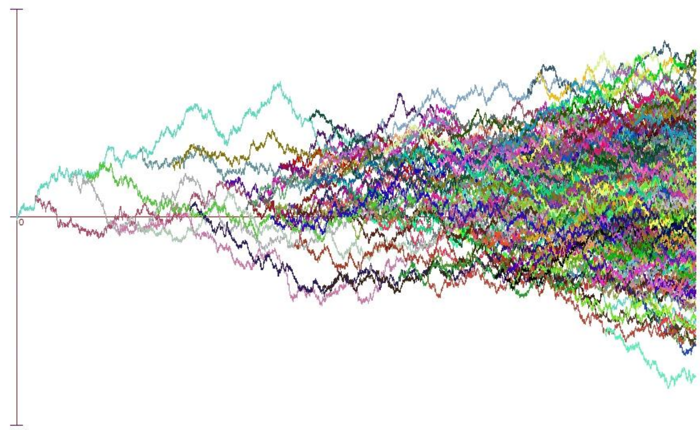


Figure 8: A Branching Brownian Motion (simulation by Matt Roberts).

This definition can be extended, as will be done in Chapter 2, to more general branching mechanisms and individual particle motions. Here we will study the behaviour of the process when we introduce a bounded domain to the model, and kill particles upon hitting the boundary. It turns out that doing this induces a phase transition in the system, and we will study what happens at criticality. However, we will not discuss this further until Section 1.3.8, and leave the precise definitions to Chapter 2.

One crucial tool in Chapter 2 will be to make us of an certain martingale associated with the process.

Lemma 1.27 (The Exponential Martingale)**.** *Let $(X_t^1, \cdots, X_t^{N_t})$ denote the positions of particles in a branching Brownian motion (in the whole of $\mathbb{R}^d$) at time t. Then for $\lambda \in \mathbb{R}$ the process*

$$W_\lambda(t) := \sum_{i=1}^{N_t} \mathrm{e}^{\lambda X_t^i + (\lambda^2/2 - \beta)t}$$

is a martingale.

Now, the discrete time counterpart of BBM, the branching random walk, turns out to be closely related to Gaussian multiplicative chaos. Indeed, the branching

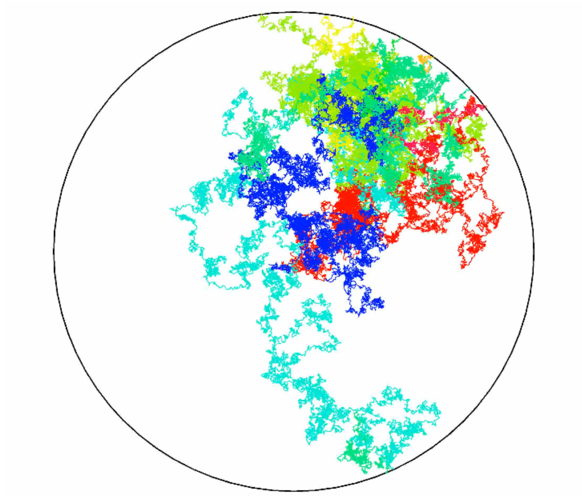


Figure 9: Branching Brownian Motion in a bounded domain (simulation by Henry Jackson).

random walk gives rise to a very similar construction of measures - known as *multiplicative cascades.*

To illustrate this, let us describe multiplicative cascade measures on the interval [0,1]. Denote the dyadic subintervals of $[0,1]$ by $\mathcal{I}_n = \{[k/2^n, (k+1)/2^n] : 0 \le k \le 2^n - 1\}$ and let $\mathcal{I} = \cup_n \mathcal{I}_n$. Take $W > 0$ any random variable with mean one and let $(W_I)_{I \in \mathcal{I}}$ be independent copies of W. Then we can define a sequences of measures on $([0,1], \mathcal{B})$ inductively, by setting μ_0 to be Lebesgue measure and defining

$$\mu_{n+1} = \sum_{I \in \mathcal{I}_n} W_I \cdot \mu_n|_I$$

for each $n \ge 0$. The following result is originally due to Kahane and Peyrière [KP76].

Theorem 1.28. *Define μ_n as above. Then*

- *$\mu_n \to \mu$ a.s. for some limiting measure μ (in the sense of weak convergence of measures.)*
- *If in addition $\mathbb{E}[W \log_2 W] < 1$, then $\mu([0,1]) > 0$ almost surely and μ almost surely has no atoms.*

Note the similarity to Theorem 1.23.

We can alternatively use the random variable W to define a branching random walk in the following way. Begin at time 0 with one particle (at 0) and after the first discrete time step, let this particle branch into two. Let each of the children of the initial particle be displaced from 0 (the position of their parent) by an independent random variable with law $\log(W)$. Repeat the procedure. It is not hard to see that the exponential martingale associated with this process (as in Lemma 1.27, when $\lambda = 1$) has exactly the same law as $\mu_n([0,1])$. Thus, the theory of multiplicative cascades boils down to the theory of exponential martingales for the branching random walk (see for example, [BK77, Kyp00, Lyo97]).

In a similar fashion, many classical techniques from the theory of branching processes can be adapted to prove results about Gaussian multiplicative chaos. This is due to the fact that the circle average process of a GFF is in fact closely linked with branching Brownian motion. We saw in Lemma 3.12 that the circle average process around a single point $z \in D$ is a Brownian motion. However, there is also an underlying branching structure. Roughly speaking, if one takes the circle averages around two different points z and w, they will be the same until the circles become disjoint, and then will move independently. For a rigorous statement along these lines see [DRSV14b, Appendix A].

We will see more connections between branching processes and LQG measures in Chapter 4. In fact, our construction of the Liouville measure using local sets is simply an analogue of the fact that branching random walk martingales also converge along certain random time sets known as *stopping lines* [BK04].

1.3.6 Schramm–Loewner evolutions

The aim of this section will be to give a brief introduction to the theory of Schramm–Loewner evolutions (SLE). These are a family of random fractal curves in the plane, that were introduced by Oded Schramm [Sch00] as the only possible candidates for the scaling limits of certain interfaces in discrete lattice models. Schramm realised that any such scaling limits would have to satisfy two defining properties (conformal invariance and a certain spatial Markov property) and was able to classify the range of possibilities as a one-parameter family.

As an example, let us consider percolation in the triangular lattice. Place a triangular lattice of mesh size δ on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$ and colour the sites of the lattice black or white with probability p, independently of one another. This model displays a phase transition when $p = 1/2$; it is the largest value of p for which all the monochromatic clusters have finite size almost surely. As with many such models, the system displays interesting behaviour at criticality.

One natural thing to consider is the collection of interfaces between monochromatic clusters, which results in a collection of discrete loops. In fact, this is rather complicated to study, and so in order to simplify things we introduce some boundary conditions. We insist that all vertices in the lattice lying the negative real line are coloured black (we assume that the lattice has been placed such that $\pm\delta/2$ are lattice points and 0 is not), and that all lattice points on the positive real line are coloured white. This induces a single interface, from the origin to ∞, that keeps black vertices on its left, and white vertices on its right. It is known as the *exploration process* for percolation.

The problem is to describe what happens to this exploration interface as the mesh size of the lattice goes to 0. It was proved by Smirnov [Smi01] that the interface converges to a continuous curve known as SLE_6. The fact that the scaling limit must be given by this curve is a consequence of properties of the discrete interfaces (conformal invariance, and a "domain Markov" property) that should be preserved in the continuum.

To define the SLE curves, we will need a few tools from complex analysis. For more details we refer the reader to the classical textbook [Law05] or to the lecture notes [BN], where one can also find proofs of all the results.

Complex $\mathbb{H}$-hulls A *complex domain* $D \subset \mathbb{C}$ is a non-empty connected open subset of the complex plane. We say that D is *simply connected* if $\mathbb{C} \setminus D$ is connected in $\mathbb{C} \cup \infty$. As we saw in Section 1.3.1, such domains are connected by a powerful theorem, the Riemann Mapping Theorem. This says that given any two simply connected domains we can choose a conformal isomorphism mapping one to the other.

We say that $K \subset \mathbb{H}$ is a *complex* $\mathbb{H}$*-hull* if K is bounded and $H := \mathbb{H} \setminus K$ is a simply connected domain. This means that K is connected to the real line, and H is a neighbourhood of ∞ in $\mathbb{H}$. For any such hull, we know by the Riemann Mapping Theorem that there must exist a conformal isomorphism $g_K : H \to \mathbb{H}$. Moreover, we can fix this isomorphism by requiring it to have a certain "hydrodynamic normalisation" at ∞. That is, we define g_K to be the unique conformal isomorphism from $H \to \mathbb{H}$, such that $g_K(z) - z \to 0$ as $z \to \infty$. One can prove that, given this choice of g_K, we have the expansion

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2})$$

as $z \to \infty$, where $a_K \geq 0$ is a real constant.

Definition 1.29 (Half-plane capacity)**.** *This constant* a_K *is known as the* half plane capacity *of* K *and denoted by* $hcap(K)$*.*

In some sense, the half-plane capacity measures the size of the hull K, when "viewed from infinity". Indeed, one can prove that

$$\text{hcap}(K) = \lim_{y \to \infty} y \mathbb{E}_{iy}[Im(B_T)]$$

where B is a Brownian motion started at iy and T is the time that it leaves the domain H. In particular, the half plane capacity increases as a hull increases: if $K \subset K'$ are two complex $\mathbb{H}$-hulls, then $\text{hcap}(K) \leq \text{hcap}(K')$.

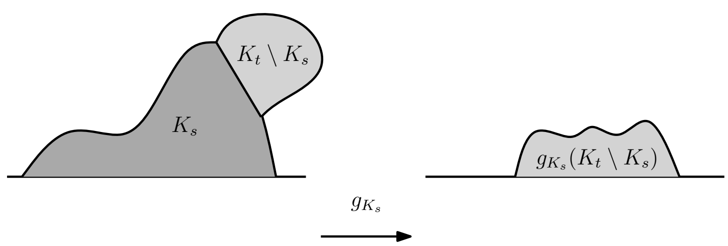
Loewner Chains

Definition 1.30 (Loewner Chain)**.** *A Loewner chain is a family $(K_t)_{t\geq 0}$ of complex $\mathbb{H}$-hulls satisfying the following two properties:*

- K_t *is increasing:* $K_s \leq K_t$ *for* $s \leq t$*, and*
- K_t *satisfies the following* local growth property*: for any* $T \geq 0$

$$\sup_{s,t\in[0,T],|s-t|\leq h} \text{rad}\left(g_{K_s}(K_t \setminus K_s)\right) \to 0 \quad \textit{as} \quad h \to 0$$

where $\text{rad}(K)$ *is the radius of the smallest semicircle that can be placed on the real line to completely encapsulate* K*.*



For any such family one can prove that, for all $t \geq 0$, there exists a unique $\xi_t \in \mathbb{C}$ with $\xi_t \in \overline{g_{K_t}(K_{t+h} \setminus K_t)}$. This process $(\xi_t)_{t\geq 0}$ turns out to be a continuous, real-valued function of time, and is known as the *Loewner transform* of the chain, or the *driving function*.

Another thing to point out is here (which of course requires some complex analysis to prove) is that the map $t \mapsto \text{hcap}(K_t)$ induces a homeomorphism of $[0, \infty)$. Thus, we may assume (by reparameterisation) that

$$\text{hcap}(t) = 2t \quad \forall t \geq 0$$

whenever we are considering a Loewner chain.

The observation of Loewner was that continuous driving functions ξ_t and Loewner chains K_t are in fact in one-to-one correspondence. Moreover, they are connected, together with the sequence of maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$, by a certain differential equation.

Theorem 1.31 (Loewner's Theorem)**.** *Let $(\xi_t)_{t\geq 0}$ be a continuous real valued function and for each $z \in \mathbb{H}$ let $g_t(z)$ be the maximal solution to the* Loewner equation

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z$$

which exists on some time interval $[0, \zeta(z)]$ by classical ODE theory. Let $K_t = \{z \in \mathbb{H} : \zeta(z) \leq t\}$.

Then $(K_t)_{t\geq 0}$ is a Loewner chain, parameterised so that $\mathrm{hcap}(K_t) = 2t$, and with driving function ξ_t. Moreover, g_t is the unique sequence of maps $\mathbb{H} \setminus K_t \to \mathbb{H}$ with the hydrodynamic normalisation at ∞.

Remark 1.32.

- *The time, $\zeta(z)$, when the solution to the Loewner equation for $g_t(z)$ fails to exist, is the time that the growing hull K_t "swallows" the point z.*
- *One can prove that continuous curves $(\gamma_t)_{t\geq 0}$ in $\mathbb{H}$ which do not cross themselves and have $|\gamma_t| \to \infty$ as $t \to \infty$ are examples of Loewner chains. In this case the map g_t sends the tip of the curve, γ_t, to the point ξ_t (where g_t is extended by continuity.)*

Chordal SLE We are now ready to define Schramm–Loewner evolutions.

Definition 1.33 (Chordal SLE in $\mathbb{H}$ from $0 \to \infty$)**.** *For $\kappa > 0$, SLE_κ in $\mathbb{H}$ from 0 to ∞ is defined to be the Loewner chain driven by $\xi_t = \sqrt{\kappa} B_t$ where B_t is a standard Brownian motion.*

Not all Loewner chains are generated by curves in the sense of Remark 1.32. Hence the obvious question: is SLE? This was answered by Rohde and Schramm for $\kappa \neq 8$ in their seminal work [RS05] on SLE. It was later completed for the case $\kappa = 8$ by Lawler, Schramm and Werner, [LSW04].

Theorem 1.34 (Rohde–Schramm, Lawler–Schramm–Werner)**.** *For every $\kappa > 0$, SLE_κ is generated by a a random continuous curve γ.*

Thus, SLE are random curves *driven* by Brownian motion. Intuitively, one can think of SLE as random curves travelling from 0 to ∞ which change direction whenever an underlying Brownian motion increases or decreases.

One of the first things to note about SLE is that, due to the scaling property of Brownian motion (B_t has the same law as $\sqrt{t} B_1$ for any t), SLE is itself scale invariant. That is, for any $r \geq 0$ if $(\gamma_t)_{t\geq 0}$ is an SLE_κ process, then the rescaled process $(r^{-1/2}\gamma_{rt})_{t\geq 0}$ also has the law of an SLE_κ.

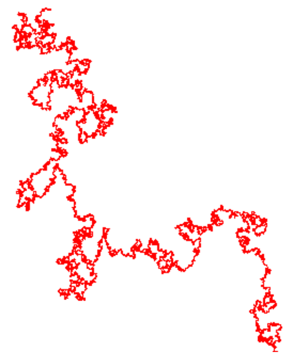


Figure 10: SLE_4 (Simulation by Tom Kennedy).

This says that SLE is invariant under conformal maps of $\mathbb{H}$ that fix 0 and ∞ (indeed scaling maps represent all such maps.) This allows us to define SLE, by conformal invariance, in any simply connected domain and between any two marked boundary points.

Definition 1.35 (Chordal SLE). *SLE_κ is a collection $(\mu_{D,a,b})$ of laws on continuous curves, indexed by triples (D, a, b) where D is a simply connected domain and a and b are two marked boundary points. The law $\mu_{\mathbb{H},0,\infty}$ is that given by Definition 1.33. For any other triple (D, a, b), $\mu_{D,a,b}$ is defined to be the image of $\mu_{\mathbb{H},0,\infty}$ under the (unique) conformal isomorphism sending $\mathbb{H}$ to D, 0 to a and ∞ to b.*

Chordal SLE defined in this way is therefore a conformally invariant process, in the following sense. Take an SLE_κ curve γ, in a domain D from a to b, and apply a conformal map $\varphi : D \to D'$ that sends a to a' and b to b'. Then $\varphi(\gamma)$ has the law of an SLE_κ in D' from a' to b'.

The other characteristic property of SLE is the following:

Definition 1.36 (Domain Markov Property). *Suppose that γ is an* SLE_κ *in D from a to b (that is, γ has the law $\mu_{D,a,b}$) and that τ is a stopping time for γ. Then, conditionally, on $\gamma[0, \tau]$, the remainder of the curve $\gamma[\tau, \infty)$ has law given by $\mu_{D\setminus\gamma[0,\tau],\gamma(\tau),b}$. This is the* domain Markov property *of $(\mu_{D,a,b})$.*

In fact, it was proved by Schramm [Sch00] that these two properties completely characterise SLE. This is what makes them the only possible candidates for certain scaling limits of discrete models.

Proposition 1.37 (Characterisation of SLE, [Sch00]). *Suppose that $(\mu_{D,a,b})$ is a family of laws on continuous curves (indexed by triples as in Definition 1.35) that is conformally invariant and satisfies the domain Markov property. Then there exists $\kappa > 0$ such that $(\mu_{D,a,b})$ is chordal* SLE_κ.

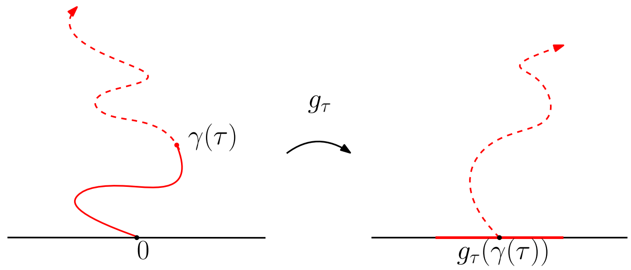


Figure 11: Domain Markov Property of SLE.

The behaviour of SLE_κ paths, of course, depends on the value of κ. Indeed, the paths should arise from physical models lying in different, so-called, universality classes. It turns out that the behaviour can be classified in to three distinct phases, [RS05].

Theorem 1.38 (Rohde–Schramm)**.** *For different values of κ the behaviour of* SLE_κ *can be described as follows:*

- *For $\kappa \leq 4$, γ is almost surely a simple curve, and $\gamma_t \in \mathbb{H}$ for all $t > 0$ (i.e. it does not hit the real line);*
- *For $\kappa \in (4, 8)$ γ has double points, and $\gamma \cap \mathbb{R} \neq \emptyset$;*
- *For $\kappa \geq 8$, γ is almost surely a space filling curve. That is, $\gamma[0, \infty) = \overline{\mathbb{H}}$.*

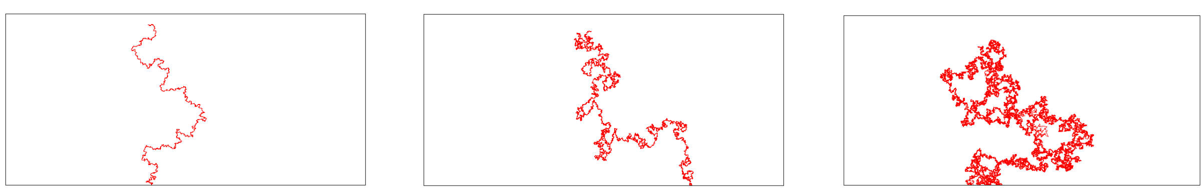


Figure 12: $\mathrm{SLE}_2, \mathrm{SLE}_4$ and SLE_6 from left to right (Simulations by Tom Kennedy).

Thus, $\kappa = 4$ and $\kappa = 4$ can be thought of as critical values for SLE. In Chapter 5 we will focus on a special connection with the Gaussian free field in the $\kappa = 4$ case.

One final thing to note about SLE, as is perhaps already visible from the pictures, is that the curves generating them have Hausdorff dimension strictly bigger than one. That is, they are fractal. The dimension of the SLE_κ curves was computed by Beffara [Bef08] and shown to be equal to $\min(2, 1 + \frac{\kappa}{8})$.

Scaling Limits As we have already mentioned several times, the defining properties of SLE mean that they are widely believed to be the scaling limits of a whole class of interfaces in discrete lattice models. The continuous parameter κ should correspond to what physicists call the "universality class" of the discrete model, and is related to the "central charge" c of conformal field theory via the relationship

$$c = \frac{(8-3\kappa)(\kappa-6)}{2\kappa}.$$

However, proving that these scaling limits exist is often extremely difficult, and results that we have obtained so far have been much celebrated:

- SLE_2 was proved by Lawler, Schramm and Werner [LSW04] to be the scaling limit of the loop erased random walk (or equivalently, by Wilson's algorithm, the branches of the uniform spanning tree);
- SLE_3 has been proved to be the limit of the interface between positive and negative spins in the critical Ising model [CS12, CDCH$^+$14];
- SLE_4 corresponds to the "level lines" of the Gaussian free field [SS09, SS13];
- $\mathrm{SLE}_{16/3}$ has been proven to arise from the exploration interface in the critical FK model [CS12, CDCH$^+$14];
- SLE_6 is the limit of the exploration path in critical percolation [CN07, Smi01]; and finally
- SLE_8 can be shown to be the scaling limit of the path separating the uniform spanning tree from its dual, [LSW04].

But there is much left to do. For example, when $\kappa = 8/3$, SLE_κ satisfies a special *restriction* property ([LSW03]) which leads us to believe that it should the scaling limit of a model known as the self-avoiding random walk. This is one of many open problems in this area.

$\mathrm{SLE}_\kappa(\rho)$ **processes.** $\mathrm{SLE}_\kappa(\rho)$ processes are a variant of SLE_κ where the curves have an additional attraction, or repulsion, from certain "force points" in the domain or on its boundary. The vector ρ tells us how strong this attraction or repulsion is, and in which direction. This is rigorously described by the addition of an extra drift term in the differential equation defining their driving functions (they are still Loewner chains, but now the driving function is the solution of an SDE related to Brownian motion rather than Brownian motion itself.)

However, the new differential equations are somewhat more tricky to deal with. It has only been shown recently, through a remarkable connection with the Gaussian free field [MS16a, WW16], that the associated Loewner chains are actually generated by continuous curves. In [PW17], which will make up Chapter 5, we generalise the notion of $\mathrm{SLE}_4(\rho)$ to a continuum of force points and show that this property still holds.

Conformal Loop Ensembles (CLE) Recall that in our earlier discussion of critical percolation, the most natural interfaces to look at were those separating *all* the monochromatic clusters in the model. Instead of a single interface from 0 to ∞, this produces a collection of loops in the upper half plane. It is believed that these should converge, as the mesh size of the lattice goes to 0, to CLE.

CLE_κ are collections of loops in complex domains (indexed by a real parameter κ), that again satisfy conformal invariance and a specific form of spatial Markov property. They were introduced by Sheffield and Werner [SW12], and can be defined either using something known as a *Brownian loop-soup*, or by using a branching variant of $\text{SLE}_\kappa(\rho)$ [She09]. This is why, locally, CLE_κ loops look like SLE_κ curves (a fact that should not be surprising given the interpretation in terms of interfaces.)

Again, it is known that conformal loop ensembles arise as scaling limits in a number of cases: $\kappa = 3$ corresponds to interfaces in the critical Ising model [CS12, CDCH$^+$14]; $\kappa = 4$ to contour lines of the Gaussian free field [MS11, ASW15]; $\kappa = 16/3$ to cluster interfaces in critical FK percolation [CS12, CDCH$^+$14]; and $\kappa = 6$ to cluster interfaces for critical percolation on the triangular lattice [CN07, Smi01]. Once more, however, the theory is far from complete.

1.3.7 Connection between SLE and the GFF

Many intriguing connections between Schramm–Loewner evolutions and the Gaussian free field have been uncovered in recent years. Although both are canonical, conformally invariant objects in the plane, the link between them is subtle, and not at all obvious.

The main connection we will focus on in this thesis, see Chapter 5 in particular, is the interpretation of SLE curves as "level lines" or "flow lines" of the GFF [Dub09, MS16a, SS09, SS13, WW16]. This was first studied by Schramm and Sheffield [SS09], who proved the following. Take a discrete GFF on a lattice approximation to a domain $D \subset \mathbb{C}$ (with certain specific boundary conditions) and extend it to a continuous function on D. Then the zero level-line of this field (that is, the line along which the field is equal to 0) converges in distribution, as the mesh size of the lattice goes to 0, to an SLE_4. Hence SLE_4 is the scaling limit of level lines of the discrete GFF.

The question follows: is it possible to make sense of SLE_4 as a level line of the continuum Gaussian free field? This was again answered by Schramm and Sheffield [SS13]. The construction they came up with is this:

Theorem 1.39 (SLE_4-GFF coupling)**.** *SLE_4 and the GFF can be coupled as follows.*

- *Take an* SLE_4 γ *in the upper half plane from* 0 *to* ∞*.*

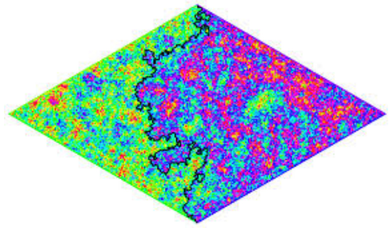


Figure 13: A level line of the discrete GFF (from [SS09]). Different colours represent varying height of the field.

- *On either side of the curve γ sample an independent GFF. On the left with boundary conditions $-\lambda$ (recall, this means that we add on the constant function $-\lambda$ to the field on the left) and on the right with boundary conditions $+\lambda$. Here, $\lambda = \frac{\pi}{2}$.* [4]
- *The law of the resulting field (we can interpret it as a field on the whole of $\mathbb{H}$, since the* SLE *curve has Lebesgue measure 0) is just another GFF. Now it has special boundary conditions given by the function $F_0 = -\lambda \mathbb{1}_{(-\infty,0)} + \lambda \mathbb{1}_{[0,\infty)}$ on the real line (recall, this means that it has the law of a zero-boundary GFF plus the harmonic extension of F_0.)*

The proof of this is by a relatively simple martingale argument, that we will sketch below.

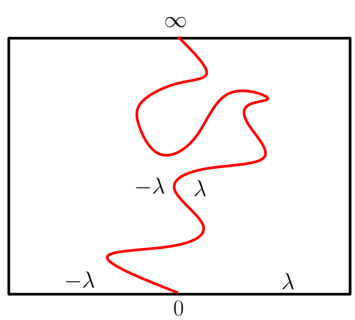


Figure 14: Construction of the coupling.

Proof of Theorem 1.39. Let $(\gamma_t)_{t\geq 0}$ be a SLE_4 curve from 0 to ∞ in $\mathbb{H}$, parameterised by half-plane capacity. By Theorem 1.38, we know that this is a continuous curve which does not touch the boundary of $\mathbb{H}$. Let η_t be the harmonic extension in $\mathbb{H} \setminus \gamma[0,t]$ of the function on the boundary given by

- $-\lambda$ on the left-hand side of $\gamma[0,t]$ and $\mathbb{R}_-$, and

[4]However, in general its specific value depends on the choice of normalisation for the Green function, and so it is usually better to leave it vague.

- $+\lambda$ on the right-hand side of $\gamma[0,t]$ and $\mathbb{R}_+$,

as in Figure 15. Recall that g_t, the unique conformal map $\mathbb{H} \setminus \gamma[0,t] \to \mathbb{H}$ with

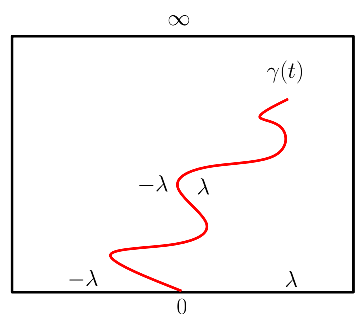


Figure 15: The boundary values of the harmonic function η_t in $\mathbb{H} \setminus \gamma[0,t]$.

hydrodynamic normalisation at ∞ satisfies, for any fixed $z \in \mathbb{H}$,

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - W_t} \quad \forall t \text{ a.s.} \tag{1.1}$$

where $W_t = 2B_t$ and B_t is a standard Brownian motion. We also set $f_t(z) := g_t(z) - W_t$ so that $f_t(\gamma(t)) = 0$.

Then we can calculate explicitly that

$$\eta_t(z) = -\arg f_t(z) + \pi/2$$

for any $z \in \mathbb{H}$. A simple application of Itô's formula, using (1.1), then gives us that $\eta_t(z)$ is a continuous local martingale. In fact, by boundedness, it is a true martingale and so has an almost sure limit $\eta_\infty(z)$ as $t \to \infty$. We also define

$$G_t(z,w) = G_{\mathbb{H}}(f_t(z), f_t(w)) \quad \text{and} \quad E_t(p) = \iint_{\mathbb{H}^2} p(z)G_t(z,w)p(w)dzdw \; ; \;\; p \in C_c^\infty(\mathbb{H}).$$

Note that $G_t(z,w)$ and $E_t(p)$ are non-decreasing in t for any $z, w \in \mathbb{H}$ and positive test function p. This means that the limits $G_\infty(z,w)$ and $E_\infty(p)$ also exist in this case.

Recall, we would like to prove that if we take a field $\tilde{h}$ which is given by the function η_∞ plus an independent Gaussian free field on either side of $\gamma([0,\infty))$, then the law of $\tilde{h}$ is that of a zero-boundary Gaussian free field plus the harmonic function η_0. To this end, it is enough to show that for any positive test function p in $C_c^\infty(\mathbb{H})$, the random variable $(\tilde{h}, p)$ is Gaussian with mean (η_0, p) and variance $E_0(p)$.

Fix such a p. We will show that for each $z \in \mathbb{H}$, (η_t, p) is a continuous martingale with quadratic variation $-E_t(p)$. This will allow us to conclude because then we have, for any $\mu > 0$

$$\begin{aligned}\mathbb{E}[\exp(-\mu(\tilde{h},p))] &= \mathbb{E}[\mathbb{E}[\exp(-\mu(\tilde{h},p))] \mid \gamma([0,\infty)]\\ &= \mathbb{E}[\exp(-\mu(\eta_\infty,p) + \frac{\mu^2}{2}E_\infty(p))]\\ &= \mathbb{E}[\exp(-\mu(\eta_0,p) + \frac{\mu^2}{2}E_0(p))]\end{aligned}$$

by the optional stopping theorem.

To verify the claim, observe that since η_t is a continuous local martingale, so is (η_t, p) by Fubini's theorem. Thus we only need to show that $(\eta_t,p)^2 + E_t(p)$ is a continuous local martingale. However, this can be seen by noting that it is a double integral of the function $\eta_t(z)\eta_t(w) + G_t(z,w)$ against $p(z)p(w)$ on $\mathbb{H}$, and again applying Fubini. The latter process is easily verified to be a martingale for each $z, w \in \mathbb{H}$, using Itô's formula and the explicit expression for $G_{\mathbb{H}}$. □

So, this provides a coupling between SLE_4 and the continuum Gaussian free field in which the curve can in some sense be interpreted as a level line of the field. In fact, we can see that it is actually more like a "level cliff"; there is a constant height gap of 2λ between either side of the curve. This, in essence, is a consequence of the roughness of the field.

One important and highly non-trivial property of this coupling, is that the Gaussian free field we get in the end actually determines the SLE_4 curve uniquely. Of course we would expect this from a continuous field and its level line, but in the case of the GFF it is not so clear. Despite the deep nature of this result, the proof in [SS13] again follows a simple and elegant argument.

So now, what about different boundary conditions? The choice of F_0 is clearly very specific. In order to make sense of this question we need a more general definition (cf. Definition 5.1.)

Definition 1.40 ([MS16a, WW16])**.** *Suppose that F is L^1 with respect to harmonic measure on $\mathbb{R}$ viewed from some point in $\mathbb{H}$ and that h is a zero boundary* GFF *in $\mathbb{H}$. If $(K_t, t \geq 0)$ is a Loewner chain and $(g_t, t \geq 0)$ is the corresponding sequence of conformal maps, set $f_t = g_t - W_t$, and let $V_t^R(x)$ (resp. $V_t^L(x)$) be the image of $x \geq 0$ (resp. $x \leq 0$) under g_t. Let η_t^0 be the bounded harmonic function on $\mathbb{H}$ with* *boundary values (see Figure 16)*

$$\begin{cases} F(f_t^{-1}(x)), & \text{if } x \geq V_t^R(0^+) - W_t, \\ \lambda, & \text{if } 0 \leq x < V_t^R(0^+) - W_t, \\ -\lambda, & \text{if } V_t^L(0^-) - W_t \leq x < 0, \\ F(f_t^{-1}(x)), & \text{if } x < V_t^L(0^-) - W_t, \end{cases}$$

and define, for $z \in \mathbb{H} \setminus K_t$,

$$\eta_t(z) = \eta_t^0(f_t(z)).$$

We say that K *is a level line of* $h + F$ *if there exists a coupling* (h, K) *such that the following domain Markov property holds: for any finite* K*-stopping time* τ*, given* K_τ*, the conditional law of* $(h+F)|_{\mathbb{H}\setminus K_\tau}$ *is equal to the law of* $h \circ f_\tau + \eta_\tau$*.*

More plainly, suppose we have a curve γ that is coupled with a Gaussian free field with boundary data F. Then we say that γ is a level line of the field if, for any stopping time τ, the curve $\gamma[0, \tau]$ is a local set for the field, and the associated harmonic function has boundary values as shown in the lefthand side of Figure 16.

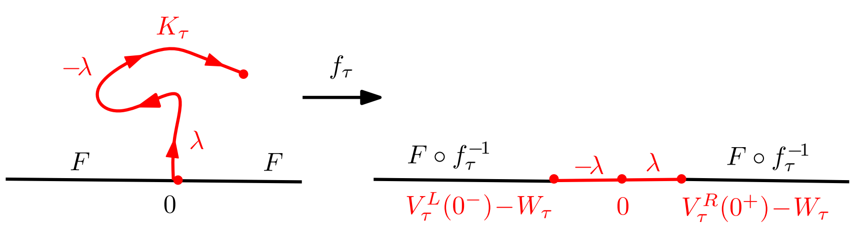


Figure 16: The lefthand side shows the boundary values of the harmonic function η_τ in $\mathbb{H} \setminus K_\tau$. This is the image under f_τ^{-1} of the harmonic function η_τ^0 in $\mathbb{H}$, whose boundary values are shown on the right hand side.

This definition is satisfied by the coupling of Schramm and Sheffield described above (by an easy adaptation of the proof given here). Moreover, the argument can be extended to show that level lines do in fact exist for a very wide class of boundary data F. Wu and Wang [WW16] proved the existence for piecewise constant functions F, and later Wu and Powell ([PW17], see Chapter 5) extended this to any function that is *regulated* (has left and right limits at every point). These couplings again satisfy the desirable property that the curve is determined by the field, and the curves themselves are given by $\text{SLE}_4(\rho)$ processes. See Chapter 5 for the specific relationship between ρ and the boundary data F.

Another point of interest is that this relationship provides a way to couple together many different $\text{SLE}_4(\rho)$ processes. That is, as level lines of a single Gaussian free field, to which lots of different boundary data is added. This results in a nice interaction between the curves; they are in fact monotonic in the corresponding boundary data [WW16, PW17]. Note that this makes sense in terms of their interpretation as level lines.

A final consequence we will mention, is that these couplings allow previously less-tractable properties of the SLE curves to be attacked from the point of view of the GFF. For example, it is not immediately obvious from their definitions in terms of Loewner chains that $\mathrm{SLE}_4(\rho)$ processes are actually generated by curves at all (although it was proved for SLE_4 in [RS05]). In general, this result comes as a consequence of their coupling with the GFF [MS16a, WW16, PW17].

Flow line coupling. When $\kappa \neq 4$ the coupling between SLE_κ and the GFF becomes even more exotic. This is the subject of the Imaginary Geometry series by Jason Miller and Scott Sheffield [MS16a, MS16b, MS16c, MS16d]. It turns out that SLE_κ curves with $\kappa \neq 4$ can be interpreted as so-called "flow lines" of the Gaussian free field. Heuristically, let h be a GFF and consider the "vector field" described by $\mathrm{e}^{ih/\chi}$ for $\chi > 0$ (although of course this is not really well defined). Then one can try to trace the flow lines of this field, that is, curves η that are solutions of the differential equation

$$\eta'(t) = \mathrm{e}^{ih(\eta(t))/\chi}\,.$$

The heuristic is that what one should get is an SLE_κ ($\chi = 2/\sqrt{\kappa} - \sqrt{\kappa}/2$).

Rigorously, one can define flow lines as curves coupled with the GFF in a similar way to Definition 1.40, but with an additional "winding" term in the definition of the harmonic function. Again, different boundary conditions of the field correspond to $\mathrm{SLE}_\kappa(\rho)$ for different values of ρ.

Coupling with CLE. We should also mention here the coupling between the Gaussian free field and CLE_4. This somehow represents a fuller description of the GFF's level lines; indeed, CLE_4 arises when one considers the scaling limit of the entire collection of contour lines of a discrete, zero-boundary condition GFF. Of course this results in a collection of loops rather than a single curve. In the continuum, the coupling can be described as follows:

- Sample a CLE_4 in a domain D.
- Define a function h_1, by taking its value inside each CLE_4 loop to be given by an independent random variable, taking values $\pm 2\lambda$ with equal probability.
- Sample independent CLE_4's inside all of these loops and repeat the procedure above in the second generation loops to obtain a function h_2'.
- Let $h_2 = h_1 + h_2'$ and iterate.

- It is known [MS11] that the limit of h_n as $n \to \infty$ is a zero-boundary Gaussian free field.

This coupling was first discovered by Miller and Sheffield [MS11], but see also [ASW15] for a proof.

Note that in all these couplings, the loops or curves define non-trivial examples of local sets for the corresponding GFF. In [ASW15] the authors construct even more general local set couplings, in which the harmonic functions can take 2 arbitrary constant values (so the coupling above with first generation CLE_4 loops corresponds to $\pm 2\lambda$.) These can be thought of as being analogous to uniformly integrable stopping times for Brownian motion.

These local sets, and the coupling between the GFF and CLE_4 will also play an important role in Chapter 4. Here we will construct the Liouville measure as a *multiplicative cascade measure* using CLE_4, in the natural way suggested by the above construction of the GFF.

Quantum Zipper. To conclude this section, we briefly mention another coupling between SLE and the GFF, due to Scott Sheffield [She16]. Roughly speaking, one takes two special Liouville quantum gravity surfaces (known as quantum wedges) and *conformally welds* them to obtain a new surface equipped with a curve. It turns out that what results is simply another Liouville quantum gravity surface, equipped with an SLE_κ ($\kappa = \gamma^2$.) This coupling can be thought of as the coupling emerging from the *real* counterpart of the martingale that produces the imaginary geometry coupling described above. For more details, see [She16] or [Ber15b].

1.3.8 Branching Processes and the Brownian CRT

This brings us to our final object of consideration. To begin, suppose we have a finite tree, that is, a finite connected graph with no cycles. Then we can view this as a metric space in a natural way, by giving each branch of the tree length one, and letting the distance between any two points on the tree be given by the length of the unique path that joins them without backtracking. Generalising this *unique path* property gives us the notion of a continuum, or real, tree.

Definition 1.41 (Real Tree)**.** *A metric space $(\mathcal{T}, d)$ is said to be a real tree if, for all $v, w \in \mathcal{T}$ the following two conditions hold:*

- *There exists a unique isometric map $\phi_{v,w} : [0, d(v,w)] \to \mathcal{T}$ with $\phi_{v,w}(0) = v$ and $\phi_{v,w}(d(v,w)) = w$.*
- *Any continuous injective map $[0,1] \to \mathcal{T}$ that joins v and w has the same image as $\phi_{v,w}$*

One way to define is a real tree is the following: take a continuous function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and define a "distance" function on $[0, \infty) \times [0, \infty)$ by

$$d_f(s,t) = f(s) + f(t) - 2 \min_{r \in [s,t]} f(r)$$

whenever $s \le t$. It is easy to verify that this defines a pseudometric on $[0, \infty)$. Thus, quotienting by the equivalence relation $\sim$ that identifies points with $d_f(\cdot, \cdot) = 0$ we obtain a metric space $(\mathcal{T}_f, d_f) := ([0, \infty)/\sim, d_f)$. One can prove, see for example [LG06], that this metric space is a real tree.

To connect this with our initial intuition, consider a finite tree, and turn it into a continuum object by joining the vertices with branches of length one. The metric described at the beginning of this section then produces a real tree. Actually, it is a real tree of the form $(\mathcal{T}_f, d_f)$, where the function f is given by the *contour function* C_t of the tree (see Figure 17) below.

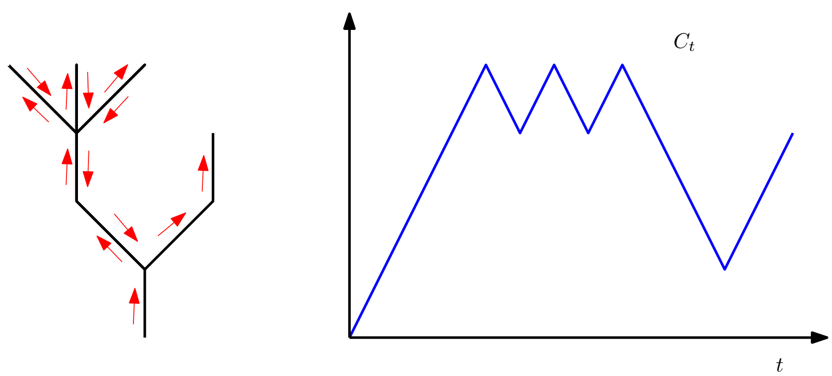


Figure 17: C_t (on the right) is the contour function of the tree on the left-hand side. This is the function that we get if we traverse the tree at speed one in a *depth-first order* with backtracking (see figure), and measure how our height is changing with time.

The Brownian continuum random tree of David Aldous [Ald91], is an example of a particularly natural probability law on real trees.

Definition 1.42 (Brownian CRT)**.** *The Brownian CRT is the metric space*

$$(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$$

where $\mathbf{e}$ *is a normalised Brownian excursion.*

By a normalised Brownian excursion we mean, intuitively, a Brownian path that is conditioned to begin at 0, stay positive on the interval $[0, 1]$ and end at 0 at time 1 (although this takes a bit of work to define rigorously). We could also consider a Brownian excursion that is conditioned to reach a fixed positive height, rather than

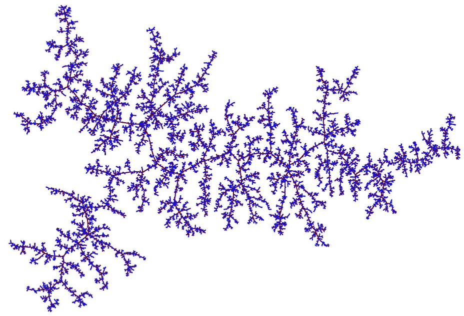


Figure 18: A Brownian CRT (Simulation by Igor Kortchemski).

to last a fixed length of time. For more rigorous definitions of these objects see, for example, [IM65].

This real tree turns out to be universal in that it arises from many different constructions. We will be mainly concerned with the view of it as a limit of conditioned critical Galton Watson trees (to be discussed shortly). However, it can also be defined as a certain law on *spanning subtrees*, or using a so-called *line breaking construction*, see [Ald91].

Let us now turn to Galton Watson trees. Let L be a random variable, with mean m and variance σ^2; this will be our *offspring distribution*. The GW tree associated with L is the tree generated by the random population growth model defined as follows:

- Initially, at time 0, there is one particle.
- At time 1 this particle dies, and gives birth to a random number of offspring, with distribution L.
- Each offspring particle then stochastically repeats the behaviour of its parent, completely independently of everything else.

It is a classical result of Kolmogorov [Kol38] that there is a phase transition in this model when the mean m of the offspring distribution is equal to 1. For $m \leq 1$ the population becomes extinct almost surely, and for $m > 1$ it survives with positive probability. We will focus here on the critical case $m = 1$.

We consider the natural tree associated with this process. It turns out [Ald93, LGD02], that if this tree is conditioned to reach a large height n (which is the same as conditioning the process to survive to generation n) then it will converge after rescaling to the Brownian CRT (in this case we mean the tree generated by a Brownian excursion conditioned to reach a fixed height). Similar statements also exist in terms of conditioning the process to have a large total progeny.

The Brownian CRT is in fact the scaling limit of a much wider class of models. For example, one generalisation of the Galton–Watson process is to take a multitype version, where the particles also have types that influence their offspring distribution. In this case, there is again a notion of criticality, and it was proved by Miermont [Mie08] that the associated critical conditioned trees converge to the CRT. In Chapter 2 we will extend this to a result concerning genealogical trees of critical branching diffusions. Take branching Brownian motion say, (see Definition 1.26), in a bounded domain. There is again a phase transition for extinction here, and criticality corresponds to a certain value of the branching rate. In [Pow17b] we prove that the genealogical tree of such a critical process, when it is conditioned to be large, also converges after rescaling to the Brownian CRT. See Chapter 2 for a proof of this fact.

We will conclude this preliminary section in what seems like a suitable manner: by mentioning one last connection. This time, between the CRT and Liouville quantum gravity. Recall the discrete approach to LQG, where one considers the scaling limits of certain random planar maps ([LG13, Mie13]). One of the key steps in the proof that this scaling limit exists is to encode the maps using a correspondence known as the Cori–Vauquelin–Schaeffer bijection. This is a correspondence between random planar maps and *labelled trees*. It turns out, in a similar vein to the above discussion, that the labelled trees corresponding to a suitable sequence of random planar maps actually converge to a Brownian CRT, together with a labelling function. The interpretation of this limit in terms of the original random planar maps is what is used to prove the existence of the limiting Brownian map.

In the continuum, the connection between LQG and the Brownian CRT was established in the paper [DMS12] of Duplantier, Miller and Sheffield. Here they find a way to "mate" a pair of correlated CRTs, that in the end produces a continuum LQG surface.

1.4 Questions

There were of course many more interesting questions that arose during this PhD, that we have so far not been able to address. Here we present a selection, that we hope will be answered at some point in the future:

- We know from Chapter 2 that the genealogical structure of critical branching diffusions in *nice, bounded domains* converge to the Brownian CRT. However, if one removes the assumption of boundedness, or regularity, then more exotic things may occur. For example, in the half line with absorption at the origin, we know by [BBS14] that at least the survival probability decays completely differently (although this is also for a slight variant of the model, where particles are given a drift towards 0.).

So, there may be *some* rough or unbounded domains where the critical genealogy *is* given by a Brownian CRT, but we expect that there should also be some where it is not (although explicit examples are not known to my knowledge). It is therefore an interesting and seemingly open problem to try and classify *all* possible scaling limits, depending on the geometry of the domain.

- Another natural extension would be to consider what happens if you allow the offspring distribution to be more general. For example, if you remove the assumption of finite variance, or allow it to depend on the spatial motion, will the behaviour remain the same?
- Some of the motivation for Chapter 4 comes from the following question: how does the Liouville measure, μ^γ, of a fixed Gaussian free field vary with γ? Current work with Juhan Aru, Avelio Sepúlveda and Xin Sun will prove that it varies continuously in the space of measures, and converges to 0 as $\gamma \to 2$. The approach will make use of the new construction of μ^γ from Chapter 4. We also hope to use this to prove that the renormalised limit $\mu^\gamma/(2-\gamma)$ converges to the critical derivative measure as $\gamma \uparrow 2$.
- On that note, can we say what happens to the Liouville measure (for some fixed γ) if we put a dynamic on the underlying GFF? For example, take the *Ornstein–Uhlenbeck dynamic* (defined by putting an independent OU process on each of the coefficients α_i from Lemma 1.17). In the subcritical case, the dynamical Liouville measure should vary continuously. But what happens at criticality?
- This leads to an analogous question regarding the level, or flow, lines of the Gaussian free field. We know from [SS13, MS16a, WW16, PW17] that the GFF determines it level lines. Furthermore, we will see in Chapter 5 that the level lines are in some sense continuous in the boundary data. But are they continuous in general (i.e. with respect to some topology on the space of fields)? What happens to the level lines or flow lines when we put an Ornstein–Uhlenbeck dynamic on the field? We believe that, at least for level lines, there should be *exceptional times* for the process. For example, there should be times when the SLE_4 curves hit the boundary of the domain. This, however, is proving to be a challenging problem, as the relationship between the GFF and its level lines remains rather mysterious.
- We can also ask what happens if we change the parameter κ in the coupling of

SLE_κ and the GFF. In this case we may also see special values of κ at which the associated curve *jumps*. As above, this seems to be a little tricky to make rigorous.

- Finally, the theory of level lines of the GFF with general boundary data, Chapter 5, is a little way from being complete. We prove the existence of level lines in the case when a certain inequality is satisfied, that corresponds to the non-existence of a *continuation threshold* for the associated $\mathrm{SLE}_4(\rho)$ process. However, we should be able to construct level lines without this assumption, at least up to the continuation threshold (i.e. until the level line hits a "bad" part of the boundary). Construction up to this time is done in [MS16a, WW16] for the case of piecewise constant boundary data, and also beyond this time in [MSW16] for the case of one force point. Extending our results to these types of situation is a work in progress with Hao Wu.

2 Invariance principles for branching diffusions in bounded domains

2.1 Introduction

This article concerns branching diffusions in a bounded domain D of $\mathbb{R}^d$. These are processes in which individual particles move according to the law of some diffusion, are killed upon exiting the domain, and branch into a random number of particles (with distribution A, independent of position) at rate $\beta > 0$. Whenever such a branching event occurs, each of the offspring then independently repeats the behaviour of its parent, starting from the point of fission. Throughout, the configuration of particles will be denoted by

$$(X_t^1, \cdots, X_t^{N_t})$$

where N_t is the number of particles alive at time t, and we will write $\mathbb{P}_x$ for the law of the process initiated from a point $x \in D$. We will always assume that the offspring distribution has mean $m > 1$ and finite variance, and that the generator $L = -\frac{1}{2}\sum_{i,j} a^{ij}\,\partial x_i\,\partial_{x_j} + \sum_i b^i\,\partial_{x_i}$ of the diffusion is uniformly elliptic and self-adjoint with smooth coefficients.

It is known [Sev58, Wat65] that such a system exhibits a phase transition in the branching rate: for large enough β there is a positive probability of survival, but for small β, including at criticality, there is almost sure extinction. The critical value of β is equal to $\frac{\lambda}{m-1}$, where λ is the first eigenvalue of L on D with Dirichlet boundary conditions. The main goal of this paper will be to study the system at criticality and find a scaling limit for the resulting genealogical tree. This is the continuous planar tree that is generated purely by the birth and death times of particles in the system, and encodes no information about the spatial movement. More precisely, for given $y > 0$, we condition the diffusion to survive until time $ny > 0$ and look at the associated genealogical tree $\mathcal{T}_n^y$, equipped with its natural distance d_n^y. Rescaling the distances by a factor n produces a sequence of random compact metric spaces $(\mathcal{T}_n^y, \frac{1}{n}d_n^y)_{n\in\mathbb{N}}$. We will prove that this sequence converges in distribution to a conditioned Brownian Continuum Random Tree as $n \to \infty$, with respect to the Gromov-Hausdorff topology. Indeed, if we write $(\mathcal{T}_{e^y}, d_{e^y})$ for the real tree whose contour function is given by e^y, a Brownian excursion conditioned to reach level y, then we obtain the following result.

Theorem 2.1. *Suppose that $D \subset \mathbb{R}^d$ is a bounded C^1 domain and that L is uniformly elliptic and self-adjoint with smooth coefficients. Further suppose that A has* *mean $m > 1$ and finite variance, and $\varphi \in C^1(\overline{D})$ where φ is the first eigenfunction of L on D. Then for any $y > 0$, and any starting point $x \in D$,*

$$(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{e^y}, d_{e^y})$$

in distribution, with respect to the Gromov-Hausdorff distance, where

$$\alpha = \sqrt{\frac{4(m-1)}{\lambda \langle 1, \varphi \rangle \mathbb{E}[A^2 - A] \int_D \varphi(y)^3 \, dy}}.$$

Remark 2.2. *One sufficient condition to ensure that the hypotheses of Theorem 2.1 are satisfied is to assume that the boundary of D is $C^{2+\lfloor d/2 \rfloor}$. Standard regularity theory of elliptic partial differential equations, see for example [Eva98, §6.3], then implies that $\varphi \in C^1(\overline{D})$. However, this is also satisfied in many other cases.*

On the way to proving Theorem 2.1 we obtain several other results on critical branching diffusions, which are interesting in their own right as well as being essential to our method. We start with the phase transition. This was first proved by Sevast'yanov [Sev58] and Watanabe [Wat65], but has also been reworked and generalised in recent years, for example in [EK04], which studies local versus global extinction in unbounded domains. The precise description is as follows:

Theorem 2.3 ([Sev58], [Wat65]). *Let $D \subset \mathbb{R}^d$ be a bounded domain, satisfying a minimal regularity assumption (see Condition 2.9). Suppose that L is a uniformly elliptic self-adjoint operator with smooth coefficients and that A is a distribution with finite mean $m > 1$. Then there are two possibilities for the long term behaviour of the branching diffusion determined by L and A in D, according to the value of the branching rate β. Namely, for any starting position $x \in D$, if λ is the principal eigenvalue of L on D with Dirichlet boundary conditions then,*

(1) for $\beta > \frac{\lambda}{m-1}$ the process survives for all time with positive probability.

(2) for $\beta \leq \frac{\lambda}{m-1}$ the process becomes extinct almost surely.

Moreover, if $\beta \leq \frac{\lambda}{m-1}$ then $\mathbb{P}_x(N_t > 0) \to 0$ uniformly in D.

In the statement above we have taken some care to specify the regularity required on the domain, which is not detailed in the earlier works. Essentially we require that the eigenfunctions of the Laplacian converge to 0 pointwise on the boundary, and that Brownian motion started from a point on the boundary leaves the domain immediately with probability one. We will provide alternative proof of this Theorem, which in contrast to the earlier more analytic proofs in [Sev58], [Wat65], uses arguments centred around martingales (also appearing in [EK04]) arising naturally from the definition of the process. This proof will show that the stated regularity assumptions are sufficient.

The rest of the paper will focus on the behaviour of the system at criticality, starting with an asymptotic for the survival probability.

Theorem 2.4. *Suppose that the domain D is C^1, that L is as in Theorem 2.3, and that A has finite variance. Then, in the critical case $\beta = \frac{\lambda}{m-1}$, for all $x \in D$ we have*

$$\mathbb{P}_x\left(N_t > 0\right) \sim \frac{1}{t} \times \frac{2(m-1)\varphi(x)}{\lambda\left(\mathbb{E}[A^2] - \mathbb{E}[A]\right)\int_D \varphi(y)^3\,dy} \tag{2.1}$$

as $t \to \infty$. Here φ is the first eigenfunction of L on D, normalised to have unit L^2 norm.

This asymptotic then allows us to study the behaviour of the system when it is conditioned to survive for a long time, which is important for the proof of Theorem 2.1. One tool that we will use is a classical *spine* change of measure, under which the process has a distinguished particle, the spine, which is conditioned to remain in D forever (as in [Pin85]). Along this spine, families of ordinary critical branching diffusions immigrate at rate $\frac{m}{m-1}\lambda$ according to a biased offspring distribution. Note that there is no extinction under this new measure, which we denote by $\mathbb{Q}_x$. We will prove that changing measure in this way is in fact somewhat close to conditioning on survival for all time, in the sense of the following Proposition.

Proposition 2.5. *Assume the hypotheses of Theorem 2.4. Then for any $T \geq 0$, $x \in D$ and $B \in \mathcal{F}_T$, where $\mathcal{F}$ is the natural filtration of the process, we have*

$$\lim_{t\to\infty} \mathbb{P}_x\left(B | N_t > 0\right) = \mathbb{Q}_x(B). \tag{2.2}$$

Furthermore, we are able to prove a Yaglom type limit theorem for the positions of the particles in the system at time t, given survival.

Theorem 2.6. *For any measurable function f on D such that $\int_D f(x)^2\varphi(x)\,dx < \infty$, we have*

$$\left(t^{-1} \sum_{i=1}^{N_t} f(X_t^i) \middle| N_t > 0 \right) \to Z$$

in distribution as $t \to \infty$, where Z is an exponential random variable with mean

$$\frac{\lambda\left(\mathbb{E}[A^2] - \mathbb{E}[A]\right)\langle\varphi, f\rangle_{L^2(D)}\int_D \varphi^3}{2(m-1)}.$$

One consequence of Theorem 2.6 (or rather its proof) is that it allows us to describe the limiting distribution of the particles in the system at time t, given survival. It turns out that this is the law with density φ, normalised to be a probability distribution.

Corollary 2.7. *Let*

$$\mu_t := \frac{1}{N_t} \sum_{i=1}^{N_t} \delta_{X_t^i}$$

be the uniform distribution on all particles alive at time t, given survival. Then, for each f as in Theorem 2.6, we have that

$$\mu_t(f) \to \mu(f)$$

in distribution, and hence in probability, as $t \to \infty$, where

$$\mu(f) = \frac{\int_D \varphi(x) f(x)\, dx}{\int_D \varphi(x)\, dx}.$$

As this paper was being completed, the author learnt that similar results to Theorem 2.4 and Theorem 2.6 have also been shown by Asmussen and Hering in [AH83]. However their proofs are completely different from those in the current paper and, more importantly, our method provides several new and crucial ingredients for the proof of Theorem 2.1.

2.1.1 Related Work

It is interesting to note the analogy between Theorems 2.3-2.6, and the classical results from the theory of Galton-Watson processes. Indeed, for critical Galton-Watson processes, Kolmogorov [Kol38] proved an asymptotic for the probability of survival up to time n;

$$\mathbb{P}(Z_n > 0) \sim \frac{c}{n}$$

where Z_n is the population size at time n, and the constant depends on the variance of the offspring distribution. Moreover, Aldous [Ald91],[Ald93] and Duquesne and Le Gall [LGD02] showed that if you condition a critical Galton-Watson process to reach a large generation or have a large total progeny, then you have a scaling limit for the resulting tree. This limit is in the Gromov-Hausdorff topology, after rescaling distances in the tree appropriately, and the limiting object is the Continuum Random Tree, [Ald91]. In fact, this result can be extended to multitype Galton-Watson processes, as in [Mie08], where the same scaling limit exists. Since a branching diffusion can be thought of as a limit of multitype Galton-Watson processes, considering the types to be positions and discretising the domain appropriately, this is the first indication we should be able to obtain a similar scaling limit.

Remark 2.8. *The constant α in Theorem 2.1 is exactly what one obtains formally from the convergence in [Mie08], considering the branching diffusion to be a scaling limit of appropriate multitype Galton-Watson processes. However, Miermont's proof strategy is to make an induction on the number of types, and the lack of uniformity in the estimates as the number of types grows means that it does not extend to the set up considered here. Instead we use a combination of probabilistic and analytic ideas; see Section 2.1.2 for a sketch of the argument.*

An asymptotic for the survival probability has also been considered previously, see [BBS14] and [Kes78], in the case of branching Brownian motion with absorption at the origin, where the branching rate is kept constant and each particle moves with a drift $-\mu$, which is varied. In this set up, there is a critical value of $\mu = \mu_c$ above which extinction occurs with probability one. The *near-critical* system, as μ approaches its critical value from below, has also been studied, and in [BBS11] a limit, as $\mu \uparrow \mu_c$, is found for the probability of survival for all time as a function of the initial position. However, these results are quite different from ours as we do not allow our domain to be unbounded. The proofs of Theorems 2.1 and 2.4 do not extend to this situation, and in fact, we would expect to see a variety of behaviours for the critical system if we remove the assumption of boundedness. It would be an interesting problem to explore the possible cases here, and classify which domains fall into the regime of Theorem 2.1 and Theorem 2.4.

2.1.2 Organisation of the Paper and Main Ideas

We begin, for completeness and in order to introduce key concepts for the latter part of the paper, by providing a full proof of Theorem 2.3 in the case of Brownian motion with binary branching. This also allows us to make precise the regularity that is required on the domain for this statement to be true, see Condition 2.9. The main idea behind the proof we will give is to exploit the existence of a certain martingale

$$M_t = \mathrm{e}^{(\lambda-\beta)t} \sum_{i=1}^{N_t} \varphi(X_t^i),$$

where φ is the first eigenfunction of $-\frac{1}{2}\Delta$ on D, with unit L^2 norm, and λ is the first eigenvalue. We show that its properties (which depend on β) change critically at the point $\beta = \lambda$. These critical features turn out to determine the long term behaviour of the entire process, and thus provide the result of the theorem.

We will then turn to the proofs of the remaining theorems. Again we will give these for binary branching Brownian motion, and wherever adaptation for general diffusions and branching mechanisms is required, we will indicate the necessary changes. Any extra arguments are in fact minor, which is why we prefer to highlight the simplest case. This allows us to keep the arguments clear and avoid introducing extra notation.

The proof of Theorem 2.4 proceeds by a combination of probabilistic arguments, and analysis of the Fisher-Kolmogorov-Petrovskii-Piscounov equation. This is a partial differential equation which is known to be satisfied by the survival probability for branching Brownian motion; first noted by McKean [McK75] in the one-dimensional case, and used as the main tool by Sevast'yanov in the original proof of Theorem 2.3. Naively, we can write the survival probability in an L^2 expansion with respect to the orthonormal basis of $L^2(D)$ given by eigenfunctions of the Laplacian. Since the survival probability satisfies the FKPP equation we get a family of coupled ODEs from the coefficients. However, this is tricky to analyse directly. Instead, we apply a probabilistic line of reasoning, changing measure by M_t/M_0 to get a spine characterisation of the system as discussed in the introduction. This allows us to deduce that the survival probability decays like $a(t)\varphi(x)$ as $t \to \infty$, where $a(t)$ is the first coefficient in our expansion. Thus, our problem is reduced to the study of a single ODE. From here elementary analysis, combined with some extra information obtained from the probabilistic arguments, yields the result. We then prove Theorem 2.6 and Corollary 2.7, using the method of moments and a *Many-to-Few* Lemma.

The remainder of the paper is devoted to the proof of Theorem 2.1. Again working primarily in the case of Brownian motion with binary branching, we take an i.i.d. sequence of critical processes and concatenate the height functions of their associated trees. We would like to find a process which approximates this, and will converge after rescaling to a reflected Brownian motion: an analogue of the *Lukasiewicz path* for Galton-Watson trees. Just as the martingale M_t roughly measures the size of our system as we increase time, exploring it in a different, *depth-first*, order provides another martingale that is a proxy for the height function. After strengthening our result, Corollary 2.7, for the conditioned system, we can prove that the quadratic variation of this martingale is essentially linear, and so we obtain an invariance principle.

We then have to prove that this martingale is indeed a good approximation to the height process. This is one of the main difficulties, as the reversibility tools that are key to proving this for the Lukasiewicz path in the Galton-Watson case are lost. Instead, we must use precise estimates, and a delicate ergodicity argument related to our spine change of measure. This is one of the reasons that our machinery from the proof of Theorem 2.4 is so essential. Tightness arguments then allow us to conclude.

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2.2 Preliminaries

2.2.1 Diffusions as Trees

As stated in the introduction, $\mathbb{P}_x$ will denote the law of our branching diffusion, initiated from the point $x \in D$. We will consider this as a law on continuous planar trees, where every vertex is also marked with a position in D. In this representation, the vertices at a given height t in the tree will correspond to the particles alive in the system at time t, and their marks will correspond to their positions. To complete the definition of the tree we also need to decide, at every branch point, how the branching subtrees are ordered from left to right. To do this, we assume that given the number of offspring at a branching event, this ordering is chosen uniformly at random. This gives us a law on planar, or equivalently labelled, rooted trees. We emphasise here that these trees, unlike Galton-Watson trees, are in *continuous time*. Note that, in Theorem 2.1, we are considering them without their marks (in fact, the marks are irrelevant to the tree considered as a metric space.) The final point to make, is that when we use our notation $(X_t^1, \cdots, X_t^{N_t})$ for the system at time t, the indices correspond to the ordering of the vertices from left to right in the tree, and the X_t^i are their positions, or marks.

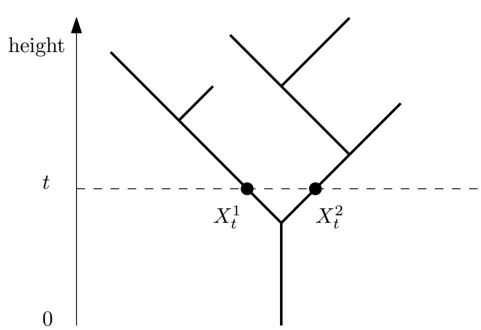

Figure 19: An example of the continuous tree generated by a branching diffusion. The two vertices at height t (marked with dots) have positions, or marks, given by X_t^1 and X_t^2.

2.2.2 Spectral Theory and Martingales

From now on we will work in the case of Brownian motion with binary branching, unless stated otherwise.

As discussed in the introduction, the behaviour of branching Brownian motion killed when leaving a given domain D will be closely related to the spectral properties of the Laplacian on that domain. Throughout, we will let $\{\lambda_i\}_{i\geq 1}$ denote the eigenvalues of $-\frac{1}{2}\Delta$ on D with Dirichlet boundary conditions, with corresponding eigenfunctions $\{\varphi_i\}_{i\geq 1}$, normalised to have unit L^2 norm. Recall that under this normalisation the eigenfunctions form an orthonormal basis for $L^2(D)$, the eigenvalues are real with

$$\lambda := \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the first eigenfunction

$$\varphi := \varphi_1$$

is strictly positive in the domain. Also note that the eigenfunctions are in $C^\infty(D)$, and assuming the domain is sufficiently regular, converge pointwise to 0 on ∂D. In particular, they are all bounded.

Condition 2.9. *For the proof of Theorem 2.3 we will assume that the domain D is regular enough that:*

(1) The eigenfunctions converge to 0 *pointwise on the boundary, and*

(2) Every point x_0 on the boundary satisfies, for every $\varepsilon > 0$, $\lim_{x\to x_0} \mathbf{P}_x(\tau_D > \varepsilon) = 0$,

where $\mathbf{P}_x$ is the law of Brownian motion started from the point x and τ_D is the first time it hits the boundary ∂D.

These are very weak regularity conditions on the domain D. For example, see [GT83, Theorem 8.29], both conditions are satisfied by any domain satisfying a uniform exterior cone condition.

To set up some notation, let $p^{\mathbb{R}^d}(t,x,y)$ be the transition density for Brownian motion on $\mathbb{R}^d$ and set, for $x, y \in D$,

$$p^D(t,x,y) = p^{\mathbb{R}^d}(t,x,y) - \mathbf{E}_x\left[p^{\mathbb{R}^d}(t-\tau_D, B_{\tau_D}, y)\,\mathbb{1}_{\{\tau_D\leq t\}}\right].$$

Then we have

$$0 \leq p^D(t,x,y) \leq p^{\mathbb{R}^d}(t,x,y) \leq \frac{1}{(2\pi t)^{d/2}} \tag{2.3}$$

and by the strong Markov property p^D is the transition density of Brownian motion *killed* when leaving the domain D. This means, in particular, that for all integrable functions f we have

$$\int_D f(y)p^D(t,x,y)\,dy = \mathbf{E}_x\left[f(B_t)\mathbb{1}_{\{\tau_D>t\}}\right] \tag{2.4}$$

for $t > 0$ and $x \in D$. It can be shown (see for example [MNV09, Remark 2.1]) that if D satisfies Condition 2.9, then for any bounded continuous function f on D which vanishes on ∂D,

$$\int_D p^D(t,x,y)f(y)dy = \mathbf{E}_x\left[f(B_t)\mathbb{1}_{\{\tau_D>t\}}\right] \tag{2.5}$$

is the unique solution of the heat equation in the domain, with initial data f and Dirichlet boundary data. One consequence of this is that for any of the eigenfunctions φ_i, we have

$$\mathbf{E}_x\left[\varphi_i(B_{t\wedge\tau_D})\right] = \mathrm{e}^{-\lambda_i t}\,\varphi_i(x) \tag{2.6}$$

for all $x \in D$. This leads to the following decomposition for functions $f \in L^2(D)$. Here and throughout the rest of the paper, $\langle\cdot,\cdot\rangle$ will represent the usual inner product on $L^2(D)$.

