Maximal regularity in l_p spaces for discrete time fractional shifted equations

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Abstract

In this paper, we are presenting a new method based on operator-valued Fourier multipliers to characterize the existence and uniqueness of ℓ_p-solutions for discrete time fractional models in the form

$$\Delta^\alpha u(n,x) = Au(n,x) + \sum_{j=1}^{k} \beta_j u(n-\tau_j,x) + f(n,u(n,x)), \;\; n \in \mathbb{Z}, x \in \Omega \subset \mathbb{R}^N, \beta_j \in \mathbb{R} \text{ and } \tau_j \in \mathbb{Z},$$

where A is a closed linear operator defined on a Banach space X and Δ^α denotes the Grünwald-Letnikov fractional derivative of order $\alpha > 0$. If X is a UMD space, we provide this characterization only in terms of the R-boundedness of the operator-valued symbol associated to the abstract model. To illustrate our results, we derive new qualitative properties of nonlinear difference equations with shiftings, including fractional versions of the logistic and Nagumo equations.

Keywords: Maximal ℓ_p-regularity; shifted equations, discrete time, Grünwald-Letnikov derivative.

1. Introduction

Evolution equations with delays arise in many areas of applied mathematics. Time delays have been incorporated by many researchers into biological models to represent resource regeneration times, maturity periods, feeding times, reaction times, etc. There has been a substantial amount of work related to this topic, as one can see consulting for example [16], [6], [27], [5] and the bibliography therein.

It is well known that the study of maximal regularity is very useful for treating semilinear and quasilinear problems. Maximal regularity of evolution equations with operator-valued Fourier multipliers began to be studied after the pioneering work of H. Amman [3] and L. Weis [33]. Some authors as Arendt and Bu [4] studied maximal regularity of periodic problems for abstract evolution equations in Banach spaces having geometrical advantages. They are called UMD-spaces. See also [11], [10] and [9] for more information on this topic and related work.

Concerning delay equations, there is an increasing number of researchers working on this topic. For instance, Poblete [29] analysed maximal regularity on vector-valued Hölder spaces. The fractional case was considered by Ponce in [30]. In [15] Fu and Li treated the well-posedness for a class of evolution

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equations with infinite delay in the scales of Lebesgue, Besov and Triebel-Lizorkin vector-valued Banach spaces. A good reference in the context of this paper is the monograph by Bátkai and Piazzera [5].

The discrete time setting naturally arises in a wide variety of applications where the temporal structure is oscillatory and possesses a discrete character as well as in the one-dimensional spatial discretization of continuous problems. Concerning literature, Hu and Li [18] examined the spatial dynamics of semidiscrete equations with a shifting habitat. Yu [39] investigated the existence of traveling waves for nonlocal semidiscrete equations with delays. In [38] and [37], Zinner et.al. investigated, respectively, the existence of traveling wave fronts for the semidiscrete Nagumo and Fisher equations. A global study of the structure of such dynamical systems was performed by Mallet-Paret [27].

However, there has been little mention of semidiscrete equations with fractional differences. Recently, Tarasov [31], [32] began to study fractional models with the Grünwald-Letnikov fractional difference. As suggested by Tarasov, these models can serve as a new microstructural basis for the fractional nonlocal continuum mechanics and physics. Fractional-order semidiscrete equations can also be used to formulate adequate models in nanomechanics [32], [34].

Fractional differences do not only exhibit the advantages of memory effects, as the continuous case does, but they also involve fewer numerical computations. Recent work of Wu, Baleanu and Xie [36] on fractional chaotic maps reveals this interest. See also the references therein. The study of the chaotic behavior of the fractional discrete logistic map with delay was recently proposed in an interesting work by Wu and Baleanu [35]. In this paper, the bifurcation diagrams are also given for various fractional orders. Since there is a discrete kernel function in the definition of $\Delta^\alpha u(n)$ by means of a discrete convolution (see definition below), the present status of $u(n)$ depends on the previous information. This is the discrete memory effect, and it has been freshly reported by Huang, Baleanu, Wu and Zeng in [19] in the case of the fractional logistic map. Roughly speaking, the discrete fractional models have some new degrees of freedom which can be used to capture the hidden aspects of real world phenomena [35].

First results concerning maximal regularity for discrete time abstract Cauchy problems in Banach spaces are due to Blunck ([7], [8]). Kalton and Portal [21] considered maximal regularity in ℓ_p spaces for the critical cases $p = 1, \infty$. In [20], Kovács, Li and Lubich showed that for a parabolic problem with maximal L_p-regularity, the time discretization by a linear multistep method has maximal ℓ_p-regularity if the method is stable. Finally, Cuevas and Vidal [13] incorporated the delay in the research of maximal ℓ_p-regularity of discrete time equations.

In this paper, we address the novel study of the existence and ℓ_p-regularity of solutions for the following abstract Cauchy problem with finite advance/delay:

$$\Delta^\alpha u(n,x) = Au(n,x) + \sum_{j=1}^{k} \beta_j u(n-\tau_j, x) + f(n, u(n,x)), \qquad n \in \mathbb{Z}, \quad x \in \Omega \subset \mathbb{R}^N, \quad \alpha > 0, \quad \tau_j \in \mathbb{Z}. \tag{1.1}$$

where A is a closed linear operator with domain $D(A)$ defined on a Banach space of functions X, and Δ^α denotes the generalized Grünwald-Letnikov derivative of order $\alpha > 0$. See Definition 2.6 below and the reference [28], where this definition is used in the context of signal processing.

In [24], the author introduced an operator-theoretical method for characterizing ℓ_p-maximal regularity of discrete time fractional linear equations. This characterization was given in terms of boundedness of the associated resolvent operator, but only in case that A is a bounded operator and $0 < \alpha \leq 1$. See also the recent paper [26] for the case $1 < \alpha \leq 2$. However, the study of maximal regularity for A unbounded was left open. Motivated by that mentioned above and the fact that such results are of interest by themselves, we focus this paper on the study of maximal ℓ_p-regularity of (1.1).

In the first part, we present a novel method based on the theory of operator-valued Fourier multipliers to handle the linear version of (1.1). Roughly speaking, this method takes into account the operator-valued symbol associated to the linear part of the model abstractly formulated in the setting of a Banach space, and then inserts it in a distributional framework related to the discrete time Fourier transform. In order to do that, the necessary preliminaries are given in the Subsection 2.1 below. Then, we introduce a new concept of ℓ_p-multiplier (on sequences) correlated with Blunck's theorem on operator-valued symbols [7]. See Definition 2.4 below. The idea of ℓ_p-multiplier resembles the variational method for PDE's, in the sense that the original model is weakly formulated. This new formulation allows us to solve the weak problem, by means of a constructive method using consecutive steps and an appropriate choice of regular (or test) functions in each step.

Using this method, we succeed in proving that under the hypothesis that X is a UMD-Banach space and $\{(1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j}\}_{t\in\mathbb{T}} \subset \rho(A)$ (the resolvent set of the operator A), the following assertions are equivalent:

(i) The operator A has maximal ℓ_p-regularity;

(ii) $N(t) := ((1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j} - A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;

(iii) The set $\{N(t) \ : \ t \in \mathbb{T}\}$ is R-bounded.

As an immediate consequence of our characterization, we deduce that the solution for the linear version of (1.1) satisfies the following discrete time maximal regularity estimate:

$$\|\Delta^\alpha u\|_{\ell_p(\mathbb{Z};X)} + |\sum_{j=1}^k \beta_j| \|u\|_{\ell_p(\mathbb{Z};X)} + \|Au\|_{\ell_p(\mathbb{Z};X)} \leq C\|f\|_{\ell_p(\mathbb{Z};X)}. \tag{1.2}$$

Concerning this last estimate, it has been recently highlighted by Akrivis, Li and Lubich [1] that the combination of discrete maximal regularity and energy estimates is very useful to derive optimal-order error bounds for the time-discrete approximation of quasilinear parabolic equations by backward difference formulas. As pointed out by Li in [22], this estimate can be regarded as the stability of the parabolic projection onto the finite element space. These results are required for instance in [23] to establish optimal $L_p((0,T);L_q)$ error estimates of finite element methods for parabolic equations. According to this research, it could be an interesting challenge to derive optimal-order error bounds for fractional order schemes using the framework given here.

