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Game characterizations and lower cones in the Weihrauch degrees (Extended Abstract*)

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Abstract. We introduce generalized Wadge games and show that each lower cone in the Weihrauch degrees is characterized by such a game. These generalized Wadge game subsume the original Wadge games, the eraser and backtrack games and well as the variants of Semmes' tree games. In particular, we propose that the lower cones in the Weihrauch degrees are the answer to Andretta's question on which classes of functions admit game characterizations. We then discuss some applications of such generalized Wadge games.

1 Introduction

The use of infinite games in set theory has a well-established tradition, going back to work by Banach, Borel, Zermelo, Kőnig, and others (see [16, §27] for a thorough historical account of the subject), and taking a prominent role in the field with the work of Gale and Stewart on the determinacy of certain types of such games [12].

In this paper, we will focus on infinite games which have been used to characterize classes of functions in descriptive set theory. Interest in this particular area began with the seminal work of Wadge [40], who introduced what is now known as the *Wadge game*, an infinite game in which two players, **I** and **II**, are given a partial function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ and play with perfect information. In each run of this game, at each round player **I** first picks a natural number and player **II** responds by either picking a natural number or passing, although she must pick natural numbers at infinitely many rounds. Thus, in the long run **I** and

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II build elements $x \in \mathbb{N}^\mathbb{N}$ and $y \in \mathbb{N}^\mathbb{N}$, respectively, and **II** wins the run if and only if $x \notin \mathrm{dom}(f)$ or $f(x) = y$. Wadge proved that this game *characterizes* the continuous functions, in the following sense.

Theorem 1 (Wadge). *A partial function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is relatively continuous iff player* **II** *has a winning strategy in the Wadge game for f.*

By adding new possibilities for player **II** at each round, one can obtain games characterizing larger classes of functions. For example, in the *eraser game* [11] characterizing the Baire class 1 functions, player **II** is allowed to erase past moves, the rule being that she is only allowed to erase each position of her output sequence finitely often. In the *backtrack game* [44] characterizing the functions which preserve the class of $\mathbf{\Sigma}^0_2$ sets under preimages, player **II** is allowed to erase *all* of her past moves at any given round, the rule in this case being that she only do this finitely many times. See [26] for a survey of these and other related results.

In his PhD thesis [39], Semmes introduced the *tree game* characterizing the (total) Borel functions in Baire space. Player **I** plays as in the Wadge game, and therefore builds some $x \in \mathbb{N}^\mathbb{N}$ in the long run, but at each round n player now **II** plays a finite *labeled tree*, i.e., a pair (T_n, ϕ_n) of a finite tree $T_n \subseteq \mathbb{N}^{<\mathbb{N}}$ and a function $\phi_n : T_n \setminus \{\langle\rangle\} \to \mathbb{N}$, where $\langle\rangle$ denotes the empty sequence. The rules are that $T_n \subseteq T_{n+1}$ and $\phi_n \subseteq \phi_{n+1}$ must hold for each n, and that the *final* labeled tree $(T, \phi) = (\bigcup_{n\in\mathbb{N}} T_n, \bigcup_{n\in\mathbb{N}} \phi_n)$ must be an infinite tree with a unique infinite branch. Player **II** then wins if the sequence of labels along this infinite branch is exactly $f(x)$. By providing suitable extra requirements on the structure of the final tree, Semmes was able to obtain the *multitape game* characterizing the classes of functions which preserve $\mathbf{\Sigma}^0_3$ under preimages, the *multitape eraser game* characterizing the class of functions for which the preimage of any $\mathbf{\Sigma}^0_2$ set is a $\mathbf{\Sigma}^0_3$ set, and a game characterizing the Baire class 2 functions.

As examples of applications of these games, Semmes found a new proof of a theorem of Jayne and Rogers characterizing the class of functions which preserve $\mathbf{\Sigma}^0_2$ under preimages, and extended this theorem to the classes characterized by the multitape and multitape eraser games, by performing a detailed analysis of the corresponding game in each case.

Given the success of such game characterizations, Andretta raised the question which classes of functions admit a characterization by a suitable game in [1]. Significant progress towards an answer was made by Motto-Ros in [23]: Starting from a general definition of a reduction game, it is shown how to construct new games from existing ones in ways that mirror the typical constructions of classes of functions (e.g. piecewise definitions, composition, pointwise limits). In particular, Motto-Ros' results show that all the usual subclasses of the Borel functions studied in descriptive set theory admit game characterizations.

In order to arrive at a full characterization of the classes of function characterizable by a game, we need to find the appropriate language to formulate such a result. Just as *nice* classes of sets can be understood as lower cones in the Wadge degrees, *nice* classes of functions are found in the lower cones in the

Weihrauch degrees. Weihrauch reducibility (in its modern form) was introduced by Gherardi and Marcone [13] and Brattka and Gherardi [2, 3] based on earlier work by Weihrauch on a reducibility between sets of functions analogous to Wadge reducibility [41, 42].

We will show that game characterizations and Weihrauch degrees correspond closely to each other. We can thus employ the results and techniques developed for Weihrauch reducibility to study function classes in descriptive set theory, and vice versa. In particular, we can use the algebraic structure available for Weihrauch degrees [6, 15] to obtain game characterizations for derived classes of functions from game characterizations for the original classes, similar to the constructions found by Motto-Ros [23].

As a further feature of our work, we should point out that our results apply to the effective setting firsthand, and are then lifted to the continuous setting via relativization. They thus follow the recipe laid out by Moschovakis in [22].

While the traditional scope of descriptive set theory is restricted to Polish spaces, their subsets and functions between them, these restrictions are immaterial for the approach presented here. Our results naturally hold for multivalued functions between represented spaces. As such, this work is part of a larger development to extend descriptive set theory to a more general setting, cf. e.g. [7, 18, 28, 32, 34].