Lemma 2.10. *If $f \in L^2(D)$ then for all $t \geq 0$ and $x \in D$*

$$\mathbf{E}_x\left[f(B_t)\mathbb{1}_{\{\tau_D>t\}}\right] = \sum_1^\infty \mathrm{e}^{-\lambda_i t}\,\varphi_i(x)\,\langle\varphi_i, f\rangle\,.$$

Proof. First note that $p^D(t,x,y)$ is bounded for all $x, y \in D$ by (2.3), and that $\left(p^D(t,x,\cdot),\varphi_i\right) = \mathrm{e}^{-\lambda_i t}\,\varphi_i(x)$ by (2.5) and (2.6). Therefore, since the φ_i's form an orthonormal basis of $L^2(D)$, we have that

$$\sum_{i=1}^n \mathrm{e}^{-\lambda_i t}\,\varphi_i(x)\varphi_i(\cdot) \to p^D(t,x,\cdot)$$

in L^2. Thus for any $f \in L^2(D)$, we may conclude that

$$\mathbf{E}_x\left[f(B_t)\mathbb{1}_{\{\tau_D>t\}}\right] = \left(p^D(t,x,\cdot),f\right) = \lim_{n\to\infty}(\sum_{i=1}^n \mathrm{e}^{-\lambda_i t}\,\varphi_i(x)\varphi_i(\cdot),f)$$

which yields the result. □

Another consequence of (2.6) is the existence of an additive martingale for the system. Indeed, a straightforward application of the branching Markov property and the Many-to-One Lemma (see for example [HH07] or [HR15]) which tells us that

$$\mathbb{E}_x[\sum_{i=1}^{N_t} f(X_t^i)] = \mathrm{e}^{\beta t}\,\mathbf{E}_x\left[f(B_t)\mathbb{1}_{\{\tau_D>t\}}\right]$$

for measurable f, provides the following:

Lemma 2.11. *The process*

$$M_t = \mathrm{e}^{(\lambda-\beta)t} \sum_{i=1}^{N_t} \varphi(X_t^i)$$

is a martingale under $\mathbb{E}_x$, for each $x \in D$.

Moreover, by positivity of φ in D, M_t is a positive martingale, and thus by the martingale convergence theorem converges almost surely to an almost surely finite limit. This is a strong indicator of the existence of a phase transition, since the properties of the limit M_∞, which depend on $\beta > 0$ and change critically at $\beta = \lambda$, determine the long term behaviour of the system.

Remark 2.12. *The above theory directly extends to more general diffusions, with generators and offspring distributions as in Theorem 2.3. In particular, Condition 2.9 will provide the required degree of regularity. Here we have that*

$$M_t = \mathrm{e}^{(\lambda-\beta(m-1))t} \sum_{i=1}^{N_t} \varphi(X_t^i)$$

defines a positive martingale, using a generalisation of the Many-to-One Lemma.

2.3 The Phase Transition

In this section we will provide a proof of Theorem 2.3.

2.3.1 The Supercritical Case

We begin by supposing that β is strictly greater than λ, in which case the proof can be summarised as follows. We already know that $M_t(\beta) \to M_\infty$ as $t \to \infty$. We will show that for this range of β, the martingale M_t is square integrable and so the limit M_∞ cannot be degenerate. Thus, since $\mathrm{e}^{(\lambda-\beta)t} \to 0$ as $t \to \infty$, it must be the case that

$$\sum_{i=1}^{N_t} \varphi(X_t^i) \to \infty$$

with positive probability. Boundedness of φ then implies that $N_t \to \infty$ on this event.

In order to prove square integrability, we need a generalisation of the Many-to-One Lemma to a Many-to-Two form. This is well know, see for example [HR15, Lemma 1] or [Wat67, Equation (2.23)].

Lemma 2.13 (Many-to-Two)**.** *If f, g are measurable functions on D then*

$$\mathbb{E}_x\left[\sum_{i=1}^{N_t} f(X_t^i)\sum_{i=1}^{N_t} g(X_t^i)\right] = \mathbf{E}_x\left[\mathrm{e}^{\beta t} f(B_t)g(B_t)\mathbb{1}_{\{\tau_D>t\}}\right] + \mathbf{E}_x\left[\int_0^{t\wedge\tau_D} 2\beta\,\mathrm{e}^{\beta s}\,\mathbf{E}_{B_s}[\sum_1^{N_{t-s}} f(X_{t-s}^i)]\mathbf{E}_{B_s}[\sum_1^{N_{t-s}} g(X_{t-s}^i)]\right]$$

where B is standard Brownian motion started at $x \in D$ and $(X_t^1, \cdots, X_t^{N_t})$ represents branching Brownian motion as usual.

Applying the Lemma with $f = g = \varphi$, and using (2.6) to rewrite the expectation terms in the integrand, along with Fubini, we see that

$$\mathbb{E}_x\left[M_t^2\right] = \mathrm{e}^{2(\lambda-\beta)t}\,\mathbf{E}_x\left[\mathrm{e}^{\beta t}\,\varphi(B_{t\wedge\tau_D})^2\right] + 2\beta\int_0^t \mathrm{e}^{2(\lambda-\beta)s}\,\mathbf{E}_x\left[\mathrm{e}^{\beta s}\,\varphi(B_{s\wedge\tau_D})^2\right] ds.$$

Since we also have the bound,

$$\mathbf{E}_x\left[\mathrm{e}^{\beta t}\,\varphi(B_{t\wedge\tau_D})^2\right] \le \|\varphi\|_\infty \mathbf{E}_x\left[\mathrm{e}^{\beta t}\,\varphi(B_{t\wedge\tau_D})\right] \le \varphi(x)\|\varphi\|_\infty\,\mathrm{e}^{(\beta-\lambda)t}$$

for all t and x, we can substitute this in and integrate to see that

$$\mathbb{E}_x\left[M_t^2\right] \le \varphi(x)\|\varphi\|_\infty\left(1 + \frac{4\beta}{\lambda-\beta}\right)$$

for all t. Thus we obtain square integrability.

Remark 2.14. *For the more general set up, with generator L and offspring distribution A as in Theorem 2.3, a generalisation of the Many-to-Two lemma, see [HR15], proves uniform integrability of the martingales. This again shows that there is a positive probability of survival in the supercritical case.*

2.3.2 The Subcritical Case

Now let us suppose that $\beta < \lambda$. The convergence of the martingale in this case, along with the fact that $\mathrm{e}^{(\lambda-\beta)t} \to \infty$ as $t \to \infty$, means that in fact

$$\sum_{i=1}^{N_t} \varphi(X_t^i) \to 0$$

almost surely as $t \to \infty$. Although this does not show that $N_t \to 0$ a.s. for all such β immediately, Lemma 2.10 ensures that $\sum_{i=1}^{N_t} \varphi(X_t^i)$ being small must mean that

N_t is small with high probability. This is made explicit by the following Lemma, which is essentially a consequence of Lemma 2.10.

Lemma 2.15. *Let $\varepsilon > 0$ be given. Then there exists $T(\varepsilon) \geq 0$ and an absolute constant K, such that for all $t \geq T(\varepsilon)$ and all $x \in D$ we have*

$$\mathbf{P}_x\left(\{\varphi(B_t) < \varepsilon\} \cap \{t < \tau_D\}\right) \leq \mathrm{e}^{-\lambda t} K\varepsilon.$$

Proof. Let $f_\varepsilon(x) = \mathbb{1}_{[0,\varepsilon)}(\varphi(x)) = \mathbb{1}_{[0,\varepsilon)} \cdot \varphi$. This is clearly in L^2 (it is bounded by 1), so we may apply Lemma 2.10 to obtain that

$$\mathbf{P}_x\left(\{\varphi(B_t) < \varepsilon\} \cap \{t < \tau_D\}\right) = \mathbf{E}_x\left[f_\varepsilon(B_t)\mathbb{1}_{\{\tau_D > t\}}\right] = \sum_{i=1}^{\infty} \mathrm{e}^{-\lambda_i t}\left(\varphi_i, f_\varepsilon\right)\varphi_i(x).$$

Intuitively, since $\lambda = \lambda_1 < \lambda_i$ for $i \geq 2$, we expect that the sum should behave roughly like $\mathrm{e}^{-\lambda t}\left(\varphi, f_\varepsilon\right)\varphi(x)$ as t becomes large. Indeed we can bound it above, using that $(\varphi, f_\varepsilon) \leq \mathrm{vol}(D)\varepsilon$, by

$$\mathrm{e}^{-\lambda t}\left(\|\varphi\|_\infty \mathrm{vol}(D)\varepsilon + \mathrm{e}^{-\gamma t}\left|\sum_{i=2}^{\infty} \mathrm{e}^{-(\lambda_i - \lambda_2)t}\left\langle\varphi_i, f_\varepsilon\right\rangle\varphi_i(x)\right|\right) \tag{2.7}$$

where $\gamma > 0$ is the spectral gap for D. Therefore, it is enough to show that the expression in the modulus above is bounded by some absolute constant, for all t large enough. To do this, observe that since $\lambda_i \to \infty$ as $i \to \infty$, there exists an N such that $(\lambda_i - \lambda_2) > \frac{\lambda_i}{2}$ for all $i \geq N$. This means that

$$\begin{aligned}\left|\sum_{i=2}^{\infty} \mathrm{e}^{-(\lambda_i-\lambda_2)t}\left\langle\varphi_i, f\right\rangle\varphi_i(x)\right| &\leq \sum_{i=2}^{\infty} \mathrm{e}^{-(\lambda_i-\lambda_2)t}\left|\left\langle\varphi_i, f\right\rangle\right|\left|\varphi_i(x)\right| \\ &\leq N\,\mathrm{vol}(D)\sup_{1\leq i\leq N}\|\varphi_i\|^2 + \sum_{N+1}^{\infty} \mathrm{e}^{-\frac{\lambda_i}{2}t}\left|\left\langle\varphi_i, f\right\rangle\right|\left|\varphi_i(x)\right|\end{aligned}$$

for all t, where the first term in the final line is a constant depending only on D. Furthermore, by Cauchy-Schwarz we have

$$\left|\sum_{N+1}^{\infty} \mathrm{e}^{-\frac{\lambda_i}{2}t}\left|\left\langle\varphi_i, f\right\rangle\right|\left|\varphi_i(x)\right|\right| \leq \sqrt{\sum_{i=N+1}^{\infty} \mathrm{e}^{-\lambda_i t}\varphi_i(x)^2}\,\sqrt{\sum_{i=N+1}^{\infty}\left\langle\varphi_i, f\right\rangle^2}$$

where the first term is less than $\|p^D(t/2, x, \cdot)\|_{L^2} \leq (\pi t)^{-d/4}$ and the second is less than $\|f\|_\infty$. This can clearly be bounded uniformly for all t large enough. $\square$

Corollary 2.16. *Let τ_D be the hitting time of ∂D, for a Brownian motion started* *at $x \in D$. Then*

$$\mathbf{P}_x\left(\tau_D > t\right) \le A\,\mathrm{e}^{-\lambda t}$$

for all $t \ge T = T(\|\varphi\|_\infty)$, where A is a constant independent of x and T.

Proof. This follows immediately from the above taking $\varepsilon = \|\varphi\|_\infty$. □

Lemma 2.17 (Expectation of the population size)**.** *Suppose $\beta \le \lambda$. Then for all $x \in D$ and all $t \ge T = T(\|\varphi\|_\infty)$, we have*

$$\mathbb{E}_x\left[N_t\right] \le A\,\mathrm{e}^{(\beta-\lambda)t}$$

where A is a constant, independent of t and x.

Proof. This is a straightforward consequence of the Many-to-One Lemma and the above Corollary. □

To prove almost sure extinction in the subcritical case, it is enough to show that $\mathbb{P}_x\left(N_t > 0\right) \to 0$ as $t \to \infty$. However, this is immediate from Lemma 2.17, since

$$\mathbb{P}_x\left(N_t > 0\right) \le \mathbb{E}_x\left[N_t\right] \le A\,\mathrm{e}^{(\beta-\lambda)t}$$

which indeed tends to 0 in the case $\beta < \lambda$.

2.3.3 The Critical Case

When β is equal to λ, it is still the case that branching Brownian motion with parameter β dies out almost surely. However, since we can no longer rely on the fact that $\mathrm{e}^{(\beta-\lambda)t} \to 0$ as $t \to \infty$, the decay of $\mathbb{E}_x\left[N_t\right]$ from Lemma 2.17 is lost, and we must apply a slightly more delicate argument. To improve the situation, we make use of the following Lemma, which can be found in [Wat65, Lemma 2.1]. We provide a proof for completeness.

Lemma 2.18 ([Wat65])**.** *For all $x \in D$*

$$\mathbb{P}_x\left(N_t \to 0 \text{ or } N_t \to \infty \text{ as } t \to \infty\right) = 1.$$

Proof. Since N_t is integer-valued it is sufficient to prove that $\mathbb{P}_x\left(N_t = k \text{ i.o.}\right) = 0$ for every $k \in \mathbb{N}$. Fix k and define a sequence of hitting and leaving times $(L_n, H_n)_{n\ge1}$, by letting L_1 be the first time t that $N_t \ne N_0$, and H_1 be the first time that $N_t = k$. Then inductively, let L_n be the first time after H_{n-1} that $N_t \ne k$, and H_n the first time after this that $N_t = k$. We have to show that $\mathbb{P}_x\left(H_n < \infty\right) \to 0$ as $n \to \infty$. Set γ to be the infinum over all $x \in D$ of the probability that a Brownian motion started at x leaves the domain before an independent exponential waiting time. It is easily verified, using for example Corollary 2.16, that $\gamma > 0$. Then we have that $\mathbb{P}_x(H_1 < \infty) \le \mathbb{P}(N_{L_1} \neq 0) \le (1-\gamma)$ and inductively, using the Markov property for the k particles alive at each time H_j, that $\mathbb{P}_x(H_n < \infty) \le (1-\gamma)(1-\gamma^k)^{n-1}$. This completes the proof. □

This, together with Lemma 2.15, provides the basis for the proof of Theorem 2.3 in the critical case. By Lemma 2.18, it will be sufficient to prove that $\mathbb{P}_x(N_t \to \infty \text{ as } t \to \infty) = 0$ for all $x \in D$. Since we know that $\sum_1^{N_t} \varphi(X_t^i) \to M_\infty < \infty$ almost surely, letting A_k be the event that $\{N_t \to \infty\} \cap \{\sum_1^{N_t} \varphi(X_t^i) \le k \text{ eventually}\}$, we can write this probability as

$$\mathbb{P}_x(N_t \to \infty) = \sup_k \mathbb{P}_x(A_k)$$

since the A_k's are increasing. Thus, it will be enough to show that $\mathbb{P}_x(A_k) = 0$ for all $k > 0$. To do this, fix k, and observe that $\mathbb{P}_x(A_k) = \lim_{m\to\infty} \mathbb{P}_x(A_k^m)$ where A_k^m is the event that $\{\{N_t \ge m\} \cap \{\sum_1^{N_t} \varphi(X_t^i) \le k\} \text{ eventually}\}$. However, for

$$\{N_t \ge m\} \cap \{\sum_1^{N_t} \varphi(X_t^i) \le k\}$$

to occur, it must be the case that one of the particles in the system at time t has $\varphi(X_t^i) \le \frac{k}{m}$ (since φ is positive.) Hence,

$$\mathbb{P}_x\left(\{N_t \ge m\} \cap \{\sum_1^{N_t} \varphi(X_t^i) \le k\}\right) \le \mathbb{P}_x\left((\sum_{i=1}^{N_t} \mathbb{1}_{\{\varphi(X_t^i) \le \frac{k}{m}\}}) \ge 1\right) \le \mathbb{E}_x\left[\sum_{i=1}^{N_t} \mathbb{1}_{\{\varphi(X_t^i) \le \frac{k}{m}\}}\right],$$

which is less than $K\frac{k}{m}$ for $t \ge T(k/m)$, as a consequence of Lemma 2.15. Since the probability of this holding for *all* large times is certainly smaller than the probability of it holding at time $T(k/m)$ say, we see that

$$\mathbb{P}_x(A_k) = \lim_{m\to\infty} \mathbb{P}_x(A_k^m) \le \lim_{m\to\infty} K\frac{k}{m} = 0$$

as required.

Remark 2.19. *The proofs given above for the subcritical and critical cases rely purely on spectral properties of the Laplacian. These still hold for our more general diffusions, so no adaptation of the arguments is required.*

To complete the proof of Theorem 2.3 we must show that the decay of the survival probability in the critical case (the subcritical case having been dealt with by Lemma 2.17) is uniform in D. However, this will follow from elementary analysis, once we have noted that the survival probability is a solution of the FKPP equation in D. This was first observed by McKean, [McK75], in one dimension, but the proof extends directly to our situation. In fact, the relationship with the FKPP equation is the key tool employed in [Wat65] and [Sev58] to prove Theorem 2.3.

Lemma 2.20. *Assume that D satisfies Condition 2.9 and let $u(t,x) := \mathbb{P}_x(N_t > 0)$ for the critical system. Then $u \in C^{2,1}(D \times (0,\infty)) \cap C(\overline{D})$ is a solution of*

$$\begin{aligned} \partial u/\partial t &= \frac{1}{2}\Delta u + \lambda(u - u^2) && \text{on } D \times (0,\infty) \\ u(x,0) &= \mathbb{1}_{\{x \in D\}} && \text{on } \overline{D} \times \{0\} \\ u(x,t) &= 0 && \text{on } \partial D \times (0,\infty). \end{aligned} \tag{2.8}$$

Proof. Conditioning on the first branching time of the process, we can write

$$u(x,t) = \mathrm{e}^{-\lambda t}\,\mathbf{P}_x(\tau_D > t) + \int_0^t \lambda\,\mathrm{e}^{-\lambda s}\,\mathbf{E}_x\left[(2u - u^2)(B_s, t-s)\mathbb{1}_{\{\tau_D > s\}}\right].$$

An easy differentiation after making the change of variables $s \leftrightarrow t-s$ in the integral then provides (2.8). Note that $u(t,x) \to 0$ as $x \to \partial D$ since

$$\begin{aligned} \mathbb{P}_x(N_t > 0) &\leq 1 - \mathbb{P}_x\,(\text{the process becomes extinct before the first branching time}) \\ &\leq 1 - \mathrm{e}^{-\lambda s_1}\,\mathbb{P}_x(\tau_D \leq s_1) \end{aligned}$$

for any s_1, where the last line can be made arbitrarily small by first taking s_1 to 0, and then using the property (2) of Condition 2.9. Thus $u \in C(\overline{D})$. ☐

To conclude the proof of 2.3, we note the continuous functions $u(x,t)$ are clearly decreasing in t and converge to the continuous function 0 for each $x \in \overline{D}$. This is a compact set and so by Dini's theorem, an elementary result from real analysis (see for example [Rud76, Theorem 7.13]), the decay must indeed be uniform.

Remark 2.21. *In the case of a more general diffusion with generator L and branching mechanism determined by offspring distribution A, the partial differential equation (2.8) becomes*

$$\frac{\partial u}{\partial t} = -Lu + \frac{\lambda}{m-1}\left((1-u) - G(1-u)\right)$$

where G is the probability generating function of A. This results in the same regularity for u.

2.4 Survival at Criticality: Proof of Theorem 2.4

Throughout this section, we will work in the critical case $\beta = \lambda$ (for binary branching Brownian motion) and also from now on assume the domain D to have C^1 boundary. We will prove the asymptotic (2.1) for the survival probability, using a combination of spine techniques, and analysis of the FKPP equation.

2.4.1 Spine Decomposition

It turns out that a helpful approach in the proof of (2.1) will be to change measure via the $\mathbb{P}_x$-martingale M_t from the previous section, see Lemma 2.11, and use a spine construction to describe the behaviour of λ-branching Brownian motion under this new measure. This is a so called *spine change of measure*, in that it changes the law of the initial particle, but then all subprocesses branching off this *spine* still behave as ordinary λ-branching Brownian motions.

To make sense of this, we extend our probability measure $\mathbb{P}_x$ to a probability measure $\overline{\mathbb{P}}_x$ on a bigger space, by choosing one distinguished line of descent which we call the *spine*. We let the initial particle be part of the spine and then, whenever a spine particle splits, the new spine particle is chosen uniformly from its children. We denote the natural filtration of this new process by $(\overline{\mathcal{F}}_t)_{t\geq 0}$, and the position of the spine by $(\xi_t)_{t\geq 0}$. Then it follows from the fact that $\mathrm{e}^{\lambda t}\,\varphi(B_t)/\varphi(x)$ is a martingale for our individual particle motion under $\mathbf{P}_x$ (a consequence of (2.6)) that the process

$$\overline{M}_t = \frac{\varphi(\xi_t)}{\varphi(x)} 2^{S_t}$$

is a martingale with respect to $\overline{\mathcal{F}}_t$, where S_t is the number of branch points along the spine before time t. For a proof of this, see [Rob10, Theorem 2.4], and see also [Rob10] and [HH09] for details of the above construction. We can therefore define a new measure $\overline{\mathbb{Q}}_x$ on the same probability space as $\overline{\mathbb{P}}_x$ via

$$\left.\frac{d\overline{\mathbb{Q}}_x}{d\overline{\mathbb{P}}_x}\right|_{\overline{\mathcal{F}}_t} = \overline{M}_t.$$

Then there is a nice characterisation of the process under $\overline{\mathbb{Q}}_x$, which follows from the classical spine theory for such changes of measure developed in [HH09]; but see also [HH07], [CR88] and [HHK12] among others. Using the fact that under the change of measure for Brownian motion defined by

$$\left.\frac{d\mathbf{Q}_x}{d\mathbf{P}_x}\right|_{\mathcal{F}_t} = \frac{\mathrm{e}^{\lambda t}\,\varphi(B_t)}{\varphi(x)} \tag{2.9}$$

we have that the new particle evolves as a Brownian motion conditioned to remain in D, see [Doo57], [Pin85], we obtain a description of the whole process under $\overline{\mathbb{Q}}_x$, as summarised in the Lemma below. We refer the reader to the papers cited above for more details, and proof of the characterisation.

Lemma 2.22. *Under the measure $\overline{\mathbb{Q}}_x$, the law of λ-branching Brownian motion with a distinguished spine can be constructed as follows:*

- *The initial particle evolves as a Brownian motion started at x and conditioned to remain in D for all time.*
- *At an accelerated rate of 2λ it splits into two particles.*
- *One of these particles, the* spine particle*, is chosen uniformly at random and goes on to repeat stochastically the behaviour of the initial ancestor.*
- *The other particle goes on to perform an independent λ-branching Brownian motion, starting from the point of fission.*

Alternatively, we can think of the process as being formed by a single spine particle, which evolves as a Brownian motion conditioned to remain in D, and along which ordinary λ-branching Brownian motions immigrate (branch off) at rate 2λ.

We also let $\mathbb{Q}_x := \overline{\mathbb{Q}}_x|_{\mathcal{F}_t}$ be the corresponding measure on the original filtration $\mathcal{F}_t$ of the branching process. Then we have that

$$\left.\frac{d\mathbb{Q}_x}{d\mathbb{P}_x}\right|_{\mathcal{F}_t} = \frac{\sum_{i=1}^{N_t} \varphi(X_t^i)}{\varphi(x)} = \frac{M_t}{\varphi(x)} \tag{2.10}$$

(again see [Rob10, Theorem 2.4]). Furthermore, the probability under $\overline{\mathbb{Q}}_x$ that a certain particle X_t^j at time t is the spine particle, given $\mathcal{F}_t$, is equal to

$$\varphi(X_t^j) / \sum_{i=1}^{N_t} \varphi(X_t^i), \tag{2.11}$$

see [Rob10] or [HR14, Remark 1.2].

Remark 2.23. *For more general diffusions and branching mechanisms, we obtain a similar characterisation of the system after changing measure by the corresponding martingale. In this case, the spine particle will move under the law of the original diffusion conditioned to remain in the domain (defined by the same change of measure as in (2.9)), and at rate $\frac{m}{m-1}\lambda$, a certain number of the original branching diffusions will immigrate. The number of these immigrants has the size-biased* *offspring distribution $\tilde{A}$, where*

$$\mathbb{P}\left(\tilde{A} = k\right) = \frac{k+1}{m}\mathbb{P}\left(A = k+1\right)$$

for $k \geq 0$. Note the spine particle itself is not *included in $\tilde{A}$.*

One important fact we will use is that the position of the spine particle in the decomposition described by Lemma 2.22 converges quickly to an equilibrium distribution, with density φ^2. In fact, if we assume that $\frac{1}{2}\Delta$ is *intrinsically ultracontractive* for the domain (for which a Lipschitz assumption is enough), it is well known that this convergence is uniform in the starting position.

Lemma 2.24. *Suppose that D is a bounded Lipschitz domain. If*

$$K^D(t, x, y) = \frac{\mathrm{e}^{\lambda t}\, p^D(t, x, y)\varphi(y)}{\varphi(x)}$$

is the transition density for Brownian motion conditioned to remain in D, then for any $\varepsilon > 0$ there exists a constant C_ε depending only on the domain such that

$$\left|\frac{K^D(t, x, y)}{\varphi(y)^2} - 1\right| \leq C_\varepsilon\, \mathrm{e}^{-\gamma t}$$

for all $t > \varepsilon$ and $x, y \in D$ where $\gamma := \lambda_2 - \lambda_1 > 0$ is the spectral gap for $-1/2$ the Laplacian on D.

Proof. See for example [DS84] or [Bañ99, Equation (1.8)]. □

Remark 2.25. *The corresponding convergence to equilibrium when the single particle motion is governed by a generator L as in Theorem 2.3 still holds whenever the domain is Lipschitz (see [Bañ99].)*

2.4.2 Asymptotics for the Survival Probability

Using this spine decomposition, and the fact that the law of the spine particle converges to an equilibrium distribution as $t \to \infty$, we may first deduce that we have an asymptotic for the survival probability which is of the correct form.

Proposition 2.26. *Uniformly in $x \in D$*

$$\mathbb{P}_x\left(N_t > 0\right) \sim a(t)\varphi(x)$$

as $t \to \infty$, where

$$a(t) := \int_D \mathbb{P}_z\left(N_t > 0\right)\varphi(z)\, dz$$

converges to 0 *as* $t \to \infty$*, and is independent of* x*.*

Proof. The key idea for the proof of this is to write, recalling (4.6),

$$\frac{\mathbb{P}_x\left(N_t > 0\right)}{\varphi(x)} = \mathbb{Q}_x\left[\frac{1}{\sum_{i=1}^{N_t} \varphi(X_t^i)}\right]$$

and then show that the right hand side essentially does not depend on x for large t. The intuition behind this is that under the new measure, the position of the spine particle will converge very quickly to equilibrium. Then, contributions to the sum in the denominator from subprocesses branching off the spine before its position has become well mixed are unlikely to occur, as these have the law of standard λ-branching Brownian motions, which we know are unlikely to survive for a long time.

To begin, for $t_0 \leq t$, write

$$\sum_{i=1}^{N_t} \varphi(X_t^i) = M_t := M_{t_0,t} + M_{0,t_0}$$

where M_{0,t_0} is the sum of all contributions to M_t from subprocesses branching off the spine before time t_0. Also define $f(r) := \mathbb{Q}_{\varphi^2}[1/\sum_{i=1}^{N_r} \varphi(X_r^i)]$, where φ^2 in the subscript indicates that the initial position is distributed according to the probability measure with density φ^2. Note that this is a function of r only. Then for any $t_0 \leq t$

$$\frac{1}{M_t} = \frac{1}{M_{t_0,t}} + \left(\frac{1}{M_t} - \frac{1}{M_{t_0,t}}\right) = \frac{1}{M_{t_0,t}} - \frac{M_{0,t_0}}{M_t M_{t_0,t}}$$

and so

$$\mathbb{Q}_x\left[\frac{1}{\sum_{i=1}^{N_t} \varphi(X_t^i)}\right] = f(t-t_0) + \left(\mathbb{Q}_x\left[\frac{1}{M_{t_0,t}}\right] - f(t-t_0)\right) - \mathbb{Q}_x\left[\frac{M_{0,t_0}}{M_t M_{t_0,t}}\right]$$

where we label the error (second and third) terms above by $\epsilon_x^1(t,t_0)$ and $\epsilon_x^2(t,t_0)$ respectively. The plan is to show that we can choose $t_0(t) < t$ such that both the error terms become small as $t \to \infty$. The reason we expect these terms to decay is as in the heuristic discussion above, since the equilibrium distribution for Brownian motion conditioned to remain in D is precisely φ^2.

Now observe that since the quantities whose expectations we are evaluating are $\mathcal{F}_t$ measurable, their $\mathbb{Q}_x$ and $\overline{\mathbb{Q}}_x$ expectations are the same, and we may work with either. Considering the definition of $M_{t_0,t}$, and conditioning on $\mathcal{G}_{t_0}$, for $(\mathcal{G}_s)_{s\geq 0}$ the filtration generated by the position of the spine up to time s (a subfiltration of

$(\overline{\mathcal{F}}_s)_{s\geq 0})$ we see that

$$\epsilon_x^1(t,t_0) = \overline{\mathbb{Q}}_x[\overline{\mathbb{Q}}_x[1/M_{t_0,t}|\mathcal{G}_{t_0}]] - f(t-t_0)$$

is simply the difference in expectation of the function of y,

$$\mathbb{Q}_y[\frac{1}{\sum_{i=1}^{N_{t-t_0}} \varphi(X^i_{t-t_0})}] = \mathbb{P}_y\left(N_{t-t_0} > 0\right)/\varphi(y)$$

under the $\overline{\mathbb{Q}}_x$ law of the spine particle at time t_0, and the law with density φ^2. As t_0 increases, we know that the law of the spine particle approaches the law with this density, and by Lemma 2.24 we can quantify this with the bound

$$\sup_{x,y\in D}\left|\frac{K^D(t_0,x,y)}{\varphi(y)^2} - 1\right| \leq C\,\mathrm{e}^{-\gamma t_0} \tag{2.12}$$

for all $t_0 > 1$ say, where $K^D(s,x,y)$ is the transition density of Brownian motion conditioned to remain in D. Hence we have the estimate, for all $t_0 > 1$:

$$|\epsilon_x^1(t,t_0)| \leq \sup_{x,y\in D}\left|\frac{K^D(t_0,x,y)}{\varphi(y)^2} - 1\right| \int_D \varphi(y)\mathbb{P}_y\left(N_{t-t_0} > 0\right)\,dy \leq C\,\mathrm{e}^{-\gamma t_0}\,f(t-t_0). \tag{2.13}$$

To bound the second term, write $A^t_{t_0}$ for the event that some subprocess branching from the spine before time t_0 survives until (total) time t. Since $M_{0,t_0}/M_tM_{t_0,t}$ is positive and less than or equal to $1/M_{t_0,t}$ we see that

$$\left|\epsilon_x^2(t,t_0)\right| \leq \overline{\mathbb{Q}}_x\left[\frac{1}{M_{t_0,t}}\mathbb{1}_{A^t_{t_0}}\right].$$

Again, to estimate this we condition; but now on $\tilde{\mathcal{G}}_{t_0}$, where $\tilde{\mathcal{G}} \supset \mathcal{G}$ is the filtration which also contains information about the branching points along the spine. The reason for doing this is that we know, given the position of the spine (ξ_s) for $0 \leq s \leq t_0$ and all its branching points, that the subprocesses branching off the spine before t_0 and the process continuing on from ξ_{t_0} are independent. Thus, the term on the right above is equal to

$$\overline{\mathbb{Q}}_x\left[\overline{\mathbb{Q}}_x\left[\mathbb{1}_{A^t_{t_0}}|\tilde{\mathcal{G}}_{t_0}\right]\overline{\mathbb{Q}}_x\left[\frac{1}{M_{t_0,t}}\middle|\tilde{\mathcal{G}}_{t_0}\right]\right] = \overline{\mathbb{Q}}_x\left[\overline{\mathbb{Q}}_x\left[\mathbb{1}_{A^t_{t_0}}|\tilde{\mathcal{G}}_{t_0}\right]\frac{\mathbb{P}_{\xi_{t_0}}\left(N_{t-t_0} > 0\right)}{\varphi(\xi_{t_0})}\right],$$

where we can now show that the conditional probability $\overline{\mathbb{Q}}_x[\mathbb{1}_{A^t_{t_0}}|\tilde{\mathcal{G}}_{t_0}]$ is small. Indeed, since the probability that any subprocess branching off the spine before t_0

survives until total time t is less than $\sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)|$, we see that

$$\overline{\mathbb{Q}}_x \left[\mathbb{1}_{A^t_{t_0}} | \tilde{\mathcal{G}}_{t_0}\right] \leq S_{t_0} \sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)|,$$

where S_{t_0} is the number of such subprocesses. Moreover, again for $t_0 > 1$ say by (2.12), we have

$$\overline{\mathbb{Q}}_x \left[\frac{\mathbb{P}_{\xi_{t_0}} (N_{t-t_0} > 0)}{\varphi(\xi_{t_0})}\right] \lesssim f(t-t_0)$$

where the implied constant depends only on D. Combining all of the above, and noting that S_{t_0} is independent of the motion under $\overline{\mathbb{Q}}_x$ with $\overline{\mathbb{Q}}_x [S_{t_0}] = 2\lambda t_0$ provides the final estimate

$$|\epsilon^2_x(t,t_0)| \leq \tilde{C} t_0 f(t-t_0) \sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)| \tag{2.14}$$

for all $t_0 > 1$, where $\tilde{C}$ is another constant. With both these error bounds in hand, we may deduce that

$$\left|\frac{\mathbb{P}_x (N_t > 0)/\varphi(x)}{f(t-t_0)} - 1\right| \leq \left|\frac{\epsilon^1_x(t,t_0)}{f(t-t_0)}\right| + \left|\frac{\epsilon^2_x(t,t_0)}{f(t-t_0)}\right| \leq C\,\mathrm{e}^{-\gamma t_0} + \tilde{C} t_0 \sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)|$$

for any $x \in D$, and $1 < t_0 < t$.

Now, since we know that $\sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)| \to 0$ as $s \to \infty$, it is possible to choose $t_0(t)$ such that both $t_0(t) \to \infty$ and $t_0(t) \sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)| \to 0$ as $t \to \infty$. Then we have, letting $c(t) = f(t - t_0(t))$, that

$$\left|\frac{\mathbb{P}_x (N_t > 0)/\varphi(x)}{c(t)} - 1\right| \to 0 \tag{2.15}$$

as $t \to \infty$, uniformly for $x \in D$. To complete the proof, we need only show that $c(t)$ must be asymptotically equivalent to $a(t) := \int_D \mathbb{P}_x (N_t > 0)\, \varphi(x)\, dx$ as $t \to \infty$. Note that $a(t)$ is less than $\sup_{w\in D} |\mathbb{P}_w (N_{t-t_0} > 0)|$, and so clearly decays with t. To see the equivalence, observe that

$$\left|\frac{a(t)}{c(t)} - 1\right| = \left|\int_D \left(\frac{\mathbb{P}_x (N_t > 0)}{c(t)} - \varphi(x)\right) \varphi(x)\, dx\right| \leq \left\|\frac{\mathbb{P}_x (N_t > 0)}{c(t)} - \varphi(x)\right\|_{L^2(D)} \tag{2.16}$$

where the inequality holds by Cauchy-Schwarz and the fact that $\int_D \varphi^2 = 1$. Multiplying (2.15) by $\varphi(x)$ and integrating tells us that the final expression converges to 0 as $t \to \infty$.

□

Remark 2.27. *The proof of this Lemma remains essentially the same in our more* *general framework, using Remark 2.23 in place of Lemma 2.24. Some more care is required to bound*

$$\overline{\mathbb{Q}}_x\left[S_{t_0}\right],$$

as multiple processes may now immigrate at each branching point on the spine, but since the size-biased distribution has finite mean (we are assuming that A has finite variance) this is again less than some constant times t_0.

2.4.3 Asymptotics for $a(t)$

Now, since by Lemma 2.20 we know that

$$\mathbb{P}_x\left(N_t > 0\right) := u(x,t)$$

is a solution of the FKPP equation in D, we can find an ODE which is satisfied by $a(t)$. This will allow us to deduce the desired asymptotic for $a(t)$ as $t \to \infty$. More precisely, we have the following:

Lemma 2.28. *Assume that D is C^1. Letting $a(t) = \int_D u(x,t)\varphi(x)\,dx$ we have that $a(t)$ is differentiable for all $t > 0$ and*

$$\frac{da}{dt}(t) = -\lambda \int_D u^2(x,t)\varphi(x)\,dx. \tag{2.17}$$

Proof. First suppose that $u(\cdot,t) \in H_0^1(D)$ and $\Delta u(\cdot,t) \in L^2(D)$ for all $t > 0$. Then, since

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta u(x,t) + \lambda(u(x,t) - u^2(x,t))$$

we see that $\frac{\partial u}{\partial t}(\cdot,s) \in L^2(D)$ for all strictly positive s, and

$$\int_D \frac{\partial u}{\partial t}(x,s)\varphi(x)\,dx = \int_D \left(\frac{1}{2}\Delta u(x,s) + \lambda(u(x,s) - u^2(x,s))\right)\varphi(x)\,dx$$

is well defined. Furthermore, we have that $\varphi \in H_0^1(D)$ (since ∂D is assumed C^1 and φ vanishes on the boundary - see [Eva98, §5.5]) and that $u \in H_0^1(D)$ by assumption. This means that we can integrate by parts against $\varphi \in H_0^1(D)$ and use that φ is an eigenfunction of the Laplacian to obtain the equality

$$\int_D \frac{du}{dt}(x,s)\varphi(x)\,ds = -\lambda \int_D u^2(x,s)\varphi(x)dx. \tag{2.18}$$

Observe that the left hand side is continuous in s (for $s > 0$) since the right hand side must be by continuity of u^2 and dominated convergence.

Hence, for any $t > 0$, letting $0 < a < t$ we see that

$$-\lambda \int_D u^2(x,t)\varphi(x)\,dx = \frac{d}{dt}\int_a^t \int_D -\lambda u^2(x,s)\varphi(x)\,dx = \frac{d}{dt}\int_a^t \int_D \frac{\partial u}{\partial t}(x,s)\varphi(x)\,dxds$$

by the continuity discussed above, and (2.18). Then, applying Fubini and using the continuity of $\frac{du}{dt}(x,s)$ in s for fixed x, we can write this as

$$\frac{d}{dt}\int_D \int_a^t \frac{\partial u}{\partial t}(x,s)ds\,\varphi(x)dx = \frac{d}{dt}\int_D u(x,t)\varphi(x) - u(x,a)\varphi(x)\,dx = \frac{da}{dt}(t).$$

Therefore, we need only show that $u \in H_0^1(D)$ and $\Delta u \in L^2(D)$. However, this is simply the regularity that u obtains by virtue of being a solution of (2.8). This is a straightforward consequence of the standard regularity theory for parabolic PDEs. Since the domain is C^1 and u vanishes on the boundary, it is enough to show that

$$\sum_{k=1}^{\infty} \lambda_k \langle f,\varphi_k\rangle^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k^2 \langle f,\varphi_k\rangle^2 < \infty \tag{2.19}$$

(see, for example, [Tho06, Lemma 3.1]). This can be proved using the Duhamel representation for u as a solution of (2.8) and a standard bootstrapping argument. We omit the straightforward calculations.

□

Remark 2.29. *For the more general branching diffusion we can apply the same arguments to show that*

$$\frac{da}{dt}(t) = -\frac{\lambda}{m-1}\int_D \left(G(1-u) + mu - 1\right)\varphi(x)\,dx \tag{2.20}$$

where G is the probability generating function of A. Continuity of the right-hand side requires an extra application of the dominated convergence theorem to first see that $G(1-u) = \mathbb{E}[(1-u)^A]$ is continuous. To bound the Sobolev norms in (2.19) one must observe that

$$\nabla(G(1-u)) = -\nabla u\,\mathbb{E}[A(1-u)^A]$$

is bounded in modulus by a constant times $|\nabla u|$.

This allows us to deduce an asymptotic for $a(t)$, which completes the proof of Theorem 2.4.

Proposition 2.30.

$$a(t) \underset{t\to\infty}{\sim} \frac{1}{\left(\lambda \int_D \varphi(y)^3\,dy\right) t}.$$

Proof. The desired asymptotic for $a(t)$ follows fairly easily from Lemma 2.28 since we have, writing $u(t,x) = a(t)\varphi(x) + v(t,x)$ and substituting this into (2.17), that

$$\frac{da}{dt}(t) = -\lambda a^2(t)\int_D \varphi(y)^3\,dy - 2\lambda a(t)\int_D v(t,x)\varphi^2(x)\,dx - \lambda\int_D v^2(t,x)\varphi(x)\,dx. \tag{2.21}$$

Then, Proposition 2.26 tells us that the second two terms are negligible compared with the first for large t. Indeed, since $|u(t,x)/a(t)\varphi(x) - 1| = |v(t,x)/a(t)\varphi(x)| \to 0$ uniformly in x, we have that

$$\frac{da}{dt}(t) = \left(-\lambda\int_D \varphi(y)^3\,dy - g(t)\right)a^2(t)$$

$$\text{for}\quad g(t) := 2\lambda\int_D \frac{v(t,x)}{a(t)}\varphi^2(x)\,dx + \lambda\int_D \frac{v(t,x)^2}{a(t)^2}\varphi(x)\,dx$$

where $g(t) \to 0$ as $t \to \infty$. Thus we obtain, denoting differentiation with respect to t by a dot, that

$$-\lambda\int_D \varphi(y)^3\,dy - |g(t)| \le \frac{\dot{a}(t)}{a^2(t)} \le -\lambda\int_D \varphi(y)^3\,dy + |g(t)|.$$

Moreover, since $\frac{da^{-1}}{dt}(t) = -\frac{\dot{a}(t)}{a^2(t)}$, integration yields that

$$\frac{1}{a(t)} \ge \left(\lambda\int_D \varphi(y)^3\,dy\right)(t-1) - \int_1^t |g(s)|\,ds + \frac{1}{a(1)}$$

$$\frac{1}{a(t)} \le \left(\lambda\int_D \varphi(y)^3\,dy\right)(t-1) + \int_1^t |g(s)|\,ds + \frac{1}{a(1)}$$

where we have started from 1 to avoid any differentiability issues at 0. Note that $\dot{a}(s)/a^2(s)$ is clearly integrable over $[1,t]$ for any t, and so we are justified in applying the fundamental theorem of calculus here. Upon dividing by $\left(\lambda\int_D \varphi(y)^3\,dy\right)t$ we see that

$$\left|\frac{\frac{1}{(\lambda\int_D \varphi(y)^3\,dy)t}}{a(t)} - 1\right| \le \frac{1}{t} + \frac{1}{(\lambda\int_D \varphi(y)^3\,dy)}\left(\frac{1}{a(1)t} + \frac{\int_0^t |g(s)|\,ds}{t}\right).$$

The first term in the brackets on the right hand side of this expression clearly converges to 0 as $t \to \infty$. Furthermore, since $|g|$ is bounded and $|g(s)| \to 0$ as $s \to \infty$, so does the second. This yields the result. □

Remark 2.31. *Note that this Proposition, combined with the proof Theorem 2.3,* *in particular (2.15), shows that*

$$\left| \frac{t\lambda \int_D \varphi(y)^3\, dy \times \mathbb{P}_x(N_t > 0)}{\varphi(x)} - 1 \right| \longrightarrow 0$$

as $t \to \infty$, uniformly in x.

Remark 2.32. *For the more general set up, observe that*

$$G(1-u) + mu - 1 = \frac{\mathbb{E}[A^2] - \mathbb{E}[A]}{2} u^2 + o(u^2)$$

by Taylor's theorem and our moment assumption on the offspring distribution A (see [DN80, Theorem A] for a general statement concerning Taylor expansions of probability generating functions). Replacing the expression for $\frac{da}{dt}$ by (2.20) and proceeding as above, we may incorporate the error term from the integral into $g(t)$, and reach the desired conclusion.

2.5 The Conditioned System

Theorem 2.4 allows us to study the law of branching Brownian motion conditioned to survive for a long time in much greater depth. One aspect of the limiting behaviour is captured by what happens to the law of the process run up to some fixed time T, if it is then conditioned to survive until a much larger time t. It turns out that this limiting description is given precisely by the evolution of the process under $\mathbb{Q}_x$, as described in Lemma 2.22.

Proof of Proposition 2.5. Recall, we would like to prove that for any $T \geq 0$, $x \in D$ and $B \in \mathcal{F}_T$, we have that

$$\lim_{t\to\infty} \mathbb{P}_x\left(B | N_t > 0\right) = \mathbb{Q}_x(B).$$

Conditioning on $\mathcal{F}_T$, we see that

$$\mathbb{P}_x\left(B|N_t > 0\right) = \frac{\mathbb{E}_x\left[\mathbb{1}_B \mathbb{P}_x\left(N_t > 0 | \mathcal{F}_T\right)\right]}{\mathbb{P}_x\left(N_t > 0\right)} := \frac{\mathbb{E}_x\left[\mathbb{1}_B Y\right]}{\mathbb{P}_x\left(N_t > 0\right)}$$

where we have defined

$$Y := \mathbb{P}_x\left(N_t > 0 | \mathcal{F}_T\right) = \sum_{i=1}^{N_T} \mathbb{P}_{X_T^i}\left(N_{t-T} > 0\right) \left(\prod_{j<i} \mathbb{P}_{X_T^j}\left(N_{t-T} = 0\right) \right).$$

Then, from our asymptotic for the survival probability and the fact that $\frac{t}{t-T} \to 1$ as $t \to \infty$, it follows that

$$\frac{\mathbb{1}_B Y}{\mathbb{P}_x (N_t > 0)} \underset{t\to\infty}{\to} \frac{\sum_{i=1}^{N_T} \varphi(X_T^i)}{\varphi(x)} \mathbb{1}_B = \frac{M_T}{M_0} \mathbb{1}_B$$

almost surely, as $t \to \infty$. Moreover, we have that $Y \le \sum_{i=1}^{N_T} \mathbb{P}_{X_T^i} (N_{t-T} > 0) \lesssim \frac{M_T}{t-T}$ for all large enough t (recalling that our asymptotic estimates were uniform in D), and so we can dominate $\mathbb{1}_B Y / \mathbb{P}_x (N_t > 0)$ by an integrable random variable, namely a constant multiple of M_T. The dominated convergence theorem then provides the result.

□

Given the asymptotic for the survival probability, it is also not too much work to prove Theorem 2.6, which gives some limiting information on the positions of particles at time t, given survival. Recall (since we are working in the branching Brownian motion case) we would like to show that for any f with $\langle f^2, \varphi \rangle < \infty$ that we have

$$(t^{-1} \sum_{i=1}^{N_t} f(X_t^i) | N_t > 0) \to Z$$

in distribution as $t \to \infty$, where $Z \sim \text{Exp}(1/\lambda \langle \varphi, f \rangle \int_D \varphi(y)^3 \, dy)$. To prove Theorem 2.6, we will use the method of moments, relying on Theorem 2.4 and the following Lemma.

Lemma 2.33. *For all $n \in \mathbb{N}$ and $x \in D$,*

$$\frac{\mathbb{E}_x \left[(\sum_{i=1}^{N_t} \varphi(X_t^i))^n \right]}{n! \varphi(x) \lambda^{n-1} (\int_D \varphi^3)^{n-1} t^{n-1}} \to 1$$

as $t \to \infty$, uniformly in x.

Proof of Lemma 2.33. The proof of this relies on the expression for $\mathbb{E}_x[(\sum_{i=1}^{N_t} \varphi(X_t^i))^n]$ that one obtains from a Many-To-Few generalisation of the Many-to-Two Lemma, see below, and then proceeds by induction. We start with $n = 1$. This case is simple, however, because we know that

$$\left| \mathbb{E}_x \left[\frac{\sum_{i=1}^{N_t} \varphi(X_t^i)}{\varphi(x)} \right] - 1 \right| = 0$$

for all t and x, since the expectation is that of our mean 1 martingale M_t / M_0.

We also record here that

$$\left|\frac{\mathrm{e}^{\lambda t}\,\mathbf{E}_x\left[\varphi^2(B_t)\mathbb{1}_{\{t>\tau_D\}}\right]}{\varphi(x)\int_D\varphi^3}-1\right| = \frac{1}{\int_D\varphi(y)^3\,dy}\int_D\frac{e^{\lambda t}p^D(t,x,y)\varphi^2(y)}{\varphi(x)}-\varphi(y)^3\,dy \tag{2.22}$$

$$\leq \sup_{y\in D}\left|\frac{K^D(t,x,y)}{\varphi(y)^2}-1\right|$$

which we know is less than $C\,\mathrm{e}^{-\gamma t}$ for all x and $t \geq 1$ say, where C is some universal constant. This fact will be crucial to the induction.

For the inductive step, we need a Many-to-Few Lemma, which is a generalisation of Lemma 2.13. This tells us, as a special case, that

$$\mathbb{E}_x[(\sum_{i=1}^{N_t}\varphi(X_t^i))^n] = \mathrm{e}^{\lambda t}\,\mathbf{E}_x\left[\varphi(B_t)^n\mathbb{1}_{\{\tau_D>t\}}\right] + \sum_{j=1}^{n-1}\binom{n}{j}\int_0^t\lambda\,\mathrm{e}^{\lambda s}\,\mathbf{E}_x\left[\mathbb{1}_{\{\tau_D>s\}}E^j(t-s,B_s)\right]\,ds \tag{2.23}$$

where $E^j(s,x) := \mathbb{E}_x[(\sum_{i=1}^{N_s}\varphi(X_s^i))^{n-j}]\mathbb{E}_x[(\sum_{i=1}^{N_s}\varphi(X_s^i))^j]$, and is proved in a very general setting in [HR15, Lemma 1]. Thus, for $n > 1$ we can break up

$$\left|\frac{\mathbb{E}_x\left[(\sum_{i=1}^{N_t}\varphi(X_t^i))^n\right]}{n!\varphi(x)\lambda^{n-1}(\int_D\varphi^3)^{n-1}t^{n-1}}-1\right|$$

into n parts, each corresponding to one of the terms in (2.23). Since we know that $\mathrm{e}^{\lambda t}\,\mathbf{E}_x\left[\varphi(B_t)^n\mathbb{1}_{\{\tau_D>t\}}\right] \leq \|\varphi\|_\infty^{n-1}\varphi(x)$, the first of these terms will tend to 0 uniformly in x as $t\to\infty$. Thus we need only show that for each $1\leq j\leq n-1$, we have

$$\left|\frac{\int_0^t\mathrm{e}^{\lambda s}\,\mathbf{E}_x\left[\mathbb{1}_{\{\tau_D>s\}}E^j(t-s,B_s)\right]\,ds}{j!(n-j)!\varphi(x)\lambda^{n-2}(\int_D\varphi^3)^{n-1}t^{n-1}}-\frac{1}{n-1}\right|\to 0 \tag{2.24}$$

uniformly in x as $t\to\infty$. In the following we set

$$Q_y^k(r) := \frac{\mathbb{E}_y\left[(\sum_{i=1}^{N_r}\varphi(X_r^i))^k\right]}{k!\lambda^{k-1}(\int_D\varphi^3)^{k-1}r^{k-1}} \tag{2.25}$$

for $y \in D$, $k \in \mathbb{N}$ and $r \geq 0$ so that (2.24) is less than

$$\left| \frac{\int_0^t \mathrm{e}^{\lambda s}(t-s)^{n-2}\mathbf{E}_x \left[\mathbb{1}_{\{\tau_D > s\}} \left(Q_{B_s}^{n-j}(t-s) Q_{B_s}^{j}(t-s) - \varphi^2(B_s) \right) \right] ds}{t^{n-1}\varphi(x) \int_D \varphi^3} \right| + \left| \frac{\int_0^t \mathrm{e}^{\lambda s}\, \mathbf{E}_x \left[\varphi^2(B_s) \mathbb{1}_{\{\tau_D > s\}} \right] (t-s)^{n-2}\, ds}{t^{n-1}\varphi(x) \int_D \varphi^3} - \frac{1}{n-1} \right|. \quad (2.26)$$

We will show that both the terms here converge to 0, uniformly in x.

The second expression is relatively easy to deal with, since $\int_0^t (t-s)^{n-2}/t^{n-1} = 1/(n-1)$. This means that we can pull the $1/(n-1)$ term, and then the modulus, inside of the integral. After doing this, and noting that $(t-s)^{n-2}/t^{n-1} \leq 1/t$ for all $s \in [0,t]$, we see that the term is bounded above by

$$\frac{1}{t} \int_0^t \left| \frac{\mathrm{e}^{\lambda s}\, \mathbf{E}_x \left[\varphi^2(B_s) \mathbb{1}_{\{\tau_D > s\}} \right]}{\varphi(x) \int_D \varphi^3} - 1 \right| ds.$$

From here the result follows by (2.22). Note that the bound $\mathbf{E}_x[\varphi^2(B_s)\mathbb{1}_{\{\tau_D > s\}}] \leq \mathrm{e}^{-\lambda s}\, \varphi(x) \|\varphi\|_\infty$ (which holds for all s and x) tells us that the integrand is uniformly bounded in x.

For the first expression, we use our induction hypothesis. This tells us that $Q_z^{n-j}(r) Q_z^j(r)/\varphi(z)^2 \to 1$ uniformly in z as $r \to \infty$. It is also clear from the definition (2.25) that for fixed n and j, $r^{n-2} Q_z^{n-j}(r) Q_z^j(r)$ is uniformly bounded on any compact interval $[0,T]$ (for example by bounding the expectation in (2.25) by the corresponding expectation for a Yule process and using that φ is bounded). With this in mind we bound (2.26) above by

$$\frac{\sup_z \sup_{r \geq T} |Q_z^{n-j}(r) Q_z^j(r)/\varphi(z)^2 - 1|}{t \int_D \varphi(y)^3\, dy} \int_0^{t-T} \mathrm{e}^{\lambda s}\, \frac{\mathbf{E}_x \left[\mathbb{1}_{\{\tau_s > 0\}} \varphi^2(B_s) \right]}{\varphi(x)}\, ds + \frac{\sup_z \sup_{r \leq T} (|r^{n-2} Q_z^{n-j}(r) Q_z^j(r)| + |r^{n-2}\varphi(z)^2|)}{t^{n-1} \int_D \varphi(y)^3\, dy} \int_{t-T}^t \mathrm{e}^{\lambda s}\, \frac{\mathbf{E}_x \left[\mathbb{1}_{\{\tau_D > s\}} \right]}{\varphi(x)}\, ds$$

where we have used that $(t-s)^{n-2}/t^{n-1} \leq 1/t$ for the first part. Since

$$e^{\lambda s} \frac{\mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}]}{\varphi(x)} = \int_D \frac{e^{\lambda s} p^D(s,x,y)}{\varphi(x)\varphi(y)} \varphi(y)\, dy$$

is the expectation of $\varphi(\xi_s)$ for ξ a Brownian motion conditioned to remain in D, and therefore bounded uniformly in s and x, we obtain the convergence to 0 by letting first $T \to \infty$ and then $t \to \infty$ in the above. □

Proof of Theorem 2.6. Lemma 2.33 combined with Theorem 2.4 tells us that for each $n > 0$,

$$\mathbb{E}_x[(t^{-1}\sum_{i=1}^{N_t}\varphi(X_t^i))^n | N_t > 0] \to n!\left(\lambda\int_D \varphi^3\right)^n$$

as $t \to \infty$, where the right hand side is the nth moment of an $\mathrm{Exp}\left(1/\lambda\int_D\varphi^3\right)$ random variable. Since convergence of the moments is enough to ensure convergence in distribution when the limiting distribution is nice enough, see for example [Bil95, Theorem 30.2], Theorem 2.6 is proved in the case $f = \varphi$. To deal with general f we write $\tilde{f} = f - \langle\varphi, f\rangle\,\varphi$. We will show that for any $\varepsilon > 0$

$$\mathbb{P}_x(|t^{-1}\sum_{i=1}^{N_t}\tilde{f}(X_t^i)| > \varepsilon | N_t > 0) \to 0$$

as $t \to \infty$ (uniformly in x), which implies the result by the above decomposition and the proof when $f = \varphi$. To do this we use Markov's inequality and the Many-to-Two Lemma. This Lemma tells us that $\mathbb{E}_x[(t^{-1}\sum_{i=1}^{N_t}\tilde{f}(X_t^i))^2|N_t > 0]$ is equal to

$$\frac{\varphi(x)}{t\mathbb{P}_x(N_t > 0)}\left(\frac{\mathrm{e}^{\lambda t}\,\mathbf{E}_x[\tilde{f}(B_t)^2\mathbb{1}_{\{\tau_D > t\}}] + 2\int_0^t\lambda\,\mathrm{e}^{\lambda s}\,\mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}\mathbb{E}_{B_s}[\sum_{i=1}^{N_{t-s}}\tilde{f}(X_{t-s}^i)]^2]\,ds}{t\varphi(x)}\right)$$

where we know that the expression outside the brackets is uniformly bounded in t and x (for t large enough). We also know that

$$\frac{\mathrm{e}^{\lambda t}\,\mathbf{E}_x[\tilde{f}(B_t)^2\mathbb{1}_{\{\tau_D > t\}}]}{t\varphi(x)} \le t^{-1}\left(1 + \sup_y\left|\frac{K^D(t,x,y)}{\varphi(y)^2} - 1\right|\right)(\varphi, \tilde{f}^2) \tag{2.27}$$

and so this term converges uniformly to 0 by the assumption on f. To conclude we use the fact that $(\varphi, \tilde{f}) = 0$, which was the reason for choosing $\tilde{f}$ as we did. This means that $\sup_z\varphi(z)^{-1}\mathbb{E}_z[\sum_{i=1}^{N_r}\tilde{f}(X_r^i)] \to 0$ as $r \to \infty$, by the Many-to-One Lemma and the same argument used for the bound in (2.27). Thus we have that

$$\begin{aligned}
&(t\varphi(x))^{-1}\int_0^t\lambda\,\mathrm{e}^{\lambda s}\,\mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}\mathbb{E}_{B_s}[\sum_{i=1}^{N_{t-s}}\tilde{f}(X_{t-s}^i)]^2]\,ds\\
&\le \frac{\int_{t-T}^t\lambda\,\mathrm{e}^{\lambda s}\,\mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}\mathbb{E}_{B_s}[\sum_{i=1}^{N_{t-s}}\tilde{f}(X_{t-s}^i)]^2]\,ds}{t\varphi(x)}\\
&+ \sup_{r\ge T}\sup_z\left(\varphi(z)^{-1}\mathbb{E}_z[\sum_{i=1}^{N_r}\tilde{f}(X_r^i)]\right)^2\frac{\int_0^{t-T}\lambda\,\mathrm{e}^{\lambda s}\,\mathbf{E}_x[\varphi(B_s)^2\mathbb{1}_{\{\tau_D > t\}}]\,ds}{t\varphi(x)}
\end{aligned}$$

where the second term on the right can be made arbitrarily small for all $t \ge T$ as long as T is large enough (using the bound $\mathrm{e}^{\lambda s}\,\mathbf{E}_x[\varphi(B_s)^2\mathbb{1}_{\{\tau_D > t\}}] \le \varphi(x)\|\varphi\|_\infty$ on the integrand). Therefore, we are left with having to show that, for fixed T, the first term converges to 0 (uniformly in x) as $t \to \infty$. However, we can apply the Many-to-one Lemma and Cauchy-Schwarz to the $\mathbb{E}_{B_s}[\cdot]$ term in the integral, and then the Markov property, to see that

$$\begin{aligned}\mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}\mathbb{E}_{B_s}[\sum_{i=1}^{N_{t-s}} \tilde{f}(X^i_{t-s})]^2] &\le \mathbf{E}_x[\mathbb{1}_{\{\tau_D > s\}}\, \mathrm{e}^{2\lambda(t-s)}\, \mathbf{E}_{B_s}[\tilde{f}^2(\tilde{B}_{t-s})] \\ &\le \mathrm{e}^{2\lambda(t-s)}\, \mathbf{E}_x[\mathbb{1}_{\{\tau_D > t\}}\tilde{f}(B_t)^2]\end{aligned}$$

where the process $\tilde{B}$ in the middle expression is just an independent Brownian motion. Using the assumption that $(\varphi, \tilde{f}^2) < \infty$ and (2.27) once more, the result follows from computing the integral.

□

Remark 2.34. *We note here that Lemma 2.33 would also hold if we started the initial particle in a random position, with density φ^2. This follows directly from the proof, since everything is uniform in x. In fact, since the asymptotic for the survival probability is also uniform in the starting point by Remark 2.31, we have that Theorem 2.6 holds when $f = \varphi$ and we start the system in this random initial position. Of course this also holds for other initial configurations, but we will only use this one in the proof.*

We conclude by explaining how one can obtain Corollary 2.7 from here, which describes the asymptotic distribution of a particle picked at random from the population, given survival.

Proof of Corollary 2.7 . To prove the Corollary we first show that for any f with $(\varphi, f^2) < \infty$

$$\mathbb{P}_x\left(\left|\frac{\sum_{i=1}^{N_t} f(X^i_t)}{\sum_{i=1}^{N_t} \varphi(X^i_t)} - (\varphi, f)\right| > \varepsilon \,\middle|\, N_t > 0\right) \to 0 \tag{2.28}$$

as $t \to \infty$. Defining $\tilde{f}$ as in the proof of Theorem 2.6, this is equal to

$$\begin{aligned}\mathbb{P}_x\left(\left|\frac{t\sum_{i=1}^{N_t} \tilde{f}(X^i_t)}{t\sum_{i=1}^{N_t} \varphi(X^i_t)}\right| > \varepsilon \,\middle|\, N_t > 0\right) &\le \mathbb{P}_x\left(t^{-1}\sum_{i=1}^{N_t} \tilde{f}(X^i_t) > \delta | N_t > 0\right) \\ &\quad + \mathbb{P}_x\left(t^{-1}\sum_{i=1}^{N_t} \varphi(X^i_t) < \delta/\varepsilon | N_t > 0\right)\end{aligned} \tag{2.29}$$

for any $\delta > 0$. From the proof of Theorem 2.6, if we take a limit as $t \to \infty$ on the right hand side, we are left with simply the probability that an exponential random variable is less than δ/ε. Taking $\delta \to 0$ proves (2.28). The Corollary then follows by applying the above with both f and the constant function 1, and writing

$\sum f(X_t^i)/N_t = \sum f(X_t^i)/\sum \varphi(X_t^i) \times \sum \varphi(X_t^i)/N_t$. $\square$

Corollary 2.35. *For f as in Corollary 2.7, and any $\varepsilon > 0$, we have that*

$$\mathbb{P}_x\left(\left|\frac{\sum_{i=1}^{N_t} f(X_t^i)}{\sum_{i=1}^{N_t} \varphi(X_t^i)} - (\varphi, f)\right| > \varepsilon \,\middle|\, N_t > 0\right) \to 0 \tag{2.30}$$

as $t \to \infty$, uniformly in the starting position x.

Proof. Note that the proof of Theorem 2.6 immediately gives us convergence of the first term in (2.29) to 0 as $t \to \infty$, for any $\delta > 0$, uniformly in x. Therefore, it is sufficient for us to prove that there exists a function $g(\delta)$ converging to 0 as $\delta \downarrow 0$, such that for every $\eta > 0$ there exists a T with

$$\mathbb{P}_x\left(t^{-1}\sum_{i=1}^{N_t} \varphi(X_t^i) < \delta/\varepsilon | N_t > 0\right) \leq g(\delta) + \eta$$

for all $t \geq T$ and all $x \in D$. To do this, we change measure to $\mathbb{Q}_x$, and use the fact that the spine particle under $\overline{\mathbb{Q}}_x$ converges uniformly to an equilibrium distribution with density φ^2. By definition of the change of measure, and Cauchy-Schwarz, we have that

$$\mathbb{P}_x\left(t^{-1}\sum_{i=1}^{N_t} \varphi(X_t^i) < \delta/\varepsilon | N_t > 0\right) \tag{2.31}$$
$$\leq \frac{\varphi(x)}{t\mathbb{P}_x(N_t > 0)}\mathbb{Q}_x\left[\left(\frac{1}{\sum_{i=1}^{N_t} \varphi(X_t^i)}\right)^2\right]^{1/2} \mathbb{Q}_x\left(\frac{\sum_{i=1}^{N_t} \varphi(X_t^i)}{t} \leq \delta/\varepsilon\right)^{1/2}$$

which is in turn less than

$$\frac{\varphi(x)}{t\mathbb{P}_x(N_t > 0)}\overline{\mathbb{Q}}_x\left[1/\varphi(\xi_t)^2\right]^{1/2}\overline{\mathbb{Q}}_x\left(\mathbb{Q}_{\xi_u}\left(\frac{\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i)}{t} \leq \delta/\varepsilon\right)\right)^{1/2}$$

where ξ_s is the position of the spine particle at time s, under $\overline{\mathbb{Q}}_x$. The second inequality holds simply by positivity of φ, and the fact that $\mathbb{Q}_x = \overline{\mathbb{Q}}_x$ on $\mathcal{F}_t$, the filtration generated by the process, but not the position of the spine, up to time t. Now, the first two terms in the product (2.31) do not depend on δ, and are uniformly bounded in t and x, for $t > 1$ say. This follows from Remark 2.31 (uniform asymptotic for the survival probability), and Lemma 2.24 (convergence of the spine.)

The final term of the product is also, by Lemma 2.24 again, less than or equal to

$$\left(Ce^{-\gamma u} + \int_D \varphi(y)^2 \mathbb{Q}_y \left(\frac{\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i)}{t} \leq \delta/\varepsilon \right) dy \right)^{1/2} \tag{2.32}$$

for all $u > 1$ say, where C does not depend on x, u or t. By changing measure back to $\mathbb{P}_y$ in the integrand, and again applying Cauchy-Schwarz, we see that the second term in the square root in (2.32) above is bounded by

$$\int_D \varphi(y)^2 \Bigg(\frac{\mathbb{P}_y(N_{t-u>0})\mathbb{E}_y[(\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i))^2 | N_{t-u} > 0]^{1/2}}{\varphi(y)} \times \\ \mathbb{P}_y \left(\frac{\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i)}{t-u} \leq \frac{t}{t-u} \delta/\varepsilon \,\middle|\, N_{t-u} > 0 \right)^{1/2} \Bigg) dy.$$

However we know that $\varphi(y)^{-1}\mathbb{P}_y(N_{t-u>0})\mathbb{E}_y[(\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i))^2|N_{t-u} > 0]^{1/2}$ is uniformly bounded in y as long as $t - u > 1$ say (using the moments we calculated in Lemma 2.33 and the asymptotic for the survival probability). Since everything is positive we can take this bound outside of the integral. Furthermore, by Remark 2.34 we know that if we let $g'(\delta)$ be the square root of the probability that an exponential random variable with mean $\lambda \int_D \varphi(y)^3 \, dy$ is less than δ/ε, then $g'(\delta) \downarrow 0$ as $\delta \downarrow 0$, and

$$\int_D \varphi(y)^2 \mathbb{P}_y \left(\frac{\sum_{i=1}^{N_{t-u}} \varphi(X_{t-u}^i)}{t-u} \leq \frac{t}{t-u} \delta/\varepsilon \,\middle|\, N_{t-u} > 0 \right)^{1/2} dy \longrightarrow g'(\delta)$$

as $t \to \infty$, for any fixed u. Putting all of this together proves the result. □

We will use this to prove a stronger version of Corollary 2.7. We know by the Corollary that the average value of $f(v)$ among all vertices v at large height in one tree, given survival, converges to $\langle f, \varphi \rangle / \langle 1, \varphi \rangle$. The next Lemma will tell us that in fact we need only look at the average over a large enough subset of these vertices. This will be helpful to us for the proof of Theorem 2.1.

Lemma 2.36. *Let f be as in Corollary 2.7. Then for any $\varepsilon, \rho > 0$ and $x \in D$*

$$\mathbb{P}_x\left(B_t(\rho) | N_t > 0\right) := \mathbb{P}_x \left(\bigcup_{\rho t \leq M \leq N_t} \left\{ \left| \frac{\sum_{i=1}^M f(X_t^i)}{M} - \frac{\langle f, \varphi \rangle}{\langle 1, \varphi \rangle} \right| > \varepsilon \right\} \,\middle|\, N_t > 0 \right) \to 0$$

as $t \to \infty$.

Recall here from Section 2.2.1 how we have chosen to enumerate the particles in the system.

Proof. We will prove the Lemma by looking at the tree conditioned to survive until time t, and dividing the set of vertices at time t into families, depending on whether or not they have the same ancestor at some earlier time. This earlier time will be chosen such that with high probability, the average value of f over the positions of any one of these families is close to $\langle f, \varphi\rangle / \langle 1, \varphi\rangle$. This will show that at many places along the tth generation (vertices at height t) the average of f, taken over the positions of all previous vertices at this height, is close to what we want. To extend this to all vertices far enough along the tth generation, we will show that the size of each of these families is very small compared to t.

