CRANFIELD UNIVERSITY

SAMEH EL SAID ABDEL AZIZ ASKAR

Symbolic Approaches And Artificial Intelligence Algorithms For Solving Multi-Objective Optimisation Problems

SCHOOL OF APPLIED SCIENCES

PhD

This thesis is submitted in partial fulfilment of the requirements

for the degree of Doctor of Philosophy

Academic: Year 2010-2011

Supervisor: Dr Ashutosh Tiwari

April 2011

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Abstract

Problems that have more than one objective function are of great importance in engineering sciences and many other disciplines. This class of problems are known as multi-objective optimisation problems (or multicriteria). The difficulty here lies in the conflict between the various objective functions. Due to this conflict, one cannot find a single ideal solution which simultaneously satisfies all the objectives. But instead one can find the set of Pareto-optimal solutions (Pareto-optimal set) and consequently the Pareto-optimal front is established. Finding these solutions plays an important role in multi-objective optimisation problems and mathematically the problem is considered to be solved when the Pareto-optimal set, i.e. the set of all compromise solutions is found. The Pareto-optimal set may contain information that can help the designer make a decision and thus arrive at better trade-off solutions. The aim of this research is to develop new multi-objective optimisation symbolic algorithms capable of detecting relationship(s) among decision variables that can be used for constructing the analytical formula of Pareto-optimal front based on the extension of the current optimality conditions.

A literature survey of theoretical and evolutionary computation techniques for handling multiple objectives, constraints and variable interaction highlights a lack of techniques to handle variable interaction. This research, therefore, focuses on the development of techniques for detecting the relationships between the decision variables (variable interaction) in the presence of multiple objectives and constraints. It attempts to fill the gap in this research by formally extending the theoretical results (optimality conditions). The research then proposes first-order multi-objective symbolic algorithm or MOSA-I and second-order multi-objective symbolic algorithm or MOSA-II that are capable of detecting the variable interaction. The performance of these algorithms is analysed and compared to a current state-of-the-art optimisation algorithm using popular test problems.

The performance of the MOSA-II algorithm is finally validated using three appropriately chosen problems from literature. In this way, this research proposes a fully tested and validated methodology for dealing with multi-objective optimisation problems.

In conclusion, this research proposes two new symbolic algorithms that are used for identifying the variable interaction responsible for constructing Pareto-optimal front among objectives in multi-objective optimisation problems. This is completed based on a development and relaxation of the first and second-order optimality conditions of Karush-Kuhn-Tucker.

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List of Publications

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Submitted papers

Askar, S.S., Tiwari, A. (2009), 'Optimality Conditions for Differentiable Multi-Objective Optimisation Problems', Journal of Applied Mathematics and Computations.

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Commonly Used Abbreviations

$\mathcal{P}^*$	**P**areto-Optimal Set
$\mathbb{R}^{n}$	n-Dimensional Euclidean space
FJ	**F**ritz-**J**ohn
GAs	**G**enetic **A**lgorithm**s**
GD	**G**enerational **D**istance Metric
$\mathbb{H}$	**H**essian Matrix
IDP	**I**nnovative **D**esign **P**rinciples
KKT	**K**arush-**K**uhn **T**ucker
LCOF	**L**inear **C**onstrained **O**bjective **F**unctions
MOEA/D	**M**ulti-**O**bjective **E**volutionary **A**lgorithm **B**ased on **D**ecomposition
MOGA	**M**ulti-**O**bjective **G**enetic **A**lgorithm
MOOP	**M**ulti-**O**bjective **O**ptimisation **P**roblem
MOSA-I	First-Order **M**ulti-**O**bjective **S**ymbolic **A**lgorithm
MOSA-II	Second-Order **M**ulti-**O**bjective **S**ymbolic **A**lgorithm
NLCOF	**N**on-**L**inear **C**onstrained **O**bjective **F**unctions
NPGA	**N**iched **P**areto **G**enetic **A**lgorithm
NSGA	**N**on-dominated **S**orting **G**enetic **A**lgorithm
NSGA-II	**F**ast **E**litist **N**on-dominated **S**orting **G**enetic **A**lgorithm
RA	**R**egression **A**nalysis
SPEA	**S**trength **P**areto **E**volutionary **A**lgorithm
SPEA2	**S**trength **P**areto **E**volutionary **A**lgorithm
VEGA	**V**ector **E**valuated **G**enetic **A**lgorithm
ZDT	**Z**itzler-**D**eb-**T**hiele
$\mathcal{PF}^*$	**P**areto-Optimal **F**ront
$\nabla f(x)$	**T**he gradient of f at x

1 INTRODUCTION

Problems that have more than one objective function are of great importance in engineering sciences and many other disciplines. This class of problems are known as multi-objective optimisation problems (or multicriteria). The difficulty here lies in the conflict between the various objective functions. Due to this conflict, one can not find a single ideal solution which simultaneously satisfies all the objectives. But instead one can find the set of the Pareto-optimal solutions (Pareto-optimal set) and consequently the Pareto-optimal front is established. Finding these solutions plays an important role in multi-objective optimisation problems and mathematically the problem is considered to be solved when the Pareto-optimal set, i.e. the set of all compromise solutions is found. The Pareto-optimal set may contain information that can help the designer make a decision and thus arrive at better trade-off solutions (Hillermeier, 2001).

The challenge posed by real-life optimisation problems (such as multiple objectives, constraints and variable interaction) has prompted the engineers to look for rigorous ways of optimising its design (Kirsch, 1981). Computational techniques are more popular in dealing with these demands. Traditional trial-and-error method of optimisation is not capable of meeting these engineering demands.

This research focuses on developing optimisation algorithms based on the extension of the current optimality conditions to address the complexities of multi-objective optimisation problems. This chapter attempts to address the following.

- *To define the problem tackled in this thesis and present a brief introduction to optimality conditions.*
- *To give a brief introduction to evolutionary computation techniques.*
- *To illustrate the problem statement and motivation of this research.*
- *To present the thesis layout.*

1.1 Problem Setting and Optimality Conditions

This section highlights the multi-objective optimisation problem tackled throughout this thesis. In addition, the first and second-order optimality conditions of Karush-Kuhn-Tucker are illustrated.

1.1.1 Problem Setting

The general description of the multi-objective optimisation problem used in this thesis is written as follows (Miettinen, 1999; Jahn, 2004):

(1.1)

Where, and are all twice continuously differentiable functions and each decision variable has a lower and upper bounds and respectively.

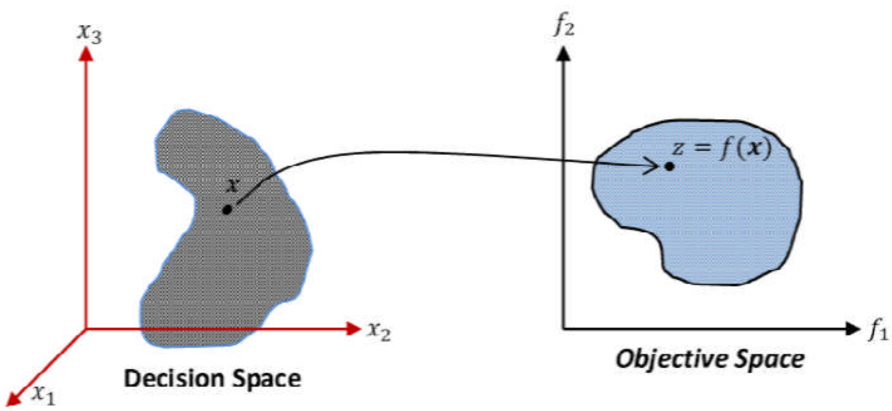

Figure 1.1: Decision and objective spaces

When dealing with multi-objective optimisation problems, two important spaces attract our attention. The first one is called the decision space and the other is called the objective space (Figure 1.1). The link between the two spaces is the map defined in Equation (1.1). Solving Equation (1.1) means seeking the solution vectors in the decision space which simultaneously satisfy the inequality constraints. These solution vectors provide the best compromised solutions between the conflicting objectives. The characteristics of these solutions are illustrated below. Before going into details on these characteristics, we begin with some basic notations.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ be any two vectors such that $\boldsymbol{x} = (x_1, x_2, \dots, x_n)^T$ and $\boldsymbol{y} = \left(y_1, y_2, \dots, y_n\right)^T$; here, the superscript T refers to the transpose of $\boldsymbol{x}$. The inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as follows:

$$\boldsymbol{x}^T \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, the following conventions are used throughout the thesis:

$$\begin{aligned} \boldsymbol{x} \leqq \boldsymbol{y} &\Leftrightarrow x_i \leqq y_i, i = 1,2, \dots, n \\ \boldsymbol{x} \leq \boldsymbol{y} &\Leftrightarrow x_i \leqq y_i, i = 1,2, \dots, n, \quad \boldsymbol{x} \neq \boldsymbol{y} \\ \boldsymbol{x} < \boldsymbol{y} &\Leftrightarrow x_i < y_i, i = 1,2, \dots, n \end{aligned}$$

Definition 1.1: A subset $\mathbb{X} \subseteq \mathbb{R}^n$ is said to be a convex set if for any two points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{X}$ the segment:

$$(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \in \mathbb{X}, \quad \alpha \in [0,1]$$

Definition 1.2: A function $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ is valid that

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \leq \alpha f(\boldsymbol{x}) + (1 - \alpha) f(\boldsymbol{y}), \quad \alpha \in [0,1]$$

Definition 1.3: A function $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\hat{\boldsymbol{x}} \in \mathbb{X}$ if:

$$\begin{gathered} f(\boldsymbol{x} + \boldsymbol{d}) - f(\hat{\boldsymbol{x}}) = \nabla f(\hat{\boldsymbol{x}})^T \boldsymbol{d} + \|\boldsymbol{d}\| \varepsilon(\hat{\boldsymbol{x}}, \boldsymbol{d}), \\ \varepsilon(\hat{\boldsymbol{x}}, \boldsymbol{d}) \to \boldsymbol{0} \; as \; \|d\| \to \boldsymbol{0} \end{gathered}$$

Definition 1.4: Let the function $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at every $\boldsymbol{x} \in \mathbb{X}$. Then it is pseudo-convex function if for all $x, y \in \mathbb{X}$:

$$\nabla f(\boldsymbol{x})^T (\boldsymbol{y} - \boldsymbol{x}) \geq \boldsymbol{0} \Longrightarrow f(\boldsymbol{y}) \geq f(\boldsymbol{x})$$

Definition 1.5: A function $f: \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be quasiconvex if for all $x, y \in \mathbb{X}$:

$$f(\alpha \boldsymbol{x} + (1 - \alpha)\boldsymbol{y}) \leq \max\left(f(\boldsymbol{x}), f(\boldsymbol{y})\right), \quad \alpha \in [0,1]$$

In addition, if the function f is differentiable at $\boldsymbol{x} \in \mathbb{X}$ with convex domain then f is quasiconvex if and only if the following implication holds:

$$\boldsymbol{y} \in \mathbb{X}, f(\boldsymbol{y}) \leq f(\boldsymbol{x}) \Longrightarrow \nabla f(\boldsymbol{x})^T (\boldsymbol{y} - \boldsymbol{x}) \leq \boldsymbol{0}$$

Definition 1.6: Let the constraints g_j of problem (1.1) be continuously differentiable at $\hat{\boldsymbol{x}} \in \mathbb{X}$. The problem satisfies the Kuhn-Tucker constraint qualification at $\hat{\boldsymbol{x}}$ if for any direction $\boldsymbol{d} \in \mathbb{R}^n$ such that $\nabla g_j(\hat{\boldsymbol{x}})\boldsymbol{d} \leq \boldsymbol{0}$ for all $j \in J(\hat{x}) = \{j \in \{1,2, \dots, J\}: g_j(\hat{x}) = 0\}$, there does exist a function $\varphi: [0,1] \longrightarrow \mathbb{R}^n$ which is continuously differentiable at 0, and some real scalar $\beta > 0$, such that $\varphi(\boldsymbol{0}) = \hat{\boldsymbol{x}}, g(\varphi(t)) \leq 0$ for all $0 \leq t \leq 1$ and $\acute{\varphi}(0) = \beta \boldsymbol{d}$.

Motzkin's Theorem 1.1: Let A and C be given matrices, then either the system of inequalities $A\boldsymbol{x} < \mathbf{0}, C\boldsymbol{x} \leq \mathbf{0}$ has a solution $\boldsymbol{x}$, or the system $A^T\boldsymbol{\lambda} + C^T\boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\lambda} \neq \mathbf{0}, \boldsymbol{\mu} \geq \mathbf{0}$ has a solution $(\boldsymbol{\lambda}, \boldsymbol{\mu})$, but never both (Miettinen, 1999).

Here, we consider the objective functions in problem (1.1) are conflicting. Because of this conflict and possible incommensurability of the objective functions, it is not possible to find a single solution that would be optimal for all the objectives simultaneously. Therefore, for problem (1.1) the solutions are redefined in terms of (weak) Pareto-optimal solutions.

Definition 1.7: A point $\hat{\boldsymbol{x}} \in \mathbb{X}$ is said to be a Pareto-optimal (or non-inferior or efficient) solution to problem (1.1) if and only if there is no $\boldsymbol{x} \in \mathbb{X}$ such that $f(\boldsymbol{x}) \leq f(\hat{\boldsymbol{x}})$.

For simplicity, if this definition is applied to a minimisation bi-objective optimisation problem, then a Pareto-optimal solution for this problem means that there is no vector solution that can cause a decrease in one objective function without causing a simultaneous increase in the other objective function value (Sawaragi et al., 1985; Ehrgott, 2000).

Definition 1.8: A point $\hat{\boldsymbol{x}} \in \mathbb{X}$ is said to be a weak Pareto optimal solution to problem (1.1) if and only if there is no $\boldsymbol{x} \in \mathbb{X}$ such that $f(\boldsymbol{x}) < f(\hat{\boldsymbol{x}})$.

Definition 1.9 (Pareto-optimal set): For a given vector of objective functions $f(x)$, the Pareto-optimal set $\boldsymbol{\mathcal{P}}^*$ is defined as (Sawaragi et al., 1985; Ehrgott, 2000):

$$\boldsymbol{\mathcal{P}}^* \coloneqq \{\boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n : \nexists \hat{\boldsymbol{x}} \in \mathbb{X} \text{ such that } f(\hat{\boldsymbol{x}}) \leq f(\boldsymbol{x})\}$$

Definition 1.10 (Pareto-optimal front): For a given vector of objective functions $f(x)$ and Pareto-optimal set $\boldsymbol{\mathcal{P}}^*$ the Pareto-optimal front $\boldsymbol{\mathcal{PF}}^*$ is defined as (Coello et al., 2002):

$$\mathcal{PF}^* \coloneqq \left\{f(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), \dots, f_M(\boldsymbol{x})\right) : \boldsymbol{x} \in \boldsymbol{\mathcal{P}}^*\right\}$$

The following definitions are according to Mishra et al. (2005a, 2005b).