We shall freely use standard concepts and notation from descriptive set theory, and refer to [17] for an introduction.

2 Preliminaries on represented spaces and Weihrauch reducibility

Represented spaces and continuous/computable maps between them form the setting for computable analysis [43]. For a comprehensive modern introduction we refer to [30].

A *represented* space $\mathbf{X} = (X, \delta_\mathbf{X})$ is given by a set X and a partial surjection $\delta_\mathbf{X} :\subseteq \mathbb{N}^\mathbb{N} \to X$. A (multivalued) function between represented spaces is just a (multivalued) function on the underlying sets. We say that a partial function $F :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ is a *realizer* for a multivalued function $f :\subseteq \mathbf{X} \to \mathbf{Y}$ (in symbols: $F \vdash f$) if $\delta_\mathbf{Y}(F(p)) \in f(\delta_\mathbf{X}(p))$ for all $p \in \mathrm{dom}(f\delta_\mathbf{X})$. We call f computable (continuous), if it admits some computable (continuous) realizer.

Represented spaces and continuous functions do indeed generalize Polish spaces and continuous functions. Let (X, τ) be some Polish space. Fix a countable dense sequence $(a_i)_{i\in\mathbb{N}}$ and a compatible metric d. Now define $\delta_\mathbf{X}$ by $\delta_\mathbf{X}(p) = x$ iff $d(a_{p(i)}, x) < 2^{-i}$ holds for all $i \in \mathbb{N}$. In words: We represent a point by a sequence of basic points converging to it with prescribed speed. It is a foundational result in computable analysis that the notion of continuity for the represented space $(X, \delta_\mathbf{X})$ coincides with that for the Polish space (X, τ).

Definition 1. *Let f and g be partial, multivalued functions between represented spaces. Say that f is* Weihrauch reducible *to g, in symbols $f \leq_\mathrm{W} g$, if there* *are computable functions* $K :\subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *and* $H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *such that whenever* $G \vdash g$*, the function* $F := (p \mapsto K(p, G(H(p))))$ *is a realizer for* f*.*

If there are computable functions $K, H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *such that whenever* $G \vdash g$ *then* $KGH \vdash f$*, then we that that* f *is* strongly Weihrauch reducible *to* g *(*$f \leq_{\mathrm{sW}} f$*). We write* $f \leq^{\mathrm{c}}_{\mathrm{W}} g$ *and* $f \leq^{\mathrm{c}}_{\mathrm{sW}} g$ *for the variations where computable is replaced with continuous.*

A multivalued function f *tightens* g, denoted by $f \preceq g$, if $\mathrm{dom}(g) \subseteq \mathrm{dom}(f)$ and whenever $x \in \mathrm{dom}(g)$, then $f(x) \subseteq g(x)$, cf. [29, 33].

Proposition 1 (e.g. [27, Chapter 4]). *Let* $f :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$ *and* $g :\subseteq \mathbf{C} \rightrightarrows \mathbf{D}$*. We have*

1. $f \leq_{\mathrm{sW}} g$ *(*$f \leq^{\mathrm{c}}_{\mathrm{sW}} g$*) iff there exist computable (continuous)* $k :\subseteq \mathbf{A} \rightrightarrows \mathbf{C}$ *and* $h :\subseteq \mathbf{D} \rightrightarrows \mathbf{B}$ *such that* $hgk \preceq f$*; and*
2. $f \leq_{\mathrm{W}} g$ *(*$f \leq^{\mathrm{c}}_{\mathrm{W}} g$*) iff there exist computable (continuous)* $k :\subseteq \mathbf{A} \rightrightarrows \mathbb{N}^\mathbb{N} \times \mathbf{C}$ *and* $h :\subseteq \mathbb{N}^\mathbb{N} \times \mathbf{D} \rightrightarrows \mathbf{B}$ *such that* $h(\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times g)k \preceq f$*.*

There are plenty of interesting operations defined on Weihrauch degrees (see e.g. the introduction of [4] for a recent overview), here we only require the sequential composition operator $\star$ from [5, 6]. Rather than defining it explicitly as in [6], we will make use of the following characterization:

Theorem 2 (Brattka & P. [6]). $f \star g \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{f' \circ g' \mid f \leq_{\mathrm{W}} f \ \wedge\ g' \leq_{\mathrm{W}} g\}$

3 Transparent cylinders

We call $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ a *cylinder* if $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times f \leq_{\mathrm{sW}} f$. Note that f is a cylinder iff $g \leq_{\mathrm{W}} f$ and $g \leq_{\mathrm{sW}} f$ are equivalent for all g. This notion is from [3].

Definition 2. *Call* $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ transparent *iff for any computable (continuous)* $g :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ *there is a computable (continuous)* $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ *with* $T \circ f \preceq g \circ T$*.*

A represented space $\mathbf{Z} = (Z, \delta_\mathbf{Z})$ is a *subspace* of $\mathbf{Y} = (Y, \delta_\mathbf{Y})$ if $Z \subseteq Y$ and $\delta_\mathbf{Z} = \delta_\mathbf{Y} \restriction \{p \in \mathrm{dom}(\delta_\mathbf{Y})\,;\, \delta_\mathbf{Y}(p) \in Z\}$.

Lemma 1. *Let* $T :\subseteq (X, \delta_\mathbf{X}) \rightrightarrows (Y, \delta_\mathbf{Y})$ *be transparent, and let* $(Z, \delta_\mathbf{Z})$ *be a subspace of* $(Y, \delta_\mathbf{Y})$*. Then* $S :\subseteq (X, \delta_\mathbf{X}) \rightrightarrows (Z, \delta_\mathbf{Z})$ *given by*

$$S = T \restriction \{x \in \mathrm{dom}(T)\,;\, T(x) \subseteq Z\}$$

is also transparent.

