Semidefinite relaxation of a class of quadratic integral inequalities

Giovanni Fantuzzi and Andrew Wynn

***Abstract*— We propose a novel technique to solve optimization problems subject to a class of integral inequalities whose integrand is quadratic and homogeneous with respect to the dependent variables, and affine in the parameters. We assume that the dependent variables are subject to homogeneous boundary conditions. Specifically, we derive rigorous relaxations of such integral inequalities in terms of semidefinite constraints, so a strictly feasible and near-optimal point for the original problem can be computed using semidefinite programming. Simple examples arising from the stability analysis of partial differential equations illustrate the potential of our method compared to existing techniques.**

I. INTRODUCTION

The study of spatially-extended systems governed by partial differential equations (PDEs) often leads to optimization problems constrained by infinite-dimensional inequalities involving quadratic forms of the system state $\boldsymbol{u}(t,\boldsymbol{x})$, where t denotes time and $\boldsymbol{x}$ the spatial coordinate (usually, $\boldsymbol{x} \in \mathbb{R}^3$ for physical systems). More specifically, such optimization problems take the form

$$\min_{\boldsymbol{\gamma}\in\Gamma} \quad \boldsymbol{c}^T\boldsymbol{\gamma} \tag{1}$$
$$\text{s.t.}\quad \mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} := \int_\Omega \partial^k\boldsymbol{u}^T\boldsymbol{F}(\boldsymbol{x};\boldsymbol{\gamma})\partial^k\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \geq 0 \quad \forall \boldsymbol{u}\in H,$$

where the vector of system parameters $\boldsymbol{\gamma}\in\Gamma\subset\mathbb{R}^s$ is the decision variable, $\boldsymbol{c}\in\mathbb{R}^s$ is the cost vector, $\partial^k\boldsymbol{u}$ is a $q(k+1)\times 1$ vector listing all partial derivatives of the function $\boldsymbol{u}:\Omega\subset\mathbb{R}^n\to\mathbb{R}^q$ up to and including order k, Ω is the spatial domain, $\boldsymbol{F}:\Omega\times\Gamma\to\mathbb{R}^{q(k+1)\times q(k+1)}$ defines the integrand, and H is a particular functional space (e.g. the space of k-times differentiable functions satisfying a set of boundary conditions).

Examples of problems that can be recast into form (1) include: establishing L^2 stability (also known as energy stability) for linear PDEs and PDEs with so-called "energy-preserving" nonlinearities [1]; studying input-output properties of systems governed by PDEs [2]; bounding space- and time-averaged properties of a class of nonlinear systems such as turbulent fluid flows (see e.g. [3], [4], [5], [6], [7], [8]).

It has recently been suggested that semidefinite programming and sum-of-squares (SOS) optimization can be applied to solve optimization problems in the form (1) efficiently. In [9], [10], integration by parts and algebraic identities are used to recast an integral inequality whose integrand is polynomial in both the dependent and independent variables as a differential matrix inequality, which after a SOS relaxation can be solved efficiently via a semidefinite program (SDP).

In the homogeneous quadratic case (1), an integral inequality can also be recast as a linear matrix inequality (LMI) using orthogonal series expansions, as demonstrated by the authors in [11], [8] for selected problem instances. In this work, we generalize the ideas of [11], [8] and propose a novel, robust approach to rigorously formulate a class of optimization problems of type (1) as SDPs. We consider functionals $\mathcal{F}$ over one-dimensional compact domains, that is we let $x\equiv\boldsymbol{x}\in\Omega\equiv[a,b]\subset\mathbb{R}$, and assume that the entries of the matrix $\boldsymbol{F}$ are polynomials of x whose coefficients are affine in $\boldsymbol{\gamma}$. Using Legendre series expansions, we show that an inner approximation to the feasible set of (1) can be described by LMIs, so a feasible point for (1) can be computed with an SDP. Moreover, although we do not provide a formal proof, the optimal solution of our SDP relaxation can be expected to converge to the optimal solution of (1) as the number of basis functions in the expansion is increased.

We remark that our approach resembles the SOS method of [9], [10] only in so far as we formulate an SDP, which can be derived in an algorithmic manner and independently of the specific problem data. However, the underlying ideas, the eventual SDPs, and the performance of the two methods are radically different. In particular, our examples show that despite its wider applicability, the SOS approach of [9], [10] may be too restrictive to achieve a good approximation for the optimal solution of (1); on the other hand, our approach allows us to compute a near-optimal solution.

Notation. Throughout this paper, $\mathcal{L}_n(x)$ denotes the Legendre polynomial of degree n,

$$\mathcal{L}_n(x) = \frac{1}{n!\,2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n. \tag{2}$$

Bold font indicates vectors and matrices; in particular, $\boldsymbol{0}$ denotes the zero matrix (the size should be clear from the context). The (block) entries of a matrix $\boldsymbol{Q}$ will be denoted by $\boldsymbol{Q}_{ij}$. The usual Euclidean norm of a vector $\boldsymbol{v}$ will be denoted by $\|\boldsymbol{v}\|$, while $\|\boldsymbol{Q}\|_F$ is the standard Frobenius norm of a matrix $\boldsymbol{Q}$. Moreover, $\mathrm{sym}(\boldsymbol{Q}) = (\boldsymbol{Q}+\boldsymbol{Q}^T)/2$ is the symmetric part of a square matrix $\boldsymbol{Q}$, and $\boldsymbol{Q}\succeq 0$ means that $\boldsymbol{Q}$ is positive semidefinite, i.e., $\boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v}\geq 0$ for any vector $\boldsymbol{v}$.

Given a function $u:\Omega\subset\mathbb{R}\to\mathbb{R}$ and $\alpha\in\mathbb{N}$, we write $\partial^\alpha u := \mathrm{d}^\alpha u/\mathrm{d}x^\alpha$. The space $C^k([-1,1],\mathbb{R}^q)$ is the usual space of k-times continuously differentiable functions with domain $[-1,1]$ and values in $\mathbb{R}^q$; we also write $C^k([-1,1])$ for $C^k([-1,1],\mathbb{R})$. Finally, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ denote the usual norms in the Lesbegue spaces $L^2(-1,1)$ and $L^\infty(-1,1)$, and $\langle\cdot,\cdot\rangle$ is the standard inner product in $L^2(-1,1)$.