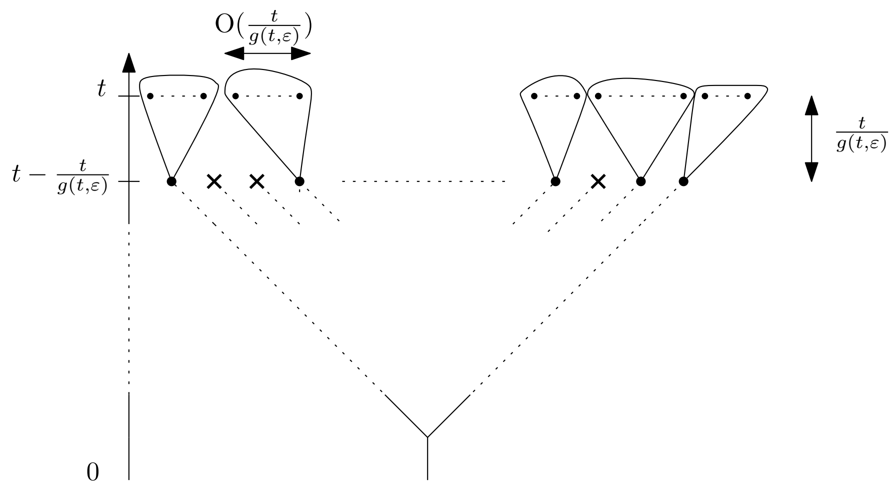

Figure 20: Sketch of the argument. There are $O(g(t,\varepsilon))$ particles at time $t-t/g(t,\varepsilon)$ with descendants at time t (marked with dots). Each of these families is likely to be *good* and has size $O(t/g(t,\varepsilon))$.

To do this, fix $\varepsilon > 0$ and write

$$p(t,\varepsilon) := \sup_{x\in D} \mathbb{P}_x \left(\left| \frac{\sum_{i=1}^{N_t} f(X_t^i)}{N_t} - \frac{\langle f,\varphi\rangle}{\langle 1,\varphi\rangle} \right| > \varepsilon/2 \,\middle|\, N_t > 0 \right),$$

which by Corollary 2.35, converges to 0 as $t \to \infty$. This means that we can choose a function $g(t,\varepsilon)$ such that $g(t,\varepsilon) \to \infty$ as $t \to \infty$, but

$$g(t,\varepsilon)\, p\left(\frac{t}{g(t,\varepsilon)}, \varepsilon\right) \to 0 \tag{2.33}$$

as $t \to \infty$. Indeed, since $\sup\{p(u,\varepsilon); u \geq t\}$ converges monotonically to 0 as $t \to \infty$, you can choose $g(t,\varepsilon)$ less than $\sqrt{t}$ but still converging to ∞, such that $g(t,\varepsilon)\sup\{p(u,\varepsilon); u \geq \sqrt{t}\} \to 0$ as $t \to \infty$. Then since $p(t/g(t,\varepsilon),\varepsilon) \leq \sup\{p(u,\varepsilon); u \geq t/g(t,\varepsilon)\} \leq \sup\{p(u,\varepsilon); u \geq \sqrt{t}\}$, the function g will satisfy (2.33).

As mentioned above, we will break up the particles at time t into families.

Two vertices will be in the same family if they have a common ancestor at time $t - t/g(t, \varepsilon)$. We let the number of these families be $\hat{N}_{t-t/g(t,\varepsilon)}$ and set m_i to be the average value of f among the ith family. Here the order of the families corresponds to the order of the ancestors at time $t-t/g(t, \varepsilon)$. The key to the proof of this Lemma will be to show that

$$\mathbb{P}_x \left(\bigcup_{i=1}^{\hat{N}_{t-t/g(t,\varepsilon)}} \left\{ \left| m_i - \frac{\langle f, \varphi \rangle}{\langle 1, \varphi \rangle} \right| > \varepsilon/2 \right\} \middle| N_t > 0 \right) \to 0 \tag{2.34}$$

as $t \to \infty$. The reason for this is that there are order $g(t, \varepsilon)$ particles that have descendants at time t, and the probability that they are bad in the sense of (2.34) is less than $p(t/g(t, \varepsilon))$ by definition. Then (2.33) provides the result.

To prove this rigorously however, it is more convenient to consider the unconditioned version of the probability in (2.34), noting that the event clearly does not occur if $N_t = 0$. To analyse this probability, we condition on the total collection of particles at time $t - t/g(t, \varepsilon)$. There are order 1 of these, uniformly in t by Markov's inequality (i.e. for any $\delta > 0$ there exists a K such that $\mathbb{P}_x(N_{t-t/g(t,\varepsilon)} > K) \leq \delta$ for all t.) Moreover, for any one of them, the probability that it

(a) has a descendant at time t, and

(b) the average value of f over all its descendants at time t is more than $\varepsilon/2$ away from $(f, \varphi)/(1, \varphi)$

is less than some constant times $\frac{g(t,\varepsilon)}{t} p\left(t/g(t, \varepsilon), \varepsilon\right)$. This follows from the definition of p and the asymptotic for the survival probability. Multiplying by t to account for the conditioning in (2.34) and applying (2.33) gives (2.34).

The upshot of (2.34) is that we now know, letting σ_i be the number of particles in the ith family at time t, that

$$\mathbb{P}_x \left(A_t | N_t > 0 \right) := \mathbb{P}_x \left(\bigcup_{i=1}^{\hat{N}_{t-t/g(t,\varepsilon)}} \left\{ \left| \frac{\sum_{j=1}^{\sigma_1 + \cdots + \sigma_i} f(X_t^j)}{\sigma_1 + \cdots + \sigma_i} - \frac{\langle f, \varphi \rangle}{\langle 1, \varphi \rangle} \right| > \varepsilon/2 \right\} \middle| N_t > 0 \right) \to 0 \tag{2.35}$$

as $t \to \infty$ (observe that the families are clearly grouped together in the ordering of generation t.) This tells us that we have many "good" vertices along generation t, where the average value of f (considered over previously visited vertices) is close to what we want. To complete the proof, we must show that gaps between these good vertices, i.e. the lengths σ_i, are not too long.

To do this, we will prove that

$$\mathbb{P}_x\left(A_t'|N_t>0\right) := \mathbb{P}_x\left(\bigcup_{i=1}^{\hat{N}_{t-t/g(t,\varepsilon)}}\left\{\sigma_i > \frac{t}{(g(t,\varepsilon))^{1/3}}\right\}\middle| N_t>0\right) \to 0 \qquad (2.36)$$

as $t\to\infty$. For this we apply a similar argument to above. We consider the unconditioned probability and condition on the system at time $t-t/g(t,\varepsilon)$. By Markov's inequality there are order 1 particles at this time, uniformly in t, and for any one of them the probability that it

(a) has a descendant at time t, and

(b) the total number of descendants at time t is greater than $t/(g(t,\varepsilon))^{1/3}$

is less than some constant times

$$\frac{g(t,\varepsilon)}{t}\times g(t,\varepsilon)^{-4/3}.$$

The first term in the product comes from the asymptotic for the survival probability. The second comes from the fact that, given survival of a process to time $t/g(t,\varepsilon)$, the total number of particles is roughly $t/g(t,\varepsilon)$ times an exponential random variable by Theorem 2.6. We use Markov's inequality to get the explicit bound (uniformly in the starting point). Again multiplying by t to account for the conditioning gives (2.36).

Let us now show that $\mathbb{P}_x(B_t(\rho)|N_t>0)\to 0$ as $t\to\infty$. By the above work, and a union bound it is enough to show that

$$B_t(\rho)\subset A_t\cup A_t'$$

for all t large enough. Suppose we are on the event $\{A_t\cup A_t'\}^c$, and for every $\rho t\le M\le N_t$, set $k(M)=\sigma_1+\cdots+\sigma_i$, where i is such that $\sigma_1+\cdots+\sigma_i\le M\le\sigma_1+\cdots+\sigma_{i+1}$. Then

$$\left|\frac{\sum_{i=1}^{k(M)}f(X_t^i)}{k(M)}-\frac{\langle f,\varphi\rangle}{\langle 1,\varphi\rangle}\right|\le\varepsilon/2$$

for all $\rho t\le M\le N_t$ simultaneously However, since we are on the event $\{A_t'\}^c$ we also have

$$\left|\frac{\sum_{i=1}^{M}f(X_t^i)}{M}-\frac{\sum_{i=1}^{k(M)}f(X_t^i)}{k(M)}\right|\le 4\frac{\langle f,\varphi\rangle}{\langle 1,\varphi\rangle}\frac{1}{\rho(g(t,\varepsilon))^{1/3}}. \qquad (2.37)$$

Thus, for all t large enough, we must also be on the event $\{B_t(\rho)\}^c$.

□

Remark 2.37. *The proof of Theorem 2.6 directly extends to the case of a general branching diffusion, by applying a generalised version of the Many-to-Few Lemma, see [HR15, Lemma 1]. Then Corollary 2.7 and Lemma 2.36 follow as above.*

2.6 Convergence to the Brownian CRT

From this point onwards, we will assume that $\varphi \in C^1(\overline{D})$ as in the statement of Theorem 2.1. In particular this means that $|\nabla\varphi|$ is bounded on D.

Recall from Section 2.2.1 that we can consider our branching diffusions as continuous planar trees, where all the vertices are marked with a position. We write $\mathcal{P}_x$ for the law of a sequence of i.i.d critical branching Brownian motion trees, each starting at $x \in D$. We will explore these trees in a *depth-first* order. This exploration is defined as follows:

- We start at the root of the first tree and move upwards (i.e. increasing height) at speed one. Whenever we reach a branching point we take the left branch.
- When we can no longer continue, we jump instantaneously to the most recent branching point that we have visited.
- We then repeat the process, starting to explore along the right branch emanating from this point.
- Whenever we can no longer continue, we jump instantaneously to the most recent branch point that we have visited, but not already jumped up to.
- When we reach the end of the first tree, we jump instantaneously to the root of the next tree and repeat.

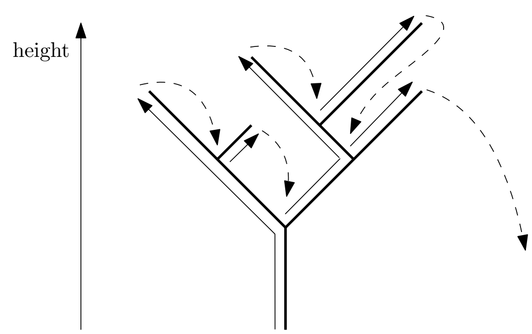

Figure 21: A sketch of the depth-first exploration of a continuous tree. Full arrowed lines represent motion at speed one in the vertical direction. Dotted arrowed lines represent instantaneous jumps.

This is the analogue of the *lexicographical* ordering for discrete trees. Recording the height of vertices as we traverse the trees in this way gives us the *height process*

H_t associated with the sequence. To show the convergence in Theorem 2.1 it will be important to show that this height process, when rescaled appropriately, looks like a reflected Brownian motion. To do this, we introduce a further process, S_t, which will turn out to be a martingale.

In the following, we will say that a vertex in the sequence of trees has been *visited* by time t if the exploration has passed through that point before time t. We will say that a vertex, that is also a branch point, has been *explored* by time t if it has been visited *and* jumped back to before time t. Recall that the mark associated with a vertex v corresponds to the spatial position of the particle it represents: we denote this by v^*. Finally, we write $Y(v)$ for the set of branch points that have been visited but not explored before the time that v is visited.

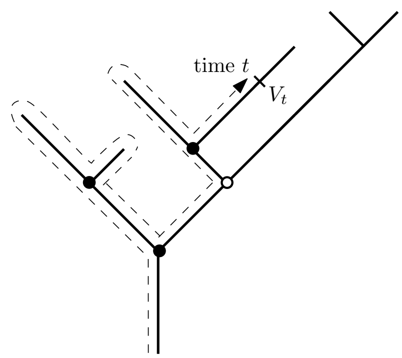

Figure 22: The exploration up to time t. Branch points with filled circles have been visited and explored before time t. Branch points with empty circles have been visited but not explored. Note that the branch point furthest to the right has neither been visited nor explored before time t. $Y(V_t)$ is the set of branch points with empty circles.

Definition 2.38. *Let V_t be the vertex that is visited at time t in the depth-first exploration, writing V_t^* as usual for its position in D. We define for $t \geq 0$*

$$S_t = \varphi(V_t^*) + \sum_{v \in Y(V_t)} \varphi(v^*) - \Lambda_t \varphi(x).$$

Here, Λ_t is the index of the tree being visited at time t.

In fact, S_t is very closely related to the martingale M_t. Essentially they are the same process, but explored in different orders, and we will see that this is enough to preserve the martingale property. We would like to approximate the height process by S_t, and then apply an invariance principle for the martingale. This is an analogous idea to that used to prove convergence of Galton-Watson processes to the CRT in [LGD02], where S_t here plays the role of the *Lukasiewicz path.* We first record a property of this process, which will be essential to showing a relationship with the height function:

Lemma 2.39. *Let $S'_t = S_t - \varphi(V_t^*)$ and $I'_t = \inf_{0\le s\le t} S'_s$. Then*

$$S'_t - I'_t = \sum_{v\in Y(V_t)} \varphi(v^*)$$

Proof. Since φ is positive, it is clear that $I'_t = -\Lambda_t\varphi(x)$. This implies the result. □

Definition 2.40. *The process on the right hand side in Lemma 2.39 also makes sense for a depth-first exploration of a single tree, and we denote such a process by $\hat{S}$. Note that this is a process that starts at 0 and is positive until the exploration of the tree is finished, i.e. an* excursion.

We also write $\hat{S}^y$ for $\hat{S}$ conditioned to reach level y. Then $\hat{S}^y$ is equal in law to $S'_t - I'_t$, restricted to the first excursion in which it exceeds y.

Observe that we can decompose S into continuous and discontinuous parts, S^c and S^d. Indeed, if we let S^d be the pure jump process that jumps up by $\varphi(v^*)$ whenever the exploration reaches a branch point vertex v, then $S^c = S - S^d$ is continuous. In fact, S^c is simply $\varphi(V_t^*)$ minus a compensating sum that makes it continuous. If the exploration reaches the end of a branch at time t, then $\lim_{s\uparrow t}\varphi(V_s^*) = 0$, but $\lim_{s\downarrow t}\varphi(V_s^*) > 0$. It is easy to verify that S^c is just $\varphi(V_t^*)$ with the jumps subtracted whenever they occur.

Remark 2.41. *For a general diffusion and branching mechanism, as in the statement of Theorem 2.1, we define the process S in almost the same way, setting φ to be the first eigenfunction of the generator as usual. In this case we will only say that a branch point has been explored when* all *of the subtrees branching from the point have been partially explored. If a branch point has been visited but not explored before time t we let $k_t(v)$ be the number of these subtrees that have* not been explored at all *before time t. We replace the sum $\sum_{v\in Y(V_t)}\varphi(v^*)$ in the definition of S by $\sum_{v\in Y(v_t)} k_t(v)\varphi(v^*)$. Lemma 2.39 then holds with this sum on the right hand side. When we decompose S into its continuous and discontinuous parts, the jumps are given by $(k-1)\varphi(v^*)$ whenever a branch point v with k branches is reached. The continuous part is again just $\varphi(V_t^*)$ with the jumps removed.*

We will return to this later, but let us now prove an invariance principle for S_t.

2.6.1 Martingale Convergence

Lemma 2.42. *Under $\mathcal{P}_x$, for any $x \in D$, $(S_t)_{t\ge 0}$ is a locally square-integrable martingale with respect to the natural filtration generated by the depth-first exploration.*

Its predictable quadratic variation is given by

$$\langle S\rangle_t = \int_0^t \lambda\varphi(V_s^*)^2 + |\nabla\varphi(V_s^*)|^2\, ds.$$

Proof. First observe that breaking up $S = S^d + S^c$ gives us the expression for the predictable quadratic variation. The first term comes from the discontinuous part, which we know jumps, at rate λ, by φ applied to the position of the vertex being visited. The second comes from the fact that we can write $dS_t^c = \varphi(V_t^*)dB_t$ for B a standard Brownian motion. This follows from the description of S_t^c, and the fact that the increments of $(V_t^*)_{t\geq 0}$ are equal in law to those of a Brownian motion.

To see that S is a martingale, we condition on the depth first exploration up to time s and notice that we can write

$$\mathcal{E}_x\left[S_t - S_s|\mathcal{F}_s\right] = \mathbb{E}_{V_s^*}[\hat{S}_\tau - \varphi(V_s^*)] + \sum_{v\in Y(V_s)} \mathbb{E}_{v^*}[\hat{S}_{v,\tau_v} - \varphi(v^*)] + \sum_{i=1}^{\infty} \mathbb{E}_x[\hat{S}_{i,\tau_i} - \varphi(x)]$$

by the Markov property, where $\hat{S}_\tau$, $(\hat{S}_{v,\tau_v})_{v\in Y(V_s)}$ and $(\hat{S}_{i,\tau_i})_{i\geq 1}$ are the processes $\hat{S}$ for the subtrees rooted at $v_s, \{v\}_{v\in Y(V_s)}$ and $\{v_i\}_{i\geq 1}$ (the roots of the remaining sequence of trees) respectively, each run up to a stopping time that *does not depend on that subtree* by conditional independence. Thus, to prove the martingale property it is enough to show that

$$\mathbb{E}_x[\hat{S}_t] = \varphi(x)$$

for any $x \in D$ and $t \geq 0$.

To do this, we will approximate $\hat{S}$ by a discrete version $(\hat{S}_n^\delta,\ n \in \mathbb{N})$, with $\delta \downarrow 0$. This process will be defined by discretising the tree using steps of size δ in the natural way. This results in a discrete tree where every vertex is marked with a position - corresponding to its spatial position in the original tree. To define $\hat{S}^\delta$, set $\hat{S}_0^\delta = \varphi(x)$ (where x is the starting position) and traverse the discrete tree in a depth-first, or lexicographical, order. If you are visiting a vertex with position y at step n, and it has children with positions $(z_j)_{1\leq j\leq J}$, then set $\hat{S}_{n+1}^\delta - \hat{S}_n^\delta = \sum_{j=1}^J \varphi(z_j) - \varphi(y)$. By considering the martingale M, it is clear that $\mathbb{E}_x[\hat{S}_n^\delta] = \varphi(x)$ for every n and δ. Moreover, for fixed $t \geq 0$ we have that $\hat{S}_{\lfloor t/\delta\rfloor}^\delta \to \hat{S}_t$ almost surely as $\delta \to 0$. This is clear after noting that:

- Almost surely the discrete tree only captures single branching events in any step, for all δ small enough.
- If vertex being visited at time $\lfloor t/\delta\rfloor$ in the discrete process corresponds to the vertex at time u in the continuous process, then $|u-t| \leq \delta(1+W_t)$, where W_t is

the total number of deaths plus branch points before time t in the exploration.

- The particle motion is almost-surely continuous.

Since the $\hat{S}^{\delta}_{\lfloor t/\delta \rfloor}$ are dominated, for example by $\|\varphi\|_{\infty}$ times the number of branch points in the continuous tree up to level t, the result follows by dominated convergence. $\square$

Remark 2.43. *If we have a generator $L = -\frac{1}{2}\sum_{i,j} a^{ij}\,\partial_{x_i x_j} + \sum_i b^i\,\partial_{x_i}$ as in Theorem 2.1 and an offspring distribution A with finite variance, then the same argument implies that S is a martingale. Letting λ, φ be the first eigenvalue/eigenfunction pair for $-L$ we can calculate that the predictable quadratic variation of S is given by*

$$\langle S \rangle_t = \int_0^t \frac{\lambda}{m-1}\mathbb{E}[A^2 - A]\varphi(V^*_s)^2 - \lambda\varphi(V^*_s)^2 + \sum_{i,j}\frac{\partial\varphi}{\partial x_i}\frac{\partial\varphi}{\partial x_j}a^{i,j}(V^*_s)\,ds.$$

The second term here comes from the single particle motion under L and the first comes from the jumps (recall the critical branching rate in this case is $\lambda/(m-1)$).

Proposition 2.44. *Let*

$$\sigma^2 = \frac{2\lambda\int_D \varphi(y)^3\,dy}{\langle 1, \varphi\rangle}.$$

Then

$$\left(\frac{S_{nt}}{\sqrt{n}}\right)_{t\geq 0} \to (\sigma B_t)_{t\geq 0}$$

in distribution as $n \to \infty$, with respect to the Skorohod topology.

Proof. Since the jumps of S are bounded, this follows from the functional central limit theorem for martingales [JS87, Theorem 3.22, Chapter VIII] once we can show that for all $t \geq 0$

$$\langle S^n \rangle_t \to \sigma t$$

in probability as $n \to \infty$. Here $(S^n_t)_{t\geq 0} = (S_{nt}/\sqrt{n})_{t\geq 0}$. However, we can write

$$\frac{\langle S^n\rangle_t}{t} = \frac{1}{nt}\int_0^{nt} \lambda\varphi(V^*_s)^2 + |\nabla\varphi(V^*_s)|^2\,ds.$$

Then, since φ and $|\nabla\varphi|$ are bounded, this follows immediately from Proposition 2.45 below. We recover σ^2 by integrating by parts to see that $\left\langle \lambda\varphi(\cdot)^2 + |\nabla\varphi(\cdot)|^2, \varphi\right\rangle = 2\lambda\int_D \varphi(y)^3\,dy$. $\square$

Proposition 2.45. *Suppose that f is a bounded, measurable function and let $(V^*_s)_{0\leq s\leq t}$ be the positions of vertices visited in the depth-first exploration before* *time t as usual. Then*

$$Q_t := \frac{1}{t}\int_0^t f(V_s^*)\,ds \to \frac{\langle f,\varphi\rangle}{\langle 1,\varphi\rangle}$$

in $\mathcal{P}_x$-probability as $t\to\infty$.

Remark 2.46. *In the general case (assuming Proposition 2.45) we can integrate by parts, using that φ is an eigenvector of L and L is self-adjoint, to see that $\langle S^n\rangle_t \to \sigma t$ as $n\to\infty$, where*

$$\sigma^2 = \frac{\lambda\mathbb{E}[A^2-A]\int_D \varphi(y)^3\,dy}{(m-1)\,\langle 1,\varphi\rangle}.$$

Since the jumps are not necessarily bounded in this case, to apply the FCLT for martingales we need to verify a Lindenberg-Feller type condition (for example, [JS87, (3.23),Theorem 3.22, Chapter VIII].) However, this follows immediately from the fact that we are assuming the offspring distribution to have finite variance. Then Proposition 2.44 holds.

Before we prove Proposition 2.45, let us record some of the consequences of Proposition 2.44.

Proposition 2.47. *Let Λ_t be the index of the tree being visited in the exploration at time t (so $\Lambda_0 = 1$). Then we have the joint convergence*

$$\left(\frac{S'_{nt}-I'_{nt}}{\sqrt{n}},\frac{\Lambda_{nt}}{\sqrt{n}}\right)_{t\geq 0} \to \left(\sigma|\beta_t|,\frac{\sigma}{\varphi(x)}L_t^0(\beta)\right)_{t\geq 0} \tag{2.38}$$

as $n\to\infty$, in distribution with respect to the Skorohod topology. Here, β is a standard Brownian motion started at 0 and $L_t^0(\beta)$ is the local time of β at 0. Furthermore, for any $y>0$, we have

$$\left(\frac{\hat{S}_{nt}^{y\sqrt{n}}}{\sqrt{n}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} \left(\sigma e_t^{y/\sigma}\right)_{t\geq 0} \tag{2.39}$$

where $e^{y/\sigma}$ is a Brownian excursion conditioned to reach height y/σ.

Proof. From the definition of S', we have that $|S'_{nt}/\sqrt{n} - S_{nt}/\sqrt{n}| \leq \|\varphi\|_\infty/\sqrt{n}$ for all $t\geq 0$ and so Proposition 2.44 implies that

$$\left(\frac{S'_{nt}}{\sqrt{n}}\right)_{t\geq 0} \to (\sigma B_t)_{t\geq 0}$$

as $n\to\infty$ as well. Writing $\underline{B}_t = \inf_{0\leq s\leq t} B_s$ this implies the joint convergence

$$\left(\frac{S'_{nt}-I'_{nt}}{\sqrt{n}},-\frac{I'_{nt}}{\sqrt{n}}\right)_{t\geq 0} \xrightarrow[n\to\infty]{(d)} (\sigma(B_t-\underline{B}_t),-\sigma\underline{B}_t)_{t\geq 0} \tag{2.40}$$

where the right-hand side by Lévy's theorem, see for example [RY91, Chapter VI, Theorem VI.2.3], is equal in distribution to

$$\left(\sigma|\beta_t|, \sigma L_t^0(\beta)\right)_{t\geq 0}.$$

However, we know that $\Lambda_t = -I_t'/\varphi(x)$, and so (2.38) follows. For the second claim of the Proposition, we follow [LGD02, Proposition 2.5.2]. It is well known that you can construct the process $e^{y/\sigma}$ from a standard Brownian motion β by taking

$$e_t^{y/\sigma} = |\beta_{(G+t)\wedge D}|$$

where $T = \inf\{t \geq 0 : |\beta_t| \geq y/\sigma\}$, $G = \sup\{t \leq T : \beta_t = 0\}$ and $D = \inf\{t \geq T : \beta_t = 0\}$. By the Skorohod representation theorem and (2.38) we also know that there exists a process

$$\left(Z_t^{(n)}, \Lambda_t^{(n)}\right)_{t\geq 0} \overset{(d)}{=} \left(\frac{S_{nt}' - I_{nt}'}{\sqrt{n}}, \frac{\Lambda_{nt}}{\sqrt{n}}\right)_{t\geq 0}$$

such that

$$\left(Z_t^{(n)}, \Lambda_t^{(n)}\right)_{t\geq 0} \underset{n\to\infty}{\longrightarrow} \left(\sigma|\beta_t|, \frac{\sigma}{\varphi(x)} L_t^0(\beta)\right)_{t\geq 0}$$

uniformly on every compact set almost surely. This is because Skorohod convergence is equivalent to local uniform convergence when the limit is continuous. Define $T^{(n)} = \inf\{t \geq 0 : Z^{(n)} \geq y\}$ for this sequence of processes, and $G^{(n)}$, $D^{(n)}$ in the same way as G and D above. By the remark in Definition 2.40 and (2.38), if we can prove that $G^{(n)} \to G$ and $D^{(n)} \to D$ almost surely, we will be done. First note that since β must exceed x/σ immediately after time T, we have that $T^{(n)} \to T$ almost surely. This implies straight away that for all $t < D$ we have $t \leq D^{(n)}$ for all n large enough almost surely. Now we must show that for all $t > D$ we have $t \geq D^{(n)}$ for all n large enough almost surely. These facts together (along with the corresponding results for G) are enough to prove the convergence. To see the final claim, we use the convergence of the local time. For any $t > D$, we have using basic properties of Brownian local time that $L_t^0 > L_D^0 = L_T^0$. The convergence of the local time therefore tells us that $\Lambda_t^{(n)} > \frac{\sigma}{\varphi(x)} L_T^0$ for all n large enough almost surely. Since

$$\Lambda_{T^{(n)}}^{(n)} \to \frac{\sigma}{\varphi(x)} L_T^0$$

almost surely, this implies that we have also have $\Lambda_t^{(n)} > \Lambda_{T^{(n)}}^{(n)}$ for all n large enough almost surely. Using the fact that $\Lambda^{(n)}$ stays constant on $[T^{(n)}, D^{(n)})$, we see that $t \geq D^{(n)}$.

$\square$

Remark 2.48. *The above proof is not affected by changing the generator or offspring distribution, since it relies only on the convergence from Proposition 2.44.*

The rest of this subsection will be devoted to the proof of Proposition 2.45, but we will first need some preliminary estimates. We write $h(v)$ for the height of a vertex v in the the exploration.

Lemma 2.49. *Let L be the total length of a branching Brownian motion process (i.e. how long it takes to traverse the tree in depth-first order). Then for all $x \in D$*

$$\mathbb{P}_x(L > t) \gtrsim \frac{\varphi(x)}{\sqrt{t}}.$$

Let U_t be a vertex picked uniformly at random from those visited before time t in the depth-first exploration under $\mathcal{P}_x$. Write U_t^* as usual for the position in D corresponding to its mark. Finally, denote by $\overline{\mathcal{P}}_x$ the law on the sequence of trees plus U_t. Then we have the following control on the height of the uniform vertex.

Lemma 2.50. *For all $x \in D$*

$$\overline{\mathcal{P}}_x\left(h(U_t) \geq C\sqrt{t}\right) \to 0$$

as $C \to \infty$, uniformly in t.

We will now show how we may deduce Proposition 2.45, and then go on to prove the Lemmas. Let N_s^i be the number of particles at level s in the ith tree of our exploration. Also write $(X_s^{i,j})_{1 \leq j \leq N_s^i}$ for the positions of these particles. Finally, let $I(t)$ be the index of the tree in which the uniform vertex lies.

Proof of Proposition 2.45. We first show that $\mathcal{E}_x[Q_t] \to \frac{\langle f,\varphi \rangle}{\langle 1,\varphi \rangle}$ as $t \to \infty$. Let m_t be the average value of f among the vertices at height $h(u_t)$ of the $I(t)$th tree, that have been visited before time t. Observe that by conditioning on m_t, we have

$$\mathcal{E}_x[Q_t] = \overline{\mathcal{E}}_x[f(U_t^*)] = \overline{\mathcal{E}}_x[m_t]$$

as $t \to \infty$. This is because, given the positions of these particles, we know that u_t is chosen uniformly from them. We will aim to show that for fixed $\varepsilon > 0$,

$$\overline{\mathcal{P}}_x\left(A_t\right) := \overline{\mathcal{P}}_x\left(\left|m_t - \frac{\langle \varphi, f \rangle}{\langle \varphi, 1 \rangle}\right| > \varepsilon\right) \to 0 \tag{2.41}$$

as $t \to \infty$. Then since m_t is bounded, the convergence in expectation will follow.

For the ith tree in our exploration and $s > 0$, let $m^i_{s,t}$ be the average value of f among the vertices of tree i at height s that are visited before time t, and $\tilde{N}^i_{s,t}$ be the number of such vertices. Also write $A^i_{s,t}$ for the event that $|m^i_{s,t} - \langle \varphi, f\rangle / \langle \varphi, 1\rangle | > \varepsilon$. Then, conditioning on the entire sequence of trees, we can write our probability as

$$\frac{1}{t}\mathcal{E}_x \left[\int_0^\infty \sum_{i=1}^\infty \mathbb{1}_{\{i \le \Lambda_t\}} \mathbb{1}_{A^i_{s,t}} \tilde{N^i_{s,t}}\, ds, \right]$$

since the probability of picking a vertex in tree i at height ds is $\tilde{N}^i_s/t\, ds$. Note that we can interchange the integral, expectation and sum as we like here by Fubini (the expression being bounded by 1.) Now, given $\delta > 0$, by Lemma 2.50 we can choose C such that $\mathbb{P}(h(U_t) \ge C\sqrt{t}) < \frac{\delta}{3}$ for all t. Similarly we can define $R(t) > 0$ such that $\overline{\mathcal{P}}_x(h(U_t) \le R(t)) = \frac{\delta}{3}$. Note that $R(t) \to \infty$ as $t \to \infty$ by the law of large numbers. Indeed, for any $K > 0$ the proportion of time spent below height K in the exploration is less than or equal to $\sum_{i=1}^{\Lambda_t} p^i_K/t$ where p^i_K is the time spent below level K in the ith tree. We also have that t is greater than or equal to $\sum_{i=1}^{\Lambda_t - 1} L^i$ where L^i is the length of the ith tree. However, p^i_K has finite variance and L^i does not, and so the strong law of large numbers allows us to conclude.

Since $\mathbb{1}_{A_i^{s,t}} \le 1$ this tells us that

$$\overline{\mathcal{P}}_x\left(A_t\right) \le \frac{2\delta}{3} + \frac{1}{t}\mathcal{E}_x \left[\int_{R(t)}^{C\sqrt{t}} \sum_{i=1}^\infty \mathbb{1}_{\{i \le \Lambda_t\}} \mathbb{1}_{A^i_{s,t}} \tilde{N}^i_s\, ds \right] \tag{2.42}$$

where by Fubini we can rewrite the expectation term as

$$\frac{1}{t}\int_{R(t)}^{C\sqrt{t}} \sum_{i=1}^\infty \mathcal{E}_x \left[\mathbb{1}_{\{i \le \Lambda_t\}} \mathbb{1}_{A^i_{s,t}} \tilde{N}^i_{s,t} \right] ds. \tag{2.43}$$

Now for each i, we condition on $\mathcal{G}_{i-1}$: the σ-algebra generated by the first $i-1$ trees. Note that $\mathcal{G}_{i-1}$ is independent of the ith tree. Moreover, the event $\{i \le \Lambda_t\}$ and the start time τ_i of the ith tree, are measurable with respect to $\mathcal{G}_{i-1}$. Thus we can write

$$\mathcal{E}_x \left[\mathbb{1}_{\{i \le \Lambda_t\}} \mathbb{1}_{A^i_{s,t}} \tilde{N}^i_{s,t} \middle| \mathcal{G}_{i-1} \right] = \mathbb{1}_{\{i \le \Lambda_t\}} \mathbb{E}_x \left[\mathbb{1}_{A_s(t-\tau_i)} N_s(t-\tau_i) \middle| N_s > 0 \right] \mathbb{P}_x(N_s > 0)$$

where the expectation is now with respect to a single branching Brownian motion tree, $N_s(r)$ is the number of particles at level s before time r in a depth-first exploration of the tree, and $A_s(r)$ is the event that the average of f among these particles is more than ε away from $\langle \varphi, f\rangle / \langle \varphi, 1\rangle$.

We note here, by Theorems 2.4 and 2.6, that there exists a K such that

$$\mathbb{E}_x\left[N_s^2\middle|\, N_s>0\right]^{1/2}\leq Ks,\quad \mathbb{P}_x(N_s>0)\leq K/s \tag{2.44}$$

for all s. We can also, by Lemma 2.49, choose this K such that

$$\mathcal{E}_x\left[\Lambda_t\right]\leq K\sqrt{t} \tag{2.45}$$

for all t. Indeed, $\mathcal{E}_x[\Lambda_t]$ is less than the expectation of geometric random variable, whose success rate is $\mathbb{P}_x(L>t)\gtrsim 1/\sqrt{t}$. Decomposing on whether or not $N_s(t-\tau_i)$ is bigger than $s\delta/6CK^2$ we have

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{1}_{A_s(t-\tau_i)}N_s(t-\tau_i)\middle|\, N_s>0\right]&\leq\frac{\delta s}{6CK^2}+\mathbb{E}_x\left[\mathbb{1}_{B_s(\delta/6CK^2)}N_s(t-\tau_i)\middle|\, N_s>0\right]\\&\leq\frac{\delta s}{6CK^2}+\mathbb{E}_x\left[N_s^2\middle|\, N_s>0\right]^{1/2}\mathbb{P}_x\left(B_s(\delta/6CK^2)\middle|\, N_s>0\right)^{1/2}\end{aligned}$$

where $B_s(\cdot)$ is the event from Lemma 2.36. Using (2.44), and the conditioning above, we have that

$$\mathcal{E}_x\left[\mathbb{1}_{\{i\leq\Lambda_t\}}\mathbb{1}_{A^i_{s,t}}\tilde{N}^i_{s,t}\right]\leq\left(\frac{\delta}{6CK}+K^2\mathbb{P}_x\left(B_s(\delta/6CK^2)\middle|\, N_s>0\right)^{1/2}\right)\mathcal{E}_x[\{i\leq\Lambda_t\}]. \tag{2.46}$$

This means that the expression in (2.43) is less than

$$\left(\frac{\delta}{6CK}+K^2\sup_{s\geq R(t)}\left\{\mathbb{P}_x\left(B_s(\delta/6K^2)\middle|\, N_s>0\right)^{1/2}\right\}\right)\frac{1}{t}\int_{R(t)}^{C\sqrt{t}}\sum_{i=1}^{\infty}\mathcal{E}_x\left[\mathbb{1}_{\{i\leq\Lambda_t\}}\right].$$

However, by Fubini and (2.45), this is less than

$$\frac{\delta}{6}+CK^3\sup_{s\geq R(t)}\left\{\mathbb{P}_x\left(B_s(\delta/6K^2)\middle|\, N_s>0\right)^{1/2}\right\}.$$

Using Lemma 2.36, and the fact that $R(t)\to\infty$, we see than this is less than $\frac{\delta}{3}$ for all t large enough. Substituting in to (2.42) proves (3.29).

To complete the proof of the Proposition, we must also show that

$$\mathcal{E}_x[Q_t^2]\to\frac{\langle f,\varphi\rangle^2}{\langle 1,\varphi\rangle^2}$$

as $t\to\infty$. However, letting U_t^1 and U_t^2 be two vertices picked independently uniformly from those visited before time t (again denoting the extended law by $\overline{\mathcal{E}}_x$) we see that

$$\mathcal{E}_x\left[Q_t^2\right]=\overline{\mathcal{E}}_x\left[f(U_t^{1*})f(U_t^{2*})\right].$$

Let m_t^i and A_t^i correspond to m_t and A_t for U_t^i, $i = 1, 2$. Then the same reasoning as above tells us that

$$\overline{\mathcal{E}}_x \left[f(U_t^{1*})f(U_t^{2*})\right] = \overline{\mathcal{E}}_x \left[m_t^1 m_t^2\right]$$

and we also have, by a union bound, that

$$\overline{\mathcal{P}}_x \left[A_t^1 \cup A_t^2\right] \to 0$$

as $t \to \infty$. The result follows in exactly the same way.

□

Proof of Lemma 2.49. We will prove this result using our asymptotic for the survival probability. Since we have

$$\mathbb{P}_x(L > t) \geq \mathbb{P}_x(L > t \,|\, N_{\sqrt{at}} > 0)\, \mathbb{P}_x(N_{\sqrt{at}} > 0) \tag{2.47}$$

and we know that the probability of survival until time $\sqrt{at}$ decays like $\varphi(x)/\sqrt{at}$, it is enough to show there exists an a such that $\mathbb{P}_x(L > t | N_{\sqrt{at}} > 0)$ is bounded below, uniformly in t. In fact, we will prove a slightly stronger statement, as it will also be of use later on. We will show that

$$\mathbb{P}_x(L \leq cs^2 | N_s > 0) \to 0 \tag{2.48}$$

as $c \to 0$, uniformly in s. Clearly this is enough to prove the Lemma.

To do this, we first fix some compact subdomain $D' \subset D$. Then, we condition on the positions X_s^i of the particles at time s that lie within D', and consider the subprocesses continuing from these points. In particular, we consider the contributions that each subprocess makes to L from its first μs generations. Note that, given the positions X_s^i, these are independent random variables, with distribution equal to that of $\int_0^{\mu s} N_u \, du$ for a branching Brownian motion started at X_s^i. This means that they all have mean $\geq C\mu s$ and variance $\leq C' \mu^3 s^3$ for some fixed $C, C' > 0$. The statement concerning the mean follows from the expression for $\mathbb{E}_y[N_u]$, which is bounded below by Lemma 2.33 for all y in the subdomain and all $u > 0$ by some C. For the variance, note that for $r < u$

$$\mathbb{E}_y[N_r N_u] = \mathbb{E}_y\left[N_r \mathbb{E}_x\left[N_u | \mathcal{F}_r\right]\right] \leq C'' \mathbb{E}_y[N_r^2]$$

for some $C'' > 0$ (not depending on r, u or y). Since $[\mathbb{E}_y[N_r^2] \lesssim r$ uniformly in y, the claim follows by integration.

By Theorem 2.6 we also know that for any $\delta > 0$ there exist m and S, such that the probability of having less than ms vertices at time s in D' (conditioned on survival) is less than δ for all $s \geq S$. On the complementary event, conditioned on the tree up to time s, we have ms independent random variables with mean $\geq C\mu s$ and variance $\leq C'\mu^3 s^3$. By the standard application of Markov's inequality to sums of independent random variables, the probability that their sum is more than $m\mu Cs^2/2$ away from its mean is less than $4C'\mu/C^2 m$ for all $s \geq S$. Since the mean is greater than $m\mu Cs^2$, this has to occur for the sum to be less than $m\mu Cs^2/2$. Taking μ to 0, and noting that L is greater than this sum, shows that the left hand side of (2.48) can be made less than δ as long as c is small enough and s is large enough. In fact, we can choose c small enough that this will hold for all s, due to the continuity in s of the probability in (2.48) (which you can prove by dominated convergence.) Thus the claim is proved. $\square$

Proof of Lemma 2.50. As explained in the proof of Proposition 2.45, Lemma 2.49 immediately implies that $\mathcal{E}_x[\Lambda_t] \lesssim \sqrt{t}$. Then using our asymptotic for the survival probability, we have

$$\overline{\mathcal{P}}_x\left(h(U_t) \geq C\sqrt{t}\right) \leq \mathcal{E}_x\left[\sum_{i=1}^{\infty} \mathbb{1}_{\{i \leq \Lambda_t\}} \mathbb{1}_{\{N^i_{C\sqrt{t}} > 0\}}\right] \lesssim \frac{1}{C\sqrt{t}} \mathcal{E}_x[\Lambda_t]$$

where the implied constant does not depend on C, x, t.

$\square$

Remark 2.51. *The proofs of Proposition 2.45 and Lemmas 2.49 and 2.50 do not need any adaptation for the more general set up.*

2.6.2 Connection with the height function

In order to make use of the above invariance principle, we must connect the martingale with the height function of our trees. In this section we will look at the height function, and the process $\hat{S}$ from Definition 2.40, for the depth-first exploration of a single tree conditioned to be large. We know by Lemma 2.39 that the value of the $\hat{S}$ when it visits a vertex v in the tree is equal to

$$\hat{S}(v) := \sum_{u \in Y(v)} \varphi(u^*).$$

We will show that for vertices with large heights, this sum is close to a constant times the height. Our approach will use an ergodicity property for the spine particle in the system under $\overline{\mathbb{Q}}_x$, and is inspired from [HR14].

In the following, given $\eta > 0$, we will say that a vertex v in a branching Brownian motion tree is *η-bad* if

$$\left| \frac{\hat{S}(v)}{h(v)} - \lambda \int_D \varphi(y)^3 \, dy \right| > \eta. \tag{2.49}$$

We also say, for given $T \geq 0$, that a vertex v is *η_T-bad* if some ancestor of the vertex v at height greater than T is η-bad. Then we have the following estimate for the proportion of η_T-bad vertices:

Proposition 2.52. *Fix $\varepsilon, \eta > 0$ and write $N_t^{\eta_T}$ for the collection of η_T-bad vertices at time t. Then we have*

$$\mathbb{P}_x \left(\frac{N_t^{\eta_T}}{N_t} > \varepsilon \middle| N_t > 0 \right) \to 0 \tag{2.50}$$

as $T \to \infty$, uniformly in $t \geq T$, for any $x \in D$.

By this we mean that for any $x \in D$, given any $\delta > 0$, there exists T' large enough that $\mathbb{P}_x(N_t^{\eta_T}/N_t > \varepsilon | N_t > 0) \leq \delta$ for all $t \geq T \geq T'$.

Proof. We will first show that for any $\varepsilon > 0$

$$\mathbb{P}_x \left(E_{T,t}^{\varepsilon} | N_t > 0 \right) := \mathbb{P}_x \left(\frac{\sum_{i=1}^{N_t} \varphi(X_t^i) \mathbb{1}_{\{X_t^i \, \eta_T-\text{bad}\}}}{\sum_{i=1}^{N_t} \varphi(X_t^i)} > \varepsilon \middle| N_t > 0 \right) \to 0$$

as $T \to \infty$, uniformly in t. To do this, we will use the spine decomposition given by Lemma 2.22. Recalling the definition of $\mathbb{Q}_x$ from this section we see that the above probability is equal to

$$\mathbb{Q}_x \left[\frac{\varphi(x)/\mathbb{P}_x(N_t > 0)}{\sum_{i=1}^{N_t} \varphi(X_t^i)} \mathbb{1}_{E_{T,t}^{\varepsilon}} \right] := \mathbb{Q}_x \left[Y_t \mathbb{1}_{E_{T,t}^{\varepsilon}} \right].$$

To see that this converges to 0 it is enough to prove that:

- $\mathbb{Q}_x(E_{T,t}^{\varepsilon}) \to 0$ as $T \to \infty$, uniformly in $t \geq T$ and
- For every $\delta > 0$, there exists T' and K positive, such that $\mathbb{Q} \left(Y_t \mathbb{1}_{|Y_t| > K} \right) \leq \delta$ for all $t \geq T'$.

The first point comes from the fact that the *spine particle* under this new law is unlikely to be η_T-bad for large T. More precisely, recall that $\overline{\mathbb{Q}}_x$ is a law on branching processes equipped with a distinguished path, the *spine*, such that

$$\mathbb{Q}_x = \overline{\mathbb{Q}}_x \big|_{\mathcal{F}_t}$$

for $\mathcal{F}_t$ the filtration generated by the process but not the distinguished path. Under $\overline{\mathbb{Q}}_x$, we know that the spine particles evolves as a Brownian motion conditioned to remain in the domain for all time, and branches at constant rate 2λ, where each branch is either to the left or right of the spine (i.e. comes before or after in the depth-first ordering) with equal probability. Due to the mixing of this Brownian motion to a stationary distribution with density φ^2, see Lemma 2.24, ergodicity tells us that the probability of the spine vertex being η_T-bad at time t converges to 0 as $T \to \infty$, uniformly in $t \geq T$.

The connection between the motion of the spine and the event $E^{\varepsilon}_{T,t}$ comes from the fact that under $\overline{\mathbb{Q}}_x$, conditioned on $\mathcal{F}_t$, the spine particle is chosen proportionally to φ, see (2.11). Indeed, since

$$\sum_{i=1}^{N_t} \varphi(X_t^i) \mathbb{1}_{\{X_t^i \, \eta_T - \text{bad}\}} / \sum_{i=1}^{N_t} \varphi(X_t^i)$$

is positive, it is enough, for the first point, to show that its $\mathbb{Q}_x$ expectation converges to 0 (uniformly in $t \geq T$ as $T \to \infty$). However, this follows directly from the above and (2.11) since

$$\frac{\sum_{i=1}^{N_t} \varphi(X_t^i) \mathbb{1}_{\{X_t^i \, \eta_T - \text{bad}\}}}{\sum_{i=1}^{N_t} \varphi(X_t^i)} = \overline{\mathbb{Q}}_x \left(\text{spine} \, \eta_T\text{-bad at time} \, t | \mathcal{F}_t \right).$$

The second point essentially says that $(Y_t)_{t \geq 0}$ is $\mathbb{Q}_x$ uniformly integrable. To prove it, one can use the change of measure between $\mathbb{Q}_x$ and $\mathbb{P}_x$ again to write

$$\mathbb{Q}_x \left[Y_t \mathbb{1}_{\{|Y_t| > K\}} \right] = \frac{\mathbb{P}_x \left(\{|Y_t| > K\} \cap \{N_t > 0\} \right)}{\mathbb{P}_x(N_t > 0)} = \mathbb{P}_x \left(|Y_t| > K | N_t > 0 \right).$$

Since $\varphi(x)/t\mathbb{P}_x(N_t > 0)$ is uniformly bounded above for $t \geq 1$ say, by Theorem 2.4, we just need to show that for any $\delta > 0$ there exists K and T' such that

$$\sup_{t \geq T'} \mathbb{P}_x \left(\frac{\sum_{i=1}^{N_t} \varphi(X_t^i)}{t} < 1/K \middle| N_t > 0 \right) \leq \delta. \tag{2.51}$$

However, this is a direct consequence of the convergence given by Theorem 2.5, since we know that for fixed K the probability above converges, as $t \to \infty$, to the probability that an exponential random variable is less than $1/K$.

We must now deduce that

$$\mathbb{P}_x \left(\frac{N_t^{\eta_T}}{N_t} > \varepsilon \middle| N_t > 0 \right) \to 0$$

uniformly in $t \geq T$ as $T \to \infty$ from the fact that

$$\mathbb{P}_x(E^{\varepsilon}_{T,t} | N_t > 0) \to 0.$$

The idea behind this is that $\sum_{i=1}^{N_t} \varphi(X^i_t) \mathbb{1}_{\{X^i_t\ \eta_T - \text{bad}\}} / \sum_{i=1}^{N_t} \varphi(X^i_t)$ is a reasonable approximation to $N^{\eta_T}_t / N_t$ on survival at large times. By Corollary 2.7, we know that for any $\delta > 0$ there exist r and T' positive such that

$$\sup_{t \geq T'} \mathbb{P}_x \left(\frac{N^{D_r}_t}{N_t} > \frac{\varepsilon}{2} \middle| N_t > 0 \right) \leq \delta/2 \tag{2.52}$$

where $D_r = \{y \in D : \varphi(y) < 1/r\}$ and $N^{D_r}_t$ is the number of particles in D_r at time t. Also, write $N^{D^c_r, \eta_T}_t$ for the number of particles that are η_T-bad and lie in D^c_r at time t. Then

$$\frac{\sum_{i=1}^{N_t} \varphi(X^i_t)}{N_t} \leq \|\varphi\|_\infty \qquad \text{and} \qquad \frac{N^{D^c_r, \eta_T}_t}{\sum_{i=1}^{N_t} \varphi(X^i_t) \mathbb{1}_{\{X^i_t\ \eta_T - \text{bad}\}}} \leq r.$$

Bounding $N^{\eta_T}_t$ above by $N^{D_r}_t + N^{D^c_r, \eta_T}_t$, and choosing $T \geq T'$ large enough that $\mathbb{P}_x(E^{(2\|\varphi\|_\infty r)^{-1}\varepsilon}_{T,t}) \leq \delta/2$, we have that

$$\mathbb{P}_x \left(\frac{N^{\eta}_t}{N_t} > \varepsilon \middle| N_t > 0 \right) \leq \delta$$

for all $t \geq T$. □

Remark 2.53. *We can extend the proof to show that for any $c > 0$ and $x \in D$,*

$$\mathbb{P}_x \left(\{\frac{N^{\eta_T}_{ct}}{N_{ct}} > \varepsilon\} \cap \{N_{ct} > 0\} \middle| N_t > 0 \right) \to 0 \tag{2.53}$$

uniformly in $ct \wedge t \geq T$ as $T \to \infty$. This follows immediately from (2.50), since

$$\mathbb{P}_x \left(\{\frac{N^{\eta_T}_{ct}}{N_{ct}} > \varepsilon\} \cap \{N_{ct} > 0\} \middle| N_t > 0 \right) = \mathbb{P}_x \left(\{\frac{N^{\eta_T}_{ct}}{N_{ct}} > \varepsilon\} \cap \{N_{ct} > 0\} \middle| N_{ct} > 0 \right) \times \frac{\mathbb{P}_x(N_{ct} > 0)}{\mathbb{P}_x(N_t > 0)}$$

where $\mathbb{P}_x(N_{ct} > 0)/\mathbb{P}_x(N_t > 0)$ is uniformly bounded in $ct \wedge t \geq T$.

Remark 2.54. *For the more general set up, the above arguments do not need to be changed, except to replace $\lambda \int_D \varphi(y)^3\, dy$ in (2.49) by*

$$\frac{\lambda \mathbb{E}[A^2 - A]}{2(m-1)} \int_D \varphi(y)^3\, dy$$

where A is the offspring distribution and φ as usual becomes the first eigenfunction of the generator. This follows from Remark 2.23, since branching occurs along the spine at rate $(m/m-1)\lambda$ and the number of younger siblings at each such point has expectation $\mathbb{E}[(A^2 - A)/2]$.

The next Lemma provides the key connection between $\hat{S}$ and the height function, for a branching Brownian motion tree that is conditioned to survive for a long time.

Lemma 2.55. *Write $\overline{\mathbb{P}}_x$ for the law of a branching Brownian motion started at x, plus a vertex u chosen uniformly from it. Then for any $\eta > 0$ we have*

$$\overline{\mathbb{P}}_x\left(u \textit{ is } \eta_T\textit{-bad} \,|\, N_t > 0\right) \to 0,$$

uniformly in $t \geq T$ as $T \to \infty$.

Proof. By conditioning on N_s and $N_s^{\eta_T}$ for all $0 \leq s < \infty$ and L, which is the total length of the tree as usual, we see that

$$\overline{\mathbb{P}}_x\left(u \,\text{is}\, \eta_T\text{-bad} \,|\, N_t > 0\right) = \mathbb{E}_x\left[\int_0^\infty \frac{N_s^{\eta_T}}{N_s}\frac{N_s}{L}\,ds\middle|\, N_t > 0\right].$$

Thus we need to show that, given $\varepsilon > 0$, we have

$$\mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{|N_s^{\eta_T}/N_s| > \varepsilon/2\}}\frac{N_s}{L}\,ds\middle|\, N_t > 0\right] \leq \varepsilon/2$$

for all $t \geq T$, whenever T is large enough. First note that by (2.48), we can pick a c such that $\mathbb{P}_x\left(L \leq ct^2\middle|\, N_t > 0\right) \leq \varepsilon/8$ for all t, and also using our asymptotic for the survival probability, can choose an R such that $\overline{\mathbb{P}}_x\left(N_{Rt} > 0|\, N_t > 0\right) \leq \varepsilon/8$ for all t. Moreover, since

$$\mathbb{E}_x\left[\int_0^s N_s\,ds\middle|\, N_t > 0\right] \leq Mst$$

for some M and all s, t, we have that $\mathbb{E}_x[\int_0^{bt} N_s\,ds|N_t > 0] \leq Mbt^2$ and can therefore choose $b > 0$ small enough that this is less than $\varepsilon ct^2/8$. Combining this with the condition on L, our problem is reduced to showing that

$$\lim_{T\to\infty}\frac{1}{ct^2}\mathbb{E}_x\left[\int_{bt}^{Rt} \mathbb{1}_{\{|N_s^{\eta_T}/N_s| > \varepsilon/2\}} N_s\,ds\middle|\, N_t > 0\right] \leq \varepsilon/8$$

uniformly in $t \geq T$. However, by Remark 2.53 we know that

$$\sup_{bt\geq T}\sup_{s\in[bt,Rt]}\mathbb{P}_x\left(\{|N_s^{\eta_T}/N_s| > \varepsilon/2\} \cap \{N_s > 0\}|\, N_t > 0\right) \to 0$$

as $T \to \infty$ (the uniformity comes straight from the proof.) Hence, applying Fubini, Cauchy-Schwarz and integrating, similarly to in the proof of Proposition 2.45, we obtain the result. ☐

We will use the above to show that, if we choose k particles $(u_1, \cdots, u_k)$ uniformly from a branching Brownian motion tree conditioned to survive up to time t and define the matrices

$$(d_t^S(u_i, u_j))_{1 \le i < j \le k} = \frac{\hat{S}(u_i) + \hat{S}(u_j) - 2\hat{S}(v_{i,j})}{t}$$

and

$$(d_t^H(u_i, u_j))_{1 \le i < j \le k} = \frac{h(u_i) + h(u_j) - 2h(v_{i,j})}{t}$$

where $v_{i,j}$ is the most recent common ancestor of u_i and u_j, then the two are essentially the same up to the multiplicative constant $\lambda \int_D \varphi(y)^3 \, dy$. Note that d_t^H is actually how we define distances in the genealogical tree (after rescaling by t) and

$$\lambda \int_D \varphi(y)^3 \, dy = \frac{\sigma}{\alpha}$$

where α is the constant from the statement of Theorem 2.1.

Proposition 2.56. *Let d_t^S, d_t^H and $(u_1, \cdots, u_k)$ be defined as above. Then for any $\varepsilon > 0$*

$$\mathbb{P}\left(\|(\frac{\sigma}{\alpha} d_t^H(u_i, u_j) - d_t^S(u_i, u_j))\| > \varepsilon \,\middle|\, N_t > 0 \right) \to 0 \tag{2.54}$$

as $t \to \infty$, where the distance is the Euclidean distance between $k \times k$ matrices.

Proof. We prove this in the case $k = 2$, the general result following by a union bound. Lemma 2.55 tells us that if we pick a vertex u uniformly at random from a tree conditioned to survive to time t, then with high probability the value of $\hat{S}(u)/h(u)$ is close to σ/α. In fact, the probability that this holds at *all* points along u's ancestry (except for at very small times) is very large. Hence, if we pick any pair of vertices uniformly, since d_t^H and d_t^S depend only on their joint ancestry and the values of $\hat{S}$ along it, they will be close (when d_t^H is multiplied σ/α) with high probability.

To make this more precise, pick an $\eta > 0$. Then we can pick T large enough that the probability of two uniformly chosen vertices u_1, u_2 from the tree under B_n^H being η_T-bad is arbitrarily small, uniformly in $t \ge T$. We can also pick R large enough the probability that $[h(z_1) \wedge h(z_2), h(z_1) \vee h(z_2)] \subset [t/R, Rt]$ is arbitrarily close to 1, uniformly in t (see the proof of Lemma 2.55.)

Then, on the event that u_1 and u_2 are both η_T-good, and $[h(u_1) \wedge h(u_2), h(u_1) \vee$

$h(u_2)] \subset [t/R, Rt]$, we have, as long as $t \geq RT$

$$\left| \frac{\hat{S}(u_i)}{h(u_i)} - \sigma/\alpha \right| \leq \eta$$

for $i = 1, 2$. The other contributions to d_t^H and d_t^S come from the height (resp. the value of $\hat{S}$) at the most recent common ancestor u of the two vertices. There are two possibilities: either $h(u) \leq T$, or not. In the second case we have that

$$\left| \frac{\hat{S}_n(u)}{h(u)} - \sigma/\alpha \right| \leq \eta$$

and so

$$\left| d_t^S(u_1, u_2) - \frac{\sigma}{\alpha} d_t^H(u_1, u_2) \right| \leq 4R\eta.$$

In the first we have

$$\left| d_t^S(u_1, u_2) - \frac{\sigma}{\alpha} d_t^H(u_1, u_2) \right| \leq 2R\eta + \left(\frac{2\sigma}{\alpha} + \sup_{s \leq T} \frac{N_s}{s} \right) \frac{T}{t}.$$

However for fixed T, we have that $\mathbb{P}_x \left(\sup_{s \leq T} |N_s/s| \geq K \middle| N_t > 0 \right)$ converges to 0 as $K \to \infty$, uniformly in t (see Proposition 2.5). This proves the convergence in probability. □

Immediately from the proof, we also get the following Corollary, which we will need later on.

Corollary 2.57. *We also have for any $\varepsilon > 0$ and $c > 0$*

$$\mathbb{P} \left(\|(\frac{\sigma}{\alpha} d_{ct}^H(u_i, u_j) - d_{ct}^S(u_i, u_j))\| > \varepsilon \middle| N_t > 0 \right) \to 0 \qquad (2.55)$$

as $t \to \infty$, where the distance is the Euclidean distance between $k \times k$ matrices.

Remark 2.58. *The proofs of Lemma 2.55 and Proposition 2.56, given all the previous work, are exactly the same for the more general set up.*

2.6.3 Convergence to the CRT

2.6.4 Preliminaries on convergence of metric measure spaces

Before we can prove Theorem 2.1, we must introduce various notions of convergence for metric spaces, and more generally, for metric measure spaces. Although our aim is to prove convergence of conditioned genealogical trees in the sense of Gromov-Hausdorff distance between metric spaces, it turns out to be helpful to go through the framework of metric measure spaces. We first recall the definition of the Gromov-Hausdorff metric on $\mathbb{X}_c$: the space of (isometry classes of) compact metric spaces.

Definition 2.59. *The* Gromov-Hausdorff *distance between* (X, r_X) *and* (Y, r_Y) *in* $\mathbb{X}_c$ *is given by*

$$d_{GH}\left((X, r_X), (Y, r_Y)\right) = \inf_{g_X, g_Y, Z} d_H^{(Z, r_Z)}(g_X(X), g_Y(Y)),$$

where the infinum is taken over all isometric embeddings g_X, g_Y *from* X *and* Y *to a common metric space* (Z, r_Z)*, and* $d_H^{(Z, r_Z)}$ *is the usual Hausdorff distance on* (Z, r_Z)*.*

Now we will briefly discuss some modes of convergence for metric measure spaces, and how they are related, both with each other and the above. For us, a *metric measure space* (X, r, μ) will be a compact metric space, equipped with a finite Borel measure. These will be considered up to isometry, where $(X, r, \mu) \sim (X', r', \mu')$ if there exists a measure preserving isometry between X and X'. We denote the set of (isometry classes) of these spaces by $\mathbb{X}$. We will be interested in the *Gromov-Prohorov* metric and the *Gromov-Hausdorff-Prohorov* metric on $\mathbb{X}$. We will begin by defining the so-called *Gromov-Weak* topology.

Definition 2.60. *[GPW09, Definition 2.3] We will call a function* $\Phi : \mathbb{X} \to \mathbb{R}$ *a* polynomial *if there exists an* $k \in \mathbb{N}$ *and a bounded continuous function* $\phi : [0, \infty)^{\binom{k}{2}} \to \mathbb{R}$ *such that*

$$\Phi((X, r, \mu)) = \int \mu^{\otimes k}(d(x_1, \cdots, x_n))\phi((r(x_i, x_j))_{1 \leq i < j \leq k}),$$

where $\mu^{\otimes k}$ *is the product measure of* μ*. Write* Π *for the set of all polynomials.*

Definition 2.61. *[GPW09, Definition 2.8] A sequence* $\mathcal{X}_n \in \mathbb{X}$ *is said to converge to* $\mathcal{X} \in \mathbb{X}$ *with respect to the* Gromov-weak topology *if and only if* $\Phi(\mathcal{X}_n)$ *converges to* $\Phi(\mathcal{X})$ *in* $\mathbb{R}$*, for all polynomials* $\Phi \in \Pi$*.*

It was proved in [GPW09, Theorem 5] that this topology is metrised by the *Gromov-Prohorov metric* defined below.

Definition 2.62. *The* Gromov-Prohorov *distance between* $\mathcal{X} = (X, r_X, \mu_X)$ *and* $\mathcal{Y} = (Y, r_Y, \mu_Y)$ *in* $\mathbb{X}$ *is given by*

$$d_{GP}(\mathcal{X}, \mathcal{Y}) = \inf_{g_X, g_Y, Z} d_{Pr}^{(Z, r_Z)}((g_X)_*(\mu_X), (g_Y)_*(\mu_Y)),$$

where the infinum is as in Definition 2.59 and $d_P^{(Z, r_Z)}$ *is the Prohorov distance between probability measures on* (Z, r_Z)*.*

Finally, we define the Gromov-Hausdorff-Prohorov metric [ADH13],[Mie09] on $\mathbb{X}$.

Definition 2.63. *Let $\mathcal{X}, \mathcal{Y}$ be as in Definition 2.62. The* Gromov-Hausdorff-Prohorov *distance between $\mathcal{X}$ and $\mathcal{Y}$ is defined by*

$$d_{GHP}(\mathcal{X}, \mathcal{Y}) = \inf_{g_X, g_Y, Z} \left(d_{Pr}^{(Z, r_Z)}((g_X)_*(\mu_X), (g_Y)_*(\mu_Y)) + d_H^{(Z, r_Z)}(g_X(X), g_Y(Y)) \right).$$

Remark 2.64. *It is clear from the above definitions that convergence in the Gromov-Hausdorff-Prohorov metric implies convergence in both the Gromov-Hausdorff metric and the Gromov-Prohorov metric.*

We will need a couple of facts for our proof:

Lemma 2.65. *[GPW09, Corollary 3.1] A sequence $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ of probability measures on $\mathbb{X}$ converges weakly to a probability measure $\mathbb{P}$ with respect to the Gromov-weak topology, if and only if*

(i) The family $\{\mathbb{P}_n\}_{n\in\mathbb{N}}$ is relatively compact in the space of probability measures on $\mathbb{X}$.

(ii) For all polynomials $\Phi \in \Pi$, $\mathbb{P}_n[\Phi] \to \mathbb{P}[\Phi]$ in $\mathbb{R}$ as $n \to \infty$.

and

Lemma 2.66. *[ADH13, Theorem 2.4], [BBI01, Theorem 7.4.15] A set $\mathbb{K} \subset \mathbb{X}$ is relatively compact with respect to the Gromov-Hausdorff-Prohorov metric if and only if*

(i) There is a constant D such that $\operatorname{diam}(\mathcal{X}) < D$ *for all $\mathcal{X} \in \mathbb{K}$.*

(ii) For every $\delta > 0$ there exists $N = N_\delta$ such that for all $\mathcal{X} \in \mathbb{K}$, X can be covered by N_δ balls of radius δ.

(iii) $\sup_{\mathcal{X}\in\mathbb{K}} \mu_X(X) < +\infty$

2.6.5 Proof of the main theorem

In this section we will assume that our critical branching Brownian motion is always started from position $x \in D$. Recall that for any $y > 0$ we let $\mathcal{T}_n^y$ be the genealogical tree generated by this process, when it is conditioned to survive until time ny. This is just our usual tree, but with the marks forgotten. Denote by d_n^y the natural metric on this tree (given by the length of the path connecting two vertices by their most recent common ancestor). Finally, write $(\mathcal{T}_{e^y}, d_{e^y})$ for the real tree whose contour function is given by e^y, a Brownian excursion conditioned to reach height y. We will often want to put a *uniform measure* on these trees, to make them into metric measure spaces. In the case of $\mathcal{T}_n^y$ the measure will be denoted by μ_n^y and will be defined by the following procedure. Let $\phi : \mathcal{T}_n^y \to \mathbb{R}$ send the vertex visited at time t in a depth-first exploration of the tree to $t \in \mathbb{R}$. Then we can put a uniform measure on the image of ϕ (which will almost surely be a finite interval) and pull this back to the tree. Note that choosing a vertex from $\mathcal{T}_n^y$ according to μ_n^y is the same as choosing a vertex uniformly from the tree, as in Section 2.6.2. To define the uniform measure μ_{e^y} for $(\mathcal{T}_{e^y}, d_{e^y})$, recall that $\mathcal{T}_{e^y}$ is the real tree encoded by the excursion e^y. This means that the metric space is (isometric to) the interval on which the excursion is supported, quotiented by the equivalence relation that $s \sim r$ if $\inf\{e_u^y; u \in [r,s]\} = e_r^y = e_s^y$, with metric $d_{e^y}(r,s) = e_r^y + e_s^y - 2\inf\{e_u^y; u \in [r,s]\}$. The uniform measure μ_{e^y} is then just the quotient measure of uniform measure on the interval.

Recall that, since we are working in the case of binary branching Brownian motion, we would like to show that for any $y > 0$

$$(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{e^y}, d_{e^y})$$

in distribution, with respect to the Gromov-Hausdorff distance, where

$$\alpha = \sqrt{\frac{2}{\lambda \langle 1, \varphi \rangle \int_D \varphi(y)^3 \, dy}}.$$

Proof. An outline of the proof is as follows:

(1) We will show that if we make $(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y})$ into a metric measure space for each n, by equipping it with the uniform measure $\mu_n^{\alpha y}$, then the family

$$\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}, \mu_n^{\alpha y}\right)_{n \geq 0}$$

is tight with respect to the Gromov-Hausdorff-Prohorov metric.

(2) Using Propositions 2.47 and 2.56, we will show convergence of the above family, with respect to the Gromov-Prohorov metric, to $(\mathcal{T}_{e^y}, d_{e^y}, \mu_{e^y})$, where μ_{e^y} is uniform measure on the real tree $\mathcal{T}_{e^y}$, as defined above.

(3) This also characterises the subsequential limits with respect to the Gromov-Hausdorff-Prohorov metric, since Gromov-Hausdorff-Prohorov convergence implies Gromov-Prohorov convergence. Thus we have the convergence in distribution

$$\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}, \mu_n^{\alpha y}\right) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{e^y}, d_{e^y}, \mu_y)$$

with respect to the Gromov-Hausdorff-Prohorov metric. Consequently, by Remark 2.64, we have that

$$\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}\right) \underset{n\to\infty}{\longrightarrow} (\mathcal{T}_{e^y}, d_{e^y})$$

with respect to the Gromov-Hausdorff metric, as desired.

All that remains is to verify the statements in (1) and (2). For Part (1), we need to show that for any $\varepsilon > 0$ there exists a relatively compact $\mathbb{K} \subset \mathbb{X}$ (wrt the Gromov-Hausdorff metric) such that

$$\inf_n \mathbb{P}((\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}) \in \mathbb{K}) \geq 1 - \varepsilon.$$

We will use the characterisation given by Lemma 2.66. Since all of our measures are probability measures, condition (iii) of this characterisation is trivial. We begin by proving the existence of $K > 0$ such that

$$\sup_n \mathbb{P}\left(\text{diam}\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y})\right) > 2K\right) < \varepsilon/2, \tag{2.56}$$

which gives condition (ii). However, diam $\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}\right)$ is less than $2/\alpha n$ times the maximum height of a branching Brownian motion process conditioned to survive until time $\alpha n y$, so this follows immediately from our asymptotic for the survival probability. To complete the proof of tightness we will consider the tree $\left(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n} d_n^{\alpha y}\right)$ cut off at height Kn, and show that there exists an $M > 0$ such that the probability of this cut off tree having a δ-net with less than M/δ^4 balls is greater than $1 - \delta\varepsilon/2$ for all $n \in \mathbb{N}$ and $\delta > 0$. By summing probabilities over the sequence $\delta_k = 2^{-k}$, and combining with (2.56) this provides the set $\mathbb{K}$ that we need. This estimate is of course very crude, but will suffice for our purposes here.