In the second part of this paper, we show some practical criteria for proving the existence of solutions on the Lebesgue space $\ell_p(\mathbb{Z};X)$ for a large class of nonlinear difference equations with advance/delay, notably including the fractional logistic and Nagumo equations.

More specifically, we provide conditions that ensure the existence of solutions in $\ell_p(\mathbb{Z};H)$ spaces for the nonlinear equation (1.1) on a Hilbert space H solely in terms of the boundedness of the Nemytskii's operator and the compactness of the unit ball of $D(A)$. Moreover, we consider the nonlinear perturbed equation

$$\Delta^\alpha u(n,x) = Au(n,x) + \sum_{j=1}^k \beta_j u(n-\tau_j,x) + G(u(n,x)) + \rho f(n,x) \tag{1.3}$$

where $0<\rho<1$, $f \in \ell_p(\mathbb{Z};X)$ and $G : \ell_p(\mathbb{Z};X) \to \ell_p(\mathbb{Z};X)$. We give sufficient conditions in terms of the R-boundedness of the operator-valued symbol associated to the linear part of equation (1.3) and the regularity of G at $u=0$ to ensure its ℓ_p-regularity for small values of the parameter ρ.

These results provide interesting consequences notably including the study of the existence of solutions in ℓ_p-spaces for the fractional logistic and Nagumo equations. We prove the existence of solutions in $\ell_p(\mathbb{Z};\mathbb{R})$ for the non-homogeneous fractional logistic map

$$\Delta^\alpha u(n) = (r-1)u(n) - ru(n)^2 + \rho f(n), \qquad r,\rho>0, \quad n\in\mathbb{Z}, \tag{1.4}$$

whenever $r \neq 1, 1 \pm \Big(\dfrac{4}{1+\tan(\frac{k\pi}{\alpha})}\Big)^{\frac{\alpha}{2}}$, $k \in \mathbb{Z}$. We also point out that in the non-fractional case we obtain ℓ_p-solutions whenever $r \neq 1, 3$, a qualitative property which seems to be new in the literature.

We finish this paper considering the one dimensional fractional discrete equation with delay

$$\Delta^\alpha u(n,t) = \frac{1}{d}\frac{\partial}{\partial t}u(n,t) + (\frac{a}{d}+1)u(n,t) - u(n-1,t) + \frac{1}{d}u(n,t)^3 - \left(\frac{1+a}{d}\right)u(n,t)^2 + \frac{1}{d}f(n,t), \tag{1.5}$$

where $a,d>0$, $t\in\mathbb{R}$ and $n\in\mathbb{Z}$, showing the existence of at least one solution in $\ell_p(\mathbb{Z};L^2(\mathbb{R}))$ for a,d being big enough to satisfy the compatibility condition $d < \frac{a}{2^\alpha}$. New insights are provided in the case $\alpha = 1$ which corresponds to the non-homogeneous discrete Nagumo equation. The main idea behind this example, and others of the same nature, is to combine the given characterization of maximal regularity, which is provided only in terms of the data of the equation, and the well established theory of strongly continuous semigroups (or C_0-semigroups). Indeed, as we will see in our last example (Example 4.5) we take advantage of the fact that in our abstract setting the differential operator $A = \frac{1}{d}\frac{\partial}{\partial t} + (\frac{a}{d}+1)I$ generates a C_0-semigroup on some Lebesgue spaces, like $L^2(\mathbb{R})$.

2. Preliminaries

In this section, we present some results that will be needed throughout the paper.

2.1. The discrete time Fourier transform in $\ell_p(\mathbb{Z};X)$

By $\mathcal{S}(\mathbb{Z};X)$ we denote the space of all vector-valued sequences $f:\mathbb{Z}\to X$ such that for each $k\in\mathbb{N}_0$ there exists a constant $C_k>0$ satisfying

$$p_k(f) := \sup_{n\in\mathbb{Z}} |n|^k \|f(n)\| < C_k.$$

Recall that $\mathcal{S}(\mathbb{Z};X)$ is norm dense in $\ell_p(\mathbb{Z},X)$ when $1\le p<\infty$. We also consider $C^n_{per}(\mathbb{R};X), n\in\mathbb{N}_0$, the space of all 2π-periodic X-valued and n-times continuously differentiable functions defined in $\mathbb{R}$.

In what follows, we will denote $\mathbb{T} := (-\pi,\pi)$ and $\mathbb{T}_0 := (-\pi,\pi)\setminus\{0\}$.

The space of test functions is the space $C^\infty_{per}(\mathbb{T};X) := \bigcap_{n\in\mathbb{N}_0} C^n_{per}(\mathbb{R};X)$. The topology of $C^\infty_{per}(\mathbb{T};X)$ is induced by the countable family of seminorms:

$$q_k(\varphi) = \max_{k\in\mathbb{N}_0} \sup_{t\in[-\pi,\pi]} \|\varphi^{(k)}(t)\|,$$

and so $C^\infty_{per}(\mathbb{T};X)$ becomes a Fréchet space. If $X=\mathbb{C}$ we simply denote $C^\infty_{per}(\mathbb{T};X) = C^\infty_{per}(\mathbb{T})$ and $\mathcal{S}(\mathbb{Z};X) = \mathcal{S}(\mathbb{Z})$.

We will also need the following spaces of vector-valued distributions:

$$\mathcal{S}'(\mathbb{Z};X) := \{T:\mathcal{S}(\mathbb{Z})\to X \ : \ T \text{ is linear and continuous}\}$$

and

$$\mathcal{D}'(\mathbb{T};X) := \{T: C^\infty_{per}(\mathbb{T})\to X \ : \ T \text{ is linear and continuous}\}.$$

It is useful to observe that for each $f\in\ell_p(\mathbb{Z};X)$ we can define

$$T_f(\psi) := \langle T_f,\psi\rangle := \sum_{n\in\mathbb{Z}} f(n)\psi(n), \quad \psi\in\mathcal{S}(\mathbb{Z}), \tag{2.1}$$

and we have $T_f\in\mathcal{S}'(\mathbb{Z},X)$.

Remark 2.1. By this mapping we identify $\ell_p(\mathbb{Z};X)$ with a subspace of $\mathcal{S}'(\mathbb{Z};X)$. When convenient and confusion seems unlikely, a function $f\in\ell_p(\mathbb{Z};X)$ is identified with $T_f\in\mathcal{S}'(\mathbb{Z},X)$.

There also exists a natural mapping that identifies $C^\infty_{per}(\mathbb{T};X)$ with a subspace of $\mathcal{D}'(\mathbb{T};X)$ which assigns to each $S\in C^\infty_{per}(\mathbb{T};X)$ the linear map

$$L_S(\varphi) := \langle L_S,\varphi\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)S(t)dt, \quad \varphi\in C^\infty_{per}(\mathbb{T}),$$

and we have $L_S\in\mathcal{D}'(\mathbb{T};X)$.

It is well known that the discrete time Fourier transform $\mathcal{F}:\mathcal{S}(\mathbb{Z};X)\to C^\infty_{per}(\mathbb{T};X)$ defined by

$$\mathcal{F}\varphi(t) \equiv \widehat{\varphi}(t) := \sum_{j=-\infty}^{\infty} e^{-ijt}\varphi(j), \qquad t\in(-\pi,\pi]$$

is an isomorphism whose inverse is given by

$$\mathcal{F}^{-1}\varphi(n) \equiv \check{\varphi}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{int}dt, \quad n \in \mathbb{Z}, \tag{2.2}$$

where $\varphi \in C_{per}^{\infty}(\mathbb{T}; X)$.