In these definitions, $b_0, b_1 : \mathbb{X} \times \mathbb{X} \times [0,1] \longrightarrow \mathbb{R}^+, b(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \lim_{\lambda \to 0} b(\boldsymbol{x}, \hat{\boldsymbol{x}}, \boldsymbol{\lambda})$ and b does not depend on $\boldsymbol{\lambda}$ if the corresponding functions are differentiable, $\varphi_0, \varphi_1 : \mathbb{R} \longrightarrow \mathbb{R}$ and $\eta : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{R}^n$ is $\boldsymbol{n}$-dimensional vector-valued function. Furthermore, φ_0, φ_1 satisfy the following condition:

$$\begin{aligned} &u \leq 0 \Rightarrow \varphi_0(u) \leq 0, \\ &u \leqq 0 \Rightarrow \varphi_1(u) \leqq 0, \\ &b_0(\boldsymbol{x}, \hat{\boldsymbol{x}}) > 0, b_1(\boldsymbol{x}, \hat{\boldsymbol{x}}) \geqq 0 \end{aligned}$$

Definition 1.11: The problem (1.1) is said to be weak strictly pseudo type I univex at $\hat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that:

$$\begin{aligned} &b_0(\boldsymbol{x}, \hat{\boldsymbol{x}})\varphi_0[f_i(\boldsymbol{x}) - f_i(\hat{\boldsymbol{x}})] \leq \mathbf{0} \Longrightarrow \nabla f_i(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) < \mathbf{0} \\ &-b_1(\boldsymbol{x}, \hat{\boldsymbol{x}})\varphi_1 g_j(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \nabla g_j(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leqq \mathbf{0} \end{aligned}$$

For all $\boldsymbol{x} \in \mathbb{X}$ and for all $i = 1,2,\dots,p$, and $j = 1,2,\dots,m$.

Definition 1.12: The problem (1.1) is said to be strong pseudoquasi type I univex at $\widehat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that:

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0[f_i(\boldsymbol{x}) - f_i(\widehat{\boldsymbol{x}})] \le \mathbf{0} \Longrightarrow \nabla f_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \le \mathbf{0}$$
$$-b_1(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_1 g_j(\widehat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \nabla g_j(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \leqq \mathbf{0}$$

For all $\boldsymbol{x} \in \mathbb{X}$ and for all $i = 1,2,\dots,p$, and $j = 1,2,\dots,m$.

Definition 1.13: The problem (1.1) is said to be weak quasistrictly pseudo type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\widehat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that:

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0[f_i(\boldsymbol{x}) - f_i(\widehat{\boldsymbol{x}})] \le \mathbf{0} \Longrightarrow \nabla f_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \leqq \mathbf{0}$$
$$-b_1(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_1 g_j(\widehat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \nabla g_j(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \le \mathbf{0}$$

For all $\boldsymbol{x} \in \mathbb{X}$ and for all $i = 1,2,\dots,p$, and $j = 1,2,\dots,m$.

Definition 1.14: The problem (1.1) is said to be weak strictly pseudo type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\widehat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that:

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0[f_i(\boldsymbol{x}) - f_i(\widehat{\boldsymbol{x}})] \le \mathbf{0} \Longrightarrow \nabla f_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) < \mathbf{0}$$
$$-b_1(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_1 g_j(\widehat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \nabla g_j(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) < \mathbf{0}$$

For all $\boldsymbol{x} \in \mathbb{X}$ and for all $i = 1,2,\dots,p$, and $j = 1,2,\dots,m$.

1.1.2 First-order Optimality Conditions

Optimality conditions for multi-objective optimisation problems are thoroughly important in optimisation. Here, we highlight the first-order optimality conditions of Karush-Kuhn-Tucker (Miettinen, 1999).

Theorem 1.2 (Karush-Kuhn-Tucker sufficient conditions for Pareto optimality): Let the objective and constraint functions of problem (1.1) are convex and continuously differentiable at a decision vector $\widehat{\boldsymbol{x}} \in \mathbb{X}$. A sufficient condition for $\widehat{\boldsymbol{x}}$ to be a Pareto-optimal solution is that there exist multipliers $\mathbf{0} < \boldsymbol{\lambda} \in \mathbb{R}^M$ and $\mathbf{0} < \boldsymbol{\mu} \in \mathbb{R}^J$ such that:

$$\sum_{i=1}^{M} \lambda_i \nabla f_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{J} \mu_j \nabla g_j(\widehat{\boldsymbol{x}}) = \mathbf{0}$$
$$\mu_j g_j(\widehat{\boldsymbol{x}}) = 0 \text{ for all } j = 1,2,\dots,J$$

Proof: is given in Miettinen, (1999).

In this thesis, a relaxation on the assumptions imposed in this theorem is considered. Furthermore, this relaxation is used to propose symbolic algorithms for handling multi-objective optimisation problems.

1.1.3 Second-order Optimality Conditions

Second-order Optimality conditions (presuming twice continuously differentiable objective and constraint functions) have been examined substantially less than first-order optimality conditions. Second-order optimality conditions provide a means of reducing the set of candidate solutions produced by the first-order optimality conditions but at the same time tighten the assumptions set of the regularity of the problem.

Definition 1.15 (Regularity): A point $\hat{\boldsymbol{x}} \in \mathbb{X}$ is said to be a regular point if the gradients of the active constraints at $\hat{\boldsymbol{x}}$ are linearly independent.

Theorem 1.3 (Second-order sufficient conditions for Pareto optimality): Let the objective and constraint functions of problem (1.1) are twice continuously differentiable at a decision vector $\hat{\boldsymbol{x}} \in \mathbb{X}$. A sufficient condition for $\hat{\boldsymbol{x}}$ to be a Pareto-optimal solution is that there exist multipliers $\mathbf{0} \leq \boldsymbol{\lambda} \in \mathbb{R}^M$ and $\mathbf{0} \leq \boldsymbol{\mu} \in \mathbb{R}^J$ for which $(\boldsymbol{\lambda}, \boldsymbol{\mu}) \neq (\mathbf{0}, \mathbf{0})$ such that

$$\sum_{i=1}^{M} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{J} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \mathbf{0}$$
$$\mu_j g_j(\hat{\boldsymbol{x}}) = 0 \text{ for all } j = 1,2,\dots,J$$
$$\boldsymbol{d}^T \left(\sum_{i=1}^{M} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{J} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}}) \right) \boldsymbol{d} > \mathbf{0}$$

for either all $\boldsymbol{d} \in \{0 \neq \boldsymbol{d} \in \mathbb{R}^n : \nabla f_i(\hat{\boldsymbol{x}})^T \boldsymbol{d} \leq 0 \text{ for all } i = 1,2,\dots,M,\ \nabla g_j(\hat{\boldsymbol{x}})^T \boldsymbol{d} \leq 0 \text{ for all } j \in J(\hat{\boldsymbol{x}})\}$ or all $\boldsymbol{d} \in \{0 \neq \boldsymbol{d} \in \mathbb{R}^n : \nabla g_j(\hat{\boldsymbol{x}})^T \boldsymbol{d} = 0 \text{ for all } j \in J^+(\hat{\boldsymbol{x}}), \nabla g_j(\hat{\boldsymbol{x}})^T \boldsymbol{d} \leq 0 \text{ for all } j \in J(\hat{\boldsymbol{x}}) \backslash J^+(\hat{\boldsymbol{x}})\}$.

where,

$$J(\hat{\boldsymbol{x}}) = \{ j \in \{1,2,\dots,J : g_j(\hat{\boldsymbol{x}}) = 0 \}$$
$$J^+(\hat{\boldsymbol{x}}) = \{ j \in J(\hat{\boldsymbol{x}}) : \mu_j > 0 \}$$

1.2 Introduction to Evolutionary Computation Techniques

In the natural world, evolution has created an unimaginably diverse range of designs, having much greater complexity than mankind could ever hope to achieve. Inspired by this, researchers have started using the evolutionary computation techniques (or EC Techniques) that use the principles of evolution to guide the optimisation process. There are a number of benefits of evolutionary-based optimisation that justify the effort invested in this area (Gershenfeld, 1999). The evolutionary-based optimisation techniques have many iterations and each iteration generates a population of multiple solutions instead of a single solution. This enables them, in principle, to identify multiple optimal solutions in their final population. These characteristics of the EC techniques also make them a suitable candidate for handling a combination of features of optimisation problems in a single run. These features include the presence of multiple objectives, constraints and interaction among decision variables. This research, therefore, uses a popular evolutionary technique, NSGA-II, for comparison with the thesis approach. It is important to mention here that any other stochastic algorithms can be used for the comparison process, however NSGA-II gives better solutions for the problems tackled in this thesis.

Genetic algorithms or GAs are wise stochastic search techniques widely used in optimisation. They are based on the mechanism of natural genetics and natural selection. The terminology used in the field of genetic algorithms based methods is mimic natural evolution (Giotis and Giannakoglou, 1999). Individuals in a population are called strings or chromosomes which represent alternative solutions to the given optimisation problem. A population of chromosome is modified by the probabilistic application of the genetic operators (selection, crossover and mutation operators) which are applied on the strings. Evaluation of each string that corresponds a solution in the search space is based on a fitness function that is problem dependent. According to the fitness function, the high performing individuals are selected for the mating pool, where they reproduce with other individuals to produce offspring. Figure 1.2 gives a schematic description of the GAs (Goldberg, 1989; Osyczka, 2002).

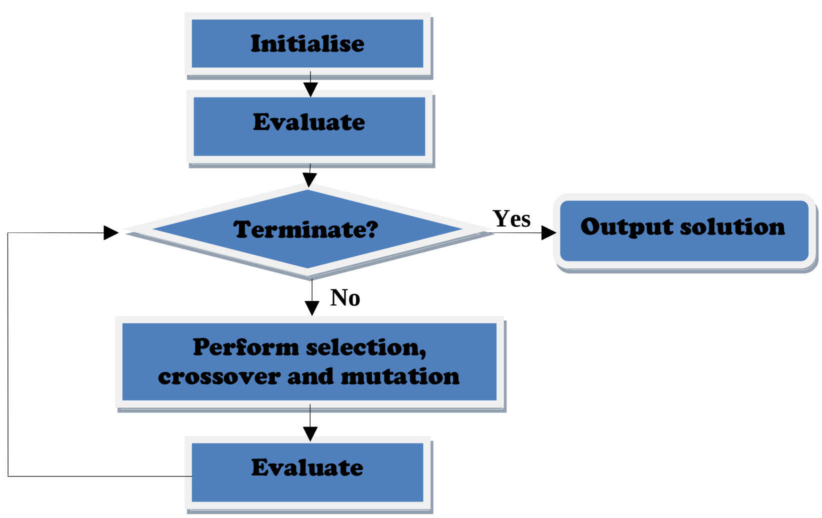

Figure 1.2: Schematic description of GAs

The majority of current implementations of evolutionary algorithms descend from four strongly related but independently developed approaches: Genetic Algorithms (GAs), Evolutionary Programming (EP), Evolution Strategies (ESs) and Genetic Programming (GP) (Back *et al.*, 1997). These approaches are defined below.

- The GAs are a type of search and optimisation algorithms that are based on the mechanics of genetics and natural selection.
- The EP, which was originally offered as an attempt to create artificial intelligence, relies on transformations depending upon a finite set of states and state transition rules.
- The ESs, which were initially designed for solving complex optimisation problems, involve the modification of behavioural traits of solutions.
- Finally, the GP is an automated method for creating a working computer program from a high-level problem statement. The GP does this by genetically breeding a population of computer programs using the principles of Darwinian natural selection and biologically inspired operations, (Deb, 2001).

Over the last decade, the GAs has been extensively used as search and optimisation tools in various problem domains, including engineering design. The primary reasons for their success over other EC techniques are their broad applicability, ease of use and global perspective (Goldberg, 1989).

1.3 Problem Statement and Motivation

The problem statement of this research is as follows.

> ***Development of symbolic optimisation algorithms that are capable of dealing with the challenges of multi-objective optimisation problems using the extension of the current theoretical results.***

The development of robust optimisation algorithms enables the handling of the features of multi-objective optimisation problems, such as multiple measures of performance (objectives), constraints and interaction among decision variables. This enhances the effectiveness of optimisation algorithms by giving them the capability of dealing with a wide variety of problems. In this way, one of the main inhibitors to their industrial use is addressed, and their popularity and relevance for industry could be enhanced. This is the main motivation for this research. Furthermore, the development of the current theoretical results (optimality conditions) for multi-objective optimisation problems supports the development of these algorithms and makes a direct contribution to research.

1.4 Thesis Layout

The layout of this thesis is developed based on the story of this research. This story, which is pictorially depicted in Figure 1.3, aids the identification of individual chapters. A brief description of these chapters is given below.

Chapter 1 discusses the background of this research, briefly explaining the aim of this research. It presents the problem statement and motivation for this research.

Chapter 2 provides an overview of optimisation techniques used for solving multi-objective optimisation problems, and the test problems used for evaluating these techniques. It presents a critical analysis of the state-of-the-art evolutionary-based optimisation techniques.

Chapter 3 gives a brief description of this research, outlining its aim, objectives and scope. It also discusses the methodology that is adopted for ensuring that the aim and objectives of this research are attained.

Chapter 4 deals with some different theoretical classes of multi-objective optimisation problems. The first-order optimality conditions for each class are separately proposed and proved. Furthermore, the duality results for each class are suggested and then some first-order duality theorems are illustrated and proved. In addition, this chapter proposes second-order optimality conditions under a generalised type of univexity functions.

Chapter 5 presents a multi-objective symbolic algorithm (MOSA-I) for solving multi-objective optimisation problems with particular aspects. These aspects include the class of all the objective functions and inequality constraints that are continuous, differentiable, and convex. The algorithm is checked by solving some multi-objective optimisation problems that have been carefully selected from literature according to the state-of-the art optimisation algorithm, NSGA-II.

Chapter 6 generalises the algorithm presented in the previous chapter. The only difference between this algorithm (MOSA-II) and the one mentioned in the previous chapter is that this one includes second-order optimality conditions that are used to reduce solutions provided by the first-order optimality conditions. This algorithm is a generalisation of the previous algorithm in such a way that it overcomes some of the drawbacks found in the previous one..This algorithm is checked by solving some multi-objective optimisation problems that are carefully selected from literature.