The transparent (singlevalued) functions on Baire space where studied by de Brecht under the name *jump operator* in [8]. These are relevant because they induce endofunctors on the category of represented spaces, which in turn can characterize function classes in DST ([31]). The term *transparent* was coined in [5]. Our extension of the concept to multivalued functions between represented spaces is rather straight-forward, but requires the use of the notion of tightening.

Note that if $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ is transparent, then for every $y \in \mathbf{Y}$ there is some $x \in \mathrm{dom}(T)$ with $T(x) = \{y\}$, i.e. T is *slim* in the terminology of [5, Definition 2.7].

Theorem 3 (Brattka & P. [6]). *For every multivalued function g there is a multivalued function $g^t \equiv_{\mathrm{W}} g$ which is a transparent cylinder.*

Proposition 2. *Let $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $S :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Z}$ be cylinders. If T is transparent then $S \circ T$ is a cylinder and $S \circ T \equiv_{\mathrm{W}} S \star T$. Furthermore, if S is also transparent then so is $S \circ T$.*

4 Generalized Wadge games

Definition 3. *A* probe *for $\mathbf{Y}$ is a computable partial function $\zeta :\subseteq \mathbf{Y} \to \mathbb{N}^{\mathbb{N}}$ such that for every computable (continuous) $f :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ there is a computable (continuous) $e :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $\zeta e \preceq f$.*

Note that being a probe is just the dual notion to be being an admissible representation as in the approach taken by Schröder in [36]. As each constant function is continuous, a probe has to be surjective. Moreover, a probe is always transparent.

The following definition generalizes the definition of a reduction game from [23, Subsection 3.1], which is recovered as the special case if all involved spaces are $\mathbb{N}^{\mathbb{N}}$, the map ζ is the identity on $\mathbb{N}^{\mathbb{N}}$ and T is a function (rather than a multivalued function).

Definition 4. *Let $\zeta :\subseteq \mathbf{Y} \to \mathbb{N}^{\mathbb{N}}$ be a probe, $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $f :\subseteq \mathbf{A} \rightrightarrows \mathbf{B}$. The* (ζ, T)-Wadge game for f *is played by two players,* **I** *and* **II**, *who take turns in infinitely many rounds. At each round of a run of the game, player* **I** *first plays a natural number and player* **II** *then either plays a natural number or passes, as long as she plays natural numbers infinitely often. Therefore, in the long run player* **I** *builds $x \in \mathbb{N}^{\mathbb{N}}$ and* **II** *builds $y \in \mathbb{N}^{\mathbb{N}}$, and player* **II** wins *the run of the game if $x \notin \mathrm{dom}(f\delta_{\mathbf{A}})$, or $y \in \mathrm{dom}(\delta_{\mathbf{B}}\zeta T \delta_{\mathbf{X}})$ and $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}}(y) \subseteq f\delta_{\mathbf{A}}(x)$.*

For example, it is easy to see that the Wadge game is the $(\mathrm{id}, \mathrm{id})$-Wadge game, the eraser game is the $(\mathrm{id}, \lim)$-Wadge game, and the backtrack game is the $(\mathrm{id}, \lim_{\Delta})$-Wadge game, where $\lim_{\Delta}(p) = \lim(p)$ with $\mathrm{dom}(\lim_{\Delta}) = \{p \in \mathbb{N}^{\mathbb{N}} \,;\, \exists n \forall m, k \geq n.\ (p)_m = (p)_k\}$.

Theorem 4. *Let T be a transparent cylinder. Then player* **II** *has a (computable) winning strategy in the (ζ, T)-Wadge game for f iff $f \leq^{\mathrm{c}}_{\mathrm{W}} T$ ($f \leq_{\mathrm{W}} T$).*

Proof. ($\Rightarrow$) Any (computable) strategy for player **II** gives rise to a continuous (computable) function $k :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. If the strategy is winning, then $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}} k \preceq f\delta_{\mathbf{A}}$, which implies $\delta_{\mathbf{B}}\zeta T\delta_{\mathbf{X}} k \delta_{\mathbf{A}}^{-1} \preceq f\delta_{\mathbf{A}}\delta_{\mathbf{A}}^{-1} = f$. Thus the continuous (computable) maps $\delta_{\mathbf{B}} \circ \zeta$ and $\delta_{\mathbf{X}} k \delta_{\mathbf{A}}^{-1}$ witness that $f \leq^{\mathrm{c}}_{\mathrm{sW}} T$ ($f \leq_{\mathrm{sW}} T$).

($\Leftarrow$) As T is a cylinder, if $f \leq_{\mathrm{W}}^{\mathrm{c}} T$ ($f \leq_{\mathrm{W}} T$), then already $f \leq_{\mathrm{sW}}^{\mathrm{c}} T$ ($f \leq_{\mathrm{sW}} T$). Thus, there are continuous (computable) h, k with $h \circ T \circ k \preceq f$. As $\delta_{\mathbf{B}} \circ \delta_{\mathbf{B}}^{-1} = \mathrm{id}_{\mathbf{B}}$, we find that $\delta_{\mathbf{B}} \circ \delta_{\mathbf{B}}^{-1} \circ h \circ T \circ k \preceq f$. Now $\delta_{\mathbf{B}}^{-1} \circ h :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is continuous (computable), so by definition of a probe, there is some continuous (computable) $e :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ with $\delta_{\mathbf{B}} \circ \zeta \circ e \circ T \circ k \preceq f$. As T is a cylinder, there is some continuous (computable) g with $e \circ T \succeq T \circ g$, thus $\delta_{\mathbf{B}} \circ \zeta \circ T \circ g \circ k \preceq f$.