In particular, we have

$$\varphi \in C_{per}^{\infty}(\mathbb{T}) \implies \check{\varphi} \in \mathcal{S}(\mathbb{Z}),$$

which follows from integration by parts. This isomorphism, allows us to define the discrete time Fourier transform (DTFT) between the spaces of distributions $\mathcal{S}'(\mathbb{Z}; X)$ and $\mathcal{D}'(\mathbb{T}; X)$ as follows:

$$\langle \mathcal{F}T, \psi\rangle \equiv \mathcal{F}(T)(\psi) := \widehat{T}(\psi) \equiv \langle T, \check{\psi}\rangle, \quad T \in \mathcal{S}'(\mathbb{Z}; X), \quad \psi \in C_{per}^{\infty}(\mathbb{T}), \tag{2.3}$$

whose inverse $\mathcal{F}^{-1} : \mathcal{D}'(\mathbb{T}; X) \to \mathcal{S}'(\mathbb{Z}; X)$ is given by

$$\langle \mathcal{F}^{-1}L, \psi\rangle \equiv \mathcal{F}^{-1}(L)(\psi) := \check{L}(\psi) \equiv \langle L, \widehat{\psi}\rangle, \quad L \in \mathcal{D}'(\mathbb{T}; X), \quad \psi \in \mathcal{S}(\mathbb{Z}).$$

In particular, we have

$$\langle \mathcal{F}T_f, \varphi\rangle = \langle T_f, \check{\varphi}\rangle = \sum_{n\in\mathbb{Z}} f(n)\check{\varphi}(n), \quad \varphi \in C_{per}^{\infty}(\mathbb{T}), \quad f \in \ell_p(\mathbb{Z}, X). \tag{2.4}$$

2.2. The Grünwald-Letnikov fractional derivative

Given $u \in \ell_p(\mathbb{Z}; X), v \in \ell_1(\mathbb{Z})$ we define the convolution product as

$$(u * v)(n) := \sum_{j=-\infty}^{n} u(n-j)v(j) = \sum_{j=0}^{\infty} u(j)v(n-j), \quad n \in \mathbb{Z}.$$

The convolution of a distribution $T \in \mathcal{S}'(\mathbb{Z}, X)$ with a function $a \in \ell_1(\mathbb{Z})$ is defined by

$$\langle T * a, \varphi\rangle := \langle T, a \circ \varphi\rangle, \quad \varphi \in \mathcal{S}(\mathbb{Z}), \tag{2.5}$$

where

$$(a \circ \varphi)(n) := \sum_{j=0}^{\infty} a(j)\varphi(j+n).$$

Observe that $a \circ \varphi \in \mathcal{S}(\mathbb{Z})$. For any $\alpha \in \mathbb{R}$, we set

$$k^{\alpha}(n) = \begin{cases} \dfrac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} & n \in \mathbb{Z}_+, \\ 0 & \textit{otherwise}, \end{cases}$$

where Γ is the Euler gamma function. Note that if $\alpha \in \mathbb{R} \setminus \{-1, -2, ..\}$, we have $k^{\alpha}(n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)}$. For any $\beta > 0$ and $f : \mathbb{Z} \to X$ a given sequence, we define the fractional sum of order β as follows

$$(\Delta^{-\beta} f)(n) := (k^{\beta} * f)(n) = \sum_{j=-\infty}^{n} k^{\beta}(n-j)f(j), \quad n \in \mathbb{Z},$$

whenever it exists. For $\alpha \in \mathbb{R}_+$, the fractional difference of order α is defined by

$$\Delta^{\alpha} f(n) := (k^{-\alpha} * f)(n) = \sum_{j=-\infty}^{n} k^{-\alpha}(n-j)f(j) = \sum_{j=0}^{\infty} k^{-\alpha}(j)f(n-j), \quad n \in \mathbb{Z}, \tag{2.6}$$

see [28, formula (27) with $h = 1$]. From [24], or directly from the definition, we have the generation formula

$$\sum_{j=0}^{\infty} k^{\beta}(j) z^j = \frac{1}{(1-z)^{\beta}}, \quad \beta \in \mathbb{R}, \quad |z| < 1,$$

see [40, p.42 formulae (1) and (8)]. In particular, for all $\alpha \in \mathbb{R}_+$ we have that the radial limit exists and

$$\widehat{k^{-\alpha}}(\omega) = \widetilde{k^{-\alpha}}(\omega) = \sum_{j=0}^{\infty} k^{-\alpha}(j) e^{-i\omega j} = \frac{1}{(1-e^{-i\omega})^{-\alpha}} = \Big(1 - e^{-i\omega}\Big)^{\alpha}, \quad \omega \in \mathbb{T}. \tag{2.7}$$

To end this section, we show the following lemma that will be crucial for the characterization of maximal regularity given in the forthcoming section.

Lemma 2.2. *Let $u, v \in \ell_p(\mathbb{Z}; X)$ be given and $a \in \ell_1(\mathbb{Z}_+)$ which is defined by 0 for negative values of n. The following assertions are equivalent:*
(i) $a * v \in \ell_p(\mathbb{Z}, X)$ and $(a * v)(n) = u(n)$ for all $n \in \mathbb{Z}$.
(ii) $\langle u, \check{\varphi} \rangle = \langle v, (\varphi \cdot \widehat{a}_-)\check{} \rangle$ for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$,
where

$$(\varphi \cdot \widehat{a}_-)\check{}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(-t) \varphi(t) e^{int} dt, \quad n \in \mathbb{Z}.$$

Proof. $(i) \implies (ii)$ By hypothesis, $a * v \in \ell_p(\mathbb{Z}, X)$ and $\langle a * v, \check{\varphi} \rangle = \langle u, \check{\varphi} \rangle$ for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. We have

$$\langle u, \check{\varphi} \rangle = \langle a * v, \check{\varphi} \rangle = \langle v, a \circ \check{\varphi} \rangle = \langle v, (\varphi \cdot \widehat{a}_-)\check{} \rangle,$$

since

$$\begin{aligned}
(a \circ \check{\varphi})(n) &= \sum_{j=0}^{\infty} a(j) \check{\varphi}(j+n) = \sum_{j=0}^{\infty} a(j) \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(j+n)t} \varphi(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big[\sum_{j=0}^{\infty} a(j) e^{ijt}\Big] e^{int} \varphi(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big[\sum_{j=-\infty}^{\infty} a(j) e^{ijt}\Big] e^{int} \varphi(t) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \widehat{a}(-t) e^{int} \varphi(t) dt.
\end{aligned}$$

$(ii) \implies (i)$ Following the above calculation in reverse order, we get

$$\langle u, \check{\varphi} \rangle = \langle v, (\varphi \cdot \widehat{a}_-)\check{} \rangle = \langle a * v, \check{\varphi} \rangle,$$

for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. Since $v \in \ell_p(\mathbb{Z}; X)$ and a is summable, we obtain $a * v \in \ell_p(\mathbb{Z}; X)$ and the identity

$$\sum_{n \in \mathbb{Z}} (a * v)(n) \check{\varphi}(n) = \sum_{n \in \mathbb{Z}} u(n) \check{\varphi}(n),$$

holds. Choosing $\varphi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$ we achieve that $(a * v)(n) = u(n)$ for all $n \in \mathbb{Z}$ and the proof is finished.

□

2.3. R-boundedness and Blunck's Theorem

Now, we recall the following Fourier multiplier theorem for operator-valued symbols given by S. Blunck [7, 2]. This theorem corresponds to the discrete version of a notable result independently proven by Weis [33] and Amann[3] which provides sufficient conditions to ensure when an operator-valued symbol is a multiplier. This theorem is established for the UMD class of Banach spaces. For more information about these spaces see [3, Section III.4.3-III.4.5].

We will first recall the notion of R-bounded sets in the space $\mathcal{B}(X, Y)$ of bounded linear operators from X into Y endowed with the uniform operator topology.

Definition 2.3. Let X and Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X,Y)$ is called R-bounded if there is a constant $c>0$ such that

$$\|(T_1x_1,...,T_nx_n)\|_R \le c\|(x_1,...,x_n)\|_R, \tag{2.8}$$

for all $T_1,...,T_n \in \mathcal{T},\ x_1,...,x_n \in X,\ n \in \mathbb{N}$, where

$$\|(x_1,...,x_n)\|_R := \frac{1}{2^n}\sum_{\epsilon_j\in\{-1,1\}^n}\Big\|\sum_{j=1}^n \epsilon_j x_j\Big\|,$$

for $x_1,...,x_n \in X$.

