The infinite simple group V of Richard J. Thompson: presentations by permutations

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Abstract

We show that one can naturally describe elements of R. Thompson's finitely presented infinite simple group V, known by Thompson to have a presentation with four generators and fourteen relations, as products of permutations analogous to transpositions. This perspective provides an intuitive explanation towards the simplicity of V and also perhaps indicates a reason as to why it was one of the first discovered infinite finitely presented simple groups: it is (in some basic sense) a relative of the finite alternating groups. We find a natural infinite presentation for V as a group generated by these "transpositions," which presentation bears comparison with Dehornoy's infinite presentation and which enables us to develop two small presentations for V: a human-interpretable presentation with three generators and eight relations, and a Tietze-derived presentation with two generators and seven relations.

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1 Introduction

In this article, we investigate R. Thompson's group V from a mostly-unexplored perspective. As a consequence we derive new, and hopefully

elegant, presentations of this well-known group and introduce a simple and dextrous notation for handling computations in V.

Recall that the group V first appears in Thompson's 1965 notes [23] and is given there as one of two "first-examples" of infinite finitely presented simple groups (along with its simple subgroup T, called "**C**" in those notes). Since then, it has been the focus of a large amount of subsequent research (see, for example, [4, 5, 6, 13, 15, 18, 24] for a small part of that research). Thompson's group V arises in various other settings, for example, Birget [2, 3] investigates connections to circuits and complexity while Lawson [19] considers links to inverse monoids and étale groupoids.

We shall demonstrate that one can consider V as a symmetric group acting, not on a finite set, but instead on a Cantor algebra (the algebra of basic clopen sets in a Cantor space). We focus upon certain well-known properties of a finite symmetric group, namely being generated by transpositions and being transitive in its natural action. Reflecting these two fundamental properties, a finite symmetric group possesses a Coxeter-type presentation, with generating set $\mathcal{T}$ corresponding to a set of appropriate transpositions and relations $t^2 = 1$ for all $t \in \mathcal{T}$, $(tu)^2 = 1$ when $t, u \in \mathcal{T}$ correspond to transpositions of disjoint support and $(tu)^3 = 1$ when $t \neq u$ but the corresponding transpositions have intersecting support. If we exploit the fact that these generators have order 2, this third type of relation can be rewritten as $t^{-1}ut = u^{-1}tu$ and indeed in the symmetric group this conjugate equals another transposition v, namely that whose support satisfies $\operatorname{supp} v = (\operatorname{supp} t)u$.

In the context of a Cantor algebra, the analogues of transpositions are piecewise affine maps which "swap" a pair of basic open sets. We shall observe that Thompson's group V is generated by such transpositions of the standard Cantor algebra and hence derive an infinite Coxeter-like presentation for V, as appears in Theorem 1.1 below.

As is well known, the standard Cantor algebra admits a natural tree-structure where the nodes correspond to the basic open sets in Cantor space $\mathfrak{C}$ and these nodes are indexed by finite words in the alphabet $X = \{0, 1\}$. Consequently, we label our transpositions by two incomparable words α and β from X^*. Indeed, for such α and β, we write $t_{\alpha,\beta}$ for the element of V that is the transposition defined in Equation (2.1). We shall also write $s_{\alpha,\beta}$ and $(\alpha\ \beta)$ for symbols representing elements in two abstract groups whose presentations we give in Theorems 1.1 and 1.2, respectively. We specifically use different notations for each of these elements so as to distinguish between the elements of each abstract group and the actual transformations of Cantor space. The thrust of our work is to demon-

strate that the two abstract groups are isomorphic to V and that under the isomorphisms these three elements $s_{\alpha,\beta}$, $(\alpha\ \beta)$ and $t_{\alpha,\beta}$ correspond. The first two of our families of relations appearing in Theorem 1.1 reflect that these $t_{\alpha,\beta}$ act as transpositions so have order 2, commute when their supports are disjoint, and conjugate in a manner analogous to transpositions in symmetric groups when their supports intersect appropriately.

Passing from the setting of actions of finite permutation groups on finite sets to the setting of corresponding actions of infinite groups on Cantor algebras has further implications for the resulting presentation. Namely, due to the self-similar nature of Cantor space, each generating transposition can be factorised. To be precise, each $t_{\alpha,\beta}$ satisfies what we call a split relation: $t_{\alpha,\beta} = t_{\alpha 0,\beta 0}\, t_{\alpha 1,\beta 1}$. This provides the third family of relations that are seen in Theorem 1.1. They have the consequence that not only is every element of V a product of our transpositions $t_{\alpha,\beta}$, but also we can re-express any such product as one that involves an even number of transpositions. Thus one can simultaneously view R. Thompson's group V as an infinite analogue of both the finite alternating groups and of the finite symmetric groups.

It follows quite easily from the presentation in Theorem 1.1 that any transposition $t_{\gamma,\delta}$ can be obtained by conjugation using only those $t_{\alpha,\beta}$ with $|\alpha|, |\beta| \leqslant 3$, for example, and this motivates an effort to find a finite presentation involving permutations and their relations, where these permutations involve only the nodes in the first three levels of the tree. Theorem 1.2 provides this presentation (involving three generators and eight relations). Note here that we depart slightly from the Coxeter-style of presentation: we exploit the presence of the symmetric group of degree 4 acting upon X^2 to reduce further the presentation, at the cost of employing a "three-cycle" as a generator. Of note, this human-interpretable presentation is much smaller than the currently known finite presentation for V (given by Thompson [23] and discussed in detail in Cannon, Floyd and Parry's survey [12]), which has four generators and fourteen relations.

As a technical exercise, we further reduce the presentation in Theorem 1.2 to a 2-generator and 7-relation presentation, found in Theorem 1.3. The resulting presentation is small, but not so readily interpretable by humans.

Our infinite presentation in Theorem 1.1 bears comparison with Dehornoy's infinite presentation for V (see [13, Proposition 3.20]). Dehornoy's presentation highlights different aspects as to why the group V can be considered as a fundamental object in group theory, and even in mathematics, bearing out, as it does, the connection of V to systems with equivalences

under associativity and commutativity.

Our viewpoint of V as a form of a symmetric or alternating group perhaps hints at why V arose as one of first two known examples of an infinite simple finitely presented group. Permuting sets is a basic activity, and the Cantor algebra represents a fundamental way to pass from a finite to an infinite context, thus it seems natural that researchers eventually noticed V. To give further background, we note there are many generalisations of V to infinite simple finitely presented groups, all of which owe their simplicity to the same fundamental idea (similar to the reason why the alternating groups are simple). One such family is the Higman–Thompson groups $G_{n,r}$ for which $V = G_{2,1}$, see [15]. (The group $G_{n,r}$ is simple for n even. When n is odd, one must pass to the commutator subgroup of index 2, reflecting the observation that the corresponding split relations in $G_{n,r}$ do not change the parity of any decomposition as a product of transpositions.) Other families include the Brin–Thompson groups nV for which $V = 1V$, see [6], and the groups $nV_{m,r}$ that generalise the previous two families, see [20], and where we have similar simplicity considerations, see [7]. The finite presentability of these groups comes from the much stronger fact that they are all in fact F_∞ groups. (There is a beautiful argument of the F_∞ nature of these groups given in [24], which applies to many of these "relatives" of V. In many specific cases, F_∞ arguments already exist for individual groups and for classes of groups in these families. See, for example, [1, 8, 9, 17].) The ideas of this paper ought to apply to all of these groups of "Thompson type" in aiding in the discovery of natural and small presentations. On the other hand, the infinite family of finitely-presented infinite simple groups arising from the Burger-Mozes construction and following related work (see, e.g., [10, 11, 21]) are of an entirely different nature, and the methods employed here do not seem appropriate to that context.

We mention here a debt to Matatyu Rubin and Matthew G. Brin. Rubin indicated to Brin a proof of the simplicity of V, which uses the generation of V by transpositions with restricted support on Cantor space. Brin set this proof out briefly in his paper [6] and developed the ideas to extend the proof to the groups nV, which he carries out in the short paper [7]. It is not a stretch to say that the current article would not exist without that thread of previous research.

A note on content

The first two sections of this article are intended for the interested mathematician and provide structure and insights into these sorts of groups. The outline of the proofs of the theorems are found towards the end of Section 2.

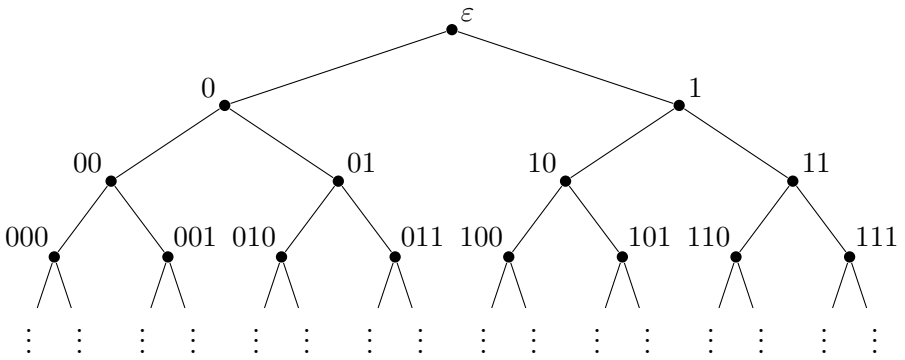


Figure 1: The infinite binary rooted tree with nodes labelled by elements of X^*

The sections that follow are more technical and verify the details required for those proofs.

Statement of results and some notation

Let $X = \{0, 1\}$. We write X^* for all finite sequences $x_1x_2 \dots x_k$ where $k \geqslant 0$ and each $x_i \in X$. In particular, we assume that X^* contains the empty word ε. We view the elements of X^* as representing the nodes on the infinite binary rooted tree with edges between nodes if they are represented by words which differ by a suffix of length 1. (Figure 1 illustrates this tree together with the nodes labelled by elements of X^*.) Similarly, we give the standard definition of the Cantor set $\mathfrak{C}$ as X^ω, the set of all infinite sequences $x_1x_2x_3 \dots$ of elements of X under the product topology (starting with X endowed with the discrete topology). Thus points in $\mathfrak{C}$ correspond to boundary points of the infinite binary rooted tree.

If $\alpha \in X^*$ and $\beta \in X^* \cup X^\omega$, then we write $\alpha\beta$ for the concatenation of the two sequences. We denote by $\alpha\mathfrak{C}$ the set of elements of $\mathfrak{C}$ with initial prefix α. This set is a basic open set in the topology on $\mathfrak{C}$ and is itself homeomorphic to $\mathfrak{C}$. We shall write $\alpha \preceq \beta$ to indicate that α is a prefix of β (including the possibility that the two sequences are equal). This notation then means that $\beta = \alpha\gamma$ for some $\gamma \in X^* \cup X^\omega$. Moreover, when $\beta \in X^*$, then α and β represent nodes on the infinite binary rooted tree such that β lies on a path descending from α (see Figure 2(i)), and therefore $\beta\mathfrak{C} \subseteq \alpha\mathfrak{C}$.

We also write $\alpha \perp \beta$ to denote that both $\alpha \not\preceq \beta$ and $\beta \not\preceq \alpha$. Then we shall say that α and β are *incomparable*. In this case, the paths to α and

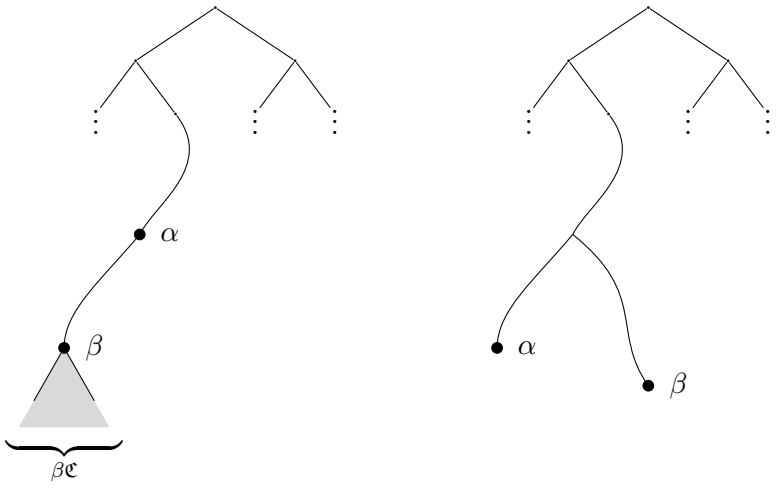


Figure 2: (i) $\alpha \preceq \beta$ (and the paths representing elements of $\beta\mathfrak{C}$); and (ii) $\alpha \perp \beta$

to β from the root separate at some node above both α and β (as shown in Figure 2(ii)), so that $\alpha\mathfrak{C} \cap \beta\mathfrak{C} = \varnothing$ (for such $\alpha, \beta \in X^*$).