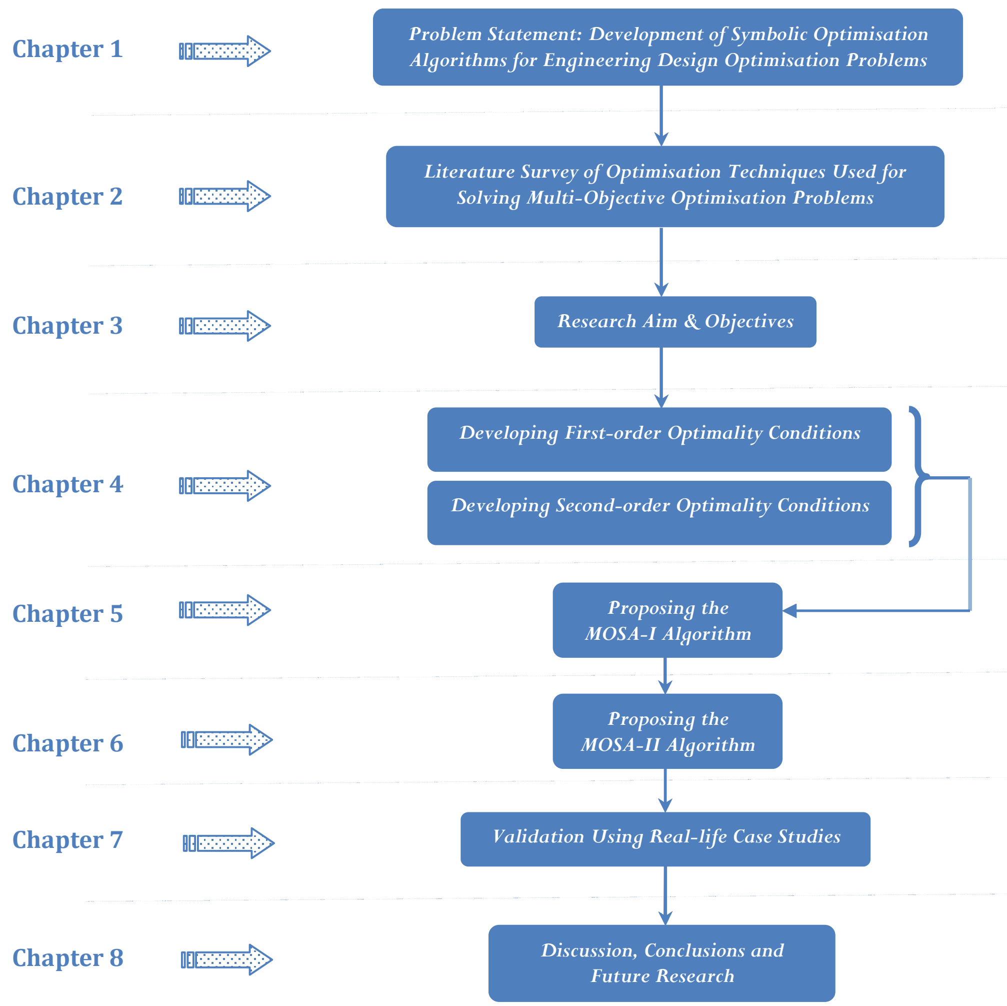

Figure 1.3: Thesis layout

Chapter 7 validates the research using three real-life case studies: design of pressure vessel, design of four-member truss and design of welded beam. The MOSA-II algorithm is applied to these problems, and the results thus obtained are analysed, compared and discussed.

Chapter 8 concludes this thesis with a discussion on the generality of this research, contribution to knowledge, and limitations of the research methodology, proposed algorithms and optimality conditions. It finally discusses the future research directions that could follow from this research.

1.5 Summary

This chapter has addressed the following.

- It has defined the problem setting used throughout the thesis.
- It has presented the first and the second-order optimality conditions of KKT.
- It has shown a brief introduction on evolutionary computation techniques.
- It has illustrated the motivation of this research.
- It has presented the thesis layout.

2 A REVIEW OF LITERATURE

In mathematics and computer science, optimisation, or mathematical programming refers to the process of choosing the best alternative from some set of available alternatives. In simple terms, this means solving problems in which one seeks one or more feasible solutions to minimise or maximise one or more objective functions by systematically choosing the solutions from within an allowed set. The need for finding such optimal solutions in a problem comes mostly from an extreme purpose, such as designing a solution for minimum possible cost of fabrication or for maximum possible reliability or others. Because of such extreme properties of optimal solutions, the optimisation algorithms are of great importance in practice, particularly in engineering design. The aim of this chapter is to give an overview of techniques used for solving multi-objective optimisation problems, and the characteristics of problems used for evaluating these techniques. It attempts to achieve the following.

- *To analyse the classical approaches used for tackling multi-objective optimisation problems*
- *To provide an overview of evolutionary computation techniques used for handling multi-objective optimisation problems and then a comparison between classical and evolutionary approaches.*
- *To provide an overview of theoretical approaches used for tackling multi-objective optimisation problems.*
- *To categorise and classify the articles used in the literature survey.*
- *To identify the current trends.*
- *To demonstrate the research gap.*

2.1 Classical Approaches to Multi-Objective Optimisation

Looking for Pareto-optimal solutions of multi-objective optimisation problems is a complicated task, particularly when one has nonlinear objectives and constraints. Therefore, finding this sort of solutions plays an important role in multi-objective optimisation and mathematically the problem is considered to be solved when the Pareto-optimal set is obtained. As a result, literature suggests a number of optimisation techniques for handling multi-objective optimisation problems.

These techniques can be classified into two broad categories: classical and evolutionary. In this section, an outline review of some popular classical optimisation methods used for this purpose is given.

2.1.1 Weighted Aggregation Approach

The weighted sum method is one of the most widely used approaches for solving multi-objective optimisation problems. This method is also called naïve approach (Coello, 1998). The simplest way to apply this technique is to take each objective function, associate a weight with the objective function and then take a weighted sum of the objective functions. Therefore, a new and unique objective function is obtained (Ehrgott, 2000; Miettinen, 1999)

Strengths and Drawbacks

The traditional weighted sum approach was the first one used in the field of multi-objective optimisation after Pareto (1906) introduced the concept of non-inferior solutions in the context of economics (http://cepa.newschool.edu/het/profiles/pareto.htm) and after that it has been very popular in many applications. Due to its wide use, researchers have identified its strengths and weaknesses as follows. From the algorithmic point of view, this technique is very efficient and easy to implement. Beside that, it is easy to understand. The main usefulness of the weighted sum approach appears when the feasible space of objective functions is convex. Nevertheless, there are a number of difficulties with this approach. As discussed in many studies (e.g. Koski, 1985; Messac et al., 2000; Das et al., 1997; Deb, 2001), this method can not discover Pareto-optimal solutions hidden in concavities and it is very sensitive to the weight parameters. Moreover, in handling mixed optimisation problems, such as those with some objectives of maximisation type and some of minimisation type, all objective functions have to be converted into one type. Although different conversion procedures can be adopted, the duality[1] principle is convenient and does not introduce any complexity. Furthermore, in nonlinear multi-objective optimisation problems, a uniformly distributed set of weights does not guarantee a uniformly distributed set of Pareto-optimal solutions. Albeit the drawbacks aforementioned, this approach is still used in applications and some modified forms of this technique have been studied recently (Kim and De Weck, 2005). For more details about the advantages and disadvantages of the weighted sum technique, see Deb (2001).

[1] The dual of the partial order set $P = (X, \leq)$ is the partial order set $P^d = (X, \geq)$. For instance, a minimum element of P will be a maximum element of P^d. Minima and maxima are dual concepts in order theory.

2.1.2 *ε-Constraint Method*

The ε-constraint approach is another well known technique for solving multi-objective optimisation problems. Instead of aggregation of criteria, only one of the objective functions is selected to be optimised, while the other objective functions are transformed to constraints by setting an upper bound to each one of them (Ehrgott, 2000; Miettinen, 1999).

Strengths and Drawbacks

As mentioned before, this technique is designed to discover Pareto-optimal solutions based on optimising one of the objective functions while treating the other objective functions as constraints bound within some allowable range. Like in the traditional weighted sum method, the problem is repeatedly solved for different values of weights until the entire Pareto-optimal set is generated. The main advantage of this technique is that it is able to identify a number of non-inferior solutions on a non-convex boundary curve which are not obtainable using other techniques. Moreover, the entire Pareto-optimal set can be obtained by varying the values (Napitupulu, 1990). Furthermore, in terms of the information needed from the user, this approach requires a vector of values representing, in some sense, the location of the Pareto-optimal solutions. The major disadvantage of this approach is that when it is used to solve multi-objective optimisation problems, it largely depends on this chosen vector. Another drawback of this approach is that it consumes a large amount of computing time, and the programming of the algorithm can be very hard to do if there are many objective functions. Also, as the number of the objectives increases, the number of elements in the vector may require extra information from the user. Inspite of these drawbacks, the relative simplicity of the method has made it popular (Collette et al., 2003; Engau and Wiecek, 2005, 2007).

2.1.3 *Hybrid Method*

The introduction of hybrid methods comes from the need to deal with complex industrial applications whose handling becomes unpractical with GA optimisation techniques. The best known hybrid method is Corley method. This approach is a combination of weighted sum and ε-constraint method. It is described in Corley (1980), and Wendell and Lee (1977).

Strengths and Drawbacks

The positive features of the weighting method and the constraint method have been combined in the hybrid approach. Explicitly, any Pareto-optimal solutions can be found independently of the convexity of the problem and one does not have to solve several problems or even think about uniqueness to guarantee the Pareto optimality of the solutions (Miettinen, 1999). Consequently, it is efficient for various kinds of optimisation problems, whether convex or not. Nevertheless, the specification of the parameter values may still be difficult since the number of parameters has been multiplied by two (Miettinen, 1999; Collette et al., 2003).

2.1.4 *Weighted Metrics Method*

The ideal point is the best outcome of a multi-objective optimisation problem. Yet, when the objectives are in conflict the ideal values are impossible to obtain. However, it can be used as a reference point, with the aim being to seek for solutions as close as possible to the ideal point (Ehrgott, 2000; Sawaragi et al., 1985). So, instead of a weighted sum of the objective functions, the weighted metrics such as ℓ_p and the distance metrics[2] ℓ_∞ are often used.

Strengths and Drawbacks

For this method to work well, a good reference point must be chosen carefully. Otherwise, the solutions obtained depending on this reference point will not be optimal. Nevertheless, this method allows one to discover solutions hidden inside concavities under some conditions as presented in Messac et al. (2000). An interested reader can find more about the robustness and the weaknesses of this approach in Miettinen (1999), Collette et al. (2003) and Deb (2001).

2.1.5 *Benson's Method*

This method has been presented by Benson (1978). This approach is to some extent similar to the weighted metrics approach. The main difference between the weighted metrics method and Benson's method is that in the later the reference solution is taken as a feasible non-Pareto-optimal solution.

[2] The weighted ℓ_p distance measure of any solution $\boldsymbol{x}$ from another solution $\boldsymbol{z}$ is $\ell_p(\boldsymbol{x}) = (\sum_{m=1}^{M} \omega_m |f_m(\boldsymbol{x}) - \boldsymbol{z}|^p)^{\frac{1}{p}}$, where M is the number of objectives.

Strengths and Drawbacks

The major strength of Benson's approach is that if one has shown an appropriate reference solution, the solutions in the non-convex region can be found (Deb, 2001). The main weakness of this procedure is the additional number of constraints needed to restrict the search in the region dominating the chosen reference solution. Moreover, the objective function is non-differentiable, thereby causing difficulties for gradient-based methods to solve the above problem (Deb, 2001).

2.2 Evolutionary Computation Techniques for Multi-objective Optimisation

Optimisation algorithms such as evolutionary swarm algorithms are heuristic techniques that have been used to deal with multi-objective optimisation problems (Collette and Siarry, 2003). They have adequately demonstrated their usefulness in finding a well-converged and a well-distributed set of near Pareto-optimal solutions (Fletcher and Powell, 1963; De Jong, 1975; Farkas and Jarmai, 1995; Deb and Kumar, 1995; Coello et al., 2002; Deb, 2001). Because of the extensive studies and the available source codes available both commercially and freely for these algorithms, they have been popularly applied in various problem-solving tasks and have received great attention (Fonseca and Fleming, 1995; Binh and Korn, 1996; Goldberg, 2002; Deb and Sundar, 2006, Dhish, 2008). However, recent studies (Jensen, 2003) have shown that multi-objective optimisation with fitness assignment based on Pareto-domination leads to long processing times for large population sizes. This has motivated a considerable amount of research and a wide variety of approaches have been suggested in the last few years (Musselman and Talavage, 1980; Osyczka and Kundu, 1995; Los and Eilben, 1996; Viennet et al., 1996, Laumanns et al., 1998; Narayanan and Azarm, 1999; Poloni et al., 2000; Tapabrata and Liew, 2002; Li and Palusinski, 2003; Shukla and Deb, 2007). Deb et al. (2007) have suggested a verification procedure based on Kaursh-Kuhn-Tucker (KKT) conditions to build confidence on the optimality of solutions obtained using an evolutionary optimisation procedure.

Most engineering design optimisation problems are multi-objective in nature since they normally have several conflicting objectives, say for example, cost and performance, which must be satisfied at the same time. This has encouraged the growth of research in the field of multi-objective optimisation using evolutionary algorithms (Schwefel, 1995; Rao, 1996; Veldhuizen, 1999; Rogero et al., 2000; Marler and Arora, 2003). This section gives an overview of some popular evolutionary-based multi-objective optimisation techniques in terms of their types and features.

2.2.1 *Vector Evaluated Genetic Algorithm (VEGA)*

This algorithm has been proposed by Schaffer (1985). In the algorithm, Schaffer has modified the selection operator of a simple genetic algorithm (SGA) so that at each generation a number of sub-populations were generated by performing proportional selection according to each objective function in turn. Schaffer realised that the solutions generated by his method were non-dominated in a local sense.

Strengths and Drawbacks

The weakness of this algorithm is that one may obtain at the final generation (at the end of optimisation process) a population composed of mean individuals with respect to all the objective functions. Such a population does not allow one to obtain a good approximation of the Pareto-optimal front. Instead, the population will be concentrated around a mean "point". Moreover, it has been shown that this algorithm does not have the ability to produce solutions in the presence of non-convex search spaces (Coello, 1999).

2.2.2 *Multi-Objective Genetic Algorithm (MOGA)*

This method is well-presented in Collette et al. (2003) and Fonseca et al. (1993). The comparison between individuals in this algorithm is based domination in the Pareto sense. The rank of an individual (an order number which allows one to rank an individual with respect to the others) is given by the number of individuals which dominate an individual. For instance, if one considers an individual x_i at generation t which is dominated by P_i^t individuals, the rank of this considered individual is given by $rank(x_i, t) = 1 + P_i^t$. The rank 1 is assigned to all the non-dominated individuals and dominated ones are penalised according to the population density of the corresponding region of the Pareto front.

Strengths and Drawbacks

The main strength of this algorithm is that it can be implemented with relative easy (Coello, 1997). In some cases, the MOGA algorithm does not allow one to obtain a good diversity of solutions for the approximation of the Pareto front.

2.2.3 Non-Dominated Sorting Genetic Algorithm (NSGA)

The Non-dominated Sorting Genetic Algorithm (NSGA) was proposed by Srinivas et al. (1994). This algorithm is based on a classification using many levels of individuals. In the first step of the algorithm, before proceeding to selection, each individual in the population is ranked on the basis of non-domination (Srinivas et al., 1994). After that, all non-dominated individuals are classified into one category. To this category, a dummy fitness value, which is proportional to the population size, is assigned to provide an equal reproductive potential for these individuals. To Maintain a good approximation of the trade-off surface, or, alternatively, a good diversity of solutions, these classified individuals are shared with dummy fitness values. Then, this group of classified individuals is ignored and another layer of non-dominated candidates are considered. The process continues until all individuals in the population are classified. Since all individuals in the first front have the maximum fitness value, they get more copies than the rest of the population. This allows the algorithm to search for non-dominated regions, and results in quick convergence of the population towards such regions.

Strengths and Drawbacks

The main efficiency of this technique lies in the fact that any number of objective functions can be handled by reducing them to a dummy fitness function using non-dominated sorting procedure. Both minimisation and maximisation problems can be handled. Some researchers have reported a drawback of this method (Coello, 1997). The main weakness of this algorithm is that it is sensitive to the value of the sharing factor. In addition, it is more inefficient computationally and in terms of quality of the Pareto-optimal fronts produced than MOGA. Other authors report that the NSGA algorithm performed well in terms of coverage of the Pareto-optimal front (Zitzler et al., 1998, 1999).