For more information about R-bounded sets and their properties see [2, Section 2.2] and [14]. We next introduce the following notion.

Definition 2.4. Let X, Y be Banach spaces, $1<p<\infty$. A function $M \in C^\infty_{per}(\mathbb{T},\mathcal{B}(X,Y))$ is an ℓ_p-multiplier (from X to Y) if there exists a bounded operator $T:\ell_p(\mathbb{Z};X)\to\ell_p(\mathbb{Z};Y)$ such that

$$\sum_{n\in\mathbb{Z}}(Tf)(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}}(\varphi\cdot M_-\check{)}(n)f(n) \tag{2.9}$$

for all $f\in\ell_p(\mathbb{Z};X)$ and all $\varphi\in C^\infty_{per}(\mathbb{T})$. Here

$$(\varphi\cdot M_-\check{)}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\varphi(t)M(-t)dt, \quad n\in\mathbb{Z}.$$

Now, we recall the following Fourier multiplier theorem for operator valued symbols given by S. Blunck [7, 2]. Since we are dealing with unbounded operators we need to state a slight modification of Blunck's theorem concerning two Banach spaces instead of one. The proof of this result follows the same steps as the case $X=Y$.

Theorem 2.5. *[7, Theorem 1.3] Let $p\in(1,\infty)$ and let X,Y be UMD spaces. Let $M\in C^\infty_{per}(\mathbb{T},\mathcal{B}(X;Y))$ such that the sets*

$$\{M(t): t\in\mathbb{T}_0\} \quad \textit{and} \quad \left\{(1-e^{it})(1+e^{it})M'(t):\ t\in\mathbb{T}_0\right\},$$

are both R-bounded. Then M is an ℓ_p-multiplier (from X to Y) for $1<p<\infty$.

The converse of Blunck's theorem also holds without any restriction on the Banach spaces X,Y as follows:

Theorem 2.6. *[7, Proposition 1.4] Let $p\in(1,\infty)$ and let X,Y be Banach spaces. Let $M:\mathbb{T}\to\mathcal{B}(X;Y)$ be an operator valued function. Suppose that there is a bounded operator $T_M: l_p(\mathbb{Z};X)\to l_p(\mathbb{Z};Y)$ such that (2.9) holds. Then the set*

$$\{M(t): t\in\mathbb{T}\}$$

is R-bounded.

3. Maximal ℓ_p-regularity of the linear shifted model

Let $\alpha\in\mathbb{R}_+,\ \beta_j\in\mathbb{R},\ \tau_j\in\mathbb{Z}$ and A be a closed linear operator defined in a Banach space X. For a given vector-valued sequence $f:\mathbb{Z}\to X$ we consider the abstract discrete equation

$$\Delta^\alpha u(n) = Au(n) + \sum_{j=1}^k \beta_j u(n-\tau_j) + f(n), \qquad n\in\mathbb{Z}. \tag{3.1}$$

Definition 3.1. Let $1<p<\infty$ be given. We say that A has maximal ℓ_p-regularity if for each $f\in\ell_p(\mathbb{Z};X)$ there exists a unique solution $u\in\ell_p(\mathbb{Z};[D(A)])$ of (3.1), where $[D(A)]$ denotes the domain of A endowed with the graph norm.

In this section, our purpose is to provide a characterization of maximal ℓ_p-regularity of equation (3.1). For the sake of simplicity, we will first obtain this characterization for the following equation

$$\Delta^\alpha u(n) = Au(n) + \beta u(n-\tau) + f(n), \qquad n \in \mathbb{Z}. \tag{3.2}$$

As a corollary, we will have a full characterization of maximal ℓ_p-regularity for the more general equation (3.1). In the following result, we show the equivalence between the R-boundedness of the operator valued symbol of the equation (3.2) defined by $((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$ and the fact that it is an ℓ_p multiplier.

Theorem 3.2. *Let X be a UMD space, $1 < p < \infty$, $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ and $\tau \in \mathbb{Z}$. Suppose*

$$\{(1-e^{-it})^\alpha - \beta e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A),$$

and denote $M(t) := ((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$, then the following assertions are equivalent:

(i) $M(t)$ is an ℓ_p-multiplier from X to $[D(A)]$.

(ii) $\{M(t)\}_{t\in\mathbb{T}}$ is R-bounded.

Proof. $(ii) \implies (i)$ Since the set $\{M(t)\}_{t\in\mathbb{T}}$ is R-bounded, it is not difficult to observe that $\{(e^{it}-1)(e^{it}+1)M'(t)\}_{t\in\mathbb{T}}$ is also R-bounded. Indeed, given $t \in \mathbb{T}$, $M(t) : X \to [D(A)]$ can be rewritten as

$$M(t) = (f_\alpha(t) - \beta e^{-it\tau} - A)^{-1}, \quad t \in \mathbb{T}, \tag{3.3}$$

where $f_\alpha(t) := (1-e^{-it})^\alpha$. Moreover,

$$M'(t) = -(f_\alpha(t) - A)^{-2}(f'_\alpha(t) + i\tau\beta e^{-it\tau}) \quad t \in \mathbb{T}. \tag{3.4}$$

Since $f'_\alpha(t) = \alpha\left(\frac{e^{it}-1}{e^{it}}\right)^{\alpha-1}\frac{i}{e^{it}} = i\alpha f_\alpha(t)\frac{1}{e^{it}-1}$, $t \in \mathbb{T}$ we obtain in (3.4) that:

$$M'(t) = (f_\alpha(t) - \beta e^{-it\tau} - A)^{-2} i\alpha f_\alpha(t)\frac{1}{1-e^{it}} - i\tau\beta e^{-it\tau}(f_\alpha(t) - \beta e^{-it\tau} - A)^{-2} \quad t \in \mathbb{T}. \tag{3.5}$$

Therefore,

$$(1-e^{it})(1+e^{it})M'(t) = i\alpha M(t)^2(1+e^{it})\left(1-e^{-it}\right)^\alpha - i\tau\beta e^{-it\tau}M(t)^2(1-e^{it})(1+e^{it}), \quad t \in \mathbb{T}.$$

From [2, Proposition 2.2.5] we conclude that the set $\{(1-e^{it})(1+e^{it})M'(t) : \ t \in \mathbb{T}\}$ is R- bounded and the claim is proven. Finally by Theorem 2.5 we obtain (i).

$(i) \implies (ii)$ By hypothesis we have that there exists a bounded operator T such that (2.9) holds. The conclusion follows from Theorem 2.6.

□

In some cases, it is necessary to have simultaneous R-bounded symbols in order to obtain ℓ_p-multipliers. This is the content of the following theorem.

Theorem 3.3. *Assume that X is a UMD space. Let $1 < p < \infty$, $\tau \in \mathbb{Z}, \beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$ be given. Assume*

$$\{(1-e^{-it})^\alpha - \beta e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A),$$

and the set $\{((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}\}_{t\in\mathbb{T}}$ is R-bounded, then the sets $N(t) := (1-e^{-it})^\alpha((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$ and $S(t) := e^{-it\tau}((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$ are ℓ_p-multipliers.

Proof. It is enough to observe that since $N(t) = f_\alpha(t)M(t)$ and $S(t) = e^{-it\tau}M(t)$ where $f_\alpha(t) := (1-e^{-it})^\alpha$, the R-boundedness of $N(t)$ and $S(t)$ follows. Then the claim holds from the equivalence between the R-boundedness of the sets $\{N(t)\}_{t\in\mathbb{T}}$ and $\{S(t)\}_{t\in\mathbb{T}}$ and the fact that $N(t)$ and $S(t)$ are ℓ_p-multipliers from X to $[D(A)]$.