If m is a positive integer, we shall also use X^m to denote the collection of all finite sequences $x_1x_2 \dots x_m$ of *length* m with $x_i \in X$ for each i. We write $|\alpha|$ for the length of the sequence $\alpha \in X^*$.

Motivated by the well-known fact (see, for example, [13]) that R. Thompson's group V has a partial action on the set of finite binary rooted trees and equally on the set X^*, we shall use the notation $\gamma \cdot (\alpha\ \beta)$, for $\alpha, \beta, \gamma \in X^*$, defined by

$$\gamma \cdot (\alpha\ \beta) = \begin{cases} \beta\delta & \text{if } \gamma = \alpha\delta \text{ for some } \delta \in X^*, \\ \alpha\delta & \text{if } \gamma = \beta\delta \text{ for some } \delta \in X^*, \\ \gamma & \text{if both } \gamma \perp \alpha \text{ and } \gamma \perp \beta, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1.1}$$

Thus $\gamma \cdot (\alpha\ \beta)$ is undefined precisely when $\gamma \prec \alpha$ or $\gamma \prec \beta$ and when it is defined it represents a prefix substitution replacing any occurrence of α by β and *vice versa*.

The notation appearing in Equation (1.1) above is motivated via our anticipated isomorphism between V and the abstract group defined by the presentation in Theorem 1.2. The map $t_{\alpha,\beta}$ is taken to the element $(\alpha\ \beta)$ and the above formula reflects the effect of applying $t_{\alpha,\beta}$ to a point in the partial action of V on X^*. Note also that we are choosing to write our maps

on the right since in our opinion this enables one to more conveniently compose a number of maps and such a convention is consistent with denoting an element of V by tree-pairs with the domain on the left and the codomain on the right. Nevertheless, we still view maps in V as being given by prefix substitutions of the infinite sequences that are elements of the Cantor space so as to be consistent with other work on these groups.

Our results are as follows:

Theorem 1.1 *Let $\mathcal{A}$ to be the set of all symbols $s_{\alpha,\beta}$ where $\alpha, \beta \in X^*$ with $\alpha \perp \beta$. Then R. Thompson's group V has an infinite presentation with generating set $\mathcal{A}$ and relations*

$$\begin{aligned} s_{\alpha,\beta}^2 &= 1 \\ s_{\gamma,\delta}^{-1}\, s_{\alpha,\beta}\, s_{\gamma,\delta} &= s_{\alpha\cdot(\gamma\ \delta),\beta\cdot(\gamma\ \delta)} \\ s_{\alpha,\beta} &= s_{\alpha 0,\beta 0}\, s_{\alpha 1,\beta 1} \end{aligned} \tag{1.2}$$

where α, β, γ and δ range over all sequences in X^* such that $\alpha \perp \beta$, $\gamma \perp \delta$, and $\alpha \cdot (\gamma\ \delta)$ and $\beta \cdot (\gamma\ \delta)$ are defined.

Our primary finite presentation for the group V has three generators a, b and c, but, as mentioned above, it is most naturally expressed in terms of a "permutation-like" notation extending the transpositions in the infinite presentation. Our generators a, b and c then correspond to permutations that we denote (00 01), (01 10 11) and (1 00), respectively, and the relations are similarly expressed in terms of elements $(\alpha\ \beta)$ that we define fully in Section 2. This "human-readable" presentation is as follows:

Theorem 1.2 *R. Thompson's group V has a finite presentation with three generators* (00 01), (01 10 11) *and* (1 00) *and eight relations*

$\mathcal{R}1$. $(00\ 01)^2 = (01\ 10\ 11)^3 = \big((00\ 01)\,(01\ 10\ 11)\big)^4 = 1$;

$\mathcal{R}2$. $(1\ 01)^{(1\ 00)} = (00\ 01)$;

$\mathcal{R}3$. $(1\ 00) = (10\ 000)\,(11\ 001)$;

$\mathcal{R}4$. $[(00\ 010), (10\ 111)] = [(00\ 011), (10\ 111)] = 1$;

$\mathcal{R}5$. $[(000\ 010), (10\ 110)] = 1$.

We shall provide words in terms of the generators a, b and c to express these relations later in Equation (2.3). Observe that the Relations $\mathcal{R}1$ tells us that (00 01) and (01 10 11) satisfy the relations of the symmetric

group S_4, so the subgroup that they generate is isomorphic to some quotient of S_4. In fact, it will turn out that this subgroup *is* isomorphic to S_4.

The element $(\alpha\ \beta)$ will correspond to the element of Thompson's group V that maps a point of the Cantor set that has prefix α to a point with prefix β and *vice versa*. Relations $\mathcal{R}4$ and $\mathcal{R}5$ then reflect the fact that certain elements of V commute because they have disjoint support.

We show that V is generated by the two elements u and v described by the tree-pairs in Figure 3. Transforming the presentation in Theorem 1.2 to one using these two generators via Tietze transformations (as described in Corollary 5.2) and reducing the nine resulting relations using the Knuth–Bendix algorithm, as implemented in the GAP package KBMAG [14, 16], results in the surprising two generator and seven relation presentation given below:

Theorem 1.3 *R. Thompson's group V has a finite presentation with two generators u and v and the seven relators*

$$u^6,\ \ v^3,\ \ (u^3v)^4,$$
$$v^{-1}u(u^2v^{-1})^2u^3vu^{-1}v^{-1}u^3vu(uvu^2(uv^{-1}u^3v)^3)^2uv^{-1}u^3v^{-1},$$
$$uv^{-1}u^3v^{-1}u^{-2}v^{-1}uvu^2v^{-1}u^{-1}vu^2v^{-1}uvu^{-1}(u^{-1}v^{-1})^2u^3vu^{-1},$$
$$v(uv^{-1}u^3v^{-1})^2u^{-1}v^{-1}u^3v^{-1}u^{-1}v^{-1}u^3v,$$
$$uvu^3vuv^{-1}u^{-2}v^{-1}u(u^2v)^2(u^2v^{-1})^2u^3vu^{-2}v^{-1}u^3v.$$

This reduction to seven relations caught the authors by surprise, but perhaps it is not so unexpected in view of the deficiency (as defined, for example, in [22, §14.1]) of the presentations in Theorems 1.2 and 1.3 both being -5.

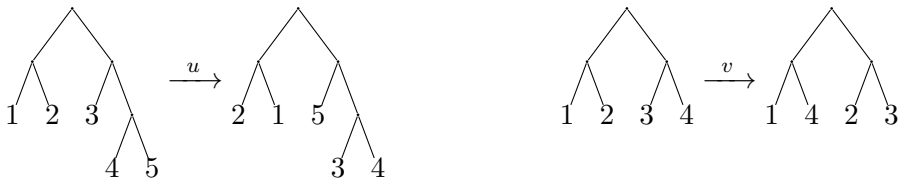


Figure 3: Two elements given by tree-pairs that generate for R. Thompson's group V

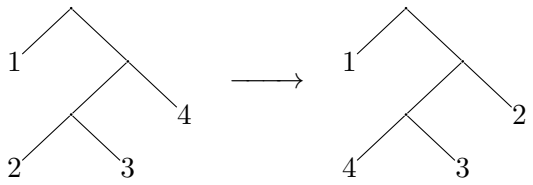


Figure 4: The map $t_{100,11}$ denoted using tree-pairs

2 Further preliminaries and the proofs of the main theorems

This section contains the heart of the mathematics within the article. We present all the remaining preliminaries required to fully understand the statements of the theorems listed in the introduction, in particular, unpacking the presentations that we use. We then provide the proofs, subject to deferring technical calculations to the sections that follow.

R. Thompson's group V

One view of Thompson's group V is as a group of certain homeomorphisms of the Cantor set $\mathfrak{C}$, namely those that are finite products of the elements $t_{\alpha,\beta}$, for $\alpha, \beta \in X^*$ with $\alpha \perp \beta$, defined as follows

$$xt_{\alpha,\beta} = \begin{cases} \beta y & \text{if } x = \alpha y \text{ for some } y \in \mathfrak{C}; \\ \alpha y & \text{if } x = \beta y \text{ for some } y \in \mathfrak{C}; \\ x & \text{otherwise} \end{cases} \tag{2.1}$$

(see Brin [6, Lemma 12.2]). Note that the map $t_{\alpha,\beta}$ has the effect of swapping those elements of $\mathfrak{C}$ that have an initial prefix α with those that have an initial prefix β and fixing all other points in $\mathfrak{C}$. A general element of V is often denoted by a pair of finite binary rooted trees representing the domain and codomain of the map. We label the leaves of these two trees by the numbers 1, 2, ..., n (for some n) and this then specifies that our element of V has the effect of substituting the prefix from X^* corresponding to the node in the domain tree labelled i by the member of X^* corresponding to the node in the codomain tree with the same label (for each i). For example, Figure 4 provides such tree-pairs for the map $t_{100,11}$ as just defined.

From the definition, it is visible that $t_{\alpha,\beta}^2 = 1$. Equations of this type (as α and β range over all incomparable pairs from X^*) will form our family of *order relations*.

If we shift our attention to conjugation, it is a straightforward calculation in V, along the lines of the familiar one concerning conjugation of permutations demonstrated to undergraduates in a first course on group theory, to verify that

$$t_{\gamma,\delta}^{-1}\, t_{\alpha,\beta}\, t_{\gamma,\delta} = t_{\alpha\cdot(\gamma\ \delta),\beta\cdot(\gamma\ \delta)}$$

whenever $\alpha \cdot (\gamma\ \delta)$ and $\beta \cdot (\gamma\ \delta)$ are both defined. We call this resultant family of relations in V our *conjugacy relations*. At this point, we also note that we will use an exponential notation for conjugation, so the left-hand side of the above relation will also be denoted by $t_{\alpha,\beta}{}^{t_{\gamma,\delta}}$ in what follows.

Our final family of relations exploit the action of our elements $t_{\alpha,\beta}$ on basic open sets of the Cantor set $\mathfrak{C}$ and the self-similar structure of this space. Specifically, if $\alpha \in X^*$, then the basic open set $\alpha\mathfrak{C}$ splits into two subsets, namely the set of all elements of $\mathfrak{C}$ with initial prefix $\alpha 0$ and those with initial prefix $\alpha 1$. In view of this, we obtain the equation

$$t_{\alpha,\beta} = t_{\alpha 0,\beta 0}\, t_{\alpha 1,\beta 1} \tag{2.2}$$

when $\alpha, \beta \in X^*$ with $\alpha \perp \beta$. We refer to the family of these relations as *split relations*.

Deriving the presentations for V

One of the presentations that we use in this article is that found by R. Thompson and discussed in Cannon–Floyd–Parry (see [12, Lemma 6.1]). This presentation has four generators A, B, C and π_0 and fourteen relations. We state these relations when we need them at the start of Section 4 below.

As described in Theorem 1.1, the first of our new presentations involves the order relations, conjugacy relations and split relations of V just described. To be precise, we define P_∞ to be the group having the infinite presentation with generating set $\mathcal{A} = \{\, s_{\alpha,\beta} \mid \alpha, \beta \in X^*,\ \alpha \perp \beta \,\}$ and the relations listed in Equation (1.2). Of course, we already know that that V is generated by the maps $t_{\alpha,\beta}$ and that these satisfy the order, conjugacy and split relations. This ensures that there is a surjective homomorphism $\phi\colon P_\infty \to V$ given by $s_{\alpha,\beta} \mapsto t_{\alpha,\beta}$ for $\alpha, \beta \in X^*$ with $\alpha \perp \beta$. When establishing Theorem 1.1, we shall be observing that ϕ is actually an isomorphism.

We now describe our primary finite presentation for V, which has three generators a, b and c and eight relations but, more importantly, can be readily understood by a human. The majority of our calculations will take

place with this presentation and so, as indicated above, we develop a helpful notation that is parallel to that used in finite permutation groups. It is a consequence of Higman [15] that all the relations that hold in V can be detected as consequences of products using tree-pairs of some bounded size. This motivates our presentation which employs essentially a finite subcollection of the order, conjugacy and split relations from P_∞ and involving only some swaps $(\alpha\ \beta)$ all satisfying $|\alpha|, |\beta| \leqslant 3$. (To aid reducing the number of relations required we encode a copy of the symmetric group of degree 4, corresponding to acting on X^2, within our presentation.)