The authors are with the Department of Aeronautics, Imperial College London, South Kensington Campus, SW7 2AZ, London, UK. E-mail: gf910@ic.ac.uk, a.wynn@imperial.ac.uk This work was supported in part by the Engineering and Physical Sciences Research Council project EP/J010537/1.

II. MOTIVATING EXAMPLE

Consider the Kuramoto–Sivashinsky equation [12], [13]

$$u_t + \gamma u_{xx} + u_{xxxx} + u u_x = 0, \tag{3}$$

where γ is the governing parameter and subscripts denote partial derivatives with respect to time t and space x. We study (3) on the interval $[-1, 1]$ with boundary conditions

$$\begin{aligned} u(-1) &= 0, & u(1) &= 0, \\ u_x(-1) &= u_x(1), & u_{xx}(-1) &= u_{xx}(1). \end{aligned} \tag{4}$$

The system has the equilibrium solution $u = 0$, which is L^2-stable at a given γ if the energy $\|u\|_2^2$ of any nonzero solution does not increase in time. Since the term γu_{xx} is destabilizing (it represents anti-diffusive effects), we are interested in the maximum γ for which L^2 stability occurs.

After computing $\mathrm{d}\|u\|_2^2/\mathrm{d}t = 2\int_{-1}^{1} u u_t \mathrm{d}x$ with the help of (3), integrating by parts using the boundary conditions, and dropping the 2 in front of the integral, we see that the maximum γ for L^2 stability is given by

$$\begin{aligned} &\max_{\gamma} \quad \gamma \\ &\text{s. t.} \quad \int_{-1}^{1} \begin{bmatrix} u \\ \partial u \\ \partial^2 u \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \partial u \\ \partial^2 u \end{bmatrix} \mathrm{d}x \geq 0, \end{aligned} \tag{5}$$

where the integral inequality must hold for all functions $u \in C^4([-1, 1])$ satisfying (4).

III. INNER SDP RELAXATIONS

A straightforward LMI approximation of the feasible region of (1) can be found if one replaces H with a finite-dimensional subspace of polynomials of degree N. With this assumption, the integral inequality can be integrated to obtain a finite-dimensional quadratic form for a finite set of unknown polynomial coefficients. Since the original inequality is affine in $\boldsymbol{\gamma}$, the non-negativity of such a quadratic form is (by definition) an LMI in $\boldsymbol{\gamma}$.

To derive an actual inner approximation of the feasible region of (1), rather than a simple approximation, we refine this simple idea by essentially keeping track of the truncation error between any function in H and its polynomial approximation of degree N.

A. Legendre Series Expansions

The key ingredient in our analysis is to expand all functions using Legendre polynomials. Given $u \in C^{k+1}([-1, 1])$, all derivatives $\partial^\alpha u$, $\alpha \in \{1, \ldots, k\}$ can in fact be represented by the uniformly convergent Legendre series

$$(\partial^\alpha u)(x) = \sum_{n=0}^{\infty} \hat{u}_n^\alpha \, \mathcal{L}_n(x), \tag{6}$$

where $\hat{u}_n^\alpha \in \mathbb{R}$ are the Legendre series coefficients [14]. Legendre expansions are convenient because the Legendre polynomials are orthogonal with respect to the uniform measure on $[-1, 1]$, which simplifies our analysis. Moreover, they take nonzero values at $x = \pm 1$, meaning that any boundary conditions on $\partial^\alpha u$ can be easily implemented.

Given $N \in \mathbb{N}$, we decompose (6) into a polynomial part plus a remainder function,

$$(\partial^\alpha u)(x) = \sum_{n=0}^{N+\alpha} \hat{u}_n^\alpha \, \mathcal{L}_n(x) + \tilde{u}_\alpha(x), \tag{7}$$

where the remainder function $\tilde{u}_\alpha$ is such that $\langle \tilde{u}_\alpha, \mathcal{L}_n \rangle = 0$ for $n \in \{0, \ldots, N+\alpha\}$. We also record the first $N+\alpha+1$ Legendre coefficients in the vector

$$\hat{\boldsymbol{u}}^\alpha = \left[\hat{u}_0^\alpha, \ldots, \hat{u}_{N+\alpha}^\alpha\right]^T \in \mathbb{R}^{N+\alpha+1}. \tag{8}$$

For $M \in \mathbb{N}$, we define the vector

$$\hat{\boldsymbol{u}} = \left[\hat{u}_0^k, \ldots, \hat{u}_{N+M+k}^k\right]^T \in \mathbb{R}^{N+M+k+1}, \tag{9}$$

and the function

$$U_k(x) := \sum_{n=N+M+k+1}^{\infty} \hat{u}_n^k \, \mathcal{L}_n(x), \tag{10}$$

so that $\partial^k u$ can be alternatively expressed as

$$\left(\partial^k u\right)(x) = \sum_{n=0}^{N+M+k} \hat{u}_n^k \, \mathcal{L}_n(x) + U_k(x). \tag{11}$$

Finally, we define a vector of boundary values

$$\hat{\boldsymbol{u}}_b = \left[u|_{-1}, \, \partial u|_{-1}, \ldots, \partial^{k-1} u|_{-1}, \, U_k|_{-1}, \, U_k|_1\right]^T \in \mathbb{R}^{k+2}. \tag{12}$$

All derivatives $\partial^\alpha u$ up to order k are uniquely described by the vectors $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{u}}_b$ and by the remainder functions $\tilde{u}_\alpha$, U_k according to the following result.

Lemma 1 (Integration): Let $u \in C^{k+1}([-1, 1])$ and $\alpha \in \{1, \ldots, k\}$. Then:

1) There exist matrices $\boldsymbol{B}_\alpha$ and $\boldsymbol{D}_\alpha$ such that $\hat{\boldsymbol{u}}^\alpha = \boldsymbol{B}_\alpha \hat{\boldsymbol{u}}_b + \boldsymbol{D}_\alpha \hat{\boldsymbol{u}}$.
2) If $s \leq N+M+\alpha$ and $r \geq k-\alpha$, there exists a matrix $\boldsymbol{D}_\alpha^{[r,s]}$ such that $[\hat{u}_r^\alpha, \ldots, \hat{u}_s^\alpha]^T = \boldsymbol{D}_\alpha^{[r,s]} \hat{\boldsymbol{u}}$.