The proof of this last assertion is analogous to the one of Theorem 3.2, taking into account the R-boundedness of the sets $\{N(t)\}_{t\in\mathbb{T}}$ and $\{S(t)\}_{t\in\mathbb{T}}$ and the identities:

$$(1-e^{it})(1+e^{it})N'(t) = i\alpha N(t)^2(1+e^{it}) - i\tau\beta S(t)N(t)(1+e^{it})(1-e^{it}) - i\alpha N(t)(1+e^{it}),$$
$$(1-e^{it})(1+e^{it})S'(t) = i\alpha S(t)N(t)(1+e^{it}) - i\tau(1+e^{it})(1-e^{it})(S(t)^2 + S(t)).$$

□

Our next result provides sufficient conditions, in terms of the spectrum of the operator A, in order to have an ℓ_p-multiplier. We point out that an additional condition on the Banach space X in this case, is not needed. In what follows we denote:

$$\delta_\tau(n) = \begin{cases} 1 & n=\tau, \\ \\ 0 & otherwise. \end{cases}$$

Theorem 3.4. *Let X be a Banach space, $1<p<\infty$, $\tau\in\mathbb{Z}$, $\beta\in\mathbb{R}$ and $\alpha\in\mathbb{R}_+$ be given. Assume*

$$\{(1-e^{-it})^\alpha - \beta e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A).$$

If A has maximal ℓ_p-regularity, then $M(t) := ((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$ is an ℓ_p-multiplier.

Proof. Let $f\in\ell_p(\mathbb{Z};X)$ be given. By hypothesis there exists a unique sequence $u_f:\mathbb{Z}\to[D(A)]$ such that $u_f\in\ell_p(\mathbb{Z};[D(A)])$ satisfies:

$$\Delta^\alpha u_f(n) = Au_f(n) + \beta u_f(n-\tau) + f(n), \quad n\in\mathbb{Z}. \tag{3.6}$$

Let $T_\alpha:\ell_p(\mathbb{Z};X)\to\ell_p(\mathbb{Z};[D(A)])$ be defined by $T_\alpha(f)=u_f$, where u_f is the unique solution of (3.6). It can be easily shown using the closed graph theorem that T_α is bounded. To finish the proof, let $\varphi\in C^\infty_{per}(\mathbb{T})$, $f\in\ell_p(\mathbb{Z};X)$ be given and set $u:=T_\alpha f$. Since $k^{-\alpha}\in\ell_1(\mathbb{Z})$, (see [40, p.42 formula (2)]) we obtain the following identities:

$$\begin{aligned}(k^{-\alpha}\circ\check{S})(n) &= \sum_{j=0}^\infty k^{-\alpha}(j)\check{S}(j+n) = \sum_{j=0}^\infty k^{-\alpha}(j)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n+j)t}S(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\Big(\sum_{j=0}^\infty e^{ijt}k^{-\alpha}(j)\Big)S(t)dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\widehat{k}^{-\alpha}(-t)S(t)dt =: (\widehat{k}^{-\alpha}_{-}\cdot S\check{)}(n),\end{aligned} \tag{3.7}$$

valid for any $S\in C^\infty_{per}(\mathbb{T},\mathcal{B}(X,Y))$. Therefore, using the hypothesis, and that $M\in C^\infty_{per}(\mathbb{T},\mathcal{B}(X,[D(A)]))$, we get

$$\begin{aligned}\langle T_\alpha f,\check{\varphi}\rangle &= \langle u,\check{\varphi}\rangle = \sum_{n\in\mathbb{Z}}\check{\varphi}(n)u(n) = \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{int}\varphi(t)u(n)dt \\ &= \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}(1-e^{it})^\alpha e^{int}\varphi(t)((1-e^{it})^\alpha-\beta e^{it\tau}-A)^{-1}u(n)dt \\ &- \beta\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{it\tau}((1-e^{it})^\alpha-\beta e^{it\tau}-A)^{-1}u(n)e^{int}\varphi(t)dt \\ &- \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}((1-e^{it})^\alpha-\beta e^{it\tau}-A)^{-1}Au(n)e^{int}\varphi(t)dt \\ &= \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{int}\widehat{k}^{-\alpha}(-t)\varphi(t)M(-t)u(n)dt \\ &- \beta\sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{int}\widehat{\delta}_\tau(t)\varphi(t)M(-t)u(n)dt - \sum_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{int}\varphi(t)M(-t)Au(n)dt \\ &= \langle u,(\widehat{k}^{-\alpha}_{-}\cdot\varphi\cdot M_-\check{)}\rangle - \beta\langle u,(\widehat{\delta}_{\tau\,-}\cdot\varphi\cdot M_-\check{)}\rangle - \langle Au,(\varphi\cdot M_-\check{)}\rangle \\ &= \langle u,k^{-\alpha}\circ(\varphi\cdot M_-\check{)}\rangle - \beta\langle u,\delta_\tau\circ(\varphi\cdot M_-\check{)}\rangle - \langle Au,(\varphi\cdot M_-\check{)}\rangle,\end{aligned}$$

where $\widehat{\delta}_\tau(t)=e^{-i\tau t}$, and in the last equality we have used (3.7) with $S=\varphi\cdot M_-$. Therefore using (2.5) and (2.6) we get

$$\langle u,\check{\varphi}\rangle = \langle k^{-\alpha}*u,(\varphi\cdot M_-\check{)}\rangle - \beta\langle\delta_\tau*u,(\varphi\cdot M_-\check{)}\rangle - \langle Au,(\varphi\cdot M_-\check{)}\rangle = \langle\Delta^\alpha u-\beta u_\tau-Au,(\varphi\cdot M_-\check{)}\rangle. \tag{3.8}$$

where $\delta_\tau * u(n) = u(n-\tau) := u_\tau(n)$. We conclude that $\langle T_\alpha f, \check{\varphi}\rangle = \langle f, (\varphi \cdot M_-\check{)}\rangle$. This proves the claim and the theorem.

□

The following theorem shows that the converse of Theorem 3.4 is also true.

Theorem 3.5. *Let X be a UMD space, and $1 < p < \infty$, $\tau \in \mathbb{Z}$, $\beta \in \mathbb{R}$, $\alpha \in \mathbb{R}_+$ be given. Suppose*

$$\{(1-e^{-it})^\alpha - \beta e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A).$$

The following assertions are equivalent:

- *(i) The operator A has maximal ℓ_p-regularity;*
- *(ii) $M(t) := ((1-e^{-it})^\alpha - \beta e^{-it\tau} - A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;*
- *(iii) The set $\{M(t) \ : \ t \in \mathbb{T}\}$ is R-bounded.*

Proof. It remains to show that (ii) implies (i). Let $f \in \ell_p(\mathbb{Z}; X)$ be given. By hypothesis, there exists $u \in \ell_p(\mathbb{Z}; [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} u(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} (\varphi \cdot M_-\check{)}(n) f(n), \tag{3.9}$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Moreover, by Theorem 3.3 there exist $v, w \in \ell_p(\mathbb{Z}; [D(A)])$ such that

$$\sum_{n\in\mathbb{Z}} v(n)\check{\psi}(n) = \sum_{n\in\mathbb{Z}} (\psi \cdot N_-\check{)}(n) f(n), \tag{3.10}$$

and

$$\sum_{n\in\mathbb{Z}} w(n)\check{\eta}(n) = \sum_{n\in\mathbb{Z}} (\eta \cdot S_-\check{)}(n) f(n), \tag{3.11}$$

for all $\psi, \eta \in C^\infty_{per}(\mathbb{T})$. Since $N(t) = \widehat{k}^{-\alpha}(t)M(t)$ and $S(t) = \widehat{\delta}_\tau(t)M(t)$, we have:

$$(\psi \cdot N_-\check{)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\psi(t)\widehat{k}^{-\alpha}(-t)M(-t)dt,$$

and

$$(\eta \cdot S_-\check{)}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\eta(t)\widehat{\delta}_\tau(-t)M(-t)dt.$$

Choosing $\varphi(t) = \psi(t)\widehat{k^{-\alpha}}(-t) \in C^\infty_{per}(\mathbb{T})$ in (3.9) we get

$$\langle v, \check{\psi}\rangle = \langle u, (\psi \cdot \widehat{k}^{-\alpha}_-\check{)}\rangle.$$

