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Orders in Artinian rings, Goldie's Theorem and the largest left quotient ring of a ring

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Abstract

This short survey is about some old and new results on left orders in left Artinian rings, new criteria for a ring to have a semisimple left quotient ring, new concepts (eg, the largest left quotient ring of a ring).

Key Words: Goldie's Theorem, orders, left Artinian ring, the left quotient ring of a ring, the largest left quotient ring of a ring, the largest regular left Ore set.

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Contents

1. Introduction.
2. New criteria for a ring to have a semisimple left quotient ring.
3. Old criteria for a ring to have a left Artinian left quotient ring.
4. Necessary and sufficient conditions for a ring to have a left Artinian left quotient ring.
5. A criterion via associated graded ring.
6. Criteria similar to Robson's Criterion.
7. A left quotient ring of a factor ring.
8. The largest denominator sets and the largest left quotient ring of a ring.
9. The maximal left quotient rings of a ring.
10. Examples.

1 Introduction

In this paper, module means a left module, all rings are associative with 1. The present paper comprises three parts.

Part I, 'New Criteria for a Ring to have a Semisimple Left Quotient Ring' (Section 2). Goldie's Theorem (1958, 1960) is an old and up to 2013 was the only example of such criteria. Four new criteria will be given that are independent of Goldie's Theorem and are based on completely new ideas and approach.

Part II, 'Left Orders in Left Artinian Rings' (Sections 3–7), deals with old and new criteria for a ring to have a left Artinian left quotient ring.

Part III, 'The Largest Left Quotient Ring of a Ring' (Sections 8–10), is about new recent concepts and results obtained in order to answer the old question:

Why does the classical left quotient ring of a ring not always exist?

A positive step in this direction is the fact that for an arbitrary ring R there always exists the largest left quotient ring $Q_l(R)$, [4], which coincides with the classical left quotient ring if the latter exists. Another new concept/fact is the existence of the maximal left quotient rings (for an arbitrary ring R). Their existence gives an affirmative answer to the following question:

given a ring R, replace the ring R by its left localization $S_1^{-1}R$ at a left denominator set (that not necessarily consists of regular elements). Then repeat the step again and again (infinitely many times of arbitrary cardinality, if necessary): $S_2^{-1}(S_1^{-1}R)$, $S_3^{-1}(S_2^{-1}(S_1^{-1}R)), \ldots$; will this process stop? (i.e. do we reach the moment we cannot invert anything new?) The answer is yes and the rings we obtain are called the maximal left quotient rings of a ring and any such a ring can be written as $S^{-1}R$ for some left denominator set S of the ring R, [4].

Goldie's Theorem (1960), which is one of the most important results in Ring Theory, is a criterion for a ring to have a semisimple left quotient ring. The aim of the paper is to give four new criteria (using a completely different approach and new ideas). The first one is based on the recently discovered fact that for an *arbitrary* ring R the set $\mathcal{M}$ of *maximal* left denominator sets of R is a non-empty set [4].

Theorem (The First Criterion). *A ring R has a semisimple left quotient ring Q iff $\mathcal{M}$ is a finite set, $\bigcap_{S\in\mathcal{M}} \mathrm{ass}(S) = 0$ and, for each $S \in \mathcal{M}$, the ring $S^{-1}R$ is a simple left Artinian ring. In this case, $Q \simeq \prod_{S\in\mathcal{M}} S^{-1}R$.*

The Second Criterion is given via the minimal primes of R and goes further than the First one in the sense that it describes explicitly the maximal left denominator sets S via the minimal primes of R. The Third Criterion is close to Goldie's Criterion but it is easier to check in applications (basically, it reduces Goldie's Theorem to the prime case). The Fourth Criterion is given via certain left denominator sets.

The conditions in old criteria for a ring R to have a left Artinian left quotient ring Q are 'strong' (like 'the ring R is a left Goldie ring') and given in terms of the ring R itself and its ideals. The conditions of the new criteria are 'weak' (like 'the ring $\overline{R}$ is a left Goldie ring' where $\overline{R} := R/\mathfrak{n}$ and $\mathfrak{n}$ is the prime radical of R) and given in terms of the ring $\overline{R}$ (rather than R) and of its finitely many explicit modules.

Goldie's Theorem [12] characterizes left orders in semi-simple rings, it is a criterion of when the left quotient ring of a ring is semi-simple (earlier, characterizations were given, by Goldie [11] and Lesieur and Croisot [16], of left orders in a simple Artinian ring). Talintyre [26] and Feller and Swokowski [10] have given conditions which are sufficient for a left Noetherian ring to have a left quotient ring. Further, for a left Noetherian ring which has a left quotient ring, Talintyre [27] has established necessary and sufficient conditions for the left quotient ring to be left Artinian. Small [21, 22], Robson [20], and latter Tachikawa [25] and Hajarnavis [13] have given different criteria for a ring to have a left Artinian left quotient ring. In this paper, three more new criteria are given (Theorem 4.1, Theorem 5.1 and Theorem 6.1).

Theorem 7.1 gives an affirmative answer to the question: *Let R be a ring with a left Artinian left quotient ring Q and I be a $\mathcal{C}$-closed ideal of R such that $I \subseteq \mathfrak{n}$. Is the left quotient ring $Q(R/I)$ of R/I a left Artinian ring?*

The set $\mathcal{C}$ of regular elements of a ring R is not always a left (or right) Ore set in R (hence, the left quotient ring $\mathcal{C}^{-1}R$ or the right quotient ring $R\mathcal{C}^{-1}$ does not always exist) but there *always* exists the *largest regular left Ore set* $S_{l,0}$ and the *largest regular right Ore set* $S_{r,0}$ in $\mathcal{C}$ of the ring R, [4]. In general, $S_{l,0} \neq S_{r,0}$, [3]. In [4], the *largest left quotient ring* $Q_l(R) := S_{l,0}^{-1}R$ and the *largest right quotient ring* $Q_r(R) := RS_{r,0}^{-1}$ are introduced. In [3], these rings are found for the ring $\mathbb{I}_1 = K\langle x, \partial, \int\rangle$ of polynomial integro-differential operators over a field K of characteristic zero, $S_{l,0}(\mathbb{I}_1) \neq S_{r,0}(\mathbb{I}_1)$ and $Q_l(\mathbb{I}_1) \not\simeq Q_r(\mathbb{I}_1)$.

Part I
New criteria for a ring to have a semisimple left quotient ring

In the paper, the following notation is fixed:

- R is a ring with 1, R^* is its group of units, $\mathfrak{n} = \mathfrak{n}_R$ is its prime radical and $\mathrm{Min}(R)$ is the set of minimal primes of R;
- $\mathcal{C} = \mathcal{C}_R$ is the set of regular elements of the ring R (i.e. $\mathcal{C}$ is the set of non-zero-divisors of the ring R);
- $Q = Q_{l,cl}(R) := \mathcal{C}^{-1}R$ is the *left quotient ring* (the *classical left ring of fractions*) of the ring R (if it exists) and Q^* is the group of units of Q;
- $\mathrm{Den}_l(R, \mathfrak{a})$ is the set of left denominator sets S of R with $\mathrm{ass}(S) = \mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R and $\mathrm{ass}(S) := \{r \in R \,|\, sr = 0 \text{ for some } s \in S\}$,
- $\mathrm{max.Den}_l(R)$ is the set of maximal left denominator sets of R (it is always a *non-empty* set, [4]).
- $\mathrm{Ore}_l(R) := \{S \,|\, S \text{ is a left Ore set in } R\}$;
- $\mathrm{Den}_l(R) := \{S \,|\, S \text{ is a left denominator set in } R\}$;
- $\mathrm{Loc}_l(R) := \{S^{-1}R \,|\, S \in \mathrm{Den}_l(R)\}$;
- $\mathrm{Ass}_l(R) := \{\mathrm{ass}(S) \,|\, S \in \mathrm{Den}_l(R)\}$;
- $S_\mathfrak{a} = S_\mathfrak{a}(R) = S_{l,\mathfrak{a}}(R)$ is the *largest element* of the poset $(\mathrm{Den}_l(R, \mathfrak{a}), \subseteq)$ and $Q_\mathfrak{a}(R) := Q_{l,\mathfrak{a}}(R) := S_\mathfrak{a}^{-1}R$ is the *largest left quotient ring associated to* $\mathfrak{a}$, $S_\mathfrak{a}$ exists (Theorem 2.1, [4]);
- In particular, $S_0 = S_0(R) = S_{l,0}(R)$ is the largest element of the poset $(\mathrm{Den}_l(R, 0), \subseteq)$, i.e. the largest regular left Ore set of R, and $Q_l(R) := S_0^{-1}R$ is the largest left quotient ring of R [4];
- $\mathrm{Loc}_l(R) := \{[S^{-1}R] \,|\, S \in \mathrm{Den}_l(R)\}$ where $[S^{-1}R]$ is an R-*isomorphism* class of the ring $S^{-1}R$ (a ring isomorphism $\sigma : S^{-1}R \to S'^{-1}R$ is called an R-*isomorphism* if $\sigma(\frac{r}{1}) = \frac{r}{1}$ for all elements $r \in R$); we usually write $S^{-1}R$ instead of $[S^{-1}R]$ if this does not lead to confusion;
- $\mathrm{Loc}_l(R, \mathfrak{a}) := \{[S^{-1}R] \,|\, S \in \mathrm{Den}_l(R, \mathfrak{a})\}$.

The largest left quotient ring of a ring. Let R be a ring. A *multiplicatively closed subset* S of R (i.e. a multiplicative sub-semigroup of $(R, \cdot)$ such that $1 \in S$ and $0 \notin S$) is said to be a *left Ore set* if it satisfies the *left Ore condition*: for each $r \in R$ and $s \in S$,

$$Sr \bigcap Rs \neq \emptyset.$$

Let S be a (non-empty) multiplicatively closed subset of R, and let

$$\mathrm{ass}(S) := \{r \in R \,|\, sr = 0 \;\text{ for some }\; s \in S\}$$

(if, in addition, S is a left Ore set then $\mathrm{ass}(S)$ is an ideal of the ring R).

Definition. Then a *left quotient ring* of R with respect to S (a *left localization* of R at S) is a ring Q together with a homomorphism $\varphi : R \to Q$ such that
(i) for all $s \in S$, $\varphi(s)$ is a unit of Q,

(ii) for all $q \in Q$, $q = \varphi(s)^{-1}\varphi(r)$ for some $r \in R$, $s \in S$, and
(iii) $\ker(\varphi) = \mathrm{ass}(S)$.

If such a ring Q exists, it is unique up to isomorphism and it is denoted by $S^{-1}R$. The condition (ii) means that the left quotient ring is as 'small' as possible in the sense that every element of it is a left fraction (the situation resembles the commutative situation). The condition (iii) means that the left quotient ring is as 'large' as possible in the sense that the elements of the ideal $\mathrm{ass}(S)$ are the *only* elements of the ring R that disappear when inverting the elements of the set S (the elements $\mathrm{ass}(S)$ are forced to disappear in any ring where the elements of the set S are units: if $sr = 0$, for some elements $r \in R$ and $s \in S$, then $0 = s^{-1}sr = r$). Recall that $S^{-1}R$ exists iff S is a left Ore set and the set $\overline{S} = \{s + \mathrm{ass}(S) \in R/\mathrm{ass}(S) \,|\, s \in S\}$ consists of regular elements ([17], 2.1.12). If the last two conditions are satisfied (i.e. those after 'iff' above) then S is called a *left denominator set*. Similarly, a *right Ore set*, the *right Ore condition*, the *right denominator set* and the *right quotient ring* RS^{-1} are defined. If both rings $S^{-1}R$ and RS^{-1} exist then they are isomorphic ([17], 2.1.4.(ii)). Recall that the left quotient ring of R with respect to the set $\mathcal{C}_R$ of all regular elements is called the left quotient ring of R. If it exists, it is denoted by $\mathrm{Frac}_l(R)$ or $Q_{cl}(R)$. Similarly, the right quotient ring, $\mathrm{Frac}_r(R) = Q_{cl}^r(R)$, is defined. If both left and right quotient rings of R exist then they are isomorphic and we write $\mathrm{Frac}(R)$ or $Q(R)$ in this case.