As $(g \circ k) :\subseteq \mathbf{A} \rightrightarrows \mathbf{X}$ is continuous (computable), it has some (continuous) computable realizer $K :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. By Theorem 1, player **II** has a winning strategy in the Wadge game for K. This strategy also wins the (ζ, T)-Wadge game for f for her. □

Corollary 1. *Let T and S be transparent cylinders. If the (ζ, T)-Wadge games characterize the class $\boldsymbol{\Gamma}$ and the (ζ', S)-Wadge games characterize the class $\boldsymbol{\Gamma}'$, then the $(\zeta', S \circ T)$-Wadge games characterize $\boldsymbol{\Gamma}' \circ \boldsymbol{\Gamma}$.*

The converse of Theorem 4 is almost true, as well:

Proposition 3. *If the (ζ, T)-Wadge games characterize a lower cone in the Weihrauch degrees, then it is the lower cone of $\zeta \circ T$, and $\zeta \circ T$ is a transparent cylinder.*

5 Using game characterizations

A main advantage of having game characterizations of some properties is realized together with determinacy: Either by choosing our set-theoretic axioms accordingly, or by restricting to simple cases and invoking e.g. Borel determinacy [21], we can conclude that if the property is false, i.e. player **II** has no winning strategy, then player **I** has a winning strategy. Thus, player **I**'s winning strategies serve as explicit witnesses of the failure of a property.

If we apply this line of reasoning to our T-Wadge games, we obtain the following corollaries of Theorem 4:

Corollary 2 (ZFC). *Let T be a transparent cylinder and ζ a probe, such that $\zeta \circ T$ is single-valued and $\mathrm{dom}(\zeta \circ T)$ is Borel. Then for $f : \mathbf{A} \rightrightarrows \mathbf{B}$ with $\mathrm{dom}(\delta_{\mathbf{A}})$ Borel and $f(x)$ being Borel for any $x \in \mathbf{A}$, we find that $f \nleq_W^{\mathrm{c}} T$ iff player **I** has a winning strategy in the (ζ, T)-Wadge game for f.*

Corollary 3 (ZF + DC + AD). *Let T be a transparent cylinder and ζ a probe. Then $f \nleq_W^{\mathrm{c}} T$ iff player **I** has a winning strategy in the (ζ, T)-Wadge game for f.*

Unfortunately, as determinacy fails in a computable setting (e.g. [10, 20]), we do not retain the computable counterparts. More generally, we are lacking a clear understanding of how these winning strategies of player **I** might look like. As pointed out to the authors by Carroy and Louveau, this holds even for the original Wadge-games, i.e. the $(\mathrm{id}_{\mathbb{N}^{\mathbb{N}}}, \mathrm{id}_{\mathbb{N}^{\mathbb{N}}})$-Wadge games. Here, we are already have a notion of explicit witnesses for discontinuity: Points of discontinuity. We can thus inquire about their relation:

Question 1. Let a point of discontinuity of a function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ be given as a sequence $(a_n)_{n\in\mathbb{N}}$, a point $a \in \mathbb{N}^\mathbb{N}$, and a word $w \in \mathbb{N}^*$ with $w \sqsubseteq f(a)$ such that $\forall n\ d(a_n, a) < 2^{-n} \wedge w \not\sqsubseteq f(a_n)$. Let Point be the multi-valued map that takes in a winning strategy for player **I** in the (id, id)-Wadge game for some function $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, and outputs a point of discontinuity for that function. Is Point computable? More generally, what is the Weihrauch degree of Point?

We can somewhat restrict the range of potential answer for the preceding question:

Theorem 5 ([3])**.** *Let player* **I** *have a computable winning strategy in the* (id, id)*-Wadge game for* $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$*. Then* f *has a computable point of discontinuity.*

A particular convenient way of exploiting determinacy of T-Wadge games could be achieved, if a more symmetric version were found. In this, we could hope for a dual principle S, where for any f either $f \leq_\mathrm{W}^c T$ or $S \leq_\mathrm{W}^c T$ holds. More generally, we hope that a better understanding of the T-Wadge games would lead to structural results about the Weihrauch lattice, similar to the results obtained by Carroy on the strong Weihrauch reducibility [9].

6 Generalized Wadge reductions

Wadge games were introduced not to characterize continuous functions, but in order to reason about a reducibility – Wadge reducibility – between sets. Given $A, B \subseteq \mathbb{N}^\mathbb{N}$, we say that $A \leq_w B$ iff there exists a continuous $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $F[B] \subseteq A$. Equivalently, we could define the multi-valued function $\frac{A}{B} : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ defined via $\frac{A}{B}(x) = A$ iff $x \in B$ and $\frac{A}{B}(x) = (\mathbb{N}^\mathbb{N} \setminus A)$ iff $x \notin B$. Now, $A \leq_w B$ iff $\frac{A}{B}$ is continuous. A famous structural result following from the determinacy of Wadge games is that for any Borel $A, B \subseteq \mathbb{N}^\mathbb{N}$, either $A \leq_w B$ or $B^C \leq_w A$. In particular, the Wadge hierarchy on the Borel sets is a strict weak order of width 2.

Both definitions immediately generalize to the case where $A \subseteq \mathbf{X}$ and $B \subseteq \mathbf{Y}$ for represented spaces $\mathbf{X}$, $\mathbf{Y}$ ([4]). However, they yield different notions, for not every continuous multi-valued function has a continuous choice function. As noted e.g. by Hertling [14], extending the former definition to the reals already introduces infinite antichains in the resulting degree structure. The second generalization was proposed by Pequignot [34] as an alternative[5].

[3] A key lemma for the proof of this theorem goes back to helpful comments by Takayuki Kihara.