From Lemma 2.2 we conclude from the above identity that

$$\Delta^\alpha u(n) = k^{-\alpha} * u(n) = v(n), \quad n \in \mathbb{Z}. \tag{3.12}$$

Now, considering $\varphi(t) = \eta(t)\widehat{\delta}_\tau(-t)$ in (3.9) we obtain $\langle w, \check{\eta}\rangle = \langle u, (\eta \cdot \widehat{\delta_\tau}_-)\check{\rangle}$. Observe that $\varphi \in C^\infty_{per}(\mathbb{T})$ because $\tau \in \mathbb{Z}$. Again, making use of Lemma 2.2, we conclude from the above identity that

$$w(n) = \delta_\tau * u(n) = u(n-\tau), \quad n \in \mathbb{Z}. \tag{3.13}$$

Since $N(t) - \beta S(t) = AM(t) + I$, after multiplication by $e^{int}\varphi(t)$ and integration over the interval $(-\pi, \pi)$, we have

$$(\varphi \cdot N_-\check{)}(n) - \beta(\varphi \cdot S_-\check{)}(n) = A(\varphi \cdot M_-\check{)}(n) + \check{\varphi} I,$$

for all $\varphi \in C^\infty_{per}(\mathbb{T})$. Then we obtain

$$\langle f, (\varphi \cdot N_-\check{)}\rangle - \beta\langle f, (\varphi \cdot S_-\check{)}\rangle = \langle f, A(\varphi \cdot M_-\check{)}\rangle + \langle f, \check{\varphi}\rangle,$$

and by replacing (3.9), (3.10) and (3.11) in the above identity we obtain

$$\sum_{n\in\mathbb{Z}} v(n)\check{\varphi}(n) - \beta\sum_{n\in\mathbb{Z}} w(n)\check{\varphi}(n) = \sum_{n\in\mathbb{Z}} Au(n)\check{\varphi}(n) + \sum_{n\in\mathbb{Z}} \check{\varphi}(n)f(n),$$

for all $\varphi \in C^{\infty}_{per}(\mathbb{T})$. Considering (3.12) and (3.13) we conclude that u satisfies the equation (3.2). We have proven the existence of a solution. It remains to prove the uniqueness.
Let $u : \mathbb{Z} \to [D(A)]$ be a solution of (3.2) with $f \equiv 0$. For all $\varphi \in C^{\infty}_{per}(\mathbb{T})$ and (3.8), we obtain

$$\langle u, \check{\varphi}\rangle = \langle \Delta^{\alpha}u - \beta u_{\tau} - Au, (\varphi \cdot M_{-}\check{)}\rangle = 0.$$

Taking $\varphi_k(t) := e^{-ikt}$, $k \in \mathbb{Z}$ we obtain $u \equiv 0$ and then the theorem is proven.

□

The following corollary follows directly from the closed graph theorem.

Corollary 3.6. *If the hypothesis of Theorem 3.5 holds true, then $u, \Delta^{\alpha}u$, $Au \in \ell_p(\mathbb{Z}; X)$ and there exists a constant $C > 0$ (independent of $f \in \ell_p(\mathbb{Z}; X)$) such that*

$$\|\Delta^{\alpha}u\|_{\ell_p(\mathbb{Z};X)} + |\beta|\|u\|_{\ell_p(\mathbb{Z};X)} + \|Au\|_{\ell_p(\mathbb{Z};X)} \leq C\|f\|_{\ell_p(\mathbb{Z};X)}. \tag{3.14}$$

We immediately obtain a characterization of maximal ℓ_p-regularity for Hilbert spaces, since each Hilbert space is UMD and, in such case, R-boundedness coincides with boundedness [2].

Corollary 3.7. *Let H be a Hilbert space, $\tau \in \mathbb{Z}$, $\beta \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Suppose*

$$\{(1 - e^{-it})^{\alpha} - \beta e^{-it\tau}\}_{t\in\mathbb{T}} \subset \rho(A).$$

The following assertions are equivalent:

(i) For all $f \in \ell_p(\mathbb{Z}; H)$ there exists a unique $u \in \ell_p(\mathbb{Z}; H)$ such that $u(n) \in D(A)$ for all $n \in \mathbb{Z}$, and u satisfies (3.2);

(ii) We have

$$\sup_{t\in\mathbb{T}} \left\| \Big((1 - e^{-it})^{\alpha} - \beta e^{-it\tau} - A \Big)^{-1} \right\| < \infty. \tag{3.15}$$

Example 3.8. Let us consider the following equation that corresponds to a discrete Lotka-Sharpe equation with delay (see e.g. [5]),

$$\Delta^2 u(n, x) = -u_x(n, x) - \mu u(n, x) + \nu u(n - 1, x) + f(n, x) \quad n \in \mathbb{Z}, \quad x \in \mathbb{R}, \tag{3.16}$$

where μ and ν are positive numbers. Equation (3.16) can be modeled as (3.2) with $\alpha = 2$, $Au = -u' - \mu u$, $\beta = \nu$ and $\tau = 1$. More concretely, we will analyse the existence of solutions for this equation in $\ell_p(\mathbb{Z}; L^2(\mathbb{R}))$.

It is well known that the operator $Bu = u'$ with domain $D(B) = W^{1,2}(\mathbb{R})$ generates a contraction semigroup on $L^2(\mathbb{R})$. Therefore, the following estimate holds true:

$$\|(\lambda - A)^{-1}\| = \|(\lambda + \mu - B)^{-1}\| \leq \frac{1}{\Re(\lambda) + \mu}.$$

It is not difficult to check that $\min_{t\in\mathbb{T}} \Re[(1 - e^{-it})^2 - \nu e^{-it}] = -\frac{1}{8}(2 + \nu)^2$. Therefore, for all μ such that

$$\frac{1}{8}(2 + \nu)^2 < \mu$$

we obtain from Corollary 3.7 that there exists a unique solution $u \in \ell_p(\mathbb{Z}; L^2(\mathbb{R}))$ of the Lotka-Sharpe equation (3.16).

As a corollary of Theorem 3.5, we immediately obtain the following characterization of maximal ℓ_p-regularity of equation (3.1).

Theorem 3.9. *Let X be a UMD space, $1 < p < \infty$, $\tau_j \in \mathbb{Z}$, $\beta_j \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Suppose*

$$\{(1 - e^{-it})^{\alpha} - \sum_{j=1}^{k} \beta_j e^{-it\tau_j}\}_{t\in\mathbb{T}} \subset \rho(A).$$

The following assertions are equivalent:

(i) The operator A has maximal ℓ_p-regularity;

(ii) $N(t) := ((1 - e^{-it})^{\alpha} - \sum_{j=1}^{k} \beta_j e^{-it\tau_j} - A)^{-1}$ is an ℓ_p-multiplier from X to $[D(A)]$;

(iii) The set $\{N(t) \ : \ t \in \mathbb{T}\}$ is R-bounded.

Remark 3.10. Below we illustrate the behavior of the set $\Omega_{\alpha,\beta,\tau} := \{(1 - e^{-it})^{\alpha} - \beta e^{-it\tau}\}_{t\in\mathbb{T}}$ for different values of α, τ and β. In Figure 1 we observe that the delay τ has the property of multiplication because there are two and three leaves representing the delay, respectively. It shows that the number of points in $\Omega_{\alpha,\beta,\tau}$ is greater in the measure that the delay increases.

Figure 1: left: $\alpha = 2; \beta \in (0, 7); \tau = 2$ right: $\alpha = 2; \beta \in (0, 7); \tau = 3$

4. ℓ_p-regularity of nonlinear shifted equations

We will first prove the following theorem that will be very useful for the analysis of the existence of solutions of some nonlinear equations.

