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ON THE EXISTENCE OF FOURIER-MUKAI FUNCTORS

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ABSTRACT. A theorem by Orlov states that any equivalence $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ between the bounded derived categories of coherent sheaves of two smooth projective varieties X and Y is isomorphic to a Fourier-Mukai transform $\Phi_E(-) = R\pi_{2*}(E \overset{L}{\otimes} L\pi_1^*(-))$, where the kernel E is in $D^b_{\mathrm{Coh}}(X \times Y)$. In the case of an exact functor which is not necessarily fully faithful, we compute some sheaves that play the role of the cohomology sheaves of the kernel, and that are isomorphic to the latter whenever an isomorphism $F \cong \Phi_E$ exists. We then exhibit a class of functors that are not full or faithful and still satisfy the above result.

1. Introduction

Let X, Y be smooth projective varieties over an algebraically closed field k. Consider an exact functor

$$F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y).$$

Orlov proved in [Orl97] that, if F is a fully faithful functor and has a right adjoint, then there exists an object $E \in D^b_{\mathrm{Coh}}(X \times Y)$ such that X is isomorphic to the Fourier-Mukai transform Φ_E, defined as

$$\Phi_E(-) = R\pi_{2*}(E \overset{L}{\otimes} L\pi_1^*(-)). \tag{1}$$

The requirement that F needs to have a right adjoint is actually unnecessary, since by [BVdB03, Theorem 1.1], every exact functor $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ has a left and a right adjoint. (For an explanation on why this is true, see for example [CS07, Remark 2.1].)

There is evidence that this theorem should generalize to the case where the functor is not full or faithful. Canonaco and Stellari, in their paper [CS07], partially following Orlov's proof, give a weaker condition for the functor for it to be still isomorphic to a Fourier-Mukai functor, i.e. of the form (1) but not necessarily fully faithful. The condition is in particular satisfied if the functor is full. This is however a bittersweet result: in fact, in [COS13] together with Orlov they also proved that when X and Y are smooth projective varieties over a field of characteristic 0, if a functor is full then it is also faithful.

Although not all functors between bounded derived categories of coherent sheaves on smooth projective varieties are isomorphic to a Fourier-Mukai functor [RVdB15][RVdB14], given any exact functor F as above, we compute sheaves $\mathscr{B}^i$ on the product $X \times Y$ that are canonically associated to F and that coincide with the cohomology sheaves of the kernel E whenever F is isomorphic to a Fourier-Mukai functor Φ_E:

Theorem 1.1. *Let X, Y be smooth projective varieties over an algebraically closed field k, and $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ an exact functor. There exist sheaves $\mathscr{B}^i \in \mathrm{Coh}(X \times Y)$, with $i \in \mathbb{Z}$, all but a finite number of them equal to zero, such that for every locally free sheaf of finite rank $\mathscr{E}$ on X and for all $n \in \mathbb{Z}$ there are functorial maps*

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to \pi_{2*}(\mathscr{B}^i \otimes \pi_1^*\mathscr{E}(n))$$

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which are isomorphisms for n sufficiently high (depending on $\mathscr{E}$). The sheaves $\mathscr{B}^i$ are independent from the projective embedding.

In particular, if $F \cong \Phi_E$, i.e. F is the Fourier-Mukai transform with kernel E, then $\mathscr{H}^i(E) \cong \mathscr{B}^i$ for all $i \in \mathbb{Z}$.

We are then able, in a special case, to construct an isomorphism between a class of functors that are not full or faithful and a Fourier-Mukai transform:

Theorem 1.2. *Let X and Y be smooth projective varieties over an algebraically closed field k, with X of dimension one, and $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ an exact functor. Assume that there exists an integer M such that the sheaves $\mathscr{B}^i$ associated to F as in Theorem 1.1 are zero for $i \neq M$, and that $\mathscr{B}^M$ is a direct sum of skyscraper sheaves $\mathscr{B}^M = \bigoplus_{i=1}^t k(p_i, q_i)$. Let Φ be the Fourier-Mukai transform associated to the complex given by the sheaf $\mathscr{B}^M$ placed in degree M, $\Phi = \Phi_{\mathscr{B}^M[-M]}$. Then there exists an isomorphism of functors $t : F \to \Phi$.*

Even when we don't know how to build a kernel out of the sheaves $\mathscr{B}^i$ obtained in Theorem 1.1, these sheaves turn out to have good properties in their own right. As an example, in section 4 we show that the analogue of the Cartan-Eilenberg Spectral Sequence converges when the dimension of X is one.

Notation. From now on, X and Y will be smooth projective varieties over an algebraically closed field k; π_1 and π_2 will be the projections $X \times Y \to X$ and $X \times Y \to Y$ respectively; $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ will be an exact functor, i.e. an additive functor that commutes with shifts and preserves triangles; and $\mathcal{O}_X(1)$ will be a very ample line bundle on X.

Given a morphism π, we will use π_* and π^* to indicate pushforward and pullback on coherent sheaves, whereas pullback and pushforward in the derived category will always be indicated with $L\pi^*$ and $R\pi_*$ unless they coincide with the regular pullback and pushforward, in which case both notations will be used interchangeably.

For a smooth variety X, $\mathrm{Coh}(X)$ will be considered as a full subcategory of $D^b_{\mathrm{Coh}}(X)$ by associating to a sheaf the complex given by that sheaf placed in degree zero. Hence we will write $F(\mathscr{E})$ to indicate $F(\mathscr{E}[0])$.

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2. Determining the cohomology sheaves of the prospective kernel

Consider an exact functor $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$. If we know that F is isomorphic to a Fourier-Mukai transform Φ_E, then we are of course able to compute the cohomology sheaves $\mathscr{B}^i = \mathscr{H}^i(E)$ corresponding to E. Even if we don't know what E is, or even if it exists, we are able to compute some sheaves on $X \times Y$ that, if the functor comes from a Fourier-Mukai transform, turn out to be the cohomology sheaves of the corresponding kernel.

Lemma 2.1. *Let X, Y be smooth projective varieties over an algebraically closed field, $\mathscr{O}_X(1)$ a very ample invertible sheaf on X.*

We define an equivalence relation $\sim$ on graded $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules $\mathscr{M} = \oplus_k \mathscr{M}_k$ by saying that $\mathscr{M} \sim \mathscr{M}'$ if there is an integer d such that $\mathscr{M}_{\geq d} \cong \mathscr{M}'_{\geq d}$, where $\mathscr{M}_{\geq d} = \oplus_{k \geq d} \mathscr{M}_k$. We say that a $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-module $\mathscr{M}$ is almost coherent if it is equivalent to a coherent $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-module.

There exists an equivalence of categories between the category of coherent sheaves on $X \times Y$ and the category of almost coherent graded $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules modulo the equivalence relation $\sim$.

Moreover, if this correspondence associates a sheaf $\oplus \mathscr{M}_n$ on Y to a sheaf $\mathscr{B}$ on $X \times Y$, there exists a functorial map of graded $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules

$$\oplus \mathscr{M}_n \xrightarrow{\psi} \oplus \pi_{2*}(\mathscr{B} \otimes \pi_1^* \mathscr{O}_X(n))$$

which is an isomorphism on the n^{th} graded piece for n sufficiently high.

Proof. This is a routine application of [Gro61, 3.2.4, 3.3.5, 3.4.3, 3.4.5, 3.3.5.1]. The fact that we get an isomorphism $\oplus \mathscr{M}_n \to \oplus \pi_{2*}(\mathscr{B} \otimes \pi_1^* \mathscr{O}_X(n))$ in large enough degree can be checked locally and hence follows by [Ser07, §3.3, Proposition 5, p.258]. □

We now move on to the proof of Theorem 1.1. We will compute the cohomology sheaves $\mathscr{B}^i$ by descending induction on i, and simultaneously show that these are the sheaves satisfying the isomorphisms in the statement of the Theorem. The idea of analyzing the cohomology sheaves of the kernel of a Fourier-Mukai transform has similarly been used in [CS12] to obtain a uniqueness result on these sheaves.

Proof of Theorem 1.1. Recall that by [BVdB03, Theorem 1.1] F has a left and right adjoint. Hence by [Orl97, Lemma 2.4], we can assume that F is bounded, i.e. that $F(\mathscr{F}) \in D^{[M,N]}_{\mathrm{coh}}(Y)$ for all coherent sheaves $\mathscr{F}$ on X; in particular, $H^i(F(\mathscr{E})) = 0$ for all $i \notin [M,N]$ and all locally free sheaves $\mathscr{E}$ of finite rank on X. Therefore we can take $\mathscr{B}^i = 0$ for $i \notin [M,N]$.

We will proceed by descending induction on the cohomology degree i. Assume we found the sheaves $\mathscr{B}^N, \mathscr{B}^{N-1}, \ldots, \mathscr{B}^{i+1}$ satisfying the conclusions of the Theorem and let us compute the sheaf $\mathscr{B}^i$. To do this we will proceed in two steps: first we will construct sheaves $\mathscr{B}^i_{\mathscr{E}}$ for all locally free sheaves $\mathscr{E}$ of finite rank, as well as maps

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to \pi_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes \pi_1^* \mathcal{O}_X(n))$$

that are isomorphisms for n sufficiently high, depending on $\mathscr{E}$ and i. Then we will show that

$$\mathscr{B}^i_{\mathscr{E}} = \mathscr{B}^i_{\mathcal{O}_X} \otimes \pi_1^* \mathscr{E}.$$

For the first step, the key is showing that the sheaf $\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}(n)))$ on Y is finitely generated for each n_0 as a $\Gamma_*(X, \mathcal{O}_X) \otimes \mathcal{O}_Y$-module. To do this, proceed as follows: let s be an integer such that we have a surjection $\mathcal{O}_X^{\oplus s} \to \mathcal{O}_X(1)$. Let $\mathscr{E}$ be a locally free sheaf of finite rank on X. Then by tensoring the map above with $\mathscr{E}$ and twisting by n we have a short exact sequence of locally free sheaves

$$0 \to K(n) \to \mathscr{E}^{\oplus s}(n) \to \mathscr{E}(n+1) \to 0.$$

Hence

$$0 \to \pi_1^*(K(n)) \to \pi_1^*(\mathscr{E}(n)^{\oplus s}) \to \pi_1^*(\mathscr{E}(n+1)) \to 0$$

is also a short exact sequence of locally free sheaves, and tensoring with $\mathscr{B}^{i+1}$ will yield another short exact sequence:

$$0 \to \mathscr{B}^{i+1} \otimes \pi_1^* K(n) \to \mathscr{B}^{i+1} \otimes \pi_1^* \mathscr{E}^{\oplus s} \to \mathscr{B}^{i+1} \otimes \pi_1^* \mathscr{E}(n+1) \to 0.$$

Moreover, since $\pi_1^* \mathcal{O}_X(1)$ is very ample with respect to $X \times Y \to Y$, for n high enough (depending on K) the pushforward to Y will still be exact:

$$0 \to \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* K(n)) \to \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* \mathcal{O}_X(n)^{\oplus s}) \to \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* \mathcal{O}_X(n+1)) \to 0.$$

Hence we get a commutative diagram

$$\begin{tikzcd}
 & \mathscr{H}^{i+1}(F(K(n))) \arrow[r] \arrow[d] & \mathscr{H}^{i+1}(F(\mathscr{E}(n)^{\oplus s})) \arrow[r] \arrow[d] & \mathscr{H}^{i+1}(F(\mathscr{E}(n+1))) \arrow[d] & \\
0 \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* K(n)) \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* \mathscr{E}(n)^{\oplus s}) \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^* \mathscr{E}(n+1)) \arrow[r] & 0
\end{tikzcd}$$

and for n high enough depending on K and $\mathscr{E}$, the vertical arrows are isomorphisms by the induction hypothesis; therefore the top sequence is also exact. Hence, moving down to the i^{th} cohomology sheaves, for n sufficiently high we also get a surjection

$$\mathscr{H}^i(F(\mathscr{E}(n)))^{\oplus s} \to \mathscr{H}^i(F(\mathscr{E}(n+1))) \to 0.$$

Since each $\mathscr{H}^i(F(\mathscr{E}(n)))$ is coherent, this is enough to conclude that the sheaf

$$\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}(n))) \tag{2}$$

is finitely generated for each n_0 as a $\Gamma_*(X, \mathcal{O}_X) \otimes \mathcal{O}_Y$-module, where the $\Gamma_*(X, \mathcal{O}_X)$-action comes from the action of $\Gamma_*(X, \mathcal{O}_X)$ on $\oplus \mathscr{E}(n)$ which gives a corresponding action on $\oplus F(\mathscr{E}(n))$ and hence on $\oplus \mathscr{H}^i(F(\mathscr{E}(n)))$. By Lemma 2.1 then, the sheaf $\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}(n)))$ on Y corresponds to a sheaf $\mathscr{B}^i_{\mathscr{E}}$ on $X \times Y$ such that the functorial map

$$\mathscr{H}^i(F(\mathscr{E}(n))) \xrightarrow{\psi} \pi_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes \pi_1^* \mathcal{O}_X(n))$$

is an isomorphisms for n sufficiently high.