[4] Note that a Wadge degree is a property of a subset of some specified space, rather than of a topological space on its own.

[5] While Pequignot only introduces the notion for quasi-Polish spaces, the extension to all represented spaces is immediate. One needs to take into account though that for general represented spaces, the Borel sets can show unfamiliar properties, e.g. even singletons can fail to be Borel (cf. also [37, 38]).

It is a natural variation to replace *continuous* in the definition of Wadge reducibility by some other class of functions (ideally closed under composition). Motto-Ros has shown that for the typical candidates of more restrictive classes of functions, the resulting degree structures will not share the nice properties of the standard Wadge degrees (they are *bad*) [24]. Larger classes of functions as reduction witnesses have been explored by Motto-Ros, Schlicht and Selivanov [25] in the setting of quasi-Polish spaces – using the generalization of the first definition of the reduction.

Definition 5. *Let T be a Weihrauch degree. We define a relation $\preceq^T$ on subsets of represented spaces as follows: For $A \subseteq \mathbf{X}$, $B \subseteq \mathbf{Y}$ let $A \preceq^T B$ hold iff $\frac{A}{B} \leq_{\mathrm{W}} T$.*

Observation 6. *If $T \star T \equiv_{\mathrm{W}} T$, then $\preceq^T$ is a quasiorder.*

The following partially generalizes [23, Theorem 6.10]:

Theorem 7. *Let $A \subseteq \mathbf{X}$, $B \subseteq \mathbf{Y}$, transparent cylinder $T : \mathbf{U} \rightrightarrows \mathbf{V}$ and probe $\zeta :\subseteq \mathbf{Y} \to \mathbb{N}^\mathbb{N}$ be such that the (ζ, T)-Wadge game for $\frac{A}{B}$ is determined. Then either $A \preceq^T B$ or $B \leq_w A^C$.*

Proof. If player **II** has a winning strategy, then by Theorem 4, we find that $\frac{A}{B} \leq_{\mathrm{W}} T$, hence by Definition 5 it follows that $A \preceq^T B$.

Otherwise, player **I** has a winning strategy. This winning strategy induces a continuous function $s : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$, such that if player **II** plays $y \in \mathbb{N}^\mathbb{N}$, then player **I** plays $s(y) \in \mathbb{N}^\mathbb{N}$. As T is a transparent cylinder and ζ a probe, there is a continuous function $t : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $(\zeta \circ T \circ \delta_\mathbf{U} \circ t) = \mathrm{id}_{\mathbb{N}^\mathbb{N}}$. Now we consider $s \circ t : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$. If $\delta_\mathbf{X}(x) \in A$, then if player **II** plays $t(x)$, player **I** needs to play some $s(t(x))$ such that $\delta_\mathbf{Y}(s(t(x))) \notin B$. Likewise, if $\delta_\mathbf{X}(x) \notin A$, then for player **I** to win, it needs to be the case that $\delta_\mathbf{Y}(s(t(x))) \in B$. Thus, $s \circ t$ is a continuous realizer of $\frac{B}{A^C}$, and $B \leq_w A^C$ follows by definition. □

Corollary 4 (ZF + DC + AD). *Let $T \star T \equiv_{\mathrm{W}} T$. Then $\prec^T$ is strict weak order of width at most 2.*

In [35], Motto-Ros has identified sufficient conditions on a generalized reduction (in a different formalism though) to ensure that its degree structure is equivalent to the Wadge degrees. We leave the task to future work to determine precisely for which T the degree structure of $\prec^T$ (restrict to subsets of $\mathbb{N}^\mathbb{N}$) is equivalent to the Wadge degrees, and which other structure types are realizable.

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References

1. Andretta, A.: The SLO principle and the Wadge hierarchy. In: Bold, S., Löwe, B., Räsch, T., van Benthem, J. (eds.) Foundations of the Formal Sciences V: Infinite Games. pp. 1–38. College Publications (2007)
2. Brattka, V., Gherardi, G.: Effective choice and boundedness principles in computable analysis. Bulletin of Symbolic Logic 1, 73 – 117 (2011), arXiv:0905.4685
3. Brattka, V., Gherardi, G.: Weihrauch degrees, omniscience principles and weak computability. Journal of Symbolic Logic 76, 143 – 176 (2011), arXiv:0905.4679
4. Brattka, V., Gherardi, G., Hölzl, R.: Probabilistic computability and choice. Information and Computation 242, 249 – 286 (2015), http://www.sciencedirect.com/science/article/pii/S0890540115000206, arXiv 1312.7305
5. Brattka, V., Gherardi, G., Marcone, A.: The Bolzano-Weierstrass Theorem is the jump of Weak König's Lemma. Annals of Pure and Applied Logic 163(6), 623–625 (2012), also arXiv:1101.0792
6. Brattka, V., Pauly, A.: On the algebraic structure of Weihrauch degrees. arXiv 1604.08348 (2016), http://arxiv.org/abs/1604.08348
7. de Brecht, M.: Quasi-Polish spaces. Annals of Pure and Applied Logic 164(3), 354–381 (2013)
8. de Brecht, M.: Levels of discontinuity, limit-computability, and jump operators. In: Brattka, V., Diener, H., Spreen, D. (eds.) Logic, Computation, Hierarchies, pp. 79–108. de Gruyter (2014), arXiv 1312.0697
9. Carroy, R.: A quasi-order on continuous functions. Journal of Symbolic Logic 78(2), 633–648 (2013)
10. Cenzer, D., Remmel, J.: Recursively presented games and strategies. Mathematical Social Sciences 24(2–3), 117 – 139 (1992)
11. Duparc, J.: Wadge hierarchy and Veblen hierarchy part i: Borel sets of finite rank. Journal of Symbolic Logic 66(1), 56–86 (2001), http://projecteuclid.org/euclid.jsl/1183746360
12. Gale, D., Stewart, F.M.: Infinite games with perfect information. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games, vol. 2, pp. 245–266. Princeton University Press (1953)
13. Gherardi, G., Marcone, A.: How incomputable is the separable Hahn-Banach theorem? Notre Dame Journal of Formal Logic 50(4), 393–425 (2009)
14. Hertling, P.: Unstetigkeitsgrade von Funktionen in der effektiven Analysis. Ph.D. thesis, Fernuniversität, Gesamthochschule in Hagen (Oktober 1996)
15. Higuchi, K., Pauly, A.: The degree-structure of Weihrauch-reducibility. Logical Methods in Computer Science 9(2) (2013)
16. Kanamori, A.: The Higher Infinite: large cardinals in set theory from their beginnings. Springer Monographs in Mathematics, Springer, second edn. (2005)
17. Kechris, A.S.: Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156. Springer-Verlag (1995)
18. Kihara, T., Pauly, A.: Point degree spectra of represented spaces. arXiv:1405.6866 (2014)
19. Le Roux, S., Pauly, A.: Weihrauch degrees of finding equilibria in sequential games. arXiv:1407.5587 (2014)
20. Le Roux, S., Pauly, A.: Weihrauch degrees of finding equilibria in sequential games. In: Beckmann, A., Mitrana, V., Soskova, M. (eds.) Evolving Computability, Lecture Notes in Computer Science, vol. 9136, pp. 246–257. Springer (2015)