Accordingly, the three generators a, b and c of the group P_3 that we define will represent cyclic permutations of basic open sets. As stated above, we shall write (00 01), (01 10 11) and (1 00) for a, b and c, respectively. This reflects the fact that, under the isomorphism that we shall establish between P_3 and V, the element a corresponds to the map $t_{00,01}$ that interchanges the basic open sets $00\mathfrak{C}$ and $01\mathfrak{C}$, b corresponds to the product $t_{01,10}\, t_{01,11}$ (inducing a 3-cycle of the sets $01\mathfrak{C}$, $10\mathfrak{C}$ and $11\mathfrak{C}$), and c corresponds to $t_{1,00}$. We extend this notation by defining elements $(\alpha\ \beta)$ that we will refer to as *swaps* below and a formula for each in terms of a, b and c will be extracted from the definitions we make. The element $(\alpha\ \beta)$ will correspond to $t_{\alpha,\beta}$ under the isomorphism. It is these swaps that appear in our list of relations found in Theorem 1.2 above.

Once we have defined the swaps below, we can translate the Relations $\mathcal{R}1$–$\mathcal{R}5$ into words in a, b and c. The result is the following restatement of Theorem 1.2:

Theorem 2.1 *R. Thompson's group V has a finite presentation with three generators a, b and c and the following eight relations:*

$$\begin{aligned} a^2 = b^3 = (ab)^4 = 1, &\qquad c^{ac} = a, \\ c = a^{bcacaa^{ba}}\, a^{b^{-1}cacaa^{b^{-1}a}}, &\qquad [a^{bcac}, a^{b^{-1}caca^b a^{b^{-1}a}}] = 1, \\ [a^{b^{-1}cac}, a^{b^{-1}caca^b a^{b^{-1}a}}] = 1, &\qquad [a^{bca^{bca}}, a^{bcaca^b a^{b^{-1}a}}] = 1. \end{aligned} \tag{2.3}$$

The careful reader will doubtless have observed that the eighth relation can be shortened by conjugating by a. The presentation as listed is a direct consequence of the interpretation of Theorem 1.2 in terms of a, b and c. No effort has been made in its statement to reduce the length of the relations.

Having found this nice presentation for V, we felt obligated to reduce the relations employing the available technology, specifically the Knuth–Bendix Algorithm. This algorithm shortens the above relations, although the results are no longer particularly transparent. To carry out these reductions, we used the implementation of the algorithm found in the freely

available KBMAG package [16] in GAP [14] in the following way. Denote the eight relators corresponding to the equations in (2.3) by r_1, r_2, …, r_8. We can construct a rewriting system associated to each of the groups $Q_i = \langle\, a, b, c \mid r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_8 \,\rangle$ for $i = 4, 5, \ldots, 8$ in sequence. The systems that KBMAG constructs are not confluent, but nevertheless enable us to replace each r_i by a Tietze-equivalent (in the group Q_i) shorter relation. This process is repeated until the resulting relations stabilise. As a consequence, the normal closure, in the free group on $\{a, b, c\}$, of the following eight relations is identical to that of our original list:

$$\begin{gathered} a^2 = b^3 = (ab)^4 = 1, \\ c^{-1}(ac)^2 a = 1, \\ (cab^{-1}aba)^2 cb(cabab^{-1}a)^2 = 1, \\ a(cb)^2 a(b^{-1}c)^2 bcabcb^{-1}cab^{-1}acb^{-1}(cb)^2 ab^{-1} = 1, \\ ab^{-1}cbc(ab^{-1})^2 cbcb^{-1}a(b^{-1}c)^2 babcb^{-1}cab^{-1} = 1, \\ ca(b^{-1}c)^2 bacabacbc(b^{-1}ca)^2 b(cb^{-1})^2 (acb)^2 cb^{-1}cab^{-1} = 1. \end{gathered} \tag{2.4}$$

We observe this mechanical process produces considerably shorter relations than our original eight in Equation (2.3).

The presentation in Theorem 1.3 is deduced in a manner that similarly depends upon the use of the Knuth–Bendix Algorithm. One first applies Tietze transformations to pass to a 2-generator presentation employing generators u and v and relations deduced from the list (2.4). We shall describe these Tietze transformations by expressing u and v in terms of a, b and c (adding extraneous generators), and expressing a, b and c in terms of u and v (removing extraneous generators). The relevant formulae are

$$u = a(a^b a^{b^{-1}})^{caca^b a^{b^{-1}a}} \qquad \text{and} \qquad v = b$$

and

$$a = u^3, \qquad b = v, \qquad c = (u^3)^{vu^{-2}vu^3}\,(u^3)^{vu^{-1}vu^3 v}.$$

These formulae can be deduced by direct calculation in V. This application of Tietze transformations is expanded upon a little in Section 5 and Corollary 5.2 provides the intermediate step to the theorem. (This corollary is established by purely theoretical means and does not rely upon computer calculation.)

We then employ the same relation reduction process using KBMAG as described earlier and this shows that two of the nine relations resulting from the Tietze transformations are extraneous. In this manner we have deduced Theorem 1.3 from Theorem 1.2.

We now proceed to formally define the swaps $(\alpha\ \beta)$ for $\alpha, \beta \in X^*$ with $\alpha \perp \beta$ and $|\alpha|, |\beta| \leqslant 3$ in terms of our generators a, b and c in order to present the group P_3. To start off, we define swaps $(\alpha\ \beta)$ for $\alpha, \beta \in X^2$ as follows:

$$\begin{aligned} &(00\ 01) = a, \quad &&(00\ 10) = a^b, \quad &&(00\ 11) = a^{b^{-1}} \\ &(01\ 10) = a^{ba}, \quad &&(01\ 11) = a^{b^{-1}a}, \quad &&(10\ 11) = a^{bab} \end{aligned} \tag{2.5}$$

Here, and in all that follows, we shall also adopt the convention that the swap $(\beta\ \alpha)$ coincides with $(\alpha\ \beta)$ whenever the latter has already been defined. We write $\mathcal{T}_2$ for the set of swaps $(\alpha\ \beta)$ with $\alpha, \beta \in X^2$.

The swaps in $\mathcal{T}_2$ and their effect when conjugating will be of fundamental importance in our calculations. Accordingly we spend a little time expanding upon the above definitions before we define the remaining swaps. In the Relations $\mathcal{R}1$, we have assumed that a and b satisfy the relations of the symmetric group S_4 and the formulae on the right-hand side of Equation (2.5) are those that correspond to transpositions in S_4. Consequently, when we multiply and conjugate elements of $\mathcal{T}_2$, they behave in exactly the same way as transpositions do. In particular, we can view individual elements of $\mathcal{T}_2$, and, by extension, products of such swaps, as transformations of the set X^2. Indeed, our notation $\gamma \cdot (\alpha\ \beta)$, as defined in Equation (1.1), when $\alpha, \beta, \gamma \in X^2$, is precisely the formulae for these maps. Our assumption of Relations $\mathcal{R}1$ justifies our using products of swaps from $\mathcal{T}_2$ as maps $X^2 \to X^2$, as we shall do explicitly, for example, in Lemma 3.4 below. Similarly, we have written $(01\ 10\ 11)$ for the generator b, since it follows from the Relations $\mathcal{R}1$ that b is equal to the product $(01\ 10)\,(01\ 11)$, which induces a 3-cycle on X^2.

To define the remaining swaps $(\alpha\ \beta)$, where $|\alpha|, |\beta| \leqslant 3$, we need one further piece of notation. If $x \in X$, we define $\bar{x}$ to be the other element in X; that is,

$$\bar{x} = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x = 1. \end{cases}$$

Then for any $x, y, z \in X$, we make our definitions in the following order:

$$(0\ 1) = (00\ 10)\,(01\ 11); \tag{2.6}$$

$$\begin{aligned} &(1\ 00) = c, \quad &&(1\ 01) = (1\ 00)^{(00\ 01)}, \\ &(0\ 1x) = (1\ 0x)^{(0\ 1)}; \end{aligned} \tag{2.7}$$

$$\begin{aligned} &(1\ 00x) = (00\ 1x)^{(1\ 00)}, \quad &&(1\ 01x) = (1\ 00x)^{(00\ 01)}, \\ &(0\ 1xy) = (1\ 0xy)^{(0\ 1)}; \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
&(00\ 01x) = (1\ 01x)^{(1\ 00)}, && (01\ 00x) = (00\ 01x)^{(00\ 01)},\\
&(1x\ 1\bar{x}y) = (0x\ 0\bar{x}y)^{(0\ 1)}, && (1x\ 0yz) = (0\bar{y}\ 0yz)^{(0\bar{y}\ 1x)},\\
&(0x\ 1yz) = (1x\ 0yz)^{(0\ 1)};
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&(000\ 001) = (1\ 000)^{(1\ 001)}, && (000\ 010) = (1\ 000)^{(1\ 010)},\\
&(000\ 011) = (1\ 000)^{(1\ 011)}, && (001\ 011) = (1\ 001)^{(1\ 011)},\\
&(xy0\ xy1) = (000\ 001)^{(00\ xy)}.
\end{aligned}
\tag{2.10}
$$

Finally, for distinct $\kappa, \lambda \in X^2$, fix a product $\rho_{\kappa\lambda}$ of swaps from $\mathcal{T}_2$ that moves 00 to κ and 01 to λ when viewed, as described above, as a map $X^2 \to X^2$. Define

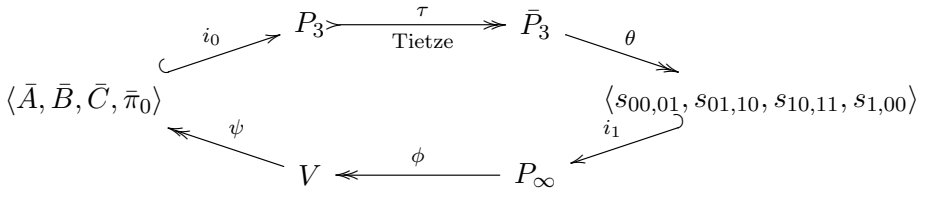
$$(\kappa x\ \lambda y) = (00x\ 01y)^{\rho_{\kappa\lambda}} \tag{2.11}$$

for $(x, y) \in \{(0,0), (0,1), (1,1)\}$. In this way, we have now defined all swaps $(\alpha\ \beta)$ where $|\alpha|, |\beta| \leqslant 3$.

Having made these definitions, it is a straightforward matter to convert the relations $\mathcal{R}1$–$\mathcal{R}5$ into the list (2.3) of actual words expressed in the generators a, b and c, completing the translation of Theorem 1.2 into Theorem 2.1.

Proofs of the main theorems

We now provide the proofs of the main theorems (that is, Theorems 1.1 and 1.2), subject to information that we shall establish in the sections of the paper that follow. Here we link the groups V, P_3 and P_∞. Specifically, we build homomorphisms between these groups as indicated in the following diagram:



(2.12)

We already know that $\phi\colon P_\infty \to V$ is a surjective homomorphism. We now describe the other parts of the hexagon of maps.

The majority of the work in Sections 3–5 involves the presentation for the group P_3, where we establish information about the swaps $(\alpha\ \beta)$ defined above. In Section 3, we verify that these swaps satisfy the order relations,

conjugacy relations and split relations of R. Thompson's group V, providing we restrict to those involving only swaps $(\alpha\ \beta)$ with $|\alpha|, |\beta| \leqslant 3$. Then in Section 4, we establish that four specific elements $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\pi}_0$ in P_3 satisfy the fourteen relations listed in Cannon–Floyd–Parry [12] that define the group V. In that section, we ensure that we only rely upon the consequences of our Relations $\mathcal{R}1$–$\mathcal{R}5$ established in Section 3. This then guarantees the existence of a surjective homomorphism $\psi\colon V \to \langle \bar{A}, \bar{B}, \bar{C}, \bar{\pi}_0 \rangle$.

In the final section, we establish that $\langle \bar{A}, \bar{B}, \bar{C}, \bar{\pi}_0 \rangle$ coincides with the group P_3 (see Proposition 5.1), from which it follows that the natural inclusion map i_0 is also surjective.