Now consider the functor

$$B : \mathrm{Vect}(X) \to \mathrm{Coh}(X \times Y)$$
$$\mathscr{E} \mapsto \mathscr{B}^i_{\mathscr{E}}$$

from the category of locally free sheaves of finite rank on X to the category of coherent sheaves on $X \times Y$. The functor B is additive and right exact. In fact, given two coherent sheaves $\mathscr{E}_1$ and $\mathscr{E}_2$,

$$\bigoplus_n \mathscr{H}^i(F((\mathscr{E}_1 + \mathscr{E}_2)(n))) = \bigoplus_n \mathscr{H}^i(F(\mathscr{E}_1(n))) \oplus \bigoplus_n \mathscr{H}^i(F(\mathscr{E}_2(n)))$$

hence the functor is additive. Moreover, given a short exact sequence of finite rank locally free sheaves $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{E}_3 \to 0$, we get a triangle $F(\mathscr{E}_1) \to F(\mathscr{E}_2) \to F(\mathscr{E}_3)$ hence for $n \gg 0$ we have (by induction hypothesis)

$$\begin{tikzcd}
& \mathscr{H}^{i+1}(F(\mathscr{E}_1(n)) \arrow[r] \arrow[d] & \mathscr{H}^{i+1}(F(\mathscr{E}_2(n))) \arrow[r] \arrow[d] & \mathscr{H}^{i+1}(F(\mathscr{E}_3(n))) \arrow[d] & \\
0 \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^*\mathscr{E}_1(n)) \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^*\mathscr{E}_2(n)) \arrow[r] & \pi_{2*}(\mathscr{B}^{i+1} \otimes \pi_1^*\mathscr{E}_3(n)) \arrow[r] & 0
\end{tikzcd}$$

and for n sufficiently high, all of the vertical maps are isomorphisms hence the top sequence is also exact for n high, say $n > n_0$.

Hence moving down to the i^{th} cohomology sheaves we get an exact sequence of $\Gamma_*(\mathcal{O}_X) \otimes \mathcal{O}_Y$-modules

$$\bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_1(n))) \to \bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_2(n))) \to \bigoplus_{n>n_0} \mathscr{H}^i(F(\mathscr{E}_3(n))) \to 0 \tag{3}$$

and so (by the equivalence of categories) get

$$\mathscr{B}^i_{\mathscr{E}_1} \to \mathscr{B}^i_{\mathscr{E}_2} \to \mathscr{B}^i_{\mathscr{E}_3} \to 0$$

hence the functor B is right exact on the full subcategory of locally free sheaves of finite rank.

Moreover, for every n, for $m \gg 0$ (depending on n) we have

$$\mathscr{H}^i(F(\mathscr{E}(n)(m))) = \pi_{2*}(\mathscr{B}^i_{\mathscr{E}(n)} \otimes \pi_1^* \mathcal{O}_X(m))$$

but also

$$\begin{aligned}
\mathscr{H}^i(F(\mathscr{E}(n)(m))) &= \mathscr{H}^i(F(\mathscr{E}(n+m))) \\
&= \pi_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes \pi_1^* \mathcal{O}_X(n+m)) \\
&= \pi_{2*}((\mathscr{B}^i_{\mathscr{E}} \otimes \pi_1^* \mathcal{O}_X(n)) \otimes \pi_1^* \mathcal{O}_X(m)),
\end{aligned}$$

hence it follows from the equivalence of categories that

$$\mathscr{B}^i_{\mathscr{E}(n)} = \mathscr{B}^i_{\mathscr{E}} \otimes \mathcal{O}_X(n).$$

Now let $\mathscr{E}$ be a locally free sheaf of finite rank on X. Then there exists a sequence

$$\oplus \mathcal{O}_X(b_j) \to \oplus \mathcal{O}_X(a_k) \to \mathscr{E} \to 0,$$

therefore since the functor B is right exact we get

$$\mathscr{B}^i_{\oplus \mathcal{O}_X(b_j)} \to \mathscr{B}^i_{\oplus \mathcal{O}_X(a_k)} \to \mathscr{B}^i_{\mathscr{E}} \to 0.$$

Since B is additive and compatible with twists, we can write

$$\oplus \mathscr{B}^i_{\mathcal{O}_X} \otimes \pi_1^* \mathcal{O}_X(b_j) \to \oplus \mathscr{B}^i_{\mathcal{O}_X} \otimes \pi_1^* \mathcal{O}_X(a_k) \to \mathscr{B}^i_{\mathscr{E}} \to 0$$

hence

$$\mathscr{B}^i_{\mathscr{E}} = \mathscr{B}^i_{\mathcal{O}_X} \otimes \pi_1^* \mathscr{E} \tag{4}$$

and the theorem follows by taking $\mathscr{B}^i = \mathscr{B}^i_{\mathcal{O}_X}$. Since there is a finite number of steps in the induction, we can find an n_0 such that for $n > n_0$ the maps

$$\mathscr{H}^i(F(\mathscr{E}(n))) \to \pi_{2*}(\mathscr{B}^i_{\mathscr{E}} \otimes \pi_1^* \mathcal{O}_X(n)) \tag{5}$$

are isomorphisms for all i.

We will now show that the $\mathscr{B}^i$'s do not depend on a choice of a very ample line bundle $\mathcal{O}_X(1)$ on X. Let $\mathscr{L}_1, \mathscr{L}_2$ be two very ample line bundles on X and we will denote by $\mathscr{B}^i(\mathscr{L}_1)$ and $\mathscr{B}^i(\mathscr{L}_2)$ the corresponding coherent sheaves on $X \times Y$. Then by (4) and (5), for n sufficiently high we have an isomorphism

$$\mathscr{H}^i(F(\mathscr{E} \otimes \mathscr{L}_i^{\otimes n})) \cong \pi_{2*}(\mathscr{B}^i(\mathscr{L}_i) \otimes \pi_1^* \mathscr{E} \otimes \pi_1^* \mathscr{L}_i^{\otimes n})$$

for $i = 1, 2$. It follows that

$$\mathscr{H}^i(F(\mathscr{L}_1^{\otimes n} \otimes \mathscr{L}_2^{\otimes n}) \cong \pi_{2*}(\mathscr{B}^i(\mathscr{L}_1) \otimes \pi_1^* \mathscr{L}_1^{\otimes n} \otimes \pi_1^* \mathscr{L}_2^{\otimes n})$$
$$\mathscr{H}^i(F(\mathscr{L}_1^{\otimes n} \otimes \mathscr{L}_2^{\otimes n}) \cong \pi_{2*}(\mathscr{B}^i(\mathscr{L}_2) \otimes \pi_1^* \mathscr{L}_1^{\otimes n} \otimes \pi_1^* \mathscr{L}_2^{\otimes n})$$

hence by the equivalence of categories of Lemma 2.1 for the line bundle $\pi_1^*(\mathscr{L}_1 \otimes \mathscr{L}_2)$, which is also very ample with respect to the projection to Y, we obtain that $\mathscr{B}^i(\mathscr{L}_1) = \mathscr{B}^i(\mathscr{L}_2)$.

In the case where $F \cong \Phi_E$, a Fourier-Mukai transform with kernel E, we have the following:

$$\begin{aligned}\mathscr{H}^i(\Phi_E(\mathscr{O}_X(n))) &= \mathscr{H}^i(R\pi_{2*}(E \overset{L}{\otimes} L\pi_1^* \mathcal{O}_X(n))) \cong \\ &\cong \mathscr{H}^i(\pi_{2*}(E \otimes \pi_1^* \mathcal{O}_X(n))) \cong \\ &\cong \pi_{2*}(\mathscr{H}^i(E \otimes \pi_1^* \mathcal{O}_X(n))) \cong \\ &\cong \pi_{2*}(\mathscr{H}^i(E) \otimes \pi_1^* \mathcal{O}_X(n))\end{aligned}$$

where for the second equality we used the fact that, $\pi_1^* \mathcal{O}_X(1)$ is very ample with respect to $X \times Y \to Y$. Therefore, by Lemma 2.1 the sheaf

$$\bigoplus_{n \geq n_0} \mathscr{H}^i(\Phi_E(\mathscr{O}_X(n))) \cong \bigoplus_{n \geq n_0} \pi_{2*}(\mathscr{H}^i(E) \otimes \pi_1^* \mathcal{O}_X(n))$$

corresponds by the equivalence of categories to the sheaf $\mathscr{H}^i(E)$ on $X \times Y$. □

While Theorem 1.1 gives a map $\mathscr{H}^i(F(\mathscr{E}(n))) \to \pi_{2*}(\mathscr{B}^i \otimes \pi_1^* \mathscr{E}(n))$ for all vector bundles $\mathscr{E}$ and all $n \in \mathbb{Z}$, in general it is only an isomorphism for n sufficiently large. In the case of the first M such that $\mathscr{H}^M(F(\mathscr{E}))$ is nonzero for some locally free sheaf of finite rank $\mathscr{E}$ we can actually say more:

Proposition 2.2. *In the situation of Theorem 1.1, assume $F(\mathscr{E}) \in D^{[M,N]}_{\mathrm{Coh}}(Y)$ for all locally free sheaves $\mathscr{E}$ of finite rank on X. Then the maps*

$$\mathscr{H}^M(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E})$$

are isomorphisms for all locally free sheaves $\mathscr{E}$ of finite rank.

As we mentioned in the proof of Theorem 1.1, the assumption that $F(\mathscr{E}) \in D^{[M,N]}_{\mathrm{Coh}}(Y)$ for all locally free sheaves $\mathscr{E}$ of finite rank on X isn't actually restrictive because of [Orl97, Lemma 2.4].

Proof. Assume we have an immersion $X \hookrightarrow \mathbb{P}^d_k$. Choose sections $s_1, \ldots, s_{d+1}$ of $\mathcal{O}_X(1)$ such that the corresponding hyperplanes have empty intersection. Then for any $m \in \mathbb{N}$ we have short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{(s_1^m, \ldots, s_{d+1}^m)} \mathcal{O}_X(m)^{d+1} \to K_m \to 0,$$

where K_m is a locally free sheaf on X.