21. Martin, D.A.: Borel determinacy. Annals of Mathematics 102(2), pp. 363–371 (1975)
22. Moschovakis, Y.N.: Classical descriptive set theory as a refinement of effective descriptive set theory. Annals of Pure and Applied Logic 162, 243–255 (2010)
23. Motto Ros, L.: Game representations of classes of piecewise definable functions. Mathematical Logic Quarterly 57(1), 95–112 (2011)
24. Motto-Ros, L.: Bad Wadge-like reducibilities on the Baire space. Fundamenta Mathematicae 224(1), 67–95 (2014)
25. Motto Ros, L., Schlicht, P., Selivanov, V.: Wadge-like reducibilities on arbitrary quasi-polish spaces. Mathematical Structures in Computer Science pp. 1–50 (11 2014), http://journals.cambridge.org/article_S0960129513000339, arXiv 1204.5338
26. Nobrega, H.: Game characterizations of function classes and Weihrauch degrees. M.Sc. thesis, University of Amsterdam (2013)
27. Pauly, A.: Computable Metamathematics and its Application to Game Theory. Ph.D. thesis, University of Cambridge (2012)
28. Pauly, A.: The descriptive theory of represented spaces. arXiv:1408.5329 (2014)
29. Pauly, A.: Many-one reductions and the category of multivalued functions. Mathematical Structures in Computer Science (2015), available at: arXiv 1102.3151
30. Pauly, A.: On the topological aspects of the theory of represented spaces. Computability 5(2), 159–180 (2016), http://arxiv.org/abs/1204.3763
31. Pauly, A., de Brecht, M.: Towards synthetic descriptive set theory: An instantiation with represented spaces. http://arxiv.org/abs/1307.1850 (2013)
32. Pauly, A., de Brecht, M.: Descriptive set theory in the category of represented spaces. In: 30th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS). pp. 438–449 (2015)
33. Pauly, A., Ziegler, M.: Relative computability and uniform continuity of relations. Journal of Logic and Analysis 5 (2013)
34. Pequignot, Y.: A Wadge hierarchy for second countable spaces. Archive for Mathematical Logic pp. 1–25 (2015), http://dx.doi.org/10.1007/s00153-015-0434-y
35. Ros, L.M.: Borel-amenable reducibilities for sets of reals. Journal of Symbolic Logic 74(1), 27–49 (2009), http://dx.doi.org/10.2178/jsl/1231082301
36. Schröder, M.: Extended admissibility. Theoretical Computer Science 284(2), 519–538 (2002)
37. Schröder, M., Selivanov, V.: Hyperprojective hierarchy of QCB_0-spaces. arXiv 1404.0297 (2014), http://arxiv.org/abs/1404.0297
38. Schröder, M., Selivanov, V.L.: Some hierarchies of QCB_0-spaces. Mathematical Structures in Computer Science pp. 1–25 (11 2014), arXiv 1304.1647
39. Semmes, B.: A game for the Borel functions. Ph.D. thesis, University of Amsterdam (2009)
40. Wadge, W.W.: Reducibility and determinateness on the Baire space. Ph.D. thesis, University of California, Berkeley (1983)
41. Weihrauch, K.: The degrees of discontinuity of some translators between representations of the real numbers. Informatik Berichte 129, FernUniversität Hagen, Hagen (Jul 1992)
42. Weihrauch, K.: The TTE-interpretation of three hierarchies of omniscience principles. Informatik Berichte 130, FernUniversität Hagen, Hagen (Sep 1992)
43. Weihrauch, K.: Computable Analysis. Springer-Verlag (2000)
44. van Wesep, R.: Wadge degrees and descriptive set theory. In: Kechris, A., Moschovakis, Y. (eds.) Cabal Seminar 76-77. Lecture Notes in Mathematics, vol. 689. Springer (1978)

A Omitted Proofs

Lemma (Lemma 1)**.** *Let $T :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (Y, \delta_{\mathbf{Y}})$ be transparent, and let $(Z, \delta_{\mathbf{Z}})$ be a subspace of $(Y, \delta_{\mathbf{Y}})$. Then $S :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (Z, \delta_{\mathbf{Z}})$ given by*

$$S = T \restriction \{x \in \mathrm{dom}(T) \,;\, T(x) \subseteq Z\}$$

is also transparent.