2.2.4 Niched Pareto Genetic Algorithm (NPGA)

This algorithm is based on the NSGA method described in the preceding sub-section. The main difference occurs during the selection process (Horn et al., 1993). In a classical genetic algorithm, the method of selection between two individuals uses a selection wheel, but in this method, the way in which one can select individuals is changed. Instead of comparing two individuals, a group of individuals (typically around 10) is used to help determine dominance. Based on whether the individuals are dominated or non-dominated, a sharing fitness is decided through the tournament selection.

Strengths and Drawbacks

As this method does not use the Pareto selection on the whole population, but just on a part at each run, its main strengths are that it is very fast and can produce good non-dominated fronts that can be kept for a large number of generations. Nevertheless, its main weak point is that besides requiring a sharing factor, this approach also requires a good choice of the size of the tournament to perform well.

2.2.5 Strength Pareto Evolutionary Algorithm (SPEA)

SPEA algorithm is well-presented in Zitzler et al. (1999). In brief, Zitzler et al. (1999) have suggested an elitist MOEA with the concept of non-domination. Starting from the initial population, they proposed maintaining an external population at every generation storing a set of non-dominated solutions (Zitzler et al., 2001a). A combined population with the external and the current population is first constructed. All non-dominated solutions in the combined population are assigned a fitness based on the number of solutions they dominate. For maintaining the diversity, Zitzler et al. (1999) assigned more fitness to a non-dominated solution having more dominated solutions in the combined population. Conversely, solutions dominated by more solutions in the combined population are also assigned by less fitness.

Strengths and Drawbacks

The advantage of this algorithm is that the clustering used is parameter-less, thereby making it attractive to use. In the absence of the clustering algorithm, the SPEA has a similar convergence to that of Rudolph's algorithm which is presented elsewhere (Deb, 2001). The fitness assignment procedure in the SPEA is similar to that of Fonseca and Fleming's MOGA (Deb, 2001) and is easy to calculate. One of the drawbacks of this algorithm is that the overall complexity needed in each generation of the SPEA is $O(MN^2)$, where M is the number of objectives and N is the size of the population. The SPEA introduces an extra parameter, the size of the external population. It is extremely important for achieving a successful working of the SPEA to balance between the regular population size and the external population size. If the external population size is large comparable to the regular population size, the selection for the elites will be large and the SPEA may not be able to converge to the Pareto-optimal front. Here too, the SPEA shares a common problem with the MOGA. The problem is that the fitness assignment has a bias to not favour all non-dominated solutions of the same rank equally.

2.2.6 Strength Pareto Evolutionary Algorithm (SPEA2)

In 2001, the strength Pareto evolutionary algorithm presented above has been improved by Zitzler et al. (2001b). The main differences in SPEA2 in comparison to SPEA are (Zitzler et al., 2001b): (i) an improved fitness assignment scheme is used, which takes into account (for each individual) how many individuals it dominates and is dominated by, (ii) a nearest neighbour density estimation technique is incorporated which allows a more precise guidance of the search process, and (iii) a new archive truncation method which guarantees the preservation of boundary solutions.

Strengths and Drawbacks

SPEA2 differs from SPEA and NSGA-II (which will be presented in the next sub-section) only in how it does fitness assignment and selection (this makes SPEA2 among the most popular multi-objective optimisation algorithms). The main strength of the SPEA2 method is that it is much better than the original SPEA in all aspects of test problems. The found solutions are closer to the outlying edges of the Pareto-optimal front, making it a weak point of this algorithm.

2.2.7 Non-Dominated Sorting Genetic Algorithm (NSGA-II)

This algorithm is currently one of the most popular multi-objective optimisation algorithms. The author of this algorithm and his students (Deb, 2000a; Deb et al., 2002) suggested an elitist non-dominated sorting GA (termed NSGA-II) in 2000. This algorithm does not have much similarity with the original NSGA, but the authors kept the name NSGA-II to emphasise its genesis and place of origin (Deb, 2001; Deb et al., 2002). The NSGA-II carries out a non-dominated sorting of a combined parent and offspring generation. Thereafter, starting from the best non-dominated solutions, each front is accepted until all population slots are filled. This makes the algorithm an elitist type. For the solutions of the last allowed front, a parameter-less crowded distance-based niching strategy is used to resolve which solutions are carried over to the new population. The algorithm also has a crowded distance metric, which makes it fast and scalable to more than two objectives.

Strengths and Drawbacks

A crowding comparison procedure is used with the tournament selection during the population reduction phase to achieve the diversity among the non-dominated solutions. Also in this algorithm, no niching parameters is required. In the absence of the crowded comparison operator, this algorithm displays a convergence proof to the Pareto-optimal solutions, but the population size would grow with the generation number. Therefore, the overall complexity will be $O(MN^2)$, where M is the number of objectives and N is the size of the population. In addition, the elitism mechanism does not allow an already found Pareto-optimal solution to be deleted. Recent study has been proposed to reduce the run time complexity of the NSGA-II and other algorithms (Mikkel, 2003). The weak point of this algorithm is that when the crowded comparison is used to restrict the population size, the algorithm loses its convergence property. Only if the size of the non-dominated set is not larger than the population size, the algorithm preserves the convergence property. However, in later generations, when more than N individuals belong to the first non-dominated set in the combined parent-offspring population, some closely packed Pareto-optimal solutions may give their places to other non-dominated but non-Pareto-optimal solutions (Deb, 2001). Additionally, a comparison between NSGA-II and SPEA2 has been presented by Lam et al. (2004). The results derived from this comparison show that the SPEA2 outperforms NSGA-II in the early generations; however NSGA-II is superior during later generations irrespective of the level of noise present in the problem. Recently, a modified version of NSGA-II has been proposed by Babbar et al. (2003). In this paper, a clustering based modification to the NSGA-II ranking scheme, that improves the performance of the algorithm in noisy environment, is presented.

2.2.8 A Multi-objective Evolutionary Algorithm Based on Decomposition (MOEA/D)

The majority of existing MOEAs are based on Pareto dominance (Deb, 2001). In these algorithms, the utility of each individual solution is mainly determined by its Pareto dominance relations with other solutions visited in the previous search. Since using Pareto dominance alone could discourage the diversity of search, some techniques such as fitness sharing and crowding have often been used as compensation in these MOEAs. NSGA-II and SPEA-II are among the most popular Pareto dominance based MOEAs. Arguably, NSGA-II (Deb, 2001) is the most popular Pareto dominance based MOEAs. The characteristic feature of NSGA-II is its fast non-dominated sorting procedure for ranking solutions in its selection.

A Pareto-optimal solution to a multi-objective optimisation problem could be an optimal solution of a scalar optimisation problem in which the objective is an aggregation function of all the individual objectives. Therefore, approximation of the Pareto-optimal front can be decomposed into a number of scalar objective optimisation subproblems. This is a basic idea behind many traditional mathematical programming methods for approximating the Pareto-optimal front. A very small number of MOEAs adopt this idea to some extent, among them the Multi-objective Evolutionary Algorithm Based on Decomposition (MOEA/D) is a very recent one (Zhang and Li, 2007). MOEA/D attempts to optimise these subproblems simultaneously. The neighborhood relations among these subproblems are defined based on the distances between their aggregation coefficient vectors. Each subproblem (i.e., scalar aggregation function) is optimized in MOEA/D by using information only from its neighboring subproblems.

Strengths and Drawbacks

One of the major advantages of MOEA/D over Pareto dominance based MOEAs is that single objective local search techniques can be readily used in MOEA/D. A comparison on some multi-objective optimisation problems has been carried out between MOEA/D and NSGA-II (Li and Zhang, 2010). The results have shown that no solution obtained in NSGA-II dominates any solutions in MOEA/D but most solutions in NSGA-II are dominated by the solutions in MOEA/D

2.3 Classical Versus Evolutionary Approaches

Both classical and evolutionary approaches have their own drawbacks in tackling multi-objective optimisation problems. Most classical methods, such as weighted sum, convert multiple objectives into one objective using different heuristics (Miettinen, 1999). Since multiple objectives are converted into one objective, the resulting solution to the single objective optimisation problem usually depends on the weights. Although this is a weak point of this approach, this approach can handle any number of objectives; say for instance five objective functions. When using evolutionary algorithms to handle multi-objective optimisation with such number of objectives, the performance of most of the evolutionary algorithms gets poor.

Classical and evolutionary approaches also suffer from serious limitations in handling the complexity of multi-objective optimisation problems (Wang and Yan, 1991). The presence of variable interaction (such as linearity and nonlinearity between decision variables) may introduce some additional features in the optimisation problems. Literature reveals that neither classical nor evolutionary approaches can handle this situation except using numerical methods such as regression analysis that can approximate a curve/surface of the Pareto-optimal front. Classical approaches, in some simple cases of multi-objective problems, can provide an exact curve for both the relationship between the decision variables and the curve of the Pareto-optimal front

(Collette et al., 2003; Arora, 2004). Despite a wide spread applicability of evolutionary optimisation algorithms over the past few decades, evolutionary researchers still face criticism about the theoretical optimality of the solutions obtained by evolutionary techniques (Deb et al., 2007).

2.4 Theoretical Approaches for Multi-objective Optimisation

Problems with two or more objectives (called multi-objective or multi-criteria) are very common in engineering and many other disciplines (Collette et al., 2003). The solutions of such problems are difficult because their objectives tend to be in conflict. In this situation, one can not find a single ideal solution which simultaneously satisfies all the decision variables across all criteria. So, a solution that is extreme with respect to one objective function requires a compromise in other objectives (Deb, 2001). Because of such properties of these solutions, optimisation methods are of great important in practice, particularly in engineering design, scientific experiments, and business decision-making (Deb, 2001). The general consensus of the engineers and mathematicians working in this area is that the Pareto-optimal set may contain information that can help the designer to make a decision and thus arrive at better trade-off solutions. Also, this information might show the designer an issue that has been ignored such as the ease of manufacture or assembly. The main goal of this section is to give an overview of current optimality conditions along with the existing analytical methods which have been used so far for solving multi-objective optimisation problems.

Optimality conditions for multi-objective optimisation problems have been studied extensively in the literature. Many efforts have been made to derive first-order necessary and/or sufficient conditions for a feasible solution to be an optimal solution. Several studies relating to first-order optimality conditions have been reported and have drawn much attention for a long time, and many beneficial results have been obtained over the past several years since the appearance of the paper by Kuhn and Tucker (1950). Herein, some important results regarding the development and extension applied to first and second-order optimality conditions are reported (Beale, 1958; Box, 1966; Lin, 1976; First et al., Kanniappan, 1983, 1990; Zalmai, 1985a; Singh, 1987; Wang, 1988; Zemin, 1996; Majumdar, 1997; Bigi and Castellani, 2000; Messac et al., 2000; Miettinen and Makela, 2001; Zheng et al., 2007; Ginchev and Ivanov, 2008). However, little work has been concerned with second-order optimality conditions for multi-objective optimisation problems.

Convexity plays a very important role in various branches of engineering in management science and operations research, in mathematical economics, in pure and applied mathematics, in optimisation theory and in statistics. In recent years, there has been an ever-increasing interest in the generalisations of basic convexity concepts with the hope of extending well known properties, characterizations and applications. Many excellent texts and hundreds of articles have appeared in recent years. The optimisation theory of Dubovitskii and Milyutin (1965) has been used by Censor (1977) to produce optimality conditions for differentiable convex vector optimisation. Benson (1978) has examined the existence of efficient and properly efficient solutions for the vector maximisation problem. Kuhn-Tucker necessary conditions (Miettinen, 1999) for a maximum point of a convex function subject to convex constraints have been stated and proved by Batson (1986). Zalmai (1985a,1985b) has adopted a geometric point of view and developed continuous-time analogues of the Fritz-John (Miettinen, 1999) and Kuhn-Tucker (Miettinen, 1999) optimality conditions in the spirit of finite-dimensional nonlinear programming. Furthermore, the relationship between the first-order stationary-point necessary optimality criteria and saddle-point optimality conditions has been discussed. A generalisation of Kuhn-Tucker sufficiency conditions has been presented by Hanson (1994). The P-norm surrogate constraint method (Li, 1996) has been applied by Li (1999) to make convex non-inferior frontier into non-convex multi-objective optimisation problems under certain conditions. Theoretical results concerning the set of non-inferior solutions and a computational algorithm for obtaining a characteristic set of non-inferior solutions have been discussed by Beeson et al. (1971). Craven (1984) has presented a modified kind of Kuhn-Tucker condition applicable to the minimisation problem but not necessarily assuming convexity.

Kuhn-Tucker saddle-point and stationary-point optimality criteria and Lagrangian duality for a class of continuous-time convex programming problems have been established by Zalmai (1985b). Mishra et al. (1996) have extended the concept of V-pseudo-invexity and V-quasi-invexity of multi-objective programming to the case of nonsmooth multi-objective programming problems. A necessary global optimality conditions for quadratic optimisation problems with binary constraints have been established by Beck et al. (2000). Three optimality concepts, weak, proper, and Pareto optimality have been presented by Miettinen et al. (2001) using cones. Svanbery (2002) has presented and investigated a class of globally convergent optimisation methods based on the concept of conservative convex separable approximations. Huy et al. (2006) have developed sufficient global optimality conditions for identifying the global minimisers of a general non-convex smooth minimisation model problem involving linear matrix inequality constraints with bounds on the variables. Jeyakumar et al. (2006) have established conditions which ensure that a feasible point is a global minimiser of a quadratic minimisation problem subject to box constraints or binary constraints. New necessary optimality conditions of Fritz-John and Karush-

Kuhn-Tucker type, called G-F-John and G-Karush-Kuhn-Tucker conditions for differentiable constrained optimisation problems have been established by Antczak (2007).

Investigation of duality theory for the field of multi-objective optimisation problems (MOOP) has also grown and become one of the most interesting results in this field. It has been further extended by the applications of varied types of generalisations of convexity concept, with and without differentiability assumptions. Mishra et al. (2005a) have extended the class of generalised type I vector-valued functions introduced by Aghezzaf and Hachimi (1999, 2000). In this study, a number of Kuhn-Tucker (KT) type sufficient optimality conditions and duality results have been discussed. Maeda-type regularity conditions (1994) for second-order Kuhn-Tucker necessary conditions have been generalised by Rizvi and Nasser (2006). Particularly, much of the recent work in this area can be found in Mishra (1996, 1998, 2000), Mishra et al. (2000a,2000b,2002), Rueda et al. (1995), and Zhian et al. (2001).

As mentioned earlier, the convexity concept plays an important role in these conditions. Many authors have extended this concept and different kinds of functions have been established, with and without differentiability assumptions. New classes of generalised convex vector functions, namely V-invex functions, have been suggested by Jeyakumar and Mond (1992). In this work, the authors have introduced a class of invex functions which preserves the optimality and duality conditions in the scalar case, and avoids the major difficulty of verifying that the inequality holds for the same function. Assuming that the objective functions to be handled in a MOOP are differentiable and second-order pseudoconvex, a notion introduced by Ginchev and Ivanov (2007) has obtained second-order optimality conditions for a multi-objective optimisation problem with inequality constraints and a set constraint in nonsmooth settings using second-order directional derivatives.