Amongst the generators for P_3 are the elements $a = (00\ 01)$ and $b = (01\ 10\ 11)$, which satisfy the relations of the symmetric group S_4 (Relations $\mathcal{R}1$). Of course, S_4 also enjoys a presentation in terms of transpositions involving only order and conjugacy relations. Accordingly, we apply Tietze transformations to convert the presentation for P_3 into one for a group $\bar{P}_3$ with generators (00 01), (01 10), (10 11) and (1 00) and some order, conjugacy and split relations (specifically translations of $\mathcal{R}2$–$\mathcal{R}5$, together with the new ones to replace $\mathcal{R}1$). Thus we have on the one hand, an isomorphism $\tau\colon P_3 \to \bar{P}_3$ and, on the other, a surjective homomorphism $\theta\colon \bar{P}_3 \to \langle s_{00,01}, s_{01,10}, s_{10,11}, s_{1,00} \rangle$ (a subgroup of P_∞), since the relations defining $\bar{P}_3$ all hold in P_∞.

We can now deduce that $P_3 = \langle \bar{A}, \bar{B}, \bar{C}, \bar{\pi}_0 \rangle$ is not trivial, since successively composing the appropriate maps in (2.12) sends, for example, $a = (00\ 01)$ to the non-identity element $t_{00,01}$ in V. Hence, from simplicity of V, we conclude $P_3 \cong V$ and therefore, subject to the work in Sections 3–5, establish Theorem 1.2.

Finally, it is relatively straightforward to observe that if α is a non-empty sequence in X^*, then there is a product w involving only the swaps (00 01), (01 10), (10 11) and (1 00) such that $00 \cdot w = \alpha$. From this, one quickly deduces, principally relying upon the conjugacy relations, that one can conjugate $s_{00,01}$ by some element of $\langle s_{00,01}, s_{01,10}, s_{10,11}, s_{1,00} \rangle$ to any generator $s_{\alpha,\beta}$ where $(\alpha, \beta) \neq (0, 1)$. This ensures that the inclusion i_1 is surjective. Hence $P_\infty \cong V$ also and we have established Theorem 1.1.

The remaining sections are perhaps technical, but carry out the deferred work just as described above. We hope that these sections will quickly impress the reader with the utility of the permutation notation $(\alpha\ \beta)$ in performing calculations within R. Thompson's group V.

Remarks

(i) If our goal had been simply to establish Theorem 1.1, then our work

in the next two sections would have been greatly reduced. Indeed, it is actually rather easy to show that the group P_∞ defined by all the relations listed in Theorem 1.1 is isomorphic to V (for example, the relations that Dehornoy [13] lists are particularly straightforward to deduce). However, from the viewpoint of the high transitivity of the action of V on the Cantor set $\mathfrak{C}$, one expects that it would be enough to restrict to relations involving swaps $(\alpha\ \beta)$ with $|\alpha|$ and $|\beta|$ bounded. Indeed, this is what leads to our presentation for the group P_3 and the small set of relations, $\mathcal{R}1$–$\mathcal{R}5$, where we are relying upon a small subset involving only swaps $(\alpha\ \beta)$ with $|\alpha|, |\beta| \leqslant 3$. Establishing that this set of relations is sufficient is the delicate business of Sections 3 and 4.

(ii) Now that we have established that the groups P_3, P_∞ and V are all isomorphic, it is safe to use the notation $(\alpha\ \beta)$ as a convenient notation for the map $t_{\alpha,\beta}$ as defined earlier. We can then perform computations within R. Thompson's group V employing this notation, for example, along the lines of those in the sections that follow, and we hope this will be of use to those working with elements in this group.

3 Verification of relations to level 3

Our principal aim is to establish that all the relations holding in R. Thompson's group V can be deduced from Relations $\mathcal{R}1$–$\mathcal{R}5$. In this section, we complete the first stage of our technical calculations by establishing essentially a subset of the infinitely many relations in the list (1.2), namely those order relations, the conjugacy relations and the split relations involving only swaps $(\alpha\ \beta)$ with $|\alpha|, |\beta| \leqslant 3$. It will turn out that these are enough to then deduce the fourteen relations for V found in [12] as we shall see in Section 4.

Accordingly, in this section, we shall verify all relations of the form

$$\begin{aligned}(\alpha\ \beta)^2 &= 1\\ (\alpha\ \beta)^{(\gamma\ \delta)} &= (\alpha{\cdot}(\gamma\ \delta)\ \ \beta{\cdot}(\gamma\ \delta))\\ (\alpha\ \beta) &= (\alpha 0\ \beta 0)\,(\alpha 1\ \beta 1)\end{aligned}$$

whenever any swap $(\kappa\ \lambda)$ appearing above satisfies $|\kappa|, |\lambda| \leqslant 3$. (So, for example, the split relation $(\alpha\ \beta) = (\alpha 0\ \beta 0)\,(\alpha 1\ \beta 1)$ needs only to be verified for $|\alpha|, |\beta| \leqslant 2$ in order that the swaps on the right-hand side of the equation fulfil this requirement.) The main focus throughout the section will be in

establishing all the conjugacy relations $\sigma^\tau = \upsilon$ and we will consider these relations when we select σ, τ and υ from the following sets:

$$\begin{aligned}
\mathcal{T}_2 &= \{\, (\alpha\ \beta) \mid (\alpha,\beta) = (0,1) \text{ or } \alpha,\beta \in X^2 \,\} \\
\mathcal{T}_3 &= \{\, (\alpha\ \beta) \mid \alpha,\beta \in X^3 \,\} \\
\mathcal{T}_{mn} &= \{\, (\alpha\ \beta) \mid \alpha \in X^m,\ \beta \in X^n \,\}, \qquad \text{where } 1 \leqslant m < n \leqslant 3
\end{aligned}$$

We start our verification by first noting that (1 00) is a conjugate of (00 01) by Relation $\mathcal{R}2$ and hence has order dividing 2 by the first relation in $\mathcal{R}1$. From this and the definitions of the swaps, we now know that $(\alpha\ \beta)^2 = 1$ whenever $\alpha \perp \beta$ with $|\alpha|, |\beta| \leqslant 3$. We will use this fact throughout the rest of this section.

We step through the various relations, essentially introducing "longer" swaps through the stages. Accordingly, we start with relations involving only swaps from $\mathcal{T}_{12} \cup \mathcal{T}_2$, then introducing swaps from $\mathcal{T}_{13}$, and so on. We need to take various side-trips from this general direction in order to successfully establish all the relations we want.

Relations involving $\mathcal{T}_{12}$ and $\mathcal{T}_2$ only

With careful analysis, one can soon establish which conjugacy relations involve only swaps selected from $\mathcal{T}_{12} \cup \mathcal{T}_2$. These are, for $x, y \in X$, the following three relations:

$$(x\ \bar{x}y)^{(0\ 1)} = (\bar{x}\ xy) \tag{3.1}$$

$$(x\ \bar{x}y)^{(\bar{x}y\ \bar{x}\bar{y})} = (x\ \bar{x}\bar{y}) \tag{3.2}$$

$$(x\ \bar{x}y)^{(x\ \bar{x}\bar{y})} = (\bar{x}y\ \bar{x}\bar{y}) \tag{3.3}$$

These are actually very easy to verify. For example, Equation (3.1) simply follows from our definition of $(0\ 1x)$ (for $x \in X$) in (2.7), while we can establish Equation (3.2) using our definition of (1 01) in (2.7) and then conjugating, if necessary, by (0 1) and using the now established Equation (3.1). We now have the first step in our verification and results along the lines of the following lemma will occur throughout our progress.

Lemma 3.1 *All conjugacy relations of the form $\sigma^\tau = \upsilon$ where $\sigma, \upsilon \in \mathcal{T}_{12}$ and $\tau \in \mathcal{T}_2$ can be deduced from Relations $\mathcal{R}1$–$\mathcal{R}5$.* □

This lemma contributes now to establishing Equation (3.3), since we observe that, for the case $x = 1$ and $y = 0$, the equation is Relation $\mathcal{R}2$ and that the general equation can then be deduced by conjugating by

a product of elements from $\mathcal{T}_2$. Specifically, conjugating by (0 1) gives $(0\ 11)^{(0\ 10)} = (10\ 11)$ and then subsequently the two equations now established by (00 01) and (10 11), respectively, establishes the final cases.

Relations involving $\mathcal{T}_{12}$, $\mathcal{T}_{13}$ and $\mathcal{T}_2$ only

When we introduce swaps from $\mathcal{T}_{13}$, in addition to those from $\mathcal{T}_{12}$ and $\mathcal{T}_2$, the relations that need to be verified are, for $x, y, z \in X$, the following:

$$(x\ \bar{x}yz)^{(0\ 1)} = (\bar{x}\ xyz) \tag{3.4}$$

$$(x\ \bar{x}yz)^{(\bar{x}y\ \bar{x}\bar{y})} = (x\ \bar{x}\bar{y}z) \tag{3.5}$$

$$(x\ \bar{x}yz)^{(x\ \bar{x}y)} = (xz\ \bar{x}y) \tag{3.6}$$

Both **Equations (3.4)** and **(3.5)** follow quickly from the definitions in (2.8). They establish:

Lemma 3.2 *All conjugacy relations of the form $\sigma^\tau = \upsilon$ where $\sigma, \upsilon \in \mathcal{T}_{13}$ and $\tau \in \mathcal{T}_2$ can be deduced from Relations $\mathcal{R}1$–$\mathcal{R}5$.* □

If we expand the definition of $(1\ 00z)$ from Equation (2.8), we obtain $(1\ 00z)^{(1\ 00)} = (00\ 1z)$ for $z \in X$. Now conjugate by an appropriate product of elements from $\mathcal{T}_2$, using Lemmas 3.1 and 3.2 similarly to the argument used for Equation (3.3), to establish Equation (3.6).

First batch of relations involving $\mathcal{T}_2$ and $\mathcal{T}_{23}$ only

We now turn to relations involving swaps from $\mathcal{T}_{23}$. There are additional relations that we shall establish later involving only swaps from $\mathcal{T}_2$ and $\mathcal{T}_{23}$, but for now we are concerned with the following equations (in particular, the first of our split relations) for $x, y \in X$, for *distinct* $\kappa, \lambda, \mu \in X^2$, and for any $\tau \in \mathcal{T}_2$ for which the first is defined:

$$(\kappa\ \lambda x)^\tau = (\kappa{\cdot}\tau\ \ (\lambda x){\cdot}\tau) \tag{3.7}$$

$$(x\ \bar{x}y) = (x0\ \bar{x}y0)\,(x1\ \bar{x}y1) \tag{3.8}$$

$$(\kappa\ \lambda x)^{(\mu\ \lambda x)} = (\kappa\ \mu) \tag{3.9}$$

We shall need the following lemma to be able to manipulate the initial swaps from $\mathcal{T}_{23}$ from which the others are built, as given in (2.9). We shall then establish Equation (3.7) in the form of Lemma 3.4 below.

Lemma 3.3 (i) $(1\ 00)$, $(10\ 000)$ *and* $(11\ 001)$ *all commute;*

(ii) $(01\ 00x)^{(1\ 00)} = (01\ 1x)$;

PROOF: (i) Using Relation $\mathcal{R}3$ and the fact that all the swaps involved have order dividing 2, we conclude that (10 000) and (11 001) commute. It then follows, again using $\mathcal{R}3$, that (1 00) also commutes with these elements.

(ii) Pull apart the formula for $(01\ 00x)$ using the definitions in (2.9), and then apply Relation $\mathcal{R}2$ and Equation (3.6) as follows:

$$(01\ 00x)^{(1\ 00)} = (1\ 01x)^{(1\ 00)\,(00\ 01)\,(1\ 00)} = (1\ 01x)^{(1\ 01)} = (01\ 1x).$$

□

Lemma 3.4 *All conjugacy relations of the form $\sigma^\tau = \upsilon$ where $\sigma, \upsilon \in \mathcal{T}_{23}$ and $\tau \in \mathcal{T}_2$ can be deduced from Relations $\mathcal{R}1$–$\mathcal{R}5$.*

PROOF: The first half of the proof deals with the case when $\sigma = (\kappa\ \lambda 0)$ for distinct $\kappa, \lambda \in X^2$. To establish this, we must first verify that the swap (00 010) commutes with (10 11). However, to achieve this, we actually work first with the swap (10 000). Indeed,

$$\begin{aligned}
(10\ 000)\,(01\ 11) &= (1\ 00)\,(11\ 001)\,(01\ 11) && \text{by Rel. } \mathcal{R}3\\
&= (1\ 00)\,(01\ 11)\,(01\ 001) && \text{by the def. of (11 001) in (2.9)}\\
&= (01\ 001)\,(01\ 11)\,(1\ 00) && \text{by Lem. 3.3(ii) twice}\\
&= (01\ 11)\,(11\ 001)\,(1\ 00) && \text{by the def. in (2.9)}\\
&= (01\ 11)\,(10\ 000) && \text{by Lem. 3.3(i) and Rel. } \mathcal{R}3.
\end{aligned}$$

As the definition of (10 000) in Equation (2.9) is $(00\ 010)^{(00\ 01)\,(01\ 10)}$, we conclude that (00 010) commutes with $(01\ 11)^{(01\ 10)\,(00\ 01)} = (10\ 11)$.