Proof. Let $f :\subseteq (Z, \delta_{\mathbf{Z}}) \rightrightarrows (Z, \delta_{\mathbf{Z}})$ be computable. Then $f :\subseteq (Y, \delta_{\mathbf{Y}}) \rightrightarrows (Y, \delta_{\mathbf{Y}})$, and therefore there exists a computable $g :\subseteq (X, \delta_{\mathbf{X}}) \rightrightarrows (X, \delta_{\mathbf{X}})$ such that

1. $\mathrm{dom}(f \circ T) \subseteq \mathrm{dom}(T \circ g)$, and
2. for all $x \in \mathrm{dom}(f \circ T)$ we have $T \circ g(x) \subseteq f \circ T(x)$.

Claim. $\mathrm{dom}(f \circ S) = \mathrm{dom}(f \circ T)$.

Indeed, the left-to-right inclusion is immediate from the definition of S. Conversely, suppose $x \in \mathrm{dom}(f \circ T)$. Therefore $x \in \mathrm{dom}(T)$ and $T(x) \subseteq \mathrm{dom}(f)$. Thus, since $\mathrm{dom}(f) \subseteq Z$, it follows that $x \in \mathrm{dom}(S)$ and $S(x) \subseteq \mathrm{dom}(f)$, as desired.

Claim. $\mathrm{dom}(f \circ S) \subseteq \mathrm{dom}(S \circ g)$.

Indeed, let $x \in \mathrm{dom}(f \circ S) = \mathrm{dom}(f \circ T)$. Then $x \in \mathrm{dom}(T \circ g)$, i.e., $x \in \mathrm{dom}(g)$ and $g(x) \subseteq \mathrm{dom}(T)$, and $T \circ g(x) \subseteq f \circ T(x) \subseteq Z$. Thus $g(x) \subseteq \mathrm{dom}(S)$, i.e. $x \in \mathrm{dom}(S \circ g)$.

Claim. For all $x \in \mathrm{dom}(f \circ S)$ we have $S \circ g(x) \subseteq f \circ S(x)$.

Indeed, let $x \in \mathrm{dom}(f \circ S) \subseteq \mathrm{dom}(S \circ g)$. We have

$$\begin{aligned} S \circ g(x) &= T \circ g(x) \\ &\subseteq f \circ T(x) \\ &= f \circ S(x) \end{aligned}$$

□

Proposition (Proposition 2)**.** *Let $T :\subseteq \mathbf{X} \rightrightarrows \mathbf{Y}$ and $S :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Z}$ be cylinders. If T is transparent then $S \circ T$ is a cylinder and $S \circ T \equiv_{\mathrm{W}} S \star T$. Furthermore, if S is also transparent then so is $S \circ T$.*

Proof. Suppose that T is transparent.

($S \circ T$ is a cylinder) As S is a cylinder, there are computable $h :\subseteq \mathbf{Z} \rightrightarrows \mathbb{N}^{\mathbb{N}} \times \mathbf{Z}$ and $k :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times S \succeq h \circ S \circ k$. Likewise, there are computable $h' :\subseteq \mathbf{Y} \rightrightarrows \mathbb{N}^{\mathbb{N}} \times \mathbf{Y}$ and $k' :\subseteq \mathbb{N}^{\mathbb{N}} \times \mathbf{X} \rightrightarrows \mathbf{X}$ such that $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times T \succeq h' \circ S \circ k'$. As composition respects tightening ([33, Lemma 2.4.1.b]), we conclude that $(\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times S) \circ (\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times T) = \mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (S \circ T) \succeq h \circ S \circ k \circ h' \circ T \circ k'$. Note that $(k \circ h') :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ is computable, and as T is transparent, there is some computable $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ with $(k \circ h') \circ T \succeq T \circ f$. But then $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (S \circ T) \succeq h \circ S \circ k \circ h' \circ T \circ k' \succeq h \circ S \circ T \circ f \circ k'$, thus h and $f \circ k'$ witness that $\mathrm{id}_{\mathbb{N}^{\mathbb{N}}} \times (S \circ T) \leq_{\mathrm{sW}} (S \circ T)$, i.e. $S \circ T$ is a cylinder.

($S \circ T \equiv_W S \star T$) The direction $S \circ T \leq_W S \star T$ is immediate. Assume $S' \leq_W S$ and $T' \leq_W T$. We need to show that $S' \circ T' \leq_W S \circ T$ (if the composition exists). As S and T are cylinders, we find that already $S' \leq_{sW} S$ and $T' \leq_{sW} T$. Let h, k witness the former and h', k' the latter. We conclude $h \circ S \circ k \circ h' \circ T \circ k' \preceq S' \circ T'$. As above, there then is some computable f with $k \circ h' \circ T \succeq T \circ f$. Then h and $f \circ k'$ witness that $S' \circ T' \leq_{sW} S \circ T$.

Now suppose that S is also transparent.

($S \circ T$ is transparent) Let $h :\subseteq \mathbf{Z} \rightrightarrows \mathbf{Z}$ be computable. By assumption that S is transparent, there is some computable $g :\subseteq \mathbf{Y} \rightrightarrows \mathbf{Y}$ such that $S \circ g \preceq h \circ S$. Then there is some computable $f :\subseteq \mathbf{X} \rightrightarrows \mathbf{X}$ with $T \circ f \preceq g \circ T$. As composition respects tightening ([33, Lemma 2.4.1.b]), we find that $h \circ S \circ T \preceq S \circ g \circ T \preceq S \circ T \circ f$, which is what we need. □

Proposition (Proposition 3)**.** *If the* (ζ, T)*-Wadge games characterize a lower cone in the Weihrauch degrees, then it is the lower cone of* $\zeta \circ T$*, and* $\zeta \circ T$ *is a transparent cylinder.*

Proof. Similar to the corresponding observation in Theorem 4, note that whenever player **II** has a (computable) winning strategy in the (ζ, T)-Wadge game for f, this induces a (strong) Weihrauch reduction $f \leq^c_{sW} \zeta \circ T$ ($f \leq_{sW} \zeta \circ T$). Conversely, by simply copying player **I**'s moves, player **II** wins the (ζ, T)-Wadge game for $\zeta \circ T$. This establishes the first claim.