Strict convexity, strong convexity, strict quasiconvexity, strict pseudo convexity etc, of functions, their properties, characterisations, applications to economics and optimisation theory are presented by Schaible and Ziemba (1981). Aleman (1985) gave generalisations of convex sets and functions while Dombi (1985) gave sufficient conditions for the existence of a strict local minimum of a quasiconvex function. Degiovanni et al. (1985) studied certain types of convexity via monotonicity of a sub-differential. Hirche (1985) derived general conditions under which sums and products of absolute values of functions are pseudoconvex while Crouzeix and Lindberg (1986) characterised additively decomposed quasiconvex functions. Martinez-Legaz (1988) characterised a class of functions which is contained in the class of quasiconvex lower semi-continuous functions. Castagnoli and Mazzoleni (1989a) studied derivatives of generalised concave functions, while, in (1989b), they investigated a unified approach to generalised convexity. Luc (1989) considered three generalisations of quasiconvexity and studied their relationships. Aze and Volle (1990) studied stability under quasiconvexity, while Borde and Crouzeix (1990) investigated continuity properties of normal cones to the level sets of quasiconvex function. Higgins and Polak (1990) studied the minimization of a pseudoconvex function over a compact set. Chavent (1991) investigated quasiconvex sets. Characterisations of classes of quasiconvex functions, differences of convex functions, the lower functions, semi-smooth functions and quasi-differentiable functions in terms of their properties of generalized Clarke's gradient were done by Ellaia and Hassouni (1991). Crouzeix, Ferland and Schaide (1992) investigated a characterisation of pseudoconvexity on affine subspaces. Properties of quasiconcave functions and their characterisations were investigated by Danao (1992) while Radzik (1991) established necessary and sufficient conditions for the existence of saddle-points of matrices that are closely related to functions of two variables and are quasiconvex/quasiconcave.

Parallel to the above development in the optimality conditions of multi-objective optimisation, there has been a very popular growth and application of the invexity theory. The concept of invexity introduced by Hanson (1981) was also generalized by Hanson and Mond (1987). They defined the so-called type I and type II objective and constraint functions. Rueda and Hanson (1988) characterised these classes of functions and presented classes of generalised type I and type II functions. Antczak (2001, 2002) extended the concepts of type I and type II objective and constraint functions and he introduced the classes of (p, r)-type I and (p, r)-type II objective and constraint functions for differentiable mathematical programming problems. Despite substituting invex for convex, many theoretical problems for differentiable programming can also be solved. But the corresponding conclusions cannot be obtained for nondifferentiable programming with the aid of invexity introduced by Hanson because the derivative is required in the definition of invexity (Hanson, 1981). While the derivative is required in the definition of invexity, there exists a generalisation of invexity to locally Lipschitz functions, with derivative replaced by Clarke generalised gradient (Craven, 1986, 1989). In these papersD, the directional derivatives have been used. Antczak (2009) has continued the extension of invexity notion into the nondifferentiable setting by defining a generalised notion of invexity for directionally differentiable vector-valued mappings. Thus, he has extended the concept of r-type objective and constraint functions introduced earlier in (Antczak, 2002) for differentiable functions. For vector optimisation problems, which constituting functions are directionally differentiable, he has defined the various classes of (generalised) d-r-type I objective and constraint functions. By considering the concept of a (weak) Pareto solution, he has proved sufficient optimality conditions and duality results in the sense of Mond-Weir for non-differentiable multi-objective programming problems with functions belonging to the classes of functions introduced in (Antczak, 2009). To prove these results, he used the Karush-Kuhn-Tucker necessary optimality conditions previously established in (Antczak, 2002) for directionally differentiable vector optimisation problems. Further, he has shown that weak duality in the sense of Wolfe is not satisfied between the considered non-differentiable multi-objective programming problem and its Wolfe duals in the case when the functions constituting these vector optimisation problems are d-r-type I objective and constraint functions. Antczak has illustrated this result by a suitable nondifferentiable multiobjective programming problem.

In the theory of constrained multi-objective optimisation problems, optimality conditions and duality results for differentiable nonlinear constrained problems are important theoretically as well as computationally and can be formulated in several different ways. In general, these criteria can be classified as either necessary or sufficient. As mentioned earlier, the best-known necessary optimality criterion for a constrained mathematical programming problem is constituted by the Karush–Kuhn–Tucker optimality conditions. However, the F-John criterion is in a sense more general. In order for the KKT optimality conditions to hold, one must impose some suitable constraint qualification (Miettinen, 1999) on the constraints of the optimisation problem. On the other hand, no such constraint qualification need be imposed on the constraints in order for the F-John optimality conditions to hold. In constrained optimisation, these necessary optimality conditions are also sufficient for optimality if the functions delimiting the mathematical programming problem are convex or satisfy certain generalised convexity properties such as pseudo-convexity or quasi-convexity (Mangasarian, 1969).

Karush-Kuhn-Tucker conditions are sufficient for optimality if the functions involved are convex. However, application of the Kuhn-Tucker conditions as sufficient conditions for optimality is not restricted to convex problems, and various generalisations of convexity have been made in order to explore the extent of this applicability. This raises the problem of finding exhaustive classes of functions which will ensure that the Kuhn-Tucker conditions are sufficient for optimality in constrained multi-objective optimisation problems. Under very general conditions Hanson and Mond (1987) have given an answer to this by introducing two new classes of functions, both closely related to, but more general than, invex functions. These two classes have been called preinvex functions. These two classes are associated with primal and dual problems respectively.

In Antczak (2007), a new class of non-convex functions, called G-invex functions has been introduced. He has extended an invexity notion (Hanson, 1981) since the defined class of functions contains many various invexity concepts. A characteristic global optimality property of various classes of invex functions has also been proved in the case of a G-invex function. It has been turned out that every stationary point of a G-invex function is its global minimum point. A sufficient condition for G-invexity has also been proved in (Antczak, 2007). In addition, some relations between the introduced class of G-invex functions with respect to η and various classes of invex functions with respect to η have been stated. Furthermore, new F- John-type and Karush–Kuhn–Tucker-type problems, called G-F-John and G-Karush–Kuhn–Tucker problems have been introduced, respectively; they were defined assuming differentiability of the functions involved. Then, Antczak (2007) has applied the G-invexity notion introduced to develop optimality conditions of F-John type and Karush–Kuhn–Tucker type for constrained differentiable mathematical programming problems. He has proved the sufficiency of the necessary optimality conditions of G-type defined for differentiable constrained optimisation problems involving G-invex functions with respect to the same function η, but not necessarily with respect to the same function G. In particular, He has finally obtained optimality conditions of F-John type and Karush–Kuhn–Tucker type that are weaker than previous conditions presented in literature. Further considerations have been devoted to duality in differentiable constrained mathematical programming problems. He has defined a new dual of Mond–Weir type, called the G-Mond–Weir dual, for the constrained optimisation problem considered. Then various duality theorems have been proved between the mathematical programming problem considered and the G-Mond–Weir dual problem introduced. The main tool in proving these duality results was the concept of G-invexity introduced. As in the case of establishing the sufficiency property for optimisation problems of this type, Antczak has assumed that all functions involved in the original optimisation problems is G-invex with respect to the same function η, but not necessarily with respect to the same function G. In addition, some examples have been given to illustrate a nature of the class of non-convex functions introduced by Antczak (2007).

2.5 Literature Review Analysis

In this section, an extensive literature survey is carried out in order to analyse and classify theoretical approaches used in tackling multi-objective optimisation problems. The literature research involves the investigation of 156 journal articles concerned with theorems and applications of multi-objective optimisation. These articles are drawn from the period 1993-2010. The articles have been classified by refereed journal articles from across some popular disciplines. Although there have been many books and conference proceedings in multi-objective optimisation, the literature survey has focused on journal articles and some selected fundamental books only. Conference proceedings are generally not included in this literature analysis. This for a simple reason that many theoretical results are regularly published in journal papers. The articles before 1993 are not included since the number of applications before 1993 is not significant in comparison with the duration after 1993. However, there exist some important theoretical works concerned with Karush-Kuhn-Tucker theorem and Fritz-John (FJ) as mentioned in the introduction.

2.5.1 Number of Articles

Figure 2.1 incorporates the references of multi-objective optimisation in the period mentioned above. Classification and analysis of these journal articles in certain areas will be discussed in detail in next sub-section. The diagram shows that there are several factors influencing this curve. Firstly, due to the importance of Pareto-optimal solutions for scientists and engineering designers, a lot of improvements on Karush-Kuhn-Tucker and Frit-John conditions have been proposed (Birbil et al., 2007; Lu et al., 2007; Li and Zhang, 2005; Vazquez aRuckmann, 2005; Shi et al., 2005; Brosowski, 2001). In these studies, different proofs of Fritz-John and Karush-Kuhn-Tucker conditions for nonlinear finite dimensional programming problems with equality and/or inequality constraints have been given. These avoid the implicit function theorem usually applied when dealing with equality constraints and use a generalisation of Farkas lemma and the Bolzano-Weierstrass property for compact sets (Birbil et al., 2007). Moreover, for an inequality constrained nonsmooth multiobjective optimisation problem, where the objective and constraint functions are locally Lipschitz (Miettinen, 1999), stronger Kuhn-Tucker type necessary optimality conditions have been derived that are expressed in terms of upper convexificators (Li and Zhang, 2006). In addition, other constraint qualifications sufficient for the nonsmooth analogue are introduced and their relationships are presented. Furthermore, a proof of Karush-Kuhn-Tucker conditions has been illustrated using Zorn's Lemma (Ehrgott, 2000). Secondly, several mathematical optimisation approaches have been devised to deal with multi-objective programming problems. For instance, and as mentioned in the previous sections, the weighted-sum approach is the most popular one used in this context. However, it has some disadvantages (Miettinen, 1999; Collette, 2003; Deb, 2001). Recently, an adaptive weighted-sum method has been proposed by Kim and De Weck (2005) to overcome some of these drawbacks. Thirdly, the existence and importance of multiple objectives in various disciplines has made researchers very keen to find out new approaches and algorithms to tackle multi-objective problems (Zhao and Dimirovski, 2004; Balbas et al., 2002).

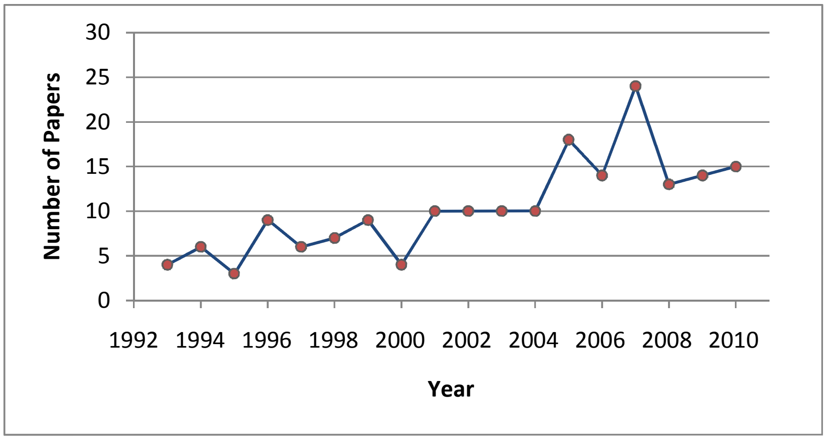

Figure 2.1: Articles in the period 1993-2010. These articles show the theoretical techniques used to solve MOOP. The evolutionary algorithms have not been included.

2.5.2 Analysis of Articles by Application Areas

In this sub-section, a breakdown of the articles in four application areas is carried out. These areas are theoretical, Operations Research (OR), Mechanical engineering, and Finance applications. The theoretical articles have the highest number among the application areas. From 156 articles in this literature analysis, there are 124 theoretical articles, i.e. 79%. This high percentage has been influenced by several factors. Firstly, this area has techniques that have been used not only in real-life applications but also in simple multi-objective optimisation problems (Zhou et al., 2007; Luo et al., 2006). For instance, it has been proven that the Kuhn–Tucker point of nonlinear programming problem is an asymptotically stable equilibrium point of a system of differential equations. Secondly, the KKT optimality conditions are valid for convex and continuously differentiable constraints and objective functions; so an extension and modification of these optimality conditions have been proposed (Maeda, 2004; Stein, 2006; Taa, 2005). In these extensions, second-order Kuhn-Tucker type necessary conditions for efficiency under constraint qualification have been introduced. In addition, optimality conditions in terms of Fritz-John and Karush-Kuhn-Tucker multipliers for vector optimisation problems when the objective and constraints are defined by the difference of convex mappings have been investigated. Thirdly, in the theoretical part, there are many papers dealing with multiple objective functions (higher dimensions in the objective space) and are involved in theorem developments (Jeyakumar et al., 2007; Lv et al., 2008; Maity and Maiti, 2007). For example, Kuhn-Tucker sufficiency conditions for possibly multi-extremal non-convex mathematical programming problems, which may have many local minimisers[3] that are not global, have been demonstrated. Regarding the other areas, an observation arising from analysing the articles is that there are some theoretical articles that have been applied in both OR and Finance application areas (Zhang and Hua, 2008; Birbil et al., 2007; Miglierina et al., 2008; Wu, 2007). The KKT conditions have been used to solve problems in these areas. However, there are only a few theoretical applications which have been used to solve even simple mechanical engineering problems (Primbs and Giannelli, 2001; Sadegh, 1997). This small number of theoretical applications applied to engineering problems is because of their complexity. Evolutionary algorithms have been used generally to optimise a larger class of engineering problems. There are a plenty of papers as mentioned in the previous sections used for this purpose. As mentioned before, the scope of the literature survey is the analytical optimisation methods not the evolutionary techniques, so any details about these evolutionary algorithms are left out from this literature as has been given previously. Real-life applications or specifically engineering applications are a significant class of multi-objective problems that might still need to be solved using these theoretical approaches to find out any mathematical relationship between decision variables that might help the engineering designer (Deb, 2003). Figure 2.2 illustrates the breakdown of the articles into four important areas of application.

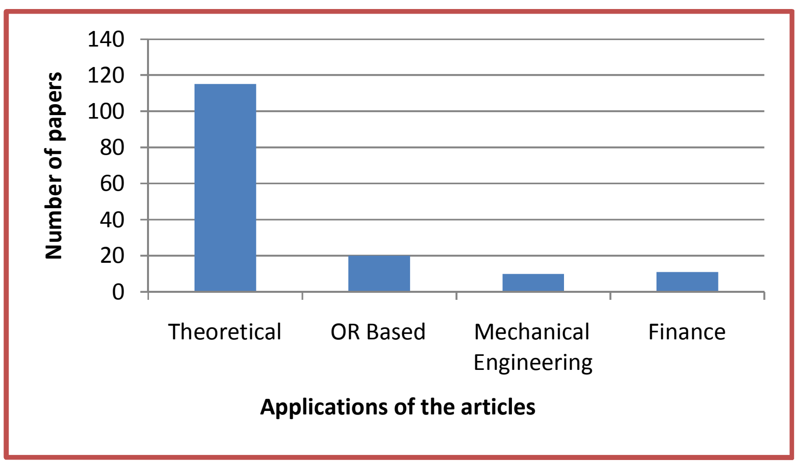

Figure 2.2: Analysis of articles by application areas

2.5.3 Itemisation of Articles by Theoretical Approaches

Figure 2.3 demonstrates the prevalence of theoretical approaches that have been applied to solve MOOP in many disciplines. The KKT and FJ conditions are the core of all studies (Li and Zhang,

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[3] Let be a function of a single variable with continuous first and second derivatives. Suppose is a stationary point of in the interior of — . If — then is a local maximiser.