Proof: See Appendix I. □

Moreover, the boundary values at $x = 1$ can be recovered according to the following result.

Lemma 2 (Boundary Conditions): If $u \in C^{k+1}([-1, 1])$, there exists a matrix $\boldsymbol{G}$ such that

$$\left[u|_{-1}, \, u|_1, \, \partial u|_{-1}, \, \ldots, \, \partial^k u|_1\right]^T = \boldsymbol{G} \begin{bmatrix} \hat{\boldsymbol{u}}_b \\ \hat{\boldsymbol{u}} \end{bmatrix}. \tag{13}$$

Proof: See Appendix II. □

B. Relaxation of a "minimal" quadratic integral functional

Let $\Omega \subset \mathbb{R}$ in (1) be a compact interval; without loss of generality, we consider $\Omega \equiv [-1, 1]$. Let $\boldsymbol{u}(x) = [u(x), v(x)]^T$ and define the space H in (1) as

$$H := \left\{ \boldsymbol{u} = [u, v]^T \in C^{k+1}(\Omega, \mathbb{R}^2) \, : \, \boldsymbol{A} \boldsymbol{u}_b = 0 \right\}, \tag{14}$$

where

$$\boldsymbol{u}_b := \left[u|_{-1}, \, u|_1, \, \partial u|_{-1}, \, \ldots, \, \partial^k u|_1, \, v|_{-1}, \, \ldots, \, \partial^k v|_1\right]^T \tag{15}$$

lists the boundary values of the components of $\boldsymbol{u}$ and their derivatives, and $\boldsymbol{A} \in \mathbb{R}^{m \times 4(k+1)}$ is a matrix of rank $m \leq 4(k+1)$ that defines m homogeneous boundary conditions.

Given $k \in \mathbb{N}$, $k \geq 1$ and $\alpha, \beta \leq k$, $\alpha + \beta < 2k$, we consider the "minimal" homogeneous quadratic functional

$$\begin{aligned}\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} := \int_{-1}^{1} \Big[f(\boldsymbol{\gamma}) \left(\partial^k u\right)^2 + 2h(\boldsymbol{\gamma})\partial^k u \, \partial^k v \\ + g(\boldsymbol{\gamma}) \left(\partial^k v\right)^2 + p(x;\boldsymbol{\gamma})\partial^\alpha u \, \partial^\beta v \Big] \mathrm{d}x, \quad (16)\end{aligned}$$

where f, g and h are affine functions of $\boldsymbol{\gamma}$ (but do *not* depend on x) and p is a polynomial of degree d_p whose coefficients are affine with respect to $\boldsymbol{\gamma}$, i.e.

$$p(x;\boldsymbol{\gamma}) = \sum_{i=0}^{d_p} \left(a_i + \boldsymbol{b}_i^T \boldsymbol{\gamma}\right) x^i, \quad a_i \in \mathbb{R}, \quad \boldsymbol{b}_i \in \mathbb{R}^s. \quad (17)$$

To begin the relaxation of the inequality $\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} \geq 0$, we choose the integers N and M such that $N \geq d_p + k - 1$ and $M \geq d_p + k$. Since $\boldsymbol{u} \in H \subset C^{k+1}([-1,1],\mathbb{R}^2)$, the functions $\partial^\alpha u$, $\partial^\beta v$, $\partial^k u$ and $\partial^k v$ in (16) can be decomposed as in Section III-A in terms of suitably defined vectors $\hat{\boldsymbol{u}}^\alpha$, $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{u}}_b$, $\hat{\boldsymbol{v}}^\beta$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{v}}_b$, and of the corresponding remainder functions $\tilde{u}_\alpha$, U_k, $\tilde{v}_\beta$, V_k. In particular, we choose to expand the terms $(\partial^k u)^2$, $(\partial^k v)^2$ and $\partial^k u \partial^k v$ using (11), while we expand the product $\partial^\alpha u \partial^\beta v$ using (7) (even when $\alpha = k$ or $\beta = k$). For notational convenience in the following, we define the vectors

$$\hat{\boldsymbol{w}} := \left[\hat{\boldsymbol{u}}_b^T, \, \hat{\boldsymbol{u}}^T, \, \hat{\boldsymbol{v}}_b^T, \, \hat{\boldsymbol{v}}^T\right]^T \in \mathbb{R}^{2(N+M+2k+3)}, \quad (18a)$$

$$\boldsymbol{W}_k := \left[U_k, \, V_k\right]^T. \quad (18b)$$

Substituting the Legendre series expansions into (16) and using the orthogonality of the Legendre polynomials we can write

$$\begin{aligned}\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \begin{bmatrix}\hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}}\end{bmatrix}^T \begin{bmatrix} f(\boldsymbol{\gamma})\boldsymbol{L} & h(\boldsymbol{\gamma})\boldsymbol{L} \\ h(\boldsymbol{\gamma})\boldsymbol{L} & g(\boldsymbol{\gamma})\boldsymbol{L}\end{bmatrix} \begin{bmatrix}\hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}}\end{bmatrix} \\ + \int_{-1}^{1} \boldsymbol{W}_k^T \boldsymbol{S}(\boldsymbol{\gamma}) \boldsymbol{W}_k \mathrm{d}x \\ + \mathcal{P}\{\boldsymbol{u};\boldsymbol{\gamma}\} + \mathcal{Q}\{\boldsymbol{u};\boldsymbol{\gamma}\} + \mathcal{R}\{\boldsymbol{u};\boldsymbol{\gamma}\},\end{aligned} \quad (19)$$

where $\boldsymbol{L}$ is defined as

$$\boldsymbol{L}_{ij} = \langle \mathcal{L}_i, \mathcal{L}_j \rangle = \frac{2\,\delta_{ij}}{2i+1}, \quad (20)$$