2 Four new criteria for a ring to have a semisimple left quotient ring

A ring is a *semiprime ring* if $\{0\}$ is the only nilpotent ideal. Let X be a non-empty subset of a ring R and $l(X) := \{r \in R \,|\, rX = 0\}$ be its *left annihilator* (it is a left ideal of R). A ring R satisfies the *ascending chain condition on left annihilators* if every ascending chain of the type

$$l(X_1) \subseteq l(X_2) \subseteq \cdots$$

stabilizers. A ring R is called a *left Goldie ring* if it satisfies the ascending chain condition on left annihilators and does not contain infinite direct sums of nonzero left ideals.

Goldie's Theorem [12] is a criterion for a ring to have a semisimple left quotient ring.

Theorem 2.1 (Goldie's Theorem, [12]) *A ring has a semisimple left quotient ring iff it is a semiprime ring that satisfies the ascending chain condition on left annihilators and does not contain infinite direct sums of nonzero left ideals.*

In [4], we introduce the following new concepts and prove their existence for an *arbitrary* ring: *the largest left quotient ring of a ring, the largest regular left Ore set of a ring, the left localization radical of a ring, a maximal left denominator set, a maximal left quotient ring of a ring, a (left) localization maximal ring.* Using an analogy with rings, the counter parts of these concepts for rings would be a maximal left ideal, the Jacobson radical, a simple factor ring. These concepts turned out to be very useful in Localization Theory and Ring Theory. They allowed us to look at old/classical results from a new more general perspective and to give new equivalent statements to the classical results using a new language and a new approach as the present paper, [4], [3], [5], [7] and [8] and several other papers under preparation demonstrate. Their universal nature naturally leads to the present criteria for a ring to have a semisimple left quotient ring. For an *arbitrary* ring R the set $\mathcal{M}$ of *maximal* left denominator sets of R is a non-empty set [4].

Theorem 2.2 *[6]* **(The First Criterion)**. *A ring R has a semisimple left quotient ring Q iff $\mathcal{M}$ is a finite set, $\bigcap_{S\in\mathcal{M}} \mathrm{ass}(S) = 0$ and, for each $S \in \mathcal{M}$, the ring $S^{-1}R$ is a simple left Artinian ring. In this case, $Q \simeq \prod_{S\in\mathcal{M}} S^{-1}R$.*

The Second Criterion is given via the minimal primes of R and certain explicit multiplicative sets associated with them. On the one hand, the Second Criterion stands between Goldie's Theorem and the First Criterion in terms how it is formulated. On the other hand, it goes further than the First Criterion in the sense that it describes explicitly the maximal left denominator sets and the left quotient ring of a ring with a semisimple left quotient ring.

Theorem 2.3 *[6]* **(The Second Criterion)**. *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a semisimple left quotient ring Q.*
2. (a) *The ring R is a semiprime ring.*
   (b) *The set $\mathrm{Min}(R)$ of minimal primes of R is a finite set.*
   (c) *For each $\mathfrak{p} \in \mathrm{Min}(R)$, the set $S_\mathfrak{p} := \{c \in R \mid c + \mathfrak{p} \in \mathcal{C}_{R/\mathfrak{p}}\}$ is a left denominator set of the ring R with $\mathrm{ass}(S_\mathfrak{p}) = \mathfrak{p}$.*
   (d) *For each $\mathfrak{p} \in \mathrm{Min}(R)$, the ring $S_\mathfrak{p}^{-1}R$ is a simple left Artinian ring.*

If one of the two equivalent conditions holds then $\mathrm{max.Den}_l(R) = \{S_\mathfrak{p} \mid \mathfrak{p} \in \mathrm{Min}(R)\}$ and $Q \simeq \prod_{\mathfrak{p}\in\mathrm{Min}(R)} S_\mathfrak{p}^{-1}R$.

So, the Second Criterion says that a ring has a semisimple left quotient ring iff all the left localizations at the minimal primes are simple Artinian rings, there are only finitely many minimal primes and the ring is semiprime.

The Third Criterion (Theorem 2.4) can be seen as a 'weak' version of Goldie's Theorem in the sense that the conditions are 'weaker' than those of Goldie's Theorem. In applications, it could be 'easier' to verify whether a ring satisfies the conditions of Theorem 2.4 compared with Goldie's Theorem as Theorem 2.4 'reduces' Goldie's Theorem essentially to the prime case and reveals the 'local' nature of Goldie's Theorem.

Theorem 2.4 *[6]* **(The Third Criterion)** *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a semisimple left quotient ring.*
2. *The ring R is a semiprime ring with $|\mathrm{Min}(R)| < \infty$ and, for each $\mathfrak{p} \in \mathrm{Min}(R)$, the ring $R/\mathfrak{p}$ is a left Goldie ring.*

Remark. This result is close to [17, Proposition 3.2.5]. The condition $|\mathrm{Min}(R)| < \infty$ in Theorem 2.4 can be replaced by any of the equivalent conditions of Theorem 2.5. For a semiprime ring R and its ideal I, the left annihilator of I in R is equal to the right annihilator of I in R and is denoted $\mathrm{ann}(I)$.

Theorem 2.5 *(*(Theorem 2.2.15, [17]*) The following conditions on a semiprime ring R are equivalent.*

1. *${}_RR_R$ has finite uniform dimension.*
2. *$|\mathrm{Min}(R)| < \infty$.*
3. *R has finitely many annihilator ideals.*
4. *R has a.c.c. on annihilator ideals.*

As far as applications are concerned, Theorem 2.6 is a very useful criterion for a ring R to have a semisimple left quotient ring.

Theorem 2.6 *[6]* **(The Fourth Criterion)** *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a semisimple left quotient ring.*

2. *There are left denominator sets $S'_1, \ldots, S'_n$ of the ring R such that the rings $R_i := S'^{-1}_i R$, $i = 1, \ldots, n$, are simple left Artinian rings and the map*

$$\sigma := \prod_{i=1}^{n} \sigma_i : R \to \prod_{i=1}^{n} R_i, \quad R \mapsto (\frac{r}{1}, \ldots, \frac{r}{1}),$$

is an injection where $\sigma_i : R \to R_i$, $r \mapsto \frac{r}{1}$.

If one of the equivalent conditions holds then the set $\max.\mathrm{Den}_l(R)$ *contains precisely the distinct elements of the set* $\{\sigma_i^{-1}(R_i^*) \,|\, i = 1, \ldots, n\}$.

The maximal denominator sets and the maximal left localizations of a ring. The set $(\mathrm{Den}_l(R), \subseteq)$ is a poset (partially ordered set). In [4], it is proved that the set $\max.\mathrm{Den}_l(R)$ of its maximal elements is a *non-empty* set.

Definition, [4]. An element S of the set $\max.\mathrm{Den}_l(R)$ is called a *maximal left denominator set* of the ring R and the ring $S^{-1}R$ is called a *maximal left quotient ring* of the ring R or a *maximal left localization ring* of the ring R. The intersection

$$\mathfrak{l}_R := \mathrm{l.lrad}(R) := \bigcap_{S \in \max.\mathrm{Den}_l(R)} \mathrm{ass}(S) \qquad (1)$$

is called the *left localization radical* of the ring R, [4]. The elements of the left localization radical $\mathrm{l.lrad}(R)$ are precisely the elements of R that 'eventually disappear' under left localizations (i.e. they are zero under localization at every 'sufficiently large' denominator set). The maximal left quotient rings of a ring will be considered in Section 9. One cannot invert anything new in such rings (Theorem 9.11).

For a ring R, there is the canonical exact sequence

$$0 \to \mathfrak{l}_R \to R \stackrel{\sigma}{\to} \prod_{S \in \max.\mathrm{Den}_l(R)} S^{-1}R, \quad \sigma := \prod_{S \in \max.\mathrm{Den}_l(R)} \sigma_S, \qquad (2)$$

where $\sigma_S : R \to S^{-1}R$, $r \mapsto \frac{r}{1}$. In general, the left localization radical $\mathfrak{l}_R$, the prime radical $\mathfrak{n}_R$ and the Jacobson radical of a ring R: are distinct. In general, even for left Artinian rings $\mathfrak{l}_R \neq \mathfrak{n}_R$, [7].

Characterization of rings R such that $R/\mathrm{l.lrad}(R)$ has a semisimple left quotient ring. Theorem 2.7 characterizes precisely the class of rings that have only finitely many maximal left denominators sets and all the left localizations at them are simple left Artinian rings.

Theorem 2.7 *[6] The following statements are equivalent.*

1. *The ring $R/\mathfrak{l}$ has a semisimple left quotient ring Q.*
2. *(a)* $|\max.\mathrm{Den}_l(R)| < \infty$.
   (b) For every $S \in \max.\mathrm{Den}_l(R)$, $S^{-1}R$ is a simple left Artinian ring.

Part II
Left Orders in Left Artinian Rings

In Part II, the following notation is fixed (in addition to that at the beginning of Part I):

- $\mathfrak{n}$ is a prime radical of R and ν is its *nilpotency degree* ($\mathfrak{n}^\nu \neq 0$ but $\mathfrak{n}^{\nu+1} = 0$);

- $\overline{R} := R/\mathfrak{n}$ and $\pi : R \to \overline{R}$, $r \mapsto \overline{r} = r + \mathfrak{n}$;
- $\overline{\mathcal{C}} := \mathcal{C}_{\overline{R}}$ is the set of regular elements of the ring $\overline{R}$ and $\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is its left quotient ring;
- $\mathcal{C}' := \pi^{-1}(\overline{\mathcal{C}}) := \{c \in R \mid c + \mathfrak{n} \in \overline{\mathcal{C}}\}$ and $Q' := \mathcal{C}'^{-1}R$ (if it exists).

3 Old criteria for a ring to have a left Artinian left quotient ring

In this section we present some old criteria for a ring to have a left Artinian left quotient ring that are due to Small (1966), Robson (1967), Tachikawa (1971) and Hajarnavis (1972). The starting point is Goldie's Theorem, [12], (1960) that gives an answer to the question: *When does a ring have a semi-simple (Artinian) left quotient ring?* Goldie's Theorem characterizes left orders in semi-simple rings, it is a criterion of when the left quotient ring of a ring is semi-simple (earlier, characterizations were given, by Goldie [11] and Lesieur and Croisot [16], of left orders in a simple Artinian ring).

Let us recall certain properties of left Artinian rings.

Proposition 3.1 (Proposition 3.1, [1]) *Let A be a left Artinian ring and $\mathfrak{r}$ be its radical. Then*

1. *The radical $\mathfrak{r}$ of A is a nilpotent ideal.*
2. *The factor ring $A/\mathfrak{r}$ is a semi-simple.*
3. *An A-module M is semi-simple iff $\mathfrak{r}M = 0$.*
4. *There is only a finite number of non-isomorphic simple A-modules.*
5. *The ring A is a left Noetherian ring.*

Definition. A ring R is called a *left Goldie ring* if it satisfies ACC (the *ascending chain condition*) for left annihilators and contains no infinite direct sums of left ideals.

Theorem 3.2 (Goldie's Theorem, [12]) *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a semi-simple (Artinian) left quotient ring.*
2. *The ring R is a semiprime left Goldie ring.*

Small's Criterion. Let W be the sum of all the nilpotent ideals of the ring R. If W is a nilpotent ideal of the ring R then $W = \mathfrak{n}$. For a subset X of the ring R, let $r(X) := \{a \in R \mid Xa = 0\}$ be its *right annihilator*. Clearly, $r(X)$ is a right ideal of the ring R. If, in addition, X is an ideal of the ring R then so is $r(X)$.