To prove the claim, we will take K as above, and for any $\delta > 0, n \in \mathbb{N}$ divide $[0, Kn]$ into intervals of length $n\delta/2 := b_{n,\delta}$ (so there are $2K/\delta$ of them.) Then we will choose M such that in any one of these intervals $[jb_{n,\delta}, (j+1)b_{n,\delta}]$ the probability of having more than $M/2K\delta^3$ particles at time $jb_{n,\delta}$ with descendants at time $(j+1)b_{n,\delta}$ is less than $\delta^2\varepsilon/4K$, independently of j, n, δ. Assuming we can do this, summing up gives that the probability of this holding for *any* of the subintervals is less than $\delta\varepsilon/2$. Moreover, on the event that it doesn't happen for any subinterval, we can put a δ ball, for each j, at every vertex in level $jb_{n,\delta}$ that has a descendant at level $(j+1)b_{n,\delta}$. This will cover the tree up to level Kn, since a δ-ball is effectively a δn ball (remember that lengths are rescaled), and the choice of positions means that every vertex with height in $[jb_{n,\delta}, (j+1)b_{n,\delta})$ for $j \geq 1$ is covered by one of the

δ balls centred at level $(j-1)b_{n,\delta}$. For $j=0$, vertices in this interval are covered by the ball placed at the root. In this covering, by definition of the event, there are less than M/δ^4 balls. Thus it remains to show that we can choose such an M, independently of δ, n. This is a result of the following observations:

- By Theorem 2.4, there exists an $R > 0$ such that the probability of a critical branching Brownian motion starting from z surviving until time $n\delta/2$ is less than $R/n\delta$ for all n, δ and z.
- Let $\tilde{N}_{jb_{n,\delta}}$ be the number of vertices at time $jb_{n,\delta}$ that have descendants at time $(j+1)b_{n,\delta}$. Then, by Wald's identity, we have for all j, n and δ

  $$\mathbb{P}_x\left(\tilde{N}_{jb_{n,\delta}} \geq \frac{M}{2K\delta^3}\right) \leq \frac{2RK\delta^2\mathbb{E}_x[N_{jb_{n,\delta}}]}{Mn} \leq \frac{2CRK\delta^2}{Mn}$$

  where $C := \sup_{z,u} \mathbb{E}_z[N_u]$ is finite by the proof of Theorem 2.6. Note that we are not conditioning on survival here.

Therefore,

$$\mathbb{P}_x\left(\tilde{N}_{jb_{n,\delta}} \geq \frac{M}{2K\delta^3}\middle| N_{\alpha yn} > 0\right) \leq \frac{2CRK\delta^2}{Mn} \times \mathbb{P}_x(N_{\alpha yn} > 0)^{-1}.$$

This is less than $\delta^2\varepsilon/4K$ if we choose

$$M = \frac{8CRK^2}{\varepsilon} \sup_{n\geq 1} \left(n\mathbb{P}_x(N_{\alpha yn} > 0)\right)^{-1}$$

where the supremum is finite by Theorem 2.4.

Now we move on to (2). We would like to show that $(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n}d_n^{\alpha y}, \mu_n^{\alpha y})$ converges to $(\mathcal{T}_{e^y}, d_{e^y}, \mu_y)$ in the Gromov-Prohorov metric, or equivalently, with respect to the Gromov-weak topology. We will consider the latter formulation, in order to use the characterisation given by Lemma 2.65. Since convergence in the Gromov-Hausdorff-Prohorov sense implies convergence in the Gromov-Prohorov/Gromov-Weak sense, the argument for (1) also shows that part (i) of the characterisation (relative compactness of the laws) is satisfied. Therefore, we need only show for any polynomial $\Phi \in \Pi$, writing $\mathbb{P}_n^H$ for the law of $(\mathcal{T}_n^{\alpha y}, \frac{1}{\alpha n}d_n^{\alpha y}, \mu_n^{\alpha y})$ and $\mathbb{P}$ for the law of $(\mathcal{T}_{e^y}, d_{e^y}, \mu_y)$, that we have $\mathbb{P}_n^H[\Phi] \to \mathbb{P}[\Phi]$ as $n \to \infty$. To do this, we use the convergence of the martingale, Proposition 2.47, and Proposition 2.56.

Fix a polynomial Φ from $[0,\infty)^{\binom{k}{2}}$ to $\mathbb{R}$. We will first introduce some notation. We let B_n^H be the event that a branching Brownian motion tree reaches height αyn, and B_n^S be the event that its associated process $\hat{S}$, recall the definition from Definition 2.40, reaches level σyn. Note that $\sigma/\alpha = \lambda \int_D \varphi(y)^3\, dy$, which is the constant from

Propositions 2.52 and 2.56 that relates the height function and $\hat{S}$. We denote by $\mathbb{P}_n^S$ the law (on metric measure spaces) which is the same as $\mathbb{P}_n^H$, except that the tree is conditioned on B_n^S rather than B_n^H. To prove that $\mathbb{P}_n^H[\Phi] \to \mathbb{P}[\Phi]$ we show that:

(i) $\mathbb{P}_x(B_n^S | B_n^H) \to 1$ and $\mathbb{P}_x(B_n^H | B_n^S) \to 1$ as $n \to \infty$.

(ii) $\mathbb{P}_n^S[\Phi] \to \mathbb{P}[\Phi]$ as $n \to \infty$.

Combining these provides the result. We start with (i). The convergence $\mathbb{P}_x(B_n^S | B_n^H) \to 1$ is in fact a direct consequence of Proposition 2.52, and the fact that given a branching Brownian motion survives until time $y\alpha n$, the probability of it surviving to time $\varepsilon y \alpha n$ and having more than two particles alive at this time, is high. For the second part, it is therefore sufficient to show that $\mathbb{P}_x(B_n^H)/\mathbb{P}_x(B_n^S) \to 1$ as $n \to \infty$. However, the convergence given by Proposition 2.47 allows us to compute an exact asymptotic for $\mathbb{P}_x(B_n^S)$, just as in for example [LG05, Section 1.4, p263]. It is easy to check that this is indeed equal to $\frac{\varphi(z)}{y\sigma n} = \frac{\varphi(x)}{\alpha y n \int_D \varphi(z)^3 \, dz}$.

For the proof of (ii) recall that we write $\hat{S}^y$ for the process $\hat{S}$ conditioned to reach y. This is where we will use Proposition 2.47. This, along with the Skorokhod representation theorem, tells us that there exists a sequence of processes $(Z_t^{(n)})_{t \geq 0}$, equal in distribution to $(\hat{S}^{y\sigma n}_{n^2 t}/n)_{t \geq 0}$, such that

$$(Z_t^{(n)})_{t \geq 0} \to (\sigma e_t^y)_{t \geq 0} \tag{2.57}$$

uniformly almost surely as $n \to \infty$. Here we set the processes identically equal to zero after they reach zero, at times that we denote by $(\tau_n)_{n \in \mathbb{N}}$ and τ. Choose k points $(z_i;\, 1 \leq i \leq k)$ uniformly from $[0, \tau]$ and for each n set $(z_i^n;\, 1 \leq i \leq k) = \frac{\tau_n}{\tau}(z_i; 1 \leq i \leq k)$. Then for each n, the law of

$$\Big((Z_t^{(n)})_{t \geq 0}, (z_i^n;\, 1 \leq i \leq k) \Big)$$

is that of $(\hat{S}^{y\sigma n}_{n^2 t}/n)_{t \geq 0}$ together with k points chosen uniformly from its length. Define the distance between z_i^n and z_j^n for $1 \leq i < j \leq k$ by

$$d_n^Z(z_i^n, z_j^n) := Z^{(n)}_{z_i^n} + Z^{(n)}_{z_j^n} - 2 \inf_{s \in [z_i^n, z_j^n]} Z_s^{(n)}. \tag{2.58}$$

Setting

$$d_{e^y}(z_i, z_j) = e^y_{z^i} + e^y_{z^j} - 2 \inf_{s \in [z_i, z_j]} e_s^y$$

(corresponding to the normal metric d_{e^y} in $\mathcal{T}_{e^y}$) it is immediate from (2.57) that

$$(z_i^n;\, 1 \leq i \leq k) \longrightarrow (z_i;\, 1 \leq i \leq k)$$

almost surely as $n \to \infty$, and so also that

$$\frac{1}{\sigma}\left(d_n^Z(z_i^n, z_j^n)\right)_{1\le i<j\le k} \longrightarrow \left(d_{e^y}(z_i, z_j)\right)_{1\le i<j\le k} \tag{2.59}$$

almost surely as $n \to \infty$. This is useful, because the law of the object on the right is the same as the law of the matrix of pairwise distances between k points chosen independently according to μ_{e^y} from $(\mathcal{T}_{e^y}, d_{e^y})$. Note that although the metric space is actually a quotient of $[0, \tau]$ in this case, the law above is not affected.

Moreover, the law of the object on the left is what we get if we choose k vertices uniformly from a branching Brownian motion tree conditioned on B_n^S, and take the matrix defined by d_n^S in Proposition 2.56. However, we know by Proposition 2.56 and Corollary 2.57 that if we condition instead on B_n^H, the difference between this matrix and the matrix of pairwise distances under $\frac{1}{\alpha n} d_n^{\alpha y}$, converges to 0 in probability as $n \to \infty$. By part (i), this is still true under conditioning on B_n^S. Since Φ is continuous and bounded, the proof of part (ii) is complete.

□

We conclude by noticing that, since the above proof does not use anything specific about the branching diffusion (except the results that have been proven earlier in the paper and some constants), the more general statement of Theorem 2.1 holds.

Remark 2.67. *The same result holds for a branching diffusion with generator L and offspring distribution A as in Theorem 2.1 where*

$$\alpha = \sqrt{\frac{4(m-1)}{\lambda \langle 1, \varphi\rangle\, \mathbb{E}[A^2 - A] \int_D \varphi(y)^3\, dy}}$$

so that

$$\frac{\sigma}{\alpha} = \frac{\lambda \mathbb{E}[A^2 - A]}{2(m-1)} \int_D \varphi(y)^3\, dy$$

as in Remark 2.54.

3 Critical Gaussian chaos: convergence and uniqueness in the derivative normalisation

3.1 Introduction

The theory of Gaussian multiplicative chaos was developed by Kahane, [Kah85], in order to rigorously define measures of the form

$$\mu^\gamma(dx) := \mathrm{e}^{\gamma h(x) - \frac{\gamma^2}{2}\mathbb{E}[h(x)^2]}\, dx$$

where h is a rough centered Gaussian field, satisfying certain assumptions, and $\gamma > 0$ is a real parameter. Since h is not defined pointwise, a regularisation procedure is required to define μ^γ. In [Kah85], it is assumed that the covariance kernel K of h is σ-positive, meaning that K can be approximated by a series of smooth positive kernels K_n. It is then possible to associate to such an approximation the sequence of measures $\mu_n(dx) := \exp\{\gamma h_n(x) - (\gamma^2/2)\mathrm{var}(h_n(x))\}dx$. Kahane proved that these measures converge as $n \to \infty$, and that the limit is independent of the choice of approximation. We call this limit the γ-chaos measure associated to h.

However, σ-positivity can be hard to check pointwise, and in recent years this theory has been significantly generalised by several authors [RV10, Ber15a, JS17, Sha16]. When K is not σ-positive, a natural way to approximate h is to convolve it with a general mollifier function θ. Writing h_ε for these regularisations, it has been shown that for log-correlated h, and under very general conditions on θ, the approximate measures

$$\mu_\varepsilon^\gamma(dx) := \mathrm{e}^{\gamma h_\varepsilon(x)}\, \mathrm{e}^{-\frac{\gamma^2}{2}\mathrm{var}(h_\varepsilon(x))}\, dx \tag{3.1}$$

converge weakly in law [RV10] and in probability [Ber15a, Sha16] as $\varepsilon \to 0$. The limit is non-zero if and only if $\gamma^2 < 2d$. Moreover, it is universal in that it does not depend on the choice of regularisation [Ber15a, JS17, Sha16].

When $\gamma^2 = 2d$, an additional renormalisation is required in order to yield a non-trivial limiting measure. Motivated by the theory of multiplicative cascades and the branching random walk [BK04, AS14] one can hope to renormalise at criticality in one of two different ways. The first is called the Seneta–Heyde renormalisation, and involves premultiplying the sequence of measures (3.1) by the deterministic sequence $\sqrt{\log(1/\varepsilon)}$. The other is a random renormalisation, which is defined by taking a derivative of the measure (3.1) in γ. It has been shown in [DRSV14a, DRSV14b] that for a special class of fields h having so-called $\star$-scale invariant kernels, and for a specific sequence of approximations to h, both procedures yield the same non-zero limiting measure (up to a constant). However, the result in these papers relies heavily on the cut-off approximation used for the kernel of h, and does not generalise to arbitrary convolution approximations. These are somewhat more natural, local approximations to the field, and the goal of the paper will be to extend the theory to this set-up.

In this paper we will be particularly, but not exclusively, interested in the specific case where the underlying field h is a 2d Gaussian free field with zero-boundary conditions. In this case the measure μ^γ (when it is defined and non-zero) is known as the Liouville measure with parameter γ. This has been an object of considerable recent interest due to its strong connection with 2d Liouville quantum gravity and the KPZ relations [DS11, RV11, Ber15b]. Recent works in the case $\gamma < 2$ include [DS11, RV11, Ber15a], which among other things make an in-depth study of its moments, multifractal structure, and universality. Recently, in [APS17], it has also been shown that these measures can be approximated using so-called local sets of the Gaussian free field. This is a particularly natural construction because it is both local and conformally invariant.

The critical case $\gamma = 2$ has also been considered for the Gaussian free field: [DRSV14b, HRV15, JS17, APS17]. In [DRSV14b], the authors generalised their construction for $\star$-scale invariant kernels to show convergence in the Seneta–Heyde and derivative renormalisations for a specific "white noise" approximation to the field. These both yield the same (up to a constant) non-trivial limiting measure μ', that we will call the critical Liouville measure. However, this proof again does not extend to convolution approximations.

The purpose of this article is to complete the picture for convolution approximations to critical chaos. We will focus specifically on the case of the 2d GFF, and fields with $\star$-scale invariant kernels. This builds on recent work of Junnila and Saksman [JS17] (and also [HRV15] in the case of the free field), who show that in either of the cases above, the critical measure can be constructed using convolution approximations in the Seneta–Heyde renormalisation.

To complete the story, therefore, it remains to show that the random "derivative" renormalisation procedure will also yield the same limit for general convolution approximations. This is the main result of the current paper. We remark that the derivative renormalisation is somewhat more natural, and in fact, it is usually easier to show convergence of this before convergence in the Seneta–Heyde renormalisation (which is then obtained by a comparison argument). Here we will reverse this procedure.

Suppose that h is a log-correlated field in $D \subset \mathbb{R}^d$ with kernel $K(x, y)$. By this we mean that $(h,\rho)_{\rho\in\mathcal{M}}$ is a centered Gaussian process, indexed by the set of signed measures ρ whose positive and negative parts $\rho^\pm$ satisfy $\iint \rho^\pm(dx)|K(x,y)|\rho^\pm(dy) < \infty$, with covariance structure

$$\mathrm{cov}((h,\rho)(h,\rho')) = \iint \rho(dx)K(x,y)\rho'(dy)$$

for $\rho, \rho' \in \mathcal{M}$. Also suppose that $\theta \in \mathcal{M}$ is a positive measure of unit mass, supported in $B(0,1)$, and such that

$$\int \frac{1}{\sqrt{|u-v|}}\,\theta(du) = \mathrm{O}(1) \tag{3.2}$$

uniformly over $v \in B(0,5)$. Then we define a sequence of θ-mollified approximations to h by setting for $\varepsilon > 0$,

$$h_\varepsilon := h \star \theta_\varepsilon(x) = (h, \theta_{\varepsilon,x}), \tag{3.3}$$

where θ_ε is the image of θ under the map $y \mapsto \varepsilon y$ and $\theta_{\varepsilon,x}$ is the image of θ_ε under the map $y \mapsto y + x$. We define the measures M_ε and D_ε associated with this approximation by setting for $\mathcal{O} \subset D$:

$$M_\varepsilon(\mathcal{O}) := \int_{\mathcal{O}} \mathrm{e}^{2h_\varepsilon(x) - 2\mathrm{var}(h_\varepsilon(x))}\,dx;$$

$$D_\varepsilon(\mathcal{O}) := \int_{\mathcal{O}} (-h_\varepsilon(x) + 2\mathrm{var}(h_\varepsilon(x)))e^{2h_\varepsilon(x) - 2\mathrm{var}(h_\varepsilon(x))}dx$$

Note that M_ε is exactly the same as $\mu_\varepsilon^{\gamma=2}$ (but we introduce the new notation to distinguish the special case $\gamma = 2$ and avoid confusing notation.) Our aim will be to prove the following:

Theorem 3.1. *Suppose that h is a 2d Gaussian free field and D_ε is defined as above, for a mollifier θ satisfying (3.2). Then D_ε converges weakly in probability as $\varepsilon \to 0$ to the critical Liouville measure μ' constructed in [DRSV14b]. In particular $\lim_\varepsilon D_\varepsilon$ does not depend on θ.*

Theorem 3.2. *Suppose that h is a Gaussian field with $\star$-scale invariant kernel and D_ε is defined as above, for a mollifier θ satisfying (3.2) and with Hölder continuous density. Then D_ε converges weakly in probability as $\varepsilon \to 0$ to a limiting measure. This measure is independent of the choice of approximation, and agrees with the critical measure constructed in [DRSV14a, JS17] (see Theorems 3.8 and 3.9).*

There is one further motivation for proving Theorem 3.1. In [APS17], the authors also construct a critical measure for the Gaussian free field, using a simple and natural approximation based on its local sets. This is closely related to the classical construction of multiplicative cascades [KP76], and we believe that this connection can be exploited to help us improve our understanding of the situation at criticality (in particular, to prove a conjecture given in [DRSV14a].) However, it is a priori hard to connect the measure of [APS17] to the measure μ' of [DRSV14b]. It turns out that Theorem 3.1 is exactly what is needed to show that they are in fact equal (for details of this argument, see [APS17]). In conclusion, Theorem 3.1 gives us a universality statement for critical Liouville quantum gravity, that is now in line with the statement for the subcritical case [Ber15a, Sha16, APS17].

Outline We will begin in Section 3.2 by giving a brief introduction to log-correlated fields, and explaining how to approximate them using general mollifiers. We will also discuss here some of the existing literature concerning subcritical and critical Gaussian multiplicative chaos, and recall some basic facts about the 3-dimensional Bessel process. These occur naturally in critical Gaussian multiplicative chaos; roughly, as the value of the field locally about a typical point, and will be instrumental in the proof of Theorems 3.1 and 3.2. In Section 3.3 we concentrate on the case when h is a 2d Gaussian free field, and prove Theorem 3.1. We begin in Section 3.3.1 by showing that certain families of "cut-off" approximations to the derivative measures (that we shall call D_ε^β) are uniformly integrable. In fact, this will not be used directly in the proof of Theorem 3.1, but is needed for the aforementioned application to [APS17], and introduces technical facts required for the rest of the proof. Section 3.3.2 contains the bulk of the proof. The main idea is to connect the derivative measures D_ε with the renormalised measures $\sqrt{\log(1/\varepsilon)}M_\varepsilon$, which we know converge by [JS17, HRV15]. To do this, we use a technique similar to that first applied in [AS14], and then in [DRSV14b], although the details of the proof are quite different. This is centred around the fact that for the circle average approximation to the free field, there is a natural "rooted measure" arising from the definition of D_ε, under which it becomes a 3d Bessel process. We can also show that for a general convolution approximation, under the corresponding rooted measure, the process is approximately a Bessel (unfortunately, this introduces many technicalities in the proof.) Properties of the Bessel process then allow us to conclude. Finally, in Section 3.4, we show how the proof can be adapted for the case of $\star$-scale invariant kernels, to give Theorem 3.2.

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3.2 Preliminaries

3.2.1 Log-correlated fields, 2d Gaussian free field and $\star$-scale invariant kernels.

Let us recap the definition of log-correlated fields from the introduction. Suppose we have a non-negative definite kernel $K(x,y)$ on $D \subset \mathbb{R}^d$ of the form

$$K(x,y) = \log(|x-y|^{-1}) + g(x,y) \tag{3.4}$$

where g is a C^1 function on $\bar{D} \times \bar{D}$. As in the introduction, we let $\mathcal{M}$ be the set of signed measures $\rho := \rho^+ - \rho^-$ whose positive and negative parts satisfy $\iint_{D\times D} |K(x,y)|\rho^\pm(dx)\rho^\pm(dy) < \infty$. The centered Gaussian field h, with covariance $K(x,y)$, is then defined as in [Ber15a] to be the unique centred Gaussian process $(h,\rho)_{\rho\in\mathcal{M}}$ indexed by $\mathcal{M}$, such that

$$\mathrm{cov}((h,\rho),(h,\rho')) = \iint_{D\times D} K(x,y)\rho(dx)\rho'(dy)$$

for all $\rho, \rho' \in \mathcal{M}$.

We say that a kernel K is $\star$-scale invariant if it takes the form

$$K(x,y) = \int_1^\infty \frac{k(u|x-y|)}{u}\, du \tag{3.5}$$

for $k : [0,\infty) \to \mathbb{R}$ a compactly supported and positive-definite C^1 function with $k(0) = 1$. One can easily check that such a K indeed has the form (3.4). Although this does not cover all kernels satisfying (3.4) it is still a natural family to consider, due to the nice scaling relations it possesses, [**?**]. Moreover, the sequence of "cut off" approximations to K given by

$$K_\varepsilon(x,y) = \int_1^{\frac{1}{\varepsilon}} \frac{k(u|x-y|)}{u}\, du$$

yields a family of approximating fields that exhibit a useful decorrelation property (see the proof of Theorem 3.2).

As mentioned in the introduction, we will also be interested in the special case when h is a 2-dimensional Gaussian free field. To define this, let $D \subset \mathbb{C}$ be a simply-connected domain. Then the zero boundary Gaussian free field h on D is defined as above, to be the log-correlated field whose kernel K is given by the Green function, G_D, for the Laplacian on D. This satisfies

$$G_D(x, y) = -\log|x - y| + g(x, y) \tag{3.6}$$

for g a smooth function on $\bar{D} \times \bar{D}$.

One feature that makes the Gaussian free field particularly nice to work with is that it satisfies the following spatial Markov property: if $A \subset D$ is a closed subset, then we can write $h = h^A + h_A$ where h^A, h_A are independent, h^A is a zero-boundary GFF on $D \setminus A$, and h_A is harmonic when restricted to $D \setminus A$. We will see how this is useful to us in Section 3.3.

In the following we will always assume, for technical reasons and without loss of generality, that our domain $D \subset \mathbb{R}^d$ contains the ball of radius 10 around the origin.

3.2.2 Convolution with mollifiers

Suppose we have a field h with kernel K satisfying (3.4). As discussed in the introduction, since h is not defined pointwise, we need to use a regularisation procedure to define its chaos measures. A natural approach is convolve h with an approximation to the identity. Let θ be a non-negative Radon measure on $\mathbb{R}^d$, satisfying the conditions described in the introduction, and define the convolution approximations $(h_\varepsilon(x))_{\varepsilon>0}$ as in (3.3). The assumption (3.2) on θ will be important to show various properties of the convolution approximations later on (cf. Lemma 3.3 and Corollary 3.6). We remark here that (3.2) is more restrictive than the condition given in [Ber15a], but includes most of the important examples. In particular, it includes the case when θ is uniform measure on the unit circle, or when θ has an L^p density with respect to Lebesgue measure for some $p > 2$.

We have the following estimate for the covariances of $(h_\varepsilon)_\varepsilon$:

Lemma 3.3 ([Ber15a]). *Suppose θ satisfies (3.2) and h_ε is defined as above. Then:*

$$\operatorname{cov}(h_\varepsilon(x), h_{\varepsilon'}(y)) = \log(1/(|x - y| \vee \varepsilon \vee \varepsilon')) + \mathrm{O}(1). \tag{3.7}$$

where by $\mathrm{O}(1)$ we mean something that is uniformly bounded in $\varepsilon, \varepsilon'$, and x, y.

Remark 3.4. *Similarly, whenever we use order notation in the sequel, we will mean the order in ε, uniformly in whatever spatial position(s) we are considering.*

3.2.3 Maxima of the mollified fields

It will also be important for us in this article to get a hold of how fast our approximations h_ε can blow up. For this we use the work of [Aco14]. Among other things, this gives us the following Lemma (in fact, we state here a slight modification of the result, that is proved in [HRV15]).

Lemma 3.5. *Let $(Y_\varepsilon^x : x \in [0,1]^d)_{\varepsilon>0}$ be a family of Gaussian fields indexed by $[0,1]^d$ for any integer d. Suppose that for some $0 < C_Y < \infty$ and all $x, y \in [0,1]^d$ we have*

(1) $|\operatorname{cov}(Y_\varepsilon^x, Y_\varepsilon^y) + \log(\max\{\varepsilon, \|x-y\|\})| \le C_Y$ *and*

(2) $\mathbb{E}\left[(Y_\varepsilon^x - Y_\varepsilon^y)^2\right] \le C_Y \sqrt{\|x-y\|}/\sqrt{\varepsilon}$ *for all* $\|x-y\| < \varepsilon$.

Then, almost surely,

$$\inf_\varepsilon \inf_{x\in[0,1]^d} \{-Y_\varepsilon^x + \sqrt{2d}\log(1/\varepsilon)\} > -\infty.$$

This Lemma, together with the assumption (3.2), allows us to deduce the following:

Corollary 3.6. *Suppose that θ satisfies our usual conditions, including (3.2), and that h has kernel K satisfying (3.23). Assume further that $\mathcal{O} \subset \mathbb{R}^d$ is bounded. Then*

$$\inf_\varepsilon \inf_{x\in\mathcal{O}} \{-h_\varepsilon(x) + 2\log(1/\varepsilon)\} > -\infty$$

almost surely.

Proof. Condition (1) of Lemma 3.5 is easy to verify. To show condition (2) we write $K(x,y) = (\log|x-y|^{-1}) + g(x,y)$ where g is C^1 on $\mathbb{R}^d \times \mathbb{R}^d$. We need only prove that $\mathbb{E}[h_\varepsilon(x)(h_\varepsilon(x) - h_\varepsilon(y))] \le C\sqrt{\|x-y\|}/\sqrt{\varepsilon}$ for all $\|x-y\| < \varepsilon$ and an absolute constant C. The result then follows by symmetry.

To show this we write

$$\begin{aligned}\mathbb{E}[h_\varepsilon(x)(h_\varepsilon(x) - h_\varepsilon(y))] &= \iint \log\left|\frac{x-y+\varepsilon(v-w)}{\varepsilon(v-w)}\right| \theta(dv)\theta(dw) \\ &+ \iint g(x+\varepsilon v, y+\varepsilon w) - g(x+\varepsilon v, x+\varepsilon w)\theta(dv)\theta(dw)\end{aligned}$$

Since g has continuous derivative and we are working on a bounded set, it is clear that the second term satisfies the required condition. To deal with the first we note that

$$\left|\log\left|\frac{x-y+\varepsilon(v-w)}{\varepsilon(v-w)}\right|\right| \le \max\left\{\log\left(1+\frac{|x-y|}{\varepsilon|v-w|}\right), \log\left(1+\frac{|x-y|}{|x-y+\varepsilon(v-w)|}\right)\right\},$$

where we have used that $|x-y|<\varepsilon$ (and that $|v-w|\leq 1$) to remove the modulus inside the log on the left-hand side. Since $\log(1+|a|)\leq\sqrt{|a|}$ for all a, we can conclude using assumption (3.2). $\square$

3.2.4 Previous works on subcritical and critical Gaussian multiplicative chaos

As discussed in the introduction, Gaussian multiplicative chaos theory is a framework we can use to make sense of measures of the form "$\mathrm{e}^{\gamma h(x)-\gamma^2/2\mathrm{var}(h(x))}\,dx$" for log-correlated Gaussian fields h. This stems from the classical martingale theory of the branching random walk [BK77, Kyp00] and multiplicative cascades [KP76], and was initiated by Kahane [Kah85] in the 1980's. In the special case where h is a 2d Gaussian free field, the Gaussian multiplicative chaos measure is often referred to as the Liouville measure [DS11]. Here we will state precisely some of the results mentioned in the introduction.

When $\gamma<\sqrt{2d}$ (the subcritical regime) there are various approximation procedures that can be used to construct the chaos measure with parameter γ. One natural choice is to use the convolution approximations h_ε described in the previous section, and define approximate measures μ_ε^γ by setting

$$\mu_\varepsilon^\gamma(dx):=\mathbb{E}[\mathrm{e}^{\gamma h_\varepsilon(x)}]^{-1}\,\mathrm{e}^{h_\varepsilon(x)}\,dx \tag{3.8}$$

for $\varepsilon>0$. Note that the normalisation factor here is equal to $\varepsilon^{\frac{\gamma^2}{2}}$ (up to a bounded constant that depends on x). We have the following result.

Theorem 3.7 ([Ber15a])**.** *For $\gamma<\sqrt{2d}$ the measures μ_ε^γ converge to a non-trivial measure μ^γ weakly in probability. Moreover, for any fixed Borel set $\mathcal{O}$ we have that $\mu_\varepsilon^\gamma(\mathcal{O})$ converges in L^1.*

We emphasise that this limit μ^γ does not depend on the choice of mollifier θ. In fact, one can approximate the field in other, completely different ways (for instance using a Karhunen–Loève expansion [Ber15a]) and find the same limit. For the case of the free field, this will even work for "non-Gaussian" approximations. Indeed, in [APS17] the authors construct (the same) Liouville measure for $\gamma<2$ using sequences of so-called "local sets" of the field.

For general h, the subcritical measures μ^γ with $\gamma<\sqrt{2d}$ are almost surely atomless, and assign positive mass to any open set. On the other hand, as discussed in the introduction, it is known that for $\gamma\geq\sqrt{2d}$, the measures μ_ε^γ converge to zero [RV10]. To define the critical (and supercritical) measures we must therefore make an additional renormalisation. These cases turn out to be much more tricky to deal with than the subcritical case, in part because the limiting measure will not possess any moments of order greater than or equal to 1. Consequently a complete theory is still lacking, but some progress has been made (see [RV14] for a survey). Here and in the rest of this paper we will discuss the critical case $\gamma = \sqrt{2d}$.

3.2.5 Critical measures

Motivated by the corresponding constructions for multiplicative cascades, [BK04, AS14], we expect to be able to obtain a non-trivial measure at criticality using either of two renormalisation procedures: one deterministic and one random. Let us outline how this should work. Suppose you have some approximations h_ε to a log-correlated field h, that are continuous fields for each ε. Then each of the following sequences should converge to the same (up to a constant) limiting measure.

- The sequence of measures $\sqrt{\log(1/\varepsilon)}\mu_\varepsilon^{\gamma=2} := \sqrt{\log(1/\varepsilon)}M_\varepsilon$, where μ_ε^γ is defined by (3.8). This is known as the Seneta–Heyde renormalisation.
- The sequence of signed "derivative" measures, obtained by taking the derivative of μ_ε^γ with respect to γ and evaluating at $\gamma = \sqrt{2d}$. That is, the sequence

$$D_\varepsilon(dz) := (-h_\varepsilon(z) + \gamma\mathbb{E}[h_\varepsilon(z)^2]) \exp\left(\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}\left[h_\varepsilon(z)^2\right]\right) dz$$

  (where we have also multiplied by -1 in order to yield a non-negative limit measure.)

This statement was verified for a specific set-up in [DRSV14a, DRSV14b].

Theorem 3.8 ([DRSV14a, DRSV14b]). *Suppose h has a $\star$-scale invariant kernel $K(x,y) = \int_1^\infty k(u|x-y|)/u\, dx$ as in (3.5) and the approximate fields h_ε have kernels given by*

$$K_\varepsilon(x,y) := \int_1^{1/\varepsilon} \frac{k(u|x-y|)}{u}\, du.$$

Then the two sequences of approximating measures described above converge weakly in probability to the same limiting measure, up to a constant $\sqrt{2/\pi}$. In particular, for any open set $\mathcal{O} \subset \mathbb{R}^d$, $\sqrt{\pi/2}\sqrt{\ln(1/\varepsilon)}M_\varepsilon(\mathcal{O})$ and $D_\varepsilon(\mathcal{O})$ converge in probability and in L^p (any $p < 1$) to the same limit.

The authors in [DRSV14a, DRSV14b] were also able to generalise this approach to the case when h is a 2d Gaussian free field, using a white-noise decomposition for the field and another specific sequence of "cut-off" approximations for the kernel. However, both of these proofs rely strongly on a martingale property satisfied by the choice of approximating fields h_ε. In particular, they do not extend to general convolution approximations.

Convolution is clearly a natural way to approximate the field h, and so we would like to have a version of Theorem 3.8 for such approximations. Using comparison techniques, Junnila and Saksman were able to do this for the Seneta–Heyde renormalisation.

Theorem 3.9 ([JS17])**.** *Let h be a $\star$-scale invariant field, and assume that in addition to (3.2), the mollifier θ has a Hölder continuous density. Then the measures $\sqrt{\log(1/\varepsilon)}M_\varepsilon$ converge to a limiting measure weakly in probability as $\varepsilon \to 0$. This limit is equal to $\sqrt{2/\pi}\mu'$ where*

- *μ' is the measure from Theorem 3.8,*
- *$\mathbb{E}[\mu'(\mathcal{O})] = \infty$ for any $\mathcal{O} \subset \mathbb{R}^d$ and*
- *$\mu'(\mathcal{O})$ is positive almost surely for any $\mathcal{O} \subset \mathbb{R}^d$.*

Again we have that for any open set $\mathcal{O} \subset \mathbb{R}^d$, $\sqrt{\pi/2}\sqrt{\ln(1/\varepsilon)}M_\varepsilon(\mathcal{O})$ and $D_\varepsilon(\mathcal{O})$ converge in probability and in L^p ($p < 1$) to the same limit.

This has also been proven for the 2d-Gaussian free field.

Theorem 3.10 ([HRV15, JS17])**.** *Let h be a 2d-GFF, and take any mollifier θ satisfying (3.2). Then the measures $\sqrt{\log(1/\varepsilon)}M_\varepsilon$ converge to a limiting measure weakly in probability as $\varepsilon \to 0$. This limit is equal to $\sqrt{2/\pi}\mu'$ where μ' is the critical Liouville measure of [DRSV14b].*

Note that Theorem 3.10 places a weaker constraint on the mollifier θ. This is due to the proof given in [HRV15]. The aim of this paper will be to prove the analogues of Theorems 3.9 and 3.10 for the derivative renormalisation.

3.2.6 Bessel processes

To conclude this introduction, we need to recall some basic properties of Brownian motion; in particular, of the 3-dimensional Bessel process. Let $\mathbf{P}$ denote the law of a standard Brownian motion B_t in $\mathbb{R}$, started from a possibly random position B_0 such that $\mathbf{P}(B_0 > 0) = 1$. Then it is easy to check that for any $\beta, \gamma > 0$, the process

$$(-B_t + \gamma\mathrm{var}(B_t) + \beta)\mathbb{1}_{\{-B_u+\gamma\mathrm{var}(B_t)+\beta>0\ \forall u\in[0,t]\}}\, \mathrm{e}^{\gamma B_t - \frac{\gamma^2}{2}\mathrm{var}(B_t)} \tag{3.9}$$

is a positive martingale. Let $\mathcal{F}_t$ be the filtration generated by the Brownian motion and define a new measure $\mathbf{Q}$ by letting its Radon–Nikodym derivative when restricted to $\mathcal{F}_t$ be given by the martingale at time t. One can check that this yields a well-defined law $\mathbf{Q}$, under which the process $(-B_t+\gamma \mathrm{var}(B_t)+\beta)_{t\geq 0}$ is a 3d Bessel process started from $-B_0+\gamma\mathrm{var}(B_0)+\beta$. Note that this starting position will also be biased, and will be positive almost surely under $\mathbf{Q}$. The next lemma records some properties of the 3d Bessel process that we will use in our proofs.

Lemma 3.11. *Let $(X_t)_{t\geq 0}$ be a 3d Bessel process started from a random (positive) position X_0 with finite variance, and law $\mathbf{Q}$. Then*

(1) $\mathbf{Q}[\frac{1}{X_t}] = \sqrt{\frac{2}{\pi t}} + \mathrm{o}(t^{-1/2})$ *where the error term is less than* $\frac{2}{\sqrt{t}}(\frac{\mathbf{Q}[X_0^2]}{t} + \frac{\mathbf{Q}[X_0]}{\sqrt{t}})$.

(2) $\mathbf{Q}[\frac{1}{X_t^2}] \leq 2/t$, *uniformly in the starting position.*

(3) $\mathbf{Q}[\frac{\sqrt{u}}{\log(2+u)^2} \leq X_u \leq (1+\sqrt{u\log(1+u)})$ *eventually*$] = 1$

(4)
$$\mathbf{Q}\left[\frac{\sqrt{u}}{R\log(2+u)^2} \leq X_u \leq R(1+\sqrt{u\log(1+u)})\ \forall u\geq 0\right] \to 1$$
as $R\to\infty$, uniformly over X_0 with $\mathbf{Q}[X_0]\leq K$ for any K.

(5) $\mathbf{Q}[\frac{1}{X_t}\mathbb{1}_{\{X_t\leq t^{1/4}\}}] \leq \frac{C}{2t}$, *uniformly in the starting position, where C is an absolute constant.*

Proof. (1),(2) and (5) are straightforward to verify using direct calculation and scaling arguments. (3) is a classical result due to Motoo [Mot58] and then (4) follows by continuity and Markov's inequality. □

3.3 Proof of Theorem 3.1

In this section we will work to prove Theorem 3.1. Recall that this concerns the case when the underlying field h is a 2d Gaussian free field in a domain $D\subset\mathbb{R}^2$. For this choice of field, there is a particular convolution approximation, when θ is uniform measure on the unit circle, that plays an important role. We call this the circle average process and distinguish it by writing $\tilde{h}_\varepsilon := h\star\theta_\varepsilon$. The Markov property of the field allows us to deduce the following:

Lemma 3.12. *For each $x\in D$ and $\delta < d(x,\partial D)$, $\{\tilde{h}_{\mathrm{e}^{-u}}(x) : u\geq\log(1/\delta)\}$ is a Brownian motion started from $\tilde{h}_\delta(x)$.*

We will also need to compare $\tilde{h}_\varepsilon$ with a general convolution approximation h_ε.

Lemma 3.13. *Let h_ε and $\tilde{h}_\varepsilon$ be the mollified and circle averages of h at a point x with $d(x,\partial D)>\varepsilon$. Then we can write*

$$h_\varepsilon(x) = \lambda_\varepsilon(x)\tilde{h}_\varepsilon(x) + Y_\varepsilon(x) \tag{3.10}$$

where $\lambda_\varepsilon(x) = 1 + \mathrm{O}(\log(1/\varepsilon)^{-1})$ *(uniformly in* x*) and* $Y_\varepsilon(x)$ *is independent of* $\tilde{h}_\varepsilon(x)$*, Gaussian, and has mean* 0 *and variance* $\mathrm{O}(1)$*.*

Proof. For this, we observe (by an easy calculation using (3.6)) that

$$\mathrm{cov}(h_\varepsilon(x), \tilde{h}_\varepsilon(x)) = \log(1/\varepsilon) + \mathrm{O}(1)$$

for any $x \in D$ and $\varepsilon < d(x, \partial D)$. Let $\lambda_\varepsilon(x) := \mathrm{cov}(h_\varepsilon(x), \tilde{h}_\varepsilon(x))/\,\mathrm{cov}(\tilde{h}_\varepsilon(x), \tilde{h}_\varepsilon(x))$, so that by direct calculation $\mathrm{cov}(h_\varepsilon - \lambda_\varepsilon \tilde{h}_\varepsilon, \tilde{h}_\varepsilon) = 0$. Then by Gaussianity, $\tilde{h}_\varepsilon$ and $Y_\varepsilon := h_\varepsilon - \tilde{h}_\varepsilon$ are independent. Using Lemmas 3.12 and 3.3, we see that the variance of Y_ε is $\mathrm{O}(1)$ and that $\lambda_\varepsilon = 1 + \mathrm{O}(\log(1/\varepsilon)^{-1})$. □

Remark 3.14. *We will often drop the* x *from* $\lambda_\varepsilon(x)$ *when it is clear from the context.*

Lemma 3.15. $Y_\varepsilon(x)$ *also has bounded covariances with* $Y_\varepsilon(y)$ *and* $\tilde{h}_\varepsilon(y)$ *for any* $x, y \in D$*. Moreover, for* $\delta \geq \varepsilon$*, we have*

$$-\rho_\delta^\varepsilon(x)/2 := \mathrm{cov}(Y_\varepsilon(x), \tilde{h}_\delta(x)) = \mathrm{O}(1),$$

uniformly in ε, δ *and* x*.*

Proof. The first claims follow using direct calculation similar to the above. For the final claim note that $\mathbb{E}[Y_\varepsilon(x)\tilde{h}_\delta(x)] = \mathbb{E}[h_\varepsilon(x)\tilde{h}_\delta(x)] - \lambda_\varepsilon(x)\mathbb{E}[\tilde{h}_\varepsilon(x)\tilde{h}_\delta(x)]$ where both expressions on the right-hand side are $\log(1/\delta) + \mathrm{O}(1)$. □

Remark 3.16. *Lemma 3.13 implies that* $\rho_\varepsilon^\varepsilon(x) = 0$ *for all* ε, x*.*

Let us now move on to the proof of Theorem 3.1. By standard arguments, see [Ber15a], we need only prove that $D_\varepsilon(\mathcal{O}) \to \mu'(\mathcal{O})$ in probability for each fixed $\mathcal{O} \subset D$. In fact, without loss of generality we may assume that $\mathcal{O} := B(0,1)$ is the unit disc. From now on we will work with this assumption.

3.3.1 A uniformly integrable family.

We know from [DRSV14b] that if μ' is the critical Liouville measure, $\mu'(\mathcal{O})$ has infinite expectation for any $\mathcal{O} \subset D$. Therefore, we cannot hope to have L^1-convergence or uniform integrability of $D_\varepsilon(\mathcal{O})$. Since we prefer to work with uniformly integrable families, we instead consider a sequence of cut-off approximations D_ε^β to D_ε. It will be very important to choose these cut-offs correctly, but for the right choice they *will* be uniformly integrable (for each β) and moreover, will converge as $\varepsilon \to 0$ (albeit in some slightly unusual sense, see Lemma 3.21). Obtaining the desired convergence in Theorem 3.1 then amounts to letting $\beta \to \infty$ and using Lemma 3.5 to see that D_ε^β is actually very close to D_ε for large enough β.

So, let us fix $\varepsilon_0 > 0$, such that $B(x,\varepsilon) \subset D$ for every $\varepsilon \leq \varepsilon_0$ and $x \in \mathcal{O}$. Then we define for $\beta > 0$ and $\varepsilon \in (0, \varepsilon_0]$, the "cut-off" approximations

$$M_\varepsilon^\beta(\mathcal{O}) := \int_{\mathcal{O}} \mathrm{e}^{2h_\varepsilon(x) - 2\mathrm{var}(h_\varepsilon(x))} \mathbb{1}_{L_\varepsilon(x)} \mathbb{1}_{\{-h_\varepsilon(x) + 2\mathrm{var}(h_\varepsilon(x)) + \beta > 1\}} \, dx; \text{ and}$$
$$D_\varepsilon^\beta(\mathcal{O}) := \int_{\mathcal{O}} (-h_\varepsilon(x) + 2\mathrm{var}(h_\varepsilon(x)) + \beta) e^{2h_\varepsilon(x) - \gamma^2/2\mathrm{var}(h_\varepsilon(x))} \mathbb{1}_{L_\varepsilon(x)} \mathbb{1}_{\{-h_\varepsilon(x) + 2\mathrm{var}(h_\varepsilon(x)) + \beta > 1\}} \, dx;$$

where

$$L_\varepsilon(x) := \{-\tilde{h}_\delta(x) + 2\lambda_\varepsilon(x)\mathrm{var}(\tilde{h}_\delta(x)) + \beta - \rho_\delta^\varepsilon(x) > 0; \, \forall \, \delta \in [\varepsilon, \varepsilon_0]\}.$$

Note that both $M_\varepsilon^\beta(\mathcal{O})$ and $D_\varepsilon^\beta(\mathcal{O})$ are positive by definition, and also that $M_\varepsilon^\beta(\mathcal{O}) \leq D_\varepsilon^\beta(\mathcal{O})$. For ease of notation we set

$$\begin{aligned}
f_{\varepsilon,\gamma}^\beta(x) &= -h_\varepsilon(x) + \gamma\mathrm{var}(h_\varepsilon(x)) + \beta \qquad & g_{\varepsilon,\gamma}(x) &= \gamma h_\varepsilon(x) - (\gamma^2/2)\mathrm{var}(h_\varepsilon(x)) \\
\tilde{f}_{\varepsilon,\gamma}^\beta(x) &= -\tilde{h}_\varepsilon(x) + \gamma\mathrm{var}(\tilde{h}_\varepsilon(x)) + \beta \qquad & \tilde{g}_{\varepsilon,\gamma}(x) &= \gamma \tilde{h}_\varepsilon(x) - (\gamma^2/2)\mathrm{var}(\tilde{h}_\varepsilon(x)) \\
f_{\varepsilon,\gamma}^{\beta,Y}(x) &= -Y_\varepsilon(x) + \gamma\mathrm{var}(Y_\varepsilon(x)) + \beta \qquad & g_{\varepsilon,\gamma}^Y(x) &= \gamma Y_\varepsilon(x) - (\gamma^2/2)\mathrm{var}(Y_\varepsilon(x))
\end{aligned}$$

recalling the definition of Y from Lemma 3.13. Then we have

$$M_\varepsilon^\beta(\mathcal{O}) := \int_{\mathcal{O}} \mathrm{e}^{g_{\varepsilon,2}(x)} \mathbb{1}_{L_\varepsilon(x)} \mathbb{1}_{\{f_{\varepsilon,2}^\beta(x) > 1\}} \, dx \text{ and } D_\varepsilon^\beta(\mathcal{O}) := \int_{\mathcal{O}} f_{\varepsilon,2}^\beta(x) e^{g_{\varepsilon,2}(x)} \mathbb{1}_{L_\varepsilon(x)} \mathbb{1}_{\{f_{\varepsilon,2}^\beta > 1\}} \, dx$$

and

$$L_\varepsilon(x) = \{\tilde{f}_{\delta,2\lambda_\varepsilon(x)}^\beta(x) - \rho_\delta^\varepsilon > 0 \, \forall \delta \in [\varepsilon, \varepsilon_0]\}$$

The decomposition

$$f_{\varepsilon,2}^\beta(\cdot) = \lambda_\varepsilon(\cdot)\tilde{f}_{\varepsilon,2\lambda_\varepsilon(\cdot)}^\beta(\cdot) + f_{\varepsilon,2}^{0,Y}(\cdot) + (1 - \lambda_\varepsilon(\cdot))\beta \text{ and } g_{\varepsilon,2}(\cdot) = \tilde{g}_{\varepsilon,2\lambda_\varepsilon(\cdot)}(\cdot) + g_{\varepsilon,2}^Y(\cdot). \tag{3.11}$$

will also come in very useful it what follows.

Proposition 3.17. *For fixed $\beta > 0$, $(D_\varepsilon^\beta(\mathcal{O}))_{\varepsilon \leq \varepsilon_0}$ is a uniformly integrable family.*

Proof. The proof of this Lemma is inspired by that of Berestycki [Ber15a], who shows uniform integrability of μ_ε^γ in the subcritical case. In analogy to his approach, for $a \geq \varepsilon > 0$ we define the *good event*

$$G_{\varepsilon,a}^R(x) :=$$

$$\left\{ \frac{\sqrt{\log(1/u)}}{R\log(2 + \log(1/u))^2} \leq \tilde{f}_{u,2\lambda_\varepsilon(x)}^\beta(x) \leq R(1 + \sqrt{\log(1/u)\log(1 + \log(1/u))}) \;\; \forall u \in [\varepsilon, a] \right\}$$

and write $D_\varepsilon^\beta(\mathcal{O}) = J_\varepsilon^\beta + \hat{J}_\varepsilon^\beta$, where J_ε^β is the integral over all "good" x, for which $G^R_{\varepsilon,\varepsilon_0}(x)$ holds. [5] The rationale behind choosing G in this way is that it separates bad points of the field, which are "too thick" and make the second moment explode, from the good points.

To conclude, it is enough to prove the following two lemmas.

Lemma 3.18. $\mathbb{E}[\hat{J}_\varepsilon^\beta] \le p(R)$ *for all* $\varepsilon \le \varepsilon_0$ *where* $p(R) \to 0$ *as* $R \to \infty$*;*

Lemma 3.19. *For fixed* R, J_ε^β *is uniformly bounded in* L^2.

We first give a very rough idea of why these should hold:

- $\mathbb{E}[\hat{J}_\varepsilon^\beta]$ corresponds to the probability of $G^R_{\varepsilon,\varepsilon_0}(x)$ *not* holding under a weighted law: specifically, under the law with Radon–Nikodym derivative (with respect to $\mathbb{P}$) proportional to

$$f^\beta_{\varepsilon,2}(x)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)>1\}}.$$

  Under this law we know that $\tilde{f}^\beta_{u,2\lambda_\varepsilon(x)}(x)$ is (approximately) a Bessel process. Thus we know by Lemma 3.11 that this probability tends to 0 as $R \to \infty$.

- Now we move on to the L^2 bound. Every time we write $\approx$ it requires a lot of justification, usually because h_ε is not exactly a Brownian motion. First note that by the Markov property of the field and the fact that (3.9) is a martingale,

$$\mathbb{E}[f^\beta_{\varepsilon,2}(x)f^\beta_{\varepsilon,2}(y)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathrm{e}^{g_{\varepsilon,2}(y)}\,\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{L_\varepsilon(y)}] \approx \mathbb{E}[f^\beta_{\delta,2}(x)f^\beta_{\delta,2}(y)\,\mathrm{e}^{g_{\delta,2}(x)}\,\mathrm{e}^{g_{\delta,2}(y)}\,\mathbb{1}_{L_\delta(x)}\mathbb{1}_{L_\delta(y)}]$$

  for $x, y \in \mathcal{O}$, where $\delta = \delta(x,y) = (|x-y|/3) \vee \varepsilon$. Now, on the event $G^R_{\varepsilon,\varepsilon_0}(x) \cap G^R_{\varepsilon,\varepsilon_0}(y)$,

$$f^\beta_{\delta,2}(x) \approx \sqrt{\log(1/\delta)},\; f^\beta_{\delta,2}(y) \approx \sqrt{\log(1/\delta)} \;\text{ and } g_{\delta,2}(y) \approx -2\sqrt{\log(1/\delta)}+2\log(1/\delta).$$

  We can use this to show that, roughly,

$$\begin{aligned}\mathbb{E}[f^\beta_{\delta,2}(x)f^\beta_{\delta,2}(y)\,\mathrm{e}^{g_{\delta,2}(x)}\,\mathrm{e}^{g_{\delta,2}(y)}\,\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(x)}\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(y)}] &\lesssim \delta^{-2}\log(1/\delta)\,\mathrm{e}^{-2\sqrt{\log(1/\delta)}}\,\mathbb{E}[\mathrm{e}^{g_{\delta,2}(x)}] \\ &= \delta^{-2}\log(1/\delta)\,\mathrm{e}^{-2\sqrt{\log(1/\delta)}}.\end{aligned}$$

  We then only need to verify that this function of $\delta(x,y)$ is integrable over $\mathcal{O} \times \mathcal{O}$.

We prove Lemmas 3.18 and 3.19 below. As already mentioned, there are several technical difficulties with making the above argument rigorous.

[5] Note that we are setting $a = \varepsilon_0$ here, but we define the more general notation $G^R_{\varepsilon,a}$ for use later on.

□

Proof of Lemma 3.18. Consider for $x \in \mathcal{O}$

$$\mathbb{E}\left[f^{\beta}_{\varepsilon,2}(x)e^{g_{\varepsilon,2}(x)}\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}(x)>1\}}\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(x)^c}\right]. \tag{3.12}$$

To prove the lemma, we need to show that this converges to 0 as $R \to \infty$, uniformly in ε and x. The strategy is to rewrite it as an expectation with respect to a different measure, under which we understand well the behaviour of $\tilde{f}_{\varepsilon,2\lambda_\varepsilon}(x)$. We set

$$\frac{d\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x}{d\mathbb{P}} = (\tilde{Z}^{\beta}_{\varepsilon}(x))^{-1}\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon}(x)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)}; \quad \tilde{Z}^{\beta}_{\varepsilon}(x) = \mathbb{E}\left[\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon}(x)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)}\right].$$

This measure will be extremely important throughout the paper because, under $\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x$, the process

$$\{\tilde{f}^{\beta}_{u,2\lambda_\varepsilon}(x) - \rho^{\varepsilon}_u(x);\, u \in [\varepsilon,\varepsilon_0]\}$$

is a time changed 3d Bessel process. To see why this is true, we split the weighting that defines $\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x$ into two steps. By decomposition (3.11) we have $g_{\varepsilon,2}(x) = \tilde{g}_{\varepsilon,2\lambda_\varepsilon}(x) + g^{Y}_{\varepsilon,2}(x)$ and so we can first consider what happens if we only weight by $\exp(g^{Y}_{\varepsilon,2}(x))$. Let us call this intermediate law $\hat{\mathbb{P}}$. By the Cameron–Martin–Girsanov theorem, and the definition $\rho^{\varepsilon}_{\delta}(x) := -2\,\mathrm{cov}(\tilde{h}_\delta(x), Y_\varepsilon(x))$, the process

$$-h_\delta(x) - \rho^{\varepsilon}_{\delta}(x)$$

is a time changed Brownian motion under $\hat{\mathbb{P}}$. For the second step in the weighting we use the definition of $L_\varepsilon(x)$, and the fact that $\rho^{\varepsilon}_{\varepsilon}(x) = 0$. This means that this second step is simply the Bessel process weighting described in Section 3.2.6, with $\gamma = 2\lambda_\varepsilon(x)$. The same argument also implies that $\tilde{Z}^{\beta}_{\varepsilon}(x)$ does not depend on ε for each x, since (3.9) is a martingale.

To prove the lemma, and we will apply this technique over and over again, we rewrite (3.12) as

$$\tilde{Z}^{\beta}_{\varepsilon}(x)\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[\frac{f^{\beta}_{\varepsilon,2}(x)}{\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon}(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}>1\}}\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(x)^c}\right]$$

where by Lemma 3.11 part (4) we know that $\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x(G^R_{\varepsilon,\varepsilon_0}(x)^c) \to 0$ as $R \to \infty$, uniformly in ε and x (using the uniform boundedness of $(\rho^{\varepsilon}_{\delta}(x))_{\delta>\varepsilon}$.)

Using the fact that

$$f^{\beta}_{\varepsilon,2}(x) = \lambda_\varepsilon(x)\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon(x)}(x) + \mathrm{O}(1) - Y_\varepsilon(x),$$

(cf. decomposition (3.11)), and Hölder's inequality, it is enough for us to show that

$$\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\left(\frac{|Y_\varepsilon(x)|+\mathrm{O}(1)}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)}\right)^{3/2}\right]=\mathrm{O}(1).$$

However, this follows by Cauchy–Schwarz, since

$$\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\left(\frac{|Y_\varepsilon(x)|+\mathrm{O}(1)}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)}\right)^{3/2}\right]^2\leq \tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{(|Y_\varepsilon(x)|+\mathrm{O}(1))^3}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)}\right]\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{1}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)^2}\right]$$

and

- $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[(\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x))^{-2}]$ is bounded by Lemma 3.11, part (2);
- $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[|Y_\varepsilon(x)|^p/\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)]\lesssim \mathbb{E}[|Y_\varepsilon(x)|^p\,\mathrm{e}^{g_{\varepsilon,2}(x)}]$ is bounded for $p=1,2,3$.

□

Proof of Lemma 3.19. First, we make the simple bound

$$\mathbb{E}[(J_\varepsilon^\beta)^2]\leq \iint_{\mathcal{O}^2}\mathbb{E}\left[|f^\beta_{\varepsilon,2}(x)||f^\beta_{\varepsilon,2}(y)|\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathrm{e}^{g_{\varepsilon,2}(y)}\,\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{L_\varepsilon(y)}\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(x)}\mathbb{1}_{G^R_{\varepsilon,\varepsilon_0}(y)}\right]dy\,dx \tag{3.13}$$

and fix some $x\in\mathcal{O}$. For this fixed x, we will break the integral over y into two parts: those with $|x-y|>3\varepsilon$, and those with $|x-y|\leq 3\varepsilon$. Let us begin with the first case. For such a y, we set $\delta=\delta(x,y):=|x-y|/3$, so that the δ-balls around x and y are disjoint. We are going to use the fact that the circle averages around x and y decorrelate after this time. More precisely, if we let $\mathcal{H}$ be the σ-algebra generated by $h|_{D\setminus(B(x,\delta)\cap B(y,\delta))}$, then we have the following observations, which we state as a lemma.

Lemma 3.20. *(1) Conditionally on $\mathcal{H}$, the processes $(\tilde{h}_{\delta\,\mathrm{e}^{-t}}(x)-\tilde{h}_\delta(x))_{t\geq 0}$ and $(\tilde{h}_{\delta\,\mathrm{e}^{-t}}(y)-\tilde{h}_\delta(y))_{t\geq 0})$ are independent Brownian motions.*

(2) We can write $Y_\varepsilon(x)=Y^1_\varepsilon(x)+Y^2_\varepsilon(x)$ and $Y_\varepsilon(y)=Y^1_\varepsilon(y)+Y^2_\varepsilon(y)$ where:

- *$Y^1_\varepsilon(x)$ and $Y^1_\varepsilon(y)$ are measurable with respect to $\mathcal{H}$;*
- *$Y^2_\varepsilon(x)$ is independent of $\mathcal{H}$, $Y^2_\varepsilon(y)$ and $(\tilde{h}_\eta(y)-\tilde{h}_\delta(y))_{\eta\leq\delta}$;*
- *$Y^2_\varepsilon(y)$ is independent of $\mathcal{H}$, $Y^2_\varepsilon(x)$ and $(\tilde{h}_\eta(x)-\tilde{h}_\delta(x))_{\eta\leq\delta}$;*
- *$Y^i_\varepsilon(x),Y^i_\varepsilon(y)$ for $i=1,2$ have bounded variance; and*
- *$2\,\mathrm{cov}(Y^2_\varepsilon(x),\tilde{h}_\eta(x)-\tilde{h}_\delta(x))=-\rho^\varepsilon_\eta(x)+\rho^\varepsilon_\delta(x)$ (similarly if x is replaced with y).*

(3) We have

$$\mathbb{E}[\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon}(x)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)}\mid\mathcal{H}] = (\tilde{f}^{\beta}_{\delta,2\lambda_\varepsilon} - \rho^{\varepsilon}_{\delta}(x))\mathbb{1}_{\{\tilde{f}_{\eta,2\lambda_\varepsilon}-\rho^{\varepsilon}_{\eta}(x)>0;\,\forall\,\eta\in[\delta,\varepsilon_0]\}} \\ \times\,\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon}(x)}\,\mathrm{e}^{2Y^{\varepsilon}_{1}(x)-2\mathrm{var}(Y^{1}_{\varepsilon}(x))}\,\mathrm{e}^{2\rho^{\varepsilon}_{\delta}(x)}.$$

(4) We also have $\mathbb{E}[|Y^2_\varepsilon(x)|\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)}\mid\mathcal{H}] \le C\,\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon}(x)}\,\mathrm{e}^{2Y^1_\varepsilon(x)}$ *where* C *is a universal constant.*

(5) Items (3) and (4) also hold when x *is replaced by* y*.*

Proof of Lemma 3.20. By the Markov property of the Gaussian free field, conditionally on $\mathcal{H}$ we can write $h = h^{\mathcal{H}} + h_{\mathcal{H}}$ where:

- $h_{\mathcal{H}}$ is measurable with respect to $\mathcal{H}$ and harmonic when restricted to $B(x,\delta)\cup B(y,\delta)$; and
- $h^{\mathcal{H}}$, independent of $\mathcal{H}$, is a sum of two independent zero boundary GFFs: one in $B(x,\delta)$ and one in $B(y,\delta)$.

We use this to prove the points in turn.

(1) This follows from the fact that $h_{\mathcal{H}}(x) = \tilde{h}_\delta(x)$ and $h_{\mathcal{H}}(y) = \tilde{h}_\delta(y)$ (by harmonicity), and the fact that the circle average process of a Gaussian free field is a Brownian motion.

(2) We have $Y_\varepsilon(x) = h_\varepsilon(x) - \lambda_\varepsilon(x)\tilde{h}_\varepsilon(x)$ by definition and so we can write $Y^1_\varepsilon(x) = (h_{\mathcal{H}},\theta_{\varepsilon,x}) - \lambda_\varepsilon(x)(h_{\mathcal{H}},\tilde{\theta}_{\varepsilon,x})$ and $Y^2_\varepsilon = (h^{\mathcal{H}},\theta_{\varepsilon,x}) - \lambda_\varepsilon(x)(h^{\mathcal{H}},\tilde{\theta}_{\varepsilon,x})$, where $\tilde{\theta}$ is uniform measure on the unit circle. The claimed properties of this decomposition are easy to see.

(3) We first take out the $\mathcal{H}$-measurable parts from the conditional expectation on the left-hand side. To this end we write for $\eta < \delta$

$$\tilde{f}^{\beta}_{\eta,2\lambda_\varepsilon}(x) = \tilde{f}^{\beta}_{\delta,2\lambda_\varepsilon}(x) - \rho^{\varepsilon}_{\delta} - (\tilde{h}_\eta(x) - \tilde{h}_\delta(x)) + 2\log(\delta/\eta) + \rho^{\varepsilon}_{\delta} \\ := W - (\tilde{h}_\eta(x) - \tilde{h}_\delta(x)) + 2\log(\delta/\eta) + \rho^{\varepsilon}_{\delta}$$

where W is $\mathcal{H}$ measurable. We can also write

$$L_\varepsilon(x) = \{\tilde{f}^{\beta}_{\eta,2\lambda_\varepsilon}(x) - \rho^{\varepsilon}_{\eta} > 0\,\forall\,\eta\in[\delta,\varepsilon_0]\}\cap\{\tilde{f}^{\beta}_{\eta,2\lambda_\varepsilon}(x) - (\rho^{\varepsilon}_{\eta} - \rho^{\varepsilon}_{\delta}) + W > 0\,\forall\,\eta\in[\varepsilon,\delta]\} \\ := L^1_\varepsilon \cap L^2_\varepsilon$$

where L^1_ε is also $\mathcal{H}$-measurable. Putting these together, and breaking up $g_{\varepsilon,2}(x)$

using (3.11) and point (2), we see that

$$\mathbb{E}[\tilde{f}^{\beta}_{\varepsilon,2\lambda_\varepsilon}(x)\,\mathrm{e}^{g_{\varepsilon,2}(x)}\,\mathbb{1}_{L_\varepsilon(x)} \mid \mathcal{H}] = \mathrm{e}^{2\rho^{\varepsilon}_{\delta}(x)}\,\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon}(x)}\,\mathrm{e}^{2Y^1_\varepsilon(x)-2\mathrm{var}(Y^1_\varepsilon(x))}\,\mathbb{1}_{L^1_\varepsilon(x)}\times$$
$$(\mathbb{E}[(W-(\tilde{h}_\eta(x)-\tilde{h}_\delta(x))+2\log(\delta/\varepsilon)+\rho^{\varepsilon}_{\delta})$$
$$\times\quad \mathrm{e}^{2(\tilde{h}_\eta(x)-\tilde{h}_\delta(x))-2\log(\delta/\eta)-2\rho^{\varepsilon}_{\delta}(x)}\,\mathrm{e}^{2Y^2_\varepsilon(x)-2\mathrm{var}(Y^2_\varepsilon(x))}\,\mathbb{1}_{L^2_\varepsilon(x)} \mid \mathcal{H}])$$

Now we can use Girsanov's theorem, as in the proof of Lemma 3.18, to get rid of the $\exp\{2Y^2_\varepsilon(x) - 2\mathrm{var}(Y^2_\varepsilon(x))\}$ term. More precisely, changing measure by $\exp\{2Y^2_\varepsilon(x)-2\mathrm{var}(Y^2_\varepsilon(x))\}$ has the effect of shifting the law of $(\tilde{h}_\eta(x)-\tilde{h}_\delta(x))_{\eta\in[\varepsilon,\delta]}$ by adding on the deterministic function $\rho^{\varepsilon}_{\delta}(x) - \rho^{\varepsilon}_{\eta}(x)$. We then see that the conditional expectation above is nothing but the expectation of the Brownian motion martingale (3.9), starting from W. The result follows.

(4) For this we bound the indicator above by 1, and take out the parts which are measurable with respect to $\mathcal{H}$ as in part (3). Then we are left with the expectation of $|Y^2_\varepsilon(x)|$ under a shifted law, where $Y^2_\varepsilon(x)$ is still a Gaussian with O(1) mean and variance (since $Y^2_\varepsilon(x)$ has bounded covariances with everything.) This proves the claim.

□

This lemma allows us to deduce that the integrand of (3.13), in the case $|x-y| > 3\varepsilon$, is less than or equal to some constant, depending on β only, times

$$\mathbb{E}\big[(\tilde{f}^{\beta}_{\delta,2\lambda_\varepsilon(x)}(x)+1+|Y^1_\varepsilon(x)|)(\tilde{f}^{\beta}_{\delta,2\lambda_\varepsilon(y)}(y)+1+|Y^1_\varepsilon(y)|)$$
$$\times\,\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(x)}(x)}\,\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(y)}(y)}\,\mathrm{e}^{2Y^1_\varepsilon(x)}\,\mathrm{e}^{2Y^1_\varepsilon(y)}\,\mathbb{1}_{G^R_{\delta,\delta}(x)}\mathbb{1}_{G^R_{\delta,\delta}(y)}\big]$$

where $\delta = \delta(x,y) = |x-y|/3$. Here we used that $\rho^{\varepsilon}_{\delta}(\cdot)$ and $\mathrm{var}(Y^1_\varepsilon(\cdot))$ are uniformly bounded, and changed $G^R_{\varepsilon,\varepsilon_0}(\cdot)$ to the larger $\mathcal{H}$-measurable event $G^R_{\delta,\delta}(\cdot)$, so that it would not interfere with the conditioning step.

Now we can use the definition of $G^R_{\delta,\delta}$. This, together with the fact that $Y^1_\varepsilon(\cdot)$ has bounded variance and covariance with everything, tells us that the above is bounded by a constant times

$$\delta^{-2}F_R(\log(1/\delta))\ ;\ \ F_R(z) := R^2(1+\sqrt{z\log(1+z)})^2\,\mathrm{e}^{-\frac{2\sqrt{z}}{R(\log(2+z))}}\ .$$

As in the sketch of this proof (given just after the statement of Lemmas 3.18 and 3.19) we have put deterministic bounds on $\tilde{f}^{\beta}_{\delta,2\lambda_\varepsilon(x)}(x)$, $\tilde{f}_{\delta,2\lambda_\varepsilon(y)}(y)$ and $\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(y)}(y)}$, and integrated over $\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(x)}(x)}$.

Hence we can bound the integral (3.13), restricted to the set $x \in \mathcal{O}$, $y \in \mathcal{O}\setminus$

$B(x,3\varepsilon)$, by a multiple of

$$\int_{x\in\mathcal{O}}\int_{y\notin B(x,3\varepsilon)}\frac{1}{|x-y|^2}F_R(-\log|x-y|)\,dy\,dx\le C\int_{x\in\mathcal{O}}\int_0^{\log(1/\varepsilon)}F_R(u)du\,dx.$$

Since F_R is integrable we see that this is uniformly bounded in ε.