Let $\mathscr{E}$ be any coherent locally free sheaf. Then by tensoring the above short exact sequence with $\mathscr{E}$ we get

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0$$

and so

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}^M(F(\mathscr{E})) \arrow[r] \arrow[d] & \mathscr{H}^M(F(\mathscr{E}(m)^{d+1})) \arrow[r] \arrow[d] & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \arrow[d] \\ 0 \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E}) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E}(m)^{d+1}) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})). \end{tikzcd}$$

Let m be high enough so that the center map is an isomorphism (this is possible by Theorem 1.1). Then the map on the left must be injective. Thus we showed: for every locally free sheaf $\mathscr{E}$ of finite rank, the map $\mathscr{H}^M(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E})$ is injective.

Now let us go back to the diagram above. By what we just showed, the map on the right $\mathscr{H}^M(F(K_m \otimes \mathscr{E})) \to \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E}))$ is injective. Hence we have

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{H}^M(F(\mathscr{E})) \arrow[r] \arrow[d, hook] & \mathscr{H}^M(F(\mathscr{E}(m)^{d+1})) \arrow[r] \arrow[d, "\cong"] & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \arrow[d, hook] \\ 0 \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E}) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E}(m)^{d+1}) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})) \end{tikzcd}$$

then by the 5 Lemma the left arrow is an isomorphism, i.e.

$$\mathscr{H}^M(F(\mathscr{E})) \xrightarrow{\cong} \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}).$$ □

This proposition in particular implies that if $F(\mathscr{E}) \in D^{[M,N]}_{\mathrm{Coh}}(Y)$ for all locally free sheaves $\mathscr{E}$ of finite rank on X, and there is at least one $\mathscr{E}$ such that $\mathscr{H}^M(F(\mathscr{E})) \neq 0$, then necessarily $\mathscr{B}^M \neq 0$.

Similarly to Proposition 2.2, we also have a stronger result than the one in Theorem 1.1 for the largest N' such that $\mathscr{B}^i \neq 0$. In this case, the map $\mathscr{H}^{N'}(F(\mathscr{E}(n)) \to \pi_{2*}(\mathscr{B}^{N'} \otimes \pi_1^*\mathscr{E}(n))$ can be constructed for all coherent sheaves on X instead of just the locally free ones:

Proposition 2.3. *In the situation of Theorem 1.1, let N' be the largest i such that $\mathscr{B}^i \neq 0$. Then for all $n \in \mathbb{Z}$, for any coherent sheaf $\mathscr{F}$ we have a map*

$$\mathscr{H}^{N'}(F(\mathscr{F}(n))) \to \pi_{2*}(\mathscr{B}^{N'} \otimes \pi_1^*\mathscr{F}(n))$$

which is an isomorphism for n sufficiently high.

Proof. First of all, notice that for any coherent sheaf $\mathscr{F}$ on X we have $\mathscr{H}^{N'+1}(F(\mathscr{F}(n))) = 0$ for $n \gg 0$. In fact, let N'' be maximal such that there exists a coherent sheaf $\mathscr{F}$ and a sequence $n_i \to \infty$ with $\mathscr{H}^{N''}(F(\mathscr{F}(n_i))) \neq 0$ for all i (N'' exists because F is bounded). Then consider a short exact sequence $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$ with $\mathscr{E}$ locally free. If $N'' > N'$, for n_i high enough by Theorem 1.1 we will have $\mathscr{H}^{N''}(F(\mathscr{E}(n_i))) = \pi_{2*}(\mathscr{B}^{N''} \otimes \pi_1^*(\mathscr{E}(n_i))) = 0$ since $\mathscr{B}^{N''} = 0$, and similarly $\mathscr{H}^{N''+1}(F(\mathscr{E}(n_i))) = 0$; hence $\mathscr{H}^{N''+1}(F(\mathscr{G}(n_i))) = \mathscr{H}^{N''}(F(\mathscr{F}(n_i))) \neq 0$, obtaining a contradiction.

Now we will repeat the argument of Theorem 1.1 just for N': in this case the argument can be applied to all coherent sheaves instead of just the locally free ones. First of all, take an integer s such that there exists a surjection $\mathcal{O}_X^{\oplus s} \to \mathcal{O}_X(1)$. By tensoring with $\mathscr{F}(n)$ we get a short exact sequence of coherent sheaves $0 \to K(n) \to \mathscr{F}^{\oplus s}(n) \to \mathscr{F}(n+1) \to 0$ and hence for $n \gg 0$ we get

$$\mathscr{H}^{N'}(F(\mathscr{F}(n)))^{\oplus s} \to \mathscr{H}^{N'}(F(\mathscr{F}(n+1))) \to 0$$

since $\mathscr{H}^{N'+1}(F(K(n))) = 0$. As in (2) this gives us a $\mathscr{B}^{N'}_{\mathscr{F}}$ and a map

$$\mathscr{H}^{N'}(F(\mathscr{F}(n))) \to \pi_{2*}(\mathscr{B}^{N'}_{\mathscr{F}} \otimes \pi_1^*(\mathscr{O}_X(n)))$$

which is an isomorphism for n big.

Now let us consider an exact sequence $\mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{F} \to 0$ with $\mathscr{E}_i$ locally free of finite rank. Since $\mathscr{E}_1$ is locally free, $\mathscr{H}^{N'+1}(F(\mathscr{E}_1(n))) = 0$ for $n \gg 0$ and the exact sequence

$$\mathscr{H}^{N'}(F(\mathscr{E}_1(n))) \to \mathscr{H}^{N'}(F(\mathscr{E}_2(n))) \to \mathscr{H}^{N'}(F(\mathscr{F}(n))) \to 0$$

gives us a an exact sequence as in (3)

$$\mathscr{B}^{N'}_{\mathscr{E}_1} \to \mathscr{B}^{N'}_{\mathscr{E}_2} \to \mathscr{B}^{N'}_{\mathscr{F}} \to 0,$$

therefore we can conclude that $\mathscr{B}^{N'}_{\mathscr{F}} = \mathscr{B}^{N'} \otimes \pi_1^*(\mathscr{F})$ for any coherent sheaf $\mathscr{F}$. $\square$

3. A special case

In this section we will give an example of a class of exact functors $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ for which we can always find an object $E \in D^b_{\mathrm{Coh}}(X \times Y)$ and an equivalence $F \cong \Phi_E$. The sheaves $\mathscr{B}^i$ will be the ones defined as in Theorem 1.1.

In what follows we will take $\dim(X)$ to be equal to one, and all but one of the $\mathscr{B}^i$'s to be equal to zero. Note that under these hypotheses we obtain an isomorphism $F(\mathscr{F}) \cong \Phi_E(\mathscr{F})$ for all objects $\mathscr{F} \in D^b_{\mathrm{Coh}}(X)$: this follows from Theorem 1.1 and the fact that locally free coherent sheaves supported in degree zero are a one-step generator for the whole derived category $D^b_{\mathrm{coh}}(X)$, i.e. the smallest full subcategory containing locally free coherent sheaves supported in degree zero and closed under finite direct sums, direct summands and shifts, and the operation of taking at most one cone is precisely $D^b_{\mathrm{coh}}(X)$. However, there is no guarantee that the isomorphism $F(\mathscr{F}) \cong \Phi_E(\mathscr{F})$ will be functorial.

Before we proceed to find an isomorphism of functors, we need to show that we can obtain an isomorphism on the cohomology sheaves. The following proposition gives a description of the cohomology sheaves of $F(\mathscr{F})$ for any $\mathscr{F}$ coherent, without needing to twist by some high n as we did in Theorem 1.1.

Proposition 3.1. *Let X, Y be smooth projective varieties over an algebraically closed field, with X of dimension one, let $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ an exact functor, and assume that the sheaves $\mathscr{B}^i$ defined as in Theorem 1.1 are zero for $i \neq M$. Assume also that $\mathscr{B}^M$ is a coherent sheaf supported at finitely many points of $X \times Y$.*

Then for any coherent sheaf $\mathscr{F}$ on X we have $\mathscr{H}^i(F(\mathscr{F})) = 0$ for $i \neq M, M-1$ and for any locally free sheaf $\mathscr{E}$ of finite rank we have $\mathscr{H}^i(F(\mathscr{E})) = 0$ for $i \neq M$.

Moreover, for each coherent sheaf $\mathscr{F}$ on X there is a functorial isomorphism

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F}).$$

Proof. Consider any torsion sheaf Q. Then we have a short exact sequence of coherent sheaves $0 \to \mathscr{E}' \to \mathscr{E} \to Q \to 0$ with $\mathscr{E}, \mathscr{E}'$ locally free. Twist $\mathscr{E}$ and $\mathscr{E}'$ by $n \gg 0$ so that $\mathscr{H}^i(F(\mathscr{E}'(n))) = \mathscr{H}^i(F(\mathscr{E}(n))) = 0$ for $i \neq M$. Since $0 \to \mathscr{E}'(n) \to \mathscr{E}(n) \to Q \to 0$ is still an exact sequence, from the long exact sequence on cohomology we can conclude that $\mathscr{H}^i(F(Q)) = 0$ for all $i \neq M, M-1$.

Now consider a locally free sheaf $\mathscr{E}$ of finite rank on X. Let $\bar{n}$ be large enough so that we know $\mathscr{H}^i(F(\mathscr{E}(\bar{n}))) = 0$ for all $i \neq M$. Then we have a short exact sequence $0 \to \mathscr{E}(\bar{n}-1) \to \mathscr{E}(\bar{n}) \to T \to 0$ where T is a torsion sheaf. A portion of the long exact sequence in cohomology gives

$$\mathscr{H}^{i-1}(F(T)) \to \mathscr{H}^i(F(\mathscr{E}(\bar{n}-1))) \to \mathscr{H}^i(F(\mathscr{E}(\bar{n})))$$

and $\mathscr{H}^{i-1}(F(T)) = \mathscr{H}^i(F(\mathscr{E}(\bar{n}))) = 0$ for $i \neq M, M+1$ hence $\mathscr{H}^i(F(\mathscr{E}(\bar{n}-1))) = 0$ for $i \neq M, M+1$. By descending induction on n we then obtain that $\mathscr{H}^i(F(\mathscr{E}(n))) = 0$ for all n and $i \neq M, M+1$. We will show at the end of the proof that $\mathscr{H}^{M+1}(F(\mathscr{E})) = 0$.

By Proposition 2.3 we know that for any coherent sheaf $\mathscr{F}$ on X we have a functorial map

$$\mathscr{H}^M(F(\mathscr{F})) \to \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F})$$

which is an isomorphism by Proposition 2.2 if $\mathscr{F}$ is locally free (notice that the hypotheses of 2.2 are satisfied by the first part of this Proposition). Moreover we also know, again by Proposition 2.3, that for any coherent sheaf $\mathscr{F}$ the map

$$\mathscr{H}^M(F(\mathscr{F}(n))) \xrightarrow{\cong} \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F}(n))$$

is an isomorphism for n sufficiently high. But if $\mathscr{F}$ is a torsion sheaf $\mathscr{F}(n) \cong \mathscr{F}$, so we get that

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F})$$

is also an isomorphism for torsion sheaves, and hence it is always an isomorphism since any coherent sheaf on X is the direct sum of a locally free part and a torsion part.