Theorem 3.3 (Small's Criterion [21, 22]) *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring.*
2. (a) *R is a left Goldie ring.*
   (b) *W is a nilpotent ideal of R.*
   (c) *For all $k \geq 1$, $R/(r(W^k) \cap W)$ is a left Goldie ring.*
   (d) *$r + W \in \mathcal{C}_{R/W} \implies r \in \mathcal{C}$ (i.e. $\mathcal{C}' \subseteq \mathcal{C}$).*

Robson's Criterion. A ring R is called $\mathfrak{n}$-*reflective* if, for $c \in R$, c is regular in R iff $c + \mathfrak{n}$ is regular in $\overline{R}$; equivalently, $\mathcal{C}' = \mathcal{C}$. A ring R is called $\mathfrak{n}$-*quorite* if, given $c \in \mathcal{C}$ and $n \in \mathfrak{n}$, there exist $c' \in \mathcal{C}$ and $n' \in \mathfrak{n}$ such that $c'n = n'c$. A left ideal I of the ring R is called a $\mathcal{C}$-*closed* if, for elements $c \in \mathcal{C}$ and $r \in R$, $cr \in I$ implies $r \in I$. Similarly, a $\mathcal{C}'$-*closed* right ideal is defined.

Theorem 3.4 (Robson, Theorem 2.10, [20]) *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring Q.*
2. (a) *The ring $\overline{R}$ is a left Goldie ring.*
   (b) *$\mathfrak{n}$ is a nilpotent ideal.*
   (c) *The ring R is $\mathfrak{n}$-reflective and $\mathfrak{n}$-quorite.*
   (d) *The ring R satisfies ACC on $\mathcal{C}$-closed left ideals.*

Hajarnavis' Criterion. This criterion is close to Small's one.

Theorem 3.5 (Hajarnavis' Criterion, [13]) *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring.*
2. (a) *R and R/W are left Goldie rings.*
   (b) *W is a nilpotent ideal of R.*
   (c) *For all $k \geq 1$, $R/r(W^k)$ has finite left uniform dimension.*
   (d) *$r + W \in \mathcal{C}_{R/W} \implies r \in \mathcal{C}$ (i.e. $\mathcal{C}' \subseteq \mathcal{C}$).*

Tachikawa's Criterion. Let $\mathcal{W}$ be an injective R-module containing the R-module R, $\mathcal{F}$ be the corresponding idempotent topologizing filter

$$\mathcal{F} := \{{}_RI \subseteq R \mid \mathrm{Hom}_R(R/I, \mathcal{W}) = 0\},$$

and H be the corresponding localization functor: for an R-module M,

$$H(M) := \lim_{I \in \mathcal{F}} \mathrm{Hom}_R(I, M/M_{\mathcal{F}})$$

where $M_{\mathcal{F}} := \{m \in M \mid I'm = 0 \text{ for some } I' \in \mathcal{F}\}$. Then $H(R)$ is a ring and $H(M)$ is an $H(R)$-module.

Theorem 3.6 (Tachikawa's Criterion, [25]) *Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring.*
2. *There exists a faithful, torsionfree, injective left R-module $\mathcal{W}$ such that the following conditions are satisfied:*
   (a) *for every left ideal J of $H(R)$ there is a left ideal I of R such that $H(I) = J$,*
   (b) *the R-module $\mathcal{W}$ satisfies the descending chain condition on annihilators,*
   (c) *the prime radical of the ring R coincides with the set of all elements $r \in R$ that annihilate $H(V)$ where V is an essential R-submodule of $\mathcal{W}$.*
3. *There exists a faithful, torsionfree, injective left R-module $\mathcal{W}$ such that the following conditions are satisfied:*
   (a) *the double centralizer Q of $\mathcal{W}$ is left Artinian,*
   (b) *$\mathcal{W}$ is a cogenerator as a left Q-module,*
   (c) *the prime radical of the ring R is equal to the intersection of R and the radical of Q.*

In the proofs of all the criteria (old and new) Goldie's Theorem is used.

4 Necessary and sufficient conditions for a ring to have a left Artinian left quotient ring

The aim of this section is to present Theorem 4.1 which is a new criterion for a ring R to have a left Artinian left quotient ring. Using Theorem 4.1 and Theorem 4.3 in combination with results of Small and P. F. Smith, criteria are obtained for a left Noetherian ring R (Corollary 4.4) and for a commutative ring R (Corollary 4.5) to have left Artinian left quotient ring.

Suppose that a ring R satisfies the condition (a) of Theorem 4.1, i.e. $\overline{R}$ is a *(semiprime) left Goldie ring.* By Goldie's Theorem, its left quotient ring $\overline{Q} := \overline{\mathcal{C}}^{-1}\overline{R}$ is a semisimple (Artinian) ring where $\overline{\mathcal{C}}$ is the set of regular elements of the ring $\overline{R}$. The ring R admits *the* $\mathfrak{n}$*-adic filtration* (*the prime radical filtration*):

$$\mathfrak{n}^0 := R \supset \mathfrak{n} \supset \cdots \supset \mathfrak{n}^i \supset \cdots \qquad (3)$$

which stops at $(\nu+1)$'th step if $\mathfrak{n}^\nu \neq 0$ but $\mathfrak{n}^{\nu+1} = 0$, i.e. ν is the nilpotency degree of the ideal $\mathfrak{n}$. The *associated graded algebra*

$$\mathrm{gr}\, R = \overline{R} \oplus \mathfrak{n}/\mathfrak{n}^2 \oplus \cdots \mathfrak{n}^i/\mathfrak{n}^{i+1} \oplus \cdots$$

is an $\mathbb{N}$-graded ring and every component $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is an $\overline{R}$-bimodule. Recall that $\overline{\mathcal{C}}$ is a left Ore set in $\overline{R}$ (by Goldie's Theorem) and that module means a left module. For each integer $i \geq 1$, let

$$\tau_i := \mathrm{tor}_{\overline{\mathcal{C}}}(\mathfrak{n}^i/\mathfrak{n}^{i+1}) := \{u \in \mathfrak{n}^i/\mathfrak{n}^{i+1} \,|\, \overline{c}u = 0 \;\text{ for some } \overline{c} \in \overline{\mathcal{C}}\} \qquad (4)$$

be the $\overline{\mathcal{C}}$*-torsion submodule* of the *left* $\overline{R}$-module $\mathfrak{n}^i/\mathfrak{n}^{i+1}$. Clearly, τ_i is an $\overline{R}$-bimodule. Then the $\overline{R}$-bimodule

$$\mathfrak{f}_i := (\mathfrak{n}^i/\mathfrak{n}^{i+1})/\tau_i \qquad (5)$$

is a $\overline{\mathcal{C}}$*-torsionfree* left $\overline{R}$-module. There is a unique ideal, say $\mathfrak{t}_i$, of the ring R such that

$$\mathfrak{n}^{i+1} \subseteq \mathfrak{t}_i \subseteq \mathfrak{n}^i \;\text{ and }\; \mathfrak{t}_i/\mathfrak{n}^{i+1} = \tau_i.$$

Clearly, $\mathfrak{f}_i \simeq \mathfrak{n}^i/\mathfrak{t}_i$.

Theorem 4.1 *[5] Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring Q.*
2. *(a) The ring $\overline{R}$ is a left Goldie ring.*
   (b) $\mathfrak{n}$ is a nilpotent ideal.
   (c) $\mathcal{C}' \subseteq \mathcal{C}$.
   (d) The left $\overline{R}$-modules $\mathfrak{f}_i$, $1 \leq i \leq 1$, contain no infinite direct sums of nonzero submodules, and
   (e) for every element $\overline{c} \in \overline{\mathcal{C}}$, the map $\cdot\overline{c} : \mathfrak{f}_i \to \mathfrak{f}_i$, $f \mapsto f\overline{c}$, is an injection; equivalently, if, for an element $a \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$, there are elements $\overline{s}$, $\overline{c} \in \overline{\mathcal{C}}$ such that $\overline{s}a\overline{c} = 0$ then $\overline{t}a = 0$ for some element $\overline{t} \in \overline{\mathcal{C}}$; equivalently, if, for an element $a \in \mathfrak{n}^i/\mathfrak{n}^{i+1}$, there is an element $\overline{c} \in \overline{\mathcal{C}}$ such that $a\overline{c} = 0$ then $\overline{t}a = 0$ for some element $\overline{t} \in \overline{\mathcal{C}}$.

Let $\overline{Q}$ be the left quotient ring of $\overline{R}$. If one of the equivalent conditions holds then $\mathcal{C} = \mathcal{C}'$, $\mathcal{C}^{-1}\mathfrak{n}$ is the prime radical of the ring Q which is a nilpotent ideal of nilpotency degree ν, and the map $Q/\mathcal{C}^{-1}\mathfrak{n} \to \overline{Q}$, $c^{-1}r \mapsto \overline{c}^{-1}\overline{r}$, is a ring isomorphism with the inverse $\overline{c}^{-1}\overline{r} \mapsto c^{-1}r$.

As an application we obtain a short proof of a known fact of when a *commutative Noetherian* ring has an Artinian quotient ring.

Corollary 4.2 *Let R be a commutative Noetherian ring. The following statements are equivalent.*

1. *The ring R has an Artinian quotient ring.*

2. *The conditions (c) of Theorem 4.1 holds, i.e.* $\mathcal{C}' \subseteq \mathcal{C}$.

3. *The associated primes of* (0) *are the minimal primes of the ring* R.

Proof. $(1 \Leftrightarrow 2)$ Theorem 4.1.
$(2 \Leftrightarrow 3)$ This equivalence was established by Small (Theorem 2.13, [21] and Theorem C, [22]).
$(1 \Leftrightarrow 3)$ Robson (Theorem 2.11, [20]). $\square$

For a ring R having a left Artinian left quotient ring Q, Theorem 4.3 provides many examples of left Ore subsets $\mathcal{C}'' \subseteq \mathcal{C}$ such that $\mathcal{C}''^{-1}R \simeq Q$.

Theorem 4.3 *Let* R *be a ring. The following statements are equivalent.*

1. *The ring* R *has a left Artinian left quotient ring* Q.
2. *The conditions (a), (b), (c'), (d) and (e) hold (see Theorem 4.1) where*
   (c') for each element $\alpha \in \overline{\mathcal{C}}$, *there exists a regular element* $c = c(\alpha) \in \mathcal{C}$ *such that* $\alpha = c + \mathfrak{n}$; *equivalently, there exists a submonoid* $\mathcal{C}'' \subseteq \mathcal{C}$ *such that* $\overline{\mathcal{C}''} = \overline{\mathcal{C}}$ *(where* $\overline{\mathcal{C}''} := \{c+\mathfrak{n} \,|\, c \in \mathcal{C}''\}$*).*

If one of the equivalent conditions holds then $\mathcal{C}''$ *is a left Ore set in* R, $\mathcal{C}''^{-1}R = Q$, $\mathcal{C}''^{-1}\mathfrak{n}$ *is the prime radical of the ring* Q *which is a nilpotent ideal of nilpotency degree* ν, *and the map* $Q/\mathcal{C}''^{-1}\mathfrak{n} \to \overline{Q}$, $c^{-1}r \mapsto \overline{c}^{-1}\overline{r}$, *is a ring isomorphism with the inverse* $\overline{c}^{-1}\overline{r} \mapsto c^{-1}r$ *where* c *is any element of* $\mathcal{C}''$ *such that* $\overline{c} = c + \mathfrak{n}$.

As an application we obtain a criterion for a *left Noetherian* ring to have a left Artinian left quotient ring.

Corollary 4.4 *Let* R *be a left Noetherian ring. The following two statements are equivalent.*

1. *The ring* R *has a left Artinian left quotient ring.*
2. $\mathcal{C}' \subseteq \mathcal{C}$.
3. *For each element* $\alpha \in \overline{\mathcal{C}}$, *there exists an element* $c = c(\alpha) \in \mathcal{C}$ *such that* $\alpha = c + \mathfrak{n}$.

Remark. $(1 \Leftrightarrow 2)$ This is due to Small [21].

In case of a commutative but not necessarily Noetherian ring R, there are several criteria when its quotient ring is an Artinian ring.