Finally, we must deal with the integral over the set $x\in\mathcal{O}$, $y\in B(x,3\varepsilon)$. By the same reasoning as above (although now we do not need to do any conditioning, since $\delta(x,y)=\varepsilon$) we see that the integrand on this region is less than some constant times $\varepsilon^{-2}F_R(\log(1/\varepsilon))$. That the integral is uniformly bounded in ε then follows from that fact that $F_R(\log(1/\varepsilon))$ is bounded, and that the area of $B(x,3\varepsilon)$ is $\mathrm{O}(\varepsilon^2)$. □

3.3.2 Convergence

We now need to show that $D_\varepsilon^\beta(\mathcal{O})$ converges (in some sense) as $\varepsilon\to 0$. To do this, we define the change of measure

$$\frac{d\mathbb{Q}^{\beta,\varepsilon}}{d\mathbb{P}}=\frac{D_\varepsilon^\beta(\mathcal{O})}{\mathbb{E}[D_\varepsilon^\beta(\mathcal{O})]}\tag{3.14}$$

for each $\varepsilon>0$. Note that this is not a martingale change of measure, but it is well defined for each $\varepsilon>0$. We will prove the following lemma (from now on we drop the dependence on $\mathcal{O}$ from our notation for compactness.)

Proposition 3.21. *For each fixed β and ε_0, and for any $\delta>0$*

$$\mathbb{Q}^{\beta,\varepsilon}\left[\left|\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\sqrt{\log(1/\varepsilon)}-\sqrt{\frac{2}{\pi}}\right|>\delta\right]\to 0$$

as $\varepsilon\to 0$.

Remark 3.22. *Since D_ε^β and M_ε^β are close to D_ε and M_ε for large β (Lemma 3.5) and $\mathbb{Q}^{\beta,\varepsilon}$ is defined by a uniformly integrable change of measure (Proposition 3.17) this is almost exactly what we need (recall that by Theorem 3.10 we have $M_\varepsilon\sqrt{\log(1/\varepsilon)}\to\sqrt{\pi/2}\mu'$ as $\varepsilon\to 0$.) Indeed, we will see that the proof of Theorem 3.1 follows in a straightforward manner once we have completed the proof of Proposition 3.21.*

Remark 3.23. *The proof of Proposition 3.21 follows the general outline of the main proof in [AS14]. However, the details of each step are somewhat different, and rely on the precise way we have constructed D_ε^β. One of the main difficulties is to make* *exact statements about the behaviour of h_ε using what we know about the behaviour of $\tilde{h}_\varepsilon$.*

Before starting the proof, we make a few remarks about the change of measure (3.14). Define

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}(dx,dh) = \frac{f^{\beta}_{\varepsilon,2}(x)\, \mathrm{e}^{g_{\varepsilon,2}(x)}\, \mathbb{1}_{\{x\in\mathcal{O}\}}\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}(x)>1\}}dx\, \mathbb{P}[dh]}{\mathbb{E}[D^{\beta}_{\varepsilon}]}$$

to be the *rooted measure* on (h,x) where h is a field and x is a point in $\mathcal{O}$. Introducing this type of measure is a classical tool for dealing with branching processes, that also comes in very useful in the context of Gaussian multiplicative chaos. We have the following description of how the point x and the field h interact under $\hat{\mathbb{Q}}^{\beta,\varepsilon}$:

- the marginal law of h under $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ is $\mathbb{E}[D^{\beta}_{\varepsilon}]^{-1}D^{\beta}_{\varepsilon}d\mathbb{P}$ (i.e. the same law as under $\mathbb{Q}^{\beta,\varepsilon}$);
- the marginal law of x under $\hat{\mathbb{Q}}^{\beta,\varepsilon}$, that we shall call $dm^{\beta,\varepsilon}(x)$, is proportional to
$$Z^{\beta}_{\varepsilon}(x) := \mathbb{1}_{\{x\in\mathcal{O}\}}\mathbb{E}[f^{\beta}_{\varepsilon,2}(x)\, \mathrm{e}^{g_{\varepsilon,2}(x)}\, \mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}(x)>1\}}].$$
- the conditional law of the field h given the point x is given by
$$\mathbf{Q}^{\beta,\varepsilon}_x := \hat{\mathbb{Q}}^{\beta,\varepsilon}(\cdot \mid x) = (Z^{\beta}_{\varepsilon}(x))^{-1} f^{\beta}_{\varepsilon,2}(x)\, \mathrm{e}^{g_{\varepsilon,2}(x)}\, \mathbb{1}_{\{x\in\mathcal{O}\}}\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}(x)>1\}}\, d\mathbb{P}.$$
- the conditional law of the point x given the field h is proportional to
$$f^{\beta}_{\varepsilon,2}(x)\, \mathrm{e}^{g_{\varepsilon,2}(x)}\, \mathbb{1}_{\{x\in\mathcal{O}\}}\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2}(x)>1\}}dx.$$

Also note that
$$\frac{d\mathbf{Q}^{\beta,\varepsilon}_x}{d\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x} = \frac{\tilde{Z}^{\beta}_{\varepsilon}(x)}{Z^{\beta}_{\varepsilon}(x)}\frac{f^{\beta}_{\varepsilon,2}(x)}{\tilde{f}^{\beta}_{\varepsilon,2}(x)}\mathbb{1}_{\{f^{\beta}_{\varepsilon,2\lambda_\varepsilon(x)}(x)\geq 1\}}$$

where we recall from the proof of Lemma 3.18 that under $\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x$ the process

$$\{(\tilde{f}^{\beta}_{e^{-u},2\lambda_\varepsilon}(x) - \rho^{\varepsilon}_{e^{-u}}(x)\, ;\, u\in[\log(1/\varepsilon_0),\log(1/\varepsilon)]\}$$

has the law of a 3d Bessel process, whose starting point is also biased (and a.s. positive.) In fact, one of the key ideas in the proof of Proposition 3.21 will be to say that $f^{\beta}_{\varepsilon}(x)$ under $\mathbf{Q}^{\beta,\varepsilon}_x$ also behaves essentially like a Bessel process. As a warm up, let us first prove the following:

Lemma 3.24.

$$\frac{Z_\varepsilon^\beta(x)}{\tilde{Z}_\varepsilon^\beta(x)} \to 1 \tag{3.15}$$

uniformly in x as $\varepsilon \to 0$.

This justifies in some sense that the measures $\mathbf{Q}_x^{\beta,\varepsilon}$ and $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}$ are similar for small ε, and is a result we will use many times.

Proof. We consider the ratio

$$\begin{aligned} Z_\varepsilon^\beta(x)/\tilde{Z}_\varepsilon^\beta(x) &= (\tilde{Z}_\varepsilon^\beta)^{-1}\mathbb{E}[f_{\varepsilon,2}^\beta(x)\, \mathrm{e}^{g_{\varepsilon,2}(x)}\, \mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f_{\varepsilon,2}^\beta(x)>1\}}] \\ &= \tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[(f_{\varepsilon,2}^\beta(x)/\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x))\mathbb{1}_{\{f_{\varepsilon,2}^\beta>1\}}\right]. \end{aligned}$$

To show this converges to 1 we write, using decomposition (3.11),

$$\frac{f_{\varepsilon,2}^\beta(x)}{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)} = 1 + \mathrm{o}(1) + \frac{\mathrm{O}(1) - Y_\varepsilon(x)}{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)}.$$

Then by exactly the same Cauchy–Schwarz argument as in the proof of Lemma 3.18, it is enough to show that

$$\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x) \le 1] \to 0 \tag{3.16}$$

uniformly in x. Since $\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)$ is close to $\sqrt{\log(1/\varepsilon)}$ with high probability under $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}$, and $f_{\varepsilon,2}^\beta(x) = \lambda_\varepsilon(x)\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x) + \mathrm{O}(1) + Y_\varepsilon(x)$, it is sufficient to control the tails of $Y_\varepsilon(x)$ under $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}$. For this we observe that, by Cauchy–Schwarz again,

$$\begin{aligned} \tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[\mathbb{1}_{\{Y_\varepsilon(x)>a\}}]^2 &\le \tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[\tilde{f}_{\varepsilon,2\lambda_\varepsilon}(x)]\, \mathbb{E}[\mathbb{1}_{\{Y_\varepsilon(x)>a\}}\, \mathrm{e}^{\tilde{g}_{\varepsilon,2\lambda_\varepsilon}(x)}\, \mathrm{e}^{g^Y_{\varepsilon,2}(x)}] \\ &\le \sqrt{\log(1/\varepsilon)}\, \mathrm{e}^{-ka} \end{aligned} \tag{3.17}$$

for some k, since $Y_\varepsilon(x)$ is Gaussian with bounded variance under $\mathbb{P}$. This allows us to conclude. □

Proof of Proposition 3.21. Our strategy to prove Proposition 3.21 is to show the following two things:

$$\mathbb{Q}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\right] = \sqrt{\frac{2}{\pi\log(1/\varepsilon)}} + \mathrm{o}\left(\frac{1}{\sqrt{\log(1/\varepsilon)}}\right) \text{ as } \varepsilon \to 0; \text{ and} \tag{3.18}$$

$$\mathbb{Q}^{\beta,\varepsilon}\left[\left(\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\right)^2\right] \le \frac{2}{\pi\log(1/\varepsilon)} + \mathrm{o}\left(\frac{1}{\log(1/\varepsilon)}\right) \text{ as } \varepsilon \to 0. \tag{3.19}$$

The result then follows using Jensen's and Markov's inequalities. (4.12) is relatively straightforward. Observe that, by the discussion preceeding this proof, we have

$$\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta} = \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{1}{f_{\varepsilon,2}^\beta(x)} \mid h\right]. \tag{3.20}$$

This means that

$$\mathbb{Q}^{\beta,\varepsilon}\left[M_\varepsilon^\beta/D_\varepsilon^\beta\right] = \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[M_\varepsilon^\beta/D_\varepsilon^\beta\right] = \hat{\mathbb{Q}}^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x)^{-1}] = \int_{\mathcal{O}} \mathbf{Q}_x^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x)^{-1}]\, dm^{\beta,\varepsilon}(x),$$

which is a useful representation, because we can write

$$\mathbf{Q}_x^{\beta,\varepsilon}[f_\varepsilon^\beta(x)^{-1}] = \frac{\tilde{Z}_\varepsilon^\beta(x)}{Z_\varepsilon^\beta(x)}\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[\tilde{f}_{\varepsilon,2}^\beta(x)^{-1}\mathbb{1}_{\{f_{\varepsilon,2}^\beta(x)>1\}}]$$

for each $x \in \mathcal{O}$. The first moment estimate then follows by (3.16), Lemma 3.15 and Lemma 3.11, parts (1) and (2).

(3.19) is rather more difficult, and requires several steps.

Step 1: *We show that restricting to an event of high probability under $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ does not affect our second moment too much. That is, we show that if we can find a sequence of events $E_\varepsilon = E_\varepsilon(x)$ with $\hat{\mathbb{Q}}^{\beta,\varepsilon}[E_\varepsilon] \to 1$ and*

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\frac{1}{f_{\varepsilon,2}^\beta(x)}\mathbb{1}_{E_\varepsilon}\right] \leq \frac{2}{\pi\log(1/\varepsilon)} + \mathrm{o}\left(\frac{1}{\log(1/\varepsilon)}\right), \tag{3.21}$$

then this will prove (3.19).

To see how this implies (3.19), take such an event E_ε and write

$$\begin{aligned}
\mathbb{Q}^{\beta,\varepsilon}\left[\left(\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\right)^2\right] &= \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\left(\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\right)^2\right] \\
&= \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\hat{\mathbb{Q}}^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x)^{-1}\mathbb{1}_{E_\varepsilon} \mid h]\right] + \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\hat{\mathbb{Q}}^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x)^{-1}\mathbb{1}_{E_\varepsilon^c} \mid h]\right] \\
&= \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\frac{1}{f_{\varepsilon,2}^\beta(x)}\mathbb{1}_{E_\varepsilon}\right] + \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\hat{\mathbb{Q}}^{\beta,\varepsilon}[f_{\varepsilon,2}^\beta(x)^{-1}\mathbb{1}_{E_\varepsilon^c} \mid h]\right].
\end{aligned}$$

We would like to show that the second term in the final expression is $\mathrm{o}(\log(1/\varepsilon)^{-1})$. For this, it is enough by Cauchy–Schwarz, to show that

- $\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[(M_\varepsilon^\beta/D_\varepsilon^\beta)^2\right] = \mathrm{O}(\log(1/\varepsilon)^{-1})$, and
- $\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\xi_\varepsilon^2\right] = \mathrm{o}(\log(1/\varepsilon)^{-1})$ where $\xi_\varepsilon := \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[f_{\varepsilon,2}^\beta(x)^{-1}\mathbb{1}_{E_\varepsilon^c} \mid h\right]$.

We deal with each point in turn. For the first point, note that by conditional Jensen's inequality we have

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[(M^\beta_\varepsilon/D^\beta_\varepsilon)^2\right] \le \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[f^\beta_{\varepsilon,2}(x)^{-2}\right] = \int_{\mathcal{O}} \mathbf{Q}^{\beta,\varepsilon}_x\left[f^\beta_{\varepsilon,2}(x)^{-2}\right] dm^{\beta,\varepsilon}(x)$$

and then by changing measure and rearranging as usual, we can write

$$\mathbf{Q}^{\beta,\varepsilon}_x\left[f^\beta_{\varepsilon,2}(x)^{-2}\right] = (\tilde{Z}^\beta_\varepsilon(x)/Z^\beta_\varepsilon(x))\,\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[\frac{1}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)f^\beta_{\varepsilon,2}(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)>1\}}\right].$$

To show that this is $\mathrm{O}(\log(1/\varepsilon)^{-1})$ we need to be a little bit careful, although heuristically it is clear from the fact that $\tilde{f}$ is a Bessel process and $Y_\varepsilon(x)$ is small. The way to make this precise is to consider the expectation on the "good" event,

$$\{\tilde{f}_{\varepsilon,2\lambda_\varepsilon} > \log(1/\varepsilon)^{1/4}\} \cap \{Y_\varepsilon(x) < (1/2)\log(1/\varepsilon)^{1/4}\}$$

and its complement separately. On the good event we have that $f^\beta_{\varepsilon,2}(x) \ge c\tilde{f}^\beta_{\varepsilon,\lambda_\varepsilon}(x)$ for some constant c, and so the expectation is $\mathrm{O}(\log(1/\varepsilon)^{-1})$ by Lemma 3.11, part (2). On the bad event, we use (3.16) and Lemma 3.11, part (5), to see that the expectation is also $\mathrm{O}(\log(1/\varepsilon)^{-1})$.

Now we treat the second point. By Jensen's inequality, and for any $a > 0$, we have

$$\begin{aligned}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\xi^2_\varepsilon\right] &\le \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[f^\beta_\varepsilon(x)^{-2}\mathbb{1}_{E^c_\varepsilon}\right] \\ &= \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{\mathbb{1}_{E^c_\varepsilon}}{f^\beta_{\varepsilon,2}(x)^2}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)\ge a\sqrt{\log(1/\varepsilon)}\}}\right] + \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{1}{f^\beta_{\varepsilon,2}(x)^2}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)< a\sqrt{\log(1/\varepsilon)}\}}\right].\end{aligned}$$

It is clear by definition that the first term is less than $\hat{\mathbb{Q}}^{\beta,\varepsilon}[E^c_\varepsilon]/(a^2\log(1/\varepsilon))$, and for the second term, we write it as

$$\int_{\mathcal{O}} \frac{\tilde{Z}^\beta_\varepsilon(x)}{Z^\beta_\varepsilon(x)}\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[\frac{1}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)f^\beta_{\varepsilon,2}(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)>1\}}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)\ge a\sqrt{\log(1/\varepsilon)}\}}\right] dm^{\beta,\varepsilon}(x). \quad (3.22)$$

Similarly to before, we consider the expectation on the event

$$\{Y_\varepsilon(x) < a\log(1/\varepsilon)^{1/4}\} \cap \{\tilde{f}^\beta_{2,\lambda_\varepsilon}(x) > a\log(1/\varepsilon)^{1/4}\}$$

and its complement separately. This allows us to bound (3.22) by some constant times $a\log(1/\varepsilon)^{-1} + \exp(-ak\log(1/\varepsilon)^{1/4})$, where $k > 0$ is the constant from (3.17). Thus, $\limsup_\varepsilon \log(1/\varepsilon)\,\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\xi^2_\varepsilon\right] \le Ca$ for any $a > 0$ and some fixed finite C. Taking

$a \to 0$, this allows us to conclude step 1.

Step 2: *We define the event E_ε, and set up the scales for the multiscale argument we will use.*

To do this we let $r_\varepsilon > \varepsilon$ be a sequence with

$$\frac{\log(1/r_\varepsilon)}{\log(1/\varepsilon)^{1/3}} \to \infty \text{ and } \frac{\log(1/r_\varepsilon)}{\log(1/\varepsilon)^{1/2}} \to 0 \tag{3.23}$$

as $\varepsilon \to 0$ (so r_ε is tending to 0 much slower than ε). Given this, we break up D and M as

$$D_\varepsilon^\beta = D_\varepsilon^{\beta,in} + D_\varepsilon^{\beta,out} \quad \text{and} \quad M_\varepsilon^\beta = M_\varepsilon^{\beta,in} + M_\varepsilon^{\beta,out}$$

where the subscript *in* refers to the integral inside $B(x, r_\varepsilon)$ and the subscript *out* refers to the integral outside of it.

The basic idea is that $D_\varepsilon^{\beta,in}$ and $M_\varepsilon^{\beta,in}$ will be small with high probability (this will be part of the definition of E_ε) and on this event, $M_\varepsilon^\beta / D_\varepsilon^\beta$ will be close to $M_\varepsilon^{\beta,out}/D_\varepsilon^{\beta,out}$. Heuristically, this occurs with high probability because the limits of M_ε and D_ε should be atomless measures, and r_ε is tending to 0. Next, we claim that $M_\varepsilon^{\beta,out}/D_\varepsilon^{\beta,out}$ is essentially independent of $f_{\varepsilon,2}^\beta(x)$. This is because r_ε is much larger than ε and f is approximately a (time changed) Bessel process, so its value at time ε is basically independent of its value at time $r_\varepsilon - \varepsilon$. From here (3.21) follows, since we already know that $M_\varepsilon^\beta / D_\varepsilon^\beta$ and $f_{\varepsilon,2}^\beta(x)^{-1}$ have (the same) expectation, of the right order.

We now choose our event E_ε, according to this plan. To do this, we first have to observe that, by the Markov property of the field, $Y_\varepsilon(x) = h_\varepsilon(x) - \lambda_\varepsilon(x)\tilde{h}_\varepsilon(x)$ can be written as

$$Y_\varepsilon(x) := Y_\varepsilon^1(x) + Y_\varepsilon^2(x),$$

where Y_ε^2 is independent of $h|_{D \setminus B(x, r_\varepsilon - \varepsilon)}$ and Y_ε^1 is measurable with respect to $h|_{D \setminus B(x, r_\varepsilon - \varepsilon)}$ (see the proof of Lemma 3.20 for a more detailed explanation.) Given this definition, we set $E_\varepsilon = E_\varepsilon^1 \cap E_\varepsilon^2$ where

$$E_\varepsilon^1 = \{D_\varepsilon^{\beta,in} \le \log(1/\varepsilon)^{-2}\}$$
$$E_\varepsilon^2 = \{\tilde{f}_{u,2\lambda_\varepsilon}^\beta(x) \in [\sqrt[3]{\log(1/u)}, \log(1/u)] \; \forall u \in [r_\varepsilon - \varepsilon, r_\varepsilon]\} \cap \{Y_\varepsilon^1(x) < (\log(1/\varepsilon))^{1/4}\}$$

(we will prove that $\hat{\mathbb{Q}}^{\beta,\varepsilon}[E_\varepsilon] \to 1$ later on.) We remark that the event E_ε^2 here is needed for the "independence" step.

Step 3: *We split the left hand side of (3.21) into two parts: one concerning the measures restricted to $\mathcal{O} \cap B(x, r_\varepsilon)$, and one concerning the measures restricted to*

$\mathcal{O} \cap (B(x,\varepsilon) \setminus B(x,r_\varepsilon))$. *We show that the first of these is negligible compared to* $\log(1/\varepsilon)$.

More precisely, we write

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\frac{1}{f_{\varepsilon,2}^\beta(x)}\mathbb{1}_{E_\varepsilon}\right] \leq \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,in}}{D_\varepsilon^\beta}\frac{1}{f_{\varepsilon,2}^\beta(x)}\mathbb{1}_{E_\varepsilon}\right] + \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}}\frac{1}{f_{\varepsilon,2}^\beta(x)}\mathbb{1}_{E_\varepsilon^2}\right]. \tag{3.24}$$

Then by definition we have that $M_\varepsilon^{\beta,in} \leq D_\varepsilon^{\beta,in}$, and so on the event E_ε it holds that $M_\varepsilon^{\beta,in} \leq \log(1/\varepsilon)^{-2}$. Moreover, we know that $f_{\varepsilon,2}^\beta(x)$ is greater than 1 under $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ and also that $\hat{\mathbb{Q}}^{\beta,\varepsilon}[1/D_\varepsilon^\beta] = \mathbb{E}[D_\varepsilon^\beta]^{-1} = \mathrm{O}(1)$. Thus, the first term is $\mathrm{o}(\log(1/\varepsilon)^{-1})$ and we need only treat the second term.

Step 4: *We condition on the field outside of $B(x, r_\varepsilon - \varepsilon)$ in order to factorise the second term on the right-hand side of (3.24). We show that the conditional expectation of $f_{\varepsilon,2}^\beta(x)^{-1}$ is of order $\sqrt{2/\pi \log(1/\varepsilon)}(1+\mathrm{o}(1))$ uniformly on E_ε^2.*

More precisely, we condition on $\mathcal{F}_{r_\varepsilon-\varepsilon}$, the σ-algebra generated by the point x and the field h restricted to $D \setminus B(x, r_\varepsilon - \varepsilon)$. Then $M_\varepsilon^{\beta,out}$, $D_\varepsilon^{\beta,out}$ and E_ε^2 are measurable with respect to $\mathcal{F}_{r_\varepsilon-\varepsilon}$, meaning that the second term on the right-hand side of (3.24) is equal to

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}}\mathbb{1}_{E_\varepsilon^2}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{1}{f_{\varepsilon,2}^\beta(x)} \mid \mathcal{F}_{r_\varepsilon-\varepsilon}\right]\right].$$

Now we write

$$\begin{aligned}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{1}{f_{\varepsilon,2}^\beta(x)}\,\middle|\,\mathcal{F}_{r_\varepsilon-\varepsilon}\right] &= \mathbf{Q}_x^{\beta,\varepsilon}\left[\frac{1}{f_{\varepsilon,2}^\beta(x)}\,\middle|\, h|_{D\setminus B(x,r_\varepsilon-\varepsilon)}\right] \\ &= \frac{\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{1}{\tilde{f}_{\varepsilon,2}^\beta(x)}\mathbb{1}_{\{f_{\varepsilon,2\lambda_\varepsilon}^\beta(x)\geq 1\}}\,\middle|\, h|_{D\setminus B(x,r_\varepsilon-\varepsilon)}\right]}{\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{f_{\varepsilon,2\lambda_\varepsilon}^\beta(x)}{\tilde{f}_{\varepsilon,2}^\beta(x)}\mathbb{1}_{\{f_{\varepsilon,2}^\beta(x)\geq 1\}}\,\middle|\, h|_{D\setminus B(x,r_\varepsilon-\varepsilon)}\right]}.\end{aligned}$$

We will show that, *on the event E_ε^2*, the numerator in the final expression is less than or equal to $\sqrt{2/(\pi\log(1/\varepsilon))} + \mathrm{o}(\log(1/\varepsilon)^{-1/2})$ and the denominator is $1+\mathrm{o}(1)$. To do this we observe that, on E_ε^2 and under the conditional law $\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}[\,\cdot \mid h_{D\setminus B(x,r_\varepsilon-\varepsilon)}]$:

- $\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)$ has the law of a Bessel process started from a position in $[\sqrt[3]{\log(1/r_\varepsilon-\varepsilon)}, \log(1/r_\varepsilon-\varepsilon)]$ and evaluated at time $\log(\varepsilon) - \log(k_\varepsilon - \varepsilon)$;
- by choice of r_ε and Lemma 3.11 part (1), this implies that the conditional expectation of $(\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x))^{-1}$ is equal to $\sqrt{2/(\pi\log(1/\varepsilon))}(1+\mathrm{o}(1))$;
- the conditional expectation of $|Y_\varepsilon^2|/\tilde{f}_{\varepsilon,2\lambda_\varepsilon}(x)$ is $\mathrm{o}(1)$, by Cauchy–Schwarz; and

- $|Y^1_\varepsilon(x)|$ times the conditional expectation of $(\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x))^{-1}$ is also $\mathrm{o}(1)$, by the second point, and definition of E^2_ε.

Together these imply that, uniformly on E^2_ε,

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[1/f^\beta_{\varepsilon,2}(x) \mid \mathcal{F}_{r_\varepsilon-\varepsilon}\right] \leq \sqrt{2/(\pi\log(1/\varepsilon))}(1+\mathrm{o}(1)).$$

Step 5: *We show that* $\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M^{\beta,out}_\varepsilon}{D^{\beta,out}_\varepsilon}\mathbb{1}_{E^2_\varepsilon}\right]$ *is also bounded above by* $\sqrt{2/(\pi\log(1/\varepsilon))}(1+\mathrm{o}(1))$. *This completes the proof of (3.21) for our choice of* E_ε.

This is the most delicate step. To do this, we define yet another event

$$E^3_\varepsilon = \{\tilde{f}^\beta_{u,2\lambda_\varepsilon}(x) \geq \sqrt[6]{\log(1/r_\varepsilon)}\ \forall u \in [\varepsilon, r_\varepsilon]\}.$$

Step 5(i): We will first show that

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M^{\beta,out}_\varepsilon}{D^{\beta,out}_\varepsilon}\mathbb{1}_{E^2_\varepsilon}\mathbb{1}_{E^3_\varepsilon}\right] \geq \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M^{\beta,out}_\varepsilon}{D^{\beta,out}_\varepsilon}\mathbb{1}_{E^2_\varepsilon}\right](1+\mathrm{o}(1)) \tag{3.25}$$

so we can instead consider the term on the left-hand side (which turns out to be easier to deal with.) To see why (3.25) is true we again condition on $\mathcal{F}_{r_\varepsilon-\varepsilon}$. This gives us that

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M^{\beta,out}_\varepsilon}{D^{\beta,out}_\varepsilon}\mathbb{1}_{E^2_\varepsilon}\mathbb{1}_{E^3_\varepsilon}\right] \geq \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M^{\beta,out}_\varepsilon}{D^{\beta,out}_\varepsilon}\mathbb{1}_{E^2_\varepsilon}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[E^3_\varepsilon \mid \mathcal{F}_{r_\varepsilon-\varepsilon}\right]\right]$$

where by changing measure as in step 4 we have

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[(E^3_\varepsilon)^c \mid \mathcal{F}_{r_\varepsilon-\varepsilon}\right] = \frac{\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[\frac{f^\beta_{\varepsilon,2}(x)}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)\geq 1\}}\mathbb{1}_{(E^3_\varepsilon)^c}\,\middle|\, h|_{D\setminus B(x,r_\varepsilon-\varepsilon)}\right]}{\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[\frac{f^\beta_{\varepsilon,2}(x)}{\tilde{f}^\beta_{\varepsilon,2\lambda_\varepsilon}(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)\geq 1\}}\,\middle|\, h|_{D\setminus B(x,r_\varepsilon-\varepsilon)}\right]}.$$

We already know that the denominator is $1+\mathrm{o}(1)$ uniformly on E^2_ε by our previous discussion. In fact, the numerator is also $\mathrm{o}(1)$ uniformly on E^2_ε. To show this, again using our observations from step 4, it is enough for us to prove that

$$\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[E^3_\varepsilon \mid h|_{D\setminus(B(x,r_\varepsilon-\varepsilon)}\right] \to 1$$

uniformly on E^2_ε. For this we again use the fact that, under this conditional law, the process $\tilde{f}^\beta_{u,2\lambda_\varepsilon}(x)$ for $u \leq r_\varepsilon-\varepsilon$ is a time-changed Bessel process (plus a small deterministic fluctuation $\rho^\varepsilon_u(x)$) starting from a position in $[\sqrt[3]{\log(1/r_\varepsilon-\varepsilon)}, \log(1/r_\varepsilon-\varepsilon)]$. Thus we need to calculate the probability that such a Bessel process, and we can clearly forget about the fluctuations, remains greater than $\log(1/r_\varepsilon)$ up to time $\log(1/\varepsilon) - \log(1/r_\varepsilon - \varepsilon)$. It is clear that this probability is smallest if we take the starting point x_0 to be $\sqrt[3]{\log(1/r_\varepsilon - \varepsilon)}$. In this case we have, writing $\mathbf{Q}_y$ for the law of a Bessel process started at y, and by scaling, that

$$\mathbf{Q}_{x_0}(X_t \geq \sqrt[6]{\log(1/r_\varepsilon)}\ \forall t \in [0, \log(1/\varepsilon) - \log(1/r_\varepsilon - \varepsilon)]) \geq \mathbf{Q}_1(X_t \geq \frac{\sqrt[6]{\log(1/r_\varepsilon)}}{\sqrt[3]{\log(1/r_\varepsilon - \varepsilon)}}\ \forall t \in [0, \infty]).$$

Taking $\varepsilon \to 0$ we see that this converges to 1: the probability that a 3d Bessel process started at 1 never hits 0.

Step 5(ii): Having done this, we can now consider the left-hand side of (3.25) and instead try to show that this is bounded above by $\sqrt{2/(\pi \log(1/\varepsilon))}(1 + \mathrm{o}(1))$. Again this will require a few arguments. We write

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}} \mathbb{1}_{E_\varepsilon^2 \cap E_\varepsilon^3}\right] \leq \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}} \mathbb{1}_{E_\varepsilon^2 \cap E_\varepsilon^3} \mathbb{1}_{E_\varepsilon^1} \mathbb{1}_{\{D_\varepsilon^\beta > \log(1/\varepsilon)^{-1}\}}\right] +$$
$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}} \mathbb{1}_{E_\varepsilon^2} \mathbb{1}_{E_\varepsilon^1} \mathbb{1}_{\{D_\varepsilon^\beta \leq \log(1/\varepsilon)^{-1}\}}\right] + \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^{\beta,out}}{D_\varepsilon^{\beta,out}} \mathbb{1}_{E_\varepsilon^2 \cap E_\varepsilon^3} \mathbb{1}_{(E_\varepsilon^1)^c}\right]$$

and will show that the first term is of the order we want, and the second and third are negligible. Indeed, it is clear that the first term is less than or equal to

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta}\right](1 + \mathrm{o}(1)) \leq \sqrt{\frac{2}{\pi \log(1/\varepsilon)}}(1 + \mathrm{o}(1))$$

by definition of the events (these imply that $D_\varepsilon^\beta / D_\varepsilon^{\beta,out} = 1 + \mathrm{o}(1)$) and our previous estimate (4.12) for the first moment. Moreover by Markov's inequality for $(1/D_\varepsilon^\beta)$, and the fact that $M_\varepsilon^\beta / D_\varepsilon^\beta \leq 1$, the second is $\mathrm{o}(\sqrt{\log(1/\varepsilon)}^{-1})$.

Step 5(iii): So we are left to deal with the third term, which we would also like to show is $\mathrm{o}(\sqrt{\log(1/\varepsilon)}^{-1})$. To do this, we first observe that we can bound it above by

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[E_\varepsilon^2 \cap E_\varepsilon^3 \cap (E_\varepsilon^1)^c\right] \leq \hat{\mathbb{Q}}^{\beta,\varepsilon}\left[E_\varepsilon^3 \cap \left\{D_\varepsilon^{\beta,in} > \frac{1}{\log(1/\varepsilon)^2}\right\}\right]. \tag{3.26}$$

Our strategy here will be to use Markov's inequality for $D_\varepsilon^{\beta,in}$. For this we have to calculate the $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ expectation of $D_\varepsilon^{\beta,in}$, which by definition is the same as calculating the $\mathbb{P}$ expectation of $D_\varepsilon^\beta \times D_\varepsilon^{\beta,in}$. Thus, we can use similar techiniques to those in the proof of uniform integrability (Lemma 3.18), where we calculated the $\mathbb{P}$ expectation of $(D_\varepsilon^\beta)^2$ on a "good" event.

As we did there, we will break up $D_\varepsilon^{\beta,in}$ into two parts: the integral over $B(x, 3\varepsilon)$, and the rest. To deal with the integral over $B(x,3\varepsilon)$ we define a further event E_ε^4 (which has high probability) and on which $f_{\varepsilon,2}^\beta$ is close to $\sqrt{\log(1/\varepsilon)}$. Crude estimates on this event, using that $|B(x,3\varepsilon)| = \mathrm{O}(\varepsilon^2)$, then give the desired expectation. To deal with the integral over $B(x,r_\varepsilon)\setminus B(x,3\varepsilon)$ we need to be more careful. Here we use the definition of E_ε^3, which allows us to control the value of $\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)$ at all times between r_ε and ε. Applying a decorrelation argument similar that in the proof of Lemma 3.18 then allows us to reach the desired conclusion.

Let us first define E_ε^4, using the following lemma:

Lemma 3.25. *There exists $p \in (0,1/2)$ such that*

$$E_\varepsilon^4 = \{f_{\varepsilon,2}^\beta(x) \in [\log(1/\varepsilon)^{1/2-p}, \log(1/\varepsilon)^{1/2+p}]\}$$

satisfies $\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[(E_\varepsilon^4)^c\right] = \mathrm{o}(\log(1/\varepsilon)^{-1/2})$.

Proof of Lemma 3.25. This is possible because for any $p>0$

$$(E_\varepsilon^4)^c \subset \{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x) \notin [2\log(1/\varepsilon)^{1/2-p}, \frac{1}{2}\log(1/\varepsilon)^{1/2+p}]\} \cup \{|Y_\varepsilon(x)| > \log(1/\varepsilon)^{1/2-p}\}$$

and then

$$\begin{aligned}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[(E_\varepsilon^4)^c\right] \le \int_{\mathcal{O}} &\mathbf{Q}_x^{\beta,\varepsilon}\left[|Y_\varepsilon(x)| > \log(1/\varepsilon)^{1/2-p}\right] \\ &+ \mathbf{Q}_x^{\beta,\varepsilon}\left[\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x) \notin [2\log(1/\varepsilon)^{1/2-p}, \frac{1}{2}\log(1/\varepsilon)^{1/2+p}]\right] dm^{\beta,\varepsilon}(x).\end{aligned}$$

It is easy to see (using the definition of the measure $\mathbf{Q}_x^{\beta,\varepsilon}$) that the first probability inside the integral decays exponentially in ε. For the second, we can write it as

$$\frac{Z_\varepsilon^\beta(x)}{\tilde{Z}_\varepsilon^\beta(x)}\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{f_{\varepsilon,2}^\beta(x)}{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)}\mathbb{1}_{\{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)\notin[2\log(1/\varepsilon)^{1/2-p},\frac{1}{2}\log(1/\varepsilon)^{1/2+p}]\}}\right].$$

which by Cauchy–Schwarz is less than or equal to

$$\begin{aligned}&\lambda_\varepsilon(x)\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\mathbb{1}_{\{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)\notin[2\log(1/\varepsilon)^{1/2-p},\frac{1}{2}\log(1/\varepsilon)^{1/2+p'}]\}}\right] + \\ &\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{(Y_\varepsilon(x)+\mathrm{O}(1))^2}{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)}\right]^{1/2}\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[\frac{\mathbb{1}_{\{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)\notin[2\log(1/\varepsilon)^{1/2-p},\frac{1}{2}\log(1/\varepsilon)^{1/2+p}]\}}}{\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)}\right]^{1/2}.\end{aligned}$$

A standard Bessel calculation, plus the fact that

$$\tilde{\mathbf{Q}}_x^{\beta,\varepsilon}\left[(\mathrm{O}(1)+Y_\varepsilon(x))^2/\tilde{f}_{\varepsilon,2\lambda_\varepsilon}^\beta(x)\right] = \mathrm{O}(1)$$

(seen by changing back to the measure $\mathbb{P}$), gives that this is $o(\log(1/\varepsilon)^{-1/2})$ for some $p < 1/2$. $\square$

Using this new event E_ε^4, and Markov's inequality, we next bound the right-hand side of (3.26) above by

$$\begin{aligned}
&o(\log(1/\varepsilon)^{-1/2}) \qquad (3.27)\\
&+2\log(1/\varepsilon)^2\,\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\mathbb{1}_{E_\varepsilon^4}\int_{w\in B(x,3\varepsilon)} f^\beta_{\varepsilon,2}(w)\mathbb{1}_{L_\varepsilon(w)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(w)>1\}}\,e^{g_{\varepsilon,2}(w)}\,dw\right]\\
&+2\log(1/\varepsilon)^2\,\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\mathbb{1}_{E_\varepsilon^3}\int_{w\in B(x,r_\varepsilon)\setminus B(x,3\varepsilon)} f^\beta_{\varepsilon,2}(w)\mathbb{1}_{L_\varepsilon(w)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(w)>1\}}\,e^{g_{\varepsilon,2}(w)}\,dw\right]
\end{aligned}$$

where the first term comes from the $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ probability of E_ε^4. Recall that, to conclude, we need to show this whole expression is $o(\log(1/\varepsilon)^{-1/2})$. Let us look at the expectation in the second term. By definition of $\hat{\mathbb{Q}}^{\beta,\varepsilon}$ this is equal to

$$\begin{aligned}
&\mathbb{E}[D^\beta_\varepsilon]^{-1}\int_{\mathcal{O}}\int_{w\in B(x,3\varepsilon)}\mathbb{E}\left[\mathbb{1}_{E_\varepsilon^4} f^\beta_{\varepsilon,2}(w)\mathbb{1}_{L_\varepsilon(w)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(w)>1\}}\,e^{g_{\varepsilon,2}(w)}\,f^\beta_{\varepsilon,2}(x)\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)>1\}}\,e^{g_{\varepsilon,2}(x)}\right]dw\,dx\\
&\le \mathbb{E}[D^\beta_\varepsilon]^{-1}\int_{\mathcal{O}}\int_{w\in B(x,3\varepsilon)}\varepsilon^{-2}\log(1/\varepsilon)^{1/2+p}\,e^{-2\log(1/\varepsilon)^{1/2-p}}\,\mathbb{E}\left[f^\beta_{\varepsilon,2}(w)\mathbb{1}_{L_\varepsilon(w)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(w)>1\}}\,e^{g_{\varepsilon,2}(w)}\right]dw\,dx\\
&\le O(1)\times\log(1/\varepsilon)^{1/2+p}\,e^{-2\log(1/\varepsilon)^{1/2-p}}
\end{aligned}$$

where the second line follows from the definition of E_ε^4. Hence, the second term of (3.27) is $o(\log(1/\varepsilon)^{-1/2})$.

We finish by dealing with the third term of (3.27). Writing $A_\varepsilon = B(x,r_\varepsilon)\setminus B(x,3\varepsilon)$, we have that the expectation in this term is equal to

$$\mathbb{E}[D^\beta_\varepsilon]^{-1}\int_{\mathcal{O}}\int_{A_\varepsilon}\mathbb{E}\left[\mathbb{1}_{E_\varepsilon^3} f^\beta_{\varepsilon,2}(w)\mathbb{1}_{L_\varepsilon(w)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(w)>1\}}\,e^{g_{\varepsilon,2}(w)}\,f^\beta_{\varepsilon,2}(x)\mathbb{1}_{L_\varepsilon(x)}\mathbb{1}_{\{f^\beta_{\varepsilon,2}(x)>1\}}\,e^{g_{\varepsilon,2}(x)}\right]dw\,dx.$$

Then, by exactly the same reasoning used in the proof of Lemma 3.18, we can deduce that this is less than or equal to some constant times

$$\int_{\mathcal{O}}\int_{A_\varepsilon}\mathbb{E}\left[e^{\tilde g_{\delta,2\lambda_\varepsilon(x)}(x)}\,e^{\tilde g_{\delta,2\lambda_\varepsilon(w)}(w)}\,\mathbb{1}_{\{\tilde f^\beta_{\delta,2\lambda_\varepsilon}(x)\ge\log(1/r_\varepsilon)^{1/6}\}}(\tilde f^\beta_{\delta,2\lambda_\varepsilon(x)}(x)+1)(\tilde f^\beta_{\delta,2\lambda_\varepsilon(w)}(w)+1)\right] \tag{3.28}$$

where $\delta(x,y)=|x-y|/3$. The final observation to make is that, by orthogonal projection, we have

$$\tilde h_\delta(w)=\alpha^\delta_{x,w}\tilde h_\delta(x)+Z^\delta_{x,w}$$

for $\alpha^\delta_{x,w}=\mathrm{cov}(\tilde h_\delta(x),\tilde h_\delta(w))/\mathrm{var}(\tilde h_\delta(x))$ where $Z^\delta_{x,w}$ is independent of $\tilde h_\delta(x)$; distributed as a centered normal random variable with variance $(1-(\alpha^\delta_{x,w})^2)\mathrm{var}(\tilde h_\delta(w))$.

The proof of this is the same as the proof of Lemma 3.13. Moreover, by Lemma 3.3, we have $\alpha^\delta_{x,w} = 1 + \mathrm{O}(\log(1/\delta)^{-1})$ uniformly in x, w.

Thus, by conditioning on $\tilde{h}_\delta(x)$, we can calculate that the integrand in (3.28) is less than or equal to a constant times

$$\mathbb{E}\left[\mathbb{1}_{\{\tilde{f}^\beta_{\delta,2\lambda_\varepsilon}(x)\geq \log(1/r_\varepsilon)^{1/6}\}} \mathrm{e}^{\tilde{g}_{\delta,\lambda_\varepsilon(x)}(x)+\tilde{g}_{\delta,2\lambda_\varepsilon(w)\alpha^\delta_{x,w}}(x)}(\tilde{f}^\beta_{\delta,2\lambda_\varepsilon(x)}(x)+1)(\alpha^\delta_{x,w}\tilde{f}^\beta_{\delta,2\lambda_\varepsilon(w)\alpha^\delta_{x,w}}(x)+1)\right]$$

which by a simple calculation can be bounded again by

$$\mathrm{e}^{-(\log(1/r_\varepsilon))^{1/6}}\,\delta^{-2}\mathbb{E}\left[\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(x)}(x)}(1+|\tilde{f}^\beta_{\delta,\lambda_\varepsilon(x)}(x)|^2)\right]$$

times some constant. This last expectation can be estimated by observing that changing measure by $\mathrm{e}^{\tilde{g}_{\delta,2\lambda_\varepsilon(x)}(x)}$ turns $\tilde{f}^\beta_{\delta,\lambda_\varepsilon(x)}(x)$ into a Gaussian random variable with mean β and variance $\log(1/\delta)+\mathrm{O}(1)$. Hence the integrand of (3.28) is less than

$$C\,\mathrm{e}^{-(\log(1/r_\varepsilon))^{1/6}}\,\delta^{-2}(1+\log(1/\delta))$$

for some constant C. Integrating over A_ε gives that the third term of (3.27) is $\lesssim \mathrm{e}^{-\log(1/r_\varepsilon)^{1/6}}\log(1/\varepsilon)^2$, and so we conclude, using our assumption (3.23) on r_ε, that (3.27) is of the correct order.

Step 6: *We show that* $\hat{\mathbb{Q}}^{\beta,\varepsilon}[E_\varepsilon] \to 1$ *as* $\varepsilon \to 0$.

Firstly, it is clear that $\hat{\mathbb{Q}}^{\beta,\varepsilon}[E^2_\varepsilon] \to 1$. Then since we have already shown, see (3.26), that $\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[E^2_\varepsilon \cap E^3_\varepsilon \cap (E^1_\varepsilon)^c\right] \to 0$ and that

$$\tilde{\mathbf{Q}}^{\beta,\varepsilon}_x\left[E^3_\varepsilon \mid h|_{D\setminus(B(x,r_\varepsilon-\varepsilon)}\right] \to 1$$

uniformly on E^2_ε, the claim follows straight away. The proof is complete. ☐

It is now relatively simple to show the convergence of $D_\varepsilon(\mathcal{O})$. To use Proposition 3.21, we must first compare $M^\beta_\varepsilon(\mathcal{O})$ and $D^\beta_\varepsilon(\mathcal{O})$ with $M_\varepsilon(\mathcal{O})$ and $D_\varepsilon(\mathcal{O})$.

Lemma 3.26. *We have*

$$\begin{aligned}\mathbb{P}(C_\beta) := \mathbb{P}\big(&\{\inf_\varepsilon \inf_{x\in D}(-\tilde{h}_\varepsilon(x)+2\mathrm{var}(\tilde{h}_\varepsilon(x))) > -(\beta+10)\}\\ &\cap\{\inf_\varepsilon \inf_{x\in D}(-h_\varepsilon(x)+2\mathrm{var}(h_\varepsilon(x))) > -(\beta+10)\}\big)\end{aligned}$$

converges to 0 *as* $\beta \to 0$*, uniformly in* ε*.*

Proof. This is a consequence of Corollary 3.6. ☐

Remark 3.27. *Note that on the event C_β we have $M_\varepsilon^\beta(\mathcal{O}) = M_\varepsilon(\mathcal{O})$ and also $D_\varepsilon^\beta(\mathcal{O}) = D_\varepsilon(\mathcal{O}) + \beta M_\varepsilon(\mathcal{O})$ for all $\mathcal{O} \subset D$.*

We are now ready to prove the main result.

Proof of Theorem 3.1. It is enough to show that for $\mathcal{O} \subset D$ and $\delta > 0$ fixed

$$\mathbb{P}\left[\left|\frac{M_\varepsilon(\mathcal{O})}{D_\varepsilon(\mathcal{O})}\sqrt{\log(1/\varepsilon)} - \sqrt{\frac{2}{\pi}}\right| > \delta\right] \to 0 \tag{3.29}$$

as $\varepsilon \to 0$. Then since $\sqrt{\log(1/\varepsilon)}M_\varepsilon(\mathcal{O}) \to \sqrt{\frac{\pi}{2}}\mu'(\mathcal{O})$ in probability, by Theorem 3.10, we also have $D_\varepsilon(\mathcal{O}) \to \mu'(\mathcal{O})$ in probability. Let us prove (3.29). By Proposition 3.21 we know that for any $\beta > 0$

$$\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\left|\frac{M_\varepsilon^\beta(\mathcal{O})}{D_\varepsilon^\beta(\mathcal{O})}\sqrt{\log(1/\varepsilon)} - \sqrt{\frac{2}{\pi}}\right| > \delta\right] \to 0$$

as $\varepsilon \to 0$, and we also know that, on the event C_β, we can compare $M_\varepsilon^\beta, D_\varepsilon^\beta$ with $M_\varepsilon, D_\varepsilon$ by Remark 3.27. With this in mind, we bound (3.29) above by

$$\mathbb{P}\left[A^1_{\beta,\varepsilon}\right] + \mathbb{P}\left[A^2_{\beta,\varepsilon} \cap (A^1_{\beta,\varepsilon})^c\right]$$

where

$$\begin{aligned}
A^1_{\beta,\varepsilon} &= \left\{\left|\frac{M_\varepsilon(\mathcal{O})}{D_\varepsilon(\mathcal{O}) + \beta M_\varepsilon(\mathcal{O})}\sqrt{\log(1/\varepsilon)} - \sqrt{\frac{2}{\pi}}\right| > \frac{\delta}{2}\right\} \text{ and} \\
A^2_{\beta,\varepsilon} &= \left\{\sqrt{\log(1/\varepsilon)}\left|\frac{M_\varepsilon(\mathcal{O})}{D_\varepsilon(\mathcal{O}) + \beta M_\varepsilon(\mathcal{O})} - \frac{M_\varepsilon(\mathcal{O})}{D_\varepsilon(\mathcal{O})}\right| > \frac{\delta}{2}\right\}.
\end{aligned}$$

In fact, the event $A^2_{\beta,\varepsilon} \cap (A^1_{\beta,\varepsilon})^c$ is deterministically non possible if ε is small enough. Thus, it is enough to show that $\mathbb{P}(A^1_{\beta,\varepsilon})$ can be made arbitrarily small by choosing β large, and then ε small. To do this, we observe by Remark 3.27 that

$$\mathbb{P}(A^1_{\beta,\varepsilon}) \leq \mathbb{P}[C_\beta^c] + \mathbb{P}\left[\left\{\left|\frac{M_\varepsilon^\beta(\mathcal{O})}{D_\varepsilon^\beta(\mathcal{O})}\sqrt{\log(1/\varepsilon)} - \sqrt{\frac{2}{\pi}}\right| > \delta/2\right\} \cap C_\beta\right].$$

Furthermore, by the definition of $\hat{\mathbb{Q}}^{\beta,\varepsilon}$, the left-hand side for any $\eta > 0$, is less than or equal to

$$\mathbb{P}[C_\beta^c] + \mathbb{P}[C_\beta \cap \{D_\varepsilon^\beta < \eta\}] + \frac{\mathbb{E}[D_\varepsilon^\beta]}{\eta}\hat{\mathbb{Q}}^{\beta,\varepsilon}\left[\left|\frac{M_\varepsilon^\beta(\mathcal{O})}{D_\varepsilon^\beta(\mathcal{O})}\sqrt{\log(1/\varepsilon)} - \sqrt{\frac{2}{\pi}}\right| > \delta\right].$$

Now note that by Markov's inequality

$$\mathbb{P}[C_\beta \cap \{D_\varepsilon^\beta < \eta\}] \leq \mathbb{P}[\sqrt{\log(1/\varepsilon)} M_\varepsilon(\mathcal{O}) < \eta^{1/4}] + \sqrt{\eta}\, \mathbb{E}[D_\varepsilon^\beta]\, \hat{\mathbb{Q}}^{\beta,\varepsilon} \left[\log(1/\varepsilon) \left(\frac{M_\varepsilon^\beta}{D_\varepsilon^\beta} \right)^2 \right].$$

Hence using Proposition 3.21 and Lemma 3.26, together with the fact that $\sqrt{\log(1/\varepsilon)} M_\varepsilon(\mathcal{O})$ converges to $\sqrt{\pi/2}\mu'(\mathcal{O})$ (which is positive almost surely) and that $\mathbb{E}[D_\varepsilon^\beta] = \int_x Z_\varepsilon^\beta(x)$ is bounded, we can conclude by letting $\beta \to \infty$, then $\eta \to 0$, and finally $\varepsilon \to 0$. □

3.4 $\star$-scale invariant kernels

In this section we prove Theorem 3.2 using a simple adaptation of our arguments from the previous section. Recalling the set-up, we have:

- $\theta : \mathbb{R}^d \to \mathbb{R}$ a mollifier, supported in $B(0,1)$, with Hölder continuous density and satisfying (3.2);
- $k : [0, \infty) \to \mathbb{R}$, a compactly supported and positive-definite C^1 function with $k(0) = 1$; and
- h a $\star$-scale invariant field on $\mathbb{R}^d$ with covariance kernel

$$K(x,y) = \int_1^\infty \frac{k(u|x-y|)}{u}\, du.$$

We would like to prove that if $h_\varepsilon(x) = h \star \theta_\varepsilon(x)$ is the θ-convolution approximation to h, the signed measures

$$D_\varepsilon(dx) := (-h_\varepsilon(x) + \sqrt{2d}\log(1/\varepsilon))\, \mathrm{e}^{\sqrt{2d}h_\varepsilon(x)}\, \varepsilon^d\, dx$$

converge weakly in probability to a limiting measure. Moreover, we would like to show that this limiting measure is equal to the measure defined in [DRSV14a, JS17] (see Theorems 3.8 and 3.9).

Proof. First pick g such that $g \star g(u) = k(u)$ (we can do this by our assumptions on k, [**?**]). Then we can define a field h with the correct covariance structure by setting

$$h(x) := \int_1^\infty \int_{\mathbb{R}^d} \frac{g(y - xu)}{\sqrt{u}} W(dy, du), \tag{3.30}$$

where $W(\cdot,\cdot)$ is a standard space-time white noise. It is then proved in [DRSV14a] that if we let

$$\tilde{h}_\varepsilon(x) := \int_1^{\frac{1}{\varepsilon}} \int_{\mathbb{R}^d} \frac{g(y-xu)}{\sqrt{u}} W(dy,du),$$

the signed derivative measures $\tilde{D}_\varepsilon(dx) := (-\tilde{h}_\varepsilon(x) + \sqrt{2d}\log(1/\varepsilon))\, \mathrm{e}^{\sqrt{2d}\tilde{h}_\varepsilon(x)}\, \varepsilon^d\, dx$ converge almost surely to a positive limiting measure μ'. It is further shown in [DRSV14b] that

$$\tilde{M}_\varepsilon(dx) := \sqrt{\log(1/\varepsilon)}\, \mathrm{e}^{\sqrt{2d}\tilde{h}_\varepsilon(x)}\, \varepsilon^d\, dx$$

converges to $\sqrt{2/\pi}\mu'$ in probability and in [JS17] that

$$M_\varepsilon := \sqrt{\log(1/\varepsilon)}\, \mathrm{e}^{\sqrt{2d}h_\varepsilon(x)}\, \varepsilon^d\, dx$$

also converges to $\sqrt{2/\pi}\mu'$ in probability.

To prove the convergence of $D_\varepsilon(dx)$ we use the same strategy as for the proof of Theorem 3.1, now letting $\tilde{h}_\varepsilon$ play the role of the circle average. In particular we need only prove Proposition 3.21 (the result then following by Corollary 3.6 and Lemma 3.26 in exactly the same way.) We observe that:

- $\tilde{h}_\varepsilon(x)$ is a (time-changed) Brownian motion for each $x \in \mathbb{R}^d$; and
- $\mathrm{cov}(h_\varepsilon(x), \tilde{h}_\delta(x)) = \log(1/(\varepsilon \wedge \delta)) + \mathrm{O}(1)$, so we can define $\lambda_\varepsilon(x)$, $Y_\varepsilon(x)$ and $\rho_\delta^\varepsilon(x)$ as in Lemmas 3.13 and 3.15, and the statements of these lemmas will still hold.

This is enough to prove (4.12) and step 1 of (3.19). For step 2 we need to explain how we define a few things. We let r_ε be chosen as before, and without loss of generality we assume that $\mathrm{supp}(k) \subset B(0,1)$. It is then easy to check using the definition of h that $h_\varepsilon(z)$ and $(\tilde{h}_\eta(x) - \tilde{h}_{r_\varepsilon-\varepsilon}(x))_{\eta \le r_\varepsilon - \varepsilon}$ are independent for all $z \notin B(x, r_\varepsilon)$. We let $\mathcal{F}_\varepsilon = \sigma(\{\tilde{h}_u(x) : u \ge r_\varepsilon - \varepsilon\}) \vee \sigma(\{h_\varepsilon(z) : z \in D \setminus B(x, r_\varepsilon)\})$ so that $(\tilde{h}_\eta(x) - \tilde{h}_{r_\varepsilon-\varepsilon}(x))_{\eta \le r_\varepsilon-\varepsilon}(x)$ is independent of $\mathcal{F}_\varepsilon$, and $M_\varepsilon^{\beta,out}/D_\varepsilon^{\beta,out}$ is measurable with respect to it. This is what we will use in place of $\mathcal{F}_{r_\varepsilon-\varepsilon}$ from the original proof. Using standard properties of Gaussian processes we see that we can also write $Y_\varepsilon(x) = Y_\varepsilon^1(x) + Y_\varepsilon^2(x)$ where $Y_\varepsilon^1(x)$ is measurable with respect to $\mathcal{F}_\varepsilon$ and $Y_\varepsilon^2(x)$ is independent of it. From this point onwards we can define everything in the same way, and steps 3 and 4 follow, using only properties of the 3d Bessel process.

To conclude, we need only complete step 5, since step 6 is a straightforward consequence of this (as in the original proof.) For this step we note that by our assumption on $\mathrm{supp}(k)$, $(\tilde{h}_{\delta+\eta}(x) - \tilde{h}_\delta(x))_{\eta \ge 0}$ and $(\tilde{h}_{\delta+\eta}(y) - \tilde{h}_\delta(y))_{\eta \ge 0}$ are independent as soon as $|y - x| \ge \delta$. Since this is the only extra property we used in this step, the proof of Proposition 3.21 goes through.

□

Remark 3.28. *We remark here that the authors in [DRSV14a, DRSV14b] suggest that their constructions should hold for more general kernels than the* $\star$-*scale invariant ones. In particular, for any positive definite kernel of the form* $K(x,y) = -\log(|x-y|) + g(|x-y|)$ *with* g *continuous, one has a white-noise decomposition for the corresponding field* h*, analogous to (3.30). This means that the theory in [DRSV14b, Appendix D] should go through, and as a consequence, the result of Theorem 3.2 should also hold. More generally we conjecture that Theorem 3.2 should hold for any* K *satisfying (3.4).*

4 Approximating the Liouville measure using local sets of the Gaussian free field

4.1 Introduction

Gaussian multiplicative chaos (GMC) theory, initiated by Kahane in the 80s [Kah85] as a generalization of multiplicative cascades, aims to give a meaning to "$\exp(\Gamma)$" for rough Gaussian fields Γ. In a simpler setting it was already used in the 70s to model the exponential interaction of bosonic fields [HK71], and over the past ten years it has gained importance as a key component in constructing probabilistic models of so-called Liouville quantum gravity in 2D [DS11, DKRV16].

One of the important cases of GMC theory is when the underlying Gaussian field is equal to $\gamma\Gamma$, for Γ a 2D Gaussian free field (GFF) [DS11] and $\gamma > 0$ a parameter. It is then possible to define random measures with area element "$\exp(\gamma\Gamma) dx \wedge dy$". These measures are sometimes also called Liouville measures [DS11] and we will do so for convenience in this article. Due to the recent work of many authors [RV10, DS11, Ber15a, Sha16] one can say that we have a rather thorough understanding of Liouville measures in the so-called subcritical regime ($\gamma < 2$). The critical regime ($\gamma = 2$) is trickier, but several constructions are also known [DRSV14a, DRSV14b, JS17, Pow17a].

Usually, in order to construct the GMC measure, one first approximates the underlying field using either a truncated series expansion or smooth mollifiers, then takes the exponential of the approximated Gaussian field, renormalizes it and shows that the limit exists in the space of measures. In [Aid15] the author proposed a different way to construct measures of multiplicative nature using nested conformally invariant loop ensembles, inspired by multiplicative cascades. He conjectured that in the subcritical and critical regime, and in the case where these loop ensembles correspond to certain same-height contour lines of the underlying GFF, the limiting measure should have the law of the Liouville measure. In this paper we confirm his conjecture. This is done by providing new constructions of the subcritical and critical Liouville measures using a certain family of so called local sets of the GFF [SS13, ASW15] and reinterpreting his construction as a special case of this general setting. Some of our local-set based constructions correspond to simple multiplicative cascades, and others in some sense to stopping lines constructions of the multiplicative cascade measures [Kyp00]. To our knowledge we provide a first "non-Gaussian" approximation of Liouville measures that is both local and conformally invariant. We also remark that our construction strongly uses the Markov property of the GFF and hence does not easily generalize to other log-correlated fields.

One simple, but important, consequence of our results is the simultaneous construction of a GFF in a simply connected domain and its associated Liouville measure using nested CLE_4 and a collection of independent coin tosses. Start with a height function $h_0 = 0$ on $\mathbb{D}$ and sample a CLE_4 in $\mathbb{D}$. Inside each connected component of its complement add either $\pm\pi$ to h_0 using independent fair coins. Call the resulting function h_1. Now repeat this procedure independently in each connected component: sample an independent CLE_4, toss coins and add $\pm\pi$ to h_1 to obtain h_2. Iterate. Then it is known [MS11, ASW15] that these piecewise constant fields h_n converge to a GFF Γ. It is also possible to show that the nested CLE_4 used in this construction (we call the complement of the nth level loops A_n) is a measurable function of Γ. Proposition 4.10 of the current article implies that one can construct the Liouville measures associated to Γ by just taking the limit of measures

$$M_n^\gamma(dz) = e^{\gamma h_n(z)}\, \mathrm{CR}(z;\mathbb{D}\setminus A^n)^{\frac{\gamma^2}{2}}\, dz.$$

Here $\mathrm{CR}(z;\mathbb{D}\setminus A^n)$ is the conformal radius of the point z inside the n-th level loop.

Observe that the above approximation is different from taking naively the exponential of h_n and normalizing it pointwise by its expectation. In fact, it is not hard to see that in this setting the latter naive procedure that is used for mollifier and truncated series approximations would not give the Liouville measure.

In the critical case, and keeping to the above concrete approximation of the GFF, regularized Liouville measures can be given by the so-called derivative approximations:

$$D_n(dz) = \int_{\mathcal{O}} \left(-h_n(z) + 2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A^n)\right) e^{2h_n(z)}\, \mathrm{CR}(z;\mathbb{D}\setminus A^n)^2\, dz.$$

As the name suggests, they correspond to the derivative of the above measure M_n^γ w.r.t. to γ, taken at the critical parameter $\gamma = 2$. We show that these approximate signed measures converge to a positive measure that agrees (up to a constant factor 2) with the limiting measure of [Aid15] described in Section 4.3.3, and also to the critical Liouville measure constructed in [DRSV14b, Pow17a].

The connection between multiplicative cascades and the Liouville measure established by our construction makes it possible to directly adapt many techniques developed in the realm of branching random walks and multiplicative cascades to the study of the Liouville measure. For example, this allows us to prove a “Seneta–Heyde” rescaling result in the critical regime by following very closely the proof for the branching random walk in [AS14] and doing minimal extra work. Finally, our proofs are robust enough to study the Liouville measure in non-simply connected domains and also to study the boundary Liouville measure.

The rest of the article is structured as follows. We start with preliminaries on the GFF, its local sets and Liouville measure. Then, we treat the subcritical regime and discuss generalizations to non-simply connected domains and to the boundary Liouville measure. Finally, we handle the critical case: we first show that our construction agrees with both a construction by E. Aidekon (up to a constant factor 2) and a mollifier construction of the critical Liouville measure; then, we consider the case of Seneta-Heyde scaling.

4.2 Preliminaries on the Gaussian free field and its local sets

Let $D \subseteq \mathbb{R}^2$ denote a bounded, open and simply connected planar domain. By conformal invariance, we can always assume that D is equal to $\mathbb{D}$, the unit disk. Recall that the Gaussian Free Field (GFF) in D can be viewed as a centered Gaussian process Γ, indexed by the set of continuous functions in D, with covariance given by

$$\mathbb{E}\left[(\Gamma, f)(\Gamma, g)\right] = \iint_{D\times D} f(x) G_D(x,y) g(y) dx dy. \tag{4.1}$$

Here G_D is the Dirichlet Green's function in D, normalized such that $G_D(x,y) \sim \log(1/|x-y|)$ as $x \to y$ for all $y \in D$.

Let us denote by ρ_z^ε the uniform measure on the circle of radius ε around z. Then for all $z \in D$ and all $\varepsilon > 0$, one can define $\Gamma_\varepsilon := (\Gamma, \rho_z^\varepsilon)$. We remark that this concrete choice of mollifying the free field is of no real importance, but is just a bit more convenient in the write-up of the critical case.

An explicit calculation then shows that:

$$\mathbb{E}\left[\varepsilon^{\frac{\gamma^2}{2}} \exp\left(\gamma(\Gamma, \rho_z^\varepsilon)\right)\right] \begin{cases} = \mathrm{CR}(z;D)^{\gamma^2/2} & \text{if } d(z,\partial D) \geq \varepsilon, \\ \leq 1 & \text{if } d(z,\partial D) < \varepsilon, \end{cases} \tag{4.2}$$

where $\mathrm{CR}(z;D)$ is the conformal radius of z in the simply-connected domain D.

The Gaussian free field satisfies a spatial Markov property, and in fact it also satisfies a strong spatial Markov property. To formalise this, the concept of local sets was introduced in [SS13]. They can be thought as the generalisation of stopping times to a higher dimension.

Definition 4.1 (Local sets)**.** *Consider a random triple* (Γ, A, Γ_A)*, where* Γ *is a GFF in* D*,* A *is a random, relatively closed subset of* D *and* Γ_A *a random distribution* *that is equal to a harmonic function, h_A, when restricted to $D \setminus A$. We say that A is a local set for Γ if conditionally on A and Γ_A, $\Gamma^A := \Gamma - \Gamma_A$ is a GFF in $D \setminus A$.*

Here, by a random closed set we mean a probability measure on the space of relatively closed subsets of D, endowed with the Hausdorff metric and its corresponding Borel $\sigma-$algebra. For simplicity, we will only work with local sets A that are measurable functions of Γ and such that $A \cup \partial D$ is connected. In particular, this implies that all connected components of $D \backslash A$ are simply-connected. We define $\mathcal{F}_A = \sigma(A) \vee \sigma(\Gamma_A)$.

Other than the Markov property apparent from the definition, we will use the following simple properties of local sets. See for instance [SS13, Wer] for further properties.

Lemma 4.2. *Let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets measurable w.r.t. Γ. Then*

1. $\mathcal{F}_{A^n} \subset \mathcal{F}_{A^{n+1}}$,
2. $\overline{\bigcup A^n}$ *is also a local set and* $\Gamma_{A_N} \to \Gamma_{\overline{\bigcup A^n}}$ *in probability as distributions as* $N \to \infty$,
3. *if* $\overline{\bigcup A^n} = \overline{D}$, *then the join of the σ-algebras $\mathcal{F}_{A^n}$ is equal to $\sigma(\Gamma)$. Moreover, $\Gamma_n := \Gamma_{A^n}$ then converges to Γ in probability in the space of distributions.*

The property (1) follows from the fact that our local sets are measurable w.r.t. Γ and the characterization of local sets found in [SS13]. Properties (2) and (3) follow from the fact that when $A^n \cup \partial D$ is connected we have that the Green's functions $G_{D\backslash A^n} \to G_{D\backslash A}$.

In other words, one can approximate the Gaussian free field by taking an increasing sequence of measurable local sets $(A^n)_{n\in\mathbb{N}}$ and for each n defining $\Gamma_n := \Gamma_{A_n}$. In some sense these give very intrinsic approximations to the GFF. For example, one could intuitively think that A^n are the sets that discover the part of the surface described by the GFF that is linked to the boundary and has height between $-n$ and n.

4.2.1 Two useful families of local sets

One useful family of local sets are the so-called two-valued local sets introduced in [ASW15] and denoted by $A_{-a,b}$. For fixed $a, b > 0$, $A_{-a,b}$ is a local set of the GFF such that: the value of $h_{A_{-a,b}}$ inside each connected component of $D \setminus A$ is constant with value either $-a$ or b; and that is thin in the sense that for all f smooth we have $(\Gamma_A, f) = \int_{D\backslash A} f(z) h_A(z)\, dz$. Equivalently, Γ_A is equal to h_A as a distribution. The prime example of such a set is CLE_4 coupled with the Gaussian free field as $A_{-2\lambda,2\lambda}$, where λ is an explicit constant equal to $\lambda = \pi/2$ in our case [MS11, ASW15]. In analogy with stopping times, they correspond to exit times of Brownian motion from the interval $[-a, b]$. We recall the main properties of two-valued sets:

Proposition 4.3. *Let us consider $-a < 0 < b$.*

1. *When $a + b < 2\lambda$, there are no local sets of Γ with the characteristics of $A_{-a,b}$.*
2. *When $a + b \geq 2\lambda$, it is possible to construct $A_{-a,b}$ coupled with a GFF Γ. Moreover, the sets $A_{-a,b}$ are*
   - *Unique in the sense that if A' is another local set coupled with the same Γ, such that for all $z \in D$, $h_{A'}(z) \in \{-a, b\}$ almost surely and A' is thin in the sense above, then $A' = A_{-a,b}$ almost surely.*
   - *Measurable functions of the GFF Γ that they are coupled with.*
   - *Monotonic in the following sense: if $[a, b] \subset [a', b']$ and $-a < 0 < b$ with $b + a \geq 2\lambda$, then almost surely, $A_{-a,b} \subset A_{-a',b'}$.*
   - *$A_{-a,b}$ has almost surely Lebesgue measure 0.*
   - *For any z, $\log \mathrm{CR}(z; D\backslash A_{-a,b}) - \log \mathrm{CR}(z; D)$ has the distribution of the hitting time of $\{-a, b\}$ by a standard Brownian motion.*

Another nice class of local sets are those that only take one value in the complement of A. We call them first passage sets and denote them by A_a (if they only take the value a). These correspond to one-sided hitting times of the Brownian motion: hence the name. They are of interest in describing the geometry of the Gaussian free field and are treated in more detail in [ALS17]. Here, we only provide one working definition and refer to [ALS17] for a more intrinsic definition, uniqueness and other properties not needed in the current paper.