Now let us show that $\mathscr{H}^{M+1}(F(\mathscr{E})) = 0$ for a locally free sheaf $\mathscr{E}$: using again the short exact sequence $0 \to \mathscr{E}(\bar{n}-1) \to \mathscr{E}(\bar{n}) \to T \to 0$, we obtain a diagram

$$\begin{tikzcd}
\mathscr{H}^M(F(\mathscr{E}(\bar{n}))) \arrow[r] \arrow[d, "\cong"] & \mathscr{H}^M(F(T)) \arrow[r] \arrow[d, "\cong"] & \mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}-1))) \arrow[r] \arrow[d] & \mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}))) = 0 \\
\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}(\bar{n})) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*T) \arrow[r] & 0 &
\end{tikzcd}$$

where the bottom sequence is right exact because π_{2*} is exact when applied to a sequence of flasque sheaves, which is the case here by the way we chose our $\mathscr{B}^M$. By the five lemma it follows that $\mathscr{H}^{M+1}(F(\mathscr{E}(\bar{n}-1))) = 0$. So we can again proceed by descending induction on n. □

Thanks to Proposition 3.1 we can now get an isomorphism of the δ-functors obtained by first applying the two functors F and $\Phi_{\mathscr{B}^M[-M]}$ and then taking their cohomology:

Proposition 3.2. *In the setting of Proposition 3.1, let Φ be the Fourier-Mukai transform associated to the complex given by the sheaf $\mathscr{B}^M$ placed in degree M, $\Phi = \Phi_{\mathscr{B}_M[-M]}$.*

Then there is an isomorphism of δ-functors

$$\mathscr{H}^i(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^i(\Phi(\cdot))$$

on the category of coherent sheaves on X, which gives an isomorphism of functors $F \to \Phi$ for the full subcategory of $D^b_{\mathrm{Coh}}(X)$ consisting of locally free sheaves placed in degree zero.

Proof. The fact that there is a functorial isomorphism

$$\mathscr{H}^M(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^M(\Phi(\cdot))$$

on the category of coherent sheaves on X follows immediately from Proposition 3.1 given that $\mathscr{H}^M(\Phi(\mathscr{F})) = \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F})$ since pushforward is exact for flasque sheaves.

Moreover, for any locally free sheaf of finite rank $\mathscr{E}$, since the only nonzero cohomology sheaf of $F(\mathscr{E})$ is in degree M,

$$\begin{aligned} F(\mathscr{E}) = \mathscr{H}^M(F(\mathscr{E}))[-M] &\xrightarrow{\cong} \mathscr{H}^M(\Phi(\mathscr{E}))[-M] = \\ &= \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E})[-M] = R\pi_{2*}(\mathscr{B}^M[-M] \overset{L}{\otimes} L\pi_1^*\mathscr{E}) = \Phi(\mathscr{E}), \end{aligned}$$

where the third equality follows again by Proposition 3.1. This gives the isomorphism of functors on the full subcategory of $D^b_{\mathrm{Coh}}(X)$ of locally free sheaves placed in degree zero.

Let us now construct the isomorphism

$$\mathscr{H}^{M-1}(F(\cdot)) \xrightarrow{\cong} \mathscr{H}^{M-1}(\Phi(\cdot)).$$

Consider a coherent sheaf Q on X which is not locally free. Then there is a short exact sequence of coherent sheaves $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to Q \to 0$ where $\mathscr{E}_1$ and $\mathscr{E}_2$ are locally free. We get a long exact sequence on cohomology

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{H}^{M-1}(F(Q)) \arrow[r] & \mathscr{H}^{M}(F(\mathscr{E}_1)) \arrow[r] \arrow[d] & \mathscr{H}^{M}(F(\mathscr{E}_2)) \arrow[d] \\
0 \arrow[r] & \mathscr{H}^{M-1}(\Phi(Q)) \arrow[r] & \mathscr{H}^{M}(\Phi(\mathscr{E}_1)) \arrow[r] & \mathscr{H}^{M}(\Phi(\mathscr{E}_2))
\end{tikzcd}$$

so we get an isomorphism $\mathscr{H}^{M-1}(F(Q)) \to \mathscr{H}^{M-1}(\Phi(Q))$. We still need to show that this map is functorial and that it does not depend on the choice of a short exact sequence. Consider a map $Q \to T$ of coherent sheaves. Then we can construct two short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_2 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{H}_1 \arrow[r] & \mathscr{H}_2 \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

with $\mathscr{E}_2, \mathscr{E}_1, \mathscr{H}_2$ and $\mathscr{H}_1$ locally free of finite rank. Then we get the following diagram on cohomology:

$$\begin{tikzcd}
& 0 \arrow[rr] & & \mathscr{H}^{M-1}(\Phi(Q)) \arrow[rr] \arrow[dd] & & \mathscr{H}^{M}(\Phi(\mathscr{E}_1)) \arrow[rr] \arrow[dd] & & \mathscr{H}^{M}(\Phi(\mathscr{E}_2)) \arrow[dd] \\
0 \arrow[r] & & \mathscr{H}^{M-1}(F(Q)) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathscr{H}^{M}(F(\mathscr{E}_1)) \arrow[rr] \arrow[ur] \arrow[dd] & & \mathscr{H}^{M}(F(\mathscr{E}_2)) \arrow[ur] \arrow[dd] & \\
& 0 \arrow[rr] & & \mathscr{H}^{M-1}(\Phi(T)) \arrow[rr] & & \mathscr{H}^{M}(\Phi(\mathscr{H}_1)) \arrow[rr] & & \mathscr{H}^{M}(\Phi(\mathscr{H}_2)) \\
0 \arrow[r] & & \mathscr{H}^{M-1}(F(T)) \arrow[rr] \arrow[ur] & & \mathscr{H}^{M}(F(\mathscr{H}_1)) \arrow[rr] \arrow[ur] & & \mathscr{H}^{M}(F(\mathscr{H}_2)) \arrow[ur] &
\end{tikzcd} \tag{6}$$

where all but possibly the leftmost shaded squares commute, hence the leftmost shaded square will also commute. This shows functoriality.

To show that the maps we chose do not depend on the choice of a short exact sequence, notice that given two short exact sequences $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to Q \to 0$ and $0 \to \mathscr{H}_1 \to \mathscr{H}_2 \to Q \to 0$ there is a short exact sequence $0 \to C \to \mathscr{E}_2 \oplus \mathscr{H}_2 \to Q \to 0$ mapping to both of them. So we just need to prove this statement for two short exact sequences with maps between them. But then we are again in the situation of diagram (6), where $T = Q$ and the two rightmost maps in the diagram are the identity. So this follows again from the commutativity of the leftmost shaded square.

Finally, since by Proposition 3.1 we have $\mathscr{H}^i(F(\mathscr{F})) = 0$ for $i \neq M-1, M$, all that is left to show is that for every short exact sequence $0 \to \mathscr{H}_1 \to \mathscr{H}_2 \to Q \to 0$ the diagram

$$\begin{tikzcd}
\mathscr{H}^{M-1}(F(Q)) \arrow[r] \arrow[d] & \mathscr{H}^{M}(F(\mathscr{H}_1)) \arrow[d] \\
\mathscr{H}^{M-1}(\Phi(Q)) \arrow[r] & \mathscr{H}^{M}(\Phi(\mathscr{H}_1))
\end{tikzcd}$$

is commutative. This follows immediately by the construction when $\mathscr{H}_1$ and $\mathscr{H}_2$ are locally free. Otherwise, construct a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_2 \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{H}_1 \arrow[r] & \mathscr{H}_2 \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

with $\mathscr{E}_2, \mathscr{E}_1$ locally free of finite rank. Then we get a diagram as in (6) with $T = Q$ and where everything commutes except possibly for the bottom leftmost square, but that follows immediately since the leftmost vertical arrow is the identity. $\square$

We are now ready to tackle the task of getting an isomorphism between the two functors in our special case. We will now assume that $\mathscr{H}^M$ is a direct sum of skyscraper sheaves. We will proceed as follows: we will first find an isomorphism for the subcategory of $D^b_{\mathrm{Coh}}(X)$ given by sheaves placed in degree zero. The isomorphism on the whole derived category will then follow by the technical Lemma 3.7.

Theorem 3.3. *Let X and Y be smooth projective varieties over an algebraically closed field, with X of dimension one, $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ an exact functor. Assume that the corresponding $\mathscr{B}^i$ defined in Theorem 1.1 are zero for $i \neq M$, and that $\mathscr{B}^M$ is a skyscraper sheaf supported at a finite number of points, $\mathscr{B}^M = \bigoplus_{j=1}^t k(p_i, q_i)$. Let Φ be the Fourier-Mukai transform associated to the sheaf $\mathscr{B}^M$ placed in degree M. Let τ be the full subcategory of coherent sheaves placed in degree 0. Then there exists an isomorphism of exact functors $t : F(\cdot)|_\tau \to \Phi(\cdot)|_\tau$.*

Before we prove the theorem, let us prove two technical lemmas that we will use in the proof.

Lemma 3.4. *Let X be a projective variety, $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Consider a surjective map $\alpha : \oplus_n \mathcal{O}_X \to Q$ where Q is a coherent sheaf on X. Then there exists an integer $h(\alpha)$ such that for all $i \leq -h(\alpha)$ and for any map $\beta : \mathcal{O}_X(i) \to Q$ there exists a map $\gamma : \mathcal{O}_X(i) \to \oplus_n \mathcal{O}_X$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{O}_X(i) \arrow[rr, dashed, "\gamma"] \arrow[dr, "\beta"'] & & \bigoplus_n \mathcal{O}_X \arrow[dl, "\alpha"] \\ & Q & \end{tikzcd}$$

Proof. We have a short exact sequence

$$0 \to \mathrm{Ker}(\alpha) \to \oplus_n \mathcal{O}_X \to Q \to 0.$$

Twist by $\mathcal{O}_X(-i)$ to get

$$0 \to \mathrm{Ker}(\alpha)(-i) \to \oplus_n \mathcal{O}_X(-i) \to Q(-i) \to 0.$$

A map $\beta : \mathcal{O}_X(i) \to Q$ is the same thing as a map $\mathcal{O}_X \to Q(-i)$, hence as an element $\beta(-i) \in H^0(X, Q(-i))$. By Serre vanishing, there exists an $h(\alpha) \geq 0$ such that $H^1(X, \mathrm{Ker}(\alpha)(-i)) = 0$ for all $i \leq -h(\alpha)$. Hence $\beta(-i)$ lifts to a section $\gamma(-i)$ of $H^0(X, \oplus_n \mathcal{O}_X(-i))$. Twist down by i to get the desired map $\gamma : \mathcal{O}_X(i) \to \oplus_n \mathcal{O}_X$. $\square$

Lemma 3.5. *Let X be a smooth projective variety over an algebraically closed field, let $p_1, \ldots, p_t \in X$ and let $\mathscr{E}$ be a locally free sheaf of rank r generated by global sections. Then there exist an open set $\mathscr{U}$ containing $p_1, \ldots, p_t$ and global sections $s_1, \ldots, s_r$ of $\mathscr{E}$ that generate the stalk $\mathscr{E}_p$ at each point $p \in \mathscr{U}$.*

Proof. Assume we found $s_1, \ldots, s_n \in \Gamma(X, \mathscr{E})$ that are linearly independent at each stalk at $p_1, \ldots, p_t$ so that we have

$$\mathcal{O}_X^{\oplus n} \to \mathscr{E} \xrightarrow{f} Q \to 0$$
$$e_j \mapsto s_j.$$

Let us find a global section of $\mathscr{E}$ such that its image in Q doesn't vanish at $p_1, \ldots, p_t$. Let $u_i \in \Gamma(X, \mathscr{E})$ such that $f(u_i)$ doesn't vanish at p_i (we can do this because f is surjective on stalks and $\mathscr{E}$ is generated by global sections). Then $u_1, \ldots, u_t$ form a sub-vector space V of $\Gamma(X, \mathscr{E})$ of dimension l for some l and, for each i, $\dim(\{u \in V : f(u)(p_i) = 0\}) \leq l - 1$. Hence

$$\{u \in V : f(u)(p_i) = 0 \text{ for some i}\} = \bigcup_i \{u \in V : f(u)(p_i) = 0\}$$

is a union of subsets of dimension less or equal to $l - 1$ and hence it is strictly contained in V since our field of definition is infinite (because it is algebraically closed). So we can find a section s_{n+1} in V such that $f(s_{n+1})$ doesn't vanish at any of the p_j. Then $s_1, \ldots, s_{n+1}$ are linearly independent at each p_j as sections of $\mathscr{E}$. We can keep doing this as long as $\operatorname{rk} Q > 0$. Then the sections $s_1, \ldots, s_r$ will generate the stalk $\mathscr{E}_p$ at each point p in an open set $\mathscr{U}$ containing $p_1, \ldots, p_t$. □

Proof of Theorem 3.3. It is actually more natural to construct the inverse isomorphism $s : F(\cdot)|_\tau \to \Phi(\cdot)|_\tau$.