Corollary 4.5 *Let* R *be a commutative ring. The following statements are equivalent.*

1. *The ring* R *has an Artinian quotient ring.*
2. (a) *The ring* $\overline{R}$ *is a Goldie ring.*
   (b) $\mathfrak{n}$ *is a nilpotent ideal.*
   (c) $\mathcal{C}' \subseteq \mathcal{C}$.
   (d) *The* $\overline{R}$*-modules* $\mathfrak{f}_i$, $1 \leq i \leq \nu$, *contain no infinite direct sums of nonzero submodules.*
3. (a) *The ring* $\overline{R}$ *is a Goldie ring.*
   (b) $\mathfrak{n}$ *is a nilpotent ideal.*
   (c) *For each element* $\alpha \in \overline{\mathcal{C}}$, *there exists an element* $c = c(\alpha) \in \mathcal{C}$ *such that* $\alpha = c + \mathfrak{n}$.
   (d) *The* $\overline{R}$*-modules* $\mathfrak{f}_i$, $1 \leq i \leq \nu$, *contain no infinite direct sums of nonzero submodules.*
4. R *is a Goldie ring and* $\mathcal{C}' \subseteq \mathcal{C}$.
5. R *is a Goldie ring and, for each element* $\alpha \in \overline{\mathcal{C}}$, *there exists an element* $c = c(\alpha) \in \mathcal{C}$ *such that* $\alpha = c + \mathfrak{n}$.

Proof. (1 ⇔ 2) Theorem 4.1.
(1 ⇔ 3) Theorem 4.3.
(1 ⇔ 4) This is due to P. F. Smith (Theorem 2.11, [13]).
(4 ⇒ 5) Trivial.
(5 ⇒ 4) The condition $\mathcal{C}' \subseteq \mathcal{C}$ is equivalent to two conditions: $\pi(\mathcal{C}) = \overline{\mathcal{C}}$ and $\mathcal{C} + \mathfrak{n} \subseteq \mathcal{C}$ where $\pi : R \to \overline{R}$, $r \mapsto \overline{r}$. By statement 5, the first condition is given. Let $c \in \mathcal{C}$ and $n \in \mathfrak{n}$. To prove the second statement we have to show that $c + n \in \mathcal{C}$. Notice that n is a nilpotent element and the ring R is a subring of $\mathcal{C}^{-1}R$. Now, the element $c + n = c(1 + c^{-1}n)$ is a unit of the ring $\mathcal{C}^{-1}R$ (as a product of two units). Therefore, $c + n \in \mathcal{C}$. □

5 A criterion via associated graded ring

The aim of this section is to give another criterion (Theorem 5.1) for a ring R to have a left Artinian left quotient ring via its associated graded ring $\mathrm{gr}\, R$ with respect to the $\mathfrak{n}$-adic filtration.

A multiplicative set S of a ring R is a *left denominator set* if it is a left Ore set and if $rs = 0$, for some elements $r \in R$ and $s \in S$, then $s'r = 0$ for some element $s' \in S$. For a left denominator S of the ring R, we can form the ring of fractions $S^{-1}R = \{s^{-1} \mid s \in S,\ r \in R\}$.

Suppose that $\overline{\mathcal{C}}$ is *a left denominator set of the associated graded ring* $\mathrm{gr}\, R = \overline{R} \oplus \mathfrak{n}/\mathfrak{n}^2 \oplus \cdots$ with respect to the $\mathfrak{n}$-adic filtration. Then the $\overline{\mathcal{C}}$-*torsion ideal* of the ring $\mathrm{gr}\, R$,

$$\tau := \mathrm{tor}_{\overline{\mathcal{C}}}(\mathrm{gr}\, R) = \oplus_{i\geq 1}\tau_i, \quad \text{where} \quad \tau_i = \mathrm{tor}_{\overline{\mathcal{C}}}(\mathfrak{n}^i/\mathfrak{n}^{i+1}), \tag{6}$$

is a homogeneous ideal of the $\mathbb{N}$-graded ring $\mathrm{gr}\, R$. The factor ring

$$\mathrm{gr}\, R/\tau = \overline{R} \oplus \mathfrak{f}_1 \oplus \mathfrak{f}_2 \oplus \cdots, \quad \text{where} \quad \mathfrak{f}_i = (\mathfrak{n}^i/\mathfrak{n}^{i+1})/\tau_i, \tag{7}$$

is an $\mathbb{N}$-graded ring ($\mathfrak{f}_i\mathfrak{f}_i \subseteq \mathfrak{f}_{i+1}$ for all $i, j \geq 1$) and a subring of the localization ring

$$\overline{\mathcal{C}}^{-1}\mathrm{gr}\, R \simeq \overline{\mathcal{C}}^{-1}(\mathrm{gr}\, R/\tau) = \overline{Q} \oplus \overline{\mathcal{C}}^{-1}\mathfrak{f}_1 \oplus \overline{\mathcal{C}}^{-1}\mathfrak{f}_2 \oplus \cdots$$

which is an $\mathbb{N}$-graded ring.

Suppose, in addition, that the nilpotency degree ν of the prime radical $\mathfrak{n}$ is *finite*. Then the prime radical $\mathfrak{n}_{\mathrm{gr}\, R/\tau}$ of the ring $\mathrm{gr}\, R/\tau$ is equal to

$$\mathfrak{n}_{\mathrm{gr}\, R/\tau} = \mathfrak{f} := \oplus_{i\geq 1}\mathfrak{f}_i. \tag{8}$$

It is a nilpotent ideal of nilpotency degree $\max\{i \geq 1 \mid \mathfrak{f}_i \neq 0\} \leq \nu$.

Theorem 5.1 *[5] Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian ring left quotient ring Q.*
2. *The set $\overline{\mathcal{C}}$ is a left denominator set in the ring $\mathrm{gr}\, R$, $\overline{\mathcal{C}}^{-1}\mathrm{gr}\, R$ is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.*
3. *The set $\overline{\mathcal{C}}$ is a left denominator set in the ring $\mathrm{gr}\, R$, the left quotient ring $Q(\mathrm{gr}\, R/\tau)$ of the ring $\mathrm{gr}\, R/\tau$ is a left Artinian ring, $\mathfrak{n}$ is a nilpotent ideal and $\mathcal{C}' \subseteq \mathcal{C}$.*

If one of the equivalent conditions holds then $\mathrm{gr}\, Q \simeq Q(\mathrm{gr}\, R/\tau) \simeq \overline{\mathcal{C}}^{-1}\mathrm{gr}\, R$ where $\mathrm{gr}\, Q$ is the associated graded ring with respect to the prime radical filtration.

6 Criteria similar to Robson's Criterion

In this section, two criteria similar to Robson's Criterion are given (Theorem 6.1 and Corollary 6.2): Robson's Criterion holds where $\mathcal{C}$ is replaced by $\mathcal{C}'$ and one of the conditions is changed accordingly (Theorem 6.1), Corollary 6.2 is a 'weaker' version of Theorem 6.1.

The next result shows that in Robson's Criterion (Theorem 3.4) the condition 'R is $\mathfrak{n}$-reflective' can be weakened.

Theorem 6.1 *[5] Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring Q.*
2. (a) *The ring $\overline{R}$ is a left Goldie ring.*
   (b) *$\mathfrak{n}$ is a nilpotent ideal.*
   (c) *$\mathcal{C}' \subseteq \mathcal{C}$.*
   (d) *If $c \in \mathcal{C}'$ and $n \in \mathfrak{n}$ then there exist elements $c_1 \in \mathcal{C}'$ and $n_1 \in \mathfrak{n}$ such that $c_1 n = n_1 c$.*
   (e) *The ring R satisfies ACC for $\mathcal{C}'$-closed left ideals.*

The next corollary shows that the condition (c) in Theorem 6.1 can be weakened.

Corollary 6.2 *[5] Let R be a ring. The following statements are equivalent.*

1. *The ring R has a left Artinian left quotient ring Q.*
2. (a) *The ring $\overline{R}$ is a left Goldie ring.*
   (b) *$\mathfrak{n}$ is a nilpotent ideal.*
   (c) *There exists a submonoid $\mathcal{C}''$ of $\mathcal{C}$ such that $\overline{\mathcal{C}''} = \mathcal{C}$.*
   (d) *If $c \in \mathcal{C}''$ and $n \in \mathfrak{n}$ then there exist elements $c_1 \in \mathcal{C}''$ and $n_1 \in \mathfrak{n}$ such that $c_1 n = n_1 c$.*
   (e) *The ring R satisfies ACC for $\mathcal{C}''$-closed left ideals.*

If one of the equivalent conditions holds then $\mathcal{C}''$ is a left Ore set in R, $\mathcal{C}''^{-1}R = Q$, $\mathcal{C}''^{-1}\mathfrak{n}$ is the prime radical of the ring Q which is a nilpotent ideal of nilpotency degree ν, and the map $Q/\mathcal{C}''^{-1}\mathfrak{n} \to \overline{Q}$, $c^{-1}r \mapsto \overline{c}^{-1}\overline{r}$, is a ring isomorphism with the inverse $\overline{c}^{-1}\overline{r} \mapsto c^{-1}r$ where c is any element of $\mathcal{C}''$ such that $\overline{c} = c + \mathfrak{n}$.

7 A left quotient ring of a factor ring

The aim of this section is to present Theorem 7.1 which, for a ring R with a left Artinian left quotient ring Q and its $\mathcal{C}$-closed ideal $I \subseteq \mathfrak{n}$, shows that the factor ring R/I has a left Artinian left quotient ring $Q(R/I)$.

Theorem 7.1 *[5] Let R be a ring with a left Artinian left quotient ring Q, and I be a $\mathcal{C}$-closed ideal of R such that $I \subseteq \mathfrak{n}$. Then*

1. *The set $\mathcal{C}_{R/I}$ of regular elements of the ring R/I is equal to the set $\{c + I \mid c \in \mathcal{C}\}$.*
2. *The ring R/I has a left Artinian left quotient ring $Q(R/I)$, $\mathcal{C}^{-1}I$ is an ideal of Q and the map $Q/\mathcal{C}^{-1}I \to Q(R/I)$, $c^{-1}r + \mathcal{C}^{-1}I \mapsto (c+I)^{-1}(r+I)$, is a ring isomorphism with the inverse $(c+I)^{-1}(r+I) \mapsto c^{-1}r + \mathcal{C}^{-1}I$.*

Part III
The Largest Left Quotient Ring of a Ring

We keep the notation of Parts I and II.

8 The largest denominator sets and the largest left quotient ring of a ring

For an arbitrary ring R, two fundamental concepts are introduced – the largest regular left Ore set $S_0(R)$ and the largest left quotient ring $Q_l(R)$. The group of units $Q_l(R)^*$ of the ring $Q_l(R)$ is found (Theorem 8.3). A criterion of when the ring $Q_l(R)$ is a semi-simple ring is given (Theorem 8.4) which is a generalization of Goldie's Theorem.

In general, the set $\mathcal{C}$ of regular elements of a ring R is neither left nor right Ore set of the ring R and as a result neither left nor right quotient ring ($\mathcal{C}^{-1}R$ and $R\mathcal{C}^{-1}$) exists. Remarkably, there exists the largest regular left Ore set $S_0 = S_{l,0} = S_{l,0}(R)$ and the largest regular right Ore set $S_{r,0}(R)$ (Theorem 8.1.(2)). This means that the set $S_{l,0}(R)$ is an Ore set of the ring R that consists of regular elements (i.e., $S_{l,0}(R) \subseteq \mathcal{C}$) and contains all the left Ore sets in R that consist of regular elements. Also, there exists the largest regular (left and right) Ore set $S_{l,r,0}(R)$ of the ring R. In general, all the sets $\mathcal{C}$, $S_{l,0}(R)$, $S_{r,0}(R)$ and $S_{l,r,0}(R)$ are distinct, for example, when $R = \mathbb{I}_1$ is the ring of polynomial integro-differential operators [3]. The ring

$$Q_l(R) := S_{l,0}(R)^{-1}R$$

(respectively, $Q_r(R) := RS_{r,0}(R)^{-1}$ and $Q(R) := S_{l,r,0}(R)^{-1}R \simeq RS_{l,r,0}(R)^{-1}$) is called the *largest left* (respectively, *right and two-sided*) *quotient ring* of the ring R. In general, the rings $Q_l(R)$, $Q_r(R)$ and $Q(R)$ are not isomorphic, for example, when $R = \mathbb{I}_1$.