So we now turn to the required relation when $\sigma = (\kappa\ \lambda 0)$ for distinct $\kappa, \lambda \in X^2$. As described earlier, we view τ as a permutation of X^2. As such a map, suppose τ maps κ and λ to μ and ν, respectively. Then by the definition in Equation (2.9) there are products ρ_1 and ρ_2 of elements from $\mathcal{T}_2$ such that

$$(\kappa\ \lambda 0) = (00\ 010)^{\rho_1} \qquad \text{and} \qquad (\mu\ \nu 0) = (00\ 010)^{\rho_2}.$$

Specifically, ρ_1 and ρ_2 are products that, when viewed as permutations of X^2, map 00 and 01 to κ and λ and to μ and ν, respectively. Then $\rho_1\tau\rho_2^{-1} \in \langle (10\ 11) \rangle$ since it fixes both 00 and 01. Hence, by our previous calcuation, (00 010) commutes with $\rho_1\tau\rho_2^{-1}$, so

$$(\kappa\ \lambda 0)^\tau = (00\ 010)^{\rho_1\tau} = (00\ 010)^{\rho_2} = (\mu\ \nu 0). \tag{3.10}$$

Thus, we have established Equation (3.7) in the case when $x = 0$.

To deduce the equation for $x = 1$, we proceed similarly and first need to establish that (00 011) commutes with (10 11). We calculate as follows:

$$\begin{aligned}
(10\ 000)^{(01\ 000)} &= (10\ 000)^{(1\ 00)\,(01\ 10)\,(1\ 00)} && \text{by Lem. 3.3(ii)} \\
&= (10\ 000)^{(01\ 10)\,(1\ 00)} && \text{by Lem. 3.3(i)} \\
&= (01\ 000)^{(1\ 00)} && \text{by Eq. (3.10)} \\
&= (01\ 10) && \text{by Lem. 3.3(ii).}
\end{aligned}$$

Conjugate by (01 10) and use Equation (3.10) again to establish the formula $(01\ 000)^{(10\ 000)} = (01\ 10)$. Now we find

$$\begin{aligned}
(11\ 001)\,(01\ 10) &= (10\ 000)\,(1\ 00)\,(01\ 10) && \text{by Rel. } \mathcal{R}3 \\
&= (10\ 000)\,(01\ 000)\,(1\ 00) && \text{by Lem. 3.3(ii)} \\
&= (01\ 10)\,(10\ 000)\,(1\ 00) && \text{as just established} \\
&= (01\ 10)\,(11\ 001) && \text{by Rel. } \mathcal{R}3.
\end{aligned}$$

Thus $[(11\ 001), (01\ 10)] = 1$, and then using the definition of (11 001) in (2.9) and the Relations $\mathcal{R}1$, we deduce $[(00\ 011), (10\ 11)] = 1$. We then proceed as in the first half of the proof, using this new equation, and we establish Equation (3.7) when $x = 1$, completing the proof of the lemma. □

Equation (3.8) now follows by conjugating Relation $\mathcal{R}3$ by an appropriate product of elements from $\mathcal{T}_2$ using Lemmas 3.1 and 3.4.

The establishment of Equation (3.9) requires an intermediate observation first. We use Lemma 3.4 to tell us that (11 001) commutes with (01 10), so

$$\begin{aligned}
(01\ 10)\,(11\ 001) &= (11\ 001)\,(01\ 10) \\
&= (10\ 000)\,(1\ 00)\,(01\ 10) && \text{by Rel. } \mathcal{R}3 \\
&= (10\ 000)\,(01\ 000)\,(1\ 00) && \text{by Lem. 3.3(ii)} \\
&= (10\ 000)\,(01\ 000)\,(10\ 000)\,(11\ 001) && \text{by Rel. } \mathcal{R}3.
\end{aligned}$$

Hence $(01\ 000)^{(10\ 000)} = (01\ 10)$. By a similar sequence of calculations we establish

$$\begin{aligned}
(10\ 000)\,(01\ 11) &= (01\ 11)\,(10\ 000) && \text{by Lem. 3.4} \\
&= (01\ 11)\,(1\ 00)\,(11\ 001) && \text{by Rel. } \mathcal{R}3 \\
&= (1\ 00)\,(01\ 001)\,(11\ 001) && \text{by Lem. 3.3(ii)}
\end{aligned}$$

$$= (10\ 000)\,(11\ 001)\,(01\ 001)\,(11\ 001) \quad \text{by Rel. } \mathcal{R}3,$$

so $(01\ 001)^{(11\ 001)} = (01\ 11)$. We now have two equations that, once we conjugate by a product of elements from $\mathcal{T}_2$, yield the general form of Equation (3.9) using Lemma 3.4 and Relations $\mathcal{R}1$.

Relations involving $\mathcal{T}_{12}$, $\mathcal{T}_2$ and $\mathcal{T}_{23}$

The relations that involve swaps from $\mathcal{T}_{12}$, $\mathcal{T}_{23}$ and $\mathcal{T}_2$ and that definitely include at least one swap from each of the first two sets are, for $x, y, z, t \in X$, the following:

$$(xy\ \bar{x}z)^{(x\ \bar{x}\bar{z})} = (\bar{x}z\ \bar{x}\bar{z}y) \tag{3.11}$$

$$(xy\ \bar{x}zt)^{(x\ \bar{x}z)} = (xt\ \bar{x}zy) \tag{3.12}$$

To establish Equation (3.11), first calculate

$$(00\ 1x)^{(1\ 01)\,(1\ 00)} = (00\ 1x)^{(1\ 00)\,(00\ 01)} = (1\ 00x)^{(00\ 01)} = (1\ 01x)$$

using Relation $\mathcal{R}2$ and the definitions in (2.8). Hence

$$(00\ 1x)^{(1\ 01)} = (1\ 01x)^{(1\ 00)} = (00\ 01x)$$

by the definitions in (2.9). This is Equation (3.11) in the case when $x = 1$ and $z = 0$. The general equation then follows by conjugating by a suitable product of elements from $\mathcal{T}_2$ and using Lemmas 3.1 and 3.4.

To establish Equation (3.12), we first deal with the case when $x = 1$ and $z = 0$. We calculate

$$\begin{aligned}(1y\ 00t)^{(1\ 00)} = (01\ 00t)^{(01\ 1y)\,(1\ 00)} &= (01\ 00t)^{(1\ 00)\,(01\ 00y)} \\ &= (01\ 1t)^{(01\ 00y)} = (1t\ 00y)\end{aligned}$$

using the definitions in (2.9), Equation (3.11) (twice) and (3.9). Now conjugate by an appropriate product of elements from $\mathcal{T}_2$, using Lemmas 3.1 and 3.4, to conclude that Equation (3.12) holds.

Intermediate relations

Our current goal at this stage is to complete the establishment of relations involving swaps from $\mathcal{T}_2$ and $\mathcal{T}_{23}$ to supplement Equations (3.7)–(3.9) already obtained. However, in order to achieve this, we need some intermediate relations involving swaps from $\mathcal{T}_3$ of the form $(\kappa 0\ \kappa 1)$ for some $\kappa \in X^2$, specifically for $x, y \in X$ and distinct $\kappa, \lambda \in X^2$:

$$(\kappa\ \lambda x)^{(\kappa\ \lambda\bar{x})} = (\lambda 0\ \lambda 1) \tag{3.13}$$

$$(\kappa\ \lambda x)^{(\lambda 0\ \lambda 1)} = (\kappa\ \lambda\bar{x}) \tag{3.14}$$

$$[(x\ \bar{x}y), (\bar{x}\bar{y}0\ \bar{x}\bar{y}1)] = 1 \tag{3.15}$$

We start by establishing a helpful lemma, analogous to Lemmas 3.3 and 3.4, concerning the swaps of the form $(\kappa 0\ \kappa 1)$.

Lemma 3.5 (i) $(000\ 001)^{(1\ 00)} = (10\ 11)$;

(ii) $(000\ 001)$ *commutes with* $(01\ 10)$, *with* $(01\ 11)$, *and with* $(10\ 11)$.

(iii) *All conjugacy relations of the form* $\sigma^\tau = \upsilon$, *where* $\sigma, \upsilon \in \mathcal{T}_3$ *have the form* $(\kappa 0\ \kappa 1)$ *for some* $\kappa \in X^2$ *and* $\tau \in \mathcal{T}_2$, *can be deduced from Relations* $\mathcal{R}1$–$\mathcal{R}5$.

Proof: (i) This follows using the definitions in (2.8) and (2.10):

$$\begin{aligned}(000\ 001)^{(1\ 00)} = (1\ 000)^{(1\ 001)\,(1\ 00)} &= (1\ 000)^{(1\ 00)\,(00\ 11)} \\ &= (00\ 10)^{(00\ 11)} = (10\ 11).\end{aligned}$$

(ii) First we know, by Lemma 3.4, that $(10\ 11)$ commutes with $(01\ 00x)$ for any $x \in X$. Conjugating by $(1\ 00)$, using part (i) and Lemma 3.3(ii), establishes that $(000\ 001)$ commutes with $(01\ 1x)$.

Then we perform the following calculation:

$$\begin{aligned}(000\ 001)\,(10\ 11) &= (1\ 00)\,(10\ 11)\,(1\ 00)\,(10\ 11) && \text{by part (i)} \\ &= (10\ 000)\,(11\ 001)\,(10\ 11)\,(1\ 00)\,(10\ 11) && \text{by Rel. } \mathcal{R}3 \\ &= (10\ 11)\,(11\ 000)\,(10\ 001)\,(1\ 00)\,(10\ 11) && \text{by Lem. 3.4} \\ &= (10\ 11)\,(1\ 00)\,(10\ 001)\,(11\ 000)\,(10\ 11) && \text{by Eq. (3.12)} \\ &= (10\ 11)\,(1\ 00)\,(10\ 11)\,(11\ 001)\,(10\ 000) && \text{by Lem. 3.4} \\ &= (10\ 11)\,(1\ 00)\,(10\ 11)\,(1\ 00) \\ &\qquad\qquad \text{by Lem. 3.3(i) and Rel. } \mathcal{R}3 \\ &= (10\ 11)\,(000\ 001) && \text{by part (i).}\end{aligned}$$

Thus $(000\ 001)$ commutes with $(10\ 11)$.

(iii) This follows by the same argument as used in Lemma 3.4, noting that if a product ρ of elements from $\mathcal{T}_2$ fixes 00 then it lies in the subgroup generated by $(01\ 10)$, $(01\ 11)$ and $(10\ 11)$ and so commutes with $(000\ 001)$ by part (ii). □

For Equation (3.13), start with the equations $(01\ 1x)^{(01\ 1\bar{x})} = (10\ 11)$ and then conjugate by $(1\ 00)$. Use Lemmas 3.3(ii) and 3.5(i) to conclude

$$(01\ 00x)^{(01\ 00\bar{x})} = (000\ 001)$$

for any $x \in X$. The required equation now follows by conjugating by a product of elements from $\mathcal{T}_2$ and using Lemmas 3.4 and Lemma 3.5(iii).

To establish Equation (3.14), first use Lemma 3.4 to conclude that $(10\ 000)^{(10\ 11)} = (11\ 000)$. Now conjugate by $(1\ 00)$ and use Equation (3.12) and Lemma 3.5(i) to establish the formula $(10\ 000)^{(000\ 001)} = (10\ 001)$. Conjugating by an appropriate product of elements from $\mathcal{T}_2$ and use of Lemmas 3.4 and 3.5(iii) establishes $(\kappa\ \lambda 0)^{(\lambda 0\ \lambda 1)} = (\kappa\ \lambda 1)$, which is sufficient to verify Equation (3.14).

We deduce Equation (3.15) by starting with $[(00\ 01), (10\ 11)] = 1$ and conjugating by $(1\ 00)$ to yield $[(1\ 01), (000\ 001)] = 1$, using Relation $\mathcal{R}2$ and Lemma 3.5(i). Conjugating by an appropriate product of elements from $\mathcal{T}_2$ and using Lemmas 3.1 and 3.5(iii) establishes the required relation.