δ_{ij} is the usual Kronecker delta,

$$\boldsymbol{S}(\boldsymbol{\gamma}) := \begin{bmatrix} f(\boldsymbol{\gamma}) & h(\boldsymbol{\gamma}) \\ h(\boldsymbol{\gamma}) & g(\boldsymbol{\gamma})\end{bmatrix}, \quad (21)$$

and

$$\mathcal{P}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \sum_{n=0}^{N+\alpha} \sum_{m=0}^{N+\beta} \hat{u}_n^\alpha \hat{v}_m^\beta \int_{-1}^{1} p(x;\boldsymbol{\gamma}) \mathcal{L}_m \mathcal{L}_n \, \mathrm{d}x, \quad (22a)$$

$$\begin{aligned}\mathcal{Q}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \sum_{n=0}^{N+\alpha} \hat{u}_n^\alpha \int_{-1}^{1} p(x;\boldsymbol{\gamma}) \mathcal{L}_n \tilde{v}_\beta \, \mathrm{d}x \\ + \sum_{n=0}^{N+\beta} \hat{v}_n^\beta \int_{-1}^{1} p(x;\boldsymbol{\gamma}) \mathcal{L}_n \tilde{u}_\alpha \, \mathrm{d}x,\end{aligned} \quad (22b)$$

$$\mathcal{R}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \int_{-1}^{1} p(x;\boldsymbol{\gamma}) \tilde{u}_\alpha \tilde{v}_\beta \, \mathrm{d}x. \quad (22c)$$

The first two terms in (19) represent the contribution of terms involving derivatives of order k only, that were expanded with (11); $\mathcal{P}\{\boldsymbol{u};\boldsymbol{\gamma}\}$ accounts for the contribution of the first $N+\alpha$ and $N+\beta$ modes of $\partial^\alpha u$ and $\partial^\beta v$, respectively; $\mathcal{Q}\{\boldsymbol{u};\boldsymbol{\gamma}\}$ accounts for the coupling between the first Legendre modes and the truncation functions $\tilde{u}_\alpha$, $\tilde{v}_\beta$; finally, $\mathcal{R}\{\boldsymbol{u};\boldsymbol{\gamma}\}$ represent the contribution coming purely from $\tilde{u}_\alpha$ and $\tilde{v}_\beta$.

Equation (19) can be simplified using the following results.

Lemma 3: There exist matrices $\boldsymbol{P}_{ij}(\boldsymbol{\gamma})$, $i, j \in \{1, 2\}$, affine in $\boldsymbol{\gamma}$, such that

$$\mathcal{P}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \begin{bmatrix}\hat{\boldsymbol{u}}_b \\ \hat{\boldsymbol{u}}\end{bmatrix}^T \begin{bmatrix}\boldsymbol{P}_{11}(\boldsymbol{\gamma}) & \boldsymbol{P}_{12}(\boldsymbol{\gamma}) \\ \boldsymbol{P}_{21}(\boldsymbol{\gamma}) & \boldsymbol{P}_{22}(\boldsymbol{\gamma})\end{bmatrix} \begin{bmatrix}\hat{\boldsymbol{v}}_b \\ \hat{\boldsymbol{v}}\end{bmatrix}. \quad (23)$$

Proof: See Appendix III. □

Lemma 4: There exists a matrix $\boldsymbol{Q}(\boldsymbol{\gamma})$, affine in $\boldsymbol{\gamma}$, such that $\mathcal{Q}\{\boldsymbol{u};\boldsymbol{\gamma}\} = \hat{\boldsymbol{u}}^T \boldsymbol{Q}(\boldsymbol{\gamma}) \hat{\boldsymbol{v}}$.

Proof: See Appendix IV. □

Lemma 5: There exist positive semidefinite matrices $\boldsymbol{R}$ and $\boldsymbol{\Sigma}$, with $\|\boldsymbol{R}\|_F \sim N^{\alpha+\beta-2k-1}$ and $\|\boldsymbol{\Sigma}\|_F \sim N^{\alpha+\beta-2k}$, such that

$$\begin{aligned}|\mathcal{R}\{\boldsymbol{u};\boldsymbol{\gamma}\}| \leq \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \hat{\boldsymbol{w}}^T \boldsymbol{R} \hat{\boldsymbol{w}} \\ + \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \int_{-1}^{1} \boldsymbol{W}_k^T \boldsymbol{\Sigma} \boldsymbol{W}_k \mathrm{d}x.\end{aligned} \quad (24)$$

Proof: The full proof is omitted for space reasons. Briefly, the result combines Hölder's and Young's inequalities with simple (but lengthy) bounds for $\|\tilde{u}_\alpha\|_2^2$ and $\|\tilde{v}_\beta\|_2^2$ in terms of $\|U_k\|_2^2$ and $\|V_k\|_2^2$. These rely on the integration formulae for Legendre expansions derived in Appendix I. □

For any $(k+1)$-times continuously-differentiable functions u and v, we therefore obtain the rigorous lower bound

$$\begin{aligned}\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} \geq \hat{\boldsymbol{w}}^T \left[\boldsymbol{M}(\boldsymbol{\gamma}) - \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \boldsymbol{R}\right] \hat{\boldsymbol{w}} \\ + \int_{-1}^{1} \boldsymbol{W}_k^T \left[\boldsymbol{S}(\boldsymbol{\gamma}) - \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \boldsymbol{\Sigma}\right] \boldsymbol{W}_k \mathrm{d}x,\end{aligned} \quad (25)$$

where $\boldsymbol{M}(\boldsymbol{\gamma})$ is defined as in (26) at the bottom of the page.

$$\boldsymbol{M}(\boldsymbol{\gamma}) := \mathrm{sym}\left(\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{P}_{11}(\boldsymbol{\gamma}) & \boldsymbol{P}_{12}(\boldsymbol{\gamma}) \\ \boldsymbol{0} & f(\boldsymbol{\gamma})\boldsymbol{L} & \boldsymbol{P}_{21}(\boldsymbol{\gamma}) & h(\boldsymbol{\gamma})\boldsymbol{L} + \boldsymbol{P}_{22}(\boldsymbol{\gamma}) + \boldsymbol{Q}(\boldsymbol{\gamma}) \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & h(\boldsymbol{\gamma})\boldsymbol{L} & \boldsymbol{0} & g(\boldsymbol{\gamma})\boldsymbol{L}\end{bmatrix}\right). \quad (26)$$