We will first construct the isomorphism on objects, starting with the subcategories of coherent sheaves on X given by locally free sheaves and torsion sheaves. This will a priori involve making non-canonical choices, but as it later turns out, the choices we are making are actually unique. Then we will prove that the isomorphisms are compatible with morphisms and this will allow us to define said isomorphism on a general coherent sheaf. Lastly, we will show that the given isomorphisms induce maps of triangles when applied to a short exact sequence of sheaves.

I. On the subcategory of locally free sheaves: Let $\mathscr{E}$ be a locally free sheaf of finite rank on X. Then by Proposition 3.2 there is a functorial isomorphism $s(\mathscr{E}) : \Phi(\mathscr{E}) \to F(\mathscr{E})$.

II. On torsion sheaves: Consider a torsion sheaf Q on X. Pick a short exact sequence $0 \to K \to \mathcal{O}_X^{\oplus n} \xrightarrow{\alpha} Q \to 0$, with K a locally free sheaf. Then we have a diagram

$$\begin{tikzcd} \Phi(K) \arrow[r] \arrow[d, "qis"', "s(K)"] & \Phi(\mathcal{O}_X^{\oplus n}) \arrow[r] \arrow[d, "qis"', "s(\mathcal{O}_X^{\oplus n})"] & \Phi(Q) \arrow[d, dashed] \\ F(K) \arrow[r] & F(\mathcal{O}_X^{\oplus n}) \arrow[r] & F(Q) \end{tikzcd}$$

hence there exists a dotted arrow $\Phi(Q) \to F(Q)$ which is a quasi-isomorphism (this dotted arrow is not necessarily unique). Choose one such arrow and call it $s(Q)$. Notice that $s(Q)$ will induce on cohomology the maps that we found in Proposition 3.2: for the M^{th} cohomohogy this follows because $\mathscr{H}^M(\mathcal{O}_X^{\oplus n}) \to \mathscr{H}^M(Q)$ is surjective and the functoriality for locally free sheaves, and the maps $\mathscr{H}^{M-1}(\Phi(Q)) \to \mathscr{H}^M(\Phi(K))$ and $\mathscr{H}^{M-1}(F(Q)) \to \mathscr{H}^M(F(K))$ are injective.

III. $s(-)$ is compatible with maps $\beta : \mathscr{E} \to Q$, $\mathscr{E}$ locally free, Q torsion: First of all we will prove the following: for any map $\beta : \mathcal{O}_X(i) \to Q$, the diagram

$$\begin{tikzcd} \Phi(\mathcal{O}_X(i)) \arrow[r, "\Phi(\beta)"] \arrow[d, "s(\mathcal{O}_X(i))"'] & \Phi(Q) \arrow[d, "s(Q)"] \\ F(\mathcal{O}_X(i)) \arrow[r, "F(\beta)"] & F(Q) \end{tikzcd}$$

commutes. Consider first the case where $i \leq -h(\alpha)$ where $h(\alpha)$ is defined as in Lemma 3.4, and α is as in **II**. By Lemma 3.4, for every map $\beta : \mathcal{O}_X(i) \to Q$ with $i \leq -h(\alpha)$ we have a diagram

$$\begin{tikzcd} \mathcal{O}_X(i) \arrow[rr, dashed] \arrow[dr, "\beta"'] & & \bigoplus_n \mathcal{O}_X \arrow[dl, "\alpha"] \\ & Q & \end{tikzcd}$$

By applying the functors F and Φ we obtain the following diagram:

$$\begin{tikzcd}
& & \Phi(\mathcal{O}_X(i)) \arrow[r, "\Phi(\beta)"] \arrow[dll] \arrow[dd] & \Phi(Q) \arrow[dll, "id"] \arrow[dd, "s(Q)"] \\
\Phi(\mathcal{O}_X^{\oplus n}) \arrow[r, "\Phi(\alpha)"'] \arrow[dd] & \Phi(Q) \arrow[dd, "s(Q)"'] & & \\
& & F(\mathcal{O}_X(i)) \arrow[r, "F(\beta)"] \arrow[dll] & F(Q) \arrow[dll, "id"] \\
F(\mathcal{O}_X^{\oplus n}) \arrow[r, "F(\delta)"'] & F(Q) & &
\end{tikzcd}$$

and the front square commutes because of how we chose $s(Q)$ in **II**, hence the back square will also commute.

Now let $i > -h(\alpha)$ and consider $\beta : \mathcal{O}_X(i) \to Q$. Then pick any map $\gamma : \mathcal{O}_X(-h(\alpha)) \to \mathcal{O}_X(i)$ such that γ is an isomorphism on an open set containing $p_1, \dots, p_t$, where $\mathscr{B}^M = \bigoplus_{j=1}^t k(p_i, q_i)$ (one such map is multiplication by a polynomial that does not vanish at the p_i's). Then the map $\Phi(\gamma)$: $\Phi(\mathcal{O}_X(-h(\alpha))) \to \Phi(\mathcal{O}_X(i))$ is an isomorphism: in fact the map $\pi_1^*(\gamma) : \pi_1^*(\mathcal{O}_X(-h(\alpha))) \to \pi_1^*(\mathcal{O}_X(i))$ is an isomorphism on an open set containing $(p_1, q_1), \dots, (p_t, q_t)$ and hence we will get an isomorphism when tensoring with a sheaf supported at $(p_1, q_1), \dots, (p_t, q_t)$. By letting $\delta = \beta \circ \gamma$ once again we get a diagram

$$\begin{tikzcd}
& & \Phi(\mathcal{O}_X(i)) \arrow[r, "\Phi(\beta)"] \arrow[dd] & \Phi(Q) \arrow[dd] \\
\Phi(\mathcal{O}_X(-h(\alpha))) \arrow[urr, "\Phi(\gamma)"] \arrow[r, "\Phi(\delta)"'] \arrow[dd] & \Phi(Q) \arrow[urr, "id"] \arrow[dd] & & \\
& & F(\mathcal{O}_X(i)) \arrow[r, "F(\beta)"] & F(Q) \\
F(\mathcal{O}_X(-h(\alpha))) \arrow[urr] \arrow[r, "F(\delta)"'] & F(Q) \arrow[urr, "id"'] & &
\end{tikzcd}$$

since $\Phi(\gamma)$ is a quasi-isomorphism and the front square is commutative, the back square will also commute.

Now consider any map $\beta : \mathscr{E} \to Q$ with $\mathscr{E}$ locally free and Q torsion. Let m be such that $\mathscr{E}(m)$ is generated by global sections, and let $r = \mathrm{rk}\mathscr{E}$. By Lemma 3.5 we can find $s_1, \dots, s_r$ global sections of $\mathscr{E}(m)$ that are linearly independent at each stalk of an open set $\mathscr{U}$ containing $p_1, \dots, p_t$. Then the corresponding map $\bigoplus_r \mathcal{O}_X \to \mathscr{E}(m)$ is injective and it is an isomorphism on $\mathscr{U}$. Twisting down by m we get a map $\gamma : \bigoplus_r \mathcal{O}_X(-m) \to \mathscr{E}$ which is an isomorphism on $\mathscr{U}$. By letting $\delta = \beta \circ \gamma$ we get again a diagram like the above one,

$$\begin{tikzcd}
 & & \Phi(\mathscr{E}) \arrow[r, "\Phi(\beta)"] \arrow[dd] & \Phi(Q) \arrow[dd, "s(Q)"] \\
\Phi(\bigoplus_r \mathcal{O}_X(-m)) \cong \bigoplus_r \Phi(\mathcal{O}_X(-m)) \arrow[r, "\Phi(\delta)"'] \arrow[urr, "\Phi(\gamma)"] \arrow[dd] & \Phi(Q) \arrow[urr, "id"] \arrow[dd, "s(Q)"'] & & \\
 & & F(\mathscr{E}) \arrow[r, "F(\beta)"] & F(Q) \\
F(\bigoplus_r \mathcal{O}_X(-m)) \cong \bigoplus_r F(\mathcal{O}_X(-m)) \arrow[r, "F(\delta)"'] \arrow[urr] & F(Q) \arrow[urr, "id"] & &
\end{tikzcd}$$

and since $\Phi(\gamma)$ is a quasi-isomorphism and the front square commutes, the back square will also commute.

IV. $s(-)$ is compatible with maps $\eta : Q \to T$, Q and T torsion: We need to show that for any map between torsion sheaves $Q \to T$, the corresponding diagram

$$\begin{tikzcd}
\Phi(Q) \arrow[r, "\Phi(\eta)"] \arrow[d, "s(Q)"'] & \Phi(T) \arrow[d, "s(T)"] \\
F(Q) \arrow[r, "F(\eta)"] & F(T)
\end{tikzcd}$$

is commutative. This will also prove that our choice of $s(Q)$ is canonical. To do this, consider a locally free sheaf $\mathscr{E} = \bigoplus_r \mathcal{O}_X$ with a surjection $f : \mathscr{E} \to Q$. For consistency we will represent this situation with a square diagram as before

$$\begin{tikzcd}
\mathscr{E} \arrow[r, "\eta\circ f"] \arrow[d, "f"'] & T \arrow[d, "id"] \\
Q \arrow[r, "\eta"] & T
\end{tikzcd}$$

Then we get the following diagram:

$$\begin{tikzcd}
 & \Phi(Q) \arrow[r, "\Phi(\eta)"] \arrow[dd] & \Phi(T) \arrow[dd] \\
\Phi(\mathscr{E}) \arrow[r, "\Phi(\eta\circ f)"'] \arrow[ur, "\Phi(f)"] \arrow[dd] & \Phi(T) \arrow[ur, "id"] \arrow[dd] & \\
 & F(Q) \arrow[r, "F(\eta)"] & F(T) \\
F(\mathscr{E}) \arrow[r, "F(\eta\circ f)"'] \arrow[ur] & F(T) \arrow[ur, "id"] &
\end{tikzcd}$$

where the front square commutes by III. Hence the back square will also commute after pre-composing with the map $\Phi(\mathscr{E}) \to \Phi(Q)$. But then we can conclude that the back square also commutes: in fact it commutes on cohomology because of Proposition 3.2, so we can apply Lemma 3.6 below.