Definition 4.4 (First passage set)**.** *Take $a \geq 0$. We say that A_a is the first passage set (FPS) of a GFF Γ, with height a, if it is given by $\overline{\bigcup_n A_{-n,a}}$.*

We need a few properties of these sets. The first follows from the definition, the second and third from calculations in [ASW15] Section 6:

- We have that $\Gamma_{A_a} = a - \nu_a$, where ν_a is a positive measure supported on A_a;
- A_a has zero Lebesgue measure;
- For any $a_n \to \infty$ we have that $\overline{\bigcup A_{a_n}} = \overline{D}$.

Note that because the circle-average of the GFF $(\Gamma, \rho_z^\varepsilon)$ is a.s. well-defined for all $z \in D$, $\varepsilon > 0$ simultaneously, it also means that $(\nu_a, \rho_z^\varepsilon)$ is a.s. well-defined and positive for all z, ε as above.

In fact these three properties characterize A_a uniquely [ALS17]. However, in this paper we only need a weaker uniqueness statement that is a consequence of the following lemma:

Lemma 4.5. *Denote $A^1 = A_{-a,a}$ with $a \geq \lambda$ and define iteratively A^n by exploring copies of $A_{-a,a}$ in each connected component of the complement of A^{n-1}. Then, almost surely for a dense countable set $z \in D$ the following holds: for $k \in \mathbb{N}$, let n_z be the first iteration when $h_{A^{n_z}}(z) = ak$, the connected component $D \backslash A^{n_z}$ containing z is equal to the connected component of $D \backslash A_{ak}$ containing z.*

Proof. The proof follows from the uniqueness of two-valued sets $A_{-a,b}$. Indeed, construct sets B^n by taking $B^1 = A^1$ and then repeating the construction of A^i only in the components where the value of h_{B^n} is not yet ak. Thus, by construction $B^n \subset A^n$. Moreover, for any z up to and including the first iteration where $\Gamma_{B^k}(z) = ak$, the connected component of the complement of A^n and B^n containing z coincide.

Now, note that for a fixed $z \in D$, n_z is almost surely finite. Thus it suffices to prove that for all $n \in \mathbb{N}$, the set B^n is contained in $A_{-\lceil an \rceil, ak}$ and that all connected components of $D \backslash B^n$ where h_{B^n} takes the value ak are connected components of $D \backslash A_{-\lceil an \rceil, ak}$ where $h_{A_{-\lceil an \rceil, ak}}$ is equal to ak. To see this, first note that $h_{B^n} \in \{-an, -a(n-1), \ldots, ak\}$. In particular, in each connected component where $h_{B^n} = c \notin \{-\lceil an \rceil, ak\}$ we can construct the two-valued sets $A_{-\lceil an \rceil - c, ak-c}$. This gives us a local set $\tilde{B}$ s.t. $h_{\tilde{B}}$ takes only values in $\{-\lceil an \rceil, k\}$. It is also possible to see that $\tilde{B}$ is thin, by noting that inside each compact set its Minkowski dimension is smaller than 2 (e.g. see [Sep16, Proposition 4.3]). Then, by uniqueness of the two-valued sets, Lemma 4.3, $\tilde{B}$ is equal to $A_{\lceil an \rceil, k}$. To finish, notice that we the connected components of $D \backslash B^n$ where h_{B^n} took the value ak are also connected components of $\tilde{B}$ with the same value. □

In particular, from this lemma it follows that we can also construct A_a in a different way: denote $A^1 = A_{-a,a}$ and define A^2 by iterating independent copies of $A_{-a,a}$ in each component of the complement of $D \setminus A^1$ where $h_{A_1} \neq a$. Repeat this procedure again in all components of the complement for which the value still differs from a. This iteration gives an increasing sequence of local sets A^n, whose limit is equal to A_a. For a concrete example, one could take $A_{-2\lambda,2\lambda}$ to be equal to CLE_4 in its coupling with the GFF, and the above procedure would yield $A_{2\lambda}$. In fact the sets $(A_{2\lambda n})_{n \in \mathbb{N}}$ are exactly the sets that the author [Aid15] proposes as a basis for the construction of the Liouville measure.

4.3 Overview of the Liouville measure and loop constructions of [Aid15]

There are many ways to define the Liouville measure in the subcritical case, the differences amounting to how one approximates the underlying GFF. We will first describe the approximations using circle averages in the subcritical case. Then we will discuss the critical regime, and finally present the nested-loop based constructions from [Aid15] that are conjectured to give the Liouville measure. From now on we will set $D = \mathbb{D}$ for simplicity.

4.3.1 Subcritical regime

Let us recall that we denote $\Gamma_\varepsilon(z) = (\Gamma, \rho_z^\varepsilon)$ the ε-circle average of the GFF around the point z as before. It is known that $\Gamma_\varepsilon(z)$ is a continuous Gaussian processes that converge to Γ a.s. in the space of distributions as $\varepsilon \to 0$. Thus, one can define approximate Liouville measures

$$\mu_\varepsilon^\gamma(dz) := \varepsilon^{\frac{\gamma^2}{2}} \exp\left(\gamma \Gamma_\varepsilon(z)\right) dz.$$

In the subcritical regime we have the following result [DS11, Ber15a]:

Theorem 4.6. *For $\gamma < 2$ the measures μ_ε^γ converge to a non-trivial measure μ^γ weakly in probability. Moreover, for any fixed Borel set $\mathcal{O} \subset \mathbb{D}$ we have that $\mu_\varepsilon^\gamma(\mathcal{O})$ converges in L^1 to $\mu(O)$.*

In fact it is known that the measure is also unique, in the sense that the same limit can be obtained using any sufficiently nice mollifier instead of the circle average. We will show that the approximations using local sets give the same measure.

4.3.2 Critical regime

It is known that for $\gamma \geq 2$, the measures μ_γ^ε converge to zero [RV10]. To define the critical measures an additional renormalization is therefore required. One way to do it is to use the so-called derivative martingale, originating from studies on branching random walks. Define

$$\nu_\varepsilon(dz) := \frac{\partial}{\partial \gamma}\bigg|_{\gamma=2} \mu_\varepsilon^\gamma(dz) = (-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon))\varepsilon^2 \exp\left(2\Gamma_\varepsilon(z)\right) dz$$

It has been recently shown in [Pow17a, Theorem 1.1] that ν_ε converges weakly in probability to a non-trivial limiting measure μ_2' as $\varepsilon \to 0$. Moreover, μ_2' coincides with the critical Liouville measure defined in [DRSV14a, DRSV14b]. We will again show that the approximations using local sets converge towards the same measure.

Another way to define the critical measure is to use the so-called Seneta-Heyde renormalization [AS14, DRSV14b]. In the case of the circle-average process the approximating measures would be defined as:

$$\bar{\nu_\varepsilon}(dz) := \sqrt{\log 1/\varepsilon}\mu_\varepsilon^2(dz).$$

It has been shown [HRV15, JS17] that $\bar{\nu_\varepsilon}$ converges in probability to $\sqrt{\frac{2}{\pi}}\mu_2'$ as $\varepsilon \to 0$. We will prove an analogous result in our setting.

4.3.3 Measures constructed using nested loops

In [Aid15] the author proposes a construction of measures, analogous to the Liouville measure, using nested conformally-invariant loop ensembles. We will now describe it in a concrete context that is related to this paper.

Consider a CLE_4, and inside each loop toss an independent fair coin. Keep the loops with heads on top, and sample new CLE_4 loops in the others. Also toss new independent coins inside these loops. Keep track of all the coin tosses for each loop and repeat the procedure inside each loop where the number of heads is not yet larger than the number of tails. Define the resulting set as $\tilde{A}^1$. Now define $\tilde{A}^k$ iteratively by sampling an independent copy of $\tilde{A}^1$ inside each connected component of $\mathbb{D} \setminus \tilde{A}^{k-1}$.

For any Borelian $\mathcal{O} \subset \mathbb{D}$ we can now define

$$\tilde{M}_k^\gamma(\mathcal{O}) = \frac{1}{\mathbb{E}\left[\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^1)^{\gamma^2/2}\right]^k} \int_{\mathcal{O} \cap \mathbb{D} \setminus \tilde{A}^k} \mathrm{CR}(z, \mathbb{D} \setminus \tilde{A}^k)^{\frac{\gamma^2}{2}} dz \tag{4.3}$$

It is shown in [Aid15] that for $\gamma < 2$ the measures defined by $\tilde{M}_k^\gamma$ converge weakly almost surely to a non-trivial measure $\tilde{M}^\gamma$. It is also conjectured there that the limiting measures coincide with the Liouville measures μ^γ. We will prove this statement below.

It is further proved in [Aid15] that for $\gamma \geq 2$, these measures converge almost surely to zero. In the critical case, however, one can again define a derivative martingale $\tilde{D}_k^\gamma$ by taking a derivative with respect to $-\gamma$. In other words one sets:

$$\tilde{D}_k^\gamma(\mathcal{O}) = -2\frac{\partial}{\partial\gamma}\tilde{M}_k^\gamma(\mathcal{O})$$

(we include the factor 2 here to be consistent with the definition in [Aid15]). It is shown in [Aid15] that the measures $\tilde{D}_k := \tilde{D}_k^2$ converge to a non-trivial positive measure $\tilde{D}_\infty$. In this paper, we prove that $\tilde{D}_\infty = 2\mu_2'$.

4.4 Local set approximations of the subcritical Liouville measure

In this section we prove that one can approximate the Liouville measure of a GFF in a simply connected domain using increasing sequences of local sets $(A^n)_{n\in\mathbb{N}}$ with $\overline{\bigcup A^n} = \overline{\mathbb{D}}$. In particular, the measure constructed in [Aid15] will fit in our framework and thus it agrees with the Liouville measure. In fact, for simplicity, we first present the proof of convergence in this specific case.

First, recall that we denote by h_A the harmonic function given by the restriction of Γ_A to $\mathbb{D}\setminus A$. For any local set A with Lebesgue measure 0 and bounded h_A, we define for any Borelian set $\mathcal{O}\subseteq D$:

$$M^\gamma(\mathcal{O}, A) := \int_{\mathcal{O}} e^{\gamma h_A}\,\mathrm{CR}(z;\mathbb{D}\setminus A)^{\gamma^2/2}\,dz.$$

Notice that as h_A is bounded, we can define it arbitrarily on the 0 Lebesgue measure set A.

Proposition 4.7. *Fix $\gamma\in[0,2)$. For $a>0$, let A_a be the a-FPS of Γ and μ^γ be the Liouville measure defined by Γ. Then for each Borelian set $\mathcal{O}\subset\mathbb{D}$,*

$$M_a^\gamma(\mathcal{O}) := M^\gamma(\mathcal{O}, A_a) = e^{\gamma a}\int_{\mathcal{O}} CR(z; D\backslash A_a)^{\gamma^2/2}dz$$

is a martingale with respect to $\mathcal{F}_{A_a}$ and converges a.s. to $\mu^\gamma(\mathcal{O})$ as $a\to\infty$. Thus, a.s. the measures M_a^γ converge weakly to μ^γ.

Before the proof, we make two remarks. First, we make the connection between our martingale and the martingales of [Aid15]:

Remark 4.8. *As a consequence of Lemma 4.5, the fact that $A_{-2\lambda,2\lambda}$ has the law of CLE$_4$ and the fact that the value of its corresponding harmonic function is independent in each connected component of $\mathbb{D}\backslash A_{-2\lambda,2\lambda}$ [MS11, ASW15], we see that $\tilde{A}^1$ of Section 4.3.3 is equal in law to $A_{2\lambda}$. Furthermore, the sequence $(\tilde{A}^k)_{k\in\mathbb{N}}$ has the same law has the sequence $(A_{2\lambda k})_{k\in\mathbb{N}}$.*

Now, by the iterative construction and conformal invariance the random variables

$$\log\mathrm{CR}(0,\mathbb{D}\setminus\tilde{A}^i) - \log\mathrm{CR}(0,\mathbb{D}\setminus\tilde{A}^{i-1})$$

with $A^0=\emptyset$ are i.i.d. Thus, $\mathbb{E}\left[\mathrm{CR}(0,\mathbb{D}\setminus\tilde{A}^1)^{\frac{\gamma^2}{2}}\right]^k = \mathbb{E}\left[\mathrm{CR}(0,\mathbb{D}\setminus\tilde{A}^k)^{\frac{\gamma^2}{2}}\right].$

Moreover, it is known from [SSW09, ASW15] that $-\log \mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^k)$ *corresponds precisely to the hitting time of* $k\pi$ *by a standard Brownian motion started from zero. In our case, when* $2\lambda = \pi$*, we therefore see that*

$$e^{\gamma 2\lambda k} = \mathbb{E}\left[\mathrm{CR}(0, \mathbb{D} \setminus \tilde{A}^1)^{\frac{\gamma^2}{2}}\right]^{-k}.$$

Furthermore, since $Leb(A_{2\lambda}) = 0$ *implies that* $M_a^\gamma(\mathcal{O} \cap A_{2\lambda}) = 0$*, we have that* $M_{2\lambda k}^\gamma$ *agrees with the measure* $\tilde{M}_k^\gamma$ *defined in* (4.3)*. Hence Proposition 4.7 confirms that the limit of* $\tilde{M}_k^\gamma$ *corresponds to the Liouville measure.*

Remark 4.9. *Second, in order to avoid repetition, we recall here as a remark the standard argument showing that the almost sure weak convergence of measures is implied by the almost sure convergence of* $M_a^\gamma(\mathcal{O})$ *over all boxes* $\mathcal{O}$ *with dyadic coordinates. This follows from two observations: first, the subspace of Radon measures on* $\overline{\mathbb{D}}$ *with bounded mass is compact and second, the boxes* $\mathcal{O}$ *with dyadic coordinates generate the Borel* σ*-algebra. Notice that we do not show that we have strong convergence of measures, i.e. we do not know whether almost surely* $\mu^\gamma(\mathcal{O})$ *is the limit of* $M_a^\gamma(\mathcal{O})$ *for all Borelian* $\mathcal{O}$*.*

Proof of Proposition 4.7. By Remark 4.9, it suffices to prove the convergence statement for $M_a^\gamma(\mathcal{O})$. When $\gamma \in [0, 2)$, we know that $\mu_\varepsilon^\gamma(\mathcal{O}) \to \mu^\gamma(\mathcal{O})$, in $\mathcal{L}^1$ as $\varepsilon \to 0$, where μ_ε^γ is as in Theorem 4.6. Thus,

$$\mathbb{E}\left[\mu^\gamma(\mathcal{O}) \mid \mathcal{F}_{A_a}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\mu_\varepsilon^\gamma(\mathcal{O}) \mid \mathcal{F}_{A_a}\right].$$

The key is to argue that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\mu_\varepsilon^\gamma(\mathcal{O}) \mid \mathcal{F}_{A_a}\right] = M_a^\gamma(\mathcal{O}). \tag{4.4}$$

Then $M_a^\gamma(\mathcal{O}) = \mathbb{E}\left[\mu^\gamma(\mathcal{O}) \mid \mathcal{F}_{A_a}\right]$ and we can conclude using the martingale convergence theorem and the fact that $\overline{\bigcup A_a} = \overline{\mathbb{D}}$.

To prove (4.4), define A_a^ε as the ε-enlargement of A_a. By writing $\Gamma = \Gamma_{A_a} + \Gamma^{A_a}$ and using that $(\Gamma_{A_a}, \rho_\varepsilon^z) = a$ for any $z \in \mathbb{D} \setminus A_a^\varepsilon$, we have

$$\mathbb{E}\left[\int_{\mathcal{O} \setminus A_a^\varepsilon} \mathrm{e}^{\gamma(\Gamma, \rho_\varepsilon^z)} \varepsilon^{\gamma^2/2}\, dz \,\middle|\, \mathcal{F}_{A_a}\right] = \int_{\mathcal{O} \setminus A_a^\varepsilon} \mathrm{e}^{\gamma a} \varepsilon^{\gamma^2/2} \mathbb{E}\left[\mathrm{e}^{(\Gamma^{A_a}, \rho_\varepsilon^z)} \,\middle|\, \mathcal{F}_{A_a}\right] dz$$

Using (4.2) we recognize that the right hand side is just $M_a^\gamma(\mathcal{O} \setminus A_a^\varepsilon)$.

But now for any fixed a, as $\mathrm{CR}(z, \mathbb{D}) \leq 1$ and A_a has zero Lebesgue measure, we have that $M_a^\gamma(\mathcal{O} \cap A_a^\varepsilon) = o_\varepsilon(1)$. On the other hand, from the fact that $(\Gamma_{A_a}, \rho_\varepsilon^z) \leq a$

for any z, and (4.2), it follows that

$$\mathbb{E}\left[\int_{\mathcal{O}\cap A_a^\varepsilon} \mathrm{e}^{\gamma(\Gamma,\rho_\varepsilon^z)}\,\varepsilon^{\gamma^2/2}\,dz\middle|\mathcal{F}_{A_a}\right] \le Leb(A_a^\varepsilon)e^{\gamma a}.$$

Thus, we conclude (4.4) and the proof. □

We now state a more general version of this result, which says that one can construct the Liouville measure using a variety of local set approximations. The proof is a simple adaptation of the proof above. We say that a generalized function T on $\mathbb{D}$, for which the circle-average process $T_\varepsilon(z) := (T, \rho_z^\varepsilon)$ can be defined, is bounded from above by K if for all $z \in D$ and $\varepsilon > 0$, we have that $T_\varepsilon(z) \le K$.

Proposition 4.10. *Fix $\gamma \in [0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}} A^n} = \overline{\mathbb{D}}$. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of positive integers. Suppose that almost surely for all $n \in \mathbb{N}$, we have that $Leb(A^n) = 0$ and that Γ_{A^n} is bounded from above by K_n.Then for any Borel $\mathcal{O} \subset \mathbb{D}$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma(\mathcal{O}) = \int_{\mathcal{O}} \mathrm{e}^{\gamma h_{A^n}(z)}\,\mathrm{CR}(z;\mathbb{D}\setminus A^n)^{\gamma^2/2}\,dz$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty} M_n^\gamma(\mathcal{O}) = \mu^\gamma(\mathcal{O})\ a.s.$$

where μ^γ is the Liouville measure defined by Γ. Thus, almost surely the measures M_n^γ converge weakly to μ^γ.

Let us mention two natural sequences of local sets for which this proposition applies. The first is when we take $a_n, b_n \nearrow \infty$ and study the sequence $(A_{-a_n,b_n})_{n\in\mathbb{N}}$. The second is when we take the sequence $(A^n_{-a,b})_{n\in\mathbb{N}}$ for some $a, b > 0$, where $A^n_{-a,b}$ is defined by iteration [6]. Note that in the case where $a = b = 2\lambda$, we recover the result described in the introduction for the iterated CLE_4.

Observe that whereas our martingale agrees with the one given in [Aid15] for the case of first-passage sets, for any cases where h_{A^n} can take more than one value, the martingales are in fact different. Yet, we can still identify the limit of the martingale $\tilde{M}_n^\gamma(\mathcal{O})$ of [Aid15], corresponding to an iterated CLE_4 (i.e. $(\mathrm{CLE}_4^n)_{n\in\mathbb{N}}$.) In this case Aidekon's martingale converges in distribution to $\eta^\gamma(\mathcal{O}) := \mathbb{E}\left[\mu^\gamma(\mathcal{O})|\mathcal{F}_\infty\right]$, where μ^γ is the Liouville measure and $\mathcal{F}_\infty$ is the σ-algebra containing only the geometric information from all iterations of the CLE_4. This σ-algebra is strictly smaller than

[6] We set $A^1_{-a,b} = A_{-a,b}$ and define $A^n_{-a,b}$ by sampling the $A_{-a,b}$ of $\Gamma^{A^{n-1}_{-a,b}}$ inside each connected component of $D\backslash A^{n-1}_{-a,b}$

$\mathcal{F}_{A^n_{-2\lambda,2\lambda}}$, which also contains information on the labels of CLE_4 in its coupling with the GFF. It is not hard to see that η^γ is not equal to μ^γ.

4.5 Generalizations

In this section, we describe some other situations where an equivalent of Proposition 4.10 can be proven using the same techniques as the proof of Proposition 4.7. In the following we do not present any new methods, but focus instead on announcing the propositions in context, so that they may be used in other works. We also make explicit the places where the results are already, or may in the future, be used.

4.5.1 Non-simply connected domains and general boundary conditions.

Here we consider the case when Γ is a GFF in an n-connected domain $D \subseteq \mathbb{D}$ (for more context see [ALS17]). First, let us note that in this set-up (4.2) becomes

$$\mathbb{E}\left[\varepsilon^{\frac{\gamma^2}{2}}\exp\left(\gamma(\Gamma,\rho_z^\varepsilon)\right)\right]\begin{cases} = e^{-\frac{\gamma^2}{2}\tilde{G}_D(z,z)} & \text{if } d(z,\partial D)\geq\varepsilon,\\ \leq 1 & \text{if } d(z,\partial D)<\varepsilon,\end{cases}$$

where we write $G_D(z,w) = -\log|z-w|+\tilde{G}_D(z,w)$, i.e. for any $z\in D$, $\tilde{G}_D(z,\cdot)$, is the bounded harmonic function that has boundary conditions $\log(|z-w|)$ for $w\in\partial D$. Additionally, if we work with local sets A such that all connected components of $A\cup\partial D$ contain an element of ∂D, then Lemma 4.2 will hold. All local sets we refer to here are assumed to satisfy this condition. These facts and assumptions are enough to prove the following proposition:

Proposition 4.11. *Fix $\gamma\in[0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}}A^n}=\mathbb{D}$. Suppose that almost surely for all $n\in\mathbb{N}$, we have that $Leb(A^n)=0$ and that Γ_{A^n} is bounded from above by K_n for some sequence of finite K_n. Then for any Borel $\mathcal{O}\subset D$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma := \int_{\mathcal{O}} e^{\gamma h_{A_n}(z)-\frac{\gamma^2}{2}\tilde{G}_{D\setminus A_n}(z,z)}\,dz,\qquad \mathcal{O}\subset\mathbb{D}$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty}M_n^\gamma(\mathcal{O})=\mu^\gamma(\mathcal{O})\ a.s.$$

where μ^γ is the Liouville measure defined by Γ. Thus, almost surely the measures M_n^γ converge weakly to μ^γ.

The equivalent of the sets $A_{-a,b}$ and A_a are defined in n-connected domains in [ALS17] and it is easy to see that their iterated versions satisfy the hypothesis of Proposition 4.11. In particular, the above construction allows the authors in [ALS17] to prove that the measure Γ_{A_a} is a measurable function of A_a.

4.5.2 Dirichlet-Neumann GFF

In this section we take Γ to be a GFF with Dirichlet-Neumann boundary conditions in $\mathbb{D}^+ = \mathbb{D} \cap \mathbb{H}$. That is, Γ satisfies (4.1), with G_D replaced by $G_{\mathbb{D}^+}$: the Green's function in $\mathbb{D}^+$ with Dirichlet boundary conditions on $\partial\mathbb{D}$ and Neumann boundary conditions on $[-1,1]$. To be more specific, we set $G_{\mathbb{D}^+}(x,y) = G_{\mathbb{D}}(x,y) + G_{\mathbb{D}}(x,\bar{y})$, with $G_{\mathbb{D}}$ as in Section 4.2. Then $G_{\mathbb{D}^+}(x,y) \sim \log(1/|x-y|)$ as $x \to y$ in the interior of $\mathbb{D}^+$ and $G_{\mathbb{D}^+}(x,y) \sim 2\log(1/|x-y|)$ when $y \in (0,1)$.

Let A be a closed subset of $\bar{\mathbb{D}}^+$. Suppose that Γ is a Dirichlet-Neumann GFF in $\mathbb{D}^+\backslash A$ with Neumann boundary conditions on $[-1,1]\backslash A$ and Dirichlet boundary conditions on the rest of the boundary. Let $z \in [-1,1]$ and define ϱ_z^ε to be the uniform measure on $\partial B(z,\varepsilon) \cap \mathbb{D}^+$. Then, in this set-up (4.2) becomes

$$\mathbb{E}\left[\varepsilon^{\gamma^2/4} \exp\left(\frac{\gamma}{2}(\Gamma, \varrho_x^\varepsilon)\right)\right] \begin{cases} = \mathrm{CR}(x;\mathbb{D}\backslash\breve{A})^{\gamma^2/4} & \text{if } d(z,\partial(\mathbb{D}\backslash\breve{A})) \geq \varepsilon, \\ \leq 1 & \text{if } d(z,\partial(\mathbb{D}\backslash\breve{A})) < \varepsilon. \end{cases} \tag{4.5}$$

Here we set $\breve{A} := A \cup \bar{A}$ for $\bar{A} = \{z \in \mathbb{C} : \bar{z} \in A\}$.

There is also a notion of local sets for this Dirichlet-Neumann GFF. We say that (Γ, A, Γ_A) describes a local set coupling if, conditionally on (A, Γ_A), $\Gamma^A := \Gamma - \Gamma_A$ is a GFF with Neumann boundary conditions on $[-1,1]\backslash A$ and Dirichlet on the rest. For connected local sets such $\partial\mathbb{D}^+ \cup A$ is connected, Lemma 4.2 still holds (by the same proof given for the 0-boundary GFF).

We are interested in the boundary Liouville measure on $[-1,1]$. Take $\gamma < 2$, $\varepsilon > 0$ and a Borel set $\mathcal{O} \subseteq [-1,1]$. We define the approximate boundary Liouville measures as follows:

$$\upsilon_\varepsilon^\gamma(\mathcal{O}) := \varepsilon^{\gamma^2/4} \int_{\mathcal{O}} \exp\left(\frac{\gamma}{2}(\Gamma, \varrho_x^\varepsilon)\right) dx$$

where here dx is the Lebesgue density on $[-1,1]$. It is known (see [DS11, Ber15a]) that $\upsilon_\varepsilon^\gamma \to \upsilon^\gamma$ in $\mathcal{L}^1$ as $\varepsilon \to 0$. Moreover, it is also easy to see that υ^γ is a measurable function of $\mathcal{F}_{[-1,1]}$ - this just comes from the fact that the Dirichlet GFF contains no information on the boundary. Thus, we have all the necessary conditions to deduce the following Proposition using exactly the same proof as in Section 4.4.

Proposition 4.12. *Fix $\gamma \in [0,2)$ and let $(A^n)_{n\in\mathbb{N}}$ be an increasing sequence of local sets for a GFF Γ with $\overline{\bigcup_{n\in\mathbb{N}} A^n} \supseteq [-1,1]$. Suppose that almost surely for all $n \in \mathbb{N}$, we have that $Leb_{[-1,1]}(A_n) = 0$ and that Γ_{A^n} restricted to A^n is bounded from above by K_n for some sequence of finite K_n. Then for any Borel $\mathcal{O} \subset [-1,1]$, $M_n^\gamma(\mathcal{O})$ defined by*

$$M_n^\gamma(\mathcal{O}) := \int_{\mathcal{O}} e^{\frac{\gamma}{2} h_{A_n}(z)} \operatorname{CR}(z; \mathbb{D}\backslash \breve{A}_n)^{\frac{\gamma^2}{4}}\, dz$$

is a martingale with respect to $\{\mathcal{F}_{A^n}\}_{n>0}$ and

$$\lim_{n\to\infty} M_n^\gamma(\mathcal{O}) = \upsilon^\gamma(\mathcal{O}) \text{ a.s.}$$

where where μ^γ is the boundary Liouville measure defined by Γ. Thus, almost surely the measures M_a^γ converge weakly to υ^γ.

It has recently been proven in [QW17] that sets satisfying the above hypothesis do exist, and that they can be used to couple the Dirichlet GFF with the Neumann GFF. Let us describe some concrete examples of these sets. If Γ is a Dirichlet-Neumann GFF, then in [QW17] it is shown that there exists a (measurable) thin local set $\tilde{A}(\Gamma)$ of the GFF such that:

- $\tilde{A}(\Gamma)$ has the law of the trace of an $\text{SLE}_4(0;-1)$ going from -1 to 1
- $h_{\tilde{A}(\Gamma)}$ is equal to 0 in the only connected component of $\mathbb{D}^+\backslash\tilde{A}(\Gamma)$ whose boundary intersects $\partial\mathbb{D}\cap\mathbb{H}$
- in the other connected components, $h_{\tilde{A}(\Gamma)}$ is equal to $\pm 2\lambda$, where conditionally on $\tilde{A}(\Gamma)$ the sign is chosen independently in each component.

There are two interesting sequences of local sets we can construct using this basic building-block. The first one is the boundary equivalent of $(A^n_{-2\lambda,2\lambda})_{n\in\mathbb{N}}$, and the second is the boundary equivalent of $(A_{2\lambda n})_{n\in\mathbb{N}}$. The first one is also described in [QW17, Section 3]. The construction goes as follows: choose $A^1 = \tilde{A}(\Gamma)$ and construct A^n by induction. In the connected components O of $\mathbb{D}\backslash A^n$ that contain an interval of $\mathbb{R}$, we have that Γ^{A_n} restricted to O is a Dirichlet-Neumann GFF (with Neumann boundary condition on $\mathbb{R}\cap\partial O$). Thus, by conformal invariance we can explore the set $\tilde{A}(\Gamma\mid_O)$ in each such component O. We define A^{n+1} to be the closed union of A^n with $\tilde{A}(\Gamma\mid_O)$ over all explored components O. Note that $h_{A^n} \in \{2\lambda k\}$ where k ranges between $-n$ and n. It is also not hard to see that A^n is thin (it follows from the fact that $h_A \in \mathcal{L}^1(\mathbb{D}\backslash A)$ and that for any compact set $K \subseteq \mathbb{D}^+$ the Minkowski dimension of $A^n \cap K$ is a.s. equal to $3/2$, see e.g. [Sep16, Proposition 4.3]). Thus we deduce that $\Gamma_{A^n} \leq 2\lambda n$. Additionally, note that by adjusting [MS11, Lemma 6.4], we obtain from the construction of A^1 that for any $z \in (-1,1)$ the law of $2(\log(\mathrm{CR}^{-1}(z, \mathbb{D}\backslash\breve{A}^1)) - \log(\mathrm{CR}^{-1}(z, \mathbb{D}))$ is equal to the first time that a BM exits $[-2\lambda, 2\lambda]$. It follows that for all $n \in \mathbb{N}$, $Leb_{\mathbb{R}}(A_n \cap [-1,1]) = 0$ and also $\overline{\bigcup_{n\in\mathbb{N}} A^n} \supseteq [-1,1]$. Hence we see that the sequence $(A^n)_{n\in\mathbb{N}}$ satisfies the conditions of Proposition 4.12.

For the second sequence of local sets, take $B^1 = \tilde{A}(\Gamma)$ and define B^{n+1} to be the closed union of B^n with all $\tilde{A}(\Gamma\mid_O)$ such that O is a connected component of $\mathbb{D}\backslash B^n$, $h_{B^n}\mid_O \leq 2\lambda$ and ∂O contains an interval of $\mathbb{R}$. Denote $A^1(\Gamma)$ the closed union of all the B^n. Due to the fact that B^n are BTLS with $h_{B^n} \leq 2\lambda$ on $[-1,1]$, we have that Γ_{A^1} restricted to $[-1,1]$ is smaller than or equal to 2λ. Additionally, note that $2(\log(\mathrm{CR}^{-1}(z, \mathbb{D}\backslash\breve{A}^1)) - \log(\mathrm{CR}^{-1}(z, \mathbb{D}))$ is distributed as the first time a BM hits 2λ. Now, we iterate to define $A^n(\Gamma)$ as the closed union of $A^{n-1}(\Gamma)$ and $A^1(\Gamma\mid_O)$, where O ranges over all connected components of $\mathbb{D}^+\backslash A^{(n-1)}$ containing an interval of $\mathbb{R}$. The sequence $(A^n)_{n\in\mathbb{N}}$ satisfies the condition of Proposition 4.12. Note that in this case the martingale simplifies and contains only information on the geometry of the sets A^n:

$$M_n^\gamma := e^{\gamma 2\lambda n} \int_{\mathcal{O}} CR(z; \mathbb{D}\backslash\breve{A}^n)^{\gamma^2/4} dz.$$

The fact that this martingale is a measurable function of A^n allows us to use the same techniques as in [ALS17] to prove that the measure $2\lambda n - \Gamma_{A^n}$ on $\mathbb{R}$ is a measurable function of A^n.

It is also explained in [QW17] that the sets A^n we have just constructed, and the definition of the boundary Liouville measure using them, might help to reinterpret an SLE-type of conformal welding first studied in [She16].

4.6 Critical and supercritical regimes

In this section it is technically simpler to restrict ourselves to the simply connected case and study a special family of sequences of local sets, though the results hold in a more general setting. Namely, we assume that our sets A^n are formed by an iterative procedure. That is, $A^1 = A(\Gamma)$ is some measurable local set coupled with the GFF Γ, and A^{n+1} is formed from A^n by, in each component O of $\mathbb{D} \setminus A^n$, exploring $A(\Gamma^{A^n})$. Notice, for example, that the iterated CLE_4 coupling described in the introduction is covered by this hypothesis, as are the couplings with A_{an} for any $a > 0$.

We first show that the martingales defined in Section 4.4 converge to zero for $\gamma \geq 2$. Then, in the critical case $\gamma = 2$, we define a derivative martingale and show it converges to the same measure as the critical measure μ_2' from [DRSV14a, DRSV14b, Pow17a], and 1/2 times the critical measure $\tilde{D}_\infty$ from [Aid15]. Finally, we show that for $A^n = A_{an}$ we can also construct the critical measure using the Seneta-Heyde rescaling (analogous to the main theorem of [AS14].) More precisely, for all Borelian $\mathcal{O} \subset \mathbb{D}$, we have that $\sqrt{an}M^2(\mathcal{O}, A^n)$ converges in probability to $\frac{4}{\sqrt{\pi}}\mu_2'(\mathcal{O})$ as $n \to \infty$.

4.6.1 (Super)critical regime.

Lemma 4.13. *Set $\gamma \geq 2$ and assume that A^n is formed by iteration as above and that $\overline{\bigcup_n A_n} = \overline{\mathbb{D}}$. Assume further that A is such that h_A is constant in each connected component of $\mathbb{D} \setminus A$ almost surely. Then $M_n^\gamma \to 0$ almost surely.*

Remark 4.14. *Due to the iterative nature of the construction, the condition $\overline{\bigcup_n A_n} = \mathbb{D}$, i.e. that in the limit the iterated sets cover the whole domain, is implied for example by a simple requirement on the local set $A = A_1$ - it suffices to have that there exists $\varepsilon, \delta > 0$ such that for any point $z \in \mathbb{D}$, the probability that $\mathrm{CR}(z; \mathbb{D} \backslash A_1) < (1-\varepsilon)\,\mathrm{CR}(z;\mathbb{D})$ is bigger than δ.*

In [Aid15], Aïdekon also considers the critical and supercritical cases for his iterated loop measures. In particular, from his results one can read out that, with the notation of Proposition 4.7, for any $a > 0$ and $\gamma \geq 2$, we have $M_{an}^\gamma \to 0$ almost surely as $n \to \infty$.

The proof follows from a classical technique stemming from the literature on branching random walks [Lyo97], but is based on the local set coupling with the GFF.

Proof. From (4.2) and a direct calculation we see that $M_n^\gamma(\mathbb{D})/M_0^\gamma(\mathbb{D})$ is a mean one martingale, where $M_0^\gamma(\mathbb{D}) = \int_{\mathbb{D}} \mathrm{CR}(z, \mathbb{D})^{\gamma^2/2}\, dz$. Let us define a new probability measure $\hat{\mathbb{P}}$ via the change of measure

$$\left.\frac{d\hat{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_{A^n}} = \frac{M_n^\gamma(\mathbb{D})}{M_0^\gamma(\mathbb{D})}. \tag{4.6}$$

It is well known, see for example [Dur], that in order to show that $M_n^\gamma(\mathbb{D}) \to 0$ almost surely under $\mathbb{P}$, it suffices to prove that $\limsup_n M_n^\gamma(\mathbb{D}) = +\infty$ a.s. under $\hat{\mathbb{P}}$.

To show this we actually consider a change of measure on an enlarged probability space. Define a measure $\mathbb{P}^*$ on $(\Gamma, (A^n)_n, Z)$ by sampling $(\Gamma, (A^n)_n)$ from $\mathbb{P}$ and then independently, sampling a random variable $Z \in \mathbb{D}$ with law proportional to Lebesgue measure. Note that under $\mathbb{P}^*$ the process

$$\xi_n = \mathrm{e}^{\gamma h_{A^n}(Z)+\gamma^2/2 \log \mathrm{CR}(Z,\mathbb{D}\setminus A^n)}$$

is a martingale with respect to the filtration $\mathcal{F}^*_{A^n} = \mathcal{F}_{A^n} \vee \sigma(Z)$. Thus we can define a new probability measure $\hat{\mathbb{P}}^*$ by

$$\left.\frac{d\hat{\mathbb{P}}^*}{d\mathbb{P}^*}\right|_{\mathcal{F}^*_{A^n}} := \frac{\xi_n}{\mathbb{E}[\xi_0]} \tag{4.7}$$

Then if $\hat{\mathbb{P}}$ is the restriction of $\hat{\mathbb{P}}^*$ to $\mathcal{F}_{A^n}$, $\hat{\mathbb{P}}$ and $\mathbb{P}$ satisfy (4.6). Therefore it suffices to prove that under $\hat{\mathbb{P}}^*$ and conditionally on Z, we have $\limsup_n M_n^\gamma(\mathbb{D}) = +\infty$ almost surely. By the Köebe-(1/4) Theorem and [Aid15, Lemma 2.4] we only need to prove that under this law almost surely

$$\limsup_n \mathrm{e}^{\gamma h_{A^n}(Z)+(\gamma^2/2+2) \log \mathrm{CR}(Z,\mathbb{D}\setminus A^n)} = +\infty.$$

However we can calculate, using (4.2) that

$$\gamma h_{A^n}(Z) + (\gamma^2/2+2)\log \mathrm{CR}(Z, \mathbb{D} \setminus A^n)$$

is a random walk with non-negative mean (started from $(\gamma^2/2+2)\log \mathrm{CR}(Z, \mathbb{D})$) under this law. This allows us to conclude.

□

4.6.2 The derivative martingale in the critical regime

We now show the convergence of the derivative martingale (when $\gamma = 2$, defined below) for the particular case of iterated $A_{-a,a}$, $a \geq \lambda$. For any Borel set $\mathcal{O} \subseteq \mathbb{D}$ and local set A, we define

$$D^\gamma(\mathcal{O}, A) := \int_{\mathcal{O}} \left(-h_A(z) + \gamma \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A)\right) e^{\gamma h_A(z)}\, \mathrm{CR}(z; \mathbb{D} \setminus A)^{\gamma^2/2}\, dz.$$

The rest of this section is devoted to proving the following proposition.

Proposition 4.15. *Assume that A^n is formed by iterating $A_{-a,a}$ n times, for $a \geq \lambda$. Then for any Borel $\mathcal{O} \subset \mathbb{D}$ we have that $\hat{D}_n(\mathcal{O}) := D^2(\mathcal{O}, A^n)$ is a martingale and converges almost surely to a finite, positive limit $\hat{D}_\infty(\mathcal{O})$ as $n \to \infty$. In particular the signed measures $\hat{D}_n(\mathcal{O})$ converge weakly to a limiting measure that is independent of the choice of $a > 0$ and agrees with the critical measure μ_2' defined in [DRSV14a, DRSV14b], and $1/2$ times the critical measure $\tilde{D}_\infty$ defined in Theorem*

1.3 of [Aid15].

Before the proof, let us first comment on the case where the set we are going to iterate is A_a; so n iterations gives A_{an}. In the case $a = 2\lambda$, observe that twice the derivative martingale $2D^2(\mathcal{O}, A_{2\lambda n})$ is equal to $\tilde{D}_n$ defined in (1.3) of [Aid15] (see Remark 4.8). Thus, we know that when we iterate $A_{2\lambda}$, its associated sequence of measures converges to a limit $\tilde{D}_\infty$. In fact it follows from [Aid15], that for all dyadic $a \geq 0$, $D_n(\mathcal{O}) := 2D^2(\mathcal{O}, A_{an})$ converges to the same limit. Doob's maximal inequality then implies that there exists a modification of $2D^2(\mathcal{O}, A_t)$ that also converges to $\tilde{D}_\infty$ as $t \to \infty$.

These martingales are not uniformly integrable (U.I.) and thus, our previous techniques do not apply directly. However, we will discuss how to pass through certain U.I. martingales to get convergence in the case of $D_n(\mathcal{O})$. We will then use this case to show convergence for $\hat{D}_n(\mathcal{O})$. We remark that these U.I. martingales, given in the proof below, are similar but not exactly the same as the analogous U.I. martingales introduced in [DRSV14a, Aid15].

We make $D_n(\mathcal{O})$ uniformly integrable via localization. To do this, let us introduce the following stopping times:

$$\tau_\beta := \inf \left\{ n \in \mathbb{N} : \inf_{z \in \mathbb{D} \setminus A_{an}} -h_{A_{an}}(z) + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_{an}) \leq -\beta \right\}.$$

Notice that $A_{a(n \wedge \tau_\beta)}$ is then also a local set of Γ and we can define $D_n^\beta(\mathcal{O}) := 2D^2(\mathcal{O}, A_{a(n\wedge\tau_\beta)})$. As always, we include the factor 2 for comparison with martingales in [Aid15]. Then $D_n^\beta(\mathcal{O})$ is a martingale, due to the fact that it is the derivative with respect to $-\gamma$ of the martingale $2M^\gamma(\mathcal{O}, A_a^{n\wedge\tau_\beta})$.

$D_n^\beta(\mathcal{O})$ is uniformly integrable. Let us first show the following claim:

Claim 4.16. *The martingale $D_n^\beta(\mathcal{O})$ is uniformly integrable for all $\beta \geq 0$ and so converges almost surely and in $\mathcal{L}^1$ to some limit $L(\mathcal{O}, \beta)$ as $n \to \infty$.*

Proof. Indeed, for $\eta > 0$, let $E_\eta(n, z)$ be the event that

$$\{-am + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^m) \geq -\eta \text{ for all } m \leq n\}.$$

Then Proposition 3.2. of [Aid15] implies that for all $\eta > 0$,

$$\bar{D}_n^\eta(\mathcal{O}) := \int_{\mathcal{O}} h_1\left(-2an + 4\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^n) + 2\eta\right) 1_{E_\eta(n,z)} e^{2an}\, \mathrm{CR}(z; \mathbb{D} \setminus A_a^n)^2\, dz \tag{4.8}$$

is a U.I. martingale. Here $h_1(u)$ is a so-called renewal function, that satisfies $h_1(u) \geq Ru$ for some $R > 0$. We conclude that the stopped martingale $\bar{D}^{\eta}_{n\wedge\tau_\beta}(\mathcal{O})$ is also U.I. Given that

$$-h_{A_a^{n\wedge\tau_\beta}}(z) + \gamma \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_a^{n\wedge\tau_\beta}) \geq -a - \beta$$

we can bound

$$|D_n^\beta(\mathcal{O})| \leq R^{-1} |\bar{D}^{2\beta+2a}_{n\wedge\tau_\beta}|.$$

The claim follows. □

Comparison with Aïdekon's limit. We first show that our UI martingales converge to Aïdekon's limit, and then use this to treat the case where A^n is formed by iterating $A_{-a,a}$.

Claim 4.17. *The martingales $D_n^\beta(\mathcal{O})$ converge to $\tilde{D}_\infty(\mathcal{O})$ as first $n \to \infty$ and then $\beta \to \infty$.*

Proof. From the definition, $D_n^\beta(\mathcal{O})$ and $2D^2(\mathcal{O}, A_{an})$ are equal on the event $\{\tau_\beta = \infty\}$. Additionally, $\mathbb{P}(\tau_\beta = \infty)$ is equal to $1 - o(\beta)$, due to the fact that almost surely

$$\inf_{z\in\mathbb{D}} \inf_{n\in\mathbb{N}} \big(-2an + 4 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_{an})\big) > -\infty. \tag{4.9}$$

This is proved [Aid15] after the statement of Proposition 3.2 (see also Remark 4.8 of the current paper).

In particular, as $2D^2(\mathcal{O}, A_{an})$ tends to $\tilde{D}_\infty(\mathcal{O})$ by [Aid15] (together with the comments after Proposition 4.15) we see that $\lim_{\beta\to\infty} L(\mathcal{O}, \beta)$ is also equal to $\tilde{D}_\infty(\mathcal{O})$ in this case. □

Now, let us come back to the case where A^n is formed by iterating $A_{-a,a}$. To do this, define $\hat{D}_n^\beta(\mathcal{O}) := D^2(\mathcal{O}, A^{n\wedge\hat{\tau}_\beta})$, for

$$\hat{\tau}_\beta := \inf_n \left\{ n \in \mathbb{N} : \inf_{z\in\mathbb{D}\setminus A^n} -h_{A^n}(z) + 2 \log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A^n) \leq -\beta \right\}.$$

First, observe that when both A^n, the iterated $A_{-a,a}$, and A_{an} are coupled with the same GFF as local sets, Lemma 4.5 implies that a.s. $\{\hat{\tau}_\beta = \infty\} = \{\tau_\beta = \infty\}$ and that for all $n \leq m$, $A^{n\wedge\hat{\tau}_\beta} \subseteq A_{a(m\wedge\tau_\beta)}$. Thus, as long as the limit on the lefthand side exists, we have

$$\lim_{\beta\to\infty} \lim_{n\to\infty} \hat{D}_n^\beta(\mathcal{O}) = \lim_{n\to\infty} \hat{D}_n(\mathcal{O}).$$

So, it suffices to argue that

$$2\hat{D}_n^\beta(\mathcal{O}) \mathbf{1}_{\{\hat{\tau}_\beta=\infty\}} \to L(\mathcal{O}, \beta) \mathbf{1}_{\{\hat{\tau}_\beta=\infty\}}, \qquad \text{as } n \to \infty.$$

To see this we will use the strategy of the proof of Proposition 4.7.

Namely, consider a local set A with zero Lebesgue measure, and such that (Γ_A, f) is bounded from above by $K \geq 0$ (in the sense explained before Proposition 4.10). Then from an explicit calculation similar to the key claim of Proposition 4.7 we have that for any $\gamma > 0$, a.s. and in $\mathcal{L}^1(\Omega)$ as $\varepsilon \to 0$,

$$\mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \mathcal{F}_A\right] \to D^\gamma(\mathcal{O}, A).$$

Now, for $n \leq m$ let $\hat{\mathcal{F}}_n$ and $\mathcal{G}_m$ be the sigma-algebras corresponding to the local sets $A^{n \wedge \hat{\tau}_\beta}$ and $A_{a(m \wedge \tau_\beta)}$ respectively. Noting that a.s. $A^{n \wedge \hat{\tau}_\beta} \subseteq A_{a(m \wedge \tau_\beta)}$,

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left[D_m^\beta(\mathcal{O}) \mid \hat{\mathcal{F}}_n\right] &= \mathbb{E}\left[\lim_{\varepsilon \to 0} \mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \mathcal{G}_m\right] \mid \hat{\mathcal{F}}_n\right] \\ &= \lim_{\varepsilon \to 0} \mathbb{E}\left[\int_{\mathcal{O}} (\gamma \log(1/\varepsilon) - \Gamma_\varepsilon(z)) e^{\gamma \Gamma_\varepsilon(z)} \varepsilon^{\frac{\gamma^2}{2}} dz \mid \hat{\mathcal{F}}_n\right] = \hat{D}_n^\beta(\mathcal{O}).\end{aligned}$$

Due to the fact that $D_m^\beta(\mathcal{O}) \to L(O, \beta)$ in L^1 we have that $\mathbb{E}\left[L(\mathcal{O}, \beta) | \hat{\mathcal{F}}_n\right] = 2\hat{D}_n^\beta(\mathcal{O})$ and so $2\hat{D}_n^\beta(\mathcal{O}) \to \mathbb{E}\left[L(\mathcal{O}, \beta) | \hat{\mathcal{F}}_\infty\right]$. However, the event $\{\hat{\tau}_\beta = \infty\}$ is $\hat{\mathcal{F}}_\infty$-measurable and on this event the limit of $A^{n \wedge \hat{\tau}_\beta}$ is $\mathbb{D}$ (by Lemma 4.3 and Remark 4.14). Similarly to Lemma 4.2, it then follows that $F(\Gamma) 1_{\{\hat{\tau}_\beta = \infty\}}$ is $\hat{\mathcal{F}}_\infty$ measurable for any measurable function of Γ. Thus, we have that $2\hat{D}_n^\beta(\mathcal{O}) \mathbf{1}_{\{\tau_\beta = \infty\}} \to L(\mathcal{O}, \beta) \mathbf{1}_{\{\tau_\beta = \infty\}}$, as required.

Comparison with mollified measures. It remains to prove the latter claim of the proposition, i.e. to show that the limiting measure $\hat{D}_\infty = \frac{1}{2}\tilde{D}_\infty$ is equal to the measure μ_2' from [DRSV14b, Pow17a], described in Section 4.3.3. We again mollify our measures using the circle average, and choose a sequence $\varepsilon_k \to 0$ such that $\nu_\varepsilon \to \mu_2'$ a.s. Whenever we write $\varepsilon \to 0$, it means that we are converging to 0 via $(\varepsilon_k)_{k \in \mathbb{N}}$. We set, for fixed $\mathcal{O} \subset \mathbb{D}$,

$$\nu_\varepsilon^\beta(\mathcal{O}) = \int_{\mathcal{O}} (-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon)) \mathbf{1}_{\{T_\beta(z) \leq \varepsilon\}} \, \mathrm{e}^{2\Gamma_\varepsilon(z) - 2\log(1/\varepsilon)}$$

where $T_\beta(z) = \sup\{\varepsilon \leq \varepsilon_0 : 2\Gamma_\varepsilon(z) - 2\log(1/\varepsilon) \leq -\beta\}$ and ε_0 is some fixed starting point such that $\mathrm{dist}(z, \partial\mathbb{D}) > \varepsilon_0$ for all $z \in \mathcal{O}$. It is shown in [Pow17a, Proposition 3.6] that $\nu_\varepsilon^\beta(\mathcal{O})$ is uniformly integrable for fixed $\beta \geq 0$. Additionally, define

$$C_\beta := \{-\Gamma_\varepsilon(z) + 2\log(1/\varepsilon) + \beta > 0 \text{ for all } z \in \mathbb{D}, 0 < \varepsilon \leq d(z, \partial\mathbb{D})\}$$

then $\mathbb{P}(C_\beta) = 1 - o(1)$ thanks to [HRV15, Theorem 6.3].

The strategy is to prove, for A_n the n-FPS of the GFF, that

$$\lim_{\beta\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}\left[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}\right]\mathbf{1}_{\{\tau_\beta=\infty\}} \tag{4.10}$$

is equal to both $\mu_2'(\mathcal{O})$ and $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})$ almost surely.

Let us first show that (4.10) is equal to $\mu_2'(\mathcal{O})$. Observe that since $\nu_\varepsilon^\beta(\mathcal{O})$ is uniformly integrable, we have by Fatou's and reverse Fatou's lemma that, if the limit in ε exists (we will show that is does in the next step)

$$\mathbb{E}\left[\liminf_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}\right] \le \lim_{\varepsilon\to 0}\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}] \le \mathbb{E}\left[\limsup_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}\right].$$

Taking the limit as $n,\beta\to\infty$ we obtain that

$$\lim_{\beta\to\infty}\liminf_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}) \le \lim_{\beta\to\infty}\lim_{n\to\infty}\lim_{\varepsilon\to 0}\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}] \le \lim_{\beta\to\infty}\limsup_{\varepsilon\to 0}\nu_\varepsilon^\beta(\mathcal{O}).$$

However, since $\nu_\varepsilon^\beta(\mathcal{O}) = \nu_\varepsilon(\mathcal{O})$ on the event C_β, and almost surely $\mathbf{1}_{C_\beta}\uparrow 1$ as $\beta\to\infty$, the right and left hand sides of the above two expressions are equal to $\mu_2'(\mathcal{O})$. Since also almost surely $\mathbf{1}_{\{\tau_\beta=\infty\}}\to 1$ as $\beta\to\infty$, we deduce that (4.10) is equal to $\mu_2'(\mathcal{O})$.

We now show that (4.10) is equal to $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})$. Write $\mathbb{E}[\nu_\varepsilon^\beta(\mathcal{O}) \mid \mathcal{F}_{A_n}] := E^1(n,\beta,\varepsilon)+E^2(n,\beta,\varepsilon)$ where

$$E^1(n,\beta,\varepsilon) := \int_{\mathcal{O}\setminus A_n^\varepsilon}\mathbb{E}^{A_n}\left[(\Gamma_\varepsilon(z)+2\log(1/\varepsilon))\mathbf{1}_{\{T_\beta(z)\le\varepsilon\}}\,\mathrm{e}^{2\Gamma_\varepsilon(z)-2\log(1/\varepsilon)}\right]dz;$$
$$E^2(n,\beta,\varepsilon) := \int_{\mathcal{O}\cap A_n^\varepsilon}\mathbb{E}^{A_n}\left[(\Gamma_\varepsilon(z)+2\log(1/\varepsilon))\mathbf{1}_{\{T_\beta(z)\le\varepsilon\}}\,\mathrm{e}^{2\Gamma_\varepsilon(z)-2\log(1/\varepsilon)}\right]dz;$$

and $\mathbb{E}^{A_n}$ is the regular conditional expectation w.r.t. $\mathcal{F}_{A_n}$. Here we used that $\Gamma_\varepsilon(z) = (\Gamma_{A_n},\rho_z^\varepsilon)+\Gamma_\varepsilon^{A_n}(z)$, where conditionally on $\mathcal{F}_{A_n}$ (i.e. under $\mathbb{P}^{A_n}$) Γ^{A_n} is a GFF in $\mathbb{D}\setminus A_n$. This implies that $\lim_{\varepsilon\to 0}E^2(n,\beta,\varepsilon)=0$ almost surely. To see why, note that $(\Gamma_{A_n},\rho_z^\varepsilon)\le n$ and that when $\varepsilon\ge d(z,A_n)$, the variance of $\Gamma_\varepsilon^{A_n}(z)$ is uniformly bounded (independently of z and ε). This implies that the integrand is of order $\varepsilon^2\log(1/\varepsilon)$ uniformly in z.

To deal with E^1, observe that if $\varepsilon\le d(A_n,z)$ then $(\Gamma_{A_n},\rho_z^\varepsilon)=n$. Additionally, due to the Markov property of the GFF and an explicit computation, we have that conditionally on $\mathcal{F}_{A_n}$, i.e., under the probability $\mathbb{P}^{A_n}$,

$$\left(-n-\Gamma_\delta^{A_n}(z)+2\log(1/\delta)\right)\mathbf{1}_{\{T_\beta(z)\le\delta\}}\,\mathrm{e}^{2n+2\Gamma_\delta^{A_n}(z)-2\log(1/\delta)}$$

is a (reverse) martingale for $0<\delta\le\delta_n(z):=d(z,\partial\mathbb{D}\setminus A_n)$. Thus, we have that

$E^1(n,\beta,\varepsilon)$ is equal to

$$\int_{\mathcal{O}\setminus A_n^\varepsilon} \mathbb{E}^{A_n}\left[(-n-\Gamma^{A_n}_{\delta_n(z)}(z)+2\log(1/\delta_n(z)))\mathbf{1}_{\{T_\beta(z)\le\delta_n(z)\}}\,\mathrm{e}^{2n+2\Gamma^{A_n}_{\delta_n(z)}(z)-2\log(1/\delta_n(z))}\right]dz.$$

Since the integrand does not depend on ε, taking the limit in ε simply yields the integral over the whole of $\mathcal{O}\setminus A_n$.

Now, we rewrite $\lim_{\varepsilon\to 0} E^1(n,\beta,\varepsilon)$ as a difference between

$$\int_{\mathcal{O}\setminus A_n} \mathbb{E}^{A_n}\left[(-n-\Gamma^{A_n}_{\delta_n(z)}(z)+2\log(1/\delta_n(z)))\,\mathrm{e}^{2n+2\Gamma^{A_n}_{\delta_n(z)}(z)-2\log(1/\delta_n(z))}\right]dz$$

and

$$\int_{\mathcal{O}\setminus A_n} \mathbb{E}^{A_n}\left[(-n-\Gamma^{A_n}_{\delta_n(z)}(z)+2\log(1/\delta_n(z)))\mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\,\mathrm{e}^{2n+2\Gamma^{A_n}_{\delta_n(z)}(z)-2\log(1/\delta_n(z))}\right]dz$$

Notice that the first of these terms is equal to $D_n(\mathcal{O})/2$. Let us further rewrite the second term. First, we use Girsanov's theorem. Since $\Gamma^{A_n}_{\delta_n(z)}$ is a normal random variable with mean 0 and variance $\log(1/\delta_n(z))-\log(1/\operatorname{CR}(z,\mathbb{D}\setminus A_n))$, we see that this term is equal to

$$\int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log\operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\,\tilde{\mathbb{E}}^{A_n}_z\left[\left(-n-\Gamma^{A_n}_{\delta_n(z)}+2\log\left(1/\delta_n(z)\right)\right)\mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\right]dz$$

where $\tilde{\mathbb{P}}^{A_n}_z$ is the measure under which the process $(\Gamma^{A_n}_\delta(z))_\delta$ has the same covariance structure as under $\mathbb{P}^{A_n}$ but with means shifted by $0\le\operatorname{cov}(\Gamma^{A_n}_\delta(z),\Gamma^{A_n}_{\delta_n(z)}(z))\le 2$.

Next, we further decompose this as a sum of $E^3(n,\beta)$ and $E^4(n,\beta)$ with:

$$E^3(n,\beta):=\int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log\operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\,\tilde{\mathbb{E}}^{A_n}_z\left[\left(-\Gamma^{A_n}_{\delta_n(z)}+2\log\left(\frac{\operatorname{CR}(z,\mathbb{D}\setminus A_n)}{\delta_n(z)}\right)\right)\mathbf{1}_{\{T_\beta(z)>\delta_n(z)\}}\right]dz;$$
$$E^4(n,\beta):=\int_{\mathcal{O}\setminus A_n} \mathrm{e}^{2n-2\log\operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\left(-n+2\log\operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)\right)\tilde{\mathbb{P}}^{A_n}_z\left(T_\beta(z)>\delta_n(z)\right)dz.$$

To bound E^3, we notice that $\Gamma^n_{\delta_n(z)}(z)$ has bounded variance under $\tilde{\mathbb{E}}^{A_n}_z$, and $\frac{\operatorname{CR}(z,\mathbb{D}\setminus A_n)}{\delta_n(z)}$ is uniformly bounded by the Köebe 1/4 Theorem. This means that the whole expression is less than some constant times $M_n(\mathcal{O})$, which we know converges to 0 a.s. as $n\to\infty$.

For E^4 first note that the law of $\Gamma^{A_n}_\delta$ under $\tilde{\mathbb{P}}^{A_n}_z$ is equal to its law under $\mathbb{P}^{A_n}$ up to a shift that is uniformly bounded by 2. Thus we have that $\tilde{\mathbb{P}}^{A_n}_z(T_\beta(z)>\delta_n(z))\le\mathbb{P}^{A_n}(C^c_{\beta-2})$. Additionally, on the event $\{\tau_\beta=\infty\}$ we also have for all $z\in\mathcal{O}$

$$-n+2\log\operatorname{CR}^{-1}(z,\mathbb{D}\setminus A_n)+\beta\ge 0.$$

This implies that

$$|E^4(n,\beta)|\mathbf{1}_{\{\tau_\beta=\infty\}} \leq (|D_n(\mathcal{O})|/2 + \beta M_n(\mathcal{O}))\mathbb{E}[C^c_{\beta-2}|\mathcal{F}_{A_n}]\mathbf{1}_{\{\tau_\beta=\infty\}}.$$

But the limit of the RHS as $n \to \infty$ is equal to $\frac{1}{2}\tilde{D}_\infty(\mathcal{O})\mathbf{1}_{C^c_{\beta-2}}\mathbf{1}_{\{\tau_\beta=\infty\}}$. As this tends to 0 as $\beta \to \infty$ we conclude.

4.6.3 Seneta-Heyde rescaling.

Finally, we show that one can also perform a so-called Seneta-Heyde rescaling for the construction of the critical Liouville measure using local sets. While this result itself is of interest, one of the other main objectives of this section is a proof to demonstrate how simple it is in this framework to transfer techniques and methods from multiplicative cascades and branching random walks to the study of the Liouville measure. We plan to make further use of this in a follow-up paper. The proof in this section follows very closely that of [AS14], so we only give an outline, point to concrete analogies, and highlight some minor differences. It might be helpful to have the article [AS14] on the side, although we aimed to make the section readable on its own too.

Proposition 4.18 (Seneta-Heyde Rescaling)**.** *For all $a \geq 0$, and Borelian $\mathcal{O} \subset \mathbb{D}$, we have that $\sqrt{an}M^2(\mathcal{O}, A_{an}) \to \frac{4}{\sqrt{\pi}}\mu_2'(O)$ in probability as $n \to \infty$. In particular the measures $\sqrt{an}M_n^2$ converge weakly in probability to $\frac{4}{\sqrt{\pi}}\mu_2'$.*

Again, by Remark 4.9, it suffices to prove the convergence statement for $\sqrt{an}M^2(\mathcal{O}, A_{an})$. For simplicity, we work in the case $a = 1$ and define $M_n := M^2(O, A_n)$. Before proving this proposition we need to define carefully a certain family $\hat{\mathbb{Q}}^\eta$ of *rooted measures*. Recall that if (Γ, Z) has the law $\hat{\mathbb{P}}^*(d\Gamma, dz)$ defined in (4.7), then the process

$$S_n := -2n + 4\log \mathrm{CR}^{-1}(Z, \mathbb{D} \setminus A_n)$$

is a random walk with mean-zero increments under the conditional law $\hat{\mathbb{P}}^*(d\Gamma|Z)$. Recall also the definition of $\bar{D_n^\eta}$ given in (4.8):

$$\bar{D}_n^\eta(\mathcal{O}) := \int_{\mathcal{O}} h_1\left(-2n + 4\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + 2\eta\right) 1_{E_\eta(n,z)} e^{2n}\, \mathrm{CR}(z; \mathbb{D} \setminus A_n)^2\, dz.$$

We already showed that $\bar{D_n^\eta}$ is a positive martingale with respect to $(\mathcal{F}_{A_n})_n$ and our initial probability measure $\mathbb{P}$. Hence we can define a new probability measure $\mathbb{Q}^\eta$ by setting it, when restricted to $\mathcal{F}_{A_n}$, to have Radon-Nikodym derivative $\bar{D}_n^\eta(\mathcal{O})/\bar{D}_0^\eta(\mathcal{O})$ with respect to $\mathbb{P}$.

Again we extend this to a *rooted* measure on the field Γ plus a distinguished point Z by setting $\hat{\mathbb{Q}}^\eta(d\Gamma, dz)$ restricted to $\mathcal{F}^*_{A_n} = \mathcal{F}_{A_n} \vee \sigma(Z)$ to be

$$h_1(-2n + 4\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_n) + 2\eta)\, \mathrm{e}^{2n - 2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_n)}\, \mathbf{1}_{E_\eta(z,n)} \frac{\mathbf{1}_{\mathcal{O}}(z)}{\bar{D}_0^\eta}\, dz\, \mathbb{P}[d\Gamma].$$

We make the following observations:

1. The marginal law of Z under $\hat{\mathbb{Q}}^\eta$ is proportional to $h_1(4\log \mathrm{CR}^{-1}(z, \mathbb{D}) + 2\eta)\, \mathrm{CR}(z, \mathbb{D})^2 \mathbf{1}_{\mathcal{O}}(z) dz$.
2. The marginal law of the field Γ under $\hat{\mathbb{Q}}^\eta$ is given by $\mathbb{Q}^\eta$.
3. Write $\hat{\mathbb{Q}}^\eta_z = \hat{\mathbb{Q}}^\eta[\cdot \mid Z = z]$ for the law of Γ given the point $Z = z$. The law of the sequence $(A_n)_n$ under this measure can be described as follows. First sample A_1 with law weighted by

$$\frac{h_1(-2 + 4\log \mathrm{CR}^{-1}(z, \mathbb{D} \setminus A_1) + 2\eta)}{h_1(\eta + 4\log \mathrm{CR}^{-1}(z, \mathbb{D}))} \mathbf{1}_{E_\eta(z,1)}\, \mathrm{e}^{2 - 2\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_1)}\,.$$

   Then given A_k for any $k \geq 1$, construct an independent copy of $(A_n)_n$ inside each component of $\mathbb{D} \setminus A_k$ that does *not* contain the point z. Inside the component containing z, let us call this $\mathcal{B}_k$, construct the components of $A_{k+1} \cap \mathcal{B}_k$ by weighting their laws by

$$\frac{h_1(-2(k+1) - 4\log \mathrm{CR}(z, \mathbb{D} \setminus A_{k+1}) + 2\eta)}{h_1(-2k - 4\log \mathrm{CR}(z, \mathbb{D} \setminus A_k) + 2\eta)} \mathbf{1}_{E_\eta(z,k+1)}\, \mathrm{e}^{2 + 2\log \mathrm{CR}(z,\mathbb{D}\setminus A_{k+1}) - 2\log \mathrm{CR}(z,\mathbb{D}\setminus A_k)}\,.$$

   This defines the law of the sets A_n, and hence also by iteration the law of Γ.

It follows directly from the above construction that the law of $S_n = -2n + 4\log \mathrm{CR}(z, \mathbb{D} \setminus A_n)$ under $\hat{\mathbb{Q}}^\eta_z$ has the same as its law under $\hat{\mathbb{P}}^*[\cdot \mid Z = z]$, but conditioned to stay above -2η.

Now, we note some useful properties of the renewal function see for example [AS14, Section 2]):.

- First, recall from Claim 4.16 that $h_1(u) \geq Ru$ for all $u \geq 0$ for some positive R.
- By the renewal theorem, $c_0 := \lim_{u\to\infty} \frac{h_1(u)}{u}$ exists and lies in $(0, \infty)$.
- Let $\theta = 2/(\sqrt{\pi} c_0)$ (in the case of A_{an}, $\theta = 2/(\sqrt{\pi a} c_0)$). Then

$$\hat{\mathbb{P}}^* \left[\min_{1\leq i\leq n} S_i \geq -u \mid z \right] \sim \frac{\theta h_1(S_0 + u)}{\sqrt{n}} \tag{4.11}$$

  as $n \to \infty$, for any $u \geq 0$. Moreover, the above holds uniformly in $u \in [0, b_n]$

for any sequence $b_n \in \mathbb{R}_+$ such that $\lim_{n\to\infty} b_n/\sqrt{n} = 0$.

Now set

$$\bar{M}_n^\eta(\mathcal{O}) := \int_{\mathcal{O}} e^{2n}\,\mathrm{CR}(x;\mathbb{D}\setminus A_n)^2 \mathbf{1}_{E_\eta(x,n)}\,dx.$$

Using the fact that $\bar{D}_n^\eta$ converges a.s. to a positive measure and that $\bar{D}_n^\eta = \bar{D}_n$ for η large enough, one can show, following exactly [AS14, Proof of Theorem 1.1], that in order to prove the Proposition 4.18 it suffices to establish the next claim:

Claim 4.19. *For any $\eta > 0$, $\sqrt{n}\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} \to \theta$ in $\mathbb{Q}^\eta$ probability as $n \to \infty$.*

Proof. The overall strategy follows very closely [AS14] and is to control the first and second moments of $\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}$ as $n \to \infty$:

$$\hat{\mathbb{Q}}^\eta\left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}\right] = \frac{\theta}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right) \quad\text{and}\quad \hat{\mathbb{Q}}^\eta\left[\left(\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}\right)^2\right] \le \frac{\theta^2}{n} + o\left(\frac{1}{n}\right) \tag{4.12}$$

These estimates prove the result by Jensen and Chebyshev's inequalities. The key observation lies in rewriting the moments using the rooted measure. Indeed, we can write for (Γ, Z) distributed under $\hat{\mathbb{Q}}^\eta$

$$\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta} = \hat{\mathbb{Q}}^\eta\left[\frac{1}{h_1(-2n + 4\log \mathrm{CR}(Z,\mathbb{D}\setminus A_n) + 2\eta)} \mid \mathcal{F}_{A_n}\right] \tag{4.13}$$

and thus we have $\hat{\mathbb{Q}}^\eta[\bar{M}_n^\eta/\bar{D}_n^\eta] = \int_z \hat{\mathbb{Q}}_z^\eta\left[1/h_1(-2n + 4\log \mathrm{CR}(z,\mathbb{D}\setminus A_n))\right] d\hat{\mathbb{Q}}^\eta[dz]$. The first moment estimate (4.12) then follows easily using estimates on the renewal function, as in [AS14, proof of Proposition 4.1, Equation (4.1)].