However, we are only interested in functions u and v that satisfy the m boundary conditions $\boldsymbol{A}\boldsymbol{u}_b = 0$. According to Lemma 2, this means that we should only consider vectors $\hat{\boldsymbol{w}}$ satisfying

$$\boldsymbol{A}'\hat{\boldsymbol{w}} = 0, \qquad \boldsymbol{A}' := \boldsymbol{A}\begin{bmatrix}\boldsymbol{G} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{G}\end{bmatrix}. \tag{27}$$

It may be verified that since $m \le 4(k+1) \le 2(N+M+2k+3)$, $\mathrm{rank}(\boldsymbol{A}') = \mathrm{rank}(\boldsymbol{A}) = m$, so there exists a matrix $\boldsymbol{\Pi} \in \mathbb{R}^{2(N+M+2k+3)\times 2(N+M+2k+3)-m}$ that spans the null space of $\boldsymbol{A}'$. Any admissible $\hat{\boldsymbol{w}}$ can then be expressed as

$$\hat{\boldsymbol{w}} = \boldsymbol{\Pi}\,\hat{\boldsymbol{z}} \tag{28}$$

for some $\hat{\boldsymbol{z}} \in \mathbb{R}^{2(N+M+2k+3)-m}$. After substituting (28) into (25) it is not difficult to deduce the following result.

Theorem 1: Sufficient conditions for $\boldsymbol{\gamma} \in \mathbb{R}^s$ to satisfy the integral inequality $\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} \ge 0$ for all $\boldsymbol{u} \in H$ are that

$$\boldsymbol{\Pi}^T\left[\boldsymbol{M}(\boldsymbol{\gamma}) - \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \boldsymbol{R}\right]\boldsymbol{\Pi} \succeq 0, \tag{29a}$$

$$\boldsymbol{S}(\boldsymbol{\gamma}) - \|p(\cdot\,;\boldsymbol{\gamma})\|_\infty \boldsymbol{\Sigma} \succeq 0. \tag{29b}$$

Remark 1: The semidefinite constraints in (29) are not LMIs due to the appearance of $\|p(\cdot\,;\boldsymbol{\gamma})\|_\infty$, but can be readily recast as LMIs. In fact, if $\hat{p}_n(\boldsymbol{\gamma})$ is the n^{th} coefficient of $p(x;\boldsymbol{\gamma})$ when expressed in the Legendre basis (cf. Appendix III), we can use the fact that $\|\mathcal{L}_n\|_\infty = 1$ to estimate

$$\|p(\cdot\,;\boldsymbol{\gamma})\|_\infty = \left\|\sum_{n=0}^{d_p}\hat{p}_n(\boldsymbol{\gamma})\mathcal{L}_n(x)\right\|_\infty \le \sum_{n=0}^{d_p}|\hat{p}_n(\boldsymbol{\gamma})|. \tag{30}$$

If we introduce a vector $\boldsymbol{t} = [t_0, \dots, t_{d_p}]^T$ of slack variables and the additional inequality constraints $|\hat{p}_n(\boldsymbol{\gamma})| \le t_n$ for all $n \in \{0, \dots, d_p\}$, we can further strengthen (29) into two LMIs for the variables $\boldsymbol{\gamma}$ and $\boldsymbol{t}$. Consequently, a feasible (and near-optimal, see Remark 2 below) $\boldsymbol{\gamma}$ for (1) can be computed using semidefinite programming.

Remark 2: The only relaxation was introduced by estimating the functional $\mathcal{R}$ as in Lemma 5. In fact, (29b) is necessary to make the right-hand side of (25) non-negative when one chooses $\hat{\boldsymbol{z}} = 0$ in (28) (this follows from an extension of [15, Theorem 6]). Moreover, an argument similar to that in [8, Appendix D] can be used to show that $\boldsymbol{W}_k$ can be chosen to make the integral on the right-hand side of (25) arbitrarily small, proving the necessity of (29a). Now, the estimates in Lemma 5 imply that the magnitude of $\mathcal{R}$ becomes vanishingly small as the number N of Legendre modes considered in the analysis is increased. Consequently, the relaxation is mild when N is large, and we expect that in practice our LMI relaxation is an accurate inner approximation of the original inequality $\mathcal{F}\{\boldsymbol{u};\boldsymbol{\gamma}\} \ge 0$ when a sufficiently large N is chosen.

IV. EXTENSIONS

Our analysis of the "minimal" functional (16) presented in the previous section can be extended to the more general case in which the integrand is a complete homogeneous quadratic polynomial of the components of $\boldsymbol{u} : [-1,1] \to \mathbb{R}^q$ and their derivatives, as long as the following assumption holds.

Assumption 1: Let k_i, $i = 1, \dots, q$ be the highest order derivative of the component u_i of $\boldsymbol{u}$ appearing in $\mathcal{F}$. The entries of the integrand matrix $\boldsymbol{F}(x;\boldsymbol{\gamma})$ in (1) corresponding to the quadratic terms $\partial^{k_i}u_i\partial^{k_j}u_j$ are independent of x.

This assumption restricts the class of integral inequalities for which a rigorous SDP relaxation can be derived; when it holds, all our proofs apply verbatim after replacing d_p with the degree of the entry of $\boldsymbol{F}(x;\boldsymbol{\gamma})$ of highest degree.

Removing Assumption 1 requires addressing the following two problems: first, the matrix $\boldsymbol{S}$ in (29b) becomes x-dependent; second, the Legendre modes of the highest order derivatives $\partial^{k_i}u_i$ are coupled, and one cannot separate the first $N+M+k$ Legendre coefficients from the remainder functions without further estimates. While the former difficulty can be resolved with a further SOS relaxation, the latter is more challenging, because the magnitude of the required estimates is commensurate to other terms even when N is large, and so might make our SDP relaxation infeasible.

When Assumption 1 does not hold, the methods of [9], [10] can be applied. Alternatively, an LMI approximation to the feasible set of problem (1) can be derived by simply truncating the Legendre series expansions, as described at the beginning of Section III. Although the optimal solution $\boldsymbol{\gamma}^\star$ of such an SDP approximation cannot be guaranteed to be feasible for (1), it can be considered a good approximation to the optimal solution of (1) if a sufficiently large N is chosen.