Small and Stafford [23] have shown that any (left and right) Noetherian ring R possesses a uniquely determined set of prime ideals $P_1, \ldots, P_n$ such that $\mathcal{C}_R = \cap_{i=1}^n \mathcal{C}(P_i)$, an irreducible intersection, where $\mathcal{C}(P_i) := \{r \in R \,|\, r + P_i \in \mathcal{C}_{R/P_i}\}$. Michler and Muller [18] mentioned that the ring R contains a unique maximal (left and right) Ore set of regular elements $S_{l,r,0}(R)$ and called the ring $Q(R)$ the *total quotient ring* of R. For certain Noetherian rings, they described the set $S_{l,r,0}(R)$ and the ring $Q(R)$. For the class of affine Noetherian PI-rings, further generalizations were given by Muller in [19].

Theorem 8.1 *[4]*

1. *For each $\mathfrak{a} \in \mathrm{Ass}_l(R)$, the set $\mathrm{Den}_l(R, \mathfrak{a})$ is an ordered abelian semigroup ($S_1S_2 = S_2S_1$, and $S_1 \subseteq S_2$ implies $S_1S_3 \subseteq S_2S_3$) where the product $S_1S_2 := \langle S_1, S_2\rangle$ is the multiplicative subsemigroup of $(R, \cdot)$ generated by S_1 and S_2.*

2. *$S_\mathfrak{a} := S_\mathfrak{a}(R) := \bigcup_{S \in \mathrm{Den}_l(R,\mathfrak{a})} S$ is the largest element (w.r.t. $\subseteq$) in $\mathrm{Den}_l(R, \mathfrak{a})$. The set $S_\mathfrak{a}$ is called the* largest left denominator set *associated to $\mathfrak{a}$.*

3. *Let $S_i \in \mathrm{Den}_l(R, \mathfrak{a})$, $i \in I$, where I is an arbitrary non-empty set. Then the set*

$$\langle S_i \,|\, i \in I\rangle := \bigcup_{\emptyset \neq J \subseteq I, |J| < \infty} \prod_{j \in J} S_j \in \mathrm{Den}_l(R, \mathfrak{a}) \tag{9}$$

is the left denominator set generated by the left denominator sets S_i, it is the least upper bound of the set $\{S_i\}_{i \in I}$ in $\mathrm{Den}_l(R, \mathfrak{a})$, i.e. $\langle S_i \,|\, i \in I\rangle = \bigvee_{i \in I} S_i$.

Definition, [3, 4]. For each ideal $\mathfrak{a} \in \mathrm{Ass}_l(R)$, the ring $Q_\mathfrak{a}(R) := S_\mathfrak{a}^{-1}R$ is called the *largest left quotient ring associated to* $\mathfrak{a}$. When $\mathfrak{a} = 0$, the ring $Q_l(R) := Q_{l,0}(R) := S_0^{-1}R$ is called the *largest left quotient ring* of R and $S_0 = S_0(R)$ is called the *largest regular left Ore set* of R.

The next obvious corollary shows that $Q_l(R)$ is a generalization of the classical left quotient ring $Q_{cl}(R)$.

Corollary 8.2 1. *If the classical left quotient ring $Q_{cl}(R) := \mathcal{C}_R^{-1}R$ exists then the set of regular elements $\mathcal{C}_R$ of the ring R is the largest regular left Ore set and $Q_l(R) = Q_{cl}(R)$.*

2. *Let $R_1, \ldots, R_n$ be rings. Then $Q_l(\prod_{i=1}^n R_i) \simeq \prod_{i=1}^n Q_l(R_i)$.*

Proof. It is obvious. □

The group of units $Q_l(R)^*$ of $Q_l(R)$. For a ring R and its largest left quotient ring $Q_l(R)$, Theorem 8.3 is used in the proof of Theorem 8.4 and gives answers to the following natural questions:

- *What is $S_0(Q_l(R))$?*
- *What is $S_0(Q_l(R)) \cap R$?*
- *What is the group $Q_l(R)^*$ of units of the ring $Q_l(R)$?*
- *Is the natural inclusion $Q_l(R) \subseteq Q_l(Q_l(R))$ an equality?*

Theorem 8.3 *[4]*

1. $S_0(Q_l(R)) = Q_l(R)^*$ and $S_0(Q_l(R)) \cap R = S_0(R)$.
2. $Q_l(R)^* = \langle S_0(R), S_0(R)^{-1}\rangle$, i.e. the group of units of the ring $Q_l(R)$ is generated by the sets $S_0(R)$ and $S_0(R)^{-1} := \{s^{-1} \mid s \in S_0(R)\}$.
3. $Q_l(R)^* = \{s^{-1}t \mid s, t \in S_0(R)\}$.
4. $Q_l(Q_l(R)) = Q_l(R)$.

Necessary and sufficient conditions for $Q_l(R)$ to be a semi-simple ring. A ring Q is called a *ring of quotients* if every element $c \in \mathcal{C}_Q$ is invertible. A subring R of a ring of quotients Q is called a *left order* in Q if $\mathcal{C}_R$ is a left Ore set and $\mathcal{C}_R^{-1}R = Q$. A ring R has *finite left rank* (i.e. *finite left uniform dimension*) if there are no infinite direct sums of nonzero left ideals in R.

The next theorem gives an answer to the question of *when $Q_l(R)$ is a semi-simple ring.* The answer is iff $Q_{cl}(R)$ is a semi-simple ring.

Theorem 8.4 *[4] The following properties of a ring R are equivalent.*

1. *$Q_l(R)$ is a semi-simple ring.*
2. *$Q_{cl}(R)$ exists and is a semi-simple ring.*
3. *R is a left order in a semi-simple ring.*
4. *R has finite left rank, satisfies the ascending chain condition on left annihilators and is a semi-prime ring.*
5. *A left ideal of R is essential iff it contains a regular element.*

If one of the equivalent conditions hold then $S_0(R) = \mathcal{C}_R$ and $Q_l(R) = Q_{cl}(R)$.

Remark. Goldie's Theorem states that $2 \Leftrightarrow 3 \Leftrightarrow 4 \Leftrightarrow 5$.

The next corollary gives an interesting criterion of when the classical quotient ring $Q_{cl}(R) = \mathcal{C}_R^{-1}R$ exists.

Corollary 8.5 *[4] If the ring $Q_l(R)$ is a left Artinian ring then $S_0(R) = \mathcal{C}_R$ and $Q_{cl}(R) = Q_l(R)$.*

Proposition 8.6 (Proposition 11.6, [24]; [15]) *Let A be a subring of a ring B. If M is a finitely generated flat A-module such that $B \otimes_A M$ is a projective B-module then M is a projective A-module.*

Corollary 8.7 *If there exists a finitely generated flat R-module M which is not projective then the ring $Q_l(R)$ is not a semi-simple ring.*

Proof. If $Q_l(R)$ were a semi-simple ring then $Q_l(R) \otimes_R M$ would be a projective $Q_l(R)$-module, and so M would be a projective R-module, by Proposition 8.6, a contradiction. □

9 The maximal left quotient rings of a ring

In this section, a new class of rings, the class of left localization maximal rings, is introduced. It is proved that, for an arbitrary ring R, the set of maximal elements of the poset $(\mathrm{Den}_l(R), \subseteq)$ is a non-empty set (Lemma 9.6.(2)), and therefore the set of maximal left quotient rings of the ring R is a non-empty set. A criterion is given (Theorem 9.11) for a left quotient ring of a ring to be a maximal left quotient ring of the ring. Many results on denominator sets are given. In particular, for each denominator set $S \in \mathrm{Den}_l(R, \mathfrak{a})$, connections are established between the left denominator sets $\mathrm{Den}_l(R, \mathfrak{a})$, $\mathrm{Den}_l(R/\mathfrak{a}, 0)$ and $\mathrm{Den}_l(S^{-1}R, 0)$. All the results in this section are taken from [4] where one can also find their proofs.

Proposition 9.1 *1. For each ring $A \in \mathrm{Loc}_l(R, \mathfrak{a})$ where $\mathfrak{a} \in \mathrm{Ass}_l(R)$, the set $\mathrm{Den}_l(R, \mathfrak{a}, A) := \{S \in \mathrm{Den}_l(R, \mathfrak{a}) \mid S^{-1}R = A\}$ is an ordered submonoid of $\mathrm{Den}_l(R, \mathfrak{a})$, and*

2. *$S(R, \mathfrak{a}, A) := \bigcup_{S \in \mathrm{Den}_l(R, \mathfrak{a}, A)} S$ is its largest element. In particular, $S_0(R) = S(R, 0, Q_l(R))$.*

3. *Let $S_i \in \mathrm{Den}_l(R, \mathfrak{a}, A)$, $i \in I$, where I is an arbitrary non-empty set. Then $\langle S_i \mid i \in I \rangle \in \mathrm{Den}_l(R, \mathfrak{a}, A)$ (see (9)). Moreover, $\langle S_i \mid i \in I \rangle$ is the least upper bound of the set $\{S_i\}_{i \in I}$ in $\mathrm{Den}_l(R, \mathfrak{a}, A)$ and in $\mathrm{Den}_l(R, \mathfrak{a})$.*

The next lemma establishes relations between denominator sets of a ring and its factor rings.

Lemma 9.2 *1. Let $S \in \mathrm{Den}_l(R, \mathfrak{a})$, $\mathfrak{b}$ be an ideal of the ring R such $\mathfrak{b} \subseteq \mathfrak{a}$, and $\pi : R \to R/\mathfrak{b}$, $a \mapsto \overline{a} = a + \mathfrak{b}$. Then $\pi(S) \in \mathrm{Den}_l(R/\mathfrak{b}, \mathfrak{a}/\mathfrak{b})$ and $S^{-1}R \simeq \pi(S)^{-1}(R/\mathfrak{b})$.*

2. *Let $S_1, S_2 \in \mathrm{Den}_l(R)$ and $S_1 \subseteq S_2$. Then*

   (a) *$\mathfrak{a}_1 := \mathrm{ass}(S_1) \subseteq \mathfrak{a}_2 := \mathrm{ass}(S_2)$; there is the R-ring homomorphism $\varphi : S_1^{-1}R \to S_2^{-1}R$, $s^{-1}a \mapsto s^{-1}a$; and $\ker(\varphi) = S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1)$.*

   (b) *Let $\pi_1 : R \to R/\mathfrak{a}_1$, $a \mapsto \overline{a} = a + \mathfrak{a}_1$, and $\widetilde{S}_2$ be the multiplicative submonoid of $(S_1^{-1}(R/\mathfrak{a}_1), \cdot)$ generated by $\pi_1(S_2)$ and $\pi_1(S_1)^{-1} = \{\overline{s}^{-1} \mid s \in S_1\}$. Then $\pi_1(S_2), \widetilde{S}_2 \in \mathrm{Den}_l(S_1^{-1}R, S_1^{-1}(\mathfrak{a}_2/\mathfrak{a}_1))$ and $\widetilde{S}_2^{-1}(S_1^{-1}R) \simeq \pi_1(S_2)^{-1}(S_1^{-1}R) \simeq S_2^{-1}R$.*

Denominator sets of a ring and its localizations. The set $(\mathrm{Loc}_l(R, \mathfrak{a}), \to)$ is a poset where $A_1 \to A_2$ if $A_1 = S_1^{-1}R$ and $A_2 = S_2^{-1}R$ for some denominator sets $S_1, S_2 \in \mathrm{Den}_l(R, \mathfrak{a})$ with $S_1 \subseteq S_2$, and $A_1 \to A_2$ is the map in Lemma 9.2.(2a). If (S_1', S_2') is another such a pair then, by Proposition 9.1.(1),

$$A_1 = S_1^{-1}R = S_1'^{-1}R = (S_1S_1')^{-1}R \to A_2 = S_2^{-1}R = S_2'^{-1}R = (S_2S_2')^{-1}R;$$

$S_1S_1', S_2S_2' \in \mathrm{Den}_l(R, \mathfrak{a})$ with $S_1S_1' \subseteq S_2S_2'$.