2006; Gutierrez et al., 2005; Gutierrez et al., 2006; Liu and Sun, 2004; Martinez and Svaiter, 2003; Ye and Zhu, 2003) that have been analysed in the literature survey. For instance in these studies, the constraint qualifications and Kuhn-Tucker type necessary optimality conditions for minimisation and maximisation problems with arbitrary set constraints or inequality constraints, where the objective and the constraint functions are real-valued and locally Lipschitz continuous, have been developed. The qualifications and the necessary optimality conditions have been stated in terms of upper or lower convexificators. Moreover, Fritz-John and Kuhn-Tucker type necessary and sufficient conditions for Helbig's approximate solutions (Gutierrez et al., 2006) to deal with approximate Pareto solutions in convex multiobjective optimisation problems have been obtained. In addition, Fritz-John and Kuhn-Tucker necessary and sufficient conditions for a nondifferentiable convex multiobjective optimisation problem whose feasible set is defined by affine[4] equality constraints, convex inequality constraints, and an abstract convex set constraint have been obtained. According to the literature survey, approximately 70 % of the articles in the literature are using KKT conditions as a basis of all the modifications needed to achieve relaxations of the existing analytical approaches utilised in solving multi-objective optimisation problems. Recently, fuzzy logic has been applied to tackle multi-objective optimisation as well (Maity and Maiti, 2007; Abo-Sinna et al., 2008; Mahapatra and Roy, 2006; Wu, 2004). For example, a fuzzy-stochastic approach has been applied to solve real-life multi-objective optimisation applications. In additions, fuzzy multi-objective mathematical programming with generalised triangular fuzzy number has been applied to a reliability problem subject to complex system design and consequently a new fuzzy multi-objective optimisation method has been introduced and used for decision making of series and complex system reliability with two objectives. Fuzzy logic has got 5.8 % of the articles used in this literature analysis. As one can see, this percentage is very small in comparison with the area of KKT. This reveals that this area is a fresh area of research and can address further challenges in the field of multi-objective optimisation problems. As can be seen from Figure 2.3, Karush-Kuhn-Tucker conditions (Sharma et al., 2007; Mijangos and Nabona, 2000; Preda and Chitescu, 2000) are still widely-used to solve MOOP. This is for a simple reason that most of multi-objective optimisation problems satisfy KKT conditions and it is a better way to check the Pareto optimality of the solutions obtained by these approaches.

[4] A function $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is called affine if there is a linear function $L: \mathbb{R}^m \rightarrow \mathbb{R}^n$ and a vector $\boldsymbol{b}$ in $\mathbb{R}^n$ such that $A(\boldsymbol{x}) = L(\boldsymbol{x}) + \boldsymbol{b}$ for all $\boldsymbol{x}$ in $\mathbb{R}^m$

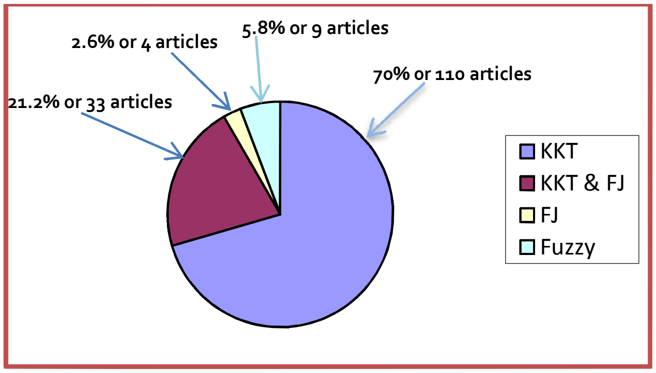

Figure 2.3: Analysis of articles by theoretical approaches

2.5.4 Analysis of Articles by Objective Functions and Constraints

In Figure 2.4, the objective functions involved in the articles are categorised. The figure shows that a large number (approximately 61.1 %) of the objective functions in literature are non-linear constrained objective functions (Li et al., 2008; Zishuang and Zheng, 2007; Antczak, 2007; Dahiya et al., 2007; Golishnikov and Izmailov, 2006; Bahahadda and Gadhi, 2006; Sach et al., 2005; Balbas et al., 2005). One reason for this high percentage is that a large number of real-life multi-objective optimisation problems are non-linear constrained problems. Another important reason is that the intricacies that appear when dealing with the analytical calculations involved in the optimisation process make this sort of multi-objective optimisation problems very significant in applications. Later on, a number of these problems will be used in the test and validation process of the proposed approaches.

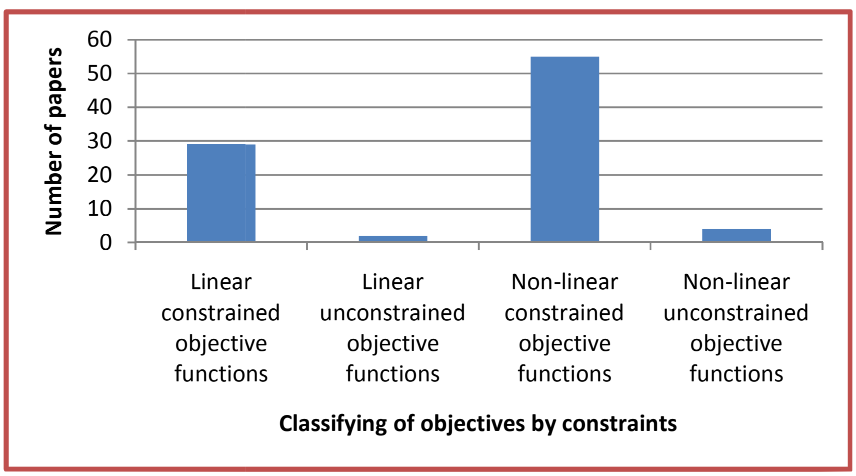

Figure 2.4: Analysis of articles by objective functions

2.5.5 Analysis of Articles by Optimisation Approaches

Figure 2.5 clearly demonstrates the optimisation techniques used in literature. Approximately 80.1% of all articles have used theoretical approaches for solving multi-objective optimisation problems. A large percentage of these theoretical approaches are dependent on the extensions of Karush-Kuhn-Tucker conditions. Out of this large percentage, there is a small number of articles that have been applied to simple multi-objective problems in order to get the Pareto-optimal solution analytically (Luo et al., 2006; Roger et al., 1993). The simulated annealing and simplex method have been employed to solve different benchmark nonlinear constrained problems. Some of these articles are also modification or extension of KKT conditions. The overall number of theoretical approaches reflects that the theoretical techniques of optimisation still need modifications and extensions to formulate and solve multi-objective optimisation problems.

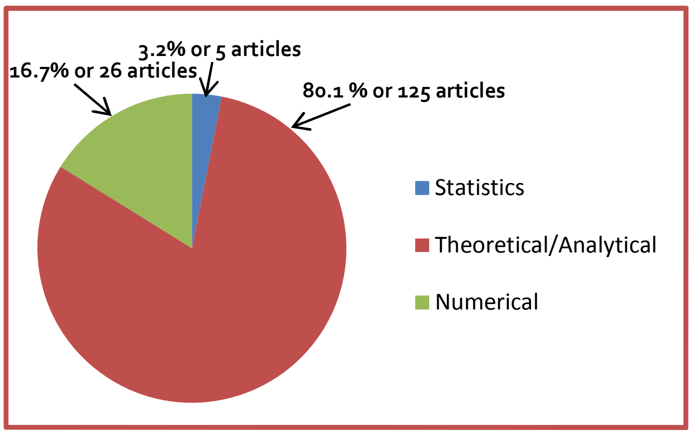

Figure 2.5: Analysis of articles by optimisation approaches

2.5.6 Analysis of Articles by Statistical/Numerical Approaches

There are only a few statistical and numerical approaches that have been used to solve multi-objective optimisation problems. These have got 3.2% and 16.7% respectively out of all the articles included in the literature. For instance, a finite series of Monte-Carlo estimators for the construction of gradient-type methods to develop implementable algorithms of stochastic programming have been developed. Furthermore, the inexact Newton approach has been used to solve the constrained system of nonlinear equations obtained by the Karush–Kuhn–Tucker (KKT) optimality conditions. There are other approaches that have been used to solve multi-objective optimisation problems numerically. Some of these approaches are gradient method, simplex method, reliability index method and interior point method. The table below gives some selected articles which have used numerical or statistical methods to deal with multi-objective optimisation problems. Within Table 2.1, LCOF means Linear Constrained Objective Functions and NLCOF means Nonlinear Constrained Objective Functions. Not all the articles using numerical and statistical approaches in the literature are involved in this table. This is just a sample of the numerical and statistical approaches used so far.

Table 2.1: Some selected papers on statistical and numerical approaches from literature

Numerical/Statistical Approaches	Type of Objectives	Application Area	Selected Articles
Simplex method	LCOF	Finance	Lu et al. (2007) Yang et al. (2002)
	LCOF	Theoretical	Luo et al. (2006)
Monte-Carlo method	LCOF	Finance	Sakalauskas (2002)
	NLCOF	Theoretical	Agarwal et al. (2004)
Gradient method	NLCOF	Theoretical	Abo-Sinna et al. (2008) Bonettini et al. (2007) Qi et al. (2004)
Reliability index method	NLCOF	Mechanical engineering	Maeda (2004)
Newton method	NLCOF NLCOF	Theoretical	Bonettini et al. (2005) Golishnikov et al. (2006) Herskovits (1998) Kanzow et al. (1995)
Interior point method	LCOF/NLCOF	Theoretical	Pacelli et al. (2001) Roger et al. (1993)

2.6 *Current Research Trends*

From analysing the literature survey, the following main research trends are identified.

- The first point to note is that there is a trend to modify the existing techniques of optimisation. For example, previous research has shown that the weighted-sum technique often produces poorly distributed solutions along Pareto-optimal front, and that it does not find Pareto-optimal solutions in non-convex regions (Miettinen, 1999; Collette, 2003). Kim and De Weck (2005) have recently proposed a modification of this approach so as to produce well-distributed solutions in a non-convex area.
- The well-known Karush-Kuhn-Tucker conditions have been extended so that they can provide necessary and sufficient conditions for multi-objective optimisation problems. For example, Lu et al. (2007) have extended the KKT approach to deal with linear referential uncooperative bi-level multi-follower decision problem. Jeyakumar et al. (2007) have presented new KKT sufficiency conditions for possibly multi-extremely non-convex mathematical optimisation problems.

- Due to the presence of non-convex objective functions that make most of the algorithms get stuck to local minima that are not global minimisers, a new direction to develop new conditions for identifying the global minimisers for such problems is thoroughly being researched (Huy et al., 2006; Jeyakumar et al., 2006).
- Since the optimality conditions and duality results for differentiable nonlinear constrained problems are extremely important (Sheng and Liu, 2006; Mishra et al., 2000b, 2008), another current research trend to introduce new class of differentiable functions (Antczak, 2007; Mishra et al., 2004, 2009) is being developed to tackle the challenges posed by multi-objective optimisation problems.

2.7 Research Gaps

Section 2.6 presented and classified the references on optimisation techniques for multi-objective optimisation. The review of theoretical and evolutionary optimisation approaches was based on the classification of the types of multi-objective optimisation problems handled using these approaches. These classifications resulted in visually highlighting a number of interesting observations and especially the lack of certain approaches.

Optimising multi-objective problems for finding the Pareto-optimal front is a significant research area and still needs extra work to challenge the established ideas and propose new innovative ideas that can add to this field. From the salient observations which have been obtained from analysing the literature articles, the following research gaps can be identified:

- Formulating and developing the optimality conditions of multi-objective optimisation, such as KKT conditions, in another way to help researchers to find out more mathematical characteristics of the Pareto-optimal set (Roghanian et al., 2008; El Maghri and Bernoussi, 2007; Rizvi et al., 2006; Zhou and Zhang, 2005).
- Developing new classes of multi-objective optimisation problems and discussing the optimality conditions and duality results for these classes.
- Developing symbolic algorithmic packages for handling multi-objective optimisation problems to overcome the challenges posed by real-life problems.
- Finding out the relationship between decision variables (variables interaction) in an analytical form for multi-objective optimisation problems. Discovering this relationship will provide exact design principles to the engineering designers and will give insights about the problem at hand.

2.8 Summary

This chapter has achieved the following.

- It has presented a literature review of the theoretical results regarding optimality conditions of multi-objective optimisation problems.
- It has presented an overview of EC techniques for solving MOOP.
- It has classified the articles identified in the literature survey.
- It has identified the current trends.
- It has finally identified the research gaps.

As mentioned in Chapter 1, this research attempts to develop symbolic optimisation algorithms that are capable of dealing with the challenges of multi-objective optimisation problems based on the extension of current theoretical results. The current chapter has given an overview of the theoretical results for handling these problems. This survey of literature enables the identification of the research aim and objectives in the next chapter.

3 RESEARCH AIM, OBJECTIVES AND METHODOLOGY

This chapter identifies and discusses the aim and objectives of this research. Based on these, the research scope and methodology are also elaborated and discussed. This chapter discusses the following:

- *Research aim.*
- *Objectives.*
- *Scope.*
- *Methodology.*

3.1 Research Aim

The research aims to develop new multi-objective optimisation symbolic algorithms capable of detecting relationship(s) among decision variables that can be used for constructing the analytical formula of Pareto-optimal front based on the extension of the current theoretical research. This research will extend the theoretical optimality conditions that are regularly used in multi-objective optimisation problems to deal with the challenges posed by certain features of multi-objective optimisation problems such as multiple objectives, constraints and interaction among decision variables. This would enhance the effectiveness of optimisation algorithms by giving them the capability of dealing with a wide variety of real-life problems.

3.2 Research Objectives

There are a number of research issues that are involved in the fulfilment of the aim of this research. The research objectives which address these issues are broken down into specific objectives as follows.

1. To construct the first-order optimality conditions of Karush-Kuhn-Tucker under univexity concept for multi-objective optimisation problems.
2. To demonstrate the duality results corresponding to the first-order optimality conditions.

3. To extend the second-order optimality conditions of Karush-Kuhn-Tucker under a relaxation of the convexity concept.
4. To propose multi-objective symbolic algorithms based on the extension of the existing theoretical results of multi-objective optimisation for extracting the relationship among the decision variables used in constructing the Pareto-optimal front.
5. To compare the performance of the proposed algorithms with a popular optimisation algorithm (NSGA-II) using selected test problems from literature.
6. To validate the performance of the proposed algorithms using three appropriately selected real-life case studies.

3.3 Research Scope

Based on the research objectives mentioned above, the scope of this research can be summarised as follows.

Context: The optimality conditions and solutions proposed in this research are with regard to multi-objective optimisation problems.

Domain: This research focuses on multi-objective optimisation problems.

Optimality Conditions: As mentioned in the previous chapter, this research concentrates on the optimality conditions of Karush-Kuhn-Tucker due to their broad applicability to multi-objective optimisation problems.

Literature Survey: The literature survey in this research concentrates on the analytical and evolutionary computation techniques used for tackling multi-objective optimisation problems with certain features: multiple objectives, constraints and variable interaction.

Development of Optimality Conditions: This research focuses on the development of optimality conditions of Karush-Kuhn-Tucker. The convexity concept in these conditions is relaxed in both the first and second-order optimality conditions. In addition, the first-order optimality conditions are established under the univexity concept.

Development of symbolic Optimisation algorithms: In this research, symbolic optimisation algorithms are proposed for handling certain characteristics of multi-objective optimisation problems: multiple objective functions, constraints and variable interaction.

Validation: In this research, the validation is carried out using multi-objective test problems and real-life case studies borrowed from literature. These case studies are analysed, and three of them are selected in such a way that a board spectrum of characteristics is analysed.