V. EXAMPLES

A. L^2 stability of the Kuramoto–Sivashinsky equation

Our SDP relaxations can be used to solve the optimization problem (5) and determine the maximum value of the parameter γ for which the equilibrium solution of the Kuramoto–Sivashinsky equation is L^2-stable (cf. Section II). Our implementation is in MATLAB®, using the parser YALMIP [16] and the SDP solver SeDuMi [17].

Table I reports the optimal value $\gamma^\star$ computed numerically for increasing values of the number N of Legendre coefficients used in our expansions, alongside the percentage error with the analytical solution $\gamma_{\text{exact}} = \pi^2$ (this can be computed using the calculus of variations, see e.g. [18]). In our computations, we chose the minimum value $M = d_p + k = 2$. The numerical solution is essentially exact when using as little as 6 Legendre coefficients. In this case (29a) and (29b) become an 8×8 and a 1×1 LMI, respectively, with 2 variables: the constant γ and the slack variable corresponding to its absolute value.

For comparison, we also solved (5) with the SOS approach of [9], [10] using polynomials of degree up to 16. In

TABLE I

OPTIMIZATION RESULTS FOR PROBLEM (5).

N	$\gamma^\star/\pi^2$	% error with γ_{exact}
2	0.9923109944	7.6890e-03
4	0.9999654586	3.4541e-05
6	0.9999999532	4.6825e-08
8	0.9999999999	9.2157e-11

contrast to the method presented in the current paper, the optimal value obtained with the SOS relaxations was only $\gamma^\star \approx 4.11 \approx 0.41\gamma_{\text{exact}}$ when polynomial of degree 10 were used, and improved by less than 0.01% for higher polynomial degrees. Moreover, the SOS relaxation of (5) using polynomials of degree 10 has 166 variables — a much larger problem than that obtained using the methods proposed in this work.

B. A modified Kuramoto–Sivashinsky equation

Consider (3) with $\gamma = \gamma(x) = \gamma_0 + \pi^2 x^2$, where γ_0 is the governing parameter, subject to the boundary conditions (4). Steps similar to Section II show that the maximum γ_0 for L^2 stability of the equilibrium solution $u = 0$ is given by

$$\begin{aligned} &\max_{\gamma_0} \quad \gamma_0 \\ &\text{s. t.} \quad \int_{-1}^{1} \begin{bmatrix} u \\ \partial u \\ \partial^2 u \end{bmatrix}^T \begin{bmatrix} 0 & -\partial\gamma(x) & 0 \\ 0 & -\gamma(x) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ \partial u \\ \partial^2 u \end{bmatrix} \mathrm{d}x \geq 0. \end{aligned} \tag{31}$$

Our numerical results are reported in Table II, and converge to $\gamma_0^\star \approx 6.90$. In contrast, solving the SOS relaxation of [9], [10] with polynomials of degree 10 gives $\gamma_0^\star \approx 3.02$ (using polynomials of higher degree improves this result by less than 0.01%); this is only approximately 43% of the optimal value obtained with our method.

TABLE II
OPTIMIZATION RESULTS FOR PROBLEM (31).

N	$\gamma^\star$
4	6.8874416462
8	6.9028539575
12	6.9028560655
16	6.9028560663

VI. CONCLUSION

To summarize, we have developed a framework based on Legendre series expansions to derive rigorous LMI representations of a class of homogeneous quadratic integral inequalities. A strictly feasible and near-optimal solution to the optimization problems subject to this type of constraints can then be computed efficiently using semidefinite programming techniques. Although our approach is not as general as the the SOS approach of [9], [10], many problems of interest fall within the class we have considered (e.g. [3], [4], [5], [6], [7], [8]). Moreover, our illustrative results regarding the L^2 stability of the Kuramoto–Sivashinsky equation demonstrate that our methods may give better results than the SOS approach of [9], [10] at a smaller computational cost. Extensions of the present techniques to more general types of functional inequalities, in particular those that do not satisfy Assumption 1, are in the interest of future work.

APPENDIX I
PROOF OF LEMMA 1

The statement is trivial when $\alpha = k$. Moreover, since $u \in C^{k+1}$, the Legendre expansions of all derivatives $\partial^\alpha u$, $0 \leq \alpha \leq k$ converge uniformly [14]. Consequently, we can use the fundamental theorem of calculus for each $\alpha \leq k-1$ to write

$$\begin{aligned} (\partial^\alpha u)(x) &= \partial^\alpha u|_{-1} + \int_{-1}^{x} \partial^{\alpha+1} u(t)\, dt \\ &= \partial^\alpha u|_{-1} + \sum_{n\geq 0} \hat{u}_n^{\alpha+1} \int_{-1}^{x} \mathcal{L}_n(t)\, dt. \end{aligned} \tag{32}$$

The last expression can be integrated recalling that $\mathcal{L}_0(x) = 1$, $\mathcal{L}_1(x) = x$, $\mathcal{L}_n(\pm 1) = (\pm 1)^n$ and using the recurrence relation for Legendre polynomials

$$(2n+1)\mathcal{L}_n(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{L}_{n+1}(x) - \mathcal{L}_{n-1}(x)\right], \tag{33}$$

which holds for $n \geq 1$ [19]. We can then rewrite (32) as

$$\begin{aligned} \partial^\alpha u(x) = \partial^\alpha u|_{-1} &+ \left[\mathcal{L}_1(x) + \mathcal{L}_0(x)\right] \hat{u}_0^{\alpha+1} \\ &+ \sum_{n\geq 1} \left[\mathcal{L}_{n+1}(x) - \mathcal{L}_{n-1}(x)\right] \frac{\hat{u}_n^{\alpha+1}}{2n+1}. \end{aligned} \tag{34}$$

Rearranging the series and comparing coefficients with the Legendre expansion of $\partial^\alpha u$ shows that

$$\hat{u}_0^\alpha = \partial^\alpha u|_{-1} + \hat{u}_0^{\alpha+1} - \frac{1}{3}\hat{u}_1^{\alpha+1}, \tag{35a}$$

$$\hat{u}_n^\alpha = \frac{\hat{u}_{n-1}^{\alpha+1}}{2n-1} - \frac{\hat{u}_{n+1}^{\alpha+1}}{2n+3}, \qquad n \geq 1. \tag{35b}$$

These relations can be applied recursively to construct the matrices $\boldsymbol{B}_\alpha$, $\boldsymbol{D}_\alpha$ and $\boldsymbol{D}_\alpha^{[r,s]}$.