V. On a general coherent sheaf on X: Let $\mathscr{F}$ be any coherent sheaf on X. Then we have a decomposition $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}_F$ where $\mathscr{F}_T$ is the canonical summand consisting of the torsion part of

$\mathscr{F}$ and $\mathscr{F}_F$ corresponds to the torsion-free part (this summand is not canonical). Then define $s(\mathscr{F}) = s(\mathscr{F}_T) \oplus s(\mathscr{F}_F)$. We need to show that this map doesn't depend on the choice of the decomposition. So consider two such decompositions $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}_F$ and $\mathscr{F} \cong \mathscr{F}_T \oplus \mathscr{F}'_F$ and call $s(\mathscr{F})$ and $s'(\mathscr{F})$ respectively the two induced maps on $\Phi(\mathscr{F})$. Then the identity $\mathscr{F} \to \mathscr{F}$ induces a map $\alpha : \mathscr{F}_F \to \mathscr{F}'_F \oplus \mathscr{F}_T$, and by I. and III. the following diagram is commutative:

$$\begin{tikzcd}
\Phi(\mathscr{F}_F) \arrow[r] \arrow[d] & \Phi(\mathscr{F}'_F) \arrow[d, "s(\mathscr{F}'_F)"'] \oplus \Phi(\mathscr{F}_T) \arrow[d, "s(\mathscr{F}_T)"] \\
F(\mathscr{F}_F) \arrow[r] & F(\mathscr{F}'_F) \oplus F(\mathscr{F}_T)
\end{tikzcd}$$

whereas the diagram for the torsion part is clearly commutative because the induced maps are just the identity. Hence every square in the following diagram is commutative:

$$\begin{tikzcd}
\Phi(\mathscr{F}) \arrow[r, "\cong"] \arrow[d, "s(\mathscr{F})"'] \arrow[rrr, bend left, "id"] & \Phi(\mathscr{F}_T) \oplus \Phi(\mathscr{F}_F) \arrow[r, "id \oplus \Phi(\alpha)"] \arrow[d, "s(\mathscr{F}_T)"', "s(\mathscr{F}_F)"] & \Phi(\mathscr{F}_T) \oplus \Phi(\mathscr{F}'_F) \arrow[r, "\cong"] \arrow[d, "s(\mathscr{F}_T)"', "s(\mathscr{F}'_F)"] & \Phi(\mathscr{F}) \arrow[d, "s'(\mathscr{F})"] \\
F(\mathscr{F}) \arrow[r, "\cong"] \arrow[rrr, bend right, "id"'] & F(\mathscr{F}_T) \oplus F(\mathscr{F}_F) \arrow[r, "id \oplus F(\alpha)"] & F(\mathscr{F}_T) \oplus F(\mathscr{F}'_F) \arrow[r, "\cong"] & F(\mathscr{F}).
\end{tikzcd}$$

It follows that the external rectangle commutes, which proves precisely that $s(\mathscr{F}) = s'(\mathscr{F})$.

VI. $s(-)$ is compatible with any maps $\mathscr{F} \to \mathscr{G}$, for $\mathscr{F}$ and $\mathscr{G}$ coherent sheaves: Given a map $f : \mathscr{F} \to \mathscr{G}$, write $\mathscr{F} = \mathscr{F}_F \oplus \mathscr{F}_T$ and $\mathscr{G} = \mathscr{G}_F \oplus \mathscr{G}_T$. Then s will be compatible with $\Phi(f)$ and $F(f)$ because it is compatible with the maps $\mathscr{F}_F \to \mathscr{G}_F$, $\mathscr{F}_F \to \mathscr{G}_T$, and $\mathscr{F}_T \to \mathscr{G}_T$.

VII. $s(-)$ is compatible with triangles of the type $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{F} \to 0$ for $\mathscr{E}_1$ and $\mathscr{E}_2$ locally free: We have to show that given a short exact sequence of coherent sheaves on X, $0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \mathscr{F} \to 0$, the maps $s(\mathscr{E}_1)$, $s(\mathscr{E}_2)$ and $s(\mathscr{F})$ give a morphism of triangles

$$\begin{tikzcd}
\Phi(\mathscr{E}_1) \arrow[r] \arrow[d, "s(\mathscr{E}_1)"] & \Phi(\mathscr{E}_2) \arrow[r] \arrow[d, "s(\mathscr{E}_2)"] & \Phi(\mathscr{F}) \arrow[r] \arrow[d, "s(\mathscr{F})"] & \Phi(\mathscr{E}_1)[1] \arrow[d, "s(\mathscr{E}_1)[1]"] \\
F(\mathscr{E}_1) \arrow[r] & F(\mathscr{E}_2) \arrow[r] & F(\mathscr{F}) \arrow[r] & F(\mathscr{E}_1)[1],
\end{tikzcd} \tag{7}$$

i.e. we need to prove that the rightmost square is commutative. First of all we will analyze the map $\Phi(\mathscr{E}_2) \to \Phi(\mathscr{F})$. We know that $\Phi(\mathscr{E}_2)$ is supported in degree M, whereas $\Phi(\mathscr{F})$ is supported in degrees M and $M-1$: hence, by [Dol60, Satz 4.7], as a complex we have $\Phi(\mathscr{F}) \cong \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \oplus \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1]$ (in a non-canonical way). The situation looks as follows:

$$\begin{tikzcd}
\Phi(\mathscr{E}_2) \arrow[r] \arrow[dr] & \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \arrow[dr] & \\
 & \oplus & \\
 & \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1] \arrow[r] & \Phi(\mathscr{E}_1)[1].
\end{tikzcd}$$

We will now show that the induced maps $\Phi(\mathscr{E}_2) \to \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1]$, as well as $\mathscr{H}^M(\Phi(\mathscr{F}))[-M] \to \Phi(\mathscr{E}_1)[1]$, are zero in $D^b_{\mathrm{Coh}}(Y)$ for some choice of a decomposition $\Phi(\mathscr{F}) \cong \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1] \oplus \mathscr{H}^M(\Phi(\mathscr{F}))[-M]$. In fact, consider a locally free resolution of $\pi_1^*\mathscr{F}$, $\bar{\mathscr{F}}_{-1} \to \bar{\mathscr{F}}_0$. Then the map $\mathscr{E}_2 \to \mathscr{F}$ induces an actual map of complexes $\pi_1^* C \to [\bar{\mathscr{F}}_{-1} \to \bar{\mathscr{F}}_0]$ since $p_1^*\mathscr{B}$ is locally free, hence an actual map of complexes

$$\begin{tikzcd} (\pi_1^*\mathscr{E}_2 \otimes \mathscr{B}^M)[-M] \arrow[r] & (\bar{\mathscr{F}}_0 \otimes \mathscr{B}^M)[-M] \\ & (\bar{\mathscr{F}}_{-1} \otimes \mathscr{B}^M)[-M+1]. \arrow[u] \end{tikzcd}$$

Now, since these complexes are direct sums of complexes of vector spaces over $k(p_i, q_i)$, we can write the complex on the right as a direct sum of its cohomology groups and get a map of complexes

$$(\pi_1^*\mathscr{E}_2 \otimes \mathscr{B}^M)[-M] \longrightarrow \begin{array}{c} (\mathscr{H}^M(\pi_1^*(\mathscr{F}) \overset{L}{\otimes} \mathscr{B}^M))[-M] \\ \oplus \\ (\mathscr{H}^{M-1}(\pi_1^*(\mathscr{F}) \overset{L}{\otimes} \mathscr{B}^M))[-M+1], \end{array}$$

and by pushing forward to Y we get a map of complexes

$$\Phi(\mathscr{E}_2) \longrightarrow \begin{array}{c} \pi_{2*}(\mathscr{H}^M(\pi_1^*(\mathscr{F}) \overset{L}{\otimes} \mathscr{B}^M))[-M] \cong \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \\ \oplus \\ \pi_{2*}(\mathscr{H}^{M-1}(\pi_1^*(\mathscr{F}) \overset{L}{\otimes} \mathscr{B}^M))[-M+1] \cong \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1]. \end{array}$$

This proves precisely that the map $\Phi(\mathscr{E}_2) \to \mathscr{H}^{M-1}(\phi(\mathscr{F}))[-M+1]$ is zero (π_{2*} is exact here because the sheaves are flasque). For the second map we can reason as follows: since the map $\Phi(\mathscr{E}_2) \to \mathscr{H}^{M-1}(\phi(\mathscr{F}))[-M+1]$ is zero, and we know that $\Phi(\mathscr{E}_2) \to \Phi(\mathscr{E}_1)[1]$ is zero, it follows that the composition $\Phi(\mathscr{E}_2) \to \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \to \Phi(\mathscr{E}_1)[1]$ is also zero. Hence the result follows if the map

$$\mathrm{Hom}(\mathscr{H}^M(\Phi(\mathscr{F}))[-M], \Phi(\mathscr{E}_1)[1]) \to \mathrm{Hom}(\Phi(\mathscr{E}_2), \Phi(\mathscr{E}_1)[1])$$

is injective, i.e. the map

$$\mathrm{Ext}^1(\mathscr{H}^M(\Phi(\mathscr{F})), \mathscr{H}^M(\Phi(\mathscr{E}_1))) \to \mathrm{Ext}^1(\mathscr{H}^M(\Phi(\mathscr{E}_2)), \mathscr{H}^M(\Phi(\mathscr{E}_1)))$$

is injective.

To show this we claim that, for any coherent sheaf $\mathscr{G}$ on X, $\mathscr{H}^M(\Phi(\mathscr{G}))$ is a direct sum of (degree one) skyscraper sheaves supported at the q_i's. This is clear for locally free sheaves and for torsion sheaves supported away from the p_i's. Let us check it for a torsion sheave $\mathscr{G}$ supported at p_i. Since $\mathscr{G} \cong \oplus \mathcal{O}_{X,p_i}/\mathfrak{m}_{p_i}^n$ we can assume we have just one such summand. Then we have a short exact sequence

$$0 \to \mathcal{O}_X(-np_i) \to \mathcal{O}_X \to \mathscr{G} \to 0$$

and hence $\pi_1^*(\mathscr{G}) = [\mathcal{O}_{X\times Y}(-(np_i \times Y)) \to \mathcal{O}_{X\times Y}]$ and $\pi_1^*(\mathscr{G}) \otimes \mathscr{B}^M = [k(p_i, q_i) \xrightarrow{0} k(p_i, q_i)]$ and since those are flasque sheaves we have $\mathscr{H}^M(\Phi(\mathscr{G})) = k(q_i)$. This proves the claim.

In particular, the surjection $\mathscr{H}^M(\Phi(\mathscr{E}_2)) \to \mathscr{H}^M(\Phi(\mathscr{F}))$ is split and hence the map

$$\mathrm{Ext}^1(\mathscr{H}^M(\Phi(\mathscr{F})), \mathscr{H}^M(\Phi(\mathscr{E}_1))) \to \mathrm{Ext}^1(\mathscr{H}^M(\Phi(\mathscr{E}_2)), \mathscr{H}^M(\Phi(\mathscr{E}_1)))$$

is injective.

We're finally ready to show that

$$\begin{tikzcd} \Phi(\mathscr{F}) \arrow[r] \arrow[d] & \Phi(\mathscr{E}_1)[1] \arrow[d] \\ F(\mathscr{F}) \arrow[r] & F(\mathscr{E}_1)[1] \end{tikzcd}$$

commutes. To do this, take the same decomposition $\Phi(\mathscr{F}) \cong \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1] \oplus \mathscr{H}^{M}(\Phi(\mathscr{F}))[-M]$ as above. We will show that the two diagrams

$$\begin{tikzcd} \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \arrow[r] \arrow[d] & \Phi(\mathscr{E}_1)[1] \arrow[d] \\ F(\mathscr{F}) \arrow[r] & F(\mathscr{E}_1)[1] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1] \arrow[r] \arrow[d] & \Phi(\mathscr{E}_1)[1] \arrow[d] \\ F(\mathscr{F}) \arrow[r] & F(\mathscr{E}_1)[1] \end{tikzcd}$$

are both commutative.

Notice that the composition $\Phi(\mathscr{E}_2) \to \Phi(\mathscr{F}) \to F(\mathscr{F}) \to F(\mathscr{E}_1)[1]$ is zero, because we already know that the central square in (7) commutes, and $F(\mathscr{E}_2) \to F(\mathscr{F}) \to F(\mathscr{E}_1)[1]$ is zero. Moreover, since $\Phi(\mathscr{E}_2) \to \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1]$ is the zero map, this means that the composition $\Phi(\mathscr{E}_2) \to \mathscr{H}^M(\Phi(\mathscr{F}))[-M] \to F(\mathscr{F}) \to F(\mathscr{E}_1)[1]$ is zero. Since we already know that on objects we have an isomorphism $F(\cdot) \cong \Phi(\cdot)$, by the same computation as above we get that

$$\mathrm{Hom}(\mathscr{H}^M(\Phi(\mathscr{F}))[-M], F(\mathscr{E}_1)[1]) \to \mathrm{Hom}(\Phi(\mathscr{E}_2), F(\mathscr{E}_1)[1])$$

is again injective hence the composition $\mathscr{H}^M(\Phi(\mathscr{F}))[-M] \to F(\mathscr{F}) \to F(\mathscr{E}_1)[1]$ is zero. In the same way, we know that $\mathscr{H}^M(\Phi(\mathscr{F}))[-M] \to \Phi(\mathscr{E}_1) \to F(\mathscr{E}_1)[1]$ is also zero. This shows that the first square commutes.