Theorem 4.1. *Let X be a UMD space, $1 < p < \infty$, $\tau \in \mathbb{Z}$, $\beta_j \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Assume*

$$\{(1 - e^{-it})^{\alpha} - \sum_{j=1}^{k} \beta_j e^{-it\tau_j}\}_{t\in\mathbb{T}} \subset \rho(A),$$

and that the set $\{(1 - e^{-it})^{\alpha} - \sum_{j=1}^{k} \beta_j e^{-it\tau_j} \ : \ t \in \mathbb{T}\}$ is R-bounded, then the operator $\mathcal{A}u(n) := \Delta^{\alpha}u(n) - Au(n) - \sum_{j=1}^{k} \beta_j u(n - \tau_j)$ with $D(\mathcal{A}) := \ell_p(\mathbb{Z}; [D(A)])$ is an isomorphism onto.

Proof. Observe that the space $\ell_p(\mathbb{Z}; [D(A)])$ becomes a Banach space under the norm $|||u||| := \|\Delta^{\alpha}u\|_p + |\sum_{j=1}^{k} \beta_j|\|u\|_p + \|Au\|_p$. By hypothesis and Corollary 3.6 the inequality (3.14) holds true, and then we get $|||u||| \leq C||\mathcal{A}u||$. On the other hand, by definition of the operator $\mathcal{A}$ we obtain $||\mathcal{A}u|| \leq |||u|||$. Therefore $\mathcal{A}$ is an isomorphism. By Theorem 3.9, the operator A has maximal ℓ_p- regularity and then $\mathcal{A}$ is onto.

□

We now proceed to study the existence of solutions of the following nonlinear equation:

$$\Delta^{\alpha}u(n) = Au(n) + \sum_{j=1}^{k} \beta_j u(n - \tau_j) + f(n, u(n)). \tag{4.1}$$

We will first introduce the following notation. Let $\mathcal{B} := \mathcal{A}^{-1} : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; D(A))$ be defined by $\mathcal{B}(f) := u$ where u is the unique solution of the equation (4.1). Let $\mathcal{N} : \ell_p(\mathbb{Z}; D(A)) \to \ell_p(\mathbb{Z}; X)$ be defined by $\mathcal{N}(u)(n) := f(n, u(n))$ the Nemytskii's operator.

Theorem 4.2. *Let H be a Hilbert space, $\tau \in \mathbb{Z}$, $\beta_j \in \mathbb{R}$ and $\alpha \in \mathbb{R}_+$. Assume that:*

(i) $\{(1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j}\}_{t\in\mathbb{T}} \subset \rho(A)$ *and* $\sup_{t\in\mathbb{T}} \left\| \left((1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j} - A \right)^{-1} \right\| < \infty$,

(ii) There exists $M > 0$ such that $\sup_{||u||\le M} ||\mathcal{N}(u)||_{\ell_2(\mathbb{Z};H)} \le \frac{M}{||\mathcal{B}||}$,

(iii) The closed unit ball of $D(A)$ is compact in H,

then the equation (4.1) has a solution $u \in \ell_2(\mathbb{Z}; H)$ and $||u||_{\ell_2(\mathbb{Z};H)} \le M$.

Proof. Since (i) holds, for all $K \in \mathbb{Z}$ we can define the operator $\mathcal{B}_K : \ell_2(\mathbb{Z}; H) \to \ell_2(\mathbb{Z}; H)$ given by

$$(\mathcal{B}_K)(g)(n) = \frac{1}{2\pi} \int_{-\pi+\frac{1}{K}}^{\pi+\frac{1}{K}} \left((1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j} - A \right)^{-1} \hat{g}(t) e^{int} dt.$$

It is clear by (iii) that B_K is compact for all $K \in \mathbb{Z}$. Moreover, since (i) holds as $K \to \infty$, $\mathcal{B}_K$ converges in norm to $\mathcal{B}$, so $\mathcal{B}$ is compact. Observe that $\mathcal{B}$ is well defined by Theorem 4.1. We can now apply the Schauder's fixed point theorem to the equation $u = \mathcal{B}\mathcal{N}(u)$ in the set $\{u \in \ell_2(\mathbb{Z}; D(A)) : ||u|| \le M\}$ and then the conclusion holds. □

We now consider the nonlinear equation:

$$\Delta^\alpha u(n) = Au(n) + \sum_{j=1}^k \beta_j u(n-\tau_j) + G(u)(n) + \rho f(n), \tag{4.2}$$

where $0 < \rho < 1$, $f \in \ell_p(\mathbb{Z}; X)$ and $G : \ell_p(\mathbb{Z}; X) \to \ell_p(\mathbb{Z}; X)$ are given. The following result shows the existence of solutions of (4.2) under some assumptions on the nonlinear term G.

Theorem 4.3. *Let X be a UMD space, $1 < p < \infty$, $\tau \in \mathbb{Z}$ and $\alpha \in \mathbb{R}_+$. Assume that*

$$\{(1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j}\}_{t\in\mathbb{T}} \subset \rho(A).$$

Suppose that

(i) The set $\{((1-e^{-it})^\alpha - \sum_{j=1}^k \beta_j e^{-it\tau_j} - A)^{-1}\}_{t\in\mathbb{T}}$ *is R-bounded,*

(ii) $G(0) = 0$, G is continuously Fréchet differentiable at $u = 0$ and $G'(0) = 0$,

then there exists $\rho^* > 0$ such that equation (4.2) is solvable for each $\rho \in [0, \rho^*)$, with solution $u := u_\rho \in \ell_p(\mathbb{Z}; X)$.

Proof. Let $\rho \in (0, 1)$ be given and let us define the following one-parameter family:

$$H[u, \rho] = -\mathcal{A}u + G(u) + \rho f.$$

By (ii), it is clear that $H[0, 0] = 0$, H is continuously differentiable at $(0, 0)$ and the partial Fréchet derivative is $H^1_{(0,0)} = -\mathcal{A}$, which is invertible by Theorem 4.1. We now apply the implicit function theorem (see e.g. [17, Theorem 17.6]), and then, we can find ρ^* such that for all $\rho \in [0, \rho^*)$ there exists $u = u_\rho \in \ell_p(\mathbb{Z}; X)$ such that $\Delta^\alpha u(n) = Au(n) + \sum_{j=1}^k \beta_j u(n-\tau_j) + G(u(n)) + \rho f(n)$ for all $n \in \mathbb{Z}$. □

We finish this paper with the following set of examples in order to illustrate our results.

Example 4.4. Let us consider the inhomogeneous logistic equation:

$$u(n+1) = ru(n)(1-u(n)) + \rho f(n), \qquad r, \rho > 0, \quad n \in \mathbb{Z}. \tag{4.3}$$

This equation can be expressed as $\Delta u(n) = (r-1)u(n) - ru(n)^2$. It fits into the scheme of equation (4.2) with $X = \mathbb{R}$, $\alpha = 1$, $G(u(n)) = -ru(n)^2$, $A = (r-1)Id$ and $\beta_j = 0$ for all $j \in \mathbb{N}$.

We now want to analyze the existence of solutions of this equation in $\ell_p(\mathbb{Z};\mathbb{R})$. It is clear that $\sup_{t\in\mathbb{T}} \left\|((1-e^{-it}) - (r-1)Id)^{-1}\right\| < \infty$, whenever $r \neq 1, 3$ and then (i) in Theorem 4.3 is satisfied. Moreover, it is clear that $G(0) = 0$ is continuously Fréchet differentiable at $u = 0$ and $G'(0) = 0$. As a consequence of Theorem 4.3, there exists $\rho^* > 0$ such that the equation (4.3) is solvable for each $\rho \in [0, \rho^*)$, with solution $u = u_\rho \in \ell_p(\mathbb{Z};\mathbb{R})$.