In the same way, the poset $(\mathrm{Loc}_l(R), \to)$ is defined, i.e. $A_1 \to A_2$ if there exists $S_1, S_2 \in \mathrm{Den}_l(R)$ such that $S_1 \subseteq S_2$, $A_1 = S_1^{-1}R$ and $A_2 = S_2^{-1}R$, $A_1 \to A_2$ stands for the map $\varphi : S_1^{-1}R \to S_2^{-1}R$ (Lemma 9.2.(2a)). The map

$$(\cdot)^{-1}R : \mathrm{Den}_l(R) \to \mathrm{Loc}_l(R), \;\; S \mapsto S^{-1}R, \tag{10}$$

is an epimorphism from the poset $(\mathrm{Den}_l(R), \subseteq)$ to $(\mathrm{Loc}_l(R), \to)$. For each ideal $\mathfrak{a} \in \mathrm{Ass}_l(R)$, it induces the epimorphism between the posets $(\mathrm{Den}_l(R, \mathfrak{a}), \subseteq)$ and $(\mathrm{Loc}_l(R, \mathfrak{a}), \to)$,

$$(\cdot)^{-1}R : \mathrm{Den}_l(R, \mathfrak{a}) \to \mathrm{Loc}_l(R, \mathfrak{a}), \;\; S \mapsto S^{-1}R. \tag{11}$$

The sets $\mathrm{Den}_l(R)$ and $\mathrm{Loc}_l(R)$ are the disjoint unions

$$\mathrm{Den}_l(R) = \bigsqcup_{\mathfrak{a} \in \mathrm{Ass}_l(R)} \mathrm{Den}_l(R, \mathfrak{a}), \;\; \mathrm{Loc}_l(R) = \bigsqcup_{\mathfrak{a} \in \mathrm{Ass}_l(R)} \mathrm{Loc}_l(R, \mathfrak{a}). \tag{12}$$

For each ideal $\mathfrak{a} \in \mathrm{Ass}_l(R)$, the set $\mathrm{Den}_l(R,\mathfrak{a})$ is the disjoint union

$$\mathrm{Den}_l(R,\mathfrak{a})) = \bigsqcup_{A\in \mathrm{Loc}_l(R,\mathfrak{a}))} \mathrm{Den}_l(R,\mathfrak{a},A). \qquad (13)$$

Let $\mathrm{LDen}_l(R,\mathfrak{a}) := \{S(R,\mathfrak{a},A)\,|\, A\in \mathrm{Loc}_l(R,\mathfrak{a})\}$, see Proposition 9.1.(2). The map

$$(\cdot)^{-1}R:\ \mathrm{LDen}_l(R,\mathfrak{a}) \to \mathrm{Loc}_l(R,\mathfrak{a}),\ \ S\mapsto S^{-1}R, \qquad (14)$$

is an isomorphism of posets.

For a left denominator set $S\in \mathrm{Den}_l(R,\mathfrak{a})$, there are natural ring homomorphisms

$$R\to R/\mathfrak{a}\to S^{-1}R.$$

Lemma 9.3 and Proposition 9.4 establish connections between the left denominator sets $\mathrm{Den}_l(R,\mathfrak{a})$, $\mathrm{Den}_l(R/\mathfrak{a},0)$ and $\mathrm{Den}_l(S^{-1}R,0)$.

Let $S,T\in \mathrm{Den}_l(R)$. The denominator set T is called *S-saturated* if $sr\in T$, for some $s\in S$ and $r\in R$, then $r\in T$, and if $r's'\in T$, for some $s'\in S$ and $r'\in R$, then $r'\in T$.

Lemma 9.3 *Let $S\in \mathrm{Den}_l(R,\mathfrak{a})$, $\pi: R\to R/\mathfrak{a}$, $a\mapsto a+\mathfrak{a}$, and $\sigma: R\to S^{-1}R$, $r\mapsto r/1$.*

1. *Let $T\in \mathrm{Den}_l(S^{-1}R,0)$ be such that $\pi(S),\pi(S)^{-1}\subseteq T$. Then $T':=\sigma^{-1}(T)\in \mathrm{Den}_l(R,\mathfrak{a})$, T' is S-saturated, $T=\{s^{-1}t'\,|\, s\in S, t'\in T'\}$, and $S^{-1}R\subseteq T'^{-1}R=T^{-1}R$.*
2. *$\pi^{-1}(S_0(R/\mathfrak{a}))=S_\mathfrak{a}(R)$, $\pi(S_\mathfrak{a}(R))=S_0(R/\mathfrak{a}))$ and $Q_\mathfrak{a}(R)=S_\mathfrak{a}(R)^{-1}R=Q_l(R/\mathfrak{a})$.*

For $S_1,S_2\in \mathrm{Den}_l(R)$ such that $S_1\subseteq S_2$,

$$[S_1,S_2]:=\{T\in \mathrm{Den}_l(R)\,|\, S_1\subseteq T\subseteq S_2\}$$

is an *interval* in the posed $\mathrm{Den}_l(R)$. If, in addition, $S_1,S_2\in \mathrm{Den}_l(R,\mathfrak{a})$ then $[S_1,S_2]\subseteq \mathrm{Den}_l(R,\mathfrak{a})$ since $S_1\subseteq S\subseteq S_2$ implies $\mathfrak{a}=\mathrm{ass}(S_1)\subseteq \mathrm{ass}(S)\subseteq \mathrm{ass}(S_2)=\mathfrak{a}$, i.e. $\mathrm{ass}(S)=\mathfrak{a}$. The next proposition establishes connections between various sets of left denominator sets of the ring R, some of its factor rings and localizations.

Proposition 9.4 *Let $S\in \mathrm{Den}_l(R,\mathfrak{a})$; $\pi: R\to R/\mathfrak{a}$, $a\to \overline{a}=a+\mathfrak{a}$; $\sigma: R\to S^{-1}R, r\to r/1$; $G:=\langle \pi(S),\pi(S)^{-1}\rangle\subseteq (S^{-1}R)^*$ (i.e. G is the subgroup of the group $(S^{-1}R)^*$ of units of the ring $S^{-1}R$ generated by $\pi(S)^{\pm 1}$).*

1. *Let $[\sigma^{-1}(G),S_\mathfrak{a}(R)]_{S-\mathrm{com}}:=\{S_1\in[\sigma^{-1}(G),S_\mathfrak{a}(R)]\,|\,\sigma^{-1}(G\pi(S_1))=S_1\}$ and $[G,S_0(S^{-1}R)]:=\{T\in \mathrm{Den}_l(S^{-1}R,0)\,|\, G\subseteq T\subseteq S_0(S^{-1}R)\}$. Then the map*

$$[\sigma^{-1}(G),S_\mathfrak{a}(R)]_{S-\mathrm{com}}\to [G,S_0(S^{-1}R)],\ \ S_1\mapsto \widetilde{S}_1:=G\pi(S_1),$$

is an isomorphism of posets and abelian monoids with the inverse map $T\mapsto \sigma^{-1}(T)$ where $G\pi(S_1)$ is the multiplicative monoid generated by G and $\pi(S_1)$ in $S^{-1}R$. In particular,

$$G\pi(S_\mathfrak{a}(R))=S_0(S^{-1}R),\ \ S_\mathfrak{a}(R)=\sigma^{-1}(S_0(S^{-1}R)),\ \ S_\mathfrak{a}(R)^{-1}R=Q_l(R/\mathfrak{a}),$$

the monoid $[\sigma^{-1}(G),S_\mathfrak{a}(R)]_{S-\mathrm{com}}$ is a complete lattice (since $[G,S_0(S^{-1}R)]$ is a complete lattice, as an interval of the complete lattice $\mathrm{Den}_l(S^{-1}R,0)$, [4]), and the map $S_1\mapsto\widetilde{S}_1$ is a lattice isomorphism.

2. *Consider the interval $[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]$ in $\mathrm{Den}_l(R/\mathfrak{a},0)$. Let $[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]_{G-\mathrm{com}}:=\{T\in[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]\,|\,GT\cap(R/\mathfrak{a})=T\}$. Then $[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]_{G-\mathrm{com}}\subseteq \mathrm{Den}_l(S^{-1}R,0)$ and the map*

$$[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]_{G-\mathrm{com}}\to [G,S_0(S^{-1}R)],\ \ T\mapsto GT,$$

is an isomorphism of posets and abelian monoids with the inverse map $T'\mapsto T'\cap(R/\mathfrak{a})$ where GT is the product in $\mathrm{Den}_l(S^{-1}R,0)$. In particular, the monoid $[G\cap(R/\mathfrak{a}),S_0(R/\mathfrak{a})]_{G-\mathrm{com}}$ is a complete lattice.

3. *The map*

$$[\sigma^{-1}(G), S_{\mathfrak{a}}(R)]_{S-\text{com}} \to [G\cap (R/\mathfrak{a}), S_0(R/\mathfrak{a})]_{G-\text{com}}, \;\; S_1 \mapsto G\pi(S_1)\cap (R/\mathfrak{a}),$$

is an isomorphism of posets and abelian monoids with the inverse map $S'\mapsto \sigma^{-1}(GS')$.

The maximal left quotient rings of a ring.

Lemma 9.5 *Let* $S_1\subseteq S_2\subseteq \cdots \subseteq S_i\subseteq \cdots$ *be an ascending chain in* $\text{Den}_l(R)$, $\mathfrak{a}_i := \text{ass}(S_i)$, $S:=\bigcup_{i\geq 1} S_i$. *Then* $\mathfrak{a}_1\subseteq \mathfrak{a}_2\subseteq \cdots \subseteq \mathfrak{a}_i\subseteq \cdots$ *is the ascending chain in* $\text{Ass}_l(R)$, $S\in \text{Den}_l(R, \mathfrak{a})$ *where* $\mathfrak{a} := \bigcup_{i\geq 1}\mathfrak{a}_i$, $S^{-1}R = \text{inj}\lim S_i^{-1}R$ *where* $S_1^{-1}R\to S_2^{-1}R\to \cdots \to S_i^{-1}R\to \cdots$ *(Lemma 9.2.(2a)).*

Proof. By Lemma 9.2.(2a), $S\in \text{Den}_l(R, \mathfrak{a})$. For each number $i=1,2,\ldots$, define the ring homomorphisms $\phi_i : S_i^{-1}R\to S^{-1}R$, $s^{-1}r\mapsto s^{-1}r$, and $\nu_i : S_i^{-1}R\to S_{i+1}^{-1}R$, $s^{-1}r\mapsto s^{-1}r$. Then $\phi_i = \phi_{i+1}\nu_i$ for all i. Hence, there is the ring homomorphism $\phi : \text{inj}\lim S_i^{-1}R\to S^{-1}R$ which is a surjection since $S=\bigcup_{i\geq 1}S_i$ and has kernel 0 since $\mathfrak{a} = \bigcup_{i\geq 1}\mathfrak{a}_i$, i.e. ϕ is an isomorphism. $\square$

Consider the poset $(\text{Den}_l(R), \subseteq)$. For each element $S\in \text{Den}_l(R)$, let $[S, \cdot) := \{S'\in \text{Den}_l(R)\,|\, S\subseteq S'\}$.

Lemma 9.6 1. *For each element* $S\in \text{Den}_l(R)$, *there exists a maximal element in the poset* $([S, \cdot), \subseteq)$.

2. *The set* $(\text{max.Den}_l(R), \subseteq)$ *of maximal elements of the poset* $(\text{Den}_l(R), \subseteq)$ *is a non-empty set.*

Proof. 1. Statement 1 follows from Lemma 9.5 and Zorn's Lemma.

2. Statement 2 follows from statement 1 and the fact that the set $\text{max.Den}_l(R)$ is the set of maximal elements of the poset $[\{1\}, \cdot) = \text{Den}_l(R)$. $\square$

Definition. An element S of the set $\text{max.Den}_l(R)$ is called a *maximal left denominator set* of the ring R and the ring $S^{-1}R$ is called a *maximal left quotient ring* (or a *maximal left localization*) of the ring R.