To show that the second square above is commutative, we just need to show that the square

$$\begin{tikzcd} \mathscr{H}^{M-1}(\Phi(\mathscr{F}))[-M+1] \arrow[r] \arrow[d] & \Phi(\mathscr{E}_1)[1] \arrow[d] \\ \mathscr{H}^{M-1}(F(\mathscr{F}))[-M+1] \arrow[r] & F(\mathscr{E}_1)[1] \end{tikzcd}$$

is commutative. But this follows from Proposition 3.2.

VIII. $s(-)$ is compatible with triangles of the type $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ for any $\mathscr{F}$ and $\mathscr{G}$: in this situation we can find $\mathscr{F}'$, $\mathscr{G}'$ locally free and a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, "a"] & \mathscr{G}' \arrow[r] \arrow[d, "b"] & \mathscr{H} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathscr{F} \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{H} \arrow[r] & 0. \end{tikzcd}$$

Then we get

$$\begin{tikzcd} \Phi(\mathscr{F}) \arrow[r] & \Phi(\mathscr{G}) \arrow[r] & \Phi(\mathscr{H}) \arrow[r, "\Phi(\delta)"] \arrow[dr, phantom, "\circlearrowleft"] & \Phi(\mathscr{F})[1] \arrow[ddd, bend left=60, "{s(\mathscr{F})[1]}"] \\ \Phi(\mathscr{F}') \arrow[r] \arrow[u] \arrow[d] & \Phi(\mathscr{G}') \arrow[r] \arrow[u] \arrow[d] & \Phi(\mathscr{H}) \arrow[r, "\Phi(\delta')"] \arrow[u, equal] \arrow[d, "s(C)"] \arrow[dr, phantom, "\circlearrowleft"] & \Phi(\mathscr{F}')[1] \arrow[u, "{\Phi(a)[1]}"'] \arrow[d, "{s(\mathscr{F}')[1]}"] \arrow[d, phantom, shift left=8ex, "\circlearrowleft"] \\ F(\mathscr{F}') \arrow[r] \arrow[d] & F(\mathscr{G}') \arrow[r] \arrow[d] & F(\mathscr{H}) \arrow[r, "F(\delta')"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & F(\mathscr{F}')[1] \arrow[d, "{F(a)[1]}"] \\ F(\mathscr{F}) \arrow[r] & F(\mathscr{G}) \arrow[r] & F(\mathscr{H}) \arrow[r, "F(\delta)"] & F(\mathscr{F})[1] \end{tikzcd}$$

where the top and bottom right squares commute because Φ and F are functors, the middle right square by part VII, and the semi-circle, which is

$$\begin{tikzcd} \Phi(\mathscr{F}')[1] \arrow[r, "\Phi(a)[1]"] \arrow[d, "s(\mathscr{F}')[1]"'] & \Phi(\mathscr{F})[1] \arrow[d, "s(\mathscr{F})[1]"] \\ F(\mathscr{F}')[1] \arrow[r, "F(a)[1]"] & F(\mathscr{F})[1] \end{tikzcd}$$

by part III since $\mathscr{F}'$ is locally free. Therefore the boundary maps commute:

$$\begin{tikzcd} \Phi(\mathscr{H}) \arrow[r, "\Phi(\delta)"] \arrow[d, "s(\mathscr{H})"'] & \Phi(\mathscr{F})[1] \arrow[d, "s(\mathscr{F})[1]"] \\ F(\mathscr{H}) \arrow[r, "F(\delta)"] & F(\mathscr{F})[1], \end{tikzcd}$$

because

$$\begin{aligned} s(\mathscr{F})[1] \circ \Phi(\delta) &= s(\mathscr{F})[1] \circ \Phi(a)[1] \circ \Phi(\delta') = F(a)[1] \circ s(\mathscr{F}')[1] \circ \Phi(\delta') \\ &= F(a)[1] \circ F(\delta') \circ s(C) = F(\delta) \circ s(C). \end{aligned}$$ □

Lemma 3.6. *In the setup of Theorem 3.3, let Q, T be two torsion coherent sheaves on X. Consider a coherent sheaf $\mathscr{E}$ on X with a surjection $\mathscr{E} \to Q$. Consider a map*

$$\xi : \Phi(Q) \to F(T)$$

that induces the zero map on all cohomology groups. If the composition $\Phi(\mathscr{E}) \to \Phi(Q) \to F(T)$ is zero, then it follows that the map ξ is zero.

Proof. We know that

$$\begin{aligned} \Phi(Q) &= R\pi_{2*}(\mathscr{B}^M[-M] \overset{L}{\otimes} \pi_1^* Q) \\ &= R\pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* Q)[-M+1] \oplus (\mathscr{B}^M \otimes \pi_1^* Q)[-M] \\ &= \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* Q))[-M+1] \oplus \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* Q)[-M] \end{aligned}$$

(since $\mathscr{B}^M$ is supported at a finite number of points and hence is flasque). Moreover, we know that $F(T)$ is isomorphic to $\Phi(T)$ by part III. of Theorem 3.3, so we also know that

$$F(T) \cong \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* T))[-M+1] \oplus \pi_{2*}(\mathscr{B}^M \otimes \pi_1^* T)[-M]$$

Fix two isomorphisms as above. Now if we know that the given map $\Phi(Q) \to F(T)$ is zero on cohomology, the map can be represented by a map

$$\pi_{2*}(\mathscr{B}^M \otimes \pi_1^* Q)[-M] \to \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* T))[-M+1]$$

i.e. an element of $\mathrm{Ext}^1(\pi_{2*}(\mathscr{B}^M \otimes \pi_1^* Q), \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* T)))$. Then it suffices to show that the map

$$\mathrm{Ext}^1(\pi_{2*}(\mathscr{B}^M \otimes \pi_1^* Q), \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* T))) \to \mathrm{Ext}^1(\pi_{2*}(\mathscr{B}^M \otimes \pi_1^* \mathscr{E}), \pi_{2*}(\mathrm{Tor}^1(\mathscr{B}^M, \pi_1^* T)))$$

is injective.

But since $\mathscr{E}$ surjects onto Q, we have a surjection $\pi_{2*}(\pi_1^*(\mathscr{E}) \otimes \mathscr{B}^M) \to \pi_{2*}(\pi_1^*(Q) \otimes \mathscr{B}^M)$ and both of these sheaves are skyscraper sheaves of degree one supported at the points $q_1, \ldots, q_t$, hence this is a surjection of vector spaces and therefore it splits. Hence the map on Ext^1 above is injective. □

To complete the proof of Theorem 1.2 we still need to extend the isomorphism to the whole derived category $D^b_{\mathrm{Coh}}(X)$. This is straightforward in our case of $\dim(X) = 1$ thanks to the following:

Lemma 3.7. *Let X, Y be smooth projective varieties with $\dim(X) = 1$. Consider two exact functors $F, \Phi : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$, and assume that there exists an isomorphism of exact functors $t : F \to \Phi$ on the full subcategory of $D^b_{\mathrm{Coh}}(X)$ given by coherent sheaves on X placed in degree zero. Then t extends to an isomorphism of exact functors on the whole $D^b_{\mathrm{Coh}}(X)$.*

Proof. Consider a complex $C^\bullet \in D^b_{\mathrm{Coh}}(X)$. Then by [Dol60, Satz 4.7] $C^\bullet \cong \oplus H^i(C^\bullet)[-i]$, in a non-canonical way. Choose one such isomorphism for each $C^\bullet$. By Theorem 3.3, since both functors are compatible with shifting, we immediately get an isomorphism $t(C^\bullet) : F(C^\bullet) \to \Phi(C^\bullet)$.

Now consider a map $C^\bullet \to D^\bullet$. This is the same as a map $\oplus H^i(C^\bullet)[-i] \to \oplus H^i(D^\bullet)[-i]$, and again since the two functors are compatible with shifting, and X has dimension 1, it is enough to show that $t(-)$ is compatible with maps $\mathscr{F} \to \mathscr{G}$ and $\mathscr{F} \to \mathscr{G}[1]$, where $\mathscr{F}$ and $\mathscr{G}$ are sheaves. The first case follows from the fact that t is an isomorphism of exact functors on $\mathrm{Coh}(X)$. A map $\alpha : \mathscr{F} \to \mathscr{G}[1]$ corresponds to an element in $\mathrm{Ext}^1(\mathscr{F}, \mathscr{G})$ so we have a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{H} \to \mathscr{F} \to 0$$

and by Theorem 3.3 we get an isomorphism of triangles

$$\begin{tikzcd} F(\mathscr{G}) \arrow[r] \arrow[d, "t(\mathscr{G})"] & F(\mathscr{H}) \arrow[r] \arrow[d, "t(\mathscr{H})"] & F(\mathscr{F}) \arrow[r, "F(\alpha)"] \arrow[d, "t(\mathscr{F})"] & \mathscr{F}(\mathscr{G})[1] \arrow[d, "t(\mathscr{G})[1]"] \\ \Phi(\mathscr{G}) \arrow[r] & \Phi(\mathscr{H}) \arrow[r] & \Phi(\mathscr{F}) \arrow[r, "\Phi(\alpha)"] & \Phi(\mathscr{G})[1] \end{tikzcd}$$

hence t is compatible with α. The fact that t is compatible with triangles is immediate. □

Proof of Theorem 1.2. This follows immediately from Theorem 3.3 and Lemma 3.7. □

Remark 3.8. *Notice that any functor satisfying the hypotheses of Theorem 1.2 will not be full and will not satisfy*

$$\mathrm{Hom}_{D^b_{\mathrm{Coh}}(Y)}(F(\mathscr{F}, \mathscr{G}[j])) = 0 \text{ if } j < 0$$

for all $\mathscr{F}, \mathscr{G} \in \mathcal{O}_X$ (take for example $\mathscr{F}$ to be supported at one of the p_i's). Hence Theorem 1.2 gives a class of functors that are Fourier-Mukai but do not satisfy the hypotheses of [CS07, Theorem 1.1].

4. A Spectral Sequence

Even when we don't know how to build a kernel out of the sheaves $\mathscr{B}^i$ that we constructed in Theorem 1.1, these sheaves still satisfy some good properties. As an example, we will show that under the same hypotheses of Theorem 1.1 the analogue of the Cartan-Eilenberg Spectral Sequence converges when the dimension of X is one, whereas the sequence of low degree terms is exact for any X, Y.