Remaining relations involving $\mathcal{T}_2$ and $\mathcal{T}_{23}$

We now establish all the relations remaining that involve just swaps from $\mathcal{T}_2$ and $\mathcal{T}_{23}$. Careful analysis shows that the ones we are currently missing are, for $x, y \in X$ and distinct $\kappa, \lambda, \mu, \nu \in X^2$, the following:

$$[(\kappa\ \lambda 0), (\mu\ \lambda 1)] = 1 \tag{3.16}$$

$$[(\kappa\ \lambda x), (\mu\ \nu y)] = 1 \tag{3.17}$$

Note that in establishing these equations we are essentially establishing that swaps from $\mathcal{T}_{23}$ that have disjoint support (or, more accurately, corresponding to maps in V with disjoint support) commute.

Equation (3.16) simply follows from Lemma 3.3(i) using Lemma 3.4.

Use of Lemma 3.4 deduces $[(\kappa\ \lambda 0), (\mu\ \nu 1)] = [(\kappa\ \lambda 1), (\mu\ \nu 1)] = 1$ for all distinct $\kappa, \lambda, \mu, \nu \in X^2$ from Relations $\mathcal{R}4$. Consequently, in the case of Equation (3.17), it remains to verify the relation in the case when $x = y = 0$. First observe that

$$(11\ 011)^{(000\ 001)} = (11\ 011)^{(10\ 000)\,(10\ 001)\,(10\ 000)} = (11\ 011)$$

by use of Equation (3.13) and then repeated use of the cases of Equation (3.17) that we already have; that is, $[(11\ 011), (000\ 001)] = 1$. Now

$$(000\ 001)^{(10\ 010)} = (000\ 001)^{(1\ 01)\,(11\ 011)} \qquad \text{by Eq. (3.8)}$$

$$
\begin{aligned}
&= (000\ 001)^{(11\ 011)} && \text{by Eq. (3.15)}\\
&= (000\ 011) && \text{as just established.}
\end{aligned}
$$

Thus, (10 010) and (000 001) commute. This means that when we conjugate the relation $[(10\ 010), (11\ 001)] = 1$, which is an instance of Equation (3.17) that we already know, by the swap (000 001), we obtain

$$[(10\ 010), (11\ 000)] = 1,$$

with use of Equation (3.14). We now make use of Lemma 3.4 in our usual way to establish the missing case of Equation (3.17), namely when $x = y = 0$.

Relations involving $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_2$ and $\mathcal{T}_{23}$

We now establish all the relations we require that involve swaps from $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_2$ and $\mathcal{T}_{23}$. In view of the relations that we have already obtained, we can assume that at least one swap from $\mathcal{T}_{13}$ and at least one from $\mathcal{T}_{23}$ occur within our relation. The relations we need to establish are therefore, for $x, y, z \in X$, the following:

$$[(x\ \bar{x}yz), (\bar{x}\bar{y}\ \bar{x}y\bar{z})] = 1 \tag{3.18}$$

$$(x\ \bar{x}y)^{(x\ \bar{x}\bar{y}z)} = (\bar{x}y\ \bar{x}\bar{y}z) \tag{3.19}$$

$$(x\ \bar{x}y)^{(\bar{x}y\ \bar{x}\bar{y}z)} = (x\ \bar{x}\bar{y}z) \tag{3.20}$$

$$(x\ \bar{x}yz)^{(x\ \bar{x}\bar{y})} = (\bar{x}\bar{y}\ \bar{x}yz) \tag{3.21}$$

For Equation (3.18), take the equation $[(00\ 10), (01\ 11)] = 1$, conjugate by (1 00) and use the definition in (2.8) and Lemma 3.3(ii) to conclude $[(1\ 000), (01\ 001)] = 1$. The required equation then follows, as usual, by use of Lemmas 3.2 and 3.4.

For Equation (3.19), we calculate as follows:

$$
\begin{aligned}
(1\ 00)^{(1\ 01z)} &= (1\ 00)^{(00\ 01)\,(1\ 00z)\,(00\ 01)} && \text{by the definition in (2.8)}\\
&= (1\ 01)^{(1\ 00z)\,(00\ 01)} && \text{by the definition in (2.7)}\\
&= (1\ 01)^{(1\ 00)\,(00\ 1z)\,(1\ 00)\,(00\ 01)} && \text{by the definition in (2.8)}\\
&= (01\ 1z)^{(1\ 00)\,(00\ 01)} && \text{using Rels. } \mathcal{R}2 \text{ and } \mathcal{R}1\\
&= (01\ 00z)^{(00\ 01)} && \text{by Eq. (3.11)}\\
&= (00\ 01z) && \text{by Lem. 3.4.}
\end{aligned}
$$

The required equation now follows using Lemmas 3.1, 3.2 and 3.4.

For Equation (3.20), our main calculation is

$$(1\ 00)^{(00\ 01z)} = (1\ 00)^{(1\ 00)\,(1\ 01z)\,(1\ 00)} = (1\ 00)^{(1\ 01z)\,(1\ 00)} \\ = (00\ 01z)^{(1\ 00)} = (1\ 01z),$$

obtained by exploiting the definition of $(00\ 01z)$ in (2.9) and Equation (3.19) above. The required equation again follows by Lemmas 3.1, 3.2 and 3.4.

Equation (3.21) follows from the definition of $(1\ 01z)$ as in Equation (2.8) with use of Lemmas 3.1, 3.2 and 3.4.

First relations involving $\mathcal{T}_2$ and $\mathcal{T}_3$ only

We now introduce swaps from $\mathcal{T}_3$ into the relations we verify. The first step is to establish that swaps from $\mathcal{T}_3$ behave well when we conjugate by one from $\mathcal{T}_2$ and then our other split relation. All other relations involving just swaps from $\mathcal{T}_2$ and $\mathcal{T}_3$ will have to wait until we have established some more intermediate relations. Accordingly, we start with the following for $x, y \in X$, distinct $\kappa, \lambda \in X^2$ and $\tau \in \mathcal{T}_2$ for which Equation (3.22) is defined:

$$(\kappa x\ \lambda y)^\tau = ((\kappa x)\cdot\tau\ \ (\lambda y)\cdot\tau) \tag{3.22}$$

$$(\kappa\ \lambda) = (\kappa 0\ \kappa 1)\,(\lambda 0\ \lambda 1) \tag{3.23}$$

As with those from $\mathcal{T}_{23}$, we begin with a lemma to manipulate the basic $\mathcal{T}_3$-swaps from the definition in (2.10). In particular, we will establish Equation (3.22) as part (iv) in the lemma.

Lemma 3.6 (i) $(000\ 01x)^{(1\ 00)} = (10\ 01x)$, *for any* $x \in X$;

(ii) $(001\ 011)^{(1\ 00)} = (11\ 011)$;

(iii) $(000\ 010)$, $(000\ 011)$ *and* $(001\ 011)$ *each commute with* $(10\ 11)$.

(iv) *All conjugacy relations of the form* $\sigma^\tau = \upsilon$, *where* $\sigma, \upsilon \in \mathcal{T}_3$ *and* $\tau \in \mathcal{T}_2$, *can be deduced from Relations* $\mathcal{R}1$–$\mathcal{R}5$.

(v) $(001\ 010)^{(1\ 00)} = (11\ 010)$;

Proof: (i) We calculate as follows:

$$\begin{aligned}(000\ 01x)^{(1\ 00)} &= (1\ 000)^{(1\ 01x)\,(1\ 00)} && \text{by the definition in (2.10)}\\ &= (1\ 000)^{(1\ 00)\,(00\ 01x)} && \text{by Eq. (3.21)}\\ &= (00\ 10)^{(00\ 01x)} && \text{by the definition in (2.8)}\end{aligned}$$

$$= (10\ 01x) \qquad \text{by Eq. (3.9).}$$

Part (ii) is established in exactly the same way.

(iii) We perform the following calculations:

$$\begin{aligned}
&(000\ 01x)^{(10\ 11)\,(1\ 00)} \\
&\quad = (10\ 01x)^{(1\ 00)\,(10\ 11)\,(1\ 00)} && \text{by part (i)} \\
&\quad = (10\ 01x)^{(10\ 000)\,(11\ 001)\,(10\ 11)\,(1\ 00)} && \text{by Rel. } \mathcal{R}3 \\
&\quad = (11\ 01x)^{(11\ 000)\,(10\ 001)\,(1\ 00)} && \text{by Lem. 3.4} \\
&\quad = (11\ 01x)^{(11\ 000)\,(1\ 00)} && \text{by Eq. (3.17) twice} \\
&\quad = (11\ 01x)^{(11\ 000)\,(10\ 000)\,(11\ 001)} && \text{by Rel. } \mathcal{R}3 \\
&\quad = (11\ 01x)^{(10\ 000)\,(10\ 11)\,(11\ 001)} && \text{by Eq. (3.9)} \\
&\quad = (11\ 01x)^{(10\ 11)\,(11\ 001)} && \text{by Eq. (3.17)} \\
&\quad = (10\ 01x)^{(11\ 001)} && \text{by Lem. 3.4} \\
&\quad = (10\ 01x) && \text{by Eq. (3.17).}
\end{aligned}$$

Thus $(000\ 01x)^{(10\ 11)} = (10\ 01x)^{(1\ 00)} = (000\ 01x)$, again by part (i). A similar argument, using (ii), shows that (001 011) commutes with (10 11).

(iv) When σ (and υ) has the form $(\kappa 0\ \kappa 1)$, this result was established in Lemma 3.5(iii). All remaining swaps in $\mathcal{T}_3$ are defined by conjugating one of (000 010), (000 011) or (001 011) by some product of elements from $\mathcal{T}_2$. The result then follows by the same argument, but now relying upon part (iii).

(v) appears to be similar to the first two parts of the lemma, but actually requires a different argument, based on what we have just established:

$$\begin{aligned}
(001\ 010)^{(1\ 00)} = (000\ 011)^{(00\ 01)\,(1\ 00)} &= (000\ 011)^{(1\ 00)\,(1\ 01)} \\
&= (10\ 011)^{(1\ 01)} = (11\ 010),
\end{aligned}$$

by part (iv), Relation $\mathcal{R}2$, part (i) and Equation (3.12). □

Equation (3.23) is now established by taking the equation $(1\ 01) = (10\ 010)\,(11\ 011)$, which is an instance of Equation (3.8), and conjugating by (1 00) to yield

$$(00\ 01) = (000\ 010)\,(001\ 011),$$

using Relation $\mathcal{R}2$ and Lemma 3.6(i) and (ii). The required relation then follows by conjugating by a product of elements from $\mathcal{T}_2$ and using the Relations $\mathcal{R}1$ and Lemma 3.6(iv).

Further intermediate relations involving $\mathcal{T}_{23}$ and $\mathcal{T}_3$

Our principal direction of travel at this stage is to complete the verification of those relations involving swaps from $\mathcal{T}_2$ and $\mathcal{T}_3$ to supplement those in Equations (3.22) and (3.23). However, to achieve this we need some intermediate results making use of swaps from $\mathcal{T}_{23}$, specifically, for $x \in X$, distinct $\kappa, \lambda, \mu \in X^2$ and distinct $\alpha, \beta \in X^3$ for which $\kappa \perp \alpha, \beta$, the following:

$$(\kappa\ \alpha)^{(\kappa\ \beta)} = (\alpha\ \beta) \tag{3.24}$$

$$(\kappa\ \alpha)^{(\alpha\ \beta)} = (\kappa\ \beta) \tag{3.25}$$

$$[(\kappa\ \lambda x), (\mu 0\ \mu 1)] = 1 \tag{3.26}$$

For Equation (3.24), first note that Equation (3.13) deals with the case when α and β share the same two-letter prefix. For the remaining cases, first observe

$$\begin{aligned}
&(10\ 000)^{(10\ 010)\,(1\ 010)} \\
&\qquad = (10\ 000)^{(10\ 010)\,(11\ 011)\,(1\ 010)} && \text{by Eqs. (3.16) and (3.17)} \\
&\qquad = (10\ 000)^{(1\ 01)\,(1\ 010)} && \text{by Eq. (3.8)} \\
&\qquad = (10\ 000)^{(01\ 10)\,(1\ 01)} && \text{by Eq. (3.6)} \\
&\qquad = (01\ 000)^{(1\ 01)} && \text{by Lem. 3.4} \\
&\qquad = (1\ 000) && \text{by Eq. (3.21).}
\end{aligned}$$

Hence $(10\ 000)^{(10\ 010)} = (1\ 000)^{(1\ 010)} = (000\ 010)$, using the definition in Equation (2.10). A similar argument establishes $(11\ 000)^{(11\ 011)} = (1\ 000)^{(1\ 011)} = (000\ 011)$. Equally we apply a variant of the argument to obtain further equations:

$$\begin{aligned}
(10\ 011)^{(10\ 000)\,(1\ 011)} &= (10\ 011)^{(1\ 00)\,(1\ 011)} && \text{arguing as before} \\
&= (10\ 011)^{(00\ 011)\,(1\ 00)} && \text{by Eq. (3.21)} \\
&= (00\ 10)^{(1\ 00)} && \text{by Eq. (3.9)} \\
&= (1\ 000) && \text{by the definition in (2.8).}
\end{aligned}$$

Thus we obtain $(10\ 011)^{(10\ 000)} = (1\ 000)^{(1\ 011)} = (000\ 011)$. Similarly we determine the equation $(11\ 011)^{(11\ 001)} = (001\ 011)$. Thus given any choice of $x, y \in X$, we have obtained one example of a relation

$$(\kappa\ \lambda x)^{(\kappa\ \mu y)} = (\lambda x\ \mu y)$$

(for some choice of distinct κ, λ and μ.) We can then obtain *all* examples by use of Lemmas 3.4 and 3.6(iv). This completes the establishment of Equation (3.24).