Let $\text{max.Ass}_l(R)$ be the set of maximal elements of the poset $(\text{Ass}_l(R), \subseteq)$. It is a subset of the set

$$\text{ass.max.Den}_l(R) := \{\text{ass}(S)\,|\, S\in \text{max.Den}_l(R)\} \tag{15}$$

which is a non-empty set, by Lemma 9.6.(2). In fact, these two sets are equal.

Proposition 9.7 $\text{max.Ass}_l(R) = \text{ass.max.Den}_l(R)\neq \emptyset$.

Let $\text{max.Loc}_l(R)$ be the set of maximal elements of the poset $(\text{Loc}_l(R), \to)$. By the very definition of $\text{Loc}_l(R)$ and by Lemma 9.3.(2),

$$\text{max.Loc}_l(R) = \{S^{-1}R\,|\, S\in \text{max.Den}_l(R)\} = \{Q_l(R/\mathfrak{a})\,|\, \mathfrak{a}\in \text{ass.max.Den}_l(R)\}. \tag{16}$$

Proposition 9.8 *Let* $\mathfrak{a}\in \text{Ass}_l(R)$, $Q:=Q_{\mathfrak{a}}(R)$, Q^* *be the group of units of the ring* Q *and* $\sigma : R\to Q_{\mathfrak{a}}(R)$, $r\mapsto \frac{r}{1}$. *Let* $T\in \text{Den}_l(Q, \mathfrak{b})$ *where* $\mathfrak{b}\in \text{Ass}_l(Q)$ *and* Q^*T *be the multiplicative sub-semigroup of* $(Q, \cdot)$ *generated by* Q^* *and* T. *Then*

1. $Q^*T\in \text{Den}_l(Q, \mathfrak{b})$.

2. *If, in addition,* $Q^*\subseteq T$ *(eg,* Q^*T *from statement 1) then*

(a) $T' := \sigma^{-1}(T) \in \mathrm{Den}_l(R, \mathfrak{b}')$ *where* $\mathfrak{b}' := \sigma^{-1}(\mathfrak{b}) \supseteq \mathfrak{a}$, $S_{\mathfrak{a}}(R) \subseteq T'$, $T = Q^*\sigma(T')$ *(i.e. the monoid* T *is generated by* Q^* *and* $\sigma(T')$*) and* $T'^{-1}R = T^{-1}Q$ *(i.e. the natural ring monomorphism* $T'^{-1}R \to T^{-1}Q$, $t^{-1}r \mapsto t^{-1}r$, *is an isomorphism).*

(b) $S_{\mathfrak{a}}(R) \subseteq S_{\mathfrak{b}'}(R)$ *and* $S_{\mathfrak{b}'}(R) = \sigma^{-1}(S_{\mathfrak{b}}(Q))$.

(c) $Q_{\mathfrak{b}'}(R) = Q_l(Q/\mathfrak{b})$, *i.e. the natural ring monomorphism* $Q_{\mathfrak{b}'}(R) \to Q_l(Q/\mathfrak{b})$, $t^{-1}r \mapsto t^{-1}r$, *is an isomorphism.*

The next theorem describes various properties of the maximal left quotient rings of a ring, in particular, their groups of units and their largest left quotient rings.

Theorem 9.9 *Let* $S \in \mathrm{max.Den}_l(R)$, $A = S^{-1}R$, A^* *be the group of units of the ring* A; $\mathfrak{a} := \mathrm{ass}(S)$, $\pi_{\mathfrak{a}} : R \to R/\mathfrak{a}$, $a \mapsto a + \mathfrak{a}$, *and* $\sigma_{\mathfrak{a}} : R \to A$, $r \mapsto \frac{r}{1}$. *Then*

1. $S = S_{\mathfrak{a}}(R)$, $S = \pi_{\mathfrak{a}}^{-1}(S_0(R/\mathfrak{a}))$, $\pi_{\mathfrak{a}}(S) = S_0(R/\mathfrak{a})$ *and* $A = S_0(R/\mathfrak{a})^{-1}R/\mathfrak{a} = Q_l(R/\mathfrak{a})$.
2. $S_0(A) = A^*$ *and* $S_0(A) \cap (R/\mathfrak{a}) = S_0(R/\mathfrak{a})$.
3. $S = \sigma_{\mathfrak{a}}^{-1}(A^*)$.
4. $A^* = \langle \pi_{\mathfrak{a}}(S), \pi_{\mathfrak{a}}(S)^{-1} \rangle$, *i.e. the group of units of the ring* A *is generated by the sets* $\pi_{\mathfrak{a}}(S)$ *and* $\pi_{\mathfrak{a}}^{-1}(S) := \{\pi_{\mathfrak{a}}(s)^{-1} \,|\, s \in S\}$.
5. $A^* = \{\pi_{\mathfrak{a}}(s)^{-1}\pi_{\mathfrak{a}}(t) \,|\, s, t \in S\}$.
6. $Q_l(A) = A$ *and* $\mathrm{Ass}_l(A) = \{0\}$. *In particular, if* $T \in \mathrm{Den}_l(A, 0)$ *then* $T \subseteq A^*$.

The next theorem is a criterion of when a ring $A \in \mathrm{Loc}_l(R, \mathfrak{a})$ is equal to $Q_{\mathfrak{a}}(R)$.

Theorem 9.10 *Let* $A \in \mathrm{Loc}_l(R, \mathfrak{a})$, *i.e.* $A = S^{-1}R$ *for some* $S \in \mathrm{Den}_l(R, \mathfrak{a})$. *Then* $A = Q_{\mathfrak{a}}(R)$ *iff* $Q_l(A) = A$.

Left localization maximal rings. We introduce a new class of rings, the left localization maximal rings, which turn out to be precisely the class of maximal left quotient rings of all rings. As a result, we have a characterization of the maximal left quotient rings of a ring (Theorem 9.11).

Definition. A ring A is called a *left localization maximal ring* if $A = Q_l(A)$ and $\mathrm{Ass}_l(A) = \{0\}$. A ring A is called a *right localization maximal ring* if $A = Q_r(A)$ and $\mathrm{Ass}_r(A) = \{0\}$. A ring A which is a left and right localization maximal ring is called a *left and right localization maximal ring* (i.e. $Q_l(A) = A = Q_r(A)$ and $\mathrm{Ass}_l(A) = \mathrm{Ass}_r(A) = \{0\}$).

Example. Let A be a simple ring. Then $Q_l(A)$ is a left localization maximal ring and $Q_r(A)$ is a right localization maximal ring.

Example. A division ring is a (left and right) localization maximal ring. More generally, a simple Artinian algebra (i.e. the matrix algebra over a division ring) is a (left and right) localization maximal ring.

The next theorem is a criterion of when a left quotient ring of a ring is a maximal left quotient ring of the ring.

Theorem 9.11 *Let a ring* A *be a left localization of a ring* R, *i.e.* $A \in \mathrm{Loc}_l(R, \mathfrak{a})$ *for some* $\mathfrak{a} \in \mathrm{Ass}_l(R)$. *Then* $A \in \mathrm{max.Loc}_l(R)$ *iff* $Q_l(A) = A$ *and* $\mathrm{Ass}_l(A) = \{0\}$, *i.e.* A *is a left localization maximal ring.*

Proof. ($\Rightarrow$) Theorem 9.9.(6).
($\Leftarrow$) Proposition 9.8. $\square$

Theorem 9.12 *Let $S \in \max.\mathrm{Den}_l(R)$, $A = S^{-1}R$, $\mathfrak{a} = \mathrm{ass}(S)$ and $\sigma_\mathfrak{a} : R \to A$, $r \mapsto \frac{r}{1}$. Then the following statements are equivalent.*

1. *A is a semi-simple ring.*
2. *$Q_{cl}(R/\mathfrak{a})$ exists and is a semi-simple ring.*

If one of these conditions holds then $A = Q_{cl}(R)$ and $S = \sigma_\mathfrak{a}^{-1}(Q_{cl}(R)^*)$.

Proof. 1. Since $A = Q_l(R/\mathfrak{a})$ and $S = \sigma_\mathfrak{a}^{-1}(A^*)$, by Theorem 9.9.(1,3), the results follow from Theorem 8.4. □

Remark. All the results/definitions of the present section are left-sided. The analogous results/definitions are true for right-sided and two-sided versions (see [4], for details). In Section 10, left, right and two-sided (i.e., left and right) versions of results are given for some classes of rings. In the two-sided definitions, the subscript 'l' in the corresponding left-sided versions is dropped. For example, $S_0(R)$ means the largest regular (left and right) Ore set of a ring R, $Q(R) := S_0(R)^{-1}R$ is the *largest* (left and right) *quotient ring* of a ring R, etc.

10 Examples

In this section, the largest (left; right; left and right) quotient ring and the maximal (left; right; left and right) quotient rings are found for the following rings: the endomorphism algebra $\mathrm{End}_K(V)$ of an infinite dimensional vector space with countable basis, semi-prime Goldie rings, the algebra $\mathbb{I}_1$ of polynomial integro-differential operators, and Noetherian commutative rings. The proofs of all the results of the present section can be found in [4].

The endomorphism algebra $\mathrm{End}_K(V)$ of an infinite dimensional vector space V with countable basis. For a vector space V, let

$$\mathcal{F} = \mathcal{F}(V) := \{\varphi \in \mathrm{End}_K(V) \,|\, \dim_K(\ker(\varphi)) < \infty,\ \dim_K(\mathrm{coker}(\varphi)) < \infty\}$$

be the set of *Fredholm* linear maps/operators in V.

Theorem 10.1 *Let V be an infinite dimensional vector space with countable basis, $R := \mathrm{End}_K(V)$ and $\mathcal{C} := \{\varphi \in R \,|\, \dim_K(\mathrm{im}(\varphi)) < \infty\}$ be the ideal of compact operators of R (this is the only proper ideal of R). Then*

1. $\mathrm{Ass}_l(R) = \mathrm{Ass}_r(R) = \mathrm{Ass}(R) = \{0, \mathcal{C}\}$.
2. $S_{l,0}(R) = S_{r,0}(R) = S_0(R) = \mathrm{Aut}_K(V)$ *and* $Q_l(R) = Q_r(R) = Q(R) = R$.
3. $S_{l,\mathcal{C}}(R) = S_{r,\mathcal{C}}(R) = S_\mathcal{C}(R) = \mathcal{F}$ *and* $Q_{l,\mathcal{C}}(R) = Q_{r,\mathcal{C}}(R) = Q_\mathcal{C}(R) = R/\mathcal{C}$.
4. $\max.\mathrm{Ass}_l(R) = \max.\mathrm{Ass}_r(R) = \max.\mathrm{Ass}(R) = \{\mathcal{C}\}$.
5. *$R/\mathcal{C}$ is a localization maximal ring and a left (resp. right; left and right) localization maximal ring.*

Semi-prime Goldie rings. Recall that a ring R is called a left Goldie ring if R has finite left uniform dimension and R satisfies ACC on left annihilators. A right Goldie ring is similarly defined. A left and right Goldie ring is called a *Goldie* ring. The reader is referred to the books [14], [17] and [24] for more details.

Corollary 10.2 *Let R be a prime Goldie ring and $\mathcal{C}_R$ be the set of regular elements of R. Then* $\mathrm{Ass}(R) = \{0\}$, $S_0(R) = \mathcal{C}_R$, $Q_0(R) = Q_{cl}(R)$ *is the only maximal localization of the ring R.*

Let R be a semi-prime Goldie ring which is not a prime ring and $\mathcal{C}_R$ be its set of regular elements. Let $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of minimal primes of the ring R. By Goldie's Theorem, $Q_{cl}(R) := \mathcal{C}_R^{-1}R \simeq R\mathcal{C}_R^{-1} \simeq \prod_{i=1}^s R_i$ is the direct product of simple Artinian rings R_i (i.e. R_i is a matrix ring over a division ring). The ring R can be identified with its image under the ring monomorphism $\sigma : R \to Q_{cl}(R)$, $r \mapsto r/1$. For each non-empty set I of the set $\{1, \ldots, s\}$, let $R_I := \prod_{i\in I} R_i$ and $CI := \{1, \ldots, s\}\backslash I$. For each $i = 1, \ldots, s$, $\mathfrak{p}_i = \sigma^{-1}(\prod_{i\neq j=1}^s R_j) = \sigma^{-1}(R_{Ci}) = R \cap R_{Ci}$ (Proposition 3.2.2, [17]). Let R_i^* be the group of units of the ring R_i. Then $R_I^* = \prod_{i\in I} R_i^*$ is the group of units of the ring R_I. A subset A of a set B is called a *proper subset* of B if $A \neq \emptyset, B$.