3.4 Research Methodology

As research methodology is one of the most significant steps required to achieve the main activities of this research, it is clarified and explained here in more details. A pictorial representation of this methodology is given in Figure 3.1, which also forms the basis for the layout of this thesis.

3.4.1 Problem Identification

As mentioned in chapter 1, this research is dedicated to discover the relationship(s) among decision variables responsible for constructing a complete curve/surface of Pareto-optimal front. The problem statement of this research is derived based on the argument found elsewhere (Deb and Srinivasan, 2008) that is Pareto-optimal solutions are not arbitrary solutions, but rather solutions which mathematically must satisfy the so-called Karush-Kuhn-Tucker conditions. Hence, this research shares the vision of proposing symbolic optimisation algorithms capable of exploiting the Karush-Kuhn-Tucker conditions for detecting these relationship(s).

3.4.2 Literature Survey

An extensive literature survey is carried out as part of this research in order to analyse and classify the analytical approaches and evolutionary computation techniques used for handling multi-objective optimisation problems. Based on the primary focus on analytical approaches and evolutionary techniques for multi-objective optimisation problems, the literature survey is carried out with respect to certain characteristics of these problems: multiple objectives, constraints and variable interaction. The literature research involves the investigation of books, peer reviewed journals and conferences, on-line articles in order to attain an in-depth knowledge. This assists in reaching a clear understanding of the existing work in terms of its strengths and weaknesses and in identifying the research gap.

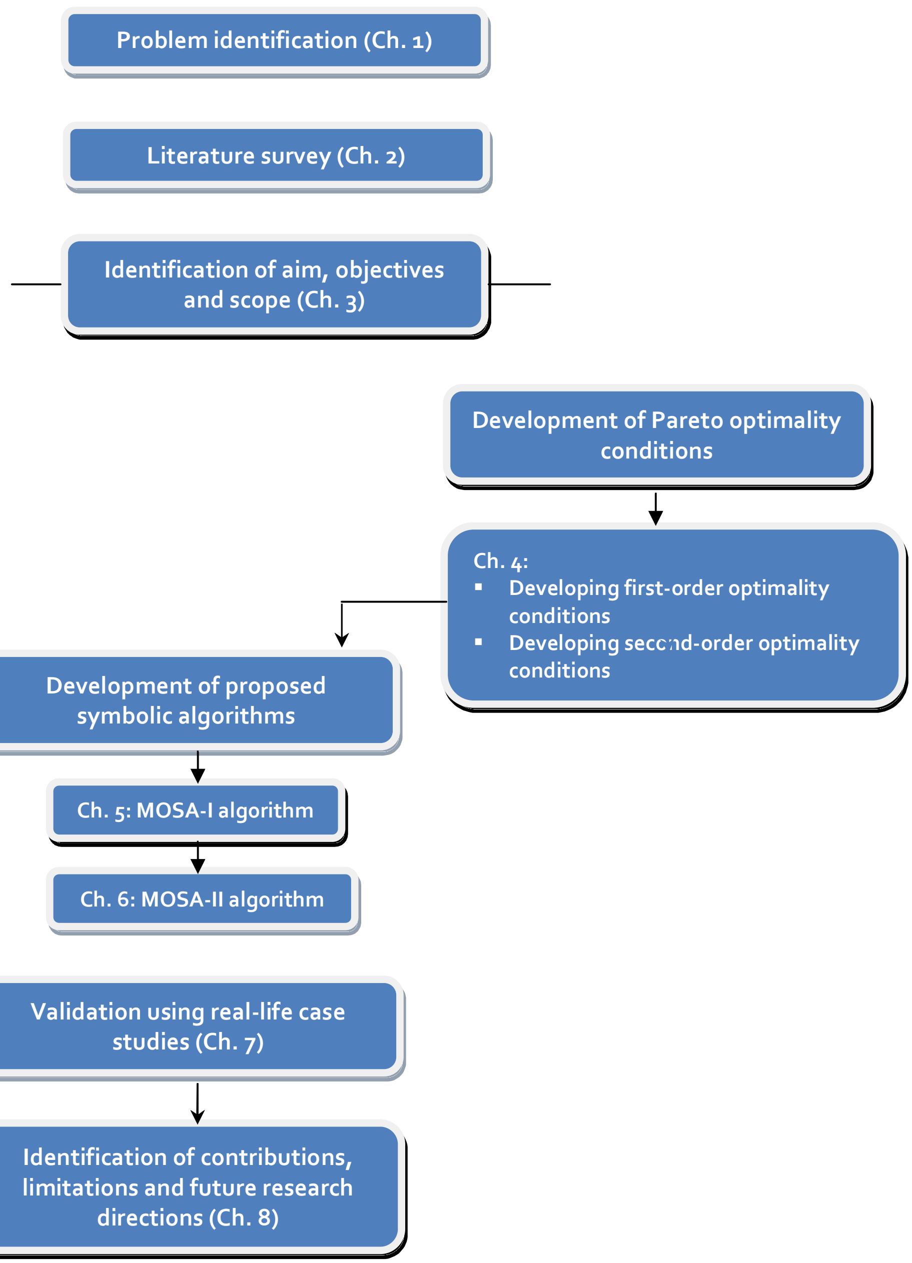

Figure 3.1: Main steps of the research methodology

3.4.3 Identification of Research Aim, Objectives and Scope

Along with the problem statement, the survey of literature provides evidence on the main research issues that need to be addressed in order to push forward the domain knowledge and provide solutions for handling optimisation problems. This enables the precise definition of the aim and objectives that this research seeks to address. The literature survey also enables the identification of the drawbacks of the evolutionary computation algorithms in tackling certain characteristics of optimisation problems. In addition, it provides the limitations of the current optimality conditions used in multi-objective optimisation. These limitations of existing evolutionary techniques and current optimality conditions define the focus of this research.

3.4.4 Development of Optimality Conditions

In this research, optimality conditions, particularly Karush-Kuhn-Tucker and Fritz-John conditions are analysed and further developed. First-order optimality conditions of Karush-Kuhn-Tucker under a generalised type of univexity functions are developed. In chapter 4, some theoretical classes of multi-objective optimisation problems are proposed. For these classes, extra conditions are imposed on the objectives and constraints to achieve univexity concept. Based on these additional conditions, the first-order optimality conditions for the problems in each class are separately proposed and proved. Furthermore, the duality results for each class are suggested and some first-order duality theorems are illustrated and proved. In addition, some sufficient second-order optimality conditions for multi-objective optimisation problems are established. Three important theorems regarding the second-order optimality conditions are proposed and proved. In these theorems, a generalised form of the convexity concept (such as strong pseudoquasi type I univex, weak strictly pseudoquasi type I univex and weak quasistrictly pseudo type I univex) is used to relax the KKT conditions. Based on these types of convexity, new second-order KKT optimality conditions are given and proved. Consequently, the applicability of KKT conditions to multi-objective optimisation is being extended.

3.4.5 Development of Symbolic Optimisation Algorithms

Based on the argument found in Deb and Srinivasan (2008), that it can be said Pareto-optimal solutions are not arbitrary solutions, but rather solutions which mathematically must satisfy the so-called Karush-Kuhn-Tucker conditions. Two multi-objective symbolic algorithms are proposed in this research to address the drawbacks of existing ones in handling multiple objective functions, constraints and variable interaction. The establishment of these algorithms is carried out in a systematic, step-by-step fashion based on the relaxation of existing optimality conditions of Karush-Kuhn-Tucker. In chapter 5, **MOSA-I** algorithm is proposed. This algorithm is used to solve multi-objective optimisation problems that involve differentiable and convex objectives. It is used to discover the relationship responsible for constructing the Pareto-optimal front. It is also used to provide an analytical formula of the relationship between the conflicting objectives. In chapter 6, a generalised version of the above algorithm, called **MOSA-II**, is introduced. This algorithm is used to deal with a widen class of multi-objective optimisation problems in which the convexity concept is generalised. This algorithm is capable of tackling multi-objective optimisation problems using second-order optimality conditions. The second-order optimality conditions are used to reduce the solutions obtained by the first-order optimality conditions included in MOSA-I. The MOSA-II algorithm is more robust than the previous one. It does not need the objectives to be convex. Hence, multi-objective optimisation problems that have differentiable objectives and constraints or even have objectives and constraints which are not convex can be solved by this algorithm. It only needs the mathematical descriptions of the objectives and constraints.

The performance of the proposed algorithms is compared, using some popular test problems that are carefully selected from literature, with a state-of-the-art optimisation technique, NSGA-II, in its standard form. Furthermore, the proposed algorithms are used for measuring the performance of NSGA-II. The performance of NSGA-II is measured using the generational distance metric (Coello et al., 2002). This is explained and shown later on in more detail in chapter 6.

3.4.6 Performance Analysis Using Test Problems

Multi-objective optimisation test problems are problems that have been artificially framed to examine and confirm the performance of algorithms. A number of these exist for multi-objective optimisation in the evolutionary multi-objective evolutionary optimisation (EMO) literature (Deb et al., 2001; Zitzler et al., 2002). The reason for developing controllable yet challenging test problems for optimisation and using them to test an optimisation methodology is to investigate the problem difficulties, for which a method performs well and problem features, for which they do not perform so well. Such problem features will enable developers and researchers to get a better insight to the working of different algorithms, a process which may help them develop better and more efficient algorithms. Since the exact Pareto-optimal fronts were known to most of these test problems, these test problems could simply be used to investigate if an algorithm is able to find well-represented set of Pareto-optimal solutions or not. If they did, the applied optimisation procedure might be considered to have overcome whatever difficulties these problems were providing. If they did not, it faces difficulties in analysing why the applied methodology could not solve the problem and what can be done to improve the algorithm for solving the problem. Here, the performance of the proposed symbolic algorithms is compared with a state-of-the-art optimisation technique. The comparison is implemented using a wide spectrum of test problems carefully chosen from literature. These selected problems are solved by the state-of-the-art stochastic algorithm, NSGA-II. The choice of NSGA-II for comparison is based on the fact that it has been shown in literature to outperform all existing techniques in tackling these selected test problems within this thesis. The reason for choosing these problems is the way they have complex inseparable function interaction leading to multi-modality in the search space, discontinuity in Pareto-optimal front and bias in the search space.

3.4.7 Validation Using Real-life Case Studies

Along with the test problems used to analyse the performance of the proposed symbolic algorithms, some real-life problems are also used in the validation process. Here, a set of real-life engineering design problems reported in literature are analysed from the perspective of the challenges that they pose for optimisation algorithms. The performance of the proposed symbolic optimisation algorithm, MOSA-II, is validated using three appropriately chosen real-life case studies. These real-life case studies are: the design of a pressure vessel, a three-member truss and a welded beam. They are selected based on some characteristics such as, the number of the objective functions, the number of the constraints, and the complexity of the search space (non-linearity, convexity, concavity, discontinuity of the Pareto-optimal front…etc.). Many criteria are used for comparison between these real-life case studies such as complexity of the objective functions (convergence to the Pareto-optimal front, diversity of solutions on the Pareto-optimal front). An elaborate description of these real-life case studies and the criteria is presented in chapter 7.

Although MOSA-I is applied to test problems, it requires the objectives and constraints to have their mathematical descriptions. In addition, the objectives and constraints should be continuous, differentiable and convex. Therefore, using multi-objective optimisation problems that do not involve the above aspects makes MOSA-I fail to tackle this sort of problems. For these reasons, MOSA-II is chosen for the validation process as it overcomes some of the above mentioned aspects. In this way, this research proposes a fully tested and validated methodology for dealing with multi-objective optimisation problems.

3.4.8 Identification of Contributions, Limitations and Future Research Directions

Finally, the contributions and limitations of the research methodology and the proposed theoretical developments and symbolic optimisation algorithms are identified. Based on this contributions to knowledge made by this thesis are established and the corresponding future research directions are proposed.

3.5 Summary

This chapter has discussed the following.

- It has stated the research aim.
- It has outlined the objectives that address the aim of this research.
- It has summarised the scope of this research based on its objectives.
- This chapter has finally discussed the methodology that has guided this research to achieve its aim and objectives. This methodology composes of seven main parts, as given below.
    - ✓ **Problem identification.**
    - ✓ **Literature survey.**
    - ✓ **Identification of research aim and objectives.**
    - ✓ **Development of Pareto-optimality conditions.**
    - ✓ **Development of symbolic optimisation algorithms.**
    - ✓ **Testing and validation.**
    - ✓ **Identification of contributions, limitations and future research directions.**

As stated in this chapter, the aim of this research is to develop symbolic optimisation algorithms that are capable of dealing with the challenges of multi-objective optimisation problems based on the extension of the current theoretical results. The next chapter develops first and second-order optimality conditions for handling these challenges.

4 FIRST AND SECOND-ORDER OPTIMALITY CONDITIONS FOR MULTI-OBJECTIVE OPTIMISATION PROBLEMS UNDER GENERALISED TYPE I UNIVEXITY FUNCTIONS

In this chapter, the univexity concept (Chapter 1) is applied to selected classes of multi-objective optimisation problems. Some different theoretical classes of multi-objective optimisation problems are proposed. For each class, extra conditions are imposed on the objective and constraint functions so that they can achieve the univexity concept. Based on these additional conditions, the first-order optimality conditions for problems in each class are proposed and proved. Furthermore, the duality results for each class are suggested and then some first-order duality theorems are illustrated and proved. In addition, the second-order optimality conditions of Karush-Kuhn-Tucker under a generalised type of univexity functions are proposed. Consequently, some sufficient second-order optimality conditions for multi-objective optimisation problems are established. Three important theorems regarding the second-order optimality conditions are proposed and proved. This chapter attempts to achieve the following.

- *To propose theoretical multi-objective optimisation problems.*
- *To propose first-order optimality conditions for the proposed multi-objective optimisation problems.*
- *To establish duality results for the above problems.*
- *To propose second-order optimality conditions.*

4.1 Proposed Multi-Objective Optimisation Theorems

In this section, some theoretical multi-objective optimisations problems are presented. To illustrate the univexity of the functions involved in these problems, additional conditions are included.

Theorem 4.1.1: Consider the function $f: \mathbb{R}^n \longrightarrow \mathbb{R}^p$ defined by

$$f(\boldsymbol{x}) = \left(f_1(\tau(\boldsymbol{x})), f_2(\tau(\boldsymbol{x})), \dots, f_p(\tau(\boldsymbol{x}))\right)^T$$

Where $f_i: \mathbb{R}^n \longrightarrow \mathbb{R}, i = 1,2, \dots, p$ are strongly pseudoconvex functions with real positive functions $\alpha_i(\boldsymbol{x}, \widehat{\boldsymbol{x}}), \tau(\boldsymbol{x}): \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is surjective[1] with $\nabla\tau(\widehat{\boldsymbol{x}})$ onto for each $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$, then the function $f(\boldsymbol{x})$ is strong pseudoquasi type I univex.