As the fractional version of the logistic equation, we consider:

$$\Delta^\alpha u(n) = (r-1)u(n) - ru(n)^2 + \rho f(n), \qquad r, \rho > 0, \quad n \in \mathbb{Z}. \tag{4.4}$$

Let us calculate the values of r such that $(1-e^{-it})^\alpha = (r-1)$, that is:

$$(2-2\cos t)^{\frac{\alpha}{2}} \cos(\alpha \arctan(\frac{\sin t}{1-\cos t})) + i(2-2\cos t)^{\frac{\alpha}{2}} \sin(\alpha \arctan(\frac{\sin t}{1-\cos t})) = r-1.$$

Since $r-1 \in \mathbb{R}$ then necessarily, $\sin(\alpha \arctan(\frac{\sin t}{1-\cos t})) = 0$ and $\frac{\sin t}{1-\cos t} = \tan(\frac{k\pi}{\alpha}), k \in \mathbb{Z}$, that is:

$$\frac{\sqrt{1-\cos t^2}}{1-\cos t} = \frac{\sin t}{1-\cos t} = \tan(\frac{k\pi}{\alpha}).$$

Solving this equation for $\cos t$ we obtain $\cos t = 1$ or $\cos t = \frac{\tan(\frac{k\pi}{\alpha})^2 - 1}{1+\tan(\frac{k\pi}{\alpha})^2}$. As a result, r will be such that:

$$r - 1 = \pm\Big(\frac{4}{1+\tan(\frac{k\pi}{\alpha})}\Big)^{\frac{\alpha}{2}}, \quad k \in \mathbb{Z} \quad \text{or} \quad r = 1.$$

Then $\sup_{t\in\mathbb{T}} \left\|((1-e^{-it})^\alpha - (r-1)Id)^{-1}\right\| < \infty$, whenever $r \neq 1, 1 \pm \big(\frac{4}{1+\tan(\frac{k\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$, $k \in \mathbb{Z}$. As a consequence, given $\alpha > 0$, Theorem 4.3 asserts the existence of $\rho^* > 0$ such that the equation (4.4) is solvable in $\ell_p(\mathbb{Z};\mathbb{R})$ for each $\rho \in [0, \rho^*)$ and $r \neq 1, 1 \pm \big(\frac{4}{1+\tan(\frac{k\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$, where $-\frac{\alpha}{2} < k < \frac{\alpha}{2}$, $k \in \mathbb{Z}$. As an example, we study the case $2 < \alpha < 4$. In Figure 2 we observe that if $2 < \alpha < 2.5$ equation (4.4) will have a solution in $\ell_p(\mathbb{Z};\mathbb{R})$ for all $r \neq 1, 1 + \big(\frac{4}{1+\tan(\frac{\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$. In contrast, if $\alpha \geq 2.5$ we can ensure the existence of solutions whenever $r \neq 1, 1 \pm \big(\frac{4}{1+\tan(\frac{\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$.

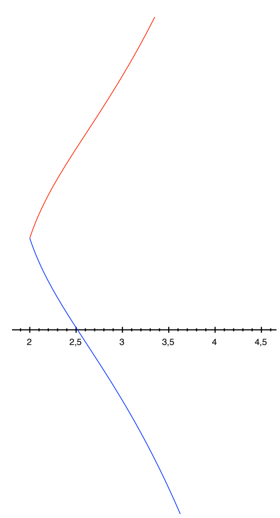

Figure 2: Red: $1 + \big(\frac{4}{1+\tan(\frac{\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$ Blue: $1 - \big(\frac{4}{1+\tan(\frac{\pi}{\alpha})}\big)^{\frac{\alpha}{2}}$

Example 4.5. The one dimensional discrete Nagumo equation:

$$\frac{\partial}{\partial t}u(n,t) = d[u(n+1,t) - 2u(n,t) + u(n-1,t)] + H(u(n,t)), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z},$$

where $H(u(n,t)) := u(n,t)(u(n,t)-a)(1-u(n,t))$, $d > 0$, $a > 0$ serves as a prototype system for investigating the properties of lattice differential equations. It has also been proposed as a model for conduction in myelinated nerve axons [12, 38]. Here we investigate the existence of ℓ_p-solutions for the inhomogeneous equation. In this case, the Nagumo equation can be rewritten as follows:

$$\Delta u(n,t) = \frac{1}{d}\frac{\partial}{\partial t}u(n,t) + (\frac{a}{d}+1)u(n,t) - u(n-1,t) + G(u(n,t)) + \frac{1}{d}f(n,t), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{4.5}$$

In Theorem 4.3 we set $\alpha = 1, A = \frac{1}{d}\frac{\partial}{\partial t} + (\frac{a}{d}+1)Id$, $\beta_1 = -1, \tau = 1, \rho = \frac{1}{d}$ and $G(u(n,t)) = \frac{1}{d}u(n,t)^3 - (\frac{1+a}{d})u(n,t)^2$. It is clear that $G(0) = 0$ is continuously Fréchet differentiable at $u = 0$ and $G'(0) = 0$. It is well known that the operator $Bu = -u'$ with domain $D(B) = W^{1,2}(\mathbb{R})$ generates a contraction C_0-group on $L^2(\mathbb{R})$. Therefore, $a \in \rho(B)$ and the estimate

$$\|(1-e^{-it})^\alpha - \beta_1 e^{-it} - A)^{-1}\| = \|(1 - (\frac{a}{d}+1) + \frac{1}{d}B)^{-1}\| = d\|(a-B)^{-1}\| \leq \frac{d}{a},$$

holds for all $a > 0$. Then Theorem 4.3 asserts the existence of a solution in $\ell_p(\mathbb{Z}; L^2(\mathbb{R}))$ of the inhomogeneous Nagumo equation (4.5) for d to be big enough.

We now consider a fractional version of the Nagumo equation given by

$$\Delta^\alpha u(n,t) = \frac{1}{d}\frac{\partial}{\partial t}u(n,t) + (\frac{a}{d}+1)u(n,t) - u(n-1,t) + G(u(n,t)) + \frac{1}{d}f(n,t), \quad t \in \mathbb{R}, \quad n \in \mathbb{Z}. \tag{4.6}$$

In this case, given $\alpha > 0$, for a, d being big enough to satisfy $d < \frac{a}{2^\alpha}$, and as consequence of Theorem 4.3, we find the existence of a solution in $\ell_p(\mathbb{Z}; L^2(\mathbb{R}))$ of (4.6).

Indeed, since

$$\begin{aligned}|(1-e^{-it})^\alpha + e^{-it}| &\leq 1 + |(2-2\cos t)^{\frac{\alpha}{2}}\cos(\alpha\arctan(\frac{\sin t}{1-\cos t})) + i(2-2\cos t)^{\frac{\alpha}{2}}\sin(\alpha\arctan(\frac{\sin t}{1-\cos t}))| = \\ &= 1 + |(2-2\cos t)^{\frac{\alpha}{2}} e^{i\alpha\arctan(\frac{\sin t}{1-\cos t})}| \leq 1 + 2^\alpha,\end{aligned}$$

we have for all $t \in \mathbb{T}$ that

$$\|(d(1-e^{-it})^\alpha + e^{-it})((a+d)-B)^{-1}\| \leq \frac{d(1+2^\alpha)}{a+d} < 1. \tag{4.7}$$

Considering now the identity

$$\|(\lambda - \mu + B)^{-1}\| \leq \|(\mu - B)^{-1}\|\|(\lambda(\mu-B)^{-1} - Id)^{-1}\|, \tag{4.8}$$

valid for all μ, $\mu - \lambda \in \rho(B)$, we have that for a, d such that $d < \frac{a}{2^\alpha}$ there exists a constant $M > 0$ verifying

$$\begin{aligned}\|((1-e^{-it})^\alpha - \beta_1 e^{-it} - A)^{-1}\| &= d\|(d(1-e^{-it})^\alpha + de^{-it} - (\frac{a}{d}+1) + \frac{1}{d}B)^{-1}\| = \\ &= d\|(d(1-e^{-it})^\alpha + de^{-it} - (a+d) + B)^{-1}\| \\ &\leq d\|((a+d)-B)^{-1}\|\|(d((1-e^{-it})^\alpha + e^{-it})((a+d)-B)^{-1} - Id)^{-1}\| \\ &\leq \frac{dM}{(a+d)},\end{aligned}$$

where we have used inequality (4.8) with $\lambda := d((1-e^{-it})^\alpha + e^{-it})$, $\mu := a+d$, $Bu = -u'$, the estimate (4.7) and the Neumann's series. We conclude that if a, d are big enough, satisfying $d < \frac{a}{2^\alpha}$, then we obtain the existence of a solution $u \in \ell_p(\mathbb{Z}; L^2(\mathbb{R}))$ for the equation (4.6).

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