We now move on to the second moment claim. Using random walk estimates and Jensen inequality, exactly as in [AS14, Lemmas 4.3-4.4], and (4.13), one can see that

$$\hat{\mathbb{Q}}_z^\eta\left[\left(\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}\right)^2\right] = \hat{\mathbb{Q}}_z^\eta\left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}\frac{1}{h_1(-2n + 4\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_n) + 2\eta)}\right] = O\left(\frac{1}{n}\right). \tag{4.14}$$

Thus, to prove the second moment bound, it suffices to find a sequence of events E_n with $\hat{\mathbb{Q}}^\eta(E_n) \to 1$ as $n \to \infty$ so that

$$\hat{\mathbb{Q}}_z^\eta\left[\frac{\bar{M}_n^\eta}{\bar{D}_n^\eta}\frac{\mathbf{1}_{E_n}}{h_1(-2n + 4\log \mathrm{CR}^{-1}(z,\mathbb{D}\setminus A_n) + 2\eta)}\right] \le \frac{\theta^2}{n} + o\left(\frac{1}{n}\right) \tag{4.15}$$

holds uniformly in z. To do this, we pick a sequence $k_n \to \infty$ such that $k_n/\sqrt{n} \to 0$ and $k_n/(\log n)^6 \to \infty$ as $n \to \infty$ (the reason for this choice will become clear later).

We then decompose $\bar{M}_n^\eta$ and $\bar{D}_n^\eta$ by writing

$$\bar{M}_n^\eta = \bar{M}_n^{\eta,[0,k_n]} + \bar{M}_n^{\eta,[k_n,n]}; \;\; \bar{D}_n^\eta = \bar{D}_n^{\eta,[k_n,n]} + \bar{D}_n^{\eta,[0,k_n]}$$

where the superscript $[0,k_n]$ refers to the integral over $\mathcal{B}_{k_n}$ and the superscript $[k_n,n]$ refers to the integral over $\mathcal{O}\setminus\mathcal{B}_{k_n}$, where $\mathcal{B}_{k_n}$ is the connected component of $D\setminus A_n$ containing z.

We now define our sequence of events E_n by setting $E_n = E_n^1 \cap E_n^2$, where

$$E_n^1 := \{\bar{D}_n^{\eta,[k_n,n]} \le 1/n^2\}; \;\; E_n^2 = \{S_{k_n} \in [k_n^{1/3}, k_n]\}.$$

Since under $\hat{\mathbb{Q}}_z^\eta$, S_n is a centered random walk conditioned to stay above -2η, it is clear at least that $\hat{\mathbb{Q}}^\eta\left[E_n^2\right] \to 1$ as $n\to\infty$. Putting aside the issue of whether or not $\hat{\mathbb{Q}}^\eta\left[E_n^1\right] \to 1$ for the moment, the next step is to bound (4.15) above by

$$\hat{\mathbb{Q}}_z^\eta\left[\frac{\bar{M}_n^{\eta,[k_n,n]}}{\bar{D}_n^\eta}\frac{\mathbf{1}_{E_n^1}}{h_1(-2n+4\log\mathrm{CR}^{-1}(z,\mathcal{B}_n)+2\eta)}\right]+\hat{\mathbb{Q}}_z^\eta\left[\frac{\bar{M}_n^{\eta,[0,k_n]}}{\bar{D}_n^{\eta,[0,k_n]}}\frac{\mathbf{1}_{E_n^2}}{h_1(-2n+4\log\mathrm{CR}^{-1}(z,\mathcal{B}_n)+2\eta)}\right].$$

Then, using that $R\bar{M}_n^{\eta,[k_n,n]} \le \bar{D}_n^{\eta,[k_n,n]} \le 1/n^2$, as in [AS14, Proof of Lemma 4.5], it can be deduced that the first term is $o(1/n)$. For the second term, we use that the two products in the expectation are conditionally independent given $\mathcal{F}^*_{A_{k_n}}$. We then have, by (4.11) and the assumption that $k_n/\sqrt{n}\to 0$, that

$$\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[h_1^{-1}(-2n+4\log\mathrm{CR}^{-1}(z,\mathcal{B}_n)+2\eta) \mid \mathcal{F}^*_{A_{k_n}}\right] = \theta/\sqrt{n}+o(1/\sqrt{n})$$

uniformly in ω and z. Since E_n^2 is $\mathcal{F}^*_{A_{k_n}}$ measurable, it therefore remains to prove that

$$\hat{\mathbb{Q}}_z^\eta\left[(\bar{M}_n^{\eta,[0,k_n]}/\bar{D}_n^{\eta,[0,k_n]})\mathbf{1}_{E_n^2}\right] \le \theta/\sqrt{n}+o(1/\sqrt{n}). \tag{4.16}$$

This is a consequence of our first moment estimate and the fact that $\bar{M}_n^{\eta,[0,k_n]}/\bar{D}_n^{\eta,[0,k_n]}$ is comparable to $\bar{M}_n^\eta/\bar{D}_n^\eta$ on the event E_n^1. The details of this claim are exactly as in [AS14, Lemma 4.5].

Thus, to finish the prove the proposition, it remains to establish that $\hat{\mathbb{Q}}^\eta\left[E_n^1\right] \to 1$ as $n\to\infty$. In fact we need to prove a stronger slightly stronger statement to also deduce (4.16) as described above:

Lemma 4.20. *Suppose that $k_n/\sqrt{n}\to 0$ and $k_n/(\log n)^6\to\infty$ as $n\to\infty$. Then there exists a deterministic sequence $p_n \nearrow 1$ such that $\mathbf{1}_{E_n^2}\hat{\mathbb{Q}}_z^\eta\left[E_n^1 \mid \mathcal{F}^*_{A_{k_n}}\right] \ge p_n$.*

It is only in the proof of this lemma that we need to do a bit of extra work. The extra work comes from the fact that, unlike in the case of multiplicative cascades, in our setting the sets at the $n-$th level have different shapes and sizes.

Proof of Lemma 4.20. Define further events E_n^3 and E_n^4 by setting

$$E_n^3 = \cap_{k_n \le j \le n} \{S_j \ge k_n^{1/6}\}; \quad E_n^4 = \cap_{k_n \le j \le n} \{\sup_{w \in \mathcal{B}_j} |z - w| \le j^c \, \mathrm{CR}(z, \mathcal{B}_j)\}$$

where c is some fixed constant to be chosen just below (as in [Aid15, Lemma 3.5]). We argue that:

(i) $\mathbf{1}_{E_n^2} \hat{\mathbb{Q}}_z^\eta \left[E_n^3 \mid \mathcal{F}^*_{A_{k_n}} \right] \ge p_n$, where $p_n \to 1$ is deterministic;

(ii) $\mathbf{1}_{E_n^2} \hat{\mathbb{Q}}_z^\eta \left[E_n^4 \mid \mathcal{F}^*_{A_{k_n}} \right] \ge q_n$, where $q_n \to 1$ is deterministic; and finally

(iii) $\hat{\mathbb{Q}}_z^\eta [\bar{D}_n^{\eta,[k_n,n]} \mathbf{1}_{E_n^3 \cap E_n^4} \mid \mathcal{F}^*_{A_{k_n}}] \le r_n$ where $r_n = o(1/n^2)$ is deterministic.

This proves the lemma by conditional Markov's inequality. For (i), one uses the fact that under the given conditional law, $(S_j - S_{k_n};\ j \ge S_{k_n})$ is a centered random walk conditioned to stay above $-S_{k_n} + 2\eta$. The details rely on estimates for the renewal function, and are as in [AS14, Proof of Lemma 4.7]. Claim (ii) follows from the proof of [Aid15, Lemma 3.5]. This proof shows that, uniformly in z,

$$\hat{\mathbb{Q}}_z^\eta \left[\sup_{w \in \mathcal{B}_j} |z - w| > j^c \, \mathrm{CR}(z, \mathcal{B}_j) \mid \mathcal{F}^*_{A_{k_n}} \right] \le c' \sqrt{j - k_n} j^{-c'c},$$

for some positive constant c' that does not depend on c (note the right-hand side is deterministic.) Choosing c large enough gives (ii). For (iii), we condition further on all the *brother loops* of the point z (that is, for each $k_n \le j \le n-1$, the components of $D \setminus A_j$ contained in $\mathcal{B}_j$ but not $\mathcal{B}_{j+1}$). After applying this conditioning, and using the description of $(A_n)_n$ given after the statement of Proposition 4.18, we see that $\hat{\mathbb{Q}}_z^\eta \left[\bar{D}_n^{\eta,[k_n,n]} \mathbf{1}_{E_n^3 \cap E_n^4} \mid \mathcal{F}^*_{A_{k_n}} \right]$ is equal to

$$\sum_{j=k_n}^{n-1} s_j + \hat{\mathbb{Q}}_z^\eta \left[\int_{\mathcal{B}_n} h_1(-2n + 4 \log \mathrm{CR}^{-1}(w, \mathcal{B}_n) + 2\eta) \, \mathrm{e}^{2n - 2 \log \mathrm{CR}^{-1}(w, \mathcal{B}_n)} \, \mathbf{1}_{E_n^3 \cap E_n^4} \, dw \mid \mathcal{F}^*_{A_{k_n}} \right],$$

where s_j is

$$\hat{\mathbb{Q}}_z^\eta \left[\mathbf{1}_{E_n^3 \cap E_n^4} \int_{\mathcal{B}_j \setminus \mathcal{B}_{j+1}} h_1(4 \log \mathrm{CR}^{-1}(w, \mathbb{D} \setminus A_{j+1}) + 2\eta - 2(j+1)) \, \mathrm{e}^{2(j+1) - 2 \log \mathrm{CR}^{-1}(w, \mathbb{D} \setminus A_{j+1})} \, dw \mid \mathcal{F}^*_{A_{k_n}} \right]$$
$$\le C \hat{\mathbb{Q}}_z^\eta \left[\mathbf{1}_{E_n^3 \cap E_n^4} \int_{\mathcal{B}_j \setminus \mathcal{B}_{j+1}} h_1(4 \log \mathrm{CR}^{-1}(w, \mathcal{B}_j) + 2\eta - 2j) \, \mathrm{e}^{2j - 2 \log \mathrm{CR}^{-1}(w, \mathcal{B}_j)} \, dw \mid \mathcal{F}^*_{A_{k_n}} \right],$$

for some constant $C \in \mathbb{R}$. The inequality here follows because $\mathrm{CR}(w, \mathbb{D} \setminus A_j)$ is decreasing in j and h_1 is bounded on either side by a linear function. Now, note that on the event $E_n^3 \cap E_n^4$, thanks to Koebe's theorem, $2j - 2 \log CR^{-1}(w, \mathcal{B}_j)$

is smaller than $S_j + 2c\log(j) + 2\log \mathrm{CR}^{-1}(z, \mathcal{B}_j)$, and the area of each B_j is also $O(\mathrm{CR}(z, \mathcal{B}_j)^2)$. This means that each s_j is $O(\exp(-\sqrt[6]{k_n}/2)n^{4c+1})$, and the assumption that $k_n/(\log n)^6 \to \infty$ therefore implies (iii). □ □

Acknowledgements.

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5 Level Lines of the Gaussian free field with general boundary data

5.1 Introduction

The relationship between Schramm–Loewner Evolution (SLE) and the two-dimensional Gaussian free field (GFF) is at the heart of recent breakthroughs in Liouville quantum gravity, imaginary geometry and more generally, random conformal geometry. Starting with the seminal papers of [Dub09], [SS13], [SS09], one key idea is to make sense of SLE-type curves as a level lines of an underlying Gaussian free field h in a domain, which we take to be the upper half plane $\mathbb{H}$ without loss of generality in the rest of the paper. When the field h is given the boundary values $\lambda := \pi/2$ on $\mathbb{R}_+$ and $-\lambda$ on $\mathbb{R}_-$, the corresponding level line is a chordal SLE_4 curve. A considerable extension of that theory is described in [MS16a], which introduced the notion of *flow lines* and *counter flow lines* of the GFF. In this case it turns out that the curves are given by SLE_κ processes with $\kappa \in (0, 4)$ and $\kappa \in (4, \infty)$ respectively.

It is also natural to wonder for which sort of boundary data the notion of level line makes sense. In [MS16a] and [WW16], the hypothesis on the boundary data is extended from the above to any arbitrary piecewise constant function on the real line. The goal of this paper will be to relax this assumption. Assuming solely that the boundary data F is a *regulated* function, i.e., the left and right limits

$$F(t^+) = \lim_{h \to 0+} F(t + h); \quad F(t^-) = \lim_{h \to 0-} F(t + h) \tag{5.1}$$

exist and are finite for all $t \in \overline{\mathbb{R}}$, and that for some $c > 0$

$$F(x) \leq \lambda - c, \quad x < 0; \quad F(x) \geq -\lambda + c, \quad x \geq 0 \tag{5.2}$$

which roughly corresponds to the non existence of a continuation threshold, we can show that the corresponding level line is well defined almost surely as a continuous transient curve. Moreover, it is almost surely determined by the field.

This also allows us, for a zero boundary GFF h, to consider the set of level lines of different heights. By this we mean the level lines of $h+F$, where F ranges over (the bounded harmonic extensions of) all regulated functions on $\mathbb{R}$. Strengthening the results of [MS16a], [WW16], we are able to prove a general monotonicity principle for the level lines, which is both a key tool in our existence proof, and an interesting result in its own right. This is deeply intertwined with the reversibility property of the level lines, which we are also able to prove in general; see Theorems 5.4 and 5.5.

A further point of interest is that we obtain some continuity in the level lines as a consequence of our proof. That is, if we take a sequence of piecewise constant functions F_n converging monotonically uniformly to some F, then the level lines of height F_n for a zero boundary GFF converge almost surely to the level line of height F. This convergence is with respect to Hausdorff distance, after conformally mapping everything to the unit disc.

We remark that our hypothesis on the boundary data is satisfied by a wide range of functions, including the special class of *functions of bounded variation.* Any such function can be described almost everywhere as the integral of a finite Radon measure ρ, and this connection allows us to deduce that the marginal law of a level line with such boundary data is given by what we call an $\mathrm{SLE}_4(\rho)$ process. This is the natural analogue of an $\mathrm{SLE}_4(\underline{\rho})$ process, where the vector $\underline{\rho}$ is replaced by a measure. Our results therefore demonstrate the existence of such processes, as well as establishing some further properties.

We first recall the definition of what it means for a curve, and more generally a Loewner chain, to be a level line. If we have a Loewner chain $(K_t, t \geq 0)$ in $\mathbb{H}$, with associated sequence of conformal maps $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$, we will often want to describe the *image under* g_t of a point x on the real line. To do this, for any $x \leq 0$ we define a process $V_t^L(x)$ by setting it equal to $g_t(x)$ if $x \notin K_t$ and if $x \in K_t$, taking it to be the image of the leftmost point of $\mathbb{R} \cap K_t$ under g_t. We define a process $V_t^R(x)$ for $x \geq 0$ analogously. The process $V_t^L(x)$ for $x \in \mathbb{R}_-$, or $V_t^R(x)$ for $x \in \mathbb{R}_+$, is what we define to be the *image of* x *under* g_t.

Definition 5.1 ([MS16a, WW16])**.** *Suppose that* F *is* L^1 *with respect to harmonic measure on* $\mathbb{R}$ *viewed from some point in* $\mathbb{H}$ *and that* h *is a zero boundary* GFF *in* $\mathbb{H}$*. If* $(K_t, t \geq 0)$ *is a Loewner chain and* $(g_t, t \geq 0)$ *is the corresponding sequence of conformal maps, set* $f_t = g_t - W_t$*, and let* $V_t^R(x)$ *(resp.* $V_t^L(x)$*) be the image of* $x \geq 0$ *(resp.* $x \leq 0$*) under* g_t*. Let* η_t^0 *be the bounded harmonic function on* $\mathbb{H}$ *with boundary values (see Figure 23)*

$$\begin{cases} F(f_t^{-1}(x)), & \text{if } x \geq V_t^R(0^+) - W_t, \\ \lambda, & \text{if } 0 \leq x < V_t^R(0^+) - W_t, \\ -\lambda, & \text{if } V_t^L(0^-) - W_t \leq x < 0, \\ F(f_t^{-1}(x)), & \text{if } x < V_t^L(0^-) - W_t. \end{cases}$$

Define, for $z \in \mathbb{H} \setminus K_t$,

$$\eta_t(z) = \eta_t^0(f_t(z)).$$

We say that K is a level line of $h+F$ if there exists a coupling (h,K) such that the following domain Markov property holds: for any finite K-stopping time τ, given K_τ, the conditional law of $(h+F)|_{\mathbb{H}\setminus K_\tau}$ is equal to the law of $h\circ f_\tau + \eta_\tau$.

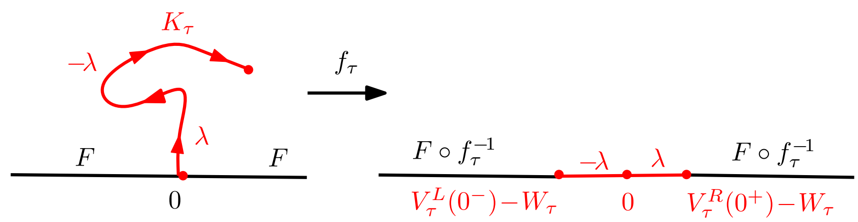

Figure 23: The left hand side shows the boundary values of the harmonic function η_τ in $\mathbb{H}\setminus K_\tau$. This is the image under f_τ^{-1} of the harmonic function η_τ^0 in $\mathbb{H}$, whose boundary values are shown on the right hand side.

Note that this definition is the same for any two functions F_1 and F_2 which are equal almost everywhere, since the harmonic extensions of such functions are necessarily equal. From Definition 5.1, we can see that the so-called level lines of the GFF have an intriguing property that distinguishes them from level lines of an ordinary smooth function. Namely, once one conditions on a level line, the conditional expectation of the field on one side of the curve differs by 2λ from the value on the other side. In a sense, a level line is more like a "level cliff" where there is a prescribed jump between the two sides of the curve.

More generally, we say that a Loewner chain $(K_t, t\geq 0)$ is a level line of a GFF h in a domain D from $a\in\partial D$ to $b\in\partial D$ if $(\varphi(K_t), t\geq 0)$ is a level line of $\varphi(h)$ as in Definition 5.1, where φ is a conformal map from $D\to\mathbb{H}$ sending a to 0 and b to ∞.

Theorem 5.2. *[Coupling] Assume the same notations as in Definition 5.1. Suppose that the function F is regulated and satisfies (5.2) for some $c>0$. Then there exists a coupling satisfying the conditions in Definition 5.1. Moreover, in this coupling, the Loewner chain K is almost surely generated by a continuous and transient curve γ with almost surely continuous driving function.*

The inequality on F in Theorem 5.2 guarantees that the corresponding level line will reach its target point ∞ before "dying" at some continuation threshold. Indeed, the level line of a GFF with piecewise constant boundary data is only defined until the first time that it hits a section of $\mathbb{R}_+$ where the boundary data is less than $-\lambda$ or a section of $\mathbb{R}_-$ where it is greater than λ. In our case, if we allowed F to approach $-\lambda$ (resp. λ) at some point in $\mathbb{R}_+$ (resp. in $\mathbb{R}_-$), then our current framework would not control the behaviour of the level line around this point (see discussion below.) Thus, we do not treat this situation here.

Theorem 5.3. *[Determination] If* (h, γ) *are coupled as in Theorem 5.2, then* γ *is almost surely determined by* h*. Moreover, the curve* γ *is almost surely simple. We call* γ *the level line of* $h + F$*.*

With this in hand, we can consider the collection of level lines determined by a given field. The following two theorems describe the interactions between the curves; corresponding to what one might expect from the level lines of a smooth function.

Theorem 5.4. *[Monotonicity] Suppose that* F, G *are functions satisfying the conditions in Theorem 5.2, and that* $F(x) \geq G(x)$ *for* $x \in \mathbb{R}$*. Suppose that* h *is a zero boundary* GFF *on* $\mathbb{H}$ *and* γ_F *(resp.* γ_G*) is the level line of* $h+F$ *(resp.* $h+G$*). Then* γ_F *lies to the left of* γ_G *almost surely.*

Theorem 5.5. *[Reversibility] Suppose that* h *is a* GFF *on* $\mathbb{H}$ *whose boundary value satisfies the conditions in Theorem 5.2. Let* γ *be the level line of* h *from* 0 *to* ∞ *and* γ' *be the level line of* $-h$ *from* ∞ *to* 0*. Then the two paths* γ *and* γ' *(viewed as sets) are equal almost surely.*

Now we will explain the relevance of Conditions (5.1) and (5.2), which we need for our approach to work. Although one can make sense of what it means to be a level line of $h+F$ for any F in L^1 (as in Definition 5.1), before this work the existence of the coupling was only known for piecewise constant boundary data. The assumption that the boundary data F is regulated corresponds precisely to the fact that F can be uniformly approximated by piecewise constant functions. Indeed, our argument will use an approximation of F by such functions, and a limit of the corresponding level lines. Thus with our current approach we are unable to say anything about functions which are not regulated. However, since Definition 5.1 still makes sense for a wider class of functions, it is an interesting question to determine the most general restrictions under which a coupling exists. For example, if one takes a GFF with boundary data which is $-\lambda$ in a neighbourhood to the left of 0 and λ in a neighbourhood to the right of 0 then one can allow much rougher boundary data away from these neighbourhoods (for example, even Neumann boundary conditions, see [KI13]), and construct a weaker form of "local coupling" with an SLE variant. Whether these types of coupling can be extended to a strong coupling as in Definition 5.1, where the curve is also determined by the field, or whether the condition near 0 can be relaxed is currently unknown.

Concerning Condition (5.2); the key to the proof of Theorem 5.2 is the continuity and transience of the approximating level lines (with piecewise constant boundary data). This allows us to use the results of [KS16] (see details in Section 5.2.2) to obtain a continuous limiting curve. If Condition (5.2) failed, the approximating level lines would only be defined up to a continuation threshold, and we would not be able to obtain such a limit. The continuity of the limiting curve is absolutely crucial to the proofs of Theorems 5.3 to 5.5. In fact, if the existence and the continuity of level lines were obtained for other boundary data, one could use similar proofs to get the corresponding theorems. However, whether continuity still holds in this set up is also a difficult open problem. Although it is natural to conjecture that for general regulated boundary data the level line will exist as a continuous curve until hitting a point on the boundary where Condition (5.2) fails, a "continuation threshold" as in [MS16a],[WW16], it is unclear whether or not the continuity will break down around this point.

Finally, we identify the law of the level lines. It is proved in [MS16a, WW16] that the level lines of GFF with piecewise constant boundary data are $\mathrm{SLE}_4(\underline{\rho})$ processes where $\underline{\rho}$ is a vector. In our context, when the boundary data is of bounded variation, the level lines turn out to be $\mathrm{SLE}_4(\rho)$ processes, where ρ is now a Radon measure. With the help of the GFF, we are able to obtain the existence, the continuity, and the reversibility of such processes, properties which are far from clear by the definition of the process through Loewner evolution.

Theorem 5.6. *Assume the same notations as in Theorem 5.2. Suppose further that F is of bounded variation. Then in the coupling (h, γ) given by Theorem 5.2, the marginal law of γ is that of an $\mathrm{SLE}_4(\rho^L; \rho^R)$ process (see Section 5.2.5) where ρ^R (resp. ρ^L) is a finite Radon measure on $\mathbb{R}_+$ (resp. on $\mathbb{R}_-$) and*

$$\begin{cases} F(x) = \lambda(1 + \rho^R([0, x])), & x \geq 0; \\ F(x) = -\lambda(1 + \rho^L((x, 0])), & x < 0 \end{cases}$$

almost everywhere. In particular, we have the following properties of the $\mathrm{SLE}_4(\rho^L; \rho^R)$ process. Suppose that there exists $c > 0$ such that

$$\rho^L((x, 0]) \geq -2 + c, \quad x < 0, \quad \rho^R([0, x]) \geq -2 + c, \quad x > 0.$$

Then

(1) There exists a law on continuous curves from 0 to ∞ in $\overline{\mathbb{H}}$ with almost surely continuous driving functions, for which the associated Loewner chain is an $ar\mathrm{SLE}_4(\rho^L; \rho^R)$ process.

(2) The above continuous curve is almost surely simple and transient.

(3) *The time reversal of the above* $\mathrm{SLE}_4(\rho^L;\rho^R)$ *process has the same law as* $\mathrm{SLE}_4(\tilde{\rho}^L;\tilde{\rho}^R)$ *,where*

$$\tilde{\rho}^R([x,\infty]) = \rho^L((x,0]), \quad x<0; \quad \tilde{\rho}^L((x,\infty]) = \rho^R([0,x]), \quad x>0.$$

Remark 5.7. *Although Theorem 5.6 gives us existence of* $\mathrm{SLE}_4(\rho^L;\rho^R)$ *processes, we do not derive uniqueness in law. That is, we have not excluded the possibility that there exists another law on Loewner chains satisfying the definition of an* $\mathrm{SLE}_4(\rho^L;\rho^R)$ *process.*

Remark 5.8. *Item (3) is the so-called reversibility of* SLE. *The reversibility was derived previously for* SLE_κ *in [Zha08b], for* $\mathrm{SLE}_\kappa(\rho)$ *where* ρ *is a vector in [Zha08a, MS16b, MS16c, WW13]. In Theorem 5.6, we derive the reversibility of* $\mathrm{SLE}_4(\rho)$ *where* ρ *is a Radon measure.*

Outline. The structure of the paper is as follows. In Section 5.2, we discuss briefly the necessary background theory, and collect some results that will be important to us. We also define the class of $\mathrm{SLE}_\kappa(\rho)$ process and generalize some of the theory from [MS16a, WW16] which will help us in the sequel. In Sections 5.3 and 5.4, we set up a general framework for the level lines of a GFF, under the assumption that they exist and are given by continuous transient curves. In particular, we show that they are monotonic in the boundary data, and describe where they can and cannot hit the boundary. Sections 5.5 and 5.6 address the existence of continuous transient curves which can be coupled as level lines of a GFF, provided the boundary data satisfies the conditions of Theorem 5.2. The proof of this is via an approximation argument; using a general theory for the weak convergence of curves, as set out in [KS16]. The key point in the proof is the monotonicity obtained in Section 5.4. In Section 5.7 we prove Theorems 5.3 to 5.5 using the ideas from Sections 5.3 and 5.4. Finally, we complete the proof of Theorem 5.6 in Section 5.8.

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5.2 Preliminaries

5.2.1 Regulated functions and functions of bounded variation

We say that a function F on $\mathbb{R}$ is *regulated* if it admits finite left and right limits

$$F(t^+) = \lim_{h\to 0+} F(t+h); \quad F(t^-) = \lim_{h\to 0-} F(t+h)$$

at every point $t \in \mathbb{R}$, including ∞. Equivalently, see [Die69, Secion 7.6], F is regulated if it can be uniformly approximated on $\mathbb{R}$ by piecewise constant functions which change value only finitely many times. It is this formulation of the definition that will be useful to us in the sequel.

Another type of function which is of particular interest in the current paper is the class of functions of *bounded variation*. Let us consider the connection (5.5) between pairs of Radon measures $(\rho^L; \rho^R)$ and functions F on the real line. We saw above that piecewise constant functions correspond to purely atomic measures. In general, finite Radon measures are in one-to-one correspondence with right-continuous functions of bounded variation.

The space of functions of bounded variation are those F which satisfy

$$\sup_{a<b}\left(\sup\left\{\sum_i |F(x_i) - F(x_{i-1})| \ : \ \{x_i\} \text{ a finite partition of } [a,b]\right\}\right) < \infty.$$

For a proof of this equivalence, see [Fol99, Theorem 3.29]. Note that these functions are clearly regulated. So, provided they satisfy the correct bounds on $\mathbb{R}_-$ and $\mathbb{R}_+$, functions of bounded variation meet the conditions of Theorem 5.2.

Furthermore, if a bounded variation function is also absolutely continuous, then the corresponding measures $(\rho^L; \rho^R)$ are absolutely continuous with respect to Lebesgue measure, and writing

$$\rho^L(dx) = f^L(x)\,dx, \quad \rho^R(dx) = f^R(x)\,dx$$

we have that the function is differentiable almost everywhere with derivative equal to $f^L(x)$ on $\mathbb{R}_-$ and $f^R(x)$ on $\mathbb{R}_+$.

5.2.2 A result on the convergence of curves

To show existence of the level line of a GFF with general boundary data as given in Theorem 5.2, we will attempt to approximate it by level lines of the field with piecewise constant boundary data. For this, a result from [KS16] on the weak convergence of curves, satisfying certain conditions on crossing probabilities, will be crucial.

In order to state the result, we need to define what we mean by *crossings of topological quadrilaterals.*

Definition 5.9. *A* topological quadrilateral $Q = (V; S_k, k = 0, 1, 2, 3)$ *consists of a domain V, along with four boundary arcs S_0, S_1, S_2, S_3, which can be mapped homeomorphically to a square in such a way that the boundary arcs are in counterclockwise order and correspond to the edges of the square. For any topological quadrilateral, there exists a unique positive L and a conformal map from Q onto the rectangle $[0, L] \times [0, 1]$, such that the boundary arcs are mapped to the edges of the quadrilateral and, in particular, S_0 is mapped to $\{0\} \times [0, 1]$. We call this unique L the* modulus *of Q, denoted by $m(Q)$.*

Definition 5.10. *We will often consider topological quadrilaterals in $\mathbb{H}$ which lie on the boundary in the sense that $S_1, S_3 \subset \mathbb{R}$ and $S_0, S_2 \subset \mathbb{H}$. If we have such a quadrilateral, then we say that a curve $\gamma : [T_0, T_1] \to \mathbb{C}$* crosses *$Q$ if there is a subinterval $[t_0, t_1] \subset [T_0, T_1]$, such that $\gamma(t_0, t_1) \subset V$ but $\gamma[t_0, t_1]$ intersects both S_0 and S_2.*

Essentially, the condition that will be required for weak convergence will be the following:

Condition 5.11. *For any simple curve γ on $\mathbb{H}$ we say that Q is a topological quadrilateral in $H_\tau := \mathbb{H} \setminus \gamma[0, \tau]$ if it is the image of the square $(0, 1)^2$ under a homeomorphism ψ. We define the sides of Q: S_0, S_1, S_2, S_3, to be the images of*

$$\{0\} \times (0, 1), \quad (0, 1) \times \{0\}, \quad \{1\} \times (0, 1), \quad (0, 1) \times \{1\}$$

under ψ. We consider Q such that the opposite sides S_1, S_3 are contained in ∂H_t and define a crossing of Q to be a curve in H_t which connects the two opposite sides S_0 and S_2. Finally, we say that Q is avoidable *if it doesn't disconnect $\gamma(\tau)$ and ∞ inside H_t.*

A family Σ of probability measures on simple curves from 0 to ∞ in $\mathbb{H}$ is said to satisfy a conformal bound on an unforced crossing *if there exists a constant $M > 0$ such that for any $\mathbb{P} \in \Sigma$, for any stopping time τ, and any avoidable quadrilateral Q of H_τ whose modulus $m(Q)$ is greater than M,*

$$\mathbb{P}\left(\gamma[\tau, \infty) \textit{ crosses } Q \mid \gamma[0, \tau]\right) \leq 1/2.$$

Now we may state the result.

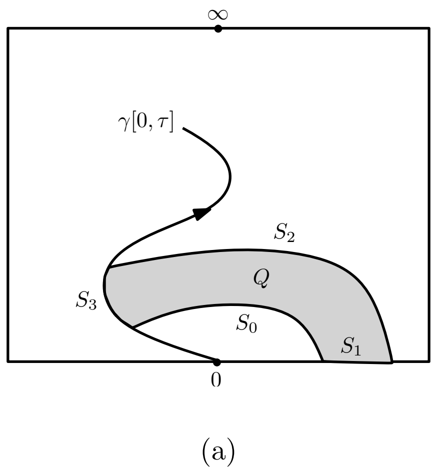


Figure 24: In the left panel, the grey part is an avoidable quadrilateral of $H_\tau = \mathbb{H} \setminus \gamma[0,\tau]$; in the right panel, the grey part is an unavoidable quadrilateral.

Proposition 5.12. *Suppose that $(W^{(n)})_{n\in\mathbb{N}}$ is a sequence of driving processes of random Loewner chains that are generated by continuous simple random cuves $(\gamma^{(n)})_{n\in\mathbb{N}}$ in $\mathbb{H}$, satisfying Condition 5.11. Suppose that the $(\gamma^{(n)})_{n\in\mathbb{N}}$ are parameterized by half plane capacity. Then*

- *$(W^{(n)})_{n\in\mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets of $[0,\infty)$.*
- *$(\gamma^{(n)})_{n\in\mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on the compact subsets of $[0,\infty)$.*

Moreover, if the sequence converges weakly in either of the topologies above, then it also converges weakly in the other and the limits agree in the sense that the law of the limiting random curve is the same as the that of the random curve generated under the law of the limiting driving process. In particular, any subsequential limit of the sequence of curves a.s. generates a Loewner chain with continuous driving function.

Proof. This may be found in [KS16] cf. Theorem 1.5 and Corollary 1.7. □

In fact, we will need to apply this theorem when the curves $(\gamma^{(n)})_{n\in\mathbb{N}}$ correspond to certain $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ processes. In this case they may hit the real line, and so are not necessarily contained in $\mathbb{H}$, as required by the Proposition. However, as discussed before the proof of Theorem 1.10 in [KS16], the result extends to curves such as ours, and so we may apply it without concern.

5.2.3 The zero boundary Gaussian free field

In this section we will describe the zero boundary Gaussian free field (GFF) in an arbitrary domain $D \subsetneq \mathbb{C}$. We will always assume that the domain has harmonically non-trivial boundary, meaning that a Brownian motion started from a point in the interior will hit the boundary almost surely.

We start with the *Green's function* G_D in D, which is the unique function in D such that

- $\Delta G_D(z, \cdot) = 2\pi\delta_z(\cdot)$ for each $z \in D$, and
- $G_D(z, w) = 0$ if z or w is in ∂D.

Explicitly,

$$G_D(z, w) = -\log|z - w| - \tilde{G}_z(w)$$

where $\tilde{G}_z(w)$ is the harmonic extension of $w \mapsto -\log|z - w|$ from ∂D to D. The Green's function is conformally invariant in the sense that for any conformal map ϕ on D, and $z, w \in D$, we have

$$G_D(z, w) = G_{\phi(D)}(\phi(z), \phi(w)).$$

Roughly speaking, the GFF will be the random Gaussian "function" on D with $\mathrm{cov}(h(z), h(w)) = G_D(z, w)$. However, it can only be made sense of rigorously as a random distribution on D. For $H_s(D)$ the space of smooth compactly supported functions on D, we let $(\cdot, \cdot)$ denote the normal L^2 inner product on $H_s(D)$. We may also endow $H_s(D)$ with the *Dirichlet inner product* defined by

$$(f, g)_\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) d^2 z$$

and we denote its Hilbert space completion under Dirichlet inner product by $H(D)$.

For $\{\phi_n\}_{n \geq 0}$ an orthonormal basis of $H(D)$, we define the *zero boundary* GFF h to be the random sum $h := \sum_n \alpha_n \phi_n$, where the α_n's are i.i.d. Gaussians with mean 0 and variance 1. This almost surely diverges in $H(D)$, but makes sense as a distribution. That is, the limit $\sum_n \alpha_n(\phi_n, p) := (h, p)$ almost surely exists for each $p \in H_s(D)$, and $p \mapsto (h, p)$ is almost surely a continuous linear functional on $H_s(D)$. Note that for any $f \in H_s(D)$ we have that $-\Delta f = p$ is also in $H_s(D)$ and so can define

$$(h, f)_\nabla := \frac{1}{2\pi}(h, p).$$

Then $(h,f)_\nabla$ is a Gaussian with mean 0 and variance

$$\frac{1}{4\pi^2}\sum_n (\phi_n, p)^2 = \sum_n (\phi_n, f)_\nabla^2 = (f,f)_\nabla.$$

In fact, this characterizes the Gaussian free field. Furthermore, noticing that for $p \in H_s(D)$

$$\Delta^{-1}p := \frac{1}{2\pi}\int_D G_D(\cdot, w)p(w)\,dw$$

is a smooth function in D whose Laplacian is p and vanishes on ∂D, we see that for any $f, g, p, q \in H_s(D)$

$$\mathrm{cov}((h,f)_\nabla, (h,g)_\nabla) = (f,g)_\nabla, \quad \mathrm{cov}((h,p),(h,q)) = \int\int_{D\times D} p(z)G_D(z,w)q(w)d^2zd^2w.$$

Proposition 5.13. *[The Markov Property] Let $W \subset D$ be open and h be a zero boundary* GFF *on D. Then we can write*

$$h = h_1 + h_2$$

where h_1 and h_2 are independent, h_1 is harmonic in W, and h_2 is a zero boundary GFF *in W.*

This tells us that, given $h|_{D\setminus W}$, the conditional law of $h|_W$ is that of a zero boundary GFF in W, plus the harmonic extension of $h|_{D\setminus W}$ to W.

Suppose that F is L^1 with respect to harmonic measure on $\mathbb{R}$ viewed from some point (hence every point) in $\mathbb{H}$; we also denote its bounded harmonic extension to $\mathbb{H}$ by F. Then the GFF *with mean* F is defined to be the sum, $h + F$, of a zero boundary GFF and F.

Proposition 5.14. *Suppose that D_1 and D_2 are two simply connected domains with non empty intersection, and h_i is a zero boundary* GFF *on D_i for $i = 1, 2$. Let F_i be harmonic on D_i, $i = 1, 2$ and $U \subset D_1 \cap D_2$ be a simply connected open domain. Then*

(1) If $\mathrm{dist}(U, \partial D_i) > 0$ for $i = 1, 2$, then the laws of

$$(h_1 + F_1)|_U \text{ and } (h_2 + F_2)|_U$$

are mutually absolutely continuous.

(2) Suppose there is a neighbourhood $\bar{U} \subset U'$ such that $D_1 \cap U' = D_2 \cap U'$ and that $F_1 - F_2$ tends to 0 along sequences approaching points in $\partial D_i \cap U'$. Then

the laws of

$$(h_1 + F_1)|_U \text{ and } (h_2 + F_2)|_U$$

are mutually absolutely continuous.

Proof. [MS16a, Proposition 3.2]. □

5.2.4 Local sets for the GFF

The theory of *local sets* for the Gaussian free field was first introduced by Schramm and Sheffield in [SS13], and we quote several of their results here. For D a simply connected domain and A a random closed subset of $\bar{D}$, we let

$$A_\delta := \{z \in D : d(z, A) \leq \delta\}$$

and $\mathcal{A}_\delta$ be the smallest σ-algebra for which A and the restriction of h to the interior of A_δ are measurable. Setting $\mathcal{A} = \cap_{\delta \in \mathbb{Q}_+} \mathcal{A}_\delta$ we obtain a σ-algebra which is intuitively the smallest such making A, and h restricted to some infinitesimal neighbourhood of A, measurable. With this in mind, we will often refer to $\mathcal{A}$ as $(A, h|_A)$.

Definition 5.15. *Suppose that (h, A) is a coupling of a* GFF *in D and a random closed subset $A \subset \overline{D}$. Then we say that A is a* local set *for h if either of the following equivalent statements hold:*

(1) For any deterministic open subset $U \subset D$ we have that, given the orthogonal projection of h onto $h^\perp(U)$, the event $\{A \cap U = \emptyset\}$ is independent of the orthogonal projection of h onto $H(U)$. This means that the conditional probability of $\{A \cap U = \emptyset\}$ given h is a measurable function of the orthogonal projection of h onto $H^\perp(U)$.

(2) Given $\mathcal{A}$, the conditional law of h is that of $h_1 + h_2$, for h a zero boundary GFF *on $D \setminus A$ and h_1 an $\mathcal{A}$-measurable random distribution which is almost surely harmonic on $D \setminus A$.*

In this case, we let $\mathcal{C}_A$ be the conditional expectation of h given $(A, h|_A)$, corresponding to h_1 in Item (2).

The interactions between local sets display some nice properties, which we will describe in the following propositions.

Proposition 5.16. *Suppose that A_1, A_2 are local sets for a* GFF *h, which are conditionally independent given h. Then $A = A_1 \cup A_2$ is also local for h and moreover,* *given $(A_1, A_2, A, h|_A)$, the conditional law of h is given by $\mathcal{C}_A$ plus an instance of the zero boundary* GFF *in $D \setminus A$.*

Proof. [SS13, Lemma 3.10]. □

Proposition 5.17. *Let A_1, A_2 be connected local sets which are conditionally independent and $A = A_1 \cup A_2$. Then $\mathcal{C}_A - \mathcal{C}_{A_2}$ is almost surely a harmonic function in $D \setminus A$ which tends to zero along any sequence converging to a limit in*

- *a connected component of $A_2 \setminus A_1$ which is larger than a singleton, or*
- *a connected component of $A_1 \cap A_2$ which is larger than a singleton, if the limit is at a positive distance from either $A_2 \setminus A_1$, or $A_1 \setminus A_2$.*

Proof. [SS13, Lemma 3.11] and [MS16a, Proposition 3.6]. □

Proposition 5.18. *Let A_1, A_2 be connected local sets which are conditionally independent and $A = A_1 \cup A_2$. Suppose that C is a $\sigma(A_1)$-measurable connected component of $D \setminus A_1$ such that $\{C \cap A_2 = \emptyset\}$ almost surely. Then $\mathcal{C}_A|_C = \mathcal{C}_{A_1}|_C$ almost surely, given A_1.*

Proof. [MS16a, Proposition 3.7]. □

Proposition 5.19. *Let h be a* GFF *and $(Z(t), t \geq 0)$ a family of closed sets such that $Z(\tau)$ is local for every Z-stopping time τ. Suppose futher that for a fixed $z \in D$, $\mathrm{CR}(z, D \setminus Z(t))$ is almost surly continuous and monotonic in t. Then, if we reparameterise time by*

$$\log \mathrm{CR}(z, D \setminus Z(0)) - \log \mathrm{CR}(z, D \setminus Z(t)),$$

the process $\mathcal{C}_{Z(t)}(z) - \mathcal{C}_{Z(0)}(z)$ has a modification which is Brownian motion until the first time that $Z(t)$ accumulates at z. In particular, $\mathcal{C}_{Z(t)}(z)$ has a modification which is almost surely continuous in t.

Proof. This is proved in [MS16a, Proposition 6.5]. Since we need the argument in the proof later, we briefly recall the proof here. For $s \geq 0$, set

$$\tau(s) := \inf\{t \geq 0 : \log \mathrm{CR}(z, D \setminus Z(0)) - \log \mathrm{CR}(z, D \setminus Z(t)) = s\}.$$

We need only show that the increments of the process $\mathcal{C}_{Z(\tau(t))}(z)$ are independent, and stationary with Gaussian distribution. By [MS16a, Lemma 6.4], we know that for any $s < t$, the conditional law of

$$\mathcal{C}_{Z(\tau(t))}(z) - \mathcal{C}_{Z(\tau(s))}(z),$$

given $(Z(\tau(s)), h|_{Z(\tau(s))})$, is a Gaussian with mean 0 and variance

$$\log \mathrm{CR}(z, D \setminus Z(\tau(s))) - \log \mathrm{CR}(z, D \setminus Z(\tau(t))) = t - s.$$

This means it must also be independent of $(Z(\tau(s)), h|_{Z(\tau(s))})$, and so of $\mathcal{C}_{Z(\tau(s))}(z)$. This completes the proof. $\square$

5.2.5 $\mathrm{SLE}_\kappa(\rho)$ processes

We call a compact set $K \subset \mathbb{H}$ an $\mathbb{H}$-hull if $H := \mathbb{H} \setminus K$ is simply connected. For any such hull one can show that there exists a unique conformal map ϕ from $H \to \mathbb{H}$ which is normalized at ∞ in the sense that

$$\phi(z) = z + \frac{2a}{z} + o(\frac{1}{z}), \quad \text{as } z \to \infty,$$

for some constant a which we call the *half-plane capacity* of K. For a continuous real-valued function $(W_t, t \geq 0)$ with $W_0 = 0$ we can define the solution $g_t(z)$ to the *chordal Loewner equation*

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

This is well defined for each $z \in \mathbb{H}$ until the first time, $\tau(z)$, that $g_t(z) - W_t$ hits 0. Setting $K_t = \{z \in \overline{\mathbb{H}} : \tau(z) \leq t\}$ and $H_t = \mathbb{H} \setminus K_t$ we find that g_t is the conformal map from H_t to $\mathbb{H}$ normalized at ∞, and the half-plane capacity of K_t is equal to $2t$. We call the family $(K_t, t \geq 0)$ the *Loewner chain driven by* $(W_t, t \geq 0)$. One class of Loewner chains that we will be particularly interested in are those generated by continuous curves; that is, those for which there exists a continuous curve γ such that K_t is the hull generated by $\gamma[0, t]$ for all t.

Chordal SLE_κ is the Loewner chain driven by $W_t = \sqrt{\kappa} B_t$, where B_t is a standard one-dimensional Brownian motion. It is characterised by the special properties of conformal invariance and the domain Markov property. Specifically, $(\mu^{-1} K_{\mu^2 t}, t \geq 0)$ has the same law as $(K_t, t \geq 0)$ for any $\mu > 0$, and for any stopping time τ, the law of $(f_\tau(K_{t+\tau}), t \geq 0)$ is the same as that of K. Here $f_\tau := g_\tau - W_\tau$.

It is known that SLE_κ is almost surely generated by a continuous curve for all κ. In the special case $\kappa \in [0, 4]$, it has also been shown that the curve is almost surely simple. Moreover we know that $\lim_{t \to \infty} \gamma(t) = \infty$ almost surely; a property we refer to as *transience*. These facts were all proved in [RS05].

Definition 5.20. *Let ρ^L and ρ^R be finite Radon measures on $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$ respectively, and $(B_t, t \geq 0)$ be a standard one-dimensional Brownian* *motion. We say that*

$$\big(W_t, (V_t^L(x))_{x\in\mathbb{R}_-}, (V_t^R(x))_{x\in\mathbb{R}_+}\big)_{t\geq 0}$$

describe an $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ *process, if they are adapted to the filtration of* B *and the following hold:*

(1) The processes W_t*,* B_t*,* $\big(V_t^L(x)\big)_{x\in\mathbb{R}_-}$ *and* $\big(V_t^R(x)\big)_{x\in\mathbb{R}^+}$ *satisfy the following SDE on time intervals where* W_t *does not collide with any of the* $V_t^{L,R}(x)$*:*

$$dW_t = \sqrt{\kappa}dB_t + \left(\int_{\mathbb{R}_-}\frac{\rho^L(dx)}{W_t - V_t^L(x)}\right)dt + \left(\int_{\mathbb{R}_+}\frac{\rho^R(dx)}{W_t - V_t^R(x)}\right)dt \tag{5.3}$$

and

$$dV_t^L(x) = \frac{2dt}{V_t^L(x) - W_t}, \quad x\in\mathbb{R}_-; \quad dV_t^R(x) = \frac{2dt}{V_t^R(x) - W_t}, \quad x\in\mathbb{R}_+. \tag{5.4}$$

(2) We have instantaneous reflection of W_t *off the* $V_t^{L,R}(x)$*, ie. it is almost surely the case that for Lebesgue almost all times* t *we have that* $W_t \neq V_t^{L,R}(x)$ *for each* $x\in\mathbb{R}$*.*

The $\mathrm{SLE}_\kappa(\rho^L;\rho^R)$ *process is then defined to be the Loewner chain driven by* W*.*

Remark 5.21. *Note that it is not immediate from the definition that such a process exists. Indeed, we will only show the existence for* $\kappa = 4$ *and a specific subset of* $(\rho^L;\rho^R)$*.*

We define the *continuation threshold* of the process to be the be the infinum of values of t for which

$$\text{either} \quad \rho^L\left(\{x\in\mathbb{R}_- : V_t^L(x) = W_t\}\right) \leq -2 \quad \text{or} \quad \rho^R\left(\{x\in\mathbb{R}^+ : V_t^R(x) = W_t\}\right) \leq -2.$$

Observe that the case $\rho^L \equiv 0, \rho^R \equiv 0$ corresponds simply to SLE_κ. Another special case is when the Radon measures are purely atomic. If this occurs we instead consider $(\rho^L;\rho^R)$ to be a pair of vectors

$$\underline{\rho}^L = (\rho_l^L, \cdots, \rho_1^L), \quad \underline{\rho}^R = (\rho_1^R, \cdots, \rho_r^R)$$

with associated force points

$$\underline{x}^L = (x_l^L < \cdots < x_1^L \leq 0), \quad \underline{x}^R = (0 \leq x_1^R < \cdots x_r^R)$$

in the obvious way. In this case, it is proved in [MS16a, Theorem 2.2] that a slightly stronger version of Definition 5.20 determines a unique law on $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ processes, defined for all time up until the continuation threshold. The additional condition they impose is that W_t, B_t, $\left(V_t^L(x)\right)_{x\in\mathbb{R}_-}$ and $\left(V_t^R(x)\right)_{x\in\mathbb{R}^+}$ in fact must satisfy (5.3) and (5.4) at all times. This ensures the uniqueness in law of these processes.

Through their connection with the GFF, which we will discuss in the next section, it was shown in [MS16a] that $\mathrm{SLE}_\kappa(\underline{\rho}^L;\underline{\rho}^R)$ processes are almost surely generated by continuous curves up to and including the continuation threshold. Moreover, on the event that the continuation threshold is not hit before the curves reach ∞, the curves are almost surely transient. One can also show that the curves are absolutely continuous with respect to SLE_κ as long as they are away from the boundary.

5.2.6 Level lines of the GFF with piecewise constant boundary data

As discussed in the introduction, the theory of level lines and flow lines of a GFF with piecewise constant boundary data has been studied previously in a number of works, including [Dub09], [MS16a],[SS13] and [WW16]. We collect in this section some results that will be useful in our article.

Suppose that F is a bounded harmonic function in $\mathbb{H}$ whose boundary value is piecewise constant on $\mathbb{R}$ and changes only finitely many times. Then F can be described almost everywhere in terms of a pair of purely atomic finite Radon measures $(\rho^L;\rho^R)$, corresponding to vectors $(\underline{\rho}^L;\underline{\rho}^R)$, via the relation

$$F(x)=\lambda(1+\rho^R([0,x])),\quad x\geq 0;\quad F(x)=-\lambda(1+\rho^L((x,0])),\quad x<0. \tag{5.5}$$

When $\kappa=4$, which corresponds to level lines of the GFF, the following results are known for any $(\underline{\rho}^L;\underline{\rho}^R)$: (see [WW16, Theorems 1.1.1 and 1.1.2])

- There exists a coupling (K,h) where K is an $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ process and h is a zero boundary GFF, such that K is a level line of $h+F$.
- If h is a zero boundary GFF and K an $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ process, coupled such that K is a level line of $h+F$, then K is almost surely determined by h.

This allows us, for any such F and an instance of the zero boundary GFF h in $\mathbb{H}$, to define *the* level line, γ, of $h+F$. It has been shown in [WW16, Theorem 1.1.3] that γ is in fact almost surely continuous up to and including the continuation threshold, and it is transient when the continuation threshold is not hit.

More generally, for any simply connected domain D and x,y in ∂D, we say that γ is the level line of a GFF h in D started at x and targeted at y, if $\phi(\gamma)$ is the level line of $h\circ\phi^{-1}$, where ϕ is any conformal map from D to $\mathbb{H}$ which sends x to 0 and y to ∞.

One nice property of the level lines is what we call *monotonicity*. Suppose that h is a GFF with piecewise constant boundary values, changing only finitely many times. For $u \in \mathbb{R}$, we define the level line of h with height u to be the level line of $h+u$, and denote it by γ_u. Then, for any $u_1 \geq u_2$, the level line γ_{u_1} lies to the left of γ_{u_2} almost surely, see [WW16, Theorem 1.1.4].

Another property of the level lines is their *reversibility*. Suppose that h is a GFF with piecewise constant boundary values changing only finitely many times. Let γ be the level line of h from 0 to ∞ and γ' be the level line of $-h$ from ∞ to 0. Then, on the event that neither hit their continuation thresholds before reaching their target points, we have $\gamma = \gamma'$ almost surely as sets. This implies the *reversibility* of the $\mathrm{SLE}_4(\underline{\rho}^R; \underline{\rho}^R)$ process: conditioned on the event that the continuation threshold is not hit, the time reversal of the process is another $\mathrm{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ process, now from ∞ to 0 in $\mathbb{H}$ with appropriate weights and force points, conditioned not to hit its continuation threshold. See [WW16, Theorem 1.1.6]. Finally, we include a list of results from [WW16] that will be useful for the later proofs.

Lemma 5.22. *Suppose that h is a zero-boundary* GFF *and F is the bounded harmonic extension of the piecewise constant boundary data which changes finitely many times. Let γ be the level line of $h+F$. We already know that γ is almost surely continuous up to and including the continuation threshold.*

(1) [WW16, Theorem 1.1.3] The curve γ is almost surely simple and is continuous up to and including the continuation threshold.

(2) [WW16, Remark 2.5.15] For any open interval I of $(-\infty, 0) \cup (0, \infty)$, assume that

$$\text{either } F(x) \geq \lambda, \quad \forall x \in I, \quad \text{or } F(x) \leq -\lambda, \quad \forall x \in I.$$

Then almost surely $\gamma \cap I = \emptyset$.

(3) [WW16, Proposition 2.5.11] For any point $x_0 \in (0, \infty)$, assume that there exists $c > 0$ such that $F \geq -\lambda + c$ in a neighborhood of $\{x_0\}$, then almost surely γ does not hit $\{x_0\}$. Symmetrically, for $x_0 \in (-\infty, 0)$, assume that there exists $c > 0$ such that $F \leq \lambda - c$ in a neighborhood of $\{x_0\}$, then almost surely γ does not hit $\{x_0\}$.

5.2.7 First generalizations to the GFF with general boundary data

In this section, we generalize some results concerning level lines with piecewise constant boundary data to general boundary data. In fact, the ideas in the proof for Lemma 5.24 when the boundary condition is piecewise constant [SS13, Lemmas

2.4-2.6] work for general boundary data with proper adjustment. In order to be self-contained, we still give a complete proof here.

Lemma 5.23. *Suppose that* $(K_t, t \geq 0)$ *is a Loewner chain driven by a continuous process* $(W_t, t \geq 0)$*. Denote by* $(g_t, t \geq 0)$ *the corresponding sequence of conformal maps and* $f_t = g_t - W_t$ *the centered conformal maps. For any fixed* $z \in \mathbb{H}$*, define*

$$C_t(z) = \log \mathrm{CR}(z, \mathbb{H}) - \log \mathrm{CR}(z, \mathbb{H} \setminus K_t).$$

Then, we have that

$$dC_t(z) = \frac{4\Im(f_t(z))^2}{|f_t(z)|^4} dt.$$

Proof. The conformal radius $\mathrm{CR}(z, \mathbb{H} \setminus K_t)$ is equal to $2/|\phi_t'(z)|$ for ϕ_t any conformal map from $\mathbb{H} \setminus K_t$ to $\mathbb{H}$ which sends z to i, an example of which is given by $m_t \circ f_t$, where $m_t : \mathbb{H} \to \mathbb{H}$ is the Möbius transformation defined by

$$m_t(w) = \frac{\Im(f_t(z))w}{\Re(f_t(z))^2 + \Im(f_t(z))^2 - \Re(f_t(z))w}.$$

This gives us that

$$C_t(z) - C_0(z) = -\log 2 + \Re(\log m_t'(f_t(z))) + \Re(\log g_t'(z)).$$

However, in this case we can calculate $m_t'(f_t(z))$ explicitly, and find that $-\Re(\log m_t'(f_t(z))) = \log \Im f_t(z)$. Since we also know that

$$dg_t'(z) = \frac{-2g_t'(z)}{f_t(z)^2} dt, \quad d\Im(f_t(z)) = \frac{-2\Im(f_t(z))}{|f_t(z)|^4} dt,$$

we can compute

$$dC_t(z) = \frac{4\Im(f_t(z))^2}{|f_t(z)|^4} dt,$$

and this implies the result. □

Lemma 5.24. *Assume the same notations as in Definition 5.1. Suppose that the Loewner chain* K *is almost surely generated by a random continuous curve* γ *on* $\overline{\mathbb{H}}$ *from 0 to* ∞ *whose driving function* W *is almost surely continuous. For* $z \in \mathbb{H}$ *and* $t \geq 0$*, set*

$$\tau(t) = \inf\{s : \log \mathrm{CR}(z, \mathbb{H}) - \log \mathrm{CR}(z, \mathbb{H} \setminus K_s) = t\}.$$

Then the pair (h, K) *can be coupled as in Definition 5.1 if and only if* $(\eta_{\tau(t)}(z), t \geq 0)$ *is a Brownian motion with respect to the filtration generated by* $(W_{\tau(t)}, t \geq 0)$ *for any* $z \in \mathbb{H}$*.*

Proof. If (h, K) is coupled as in Definition 5.1, by Proposition 5.19, we know that $(\eta_{\tau(t)}(z), t \geq 0)$ is a Brownian motion. Moreover, by the proof of Proposition 5.19, we see that, for $s < t$, the variable $\eta_{\tau(t)}(z) - \eta_{\tau(s)}(z)$ is independent of $K_{\tau(s)}$ and has the law of Gaussian with mean zero and variance $t - s$. This implies that $(\eta_{\tau(t)}(z), t \geq 0)$ is a Brownian motion with respect to the filtration generated by $(W_{\tau(t)}, t \geq 0)$.

For the converse, assume that, for each $z \in \mathbb{H}$, the process $(\eta_{\tau(t)}(z), t \geq 0)$ is a Brownian motion with respect the filtration generated by $(W_{\tau(t)}, t \geq 0)$. We will begin by showing that there exists a Brownian motion $(B_t, t \geq 0)$ (with respect to the filtration of $(W_t, t \geq 0)$) such that, for all z, we have

$$d\eta_t(z) = \Im \frac{2}{f_t(z)} dB_t. \tag{5.6}$$

Define

$$U_t(z) = \eta_t(z) + \arg(f_t(z)). \tag{5.7}$$

We have the following observations.

- By the definition of $\eta_t(\cdot)$, we know that $U_t(\cdot)$ is the bounded harmonic function on $\mathbb{H} \setminus K_t$ with the boundary values given by $F + 2\lambda$ on $\mathbb{R}_- \setminus K_t$, λ along the boundary of K_t, and F on $\mathbb{R}_+ \setminus K_t$. Therefore, for fixed z, process $(U_t(z), t \geq 0)$ is of bounded variation and is measurable with respect to the filtration generated by $(W_t, t \geq 0)$.
- By the assumption, for fixed z, the process $(\eta_t(z), t \geq 0)$ is a Brownian motion when parameterized by

  $$C_t(z) = \log \mathrm{CR}(z, \mathbb{H}) - \log \mathrm{CR}(z, \mathbb{H} \setminus K_t).$$

  Thus, by Lemma 5.23, we see that

  $$d\langle \eta_t(z) \rangle = dC_t(z) = 4 \left(\Im \frac{1}{f_t(z)} \right)^2 dt.$$

  Moreover, since $(\eta_{\tau(t)}(z), t \geq 0)$ is a Brownian motion with respect to the filtration of $(W_{\tau(t)}, t \geq 0)$, we know that, for any $s < t$, the variable $\eta_t(z) - \eta_s(z)$ is independent of $(W_u, u \leq s)$ (we implicitly use the fact that K is a Loewner chain generated by a continuous curve with continuous driving function), thus the process $(\eta_t(z), t \geq 0)$ is a local martingale with respect to the filtration of $(W_t, t \geq 0)$.

Combining these two facts, we know that $\arg(f_t(z)) = \arg(g_t(z) - W_t)$ is a semimartingale, and hence W_t is a semimartingale at least up to the first time that z is swallowed by K_t. Note that

$$d\arg(f_t(z)) = \Im\frac{-1}{f_t(z)}dW_t + \Im\left(\frac{2dt}{(f_t(z))^2} - \frac{d\langle W_t\rangle}{2(f_t(z))^2}\right),$$

and therefore, we have $d\langle W_t\rangle = 4dt$ for all such t. Note however that the process W does not depend on z, and since we can always choose z far away as we want, we can argue that $(W_u, 0 \le u \le t)$ is a semimartingale up to time t for any $t > 0$, with $d\langle W_t\rangle = 4dt$. Thus, there exists a Brownian motion $(B_t, t \ge 0)$ and a process of bounded variation $(V_t, t \ge 0)$ such that $W_t = 2B_t - V_t$. We emphasize that the processes B and V do not depend on z. Plugging in Equation (5.7), we have that

$$d\eta_t(z) = \Im\frac{2}{f_t(z)}dB_t, \quad dU_t(z) = \Im\frac{1}{f_t(z)}dV_t, \tag{5.8}$$

as desired.

With (5.6) in hand, we know that for $z, w \in \mathbb{H}$, we have

$$d\langle \eta_t(z), \eta_t(w)\rangle = \Im\left(\frac{1}{f_t(z)}\right)\Im\left(\frac{1}{f_t(w)}\right)dt.$$

We know that, for each z, $\eta_t(z)$ is a continuous martingale. We can also extend the definition of $\eta_t(z)$ by setting it equal to its limit as $s \uparrow \tau(z)$ at all times after $\tau(z)$. We further define for $z, w \in \mathbb{H}$ and $t \le \tau(z) \wedge \tau(w)$,

$$G_t(z,w) := G_{\mathbb{H}}(f_t(z), f_t(w))$$

where we again extend this to all times after $\tau(z)\wedge\tau(w)$, by setting it constant and equal to its limit as $t \uparrow \tau(z)\wedge\tau(w)$. Observe that, in each connected component of $\mathbb{H} \setminus \gamma[0,t]$, the function η_t is the bounded harmonic function with boundary values shown in Figure 25. We also know that $G_t(z,w)$ is non-decreasing in t for any fixed

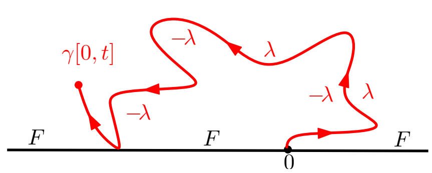


Figure 25: The function $\eta_t(\cdot)$ is a harmonic function in each connected component of $\mathbb{H} \setminus \gamma[0,t]$ with boundary values as above.

z, w.

Putting all of the above together, we can deduce by stochastic calculus that for any $p \in H_s(\mathbb{H})$, (η_t, p) is a continuous martingale with

$$d\langle(\eta_t, p)\rangle = -dE_t(p), \quad \text{where} \quad E_t(p) := \int\int p(z)p(w)G_t(z,w)\, d^2z d^2w.$$

Now we are ready to show that the pair (h, K) is coupled as in Definition 5.1. Since for each $z, w \in \mathbb{H}$ and non-negative $p \in H_s(\mathbb{H})$ we have that $\eta_t(z)$ is a martingale and $G_t(z,w)$, $E_t(p)$ are non-decreasing, it must be that all the limits $\eta_\infty(z)$, $G_\infty(z,w)$ and $E_\infty(p)$ exist. We let $\tilde{h}$ be equal to $\eta_\infty - \eta_0$ plus a sum of independent zero boundary GFF's; one in each connected component of $\mathbb{H} \setminus \gamma$. To show that $(K, \tilde{h})$ are coupled in the correct way we must verify that the marginal law of $\tilde{h}$ is that of a zero boundary GFF in $\mathbb{H}$, and that $(K, \tilde{h})$ satisfies the correct domain Markov property. This amounts to showing that for each non-negative $p \in H_s(\mathbb{H})$:

- $(\tilde{h}, p)$ is a Gaussian with mean 0 and variance $E_0(p)$.
- For any K-stopping time τ, the conditional law of $\left((\tilde{h} + \eta_0)|_{\mathbb{H}\setminus K_\tau}, p\right)$ given K_τ is a Gaussian with mean (η_τ, p) and variance $E_\tau(p)$.

To see the first point, for any $\mu > 0$ we calculate

$$\begin{aligned}
\mathbb{E}[\exp(-\mu(\tilde{h}, p))] &= \mathbb{E}[\mathbb{E}[\exp(-\mu(\tilde{h}, p))|K]] \\
&= \mathbb{E}\left[\exp\left(-\mu(\eta_\infty - \eta_0, p) - \frac{\mu^2}{2}E_\infty(p)\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\mu(\eta_\infty - \eta_0, p) + \frac{\mu^2}{2}(E_0(p) - E_\infty(p))\right)\right]\exp\left(-\frac{\mu^2}{2}E_0(p)\right) \\
&= \exp\left(-\frac{\mu^2}{2}E_0(p)\right),
\end{aligned}$$

where the last line follows from the fact that (η_t, p) is a continuous bounded martingale with mean $\eta_0(p)$ and quadratic variation $E_0(p) - E_\infty(p)$. The second point follows similarly, replacing the initial expectation with a conditional one. □

5.3 Non-boundary intersecting regime

All the conclusions in Sections 5.3 and 5.4 are proved in [MS16a, WW16] for level lines with piecewise constant boundary data and constant height difference. Although many of the ideas from these papers are fundamental to our proofs, there are several places where they fail for general boundary data. Therefore, we treat the general case here and give complete proofs in the next two sections.

Lemma 5.25. *Suppose that γ is a random continuous curve from 0 to some γ-stopping time T with almost surely continuous driving function. Assume that γ is* *coupled with a zero boundary* GFF h *as a level line of* $h+F$ *up to time* T *where*

$$F(x) \geq -\lambda, \quad \forall x < 0; \quad F(x) \geq \lambda, \quad \forall x \geq 0.$$

Then almost surely $\gamma[0,T] \cap (0,\infty) = \emptyset$.

Proof. Assume the same notations as in Definition 5.1. First, for any $z \in \mathbb{H}$, define $U_t(z)$ in the same way in Equation (5.7), and we will explain that the process $(U_t(z), 0 \leq t \leq T)$ is non-increasing. By the definition of $\eta_t(z)$, we know that $U_t(\cdot) - \lambda$ is the harmonic function on $\mathbb{H} \setminus K_t$ with the boundary values given by $F + \lambda \geq 0$ on $\mathbb{R}_- \setminus K_t$, zero along the boundary of K_t, and $F - \lambda \geq 0$ on $\mathbb{R}_+ \setminus K_t$. This harmonic function is non-increasing in t, and thus $U_t(z)$ is non-increasing.

Next, we will show that the process

$$Z_t := V_t^R(0^+) - W_t$$

cannot hit zero. By the proof of Lemma 5.24, we know that there exist a Brownian motion B and a process of bounded variation V such that $W = 2B - V$ and Equation (5.8) holds. Since $U_t(z)$ in non-increasing in t, the process V_t is non-decreasing in t up to the time that z is swallowed. However, the process V does not depend on z, and since we can always choose z far away, we have that $(V_u, 0 \leq u \leq t)$ is non-decreasing in u for any $t > 0$. Then we have, for all t,

$$dZ_t \geq -2dB_t + \frac{2dt}{Z_t}.$$

We can compare $Z_t/2$ with a Bessel process of dimension 2, and so may conclude that Z_t cannot hit 0. This implies that the curve cannot hit $(0,\infty)$. □

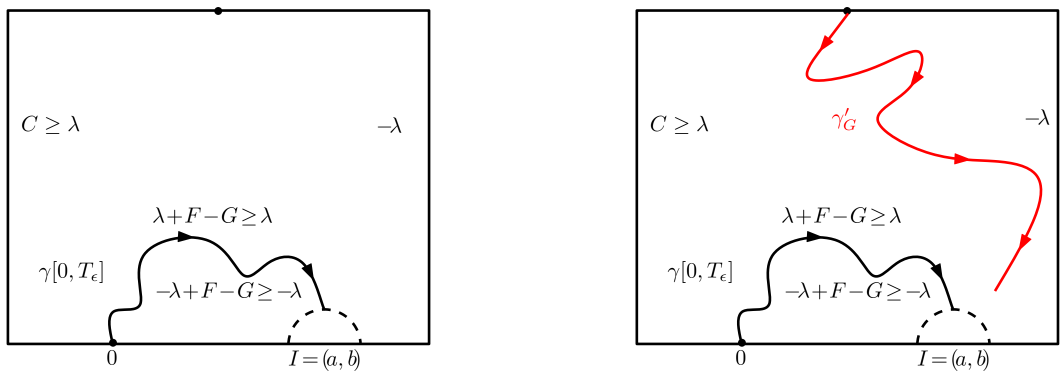
Remark 5.26. *The proof of Lemma 5.25 also applies to the case when*

$$F(x) \leq -\lambda, \quad x < 0; \quad F(x) \leq \lambda, \quad x \geq 0$$

by symmetry. In this case we see that for γ *satisfying the same conditions as in Lemma 5.25, we have* $\gamma[0,T] \cap (-\infty, 0) = \emptyset$ *almost surely.*

Lemma 5.27. *Suppose that* γ *is a random continuous curve from 0 to some* γ*-stopping time* T *with almost surely continuous driving function. Assume that* γ *is coupled with a zero boundary* GFF h *as a level line of* $h+F$ *up to time* T *where*

$$F(x) \leq -\lambda, \quad \forall x < 0; \quad F(x) \geq \lambda, \quad \forall x \geq 0.$$



(a) The boundary value of $-h - G$ given $\gamma[0, T_\varepsilon]$.