APPENDIX II
PROOF OF LEMMA 2

We only need to show that the boundary values of all derivatives $\partial^\alpha u$ at $x = 1$ can be recovered from the vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}_b$. Applying the fundamental theorem of calculus as in the proof of Lemma 1 it may be shown that

$$\partial^\alpha u|_1 = \partial^\alpha u|_{-1} + 2\hat{u}_0^{\alpha+1} \tag{36}$$

for all $\alpha \in \{0, \ldots, k-1\}$. Applying Lemma 1 to the second term, we see that it is possible to express $\partial^\alpha u|_1$ as a linear combination of the vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}_b$. Moreover, since $\mathcal{L}_n(\pm 1) = (\pm 1)^n$ we have

$$\partial^k u|_{\pm 1} = \sum_{n=0}^{N+M+k} (\pm 1)^n \hat{u}_n^k + U_k(\pm 1), \tag{37}$$

so the boundary values of $\partial^k u$ can also be written as a linear combination of the vectors $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}_b$. Using these identities we can construct the matrix $\boldsymbol{G}$ of the Lemma.

APPENDIX III
PROOF OF LEMMA 3

The polynomial $p(x; \boldsymbol{\gamma})$, of degree d_p, can be represented in terms of Legendre polynomials as

$$p(x; \boldsymbol{\gamma}) = \sum_{l=0}^{d_p} \hat{p}_l(\boldsymbol{\gamma}) \mathcal{L}_l(x), \tag{38}$$

where the coefficients $\hat{p}_l$ are affine with respect to $\boldsymbol{\gamma}$ — a straightforward consequence of (17). Consequently, we may write

$$\mathcal{P}\left\{\boldsymbol{u};\boldsymbol{\gamma}\right\} = \sum_{l=0}^{d_p}\sum_{n=0}^{N+\alpha}\sum_{m=0}^{N+\beta} \hat{p}_l(\boldsymbol{\gamma})\hat{u}_n^\alpha\,\hat{v}_m^\beta \int_{-1}^{1} \mathcal{L}_l\mathcal{L}_n\mathcal{L}_n \mathrm{d}x. \tag{39}$$

Using the results of [20] to compute the integral on the right-hand side, we can construct a family of matrices $\boldsymbol{X}_l$, $l \in \{0, \ldots, d_p\}$, such that

$$(\boldsymbol{X}_l)_{nm} = \int_{-1}^{1} \mathcal{L}_l\mathcal{L}_m\mathcal{L}_n \mathrm{d}x. \tag{40}$$

Part 1 of Lemma 1 then allows us to write

$$\begin{aligned}\mathcal{P}\left\{\boldsymbol{u};\boldsymbol{\gamma}\right\} &= \sum_{l=0}^{d_p} \hat{p}_l(\boldsymbol{\gamma})\hat{\boldsymbol{u}}^\alpha \boldsymbol{X}_l \hat{\boldsymbol{v}}^\beta \\ &= \sum_{l=0}^{d_p} \hat{p}_l(\boldsymbol{\gamma}) \begin{bmatrix}\hat{\boldsymbol{u}}_b \\ \hat{\boldsymbol{u}}\end{bmatrix}^T \begin{bmatrix}\boldsymbol{B}_\alpha^T \\ \boldsymbol{D}_\alpha^T\end{bmatrix} \boldsymbol{X}_l \begin{bmatrix}\boldsymbol{B}_\beta & \boldsymbol{D}_\beta\end{bmatrix} \begin{bmatrix}\hat{\boldsymbol{v}}_b \\ \hat{\boldsymbol{v}}\end{bmatrix} \\ &=: \begin{bmatrix}\hat{\boldsymbol{u}}_b \\ \hat{\boldsymbol{u}}\end{bmatrix}^T \begin{bmatrix}\boldsymbol{P}_{11}(\boldsymbol{\gamma}) & \boldsymbol{P}_{12}(\boldsymbol{\gamma}) \\ \boldsymbol{P}_{21}(\boldsymbol{\gamma}) & \boldsymbol{P}_{22}(\boldsymbol{\gamma})\end{bmatrix} \begin{bmatrix}\hat{\boldsymbol{v}}_b \\ \hat{\boldsymbol{v}}\end{bmatrix}. \end{aligned}\tag{41}$$

Since the coefficients $\hat{p}_l(\boldsymbol{\gamma})$ are affine in $\boldsymbol{\gamma}$, so are the matrices $\boldsymbol{P}_{ij}$. Note that generally $\boldsymbol{P}_{12} \neq \boldsymbol{P}_{21}^T$.

APPENDIX IV
PROOF OF LEMMA 4

The functions $\tilde{u}_\alpha$ and $\tilde{v}_\beta$ can be represented by the Legendre expansions

$$\tilde{u}_\alpha(x) = \sum_{m=N+\alpha+1}^{+\infty} \hat{u}_m^\alpha \mathcal{L}_m(x), \tag{42a}$$

$$\tilde{v}_\beta(x) = \sum_{m=N+\beta+1}^{+\infty} \hat{v}_m^\beta \mathcal{L}_m(x). \tag{42b}$$

Substituting these expansions in (22b), expanding p as in (38) and recalling from [20] that

$$\int_{-1}^{1} \mathcal{L}_l\mathcal{L}_n\mathcal{L}_m \mathrm{d}x \neq 0 \quad \Leftrightarrow \quad l+n-m \geq 0, \tag{43}$$

we can write

$$\begin{aligned}\mathcal{Q}\left\{\boldsymbol{u};\boldsymbol{\gamma}\right\} &= \sum_{l=0}^{d_p} \hat{p}_l(\boldsymbol{\gamma}) \begin{bmatrix}\hat{u}^\alpha_{N+\beta+1-d_p} \\ \vdots \\ \hat{u}^\alpha_{N+\alpha}\end{bmatrix}^T \boldsymbol{Y}_l^{\alpha,\beta} \begin{bmatrix}\hat{v}^\beta_{N+\beta+1} \\ \vdots \\ \hat{v}^\beta_{N+\alpha+d_p}\end{bmatrix} \\ &+ \sum_{l=0}^{d_p} \hat{p}_l(\boldsymbol{\gamma}) \begin{bmatrix}\hat{v}^\beta_{N+\alpha+1-d_p} \\ \vdots \\ \hat{v}^\alpha_{N+\beta}\end{bmatrix}^T \boldsymbol{Y}_l^{\beta,\alpha} \begin{bmatrix}\hat{u}^\alpha_{N+\alpha+1} \\ \vdots \\ \hat{u}^\alpha_{N+\beta+d_p}\end{bmatrix}. \end{aligned}\tag{44}$$