Theorem 10.3 *Let R be a semi-prime Goldie ring which is not a prime ring and $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of its minimal prime ideals. Then*

1. $\mathrm{Ass}(R) := \{0\} \cup \{\mathfrak{a}(I) := \bigcap_{i\in CI} \mathfrak{p}_i = R \cap R_I \,|\, \emptyset \subsetneqq I \subsetneqq \{1, \ldots, s\}\}$ *where* $CI = \{1, \ldots, s\}\backslash I$.
2. *For each proper subset I of $\{1, \ldots, s\}$, $S_{\mathfrak{a}(I)}(R) = \sigma^{-1}(R_{CI}^* \times R_I) = R \cap R_{CI}^* \times R_I$ and $Q_{\mathfrak{a}(I)}(R) = Q_{cl}(R)/R_I \simeq R_{CI}$.*
3. *$S_0(R) = \mathcal{C}_R \subseteq S_{\mathfrak{a}}(R)$ for all $\mathfrak{a} \in \mathrm{Ass}(R)$.*
4. *$S_{\mathfrak{a}(I)}(R) \subseteq S_{\mathfrak{a}(J)}(R)$ iff $I \subseteq J$ where I and J are proper subsets of $\{1, \ldots, s\}$.*
5. *If I and J are proper subsets of $\{1, \ldots, s\}$ such that $I \subseteq J$ then, by statement 4 and the universal property of localization, there is the unique ring homomorphism $Q_{\mathfrak{a}(I)}(R) = \prod_{i\in CI} R_i \to Q_{\mathfrak{a}(J)}(R) = \prod_{j\in CJ} R_j$ which is necessarily the projection onto $\prod_{j\in CJ} R_j$ in $\prod_{i\in CI} R_i$.*
6. $\mathrm{max.Ass}(R) = \{\mathfrak{p}_i, \,|\, i = 1, \ldots, s\}$, *$\{Q_{\mathfrak{p}_i}(R) = R_i \,|\, i = 1, \ldots, s\}$ is the set of maximal localizations of the ring R and $\mathrm{ass}(Q_{\mathfrak{p}_i}(R)) = 0$ for all $i = 1, \ldots, s$.*

The algebra $\mathbb{I}_1$ of polynomial integro-differential operators. Let us collect some facts for the algebra $\mathbb{I}_1$ which are necessary to understand the results of this subsection, in particular, Theorem 10.5 and Proposition 10.6. For the details the reader is referred to [2] or [3]. Throughout,

- K is a field of characteristic zero and K^* is its group of units;
- $P_1 := K[x]$ is a polynomial algebra in one variable x over K;
- $\partial := \frac{d}{dx}$;
- $\mathrm{End}_K(P_1)$ is the algebra of all K-linear maps from P_1 to P_1, and $\mathrm{Aut}_K(P_1)$ is its group of units (i.e. the group of all the invertible linear maps from P_1 to P_1);
- the subalgebras $A_1 := K\langle x, \partial\rangle$ and $\mathbb{I}_1 := K\langle x, \partial, \int\rangle$ of $\mathrm{End}_K(P_1)$ are called the (first) *Weyl algebra* and the *algebra of polynomial integro-differential operators* respectively where $\int : P_1 \to P_1$, $p \mapsto \int p\, dx$, is the *integration*, i.e. $\int : x^n \mapsto \frac{x^{n+1}}{n+1}$ for all $n \in \mathbb{N}$.

The algebra $\mathbb{I}_1$ is neither left nor right Noetherian and not a domain. Moreover, it contains infinite direct sums of nonzero left and right ideals, [2].

The algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations (Proposition 2.2, [2]):

$$\partial \int = 1, \;\; [H, \int] = \int, \;\; [H, \partial] = -\partial, \;\; H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \;\; i, j \in \mathbb{N}, \tag{17}$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00}\partial^j$.

The algebra $\mathbb{I}_1 = \bigoplus_{i\in\mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i,j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|}D_1 = D_1\partial^{|i|} & \text{if } i < 0, \end{cases} \tag{18}$$

the algebra $D_1 := K[H] \bigoplus \bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is a commutative non-Noetherian subalgebra of $\mathbb{I}_1$, $He_{ii} = e_{ii}H = (i+1)e_{ii}$ for $i \in \mathbb{N}$ (notice that $\bigoplus_{i\in\mathbb{N}} Ke_{ii}$ is the direct sum of non-zero ideals of D_1); $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; ${}_{D_1}(D_1\partial^i) \simeq D_1$, $d\partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \to D_1 \int^i$, $d \mapsto d\int^i$, and $\partial^i \cdot : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} Ke_{jj}$.

Each element a of the algebra $\mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i>0} a_{-i}\partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j\in\mathbb{N}} \lambda_{ij}e_{ij} \tag{19}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [2]. The algebra $\mathbb{I}_1$ has the only proper ideal

$$F = \bigoplus_{i,j\in\mathbb{N}} Ke_{ij} \simeq M_\infty(K) \text{ and } F^2 = F.$$

The factor algebra $\mathbb{I}_1/F$ is canonically isomorphic to the skew Laurent polynomial algebra $B_1 := K[H][\partial, \partial^{-1}; \tau]$, $\tau(H) = H + 1$, via $\partial \mapsto \partial$, $\int \mapsto \partial^{-1}$, $H \mapsto H$ (where $\partial^{\pm 1}\alpha = \tau^{\pm 1}(\alpha)\partial^{\pm 1}$ for all elements $\alpha \in K[H]$). The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1}H$). Therefore, they have common skew field of fractions, $\text{Frac}(A_1) = \text{Frac}(B_1)$, the *first Weyl skew field*.

The algebra $\mathbb{I}_1$ admits the involution $*$ over the field K:

$$\partial^* = \int, \quad \int^* = \partial \quad \text{and} \quad H^* = H,$$

i.e. it is a K-algebra *anti-isomorphism* ($(ab)^* = b^*a^*$) such that $a^{**} = a$. Therefore, the algebra $\mathbb{I}_1$ is *self-dual*, i.e. it is isomorphic to its opposite algebra $\mathbb{I}_1^{op}$. As a result, the left and right properties of the algebra $\mathbb{I}_1$ are the same. Clearly, $e_{ij}^* = e_{ji}$ for all $i,j \in \mathbb{N}$, and so $F^* = F$.

The next theorem describes the largest regular left and right Ore sets and the largest left and right quotient rings of the algebra $\mathbb{I}_1$.

Theorem 10.4 (Theorem 9.7, [3])

1. *$S_{r,0}(\mathbb{I}_1) = \mathbb{I}_1 \bigcap \text{Aut}_K(K[x])$ and the largest regular right quotient ring $Q_r(\mathbb{I}_1)$ of $\mathbb{I}_1$ is the subalgebra of $\text{End}_K(K[x])$ generated by $\mathbb{I}_1$ and $S_{r,0}(\mathbb{I}_1)^{-1} := \{s^{-1} \mid s \in S_{r,0}(\mathbb{I}_1)\}$.*
2. *$S_{l,0}(\mathbb{I}_1) = S_{r,0}(\mathbb{I}_1)^*$ and $S_{l,0}(\mathbb{I}_1) \neq S_{r,0}(\mathbb{I}_1)$.*
3. *The rings $Q_l(\mathbb{I}_1)$ and $Q_r(\mathbb{I}_1)$ are not isomorphic.*

The next theorem describes the largest regular (two-sided) Ore set and the largest (two-sided) quotient ring of the algebra $\mathbb{I}_1$. These objects are tiny comparing with their one-sided counterparts.

Theorem 10.5 *Let $\mathcal{M} := (K[H] + F) \bigcap \text{Aut}_K(K[x])$. Then*

1. *$S_0(\mathbb{I}_1) = S_{l,0}(\mathbb{I}_1) \bigcap S_{r,0}(\mathbb{I}_1)$, $S_0(\mathbb{I}_1)$ is a proper subset of the sets $S_{l,0}(\mathbb{I}_1)$ and $S_{r,0}(\mathbb{I}_1)$, and $S_0(\mathbb{I}_1)^* = S_0(\mathbb{I}_1)$ where $*$ is the involution of the algebra $\mathbb{I}_1$.*

2. $S_{l,0}(\mathbb{I}_1)\bigcap S_{r,0}(\mathbb{I}_1)=\mathcal{M}$ *and* $\mathcal{M}$ *is the set of regular elements of the algebra* $K[H]+F$.

3. *Let* $\mathcal{M}_0:=D_1\bigcap \mathrm{Aut}_K(K[x])$. *Then* $\mathcal{M}_0\subseteq\mathcal{M}$, $\mathcal{M}=\mathcal{M}_0(1+F)^*=(1+F)^*\mathcal{M}_0$ *and* $\mathcal{M}_0\bigcap(1+F)^*=(1+F_0)^*$ *where* $F_0:=\bigoplus_{i\in\mathbb{N}}Ke_{ii}$.

4. $\mathcal{M}_0$ *is the set of regular elements of the commutative (non-Noetherian) algebra* D_1; $D_1=\mathcal{M}_0(1+F_0)\coprod F_0=\mathcal{M}_0\cup\{0\}+F_0$, $Q_{cl}(D_1):=\mathcal{M}_0^{-1}D_1=\mathcal{M}_0^{-1}\mathcal{M}_0(1+F_0)\coprod F_0=\mathcal{M}_0^{-1}\mathcal{M}_0\cup\{0\}+F_0$.

5. $Q(\mathbb{I}_1)=S_0(\mathbb{I}_1)^{-1}\mathbb{I}_1=\sum_{i\in\mathbb{Z}}Q_{cl}(D_1)v_i+F=\sum_{i\in\mathbb{Z}}(\mathcal{M}_0^{-1}\mathcal{M}_0\cup\{0\})v_i+F=\sum_{i\in\mathbb{Z}}v_iQ_{cl}(D_1)+F=\sum_{i\in\mathbb{Z}}v_i(\mathcal{M}_0^{-1}\mathcal{M}_0\cup\{0\})+F$ *where* $Q_{cl}(D_1)$ *is the classical ring of fractions of the commutative ring* D_1 *and*
$$v_i:=\begin{cases}\int^i & \text{if } i\geq 1,\\ 1 & \text{if } i=0,\\ \partial^{|i|} & \text{if } i\leq -1.\end{cases}$$

6. $Q(\mathbb{I}_1)\subsetneqq Q_l(\mathbb{I}_1)$ *and* $Q(\mathbb{I}_1)\subsetneqq Q_r(\mathbb{I}_1)$.

Proposition 10.6 *1.* $\mathrm{Ass}_l(\mathbb{I}_1)=\mathrm{Ass}_r(\mathbb{I}_1)=\mathrm{Ass}(\mathbb{I}_1)=\{0,F\}$ *and* $\max.\mathrm{Ass}_l(\mathbb{I}_1)=\max.\mathrm{Ass}_r(\mathbb{I}_1)=\max.\mathrm{Ass}(\mathbb{I}_1)=\{F\}$.

2. $S_{l,F}(\mathbb{I}_1)=S_{r,F}(\mathbb{I}_1)=S_F(\mathbb{I}_1)=\mathbb{I}_1\backslash F$ *and* $Q_{l,F}(\mathbb{I}_1)=Q_{r,F}(\mathbb{I}_1)=Q_F(\mathbb{I}_1)=\mathrm{Frac}(B_1)=\mathrm{Frac}(A_1)$.

3. $\max.\mathrm{Den}_l(\mathbb{I}_1)=\max.\mathrm{Den}_r(\mathbb{I}_1)=\max.\mathrm{Den}(\mathbb{I}_1)=\{\mathbb{I}_1\backslash F\}$.

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