Equation (3.14) is already Equation (3.25) in the case when α and β share the same two-letter prefix. For the remaining cases, use Equation (3.24) to tell us $(1x\ 000)^{(1x\ 01y)} = (000\ 01y)$ for any $x, y \in X$. Conjugate this equation by $(1\ 00)$ and use Equation (3.12) and parts of Lemma 3.6 to establish $(10\ 00x)^{(00x\ 01y)} = (10\ 01y)$. All cases of Equation 3.25 now follow by conjugating by an appropriate product of elements from $\mathcal{T}_2$.

For Equation (3.26), start with the fact that $(10\ 11)$ commutes with $(000\ 01x)$ for any $x \in X$ (Lemma 3.6(iii)). Conjugate by $(1\ 00)$ and use Lemmas 3.5(i) and 3.6(i) to conclude $[(000\ 001), (01\ 01x)] = 1$. Then conjugating by a product of swaps from $\mathcal{T}_2$ establishes the required equation.

Remaining relations involving only $\mathcal{T}_2$, $\mathcal{T}_{23}$ and $\mathcal{T}_3$

Having established the intermediate relations, we can now establish all remaining relations involving swaps only from $\mathcal{T}_2$ and $\mathcal{T}_3$. We obtain those also involving swaps from $\mathcal{T}_{23}$ at the same time. When one analyses the relations required, we find that they are, for $x, y \in X$, distinct $\kappa, \lambda \in X^2$ and distinct $\alpha, \beta, \gamma, \delta \in X^3$, the following:

$$[(\kappa 0\ \kappa 1), (\lambda x\ \mu y)] = 1 \tag{3.27}$$

$$(\kappa x\ \lambda y)^{(\kappa 0\ \kappa 1)} = (\kappa \bar{x}\ \lambda y) \tag{3.28}$$

$$[(\kappa\ \alpha), (\beta\ \gamma)] = 1 \qquad \text{for } \kappa \perp \alpha, \beta, \gamma \tag{3.29}$$

$$(\alpha\ \beta)^{(\alpha\ \gamma)} = (\beta\ \gamma) \tag{3.30}$$

$$[(\alpha\ \beta), (\gamma\ \delta)] = 1 \tag{3.31}$$

We establish Equation (3.27) by choosing ν to be the element in $X^2 \setminus \{\kappa, \lambda, \mu\}$, using Equation (3.24) to tell us $(\lambda x\ \mu y) = (\nu\ \lambda x)^{(\nu\ \mu y)}$ and then, as Equation (3.26) says that both $(\nu\ \lambda x)$ and $(\nu\ \mu y)$ commute with $(\kappa 0\ \kappa 1)$, we establish Equation (3.27).

By Lemma 3.4, $(1x\ 01y)^{(10\ 11)} = (1\bar{x}\ 01y)$ for any $x, y \in X$. Conjugate by $(1\ 00)$ and use Lemma 3.5(i) and parts of Lemma 3.6 to conclude $(00x\ 01y)^{(000\ 001)} = (00\bar{x}\ 01y)$. Equation (3.28) then follows.

Our final equation involving swaps from $\mathcal{T}_{23}$ and $\mathcal{T}_3$ is Equation (3.29). We need to establish this in a number of stages. If $x \in X$, we know that $(10\ 11)$ commutes with the swap $(000\ 01x)$ by Lemma 3.6(iii). If we conjugate by $(1\ 00)$ and use Lemma 3.5(i) and Lemma 3.6(i), we conclude $[(10\ 01x), (000\ 001)] = 1$. We then deduce Equation (3.29) in the case when β and γ share the same two-letter prefix in our now established manner.

The second case of the equation is when α and β share their two-letter prefix. Equation (3.17) tells us that $[(1x\ 000), (1\bar{x}\ 01y)] = 1$ for any $x, y \in X$. Now conjugate by (1 00) and use Equation (3.12) and parts of Lemma 3.6 to conclude $[(10\ 00x), (00\bar{x}\ 01y)] = 1$. Equation (3.29) when α and β share their two-letter prefix then follows.

It remains to deal with the case when α, β and γ have distinct two-letter prefixes. One particular case is our Relation $\mathcal{R}5$: $[(10\ 110), (000\ 010)] = 1$. Conjugating by (110 111), using Equation (3.14) and (3.27), we deduce $[(10\ 111), (000\ 010)] = 1$. Similarly, conjugating what we now have, $[(10\ 11x), (000\ 010)] = 1$ for any $x \in X$, by (000 001) and use (3.26) and (3.28) to now conclude that $[(10\ 11x), (00y\ 010)] = 1$ for all $x, y \in X$. Finally use the same argument, conjugating by (010 011), to conclude $[(10\ 11x), (00y\ 01z)] = 1$ for all $x, y, z \in X$. The remaining case of Equation (3.29) now follows.

One case of Equation (3.30), namely when α and γ share the same two-letter prefix, has already been established as Equation (3.28). For the case when α and β share the same two-letter prefix, start with the equation $(10\ 11)^{(1x\ 01y)} = (1\bar{x}\ 01y)$, for $x, y \in X$, as given by Equation (3.9). Conjugate by (1 00) and use Lemma 3.5(i) and parts from Lemma 3.6 to conclude $(000\ 001)^{(00x\ 01y)} = (00\bar{x}\ 01y)$. From this the general formula $(\kappa 0\ \kappa 1)^{(\kappa x\ \lambda y)} = (\kappa\bar{x}\ \lambda y)$ follows for distinct $\kappa, \lambda \in X^2$.

For the case when α, β and γ have distinct two-letter prefixes, say $\alpha = \kappa x$, $\beta = \lambda y$ and $\gamma = \mu z$, choose ν to be the other element of X^2. Then

$$\begin{aligned}(\alpha\ \beta)^{(\alpha\ \gamma)} = (\kappa x\ \lambda y)^{(\kappa x\ \mu z)} &= (\kappa x\ \lambda y)^{(\nu\ \kappa x)\,(\nu\ \mu z)\,(\nu\ \kappa x)} \\ &= (\nu\ \lambda y)^{(\nu\ \mu z)\,(\nu\ \kappa x)} \\ &= (\lambda y\ \mu z)^{(\nu\ \kappa x)} = (\lambda y\ \mu z) = (\beta\ \gamma),\end{aligned}$$

by Equations (3.24) (used three times) and (3.29).

Part of Equation (3.31) has already been established as Equation (3.27), but we shall deal with our required relation in full generality. Indeed, first assume that α, β, γ and δ have between them at most three distinct two-letter prefixes. Let $\nu \in X^2$ be different from those two-letter prefixes. Then write $(\gamma\ \delta)$ as $(\nu\ \gamma)\,(\nu\ \delta)\,(\nu\ \gamma)$, by Equation (3.24), and observe this commutes with $(\alpha\ \beta)$ using Equation (3.29).

The case when α, β, γ and δ have four distinct two-letter prefixes can then be deduced as follows. Suppose as $\alpha = \kappa x$ for some $\kappa \in X^2$ and $x \in X$. By the previous case, $(\alpha\ \beta)$ commutes with both $(\kappa\bar{x}\ \gamma)$ and $(\kappa\bar{x}\ \delta)$, and hence it also commutes with $(\gamma\ \delta) = (\kappa\bar{x}\ \gamma)^{(\kappa\bar{x}\ \delta)}$, using Equation (3.30).

Relations involving $\mathcal{T}_3$, at least one of $\mathcal{T}_{12}$, $\mathcal{T}_{13}$ and $\mathcal{T}_{23}$, and possibly $\mathcal{T}_2$

We now establish the final batch of relations for this section. These involve swaps from $\mathcal{T}_3$ and at least one of $\mathcal{T}_{12}$, $\mathcal{T}_{13}$ and $\mathcal{T}_{23}$, and are the following for $x, y, z, t \in X$ and distinct $\alpha, \beta, \gamma \in X^3$ satisfying $x \not\prec \alpha, \beta, \gamma$:

$$(\bar{x}y0\ \bar{x}y1)^{(x\ \bar{x}y)} = (x0\ x1) \tag{3.32}$$

$$[(x\ \alpha), (\beta\ \gamma)] = 1 \tag{3.33}$$

$$(x\ \alpha)^{(\alpha\ \beta)} = (x\ \beta) \tag{3.34}$$

$$(\alpha\ \beta)^{(x\ \alpha)} = (x\ \beta) \tag{3.35}$$

$$(\bar{x}yz\ \bar{x}\bar{y}t)^{(x\ \bar{x}y)} = (xz\ \bar{x}\bar{y}t) \tag{3.36}$$

Equation (3.32) follows from Lemma 3.5(i) by our standard $\mathcal{T}_2$-conjugation argument.

Equation (3.33) follows by taking the relation $[(00\ 1y), (1\bar{y}\ 01z)] = 1$ and the relation $[(00\ 1y), (010\ 011)] = 1$, which hold by Lemmas 3.4 and 3.6(iv) respectively, and then conjugating by $(1\ 00)$ and proceeding as in previous arguments to conclude that $[(x\ \kappa y), (\kappa\bar{y}\ \lambda z)] = 1$ and $[(x\ \kappa y), (\lambda 0\ \lambda 1)] = 1$ for any $x, y, z \in X$ and any distinct $\kappa, \lambda \in X^2$ with $x \not\prec \kappa, \lambda$.

To establish Equation (3.34), conjugate the already established equations $(00\ 1y)^{(10\ 11)} = (00\ 1\bar{y})$ and $(00\ 1y)^{(1y\ 01z)} = (00\ 01z)$, for $y, z \in X$, by $(1\ 00)$. This yields $(1\ 00y)^{(000\ 001)} = (1\ 00\bar{y})$ and $(1\ 00y)^{(00y\ 01z)} = (1\ 01z)$, which now yields the two forms of Equation (3.34): $(x\ \kappa y)^{(\kappa 0\ \kappa 1)} = (x\ \kappa\bar{y})$ and $(x\ \kappa y)^{(\kappa y\ \lambda z)} = (x\ \lambda z)$ for any $x, y, z \in X$ and distinct $\kappa, \lambda \in X^2$ with $x \not\prec \kappa, \lambda$.

For Equation (3.35), we shall show $(000\ 001)^{(1\ 00x)} = (1\ 00\bar{x})$ and $(00x\ 01y)^{(1\ 01y)} = (1\ 00x)$ for any $x, y \in X$. Four of these occurrences are found in the definitions in (2.10), while the other two are deduced by conjugating $(10\ 11)^{(00\ 10)} = (00\ 11)$ and $(11\ 010)^{(00\ 010)} = (00\ 11)$ by $(1\ 00)$. The required equation then follows.

Finally, for Equation (3.36), from Lemma 3.6: $(00z\ 01t)^{(1\ 00)} = (1z\ 01t)$ for any $z, t \in X$. Then as in the previous equations we conjugate by products of elements from $\mathcal{T}_2$.

We have now established all required relations of the form $\sigma^\tau = \upsilon$ where σ, τ and υ come from the sets $\mathcal{T}_2$, $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_{23}$ and $\mathcal{T}_3$. This is the first stage in establishing the existence of the homomorphism ψ in Diagram (2.12) and, in particular, the verification of Theorem 1.2.