Let X, Y be smooth projective varieties over an algebraically closed field, and consider a Fourier-Mukai functor Φ_E with $E \in D^b_{\mathrm{Coh}}(X \times Y)$ (we remind the reader that a Fourier-Mukai functor is not necessarily an equivalence) . Then for each locally free sheaf $\mathscr{E} \in \mathrm{Coh}(X)$ the Cartan-Eilenberg Spectral Sequence gives

$$E_2^{pq} = R^p\pi_{2*}(\mathscr{H}^q(E) \otimes \pi_1^*\mathscr{E}) \Rightarrow \mathscr{H}^{p+q}(\Phi_E(\mathscr{E}))$$

Now consider an exact functor $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$. Suppose we computed the cohomology sheaves $\mathscr{B}^i$ of the prospective kernel in $D^b_{\mathrm{Coh}}(X \times Y)$ as in Theorem 1.1. Then we can replace $\mathscr{H}^q(E)$ with $\mathscr{B}^q$ in the above and set

$$E_2^{pq} = R^p\pi_{2*}(\mathscr{B}^q \otimes \pi_1^*\mathscr{E})$$

for any locally free sheaf $\mathscr{E}$ on X. The corresponding sequence of low degree terms is exact:

Proposition 4.1. *Let X, Y be smooth projective varieties over an algebraically closed field k, $F : D^b_{\mathrm{Coh}}(X) \to D^b_{\mathrm{Coh}}(Y)$ an exact functor. Assume that $F(\mathscr{E}) \in D^{[M,N]}_{\mathrm{Coh}}(Y)$ for all locally free sheaves $\mathscr{E}$ of finite rank on X. Let $\mathscr{B}^i$ be the sheaves computed in Theorem 1.1. Then for any locally free sheaf $\mathscr{E}$ of finite rank on X the following sequence is exact:*

$$\begin{aligned} 0 \to R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to \\ \to \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}) \to R^2\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+2}(F(\mathscr{E})) \end{aligned} \tag{8}$$

Proof. Assume that there is an embedding $X \to \mathbb{P}^d_k$. Then for every $m > 0$ we have a short exact sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(m)^{\oplus(d+1)} \to K_m \to 0$, where K_m is a locally free sheaf.

Let $\mathscr{E}$ be a locally free sheaf of finite rank on X. Then by tensoring the sequence above with $\mathscr{E}$ we get a short exact sequence

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0. \tag{9}$$

Choose m high enough so that $R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(\mathscr{E}(m))) = 0$ and that $\mathscr{H}^{M+1}(F(\mathscr{E}(m))) \cong \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}(m))$. By applying the functor F and then taking cohomology we get a long exact sequence

$$0 \to \mathscr{H}^M(F(\mathscr{E})) \to \mathscr{H}^M(F(\mathscr{E}(m)))^{\oplus(d+1)} \to \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to \dots$$

By Proposition 2.2, for any locally free sheaf of finite rank $\mathscr{F}$ we have a functorial isomorphism

$$\mathscr{H}^M(F(\mathscr{F})) \xrightarrow{\cong} \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{F}).$$

Moreover, by Theorem 1.1 we also have a map $\mathscr{H}^{M+1}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E})$. Then we get the following diagram:

$$\begin{tikzcd}
\mathscr{H}^M(F(\mathscr{E}(m)))^{\oplus(d+1)} \arrow[r] \arrow[dd, "\cong"] & \mathscr{H}^M(F(K_m \otimes \mathscr{E})) \arrow[r] \arrow[dd, "\cong"] & \mathscr{H}^{M+1}(F(\mathscr{E})) \arrow[r] \arrow[d] & \mathscr{H}^{M+1}(F(\mathscr{E}(m)))^{\oplus(d+1)} \arrow[d, "\cong"] \\
 & 0 \arrow[r] & \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*(\mathscr{E})) \arrow[r] & \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}(m)^{\oplus(d+1)}) \\
\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}(m)^{\oplus(d+1)}) \arrow[r] & \pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})) \arrow[r] & R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \arrow[r] & 0
\end{tikzcd} \tag{10}$$

By chasing diagram (10), we obtain a map

$$R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})),$$

and we also obtain that the sequence

$$0 \to R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E})$$

is exact. This is the first part of our sequence.

Now since the sequence above is exact for any $\mathscr{E}$ locally free sheaf of finite rank on X, it will also be exact for $K_m \otimes \mathscr{E}$. So we have the following diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})) \arrow[d] & \\
\dots \arrow[r] & \mathscr{H}^{M+1}(F(\mathscr{E})) \arrow[r] \arrow[d] & \mathscr{H}^{M+1}(F(\mathscr{E}(m)))^{\oplus(d+1)} \arrow[r] \arrow[d, "\cong"] & \mathscr{H}^{M+1}(F(K_m \otimes \mathscr{E})) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}) \arrow[r] & \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}(m)^{\oplus(d+1)}) \arrow[r] & \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \arrow[r] & \dots
\end{tikzcd}$$

by diagram chasing we get a map

$$\pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}) \to R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E}))$$

This has an obvious map to $\mathscr{H}^{M+2}(F(\mathscr{E}))$ given by the composition

$$R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+1}(F(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+2}(F(\mathscr{E}))$$

but since $R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}(m)) = R^2\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}(m)) = 0$, we know that

$$R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*(K_m \otimes \mathscr{E})) \cong R^2\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E})$$

this gives the second part of our sequence,

$$\pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}) \to R^2\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+2}(F(\mathscr{E}))$$

Exactness of the whole sequence again follows by diagram chasing. □

Proposition 4.2. *In the setting of Proposition 4.1, assume* $\dim(X) = 1$*. Then for all locally free sheaves of finite rank* $\mathscr{E}$ *on* X *there is a spectral sequence*

$$E_2^{pq} = R^p\pi_{2*}(\mathscr{B}^q \otimes \pi_1^*\mathscr{E}) \Rightarrow \mathscr{H}^{p+q}(F(\mathscr{E}))$$

Proof. Since $\dim X = 1$, the only nonzero terms of the spectral sequence are $E_2^{0,q}$ and $E_2^{1,q}$. Therefore all the differentials are zero and to show that the spectral sequence converges we need to show:

- There exists a map $E_2^{1,q-1} = R^1\pi_{2*}(\mathscr{B}^{q-1} \otimes \pi_1^*\mathscr{E}) \hookrightarrow \mathscr{H}^q(F(\mathscr{E}))$
- $E_2^{0,q} = \pi_{2*}(\mathscr{B}^q \otimes \pi_1^*\mathscr{E}) \cong \mathscr{H}^q(F(\mathscr{E}))/R^1\pi_{2*}(\mathscr{B}^{q-1} \otimes \pi_1^*\mathscr{E})$.

Since $\dim X = 1$ we have $R^2\pi_{2*}(\mathscr{B}^q \otimes \pi_1^*\mathscr{E}) = 0$. Therefore the exact sequence (8) of Proposition 4.1 becomes a short exact sequence

$$0 \to R^1\pi_{2*}(\mathscr{B}^M \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+1}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+1} \otimes \pi_1^*\mathscr{E}) \to 0. \tag{11}$$

Choose m high enough so that $R^p\pi_{2*}(\mathscr{B}^q \otimes \pi_1^*(\mathscr{E}(m))) = 0$ for all q and all $p > 0$ (this can be done because for m high enough $\pi_1^*\mathcal{O}_X(1)$ is very ample with respect to $X \times Y \to Y$), and such that $\mathscr{H}^i(F(\mathscr{E}(m))) \cong \pi_{2*}(\mathscr{B}^i \otimes \pi_1^*\mathscr{E}(m))$ for all i (this can be done by Theorem 1.1). Then using the short exact sequence (9)

$$0 \to \mathscr{E} \to \mathscr{E}(m)^{\oplus(d+1)} \to K_m \otimes \mathscr{E} \to 0,$$

we get a long exact sequence

$$0 = R^1\pi_{2*}(\mathscr{B}^i \otimes \pi_1^*(\mathscr{E}(m)^{\oplus d+1})) \to R^1\pi_{2*}(\mathscr{B}^i \otimes \pi_1^*(\mathscr{E} \otimes K_m)) \to R^2\pi_{2*}(\mathscr{B}^i \otimes \pi_1^*\mathscr{E}) = 0$$

hence $R^1\pi_{2*}(\mathscr{B}^i \otimes \pi_1^*(\mathscr{E} \otimes K_m)) = 0$ for all i.

Now assume by induction that we get the same short exact sequence as (11) starting with $\mathscr{B}^{M+n-1}$ for any locally free sheaf of finite rank $\mathscr{E}$:

$$0 \to R^1\pi_{2*}(\mathscr{B}^{M+n-1} \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+n}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*\mathscr{E}) \to 0$$

then the same exact sequence will hold if we substitute $\mathscr{E}$ with $K_m \otimes \mathscr{E}$, since K_m is also locally free:

$$0 \to R^1\pi_{2*}(\mathscr{B}^{M+n-1} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \to 0.$$

But since $R^1\pi_{2*}(\mathscr{B}^i \otimes \pi_1^*(\mathscr{E} \otimes K_m)) = 0$ this gives an isomorphism

$$\mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) \cong \pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*(K_m \otimes \mathscr{E})).$$

Using the short exact sequence (9) and Theorem 1.1, we get the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathscr{H}^{M+n}(F(K_m \otimes \mathscr{E})) \arrow[rr] \arrow[dd, "\cong"] & & \mathscr{H}^{M+n+1}(F(\mathscr{E})) \arrow[r] \arrow[d] & \dots \\
& & 0 \arrow[r] & \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*\mathscr{E}) \arrow[r] & \dots \\
\dots \arrow[r] & \pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \arrow[r] & R^1\pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*\mathscr{E}) \arrow[r] & 0 &
\end{tikzcd} \tag{12}$$

and from this we get a sequence

$$0 \to R^1\pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*\mathscr{E}) \to \mathscr{H}^{M+n+1}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*\mathscr{E}), \tag{13}$$

which is exact by chasing diagram (12). Again, we also have the corresponding exact sequence for the locally free sheaf $K_m \otimes \mathscr{E}$:

$$0 \to R^1\pi_{2*}(\mathscr{B}^{M+n} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \to \mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*(K_m \otimes \mathscr{E}))$$

and the first term of the sequence is zero, i.e. the map $\mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*(K_m \otimes \mathscr{E}))$ is injective. This is reflected in the following diagram:

$$\begin{tikzcd}
\dots \arrow[r] & \mathscr{H}^{M+n+1}(F(\mathscr{E})) \arrow[r] \arrow[d] & \mathscr{H}^{M+n+1}(F(\mathscr{E}(m)))^{\oplus(d+1)} \arrow[r] \arrow[d, "\cong"] & \mathscr{H}^{M+n+1}(F(K_m \otimes \mathscr{E})) \arrow[r] \arrow[d, hook] & \dots \\
0 \arrow[r] & \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*\mathscr{E}) \arrow[r] & \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*\mathscr{E}(m)^{\oplus(d+1)}) \arrow[r] & \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*(K_m \otimes \mathscr{E})) \arrow[r] & \dots
\end{tikzcd}$$

By diagram chasing this tells us that the map

$$\mathscr{H}^{M+n+1}(F(\mathscr{E})) \to \pi_{2*}(\mathscr{B}^{M+n+1} \otimes \pi_1^*\mathscr{E})$$

is actually surjective, hence (13) becomes a short exact sequence, and this completes the proof. □

References

[BVdB03] A. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36.

[COS13] A. Canonaco, D. O. Orlov, and P. Stellari, *Does full imply faithful?*, J. Noncommut. Geom. **7** (2013), no. 2, 357–371.

[CS07] A. Canonaco and P. Stellari, *Twisted Fourier-Mukai functors*, Adv. Math. **212** (2007), no. 2, 484–503.

[CS12] ———, *Non-uniqueness of fourier-mukai kernels*, Math. Z. **272** (2012), 577–588.

[Dol60] A. Dold, *Zur Homotopietheorie der Kettenkomplexe*, Mathematische Annalen **140** (1960), 278–298.

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 8.

[Orl97] D. O. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381.

[RVdB14] A. Rizzardo and M. Van den Bergh, *An example of a non-fourier-mukai functor between derived categories of coherent sheaves*, preprint, available at http://arxiv.org/abs/1410.4039.

[RVdB15] ———, *Scalar extensions of derived categories and non-fourier-mukai functors*, Advances in Mathematics (2015), 1100–1144.

[Ser07] J-P. Serre, *Faisceaux algébriques cohérents*, vol. 61, The Annals of Mathematics , Second Series, 2007.

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