(b) γ'_G cannot hit the left side of $\gamma[0, T_\varepsilon]$ or $\mathbb{R}_-$.

Figure 26: Explanation of the boundary values in the proof of Lemma 5.27.

Then, almost surely, the curve $(\gamma(t), 0 \leq t \leq T)$ *does not hit the boundary except the two end points.*

Proof. It is sufficient to show that, for any $0 < a < b < \infty$, the curve γ does not hit the interval $I = (a, b)$. We prove by contradiction.

Suppose that γ does hit I with positive probability, and on this event, define T_ε to be the first time that γ gets within ε of I. Since F is bounded, suppose that $F \geq -C$ for some finite $C \geq \lambda$. Let G be the bounded harmonic extension of the function which is equal to $-C$ on $\mathbb{R}_-$ and is equal to λ on $\mathbb{R}_+$. Note that $F \geq G$. Let γ'_G be the level line of $-h - G$ from ∞ to 0. By Lemma 5.22(1) and (2), we know that γ'_G is almost surely continuous and transient; and that γ'_G almost surely does not hit I.

Let $\tilde{h}$ be h restricted to the unbounded connected component of $\mathbb{H} \setminus \gamma[0, T_\varepsilon]$, then conditionally on $\gamma[0, T_\varepsilon]$, the field $-\tilde{h} - G$ is a GFF with boundary data as shown in Figure 26(a). Moreover, given $\gamma[0, T_\varepsilon]$, the curve γ'_G is coupled with $\tilde{h}$ so that it is a level line of $-\tilde{h} - G$ up until the first time that γ'_G hits $\gamma[0, T_\varepsilon]$ (by Propositions 5.16 to 5.18). Since $F - G$ is positive on $\mathbb{H}$, we see from Lemma 5.25 that γ'_G cannot hit the left side of $\gamma[0, T_\varepsilon]$ or $(-\infty, 0]$ before hitting the right side of $\gamma[0, T_\varepsilon]$ or the tip $\gamma(T_\varepsilon)$, see Figure 26(b). In any case, this implies that γ'_G has to get within ε of I. Since this holds for any $\varepsilon > 0$ on the event that γ hits I and γ'_G is continuous, we can conclude that γ'_G hits I with positive probability, contradiction. □

Lemma 5.28. *Assume the same notations as in Lemma 5.27. Then* γ *is almost surely simple.*

Proof. First, we argue that, for any γ-stopping time τ, we have $\gamma[0, \tau) \cap \gamma(\tau, T) = \emptyset$

almost surely. Given $\gamma[0,\tau]$, denote by $\tilde{h}$ the restriction of $h+F$ to $\mathbb{H}\setminus\gamma[0,\tau]$ (since γ does not hit the boundary, this set only has one connected component). By the domain Markov property in Definition 5.1, we know that, given $\gamma[0,\tau]$, the curve $\gamma|_{[\tau,T)}$ is coupled with $\tilde{h}$ as its level line. Note that the boundary value of $\tilde{h}+F$ is $F \le -\lambda$ on $\mathbb{R}_-$, is $-\lambda$ along the left side of $\gamma[0,\tau]$, is λ along the right side of $\gamma[0,\tau]$, and is $F \ge \lambda$ along $\mathbb{R}_+$. By Lemma 5.27, we know that $\gamma(\tau, T)$ cannot hit $\gamma(0,\tau)$.

Next, we show that γ is almost surely simple. For any $q > 0$, define A_q to be the event that $\gamma(0,q)\cap\gamma(q,T)\neq\emptyset$. If γ has double point, then A_q happens for some positive rational q, since γ is continuous. However, by the above argument, we know that $\cup_{q\in\mathbb{Q}_+}A_q$ has zero probability. Therefore, γ is almost surely simple. $\square$

Proposition 5.29. *Suppose that h is a zero boundary* GFF *and that F is bounded and satisfies*

$$F(x) \le -\lambda, \quad \forall x < 0; \quad F(x) \ge \lambda, \quad \forall x \ge 0.$$

Suppose that γ (resp. γ') is a random continuous transient curve from 0 to ∞ (resp. from ∞ to 0) with almost surely continuous driving function.

Assume that γ is coupled with h as a level line of $h+F$, that γ' is coupled with h as a level line of $-h-F$, and that the triple (h,γ,γ') are coupled so that γ and γ' are conditionally independent given h. Then almost surely γ' equals γ. In particular, this implies that γ is almost surely determined by h.

Proof. First, we argue that, for any γ'-stopping time τ', given $\gamma'[0,\tau']$, the curve γ almost surely first exits $\mathbb{H}\setminus\gamma'[0,\tau']$ at $\gamma'(\tau')$. Denote by $\tilde{h}$ the restriction of h to $\mathbb{H}\setminus\gamma'[0,\tau']$. Given $\gamma'[0,\tau']$, the curve γ is coupled with h as a level line of $\tilde{h}+F$. The boundary value of $\tilde{h}+F$ is $F\le-\lambda$ on $\mathbb{R}_-$, is $-\lambda$ along the left side of $\gamma'[0,\tau']$, is λ along the right side of $\gamma'[0,\tau']$, and is $F\ge\lambda$ on $\mathbb{R}_+$. Thus, by Lemma 5.27, we know that γ must exit $\mathbb{H}\setminus\gamma'[0,\tau']$ at $\gamma'(\tau')$.

Next, we show that γ and γ' are equal. Since γ hits $\gamma'[0,\tau']$ for the first time at $\gamma'(\tau')$ for any γ'-stopping time τ', we know that γ hits a dense countable set of points along γ' in reverse chronological order. By symmetry, γ' hits a dense countable set of points along γ. Since both γ and γ' are continuous simple curves, the two curves (viewed as sets) are equal. $\square$

5.4 Monotonicity

Lemma 5.30. *Suppose that h is a zero boundary* GFF *and that F is bounded. Suppose that γ is a random continuous curve from 0 to some γ-stopping time T with almost surely continuous driving function. Assume that γ is coupled with h as a level line of $h+F$ up to time T.*

(1) Then the curve $(\gamma(t), 0 \leq t \leq T)$ almost surely does not intersect any open interval I of $(0,\infty)$ such that

$$F(x) \geq \lambda \quad \forall x \in I.$$

Symmetrically, it does not intersect any open interval of $(-\infty, 0)$ where $F(x) \leq -\lambda$.

(2) In addition, if $(\gamma(t), 0 \leq t \leq T)$ is almost surely simple, then it does not hit any open interval I of $(-\infty, 0)$ where $F(x) \geq \lambda$. Symmetrically, it does not intersect any open interval of $(0,\infty)$ where $F(x) \leq -\lambda$.

Proof of Lemma 5.30, Item (1). We first show the conclusion when $I = (a, b)$ for $0 < a < b$ and $F(x) \geq \lambda, \forall x \in I$. Pick $\tilde{a}, \tilde{b}$ such that $a < \tilde{a} < \tilde{b} < b$. It is sufficient to show that, for any such $\tilde{a}, \tilde{b}$, the curve $(\gamma(t), 0 \leq t \leq T)$ does not hit the interval $\tilde{I} = [\tilde{a}, \tilde{b}]$. We prove by contradiction. Suppose that the curve $(\gamma(t), 0 \leq t \leq T)$ hits $\tilde{I}$ with positive probability. Since F is bounded, we have that $F \geq -C$ for some $C \geq \lambda$. Let G be the bounded harmonic extension of the function which is equal to $-C$ on $\mathbb{R}_- \cup (0, a) \cup (b, \infty)$ and λ on (a, b). Note that $F \geq G$. Let γ'_G be the level line of $-h - G$ from b to a. Note that since G is piecewise constant we know by Lemma 5.22(1) that the curve γ'_G is continuous from b to a, and the boundary data also means, by Lemma 5.22(2), that it does not hit $\tilde{I}$. This means we can repeat the same argument as in the proof of Lemma 5.27 to show that γ'_G hits $\tilde{I}$ with positive probability and obtain a contradiction. □

Proof of Lemma 5.30, Item (2). Now, let $I = (a, b)$ for $a < b < 0$, and suppose that $F(x) \geq \lambda, \forall x \in I$ and $(\gamma(t), 0 \leq t \leq T)$ is almost surely simple. It will be sufficient for us to prove that γ does not hit $\tilde{I} = [\tilde{a}, \tilde{b}]$ for any $a < \tilde{a} < \tilde{b} < b$. First note that if γ hits $[-\infty, a]$ before hitting I, since γ grows towards ∞, it can never hit $\tilde{I}$ thereafter and we are done. If not, let φ be the Möbius transform of $\mathbb{H}$ that sends the triplet $(b, 0, \infty)$ to $(\infty, 0, 1)$. Then $(\varphi(\gamma(t)), 0 \leq t \leq T)$ is a continuous curve, coupled with a zero-boundary GFF $\tilde{h}$ as a level line of $\tilde{h} + F \circ \varphi^{-1}$ until the first time it hits $[1, \infty]$. By Item (1), we know that $(\varphi(\gamma(t)), 0 \leq t \leq T)$ cannot hit the interval $(\varphi(a), \infty)$ before this time. Thus $(\gamma(t), 0 \leq t \leq T)$ cannot hit $\tilde{I}$ without first hitting the point b. Let τ be the time at which γ hits b, setting $\tau = T$ if this never happens, and τ' be the first time at which γ hits $\tilde{I}$, again setting $\tau' = T$ if necessary. By the previous reasoning, if we do not have $\{\tau < \tau' < T\}$ then we are done, so assume this occurs with positive probability. On this event, since γ is a continuous curve with continuous Loewner driving function, we see that $\{\tau \leq t \leq \tau' : \gamma(t) \in \mathbb{R}\}$ has Lebesgue measure 0, and so there exists a time $\tau < \sigma < \tau'$ with $\gamma(\sigma) \notin \mathbb{R}$. Let

w be the left-most point in $(\gamma(t), 0 \le t \le \sigma) \cap (-\infty, 0)$. Then applying the same argument as above, now to $(\gamma(t), \sigma \le t \le T)$ in the domain $\mathbb{H} \setminus (\gamma(t), 0 \le t \le \sigma)$ with (a, b) replaced by (a, w), we see that $(\gamma(t), \sigma \le t \le T)$ must first hit w before it can hit $\tilde{I}$. This is a contradiction to the simplicity of γ. $\square$

Lemma 5.31. *Suppose that h is a zero boundary* GFF *and that F is bounded. Suppose that γ is a random continuous curve from 0 to some γ-stopping time T with almost surely continuous driving function. Assume that γ is coupled with h as a level line of $h + F$ up to time T.*

1. *For any fixed point $x_0 \in (0, \infty)$, if there exists $c > 0$ such that $F \ge -\lambda + c$ in a neighborhood of x_0, then the curve $(\gamma(t), 0 \le t \le T)$ almost surely does not hit $\{x_0\}$. Symmetrically, for any fixed point $x_0 \in (-\infty, 0)$, if there exists $c > 0$ such that $F \le \lambda - c$ in a neighborhood of x_0, then the curve $(\gamma(t), 0 \le t \le T)$ almost surely does not hit $\{x_0\}$.*
2. *If there exists $X \in (0, \infty)$ and $c > 0$ such that*

$$F(x) \ge \lambda, \quad \forall x \in (-X, \infty); \quad F(x) \ge -\lambda + c, \quad \forall x \in (X, \infty)$$

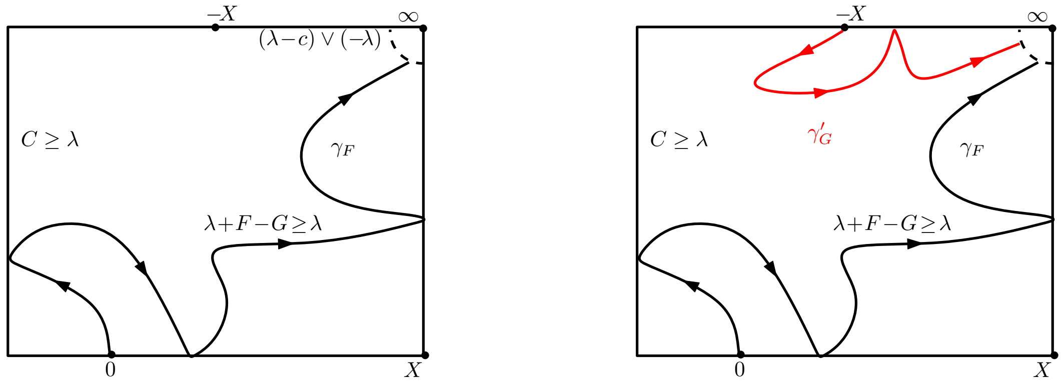
   and in addition the curve $(\gamma(t), 0 \le t \le T)$ is almost surely simple, then $(\gamma(t), 0 \le t \le T)$ almost surely does not hit ∞. Symmetrically, if there exists $X \in (0, \infty)$ and $c > 0$ such that

$$F(x) \le -\lambda, \quad \forall x \in (X, \infty); \quad F(x) \le \lambda - c, \quad \forall x \in (-X, \infty)$$

   and the curve $(\gamma(t), 0 \le t \le T)$ is almost surely simple, then $(\gamma(t), 0 \le t \le T)$ almost surely does not hit ∞

We point out that Item (2) is not a consequence of Item (1) in Lemma 5.31. In fact, if F is piecewise constant and $F(x) \in (-\lambda + c, \lambda - c)$ on $(-\infty, -X) \cup (X, \infty)$ for $c \in (0, \lambda)$, then the level line of $h + F$ is transient, and hence hits ∞ almost surely.

Proof of Lemma 5.31, Item (1). We may assume that $F \ge -\lambda + c$ on (a, b) where $0 < a < x_0 < b$ and again prove by contradiction. Suppose that the curve $(\gamma(t), 0 \le t \le T)$ does hit $\{x_0\}$ with some positive probability. Since F is bounded, suppose that $F \ge -C$ for some $C \ge \lambda$. Let G be the function which is equal to $-C$ on $\mathbb{R}_- \cup (0, a) \cup (b, \infty)$, and is $(-\lambda + c) \wedge \lambda$ on (a, b). Note that $F \ge G$. Let γ'_G be the level line of $-h - G$ from b to a. Note that γ'_G is continuous and does not the point $\{x_0\}$ by Lemma 5.22(3). Thus we can repeat the same argument as in the proof of Lemma 5.27 and show that γ'_G hits $\{x_0\}$ with positive probability, which is a contradiction. $\square$



(a) Suppose that γ_F hits ∞ with positive probability. Let T_ε be the first time that it enters $\{z : |z| > 1/\varepsilon\}$. Since $F \geq \lambda$ on $(-\infty, -X)$, the curve γ_F can never hit $(-\infty, -X)$. Given $\gamma_F[0, T_\varepsilon]$, the boundary data of $-h - G$ is shown in this figure.

(b) By the choice of G, we see that γ'_G cannot hit the union of $(-X, 0)$ and the left side of $\gamma_F[0, T_\varepsilon]$ before hitting $\gamma_F[0, T_\varepsilon]$. Therefore, γ'_G has to enter $\{z : |z| > 1/\varepsilon\}$. This holds for all $\varepsilon > 0$, thus γ'_G hits ∞ with positive chance, contradiction.

Figure 27: Explanation of the boundary values in the proof of Lemma 5.31, Item (2).

Proof of Lemma 5.31, Item (2). We prove by contradiction. Suppose that γ does hit ∞ with positive probability. Since F is bounded, suppose that $F \geq -C$ for some $C \geq \lambda$. Let G be the function which is equal to $-C$ on $(-X, 0) \cup (0, X)$ and is $(-\lambda + c) \wedge \lambda$ on $(X, \infty) \cup (-\infty, -X)$. Note that $F \geq G$. Let γ'_G be the level line of $-h - G$ from $-X$ to X. Since G is piecewise constant, we know the curve γ'_G is continuous and does not hit the point ∞. By Lemma 5.30(2), since γ is almost surely simple, we know that γ cannot hit $(-X, \infty)$ before it hits ∞. Thus, we can repeat the same argument as in the proof of Lemma 5.27 and show that γ'_G hits ∞ with positive probability, contradiction. See more details in Figure 27. $\square$

Lemma 5.32. *Suppose that h is a zero boundary* GFF *and that F is bounded and satisfies Condition (5.2). Suppose that γ is a random continuous transient curve from 0 to ∞ with almost surely continuous driving function. Assume that γ is coupled with h as a level line of $h + F$. Then γ is almost surely simple.*

Proof. We can repeat the same argument as in the proof of Lemma 5.28 replacing Lemma 5.27 by Lemmas 5.30(1) and 5.31(1). $\square$

Lemma 5.33. *Suppose that F and G are bounded, F satisfies Condition (5.2), and that*

$$F(x) \geq G(x), \quad \forall x \in \mathbb{R}.$$

Suppose that γ_F (resp. γ'_G) is a random continuous transient curve from 0 to ∞ (resp. from ∞ to 0) with almost surely continuous driving function.

Assume that γ_F is coupled with a zero boundary GFF *h as a level line of $h+F$ from 0 to ∞ and that γ'_G is coupled with h as a level line of $-h-G$ from ∞ to 0, and that the triple (h, γ_F, γ'_G) is coupled so that γ_F and γ'_G are conditionally independent given h. Then almost surely γ_F stays to the left of γ'_G.*

Proof. Note that, by Lemma 5.32, γ_F is almost surely simple. It is sufficient to show that, for any γ'_G-stopping time τ', the point $\gamma'_G(\tau')$ is to the right of γ_F.

Let $\tilde{h}$ be h restricted to $\mathbb{H} \setminus \gamma'_G[0, \tau']$. Then we know that, given $\gamma'_G[0, \tau']$, the conditional law of $\tilde{h} + F$ is that of a GFF with boundary data as shown in Figure 28(a). Moreover, γ_F is coupled as a level line of $\tilde{h} + F$ up until the first time it hits $\gamma'_G[0, \tau']$. Given $\gamma'_G[0, \tau']$, let τ be the first time that γ_F hits $\gamma'_G[0, \tau']$.

Consider the set $\gamma'_G[0, \tau']$, there are two possibilities for the intersection $\gamma'_G[0, \tau'] \cap (0, \infty)$: Case (a), the intersection is nonempty, and in this case we denote by x_G the last point in the intersection; Case (b), the intersection is empty and in this case we set $x_G = +\infty$ ie. to the right of $\gamma'_G[0, \tau']$.

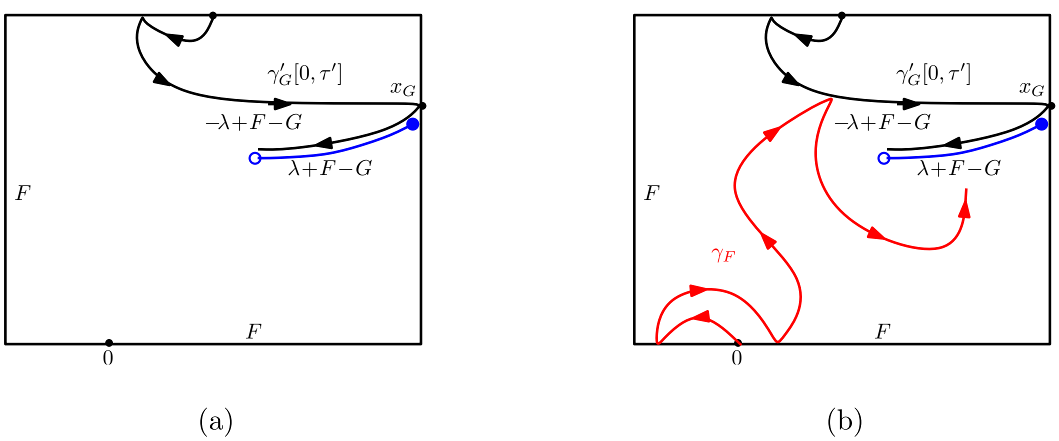


Figure 28: The curve γ_F cannot hit the blue section in the figure.

Since the boundary data on the right hand side of $\gamma'_G[0, \tau']$ is greater than λ and the boundary data is bounded away from $-\lambda$ in a neighborhood of x_G, by Lemma 5.30 we know that γ_F cannot hit the right hand side of $\gamma'_G[0, \tau']$ before hitting the left side of $\gamma'_G[0, \tau']$ or exiting $\mathbb{H}$ at ∞, approaching from the left. We also know that γ_F cannot hit $\{x_G\}$ before this time, by Lemma 5.31(1) in Case (a) and by

Lemma 5.31(2) in Case (b). Therefore γ_F cannot hit the union of the right hand side of $\gamma'_G[0,\tau']$ and $\{x_G\}$ (i.e. the blue section in Figure 28(a)) of the boundary before hitting the left hand side of $\gamma'_G[0,\tau']$ or exiting $\mathbb{H}$ at ∞, approaching from the left. In the latter case, we are done. In the former case, γ_F first hits $\gamma'_G[0,\tau']$ from its left hand side at time τ. If $\gamma'_G(\tau')$ is strictly to the left of γ_F, then it must be the case that after time τ, γ_F wraps around $\gamma'_G[0,\tau']$ and then hits the right hand side of $\gamma'_G[0,\tau']$ or exits at x_G. Let τ_δ be the first time after τ that γ_F is in the right connected component of $\mathbb{H}\setminus(\gamma_F[0,\tau]\cup\gamma'_G[0,\tau'])$ and $\mathrm{dist}(\gamma_F(t),\gamma'_G[0,\tau])\geq\delta$, setting $\tau_\delta=\infty$ if this never happens. If $\gamma'_G(\tau')$ is strictly to the left of γ_F (so in particular not on the curve γ_F) with positive probability then we know that $\{\tau_\delta<\infty\}$ occurs with strictly positive probability. However, given $\gamma'_G[0,\tau']\cup\gamma_F[0,\tau_\delta]$, the conditional law of $h+F$ is that of a GFF with boundary values as shown in Figure 28(b), and $(\gamma_F(t), t\geq\tau_\delta)$ is a level line of this field (by Propositions 5.16 to 5.18.) Therefore, by Lemmas 5.30 and 5.31 again, we know that it cannot hit the right hand side of $\gamma'_G[0,\tau']$ or exit at x_G, and hence cannot reach ∞. Thus we obtain a contradiction. □

Lemma 5.34. *Suppose that F is bounded and satisfies Condition (5.2). Suppose that γ_F (resp. γ'_F) is a random continuous transient curve from 0 to ∞ (resp. from ∞ to 0) with almost surely continuous driving functions.*

Assume that γ_F is coupled with a zero boundary GFF *h as a level line of $h+F$ from 0 to ∞, that γ'_F is coupled with h as a level line of $-h-F$ from ∞ to 0, and that the triple (h,γ_F,γ'_F) is coupled so that γ_F and γ'_F are conditionally independent given h. Then almost surely $\gamma_F=\gamma'_F$. In particular, γ_F is almost surely determined by h.*

Proof. By Lemma 5.33, we know that γ_F almost surely stays to the left of γ'_F and (by the same arguments) that γ_F almost surely stays to the right of γ'_F. Combining with the fact that γ_F,γ'_F are simple by Lemma 5.32, we know that almost surely $\gamma_F=\gamma'_F$. Since γ_F and γ'_F are coupled with h so that they are conditionally independent given h, $\gamma_F=\gamma'_F$ implies that γ_F must be almost surely determined by h. □

Lemma 5.35. *Suppose that F and G are bounded, and satisfy Condition (5.2). Suppose further that*

$$F(x)\geq G(x),\quad x\in\mathbb{R}.$$

Suppose that γ_F,γ_G (resp. γ'_G) are random continuous transient curves from 0 to ∞ (resp. from ∞ to 0) with almost surely continuous driving functions.

Assume that γ_F (resp. γ_G) is coupled with a zero boundary GFF *h as a level line of $h+F$ (resp. $h+G$), that γ'_G is coupled with h as a level line of $-h-G$ from*

∞ to 0, and that $(h, \gamma_F, \gamma_G, \gamma'_G)$ is coupled so that γ_F, γ_G and γ'_G are conditionally independent given h. Then almost surely γ_F stays to the left of γ_G.

Proof. We have the following observations.

- By Lemma 5.33, we know that γ_F stays to the left of γ'_G.
- By Lemma 5.34, we know that $\gamma_G = \gamma'_G$.

Combining these two facts, we see that γ_F stays to the left of γ_G. □

Corollary 5.36. *Suppose that F and G are piecewise constant functions changing value only finitely many times and that they satisfy Condition (5.2). Suppose further that*

$$F(x) \geq G(x), \quad x \in \mathbb{R}.$$

Let γ_F (resp. γ_G) be the level line of $h+F$ (resp. $h+G$) for h a zero boundary GFF *as in Section 5.2.6. Then almost surely γ_F stays to the left of γ_G.*

Proof. From the results in Section 5.2.6, we have the existence, the continuity and transience of γ_F and γ_G, and also γ'_G which is the level line of $-h-G$ from ∞ to 0. Moreover, we know that each of γ_F, γ_G and γ'_G is almost surely determined by h. By Lemma 5.35, we know that γ_F stays to the left of γ_G almost surely. □

5.5 Estimates on crossing probabilities

In this section, we will consider $\mathrm{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ processes for vectors

$$\underline{\rho}^L = (\rho_l^L, \cdots, \rho_1^L), \qquad \underline{\rho}^R = (\rho_1^R, \cdots, \rho_r^R),$$

with associated force points

$$\underline{x}^L = (x_l^L < \cdots < x_1^L \leq 0), \qquad \underline{x}^R = (0 \leq x_1^R < \cdots x_r^R),$$

such that for some $c > 0, C < \infty$,

$$-2+\frac{c}{\lambda} \leq \sum_{i=1}^{j} \rho_i^L \leq -1+\frac{C}{\lambda}, \quad 1 \leq j \leq l, \quad -2+\frac{c}{\lambda} \leq \sum_{i=1}^{k} \rho_i^R \leq -1+\frac{C}{\lambda}, \quad 1 \leq k \leq r. \tag{5.9}$$

We will show that if $(\gamma^{(n)})_{n\in\mathbb{N}}$ are a family $\mathrm{SLE}_4(\underline{\rho}^L; \underline{\rho}^R)$ processes as above (with the same c, C), then they satisfy Condition 5.11. Here, we know that the processes are generated by continuous curves, due to the results of [MS16a, WW16].

Note that these processes correspond to level lines of $(h + F_n)_{n\in\mathbb{N}}$ for h a zero boundary GFF, where Condition (5.9) means that the F_n's are uniformly bounded (lying in $(-C, C)$) and satisfy, for all $n \geq 0$,

$$F_n(x) \leq \lambda - c, \quad x < 0; \quad F_n(x) \geq -\lambda + c, \quad x \geq 0.$$

These are the same conditions we require on F in Theorem 5.2. Therefore, the tactic will be to approximate such an F by piecewise constant functions F_n on $\mathbb{R}$, and show that the laws of the corresponding $\mathrm{SLE}_4(\underline{\rho}^{L,n}; \underline{\rho}^{R,n})$ processes converge weakly using Proposition 5.12. This limiting law will be our candidate for the level line of $h + F$.

Lemma 5.37. *Suppose that $(\gamma^{(n)})_{n \in \mathbb{N}}$ are a family of $\mathrm{SLE}_4(\underline{\rho}^L, \underline{\rho}^R)$ processes satisfying Condition (5.9) for some $c > 0, C < \infty$ and all n. Then they satisfy Condition 5.11.*

Proof. Recall, we would like to show that our family satisfies a *conformal bound on an unforced crossing*. That is, that there exists a constant $M > 0$, such that for any of our processes $\gamma^{(n)}$, any stopping time τ and any avoidable quadrilateral of $H_\tau = \mathbb{H} \setminus K_\tau^{(n)}$ whose modulus $m(Q)$ is greater than M,

$$\mathbb{P}\left(\gamma^{(n)}[\tau, \infty) \,\text{crosses}\, Q \mid \gamma^{(n)}[0, \tau]\right) \leq 1/2.$$

Here $(K_t^{(n)}, t \geq 0)$ denotes the sequence of hulls generated by $\gamma^{(n)}$.

For $n \geq 0$, the law of $\gamma^{(n)}$ is that of an $\mathrm{SLE}_4(\underline{\rho}^{L,n}; \underline{\rho}^{R,n})$ process with force points located at $(\underline{x}^{L,n}; \underline{x}^{R,n})$. Denote its driving function by $W^{(n)}$, its sequence of conformal mappings by $g^{(n)}$, and set $f^{(n)} = g^{(n)} - W^{(n)}$. By the results of [WW16], we know that $\gamma^{(n)}$ can be coupled with h a zero boundary GFF in $\mathbb{H}$, as the level line of $h + F_n$, for

$$F_n(x) = \begin{cases} -\lambda \left(1 + \sum_{\{i : x_i^{L,n} \geq x\}} \rho_i^{L,n}\right), & x < 0 \\ \lambda \left(1 + \sum_{\{i : x_i^{R,n} \leq x\}} \rho_i^{R,n}\right), & x \geq 0. \end{cases}$$

Moreover, for any stopping time τ, we know by the domain Markov property that, conditionally on $\gamma^{(n)}[0, \tau]$, the curve evolves from time τ onwards as a level line of a GFF with boundary conditions $\eta_\tau^{(n)}$ in the remaining domain. Here $\eta^{(n)}$ is defined corresponding to F_n as in Definition 5.1. The important thing to notice is that, as a result of the condition (5.9), we have

$$\eta_\tau^{(n)} \circ (f_\tau^{(n)})^{-1} \geq -C, \quad \text{on } (-\infty, 0); \quad \eta_\tau^{(n)} \circ (f_\tau^{(n)})^{-1} \geq -\lambda + c \quad \text{on } [0, \infty)$$

for any τ and n. Therefore, if we set

$$G_1(x) := \begin{cases} -C, & x < 0 \\ -\lambda + c, & x \geq 0 \end{cases}$$

we have that

$$G_1 \leq \eta_\tau^{(n)} \circ (f_\tau^{(n)})^{-1}, \quad \text{on } \mathbb{R}.$$

Similarly if we set

$$G_2(x) := \begin{cases} \lambda - c, & x < 0 \\ C, & x \geq 0 \end{cases}$$

then

$$G_2 \geq \eta_\tau^{(n)} \circ (f_\tau^{(n)})^{-1}, \quad \text{on } \mathbb{R}.$$

Now, consider an avoidable topological quadrilateral Q of H_τ. The avoidability assumption means that, when we map it to $\mathbb{H}$ via $f_\tau^{(n)}$, its image Q' is a topological quadrilateral in $\mathbb{H}$ as in Definition 5.9 with S'_1, S'_3 (the arcs touching the boundary) either both lying in $[0, \infty)$, or both in $(-\infty, 0]$.

Suppose we are in the first case. We would like to bound above the probability of $\gamma^{(n)}[\tau, \infty)$ crossing Q, where Q has modulus greater than M for some positive M. Equivalently, we must bound the probability of $f_\tau^{(n)}(\gamma^{(n)}[\tau, \infty))$ crossing Q', noting by conformal invariance that Q' also has modulus greater than M. If $Q' = (V', (S'_k)_{1\leq k\leq 4})$, we let $Q'' = (V'', S''_0, S''_2)$ be the doubly connected domain where V'' is the interior of the closure of $V' \cup V'*$ ($V'*$ the reflection of V' in the real line) and S''_0, S''_2 are it's inner and outer boundary. Following the arguments in the proof of [KS16, Theorem 1.10], we let $x = \min(\mathbb{R} \cap S''_0) > 0$ and $r = \max\{|z - x| : z \in S''_0\} > 0$. We see that Q'' is a doubly connected domain separating x and a point on $\partial B(x, r)$ from 0 and ∞ (see Figure 29.) However, [Ahl73, Thoerem 4.7] tells us that among all such domains, the one with the largest modulus (here defined as the extremal length of the curve family connecting S''_0 and S''_2 in V'', which satisfies $m(Q'') = m(Q')/2$) is the domain formed by removing $(-\infty, 0] \cup [x, x + r]$ from the complex plane. This modulus is also calculated explicitly in [Ahl73] and so we may deduce that

$$\exp(2\pi m(Q'')) \leq 16\left(\frac{x}{r} + 1\right).$$

Since

$$m(Q'') = \frac{m(Q)}{2} \geq \frac{M}{2},$$

this means that $r \le \nu x$ for

$$\nu = \left(\frac{1}{16} \exp(\pi M) - 1 \right)^{-1}. \tag{5.10}$$

Note that ν can be made as small as we like by choosing M large.

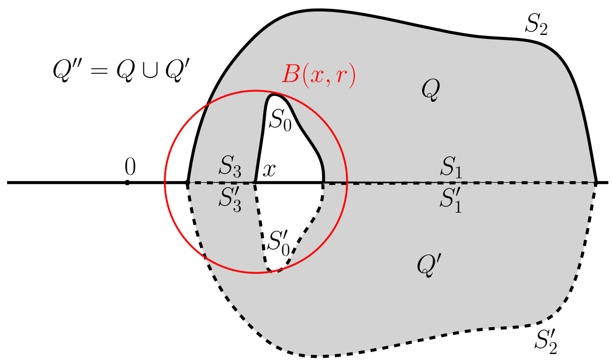


Figure 29: Since Q'' separates x and a point on $\partial B(x,r)$ from ∞, we obtain a lower bound on $m(Q'') = m(Q)/2$. Since $m(Q) \ge M$ this gives us an upper bound on r/x. Moreover, we know that for a curve to cross Q it must necessarily intersect $\overline{B(x,r)}$.

It is also clear that for $f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty))$ to cross Q', it must necessarily intersect $\overline{B(x,r)}$. However, the law of $f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty))$ is that of the level line of $\tilde{h} + \eta_\tau^{(n)} \circ (f_\tau^{(n)})^{-1}$, for $\tilde{h}$ a zero boundary GFF in $\mathbb{H}$. By the monotonicity result Corollary 5.36, we see that this level line lies to the left of the level line of $\tilde{h} + G_1$ almost surely (see Figure 30.) Thus, the probability of $f_\tau^{(n)}(\gamma^{(n)}[0,\tau])$ intersecting $\overline{B(x,r)}$ is less than the probability of an $\mathrm{SLE}_4(\rho^L;\rho^R)$ process with

$$\rho^L = -1 + \frac{C}{\lambda}; \quad \rho^R = -2 + \frac{c}{\lambda} \tag{5.11}$$

(left and right force points at the origin), intersecting it.

Therefore, we have

$$\begin{aligned}
&\mathbb{P}\left(\gamma^{(n)}[\tau,\infty) \text{ crosses } Q \,|\, \gamma^{(n)}[0,\tau]\right) \\
&\quad \le \mathbb{P}\left(f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty)) \text{ crosses } Q' \,|\, \gamma^{(n)}[0,\tau]\right) \\
&\quad \le \mathbb{P}\left(f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty)) \text{ intersects } \overline{B(x,r)} \,|\, \gamma^{(n)}[0,\tau]\right) \\
&\quad \le \mathbb{P}\left(\mathrm{SLE}_4(\rho^L;\rho^R) \text{ intersects } \overline{B(x,r)}\right) \quad (\rho^L, \rho^R \text{ are defined in Equation (5.11)}) \\
&\quad = \mathbb{P}\left(\mathrm{SLE}_4(\rho^L;\rho^R) \text{ intersects } \overline{B(1,r/x)}\right) \qquad \text{(by scaling invariance)} \\
&\quad \le \mathbb{P}\left(\mathrm{SLE}_4(\rho^L;\rho^R) \text{ intersects } \overline{B(1,\nu)}\right). \qquad (\nu \text{ is defined in Equation (5.10)})
\end{aligned}$$

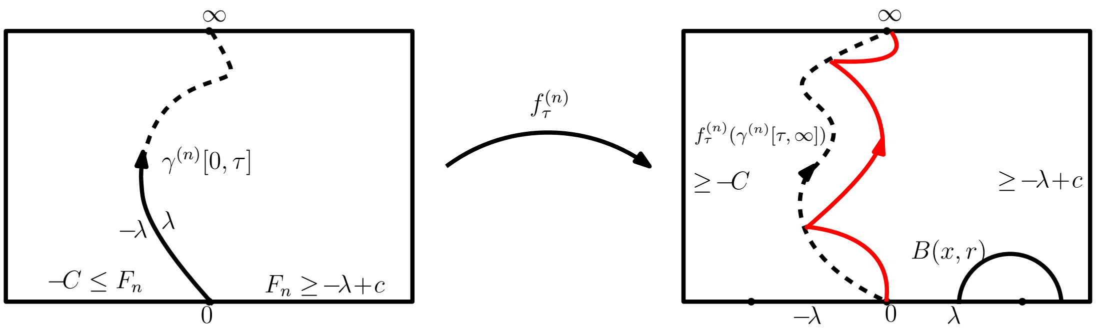


Figure 30: The boundary values of $h+F_n$ given $\gamma^{(n)}[0,\tau]$ are marked in the left panel. Thus $f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty])$ is the level line of a zero boundary GFF $\tilde{h}$ + boundary data as depicted in the right panel. By monotonicity, it must therefore lie to the left of the level line of $\tilde{h}+G_1$ (marked in red.) Consequently, the probability that $f_\tau^{(n)}(\gamma^{(n)}[\tau,\infty])$ intersects $\overline{B(x,r)}$ is less than the probability that the red level line does.

Since we know that the $\mathrm{SLE}_4(\rho^L;\rho^R)$ process with left and right force points at 0 almost surely does not hit the point 1 (in fact, there is exact estimate on this event, see for instance [MW16, Theorem 1.8]), we see that by choosing M large enough, and so ν small enough, we can make the right hand side less than 1/2. Thus there exists an M such that the left hand side is bounded above uniformly by 1/2 whenever $m(Q) \geq M$.

For the second case, when the boundary arcs S_1', S_3' of Q' both lie on the negative real line, we may use symmetrical arguments, replacing G_1 by G_2. □

Corollary 5.38. *Suppose that $(\gamma^{(n)})_{n\in\mathbb{N}}$ are a family of $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ processes satisfying Condition (5.9) for all n. Suppose further than they are all parameterised by half plane capacity and that $(W^{(n)})_{n\in\mathbb{N}}$ are the corresponding family of driving functions. Then*

- *$(W^{(n)})_{n\in\mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets of $[0,\infty)$.*
- *$(\gamma^{(n)})_{n\in\mathbb{N}}$ is tight in the metrisable space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on the compact subsets of $[0,\infty)$.*

Moreover, if the sequence converges weakly in either of the topologies above, then it also converges weakly in the other and the limits agree in the sense that the law of the limiting random curve is the same as the that of the random curve generated under the law of the limiting driving process.

Proof. This is a direct consequence of Proposition 5.12 and the remarks in Theorem 1.10 of [KS16]. □

5.6 Existence of the coupling—proof of Theorem 5.2

In this section we will show existence of the coupling described by Theorem 5.2. Recall we would like to prove that for F on $\mathbb{R}$ which is regulated, so can be approximated uniformly by piecewise constant functions changing value only finitely many times, and which satisfies Condition (5.2), there exists a coupling of a Loewner chain K with a zero boundary GFF h, such that K is a level line of $h+F$. Moreover, we will show that K is almost surely generated by a continuous and transient curve γ.

To do this we will take a sequence of piecewise constant functions F_n (changing value only finitely many times), which uniformly approximate F, and consider the level lines, denoted by $\gamma^{(n)}$, of $h+F_n$ for a zero boundary GFF h. Observe that we can choose the F_n so that the level lines are a family of $\mathrm{SLE}_4(\underline{\rho}^L;\underline{\rho}^R)$ processes satisfying the conditions of Corollary 5.38. Thus, the tightness given by the corollary will allow us to extract a subsequential limit.

Proposition 5.39. *Let F satisfy the conditions of Theorem 5.2. Suppose that $(F_n)_{n\in\mathbb{N}}$ are piecewise constant functions on $\mathbb{R}$, changing value only finitely many times. Let h be a zero boundary* GFF *and $\gamma^{(n)}$ be the level line of $h+F_n$ for each n. Suppose further that they are all parameterized by half plane capacity and that $(W^{(n)})_{n\in\mathbb{N}}$ are the corresponding family of driving functions.*

Then, if the (F_n) converge uniformly to F on $\mathbb{R}$, we have that:

(1) There exists a subsequence of the $\gamma^{(n)}$ which converges weakly in the space of continuous functions on $[0,\infty)$ with the topology of uniform convergence on compact subsets of $[0,\infty)$.

(2) The limiting law describes a continuous curve from 0 to ∞ in $\mathbb{H}$ which generates a Loewner chain with a.s. continuous driving function.

(3) The limiting curve can be coupled with a zero boundary GFF *h, as a level line of $h+F$.*

Remark 5.40. *We will later see that this limiting law does not depend on the choice of approximation, as any continuous curve which can be coupled with a zero boundary* GFF *as a level line of $h+F$ must have a unique law: see Remark 5.44. In particular, this tells us that we actually have convergence of the whole sequence in distribution.*

Proof of Theorem 5.2. Theorem 5.2 is a direct consequence of Proposition 5.39. □

Proof of Proposition 5.39, Items (1), (2). Note that the weak convergence directly follows from Corollary 5.38, as does the fact that the limiting law corresponds to a continuous curve generating a Loewner chain with almost surely continuous driving function. □

Definition 5.41. *Suppose that F is L^1 with respect to harmonic measure on $\mathbb{R}$ and that γ is a continuous curve with continuous Loewner driving function. We set η_t^0 in the same way as in Definition 5.1. Then for any $z \in \mathbb{H}$ we can define, for t less than the first time that γ swallows z,*

$$\eta_t(F, \gamma, z) = \eta_t^0(f_t(z))$$

as in Definition 5.1, emphasising the dependence on F and γ. Let

$$C_t(\gamma, z) = \log \mathrm{CR}(z, \mathbb{H}) - \log \mathrm{CR}(z, \mathbb{H} \setminus K_t),$$

$$\tilde{\eta}_t(F, \gamma, z) = \eta_{\tau(t)}(F, \gamma, z), \quad \text{where } \tau(t) := \inf\{s \geq 0 : C_s(\gamma, z) = t\}.$$

Finally, define $(\tilde{W}_t(\gamma, z))_{t \geq 0}$ for γ to be the driving function of γ reparameterised by $C_t(\gamma, z)$.

To prove Proposition 5.39, Item (3), i.e. to see that the limiting curve can be coupled as a level line in the way we want, we will use Lemma 5.24. This tells us that if we define $\tilde{\eta}_t(F, \gamma, z)$ as above for our limiting curve γ, we need only show that for each $z \in \mathbb{H}$, the process $(\tilde{\eta}_t(F, \gamma, z), t \geq 0)$ is a Brownian motion with respect to the filtration generated by $(\tilde{W}_t(\gamma, z), t \geq 0)$.

Lemma 5.42. *Let $(\gamma^{(n_k)})$ be a subsequence of the random curves in Proposition 5.39, parameterised by half plane capacity, which converge weakly to some γ in the space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on compacts. Then for every $z \in \mathbb{H}$,*

$$\left(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z)\right) \xrightarrow{d} \left(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z)\right)$$

in $C([0, \infty); \mathbb{R}) \times C([0, \infty); \mathbb{R})$ with respect to the product topology of uniform convergence on compacts.

We postpone the proof of Lemma 5.42 and first tell the readers how we obtain Proposition 5.39 from Lemma 5.42.

Lemma 5.43. *Let $(\gamma^{(n_k)})$ be a subsequence of the random curves in Proposition 5.39, parameterised by half plane capacity, which converge weakly to some γ in the space of continuous functions on $[0, \infty)$ with the topology of uniform convergence on* *compacts. Then for every $z \in \mathbb{H}$,*

$$\left(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F_{n_k}, \gamma^{(n_k)}, z)\right) \xrightarrow{d} \left(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z)\right)$$

in $C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R})$ with respect to the product topology of uniform convergence on compacts.

Proof. By Lemma 5.42, we have that

$$\left(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z)\right) \xrightarrow{d} \left(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z)\right) \tag{5.12}$$

with respect to the product topology of uniform convergence on compacts. It is also clear that, for all t, z and any curve γ'

$$\left|\tilde{\eta}_t(F_{n_k}, \gamma', z) - \tilde{\eta}_t(F, \gamma', z)\right| \leq \sup_{x \in \mathbb{R}} |F(x) - F_{n_k}(x)|.$$

Indeed, $\tilde{\eta}_t(F_{n_k}, \gamma', \cdot)$ and $\tilde{\eta}_t(F, \gamma', \cdot)$ are by definition harmonic extensions of functions whose boundary values differ by at most the right hand side. Since $\sup_{x\in\mathbb{R}} |F(x) - F_{n_k}(x)| \to 0$ by assumption, we may conclude that, for any $T > 0$, almost surely as $k \to \infty$,

$$\sup_{t \in [0,T]} \left|\tilde{\eta}_t(F_{n_k}, \gamma^{(n_k)}, z) - \tilde{\eta}_t(F, \gamma^{(n_k)}, z)\right| \to 0. \tag{5.13}$$

Combining Equations (5.12) and (5.13), we obtain the conclusion. □

Proof of Proposition 5.39, Item (3). Fix $z \in \mathbb{H}$. Since $\gamma^{(n_k)}$ is coupled as a level line of $h + F_{n_k}$ we know by Lemma 5.24 that $\left(\tilde{\eta}_t(F_{n_k}, \gamma^{(n_k)}, z), t \geq 0\right)$ is a Brownian motion for each k, with respect to the filtration of $\left(\tilde{W}_t(\gamma^{(n_k)}, z), t \geq 0\right)$. Therefore, by the weak convergence in Lemma 5.43, we have that if γ is the limiting law of the $\gamma^{(n_k)}$'s, the process $\tilde{\eta}_t(F, \gamma, z)$ must also have the law of Brownian motion, with respect to the filtration of $\left(\tilde{W}_t(\gamma, z), t \geq 0\right)$. Applying Lemma 5.24 again proves the proposition. □

Proof of Lemma 5.42. Fix $z \in \mathbb{H}$. We will show that the laws of $(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z))$ converge weakly in k to the law of $(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z))$. To do this, we begin by showing that this family of laws is tight in $C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R})$ with respect to the product topology of uniform convergence on compacts. This allows us to extract a further subsequence along which the $(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z))$'s converge. We then argue that the limit of this subsequence must be equal to that of $(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z))$, so in fact our whole original subsequence converged, and the limit is $(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z))$. Note that the proof of this lemma would be trivial if $(\tilde{W}(\cdot, z), \tilde{\eta}(F, \cdot, z))$ was a continuous function on the set of curves, however, this is not quite the case. It is essentially a continuous function when restricted to a set in which the $\gamma^{(n_k)}$'s lie with high probability.

By the proof of [KS16, Theorem 1.5], we know that for every $M > 0$ we can find a subset E of the space of continuous curves in $\mathbb{H}$ such that

$$\inf_k \mathbb{P}(\gamma^{(n_k)} \in E) \geq 1 - \frac{1}{M} \tag{5.14}$$

when the $(\gamma^{(n_k)})$ are parameterised by half plane capacity, and

- E is relatively compact with respect to the topology of uniform convergence on compacts,
- curves in E correspond to Loewner chains with continuous driving functions parameterised by half plane capacity, and
- if a sequence of curves in E converges with respect to uniform convergence on compacts, then their driving functions also converge uniformly on compacts along a further subsequence, and the limits agree.

For the construction of such an E, see Section 3.5 of [KS16], in particular the definition (60) and the discussion in the closing paragraphs. See also the opening paragraph of Section 3.6.

We argue that the set $\{(\tilde{W}(\gamma', z), \tilde{\eta}(F, \gamma', z)) : \gamma' \in E\}$ is a relatively compact subset of $C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R})$ with respect to the product topology of uniform convergence on compacts. Thus by (5.14) the laws of the

$$\left(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z)\right)$$

are tight in this topology. It is sufficient to verify the following claim: if $\gamma'_n \to \gamma'$ is any convergent sequence of curves in E, whose driving functions also converge uniformly on compacts, then for any $T > 0$, as $n \to \infty$,

$$\sup_{t \in [0,T]} \left| \tilde{\eta}_t(F, \gamma'_n, z) - \tilde{\eta}_t(F, \gamma', z) \right| \to 0 \tag{5.15}$$

and

$$\sup_{t \in [0,T]} \left| \tilde{W}_t(\gamma'_n, z) - \tilde{W}_t(\gamma', z) \right| \to 0. \tag{5.16}$$

Relative compactness then follows because the choice of E means that any sequence of curves in E has a convergent subsequence along which the driving functions also converge.

We will prove the above claim now. We let K_t (resp. K_t^n) be the hull generated by γ' (resp. γ_n') in the capacity parameterisation and W_t, g_t (resp. W_t^n, g_t^n) be the corresponding driving functions, and functions $\mathbb{H}\backslash K_t$ (resp. $\mathbb{H}\backslash K_t^n$) to $\mathbb{H}$, normalised at ∞. We define $f_t = g_t - W_t$ and $f_t^n = g_t^n - W_t^n$ as usual, and consider these to be extended to the boundary, also writing $f_t(0^+)$ for $V_t^R(0^+) - W_t$. Write

$$C_t = C_t(\gamma', z), \quad C_t^n = C_t(\gamma_n', z);$$

$$\tau(t) := \inf\{s \geq 0 : C_s = t\}, \quad \tau^n(t) := \inf\{s \geq 0 : C_s^n = t\}.$$

First, we will show that for any $T > 0$ before the first time that γ' swallows z, as $n \to \infty$,

$$\sup_{t \in [0,T]} |C_t - C_t^n| \to 0. \tag{5.17}$$

We have the following observations.

- By Lemma 5.23, and since $C_0 = C_0^n = 0$, we have

$$C_t = \int_0^t \frac{-4\Im(f_s(z))^2}{|f_s(z)|^4} ds; \quad C_t^n = \int_0^t \frac{-4\Im(f_s^n(z))^2}{|f_s^n(z)|^4} ds.$$

- $W_t^n \to W_t$ uniformly on $[0, T]$.
- $g_t^n \to g_t$ uniformly on $\{(t, z) \in [0, T] \times \overline{\mathbb{H}} : d(z, K_t) > \delta\}$ for any $\delta > 0$. (See for instance [KS16, Lemmas A.3 and A.4])

Combining these three facts, we obtain Equation (5.17).

Second, we show that, for any $T > 0$ before γ' swallows z, as $n \to \infty$,

$$\sup_{t \in [0,T]} \left|\eta_{\tau^n(t)}(F, \gamma', z) - \eta_{\tau(t)}(F, \gamma', z)\right| \to 0. \tag{5.18}$$

By Equation (5.17), we have that $[0, \tau^n(T) \vee \tau(T)] \subset [0, \tau(S)]$ for n large enough, where $S > T$, and $\tau(S)$ is a time before z is swallowed by γ'. By (5.17) again, we therefore have that, as $n \to \infty$

$$c_n := \sup_{t \in [0,T]} |C_{\tau^n(t)} - t| \leq \sup_{t \in [0, \tau^n(T) \vee \tau(T)]} |C_t - C_t^n| \to 0.$$

Since

$$\sup_{t \in [0,T]} \left|\eta_{\tau^n(t)}(F, \gamma', z) - \eta_{\tau(t)}(F, \gamma', z)\right| \leq \sup_{s,t \in [0,S], |s-t| \leq c_n} \left|\tilde{\eta}_t(F, \gamma', z) - \tilde{\eta}_s(F, \gamma', z)\right|,$$

and $\tilde{\eta}_t(F, \gamma', z)$ is uniformly continuous on $[0, S]$, we see that it must converge to 0.

Third, we show that, for any $T > 0$ before γ' swallows z, as $n \to \infty$,

$$\sup_{t \in [0,T]} \left| \eta_{\tau^n(t)}(F, \gamma'_n, z) - \eta_{\tau^n(t)}(F, \gamma', z) \right| \to 0. \tag{5.19}$$

We need only show that, on any time interval $[0, S]$ such that S is strictly less than the time γ' swallows z, the quantity $|\eta_t(F, \gamma'_n, z) - \eta_t(F, \gamma', z)|$ converges uniformly to 0. We have the following observations.

- By Definition 5.41, we know that $\eta_t(F, \gamma', \cdot)$ (resp. $\eta_t(F, \gamma'_n, \cdot)$) is the bounded harmonic function with boundary values equal F on $\mathbb{R} \setminus K_t$ (resp. on $\mathbb{R} \setminus K_t^n$), $-\lambda$ on the left side of K_t (resp. K_t^n), and λ on the right side of K_t (resp. K_t^n).
- $W_t^n \to W_t$ uniformly on $[0, S]$.
- $g_t^n \to g_t$ uniformly on $\{(t, z) \in [0, S] \times \overline{\mathbb{H}} : d(z, K_t) > \delta\}$ for any $\delta > 0$. Same reason as above.

Combining these three facts, we have that the quantity $|\eta_t(F, \gamma'_n, z) - \eta_t(F, \gamma', z)|$ converges uniformly to 0 on $t \in [0, S]$, implying Equation (5.19).

Combining Equations (5.17), (5.18) and (5.19), we obtain Equation (5.15) by noting that

$$\begin{aligned} \sup_{t \in [0,T]} \left| \eta_{\tau^n(t)}(F, \gamma'_n, z) - \eta_{\tau(t)}(F, \gamma', z) \right| &\leq \sup_{t \in [0,T]} \left| \eta_{\tau^n(t)}(F, \gamma'_n, z) - \eta_{\tau^n(t)}(F, \gamma', z) \right| \\ &\quad + \sup_{t \in [0,T]} \left| \eta_{\tau^n(t)}(F, \gamma', z) - \eta_{\tau(t)}(F, \gamma', z) \right|. \end{aligned} \tag{5.20}$$

We obtain Equation (5.16) by the same method as above, which is much simpler in this case, and so we omit the details.

Finally, we show that if $(\gamma^{(n_k)})_{k \in \mathbb{N}}$ converges weakly, and there exists a further subsequence along which $(\tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z))$ converges, then the limit must be $(\tilde{W}(\gamma, z), \tilde{\eta}(F, \gamma, z))$. To do this, for any $M \in \mathbb{N}$ take E relatively compact such that (5.14) holds, and note that by the above claim we have that

$$A_E := \left\{ (\gamma', \tilde{W}(\gamma', z), \tilde{\eta}(F, \gamma', z)) : \gamma' \in E \right\}$$

is relatively compact in $C([0, \infty); \mathbb{C}) \times C([0, \infty); \mathbb{R}) \times C([0, \infty); \mathbb{R})$, and its closure is equal to

$$\left\{ (\gamma', \tilde{W}(\gamma', z), \tilde{\eta}(F, \gamma', z)) : \gamma' \in \overline{E} \right\}.$$

This means that the joint laws of $(\gamma^{(n_k)}, \tilde{W}(\gamma^{(n_k)}, z), \tilde{\eta}(F, \gamma^{(n_k)}, z))$ are also tight, and thus we can extract an even further subsequence along which we have joint convergence. If $\mathbb{P}^*$ is the law of this joint limit then,

$$\mathbb{P}^*\left(\overline{A_E}\right) \geq \inf_k \mathbb{P}\left(\gamma^{(n_k)} \in \overline{E}\right) \geq 1 - \frac{1}{M}$$

and so we see that the probability of our marginal laws agreeing in the sense we want must be greater than $1 - \frac{1}{M}$. Since this holds for every M, agreement must hold almost surely, and as these marginal laws are equal to the limiting laws of the individually convergent sequences, the result follows. □

5.7 Proof of Theorems 5.3 to 5.5

Proof of Theorem 5.4. Suppose that γ_F and γ_G are continuous transient curves from 0 to ∞ in $\mathbb{H}$, coupled with a zero-boundary GFF h as level lines of $h+F$ and $h+G$ respectively. Suppose further that γ'_G is a continuous transient curve from ∞ to 0 and is coupled with h as a level line of $-h-G$ from ∞ to 0, such that the four objects $h, \gamma_F, \gamma_G, \gamma'_G$ are coupled with $\gamma_F, \gamma_G, \gamma'_G$ are conditionally independent given h. From Theorem 5.2, we have the existence of γ_F, γ_G and γ'_G. By Lemma 5.35, we know that γ_F stays to the left of γ_G almost surely. □

Proof of Theorems 5.3 and 5.5. Suppose that γ_F is a continuous transient curve which is coupled with h as a level line of $h+F$ from 0 to ∞, as in Theorem 5.2. Let γ'_F be a continuous curve coupled with h as a level line of $-h-F$ from ∞ to 0, such that γ_F and γ'_F are conditionally independent given h. The existence of γ'_F is given by Theorem 5.2. Lemma 5.34 then tells us that $\gamma_F = \gamma'_F$ almost surely. In particular, γ_F is almost surely determined by h. □

Remark 5.44. *By applying Theorem 5.3, we see that if γ is the weak limit of any sequence of level lines as in Proposition 5.39, then γ can be coupled as the level line of a* GFF *and is moreover determined by the* GFF *in this coupling. Thus, the law of γ is uniquely determined. In particular, it does not depend on the sequence of approximating level lines.*

Lemma 5.45. *Let F be as in Theorem 5.2. Suppose that $F_n \downarrow F$ approximate F uniformly on the real line, where the F_n are decreasing, and are piecewise constant with value changing only finitely many times.*

Let h be a zero boundary GFF *in $\mathbb{H}$, γ_n be the level line of $h+F_n$ for each n, and γ be the level line of $h+F$. Denote by H_n the open sets corresponding to the strict right hand sides of γ_n. By monotonicity these are almost surely decreasing.*

Define

$$H = \cap_n \overline{H_n}.$$

Then ∂H coincides with γ almost surely. In other words, the sequence of curves γ_n converges to γ almost surely.

Proof. First, we show that ∂H has the same law as γ_F. We use a conformal mapping to take everything to the unit disc, as it will be more convenient to work in a space where our sets are compact. We endow $\mathbb{H}$ with the metric it inherits from the unit disc $\mathbb{U}$ via the map $\varphi(z) = (z-i)/(z+i)$. Namely, let $d_*(\cdot,\cdot)$ denote the metric on $\mathbb{H}$ given by

$$d_*(z,w) = |\varphi(z) - \varphi(w)|.$$

We write $\overline{\mathbb{H}}$ for the completion of $\mathbb{H}$ with respect to d_*. For compact sets $A, B \subset \overline{\mathbb{H}}$, we have the d_*-induced Hausdorff distance

$$d_*^H(A,B) = \inf\{\varepsilon > 0 : A \subset B^{(\varepsilon)}, B \subset A^{(\varepsilon)}\},$$

where $A^{(\varepsilon)}$ denotes the open ε-neighborhood of A with respect to the metric d_*. Note that d_*^H makes the set of all non-empty compact subsets of $\overline{\mathbb{H}}$ (with metric d_*) into a compact metric space. We have the following observations.

- The sets $\overline{H_n}$ form an almost surely decreasing sequence of compact subsets of $\overline{\mathbb{H}}$, which therefore converge to H with respect to d_*^H. This implies that $\gamma_n = \partial H_n$ almost surely converges to ∂H with respect to d_*^H.
- By the assumptions on F_n, we know that the laws of γ_n fall in to the framework of Proposition 5.39. This means that we can extract a subsequence which converges weakly in the space of continuous functions on $[0,\infty)$ with respect to uniform convergence on compacts. Moreover, the limiting curve can be coupled with a zero boundary GFF $\tilde{h}$ as the level line of $\tilde{h} + F$. Furthermore, the subsequence converges weakly, to the same limit, in the space of curves from $[0,1] \to \overline{\mathbb{H}}$ with respect to the topology of uniform convergence modulo reparameterisation, where the metric on $\overline{\mathbb{H}}$ is given by d_*. This requires a slight extension of Proposition 5.12, which was stated here, but is nonetheless still true, by the extended version given in [KS16, Corollary 1.7]. By continuity, we therefore have that along this subsequence the curves also converge weakly to the same limit with respect to d_*^H. Thus ∂H has the law of a continuous curve which can be coupled with a zero boundary GFF $\tilde{h}$ as the level line of $\tilde{h} + F$.
- By Theorem 5.3, we know that the law on continuous curves which can be coupled with a GFF $\tilde{h}$ as a level line of $\tilde{h} + F$, is unique.

Combining these three facts, we may conclude that ∂H has the same law as γ_F.

Next, we show that ∂H coincides with γ almost surely. We have the following observations.

- By the above analysis, we know that ∂H has the same law as γ.
- By Theorem 5.4, we know that ∂H lies to the left of γ almost surely.

Combining these two facts, we obtain that ∂H coincides with γ almost surely. □

5.8 Proof of Theorem 5.6 and concluding remarks

In this section, we prove Theorem 5.6: the key ingredient being the proof of Lemma 5.46. This lemma is proved in [MS16a, WW16] for $\mathrm{SLE}_4(\rho)$ process when ρ is a vector. The proof given in these papers will work with minor modifications for the case when ρ is a Radon measure but, to be self-contained, we still give a complete proof here.

Lemma 5.46. *Suppose we are given a random continuous curve in $\overline{\mathbb{H}}$ from 0 to ∞ whose Loewner driving function W is almost surely continuous. If $(\rho^L;\rho^R)$ are a pair of finite Radon measures on $\mathbb{R}_-,\mathbb{R}_+$ and F is the corresponding function of bounded variation, define $(\eta_t, t\ge 0)$ as in Definition 5.1. For $z\in\mathbb{H}$ and $t\ge 0$, define*

$$\tau(t)=\inf\{s:\log\mathrm{CR}(z,\mathbb{H})-\log\mathrm{CR}(z,\mathbb{H}\setminus K_s)=t\}.$$

Then $\big(W,(V^L(x))_{x\in\mathbb{R}_-},(V^R(x))_{x\in\mathbb{R}_+}\big)$ can be coupled with a standard Brownian motion to describe an $\mathrm{SLE}_4(\rho^L;\rho^R)$ process if $(\eta_{\tau(t)}(z), t\ge 0)$ evolves as a Brownian motion with respect to the filtration generated by $(W_{\tau(t)}, t\ge 0)$ for any $z\in\mathbb{H}$.

Proof. Suppose that $(\eta_{\tau(t)}(z), t\ge 0)$ is a Brownian motion with respect to the filtration generated by $(W_{\tau(t)}, t\ge 0)$ for each $z\in\mathbb{H}$. This implies that $(\eta_t(z), t\ge 0)$ is a local martingale with respect to the filtration generated by $(W_t, t\ge 0)$. Our first step will be to show that W_t is a continuous semi-martingale. By the definition of $\eta_t(\cdot)$, we know that, for each $z\in\mathbb{H}$,

$$\begin{aligned}2\eta_t(z)=&-\int_{\mathbb{R}_-}\arg(g_t(z)-V_t^L(x))\,\rho^L(dx)-\arg(g_t(z)-W_t)\\&+(\pi-\arg(g_t(z)-W_t))+\int_{\mathbb{R}_+}\big(\pi-\arg(g_t(z)-V_t^R(x))\big)\,\rho^R(dx).\end{aligned}\qquad(5.21)$$

This follows from the integration by parts formula for functions of bounded variation, and the integral expression for the harmonic extension of a bounded function on the real line. Note here that the integrals are well defined, since for each fixed

t, z the integrands are continuous, bounded functions in x, and ρ^L, ρ^R are assumed to be finite measures. Indeed, $g_t(z)$ and $(V_t^{L,R}(x))_{x\in\mathbb{R}}$ are adapted and differentiable, and we may also differentiate under the integral in (5.21) by finiteness of ρ^L, ρ^R. Therefore, we can deduce that all the terms in (5.21) apart from the only one, $\arg(g_t(z) - W_t)$, involving W_t, are semi-martingales. Since $\eta_t(z)$ is itself a local martingale, this means that $\arg(g_t(z) - W_t)$ must also be a semi-martingale. Now, note that by Schwartz's formula, we can write $\log(g_t(z) - W_t)$, up to a constant, as a linear functional (an integral against a test function) of $\arg(g_t(z) - W_t)$. So $\log(g_t(z) - W_t)$ is also a semi-martingale, and thus it's exponential, and consequently W_t itself, must be a semi-martingale also. Hence we can write $W_t := M_t - V_t$ for M a local martingale and V of bounded variation.

Substituting this into the expression (5.21) we see that, on intervals where W_t does not collide with the $V_t^{L,R}$, the drift of $2\eta_t$ is equal to the imaginary part of

$$\frac{\int_{\mathbb{R}_-} \frac{2\rho^L(dx)}{V_t^L(x)-W_t}}{g_t(z) - W_t} dt + \frac{-2dV_t}{(g_t(z) - W_t)} + \frac{d\langle W_t\rangle - 4dt}{(g_t(z) - W_t)^2} + \frac{\int_{\mathbb{R}_+} \frac{2\rho^R(dx)}{V_t^R(x)-W_t}}{g_t(z) - W_t} dt,$$

which of course must vanish. Therefore, multiplying by $(g_t(z) - W_t)^2$ and evaluating at z such that $g_t(z) - W_t$ is arbitrarily close to 0, we can deduce that $d\langle W_t\rangle = 4dt$. On subsequently removing the third term, we also find an expression for dV_t, and can conclude that W_t satisfies (5.3) in Definition 5.20 on intervals where W_t does not collide with the $V_t^{L,R}$.

All that remains is to show that we have instantaneous reflection of W_t off the $V_t^{L,R}(x)$. It suffices to show that the number of times the curve γ hits the real line has Lebesgue measure 0. However, this is always the case for a continuous curve with continuous driving function, which we know for example by [MS16a, Lemma 2.5]. □

Remark 5.47. *We believe that Lemma 5.46 could be made into an if and only if statement if we strengthened Definition 5.20 of an $SLE_\kappa(\rho^L; \rho^R)$ process to also require that, almost surely,*

$$V_t^L(x) = x + \int_0^t \frac{2ds}{V_s^L(x) - W_s}, \quad x \in \mathbb{R}_-; \quad V_t^R(x) = x + \int_0^t \frac{2ds}{V_s^R(x) - W_s}, \quad x \in \mathbb{R}_+ \tag{5.22}$$

and

$$W_t = \sqrt{\kappa} B_t + \int_0^t ds \int_{\mathbb{R}_-} \frac{\rho^L(dx)}{W_s - V_s^L(x)} + \int_0^t ds \int_{\mathbb{R}_+} \frac{\rho^R(dx)}{W_s - V_s^R(x)}, \tag{5.23}$$

as in [WW16] and [MS16a].

That is, using the stronger definition, we could show that any such process can always be coupled with the Gaussian Free Field as generalized level line. This would also give us uniqueness in law for the $\mathrm{SLE}_4(\rho^L;\rho^R)$ *process among continuous curves. However, it seems that (5.22) and (5.23) are hard to verify assuming only that* $(\eta_{\tau(t)}(z), t \geq 0)$ *evolves as a Brownian motion.*

Proof of Theorem 5.6. Combining Theorem 5.2 with Lemmas 5.24 and 5.46 in the case that F is of bounded variation, we know that in the coupling (h, γ) given by Theorem 5.2, the marginal law of γ is that of an $\mathrm{SLE}_4(\rho^L;\rho^R)$ process. This gives us existence of the process. Moreover, we know the curve γ is almost surely continuous and transient and also satisfies the reversibility property (3) of Theorem 5.6, by Theorem 5.5. □

Remark 5.48. *We can also generalize the construction of flow lines and counterflow lines to* GFF *with general boundary data. A similar approximating idea works for flow lines and counterflow lines. Since the flow lines and counterflow lines have a duality property, instead of reversibility as for level line case, some extra work is neeeded for the proof of monotonicity as in Section 5.4. The details are left to interested readers.*

Remark 5.49. *As explained in the introduction, we restrict to boundary values satisfying Condition (5.2) throughout the paper. This condition guarantees that there is no continuation threshold. The continuity of the level lines when there does exist a continuation threshold is still open.*

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