4 Verifying the Cannon–Floyd–Parry relations

In this section we describe how to verify that all the relations that hold in R. Thompson's group V can be deduced from those assumed in Relations $\mathcal{R}1$–$\mathcal{R}5$. We shall rely upon the work in the previous section. One might wonder whether it is possible to proceed more directly to show, for example, that all relations holding in V can be deduced from the infinitely many in the presentation in Theorem 1.1. It is a consequence of our results that this can be done, but our own attempt to do so resulted in overly long arguments replicating those already found in Section 6 of [12]. We have chosen the more direct method of verifying the finite set of relations known already to define V.

In their paper (see [12, Lemma 6.1]), Cannon–Floyd–Parry provide the following presentation for V. It has generators A, B, C and π_0 and relations

CFP1.	$[AB^{-1}, X_2] = 1;$	CFP8.	$\pi_1\pi_3 = \pi_3\pi_1;$
CFP2.	$[AB^{-1}, X_3] = 1;$	CFP9.	$(\pi_2\pi_1)^3 = 1;$
CFP3.	$C_1 = BC_2;$	CFP10.	$X_3\pi_1 = \pi_1 X_3;$
CFP4.	$C_2X_2 = BC_3;$	CFP11.	$\pi_1 X_2 = B\pi_2\pi_1;$
CFP5.	$C_1A = C_2^2;$	CFP12.	$\pi_2 B = B\pi_3;$
CFP6.	$C_1^3 = 1;$	CFP13.	$\pi_1 C_3 = C_3\pi_2;$
CFP7.	$\pi_1^2 = 1;$	CFP14.	$(\pi_1 C_2)^3 = 1;$

where the elements appearing here are defined by the following formulae $C_n = A^{-n+1}CB^{n-1}$, $X_n = A^{-n+1}BA^{n-1}$ both for $n \geqslant 1$, $\pi_1 = C_2^{-1}\pi_0 C_2$ and $\pi_n = A^{-n+1}\pi_1 A^{n-1}$ for $n \geqslant 2$.

Recall from Section 2 that P_3 is the group presented by generators $a = (00\ 01)$, $b = (01\ 10)\,(01\ 11)$ and $c = (1\ 00)$ subject to relations $\mathcal{R}1$–$\mathcal{R}5$. Define four new elements of P_3 by

$$\bar{A} = (0\ 1)\,(0\ 10)\,(10\ 11); \qquad \bar{B} = (10\ 11)\,(10\ 110)\,(110\ 111);$$
$$\bar{C} = (10\ 11)\,(0\ 10); \qquad \bar{\pi}_0 = (0\ 10),$$

and then new elements $\bar{C}_n$, $\bar{X}_n$ and $\bar{\pi}_n$ for $n \geqslant 1$ defined in terms of these four by same formulae used when defining the relations for V.

It is a consequence of the relations involving swaps from the sets $\mathcal{T}_2$, $\mathcal{T}_{12}$, $\mathcal{T}_{13}$, $\mathcal{T}_{23}$ and $\mathcal{T}_3$ established in the previous section (i.e., Equations (3.1)–(3.36)) that the elements $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\pi}_0$ satisfy CFP1–CFP14. To verify this is a sequence of calculations. Below we present the verification of CFP1 for these elements. For the entertainment of the reader, Relation CFP2 required the longest calculation whilst the others are more straightforward. The following formulae are useful for this work.

Lemma 4.1 *The following formulae hold in G:*

(i) $\bar{A}\bar{B}^{-1} = (00\ 01)\,(01\ 10)\,(0\ 10)$;

(ii) $\bar{X}_2 = (0\ 11)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 11)$;

(iii) $\bar{X}_3 = (0\ 111)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 111)$;

(iv) $\bar{C}_2 = (0\ 10)\,(0\ 111)\,(110\ 111)$;

(v) $\bar{C}_3 = (0\ 110)\,(10\ 111)\,(0\ 100)\,(0\ 101)\,(10\ 110)\,(110\ 111)$;

(vi) $\bar{\pi}_1 = (10\ 110)$;

(vii) $\bar{\pi}_2 = (0\ 11)\,(00\ 010)\,(0\ 11)$;

(viii) $\bar{\pi}_3 = (0\ 111)\,(00\ 010)\,(0\ 111)$.

PROOF: We verify the two formulae, (i) and (ii), required to verify CFP1. Below, we principally rely upon the conjugacy relations, although a split relation is applied in one step. The other formulae listed are established similarly.

(i) We calculate

$$\begin{aligned}\bar{A}\bar{B}^{-1} &= (0\ 1)\,(0\ 10)\,(10\ 11)\cdot(110\ 111)\,(10\ 110)\,(10\ 11)\\ &= (0\ 1)\,(0\ 10)\,(100\ 101)\,(11\ 100)\\ &= (0\ 1)\,(00\ 01)\,(00\ 11)\,(0\ 10)\\ &= (00\ 10)\,(01\ 11)\,(00\ 01)\,(00\ 11)\,(0\ 10)\\ &= (00\ 01)\,(01\ 10)\,(0\ 10).\end{aligned}$$

(ii) We start with the definition of $\bar{A}$ and $\bar{B}$:

$$\begin{aligned}\bar{X}_2 &= \bar{A}^{-1}\bar{B}\bar{A}\\ &= (10\ 11)\,(0\ 10)\,(0\ 1)\cdot(10\ 11)\,(10\ 110)\,(110\ 111)\cdot(0\ 1)\,(0\ 10)\,(10\ 11)\\ &= (10\ 11)\,(0\ 10)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 10)\,(10\ 11)\\ &= (0\ 11)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 11).\end{aligned}$$

□

We now verify that the elements $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\pi}_0$ of our P_3 satisfy the relation CFP1:

$$\begin{aligned}[\bar{A}\bar{B}^{-1}, \bar{X}_2] = {} & (0\ 10)\,(01\ 10)\,(00\ 01)\cdot(0\ 11)\,(010\ 011)\,(00\ 010)\,(00\ 01)\,(0\ 11)\\ & \cdot(00\ 01)\,(01\ 10)\,(0\ 10)\cdot(0\ 11)\,(00\ 01)\,(00\ 010)\,(010\ 011)\\ & \cdot(0\ 11)\end{aligned}$$

$$
\begin{aligned}
&= (0\ 11)\,(10\ 11)\,(10\ 111)\,(110\ 111)\,(010\ 011)\,(00\ 010)\,(00\ 01)\\
&\quad\cdot (0\ 11)\,(00\ 01)\,(01\ 10)\,(0\ 10)\,(0\ 11)\,(00\ 01)\,(00\ 010)\\
&\quad\cdot (010\ 011)\,(0\ 11)\\
&= (0\ 11)\,(10\ 11)\,(10\ 111)\,(110\ 111)\,(010\ 011)\,(00\ 010)\,(00\ 01)\\
&\quad\cdot (110\ 111)\,(10\ 111)\,(10\ 11)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 11)\\
&= (0\ 11)\,(11\ 101)\,(100\ 101)\,(010\ 011)\,(00\ 010)\,(00\ 01)\,(100\ 101)\\
&\quad\cdot (11\ 101)\,(00\ 01)\,(00\ 010)\,(010\ 011)\,(0\ 11)\\
&= (0\ 11)\,(11\ 101)\,(100\ 101)\,(010\ 011)\,(00\ 010)\,(100\ 101)\\
&\quad\cdot (11\ 101)\,(00\ 010)\,(010\ 011)\,(0\ 11)\\
&= (0\ 11)\,(11\ 101)\,(010\ 011)\,(00\ 010)\,(11\ 101)\,(00\ 010)\,(010\ 011)\\
&\quad\cdot (0\ 11)\\
&= (0\ 11)\,(11\ 101)\,(010\ 011)\,(11\ 101)\,(010\ 011)\,(0\ 11)\\
&= 1
\end{aligned}
$$

(by first collecting (0 11) to the left, then conjugating some swaps by (0 11), some by (10 11), some by (00 01), some by (100 101), then single swaps by (00 010) and by (11 101), and finally exploiting the fact our swaps have order 2).

Once we have established the fourteen relations CFP1–CFP14, it follows that there is indeed a surjective homomorphism $\psi\colon V \to \langle \bar{A}, \bar{B}, \bar{C}, \bar{\pi}_0 \rangle$, as indicated in the Diagram (2.12) and used in the proof of Theorems 1.1 and 1.2.

5 Final details for the proofs

In this section, we complete the technical details relied upon in the proofs given in Section 2.

We first need to show that the subgroup $\langle \bar{A}, \bar{B}, \bar{C}, \bar{\pi}_0 \rangle$ coincides with the group P_3. Indeed observe this subgroup contains all the following elements:

$$
\begin{aligned}
\bar{\pi}_0 &= (1\ 00) = c\\
\bar{C}\bar{\pi}_0 &= (10\ 11)\\
(\bar{C}\bar{\pi}_0)^{\bar{A}\bar{C}} &= (10\ 11)^{(0\ 1)} = (00\ 01) = a\\
\bar{\pi}_0^{\bar{B}^{-1}\bar{C}} &= (0\ 10)^{(110\ 111)\,(10\ 110)\,(0\ 10)} = (10\ 110)\\
\bar{\pi}_0^{\bar{B}^{-1}\bar{C}}\bar{C}\bar{\pi}_0\bar{B} &= (110\ 111)
\end{aligned}
$$

and

$$
(10\ 110)^{(110\ 111)\,(10\ 11)\,(0\ 10)\,(10\ 11)} = (01\ 10).
$$

The Relations $\mathcal{R}1$ ensure that $b \in \langle (00\ 01), (01\ 10), (10\ 11)\rangle$, and so we now conclude that this subgroup generated by $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\pi}_0$ is the whole group P_3. This establishes the following result and means that we have now completed all the details required for the proofs of Theorem 1.1 and 1.2.

Proposition 5.1 *The elements $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{\pi}_0$, defined earlier, generate the group P_3.* □

Finally, we deduce a 2-generator presentation for V from Theorem 1.2. As noted in Section 2, the following is the intermediate step used to establish Theorem 1.3. Although the latter depends on computer calculation, the following is established by purely theoretical methods in line with our proofs of Theorems 1.1 and 1.2.

Corollary 5.2 *R. Thompson's group V has a finite presentation with two generators and nine relations.*

Proof: We work in the group P_3. Define u and v to be the elements

$$u = (00\ 01)\,(10\ 110)\,(10\ 111) \qquad \text{and} \qquad v = b = (01\ 10\ ).$$

(When interpreted via the isomorphism from P_3 to V, obtained by composing the maps specified in the diagram (2.12), these two elements correspond to the element of V given by tree-pairs in Figure 3 in the Introduction.)

If we rely upon the relations that hold in P_3 (i.e., simply calculating within R. Thompson's group V, as, by this stage, we have completed all steps in establishing Theorem 1.2), then we can obtain a formula for u as a product of the generators a, b and c, for example,

$$u = w(a,b,c) = a(a^b a^{b^{-1}})^{caca^b a^{b^{-1}a}}$$

and the following formulae:

$$\begin{aligned} a = (00\ 01) &= u^3, \\ (10\ 000) &= (u^3)^{vu^{-2}vu^3}, \\ (11\ 001) &= (u^3)^{vu^{-1}vu^3v} \end{aligned}$$

Therefore $c = \gamma(u,v) = (u^3)^{vu^{-2}vu^3}\,(u^3)^{vu^{-1}vu^3v}$. If $r_1(a,b,c)$, $r_2(a,b,c)$, $\ldots$, $r_8(a,b,c)$ denote the words in a, b and c that define our relations (see the Equations (2.4) in Section 2, or alternatively, Equations (2.3)), then applying Tietze transformations shows that

$$V \cong \big\langle\, u, v \;\big|\; r_1\big(u^3, v, \gamma(u,v)\big) = r_2\big(u^3, v, \gamma(u,v)\big) = \cdots = r_8\big(u^3, v, \gamma(u,v)\big) = 1, \quad u = w\big(u^3, v, \gamma(u,v)\big) \,\big\rangle\,.$$

This establishes the corollary. □

As we described in Section 2, Theorem 1.3 is deduced from this presentation. We produce the formulae $r_i\big(u^3, v, \gamma(u,v)\big)$, for $i = 1, 2, \ldots, 8$, by taking r_i to be the formulae in (2.4). We then apply the process of producing equivalent relations for this 2-generator presentation by repeatedly using the Knuth–Bendix Algorithm as described in Section 2. It is during this process that we discover that two of the relations can be omitted since the relevant word reduces to the identity. The remaining seven relations reduce to those listed in Theorem 1.3.

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