Now, as $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T) \leq_W \zeta \circ T$, the assumption that the (ζ, T)-Wadge game characterize a lower cone in the Weihrauch degrees implies that player **II** wins the (ζ, T)-Wadge game for $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T)$. Thus, $\mathrm{id}_{\mathbb{N}^\mathbb{N}} \times (\zeta \circ T) \leq_{sW} \zeta \circ T$ follows, and we find $\zeta \circ T$ to be a cylinder.

For the remaining claim that $\zeta \circ T$ is transparent, let $G :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ be continuous (computable). Then $G \circ \zeta \circ T \leq^c_W \zeta \circ T$ ($G \circ \zeta \circ T \leq_W \zeta \circ T$), hence player **II** has a (computable) winning strategy in the (ζ, T)-Wadge game for $G \circ \zeta \circ T$. This strategy induces some continuous (computable) $H :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ with $G \circ \zeta \circ T \circ \delta_\mathbf{X} \succeq \zeta \circ T \circ \delta_\mathbf{X} \circ H$. Thus, $\delta_\mathbf{X} \circ H \circ \delta_\mathbf{X}^{-1}$ is the desired witness. □

Theorem (Theorem 5)**.** *Let player* **I** *have a computable winning strategy in the* (id, id)*-Wadge game for* $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$*. Then* f *has a computable point of discontinuity.*

The proof of the theorem will require some recursion theoretic preparations. Given $p, q \in \mathbb{N}^\mathbb{N}$, let $[p \mid q] \in \mathbb{N}^\mathbb{N}$ be defined as $[p \mid q] = 0^{q(0)}(p(0)+1)0^{q(1)}(p(1)+1)0^{q(2)}\ldots$, i.e. $[p \mid q]$ increases each number in p by 1, and then intersperses zeros between the entries, with the number of repetitions being provided by q. Now, given $r \in \mathbb{N}^\mathbb{N}$ and some $A \subseteq \mathbb{N}^\mathbb{N}$, let $A^{+r} := \{[p \mid q] \mid p \in A \wedge q \geq r\}$, where $q \geq r$ denotes component-wise comparison.

Lemma 2 ([6])**.** *Let* $F : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ *be computable,* $r \in \mathbb{N}^\mathbb{N}$*,* $A, B \subseteq \mathbb{N}^\mathbb{N}$*,* $B \neq \emptyset$ *be such that* $F[B^{+r}] \subseteq A$ *and* A *is* Σ^0_2*. Then* A *contains a computable point.*

[6] The proof of this lemma is based on helpful comments by Takayuki Kihara.

Proof. Let $A = \bigcup_{n \in \mathbb{N}} Q_n$ with Π^0_1-sets Q_n. For the sake of a contradiction, assume that A and thus all Q_n contain no computable points. Pick some $p \in B$.

As $F(0^{\mathbb{N}})$ is computable, we find $F(0^{\mathbb{N}}) \notin Q_0$. As Q_0 is Π^0_1 and F computable, there is some $m_0 \geq r(0)$ such that $F[0^{m_0}\mathbb{N}^{\mathbb{N}}] \cap Q_0 = \emptyset$. Next, consider $F(0^{m_0}p(0)0^{\mathbb{N}})$. Again, this is a computable point, hence there is some $m_1 \geq r(1)$ such that $F[0^{m_0}p(p)0^{m_1}\mathbb{N}^{\mathbb{N}}] \cap Q_1 = \emptyset$. We proceed in this manner to chose all m_i, and then define $q \in \mathbb{N}^{\mathbb{N}}$ via $q(i) = m_i$. Note that $q \geq r$. Then $[p \mid q] \in B^{+r}$, but $F([p \mid q]) \notin A$ by construction, hence we derive the desired contradiction and conclude that A contains a computable point. □

of Theorem 5. Let us assume that player **I** has a winning strategy in the $(\mathrm{id}, \mathrm{id})$-Wadge game for $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$. We describe how player **II** can coax player **I** into playing a point of discontinuity of f. Player **II** starts passing, causing player **I** to produce longer and longer prefixes of some $p \in \mathbb{N}^{\mathbb{N}}$. If player **I** ever produces a prefix $p_{\leq n_0}$ such that $\exists k_0 \ f[p_{\leq n}\mathbb{N}^{\mathbb{N}}] \subseteq k_0\mathbb{N}^{\mathbb{N}}$, then player **II** will play k_0, and then goes back to passing. If subsequently, there is some n_1, such that $\exists k_1 \ f[p_{\leq n_1}\mathbb{N}^{\mathbb{N}}] \subseteq k_0k_1\mathbb{N}^{\mathbb{N}}$, then player **II** plays k_1, and starts passing again, etc. If f is continuous at p, then player **II** will play a correct response to f, hence contradict the assumption that player **I** is following a winning strategy. Thus, p has to be a point of discontinuity of f.

Note that if player **II** passes even more than necessary, this does not change the argument at all. Thus, we find that there is some non-empty set B and $r \in \mathbb{N}^{\mathbb{N}}$ such that the computable response function $S : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ maps B^{+r} into the set of points of discontinuity of f. The latter is a Σ^0_2-set, hence Lemma 2 implies that it contains a computable point. □