In the last expression, the matrices $\boldsymbol{Y}_l^{\alpha,\beta}$ and $\boldsymbol{Y}_l^{\beta,\alpha}$ contain the integrals $\int_{-1}^{1} \mathcal{L}_l\mathcal{L}_n\mathcal{L}_m \mathrm{d}x$. Note that we have assumed that α, β and d_p are such that

$$1 - d_p \leq \alpha - \beta \leq d_p - 1, \tag{45}$$

so that the vectors in (44) are well-defined. If the left (respectively, the right) inequality is not satisfied, then the first (respectively, the second) term in (44) vanishes.

Our assumptions that $N \geq d_p + k - 1$ and $M \geq d_p + k$ guarantee that we can apply part 2 of Lemma 1 and write the vectors of Legendre coefficients in (44) in terms of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ only. This means that it is possible to find a matrix $\boldsymbol{Q}(\boldsymbol{\gamma})$ such that

$$\mathcal{Q}\left\{\boldsymbol{u};\boldsymbol{\gamma}\right\} = \hat{\boldsymbol{u}}^T \boldsymbol{Q}(\boldsymbol{\gamma})\hat{\boldsymbol{v}}. \tag{46}$$

Note that the entries of $\boldsymbol{Q}(\boldsymbol{\gamma})$ are affine with respect to $\boldsymbol{\gamma}$ because they are linear combinations of the Legendre coefficients $\hat{p}_l(\boldsymbol{\gamma})$, $l \in \{0, \ldots, d_p\}$.

REFERENCES

[1] B. Straughan, *The Energy Method, Stability, and Nonlinear Convection*, 2nd ed., ser. Applied Mathematical Sciences. Springer-Verlag New York, 2004.

[2] M. Ahmadi, G. Valmorbida, and A. Papachristodoulou, "Dissipation inequalities for the analysis of a class of PDEs," *Automatica*, vol. 66, pp. 163–171, 2016.

[3] C. R. Doering and P. Constantin, "Energy dissipation in shear driven turbulence," *Phys. Rev. Lett.*, vol. 69, no. 11, pp. 1648–1651, 1992.

[4] ——, "Variational bounds on energy dissipation in incompressible flows: Shear flow," *Phys. Rev. E*, vol. 49, no. 5, pp. 4087–4099, 1994.

[5] P. Constantin and C. R. Doering, "Variational bounds on energy dissipation in incompressible flows. II. Channel flow," *Phys. Rev. E*, vol. 51, no. 4, pp. 3192–3198, 1995.

[6] C. R. Doering and P. Constantin, "Variational bounds on energy dissipation in incompressible flows. III. Convection," *Phys. Rev. E*, vol. 53, no. 6, pp. 5957–5981, 1996.

[7] G. I. Hagstrom and C. R. Doering, "Bounds on Surface Stress-Driven Shear Flow," *J. Nonlinear Sci.*, vol. 24, no. 1, pp. 185–199, 2014.

[8] G. Fantuzzi and A. Wynn, "Optimal bounds with semidefinite programming: An application to stress driven shear flows," *Phys. Rev. E*, vol. 93, no. 4, p. 043308, 2016.

[9] G. Valmorbida, M. Ahmadi, and A. Papachristodoulou, "Stability Analysis for a Class of Partial Differential Equations via Semidefinite Programming," *IEEE Trans. Automat. Contr.*, vol. 61, no. 6, pp. 1649–1654, 2016.

[10] G. Valmorbida and A. Papachristodoulou, "Introducing INTSOSTOOLS : A SOSTOOLS plug-in for integral inequalities," in *Proc. 2015 Eur. Control Conf.*, Linz, Austria, 2015, pp. 1231–1236.

[11] G. Fantuzzi and A. Wynn, "Construction of an optimal background profile for the Kuramoto–Sivashinsky equation using semidefinite programming," *Phys. Lett. A*, vol. 379, no. 1-2, pp. 23–32, 2015.

[12] Y. Kuramoto and T. Tsuzuki, "Persistent propagation of concentration waves in dissipative media far from thermal equilibrium," *Prog. Theor. Phys.*, vol. 55, no. 2, pp. 356–369, 1976.

[13] G. I. Sivashinsky, "On Flame Propagation Under Conditions Of Stoichiometry," *J. Appl. Math.*, vol. 39, no. 1, pp. 67–82, 1980.

[14] D. Jackson, *The Theory of Approximation*, ser. Am. Math. Soc. Colloq. Publ. vol. 11. New York: American Mathematical Society, 1930.

[15] A. Papachristodoulou and M. M. Peet, "On the Analysis of Systems Described by Classes of Partial Differential Equations," San Diego, CA, USA, 2006, pp. 747–752.

[16] J. Lofberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *IEEE Int. Symp. Comput. Aided Control Syst. Des.*, Taipei, Taiwan, 2004, pp. 284 – 289.

[17] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optim. methods Softw.*, vol. 11, no. 1-4, pp. 625–653, 1999.

[18] M. Giaquinta and S. Hildebrandt, *Calculus of Variations I*, ser. Grundlehren der mathematischen Wissenschaften. Springer-Verlag Berlin Heidelberg, 1996, vol. 310.

[19] R. P. Agarwal and D. O'Regan, *Ordinary and Partial Differential Equations, With Special Functions, Fourier Series and Boundary Value Problems*, ser. Universitext. Springer-Verlag New York, 2009.

[20] J. Dougall, "The product of two Legendre polynomials," *Proc. Glas. Math. Assoc.*, vol. 1, no. 3, pp. 121–125, 1953.