Proof:

To see that, let $\boldsymbol{x}, \widehat{\boldsymbol{x}} \in \mathbb{R}^n$ and with the surjective of τ there exist $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ such that $\tau(\boldsymbol{x}) = \boldsymbol{u}, \tau(\widehat{\boldsymbol{x}}) = \boldsymbol{v}$. By the strong pseudo convexity one can get,

$$\begin{aligned} f_i(\tau(\boldsymbol{x})) - f_i(\tau(\widehat{\boldsymbol{x}})) &= f_i(\boldsymbol{u}) - f_i(\boldsymbol{v}) \\ &\geq \alpha_i(\boldsymbol{u}, \boldsymbol{v})\nabla f_i(\boldsymbol{v})(\boldsymbol{u} - \boldsymbol{v}) \end{aligned}$$

As $\nabla\tau(\widehat{\boldsymbol{x}})$onto for each $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$, then $\boldsymbol{u} - \boldsymbol{v}$=$\nabla\tau(\widehat{\boldsymbol{x}})\eta(\boldsymbol{x}, \widehat{\boldsymbol{x}})$ can be solved for some $\boldsymbol{\eta}(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \in \mathbb{R}^n$. Hence,

$$\begin{aligned} f_i(\tau(\boldsymbol{x})) - f_i(\tau(\widehat{\boldsymbol{x}})) &= f_i(\boldsymbol{u}) - f_i(\boldsymbol{v}) \\ &\geq \alpha_i(\boldsymbol{u}, \boldsymbol{v})\nabla f_i(\boldsymbol{v})\nabla\tau(\widehat{\boldsymbol{x}})\eta(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \\ &= \bar{\alpha}_i(\boldsymbol{x}, \widehat{\boldsymbol{x}})\nabla f_i\big(\tau(\widehat{\boldsymbol{x}})\big)\nabla\tau(\widehat{\boldsymbol{x}})\eta(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \\ &= \bar{\alpha}_i(\boldsymbol{x}, \widehat{\boldsymbol{x}})\nabla\left(f_i\big(\tau(\widehat{\boldsymbol{x}})\big)\right)\eta(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \end{aligned}$$

Where, $\overline{\alpha}_i(\boldsymbol{x}, \widehat{\boldsymbol{x}}) = \alpha_i(\boldsymbol{u}, \boldsymbol{v}) = \alpha_i\big(\tau(\boldsymbol{x}), \tau(\widehat{\boldsymbol{x}})\big) > 0$. Assuming that φ_0 is a positive function, we have

$$b_0(\boldsymbol{x}, \widehat{\boldsymbol{x}})\varphi_0[f_i(\tau(\boldsymbol{x})) - f_i(\tau(\widehat{\boldsymbol{x}}))] \geq b_0(\boldsymbol{x}, \widehat{\boldsymbol{x}})\varphi_0\bar{\alpha}_i(\boldsymbol{x}, \widehat{\boldsymbol{x}})\nabla\left(f_i\big(\tau(\widehat{\boldsymbol{x}})\big)\right)\boldsymbol{\eta}(\boldsymbol{x}, \widehat{\boldsymbol{x}})$$

Therefore,

$$b_0(\boldsymbol{x}, \widehat{\boldsymbol{x}})\varphi_0[f_i(\tau(\boldsymbol{x})) - f_i(\tau(\widehat{\boldsymbol{x}}))] \leq \boldsymbol{0} \Rightarrow \nabla\left(f_i\big(\tau(\widehat{\boldsymbol{x}})\big)\right)\eta(\boldsymbol{x}, \widehat{\boldsymbol{x}}) \leq \boldsymbol{0}$$

Theorem 4.1.2: Consider the following composite vector function

$$f(\boldsymbol{x}) = \left(f_1(\tau_1(\boldsymbol{x})), f_2(\tau_2(\boldsymbol{x})), \dots, f_p(\tau_p(\boldsymbol{x}))\right)^T$$

Where for each $i = 1,2, \dots, p, \tau_i: \mathbb{X} \longrightarrow \mathbb{R}$ is continuously differentiable and pseudolinear with the positive proportional function $\alpha_i(\boldsymbol{x}, \widehat{\boldsymbol{x}})$ and $f_i: \mathbb{R} \longrightarrow \mathbb{R}$ being convex. Then, the function $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with $\boldsymbol{\eta}(\boldsymbol{x}, \widehat{\boldsymbol{x}}) = \boldsymbol{x} - \widehat{\boldsymbol{x}}$.

Proof:

This can be illustrated as follows. From the convexity of f_i and the pseudolinearity of τ_i one gets,

[1] A function $f: \mathbb{X} \rightarrow \mathbb{Y}$ is surjective if and only if for every $y \in \mathbb{Y}$ there is at least one $x \in \mathbb{X}$ such that $f(x) = y$

$$\begin{aligned} f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\widehat{\boldsymbol{x}})) &\geq \nabla\left(f_i\left(\tau(\widehat{\boldsymbol{x}})\right)\right)(\tau_i(\boldsymbol{x}) - \tau_i(\widehat{\boldsymbol{x}})) \\ &= \alpha_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla f_i(\tau_i(\widehat{\boldsymbol{x}}))\nabla\tau_i(\widehat{\boldsymbol{x}})(\boldsymbol{x} - \widehat{\boldsymbol{x}}) \\ &= \alpha_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla(f_i \circ \tau_i)(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \end{aligned}$$

Assuming that φ_0 is a positive function, we have

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0[f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\widehat{\boldsymbol{x}}))] \geq b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\alpha_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla(f_i \circ \tau_i)(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})$$

Therefore,

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0[f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\widehat{\boldsymbol{x}}))] \leq \boldsymbol{0} \Rightarrow \nabla(f_i \circ \tau_i)(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \leq \boldsymbol{0}$$

Theorem 4.1.3: Consider the following function

$$f(\boldsymbol{x}) = \left(f_1\big((\tau_1 \circ \theta)(\boldsymbol{x})\big), f_2\big((\tau_2 \circ \theta)(\boldsymbol{x})\big), \dots, f_p\left((\tau_p \circ \theta)(\boldsymbol{x})\right)\right)^T$$

Where $\tau_i: \mathbb{R}^n \longrightarrow \mathbb{R}, i = 1,2,\dots,p$ is pseudolinear with proportional functions $\alpha_i(\boldsymbol{x},\widehat{\boldsymbol{x}})$. $\theta: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable onto function such that $\nabla\theta(\widehat{\boldsymbol{x}})$ is surjective for each $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$, and $f_i: \mathbb{R} \longrightarrow \mathbb{R}$ is convex for each i. Then, the function $\boldsymbol{f}(\boldsymbol{x})$ is strong pseudoquasi type I univex.

Proof:

To see that, let $\theta(\boldsymbol{x}) = \boldsymbol{u}, \theta(\widehat{\boldsymbol{x}}) = \boldsymbol{v}$. Then, by the pseudolinearity,

$$\begin{aligned} \tau_i(\theta(\boldsymbol{x})) - \tau_i(\theta(\widehat{\boldsymbol{x}})) &= \tau_i(\boldsymbol{u}) - \tau_i(\boldsymbol{v}) \\ &= \alpha_i(\boldsymbol{u},\boldsymbol{v})\nabla\tau_i(\boldsymbol{v})(\boldsymbol{u} - \boldsymbol{v}) \end{aligned}$$

As $\nabla\theta(\widehat{\boldsymbol{x}})$ **is** onto for each $\widehat{\boldsymbol{x}} \in \mathbb{R}^n$, $\boldsymbol{u} - \boldsymbol{v} = \nabla\theta(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})$ can be solved for some $\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \in \mathbb{R}^n$. Hence,

$$\begin{aligned} \tau_i(\theta(\boldsymbol{x})) - \tau_i(\theta(\widehat{\boldsymbol{x}})) &= \tau_i(\boldsymbol{u}) - \tau_i(\boldsymbol{v}) \\ &= \alpha_i(\boldsymbol{u},\boldsymbol{v})\nabla\tau_i(\boldsymbol{v})(\boldsymbol{u} - \boldsymbol{v}) \\ &= \alpha_i(\boldsymbol{u},\boldsymbol{v})\nabla\tau_i(\boldsymbol{v})\nabla\theta(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \\ &= \bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla\tau_i\big(\theta(\widehat{\boldsymbol{x}})\big)\nabla\theta(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \\ &= \bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla(\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}), \end{aligned}$$

Where $\bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}}) = \alpha_i(\boldsymbol{u},\boldsymbol{v}) = \alpha_i\big(\tau(\boldsymbol{x}),\tau(\widehat{\boldsymbol{x}})\big) > 0$. Using the convexity of f_i, we get

$$\begin{aligned} f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) - f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big) &\geq \nabla f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big)\big((\tau_i \circ \theta)(\boldsymbol{x}) - (\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big) \\ &= \nabla f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big)\bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla(\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \\ &= \bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla\left(f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big)\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \end{aligned}$$

Assuming that φ_0 is a positive function, we have

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\big[f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) - f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big)\big] \geq b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\bar{\alpha}_i(\boldsymbol{x},\widehat{\boldsymbol{x}})\nabla\left(f_i\big((\tau_i \circ \theta)(\widehat{\boldsymbol{x}})\big)\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})$$

Therefore,

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\left[f_i\left((\tau_i\circ\theta)(\boldsymbol{x})\right)-f_i\left((\tau_i\circ\theta)(\widehat{\boldsymbol{x}})\right)\right]\leq\mathbf{0}\Rightarrow\nabla\left(f_i\left((\tau_i\circ\theta)(\widehat{\boldsymbol{x}})\right)\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\leq\mathbf{0}$$

Theorem 4.1.4: Consider the following function

$$f(\boldsymbol{x})=\left(\frac{(\rho_1(\boldsymbol{x}))^2}{\sigma_1(\boldsymbol{x})},\dots,\frac{(\rho_r(\boldsymbol{x}))^2}{\sigma_r(\boldsymbol{x})}\right)^T$$

Where, $\rho_i:\mathbb{X}\longrightarrow\mathbb{R},\sigma_i:\mathbb{X}\longrightarrow\mathbb{R}, i=1,2,\dots,r$. Assuming that $\mathbb{X}$ is convex set over $\mathbb{R}$ and $\rho_i(\boldsymbol{x})\geq 0,\sigma_i(\boldsymbol{x})>0$ for each $\boldsymbol{x}\in\mathbb{X}$. Let ρ_i is convex and σ_i is concave, then the function $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with $\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})=\boldsymbol{x}-\widehat{\boldsymbol{x}}$.

Proof:

To see that, consider the following

$$\begin{aligned}\sigma_i(\widehat{\boldsymbol{x}})\left(\rho_i(\boldsymbol{x})\right)^{\mathbf{2}}-\sigma_i(\boldsymbol{x})\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}&=\sigma_i(\widehat{\boldsymbol{x}})\left\{\left(\rho_i(\boldsymbol{x})\right)^{\mathbf{2}}-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\right\}-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\{\sigma_i(\boldsymbol{x})-\sigma_i(\widehat{\boldsymbol{x}})\}\\&=\sigma_i(\widehat{\boldsymbol{x}})\{\left(\rho_i(\boldsymbol{x})-\rho_i(\widehat{\boldsymbol{x}})\right)\left(\rho_i(\boldsymbol{x})+\rho_i(\widehat{\boldsymbol{x}})\right)\}-\\&\quad-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\{\sigma_i(\boldsymbol{x})-\sigma_i(\widehat{\boldsymbol{x}})\}\end{aligned}$$

As ρ_i is convex and σ_i is concave, we have

$$\begin{aligned}\sigma_i(\widehat{\boldsymbol{x}})\left(\rho_i(\boldsymbol{x})\right)^{\mathbf{2}}-\sigma_i(\boldsymbol{x})\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}&\geq\sigma_i(\widehat{\boldsymbol{x}})\nabla\rho_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\rho_i(\widehat{\boldsymbol{x}})-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\nabla\sigma_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\\&\geq 2\sigma_i(\widehat{\boldsymbol{x}})\nabla\rho_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\rho_i(\widehat{\boldsymbol{x}})-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\nabla\sigma_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\\&=\left(\sigma_i(\widehat{\boldsymbol{x}})\right)^2\left\{\frac{2\sigma_i(\widehat{\boldsymbol{x}})\nabla\rho_i(\widehat{\boldsymbol{x}})\rho_i(\widehat{\boldsymbol{x}})-\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}\nabla\sigma_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})}{\left(\sigma_i(\widehat{\boldsymbol{x}})\right)^2}\right\}\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\\&=\left(\sigma_i(\widehat{\boldsymbol{x}})\right)^2\nabla\left\{\frac{\left(\rho_i(\widehat{\boldsymbol{x}})\right)^{\mathbf{2}}}{\sigma_i(\widehat{\boldsymbol{x}})}\right\}\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\end{aligned}$$

Now, dividing the inequality by $\sigma_i(\boldsymbol{x})\sigma_i(\widehat{\boldsymbol{x}})>0$, one gets

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\left(\frac{(\rho_i(\boldsymbol{x}))^2}{\sigma_i(\boldsymbol{x})}-\frac{(\rho_i(\widehat{\boldsymbol{x}}))^2}{\sigma_i(\widehat{\boldsymbol{x}})}\right)\geq b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\left(\frac{\sigma_i(\widehat{\boldsymbol{x}})}{\sigma_i(\boldsymbol{x})}\right)\nabla\left(\frac{(\rho_i(\widehat{\boldsymbol{x}}))^2}{\sigma_i(\widehat{\boldsymbol{x}})}\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})$$

Hence,

$$b_0(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_0\left(\frac{(\rho_i(\boldsymbol{x}))^2}{\sigma_i(\boldsymbol{x})}-\frac{(\rho_i(\widehat{\boldsymbol{x}}))^2}{\sigma_i(\widehat{\boldsymbol{x}})}\right)\leq\mathbf{0}\Rightarrow\nabla\left(\frac{(\rho_i(\widehat{\boldsymbol{x}}))^2}{\sigma_i(\widehat{\boldsymbol{x}})}\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})\leq\mathbf{0}$$

Theorem 4.1.5: This problem can be dealt with as the previous one. It has the following description:

$$f(\boldsymbol{x})=\left(\frac{\rho_1(\boldsymbol{x})}{\sigma_1(\boldsymbol{x})},\dots,\frac{\rho_r(\boldsymbol{x})}{\sigma_r(\boldsymbol{x})}\right)^T$$

Where, $\rho_i:\mathbb{X}\longrightarrow\mathbb{R},\sigma_i:\mathbb{X}\longrightarrow\mathbb{R}, i=1,2,\dots,r$ and $g:\mathbb{X}\longrightarrow\mathbb{R}^m$.

Proof:

Assuming that $\mathbb{X}$ is convex set over $\mathbb{R}$ and $\rho_i(\boldsymbol{x}) \geq 0, \sigma_i(\boldsymbol{x}) > 0$ for each $\boldsymbol{x}$ on the feasible set $\Gamma = \{\boldsymbol{x} \in \mathbb{X}: g(\boldsymbol{x}) \leq \mathbf{0}\}$. The problem (4.1.5) is said to be a convex-concave fractional problem if ρ_i is convex and σ_i is concave. It is said to be strong pseudoquasi type I univex at $\hat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that:

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\left(\frac{\rho_i(\boldsymbol{x})}{\sigma_i(\boldsymbol{x})} - \frac{\rho_i(\hat{\boldsymbol{x}})}{\sigma_i(\hat{\boldsymbol{x}})}\right) \leq \mathbf{0} \Rightarrow \nabla\left(\frac{\rho_i(\hat{\boldsymbol{x}})}{\sigma_i(\hat{\boldsymbol{x}})}\right)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leq \mathbf{0}$$
$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

This definition is equivalent to,

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\big(\rho_i(\boldsymbol{x}) - \rho_i(\hat{\boldsymbol{x}})\big) \leq \mathbf{0} \Rightarrow \nabla\big(\rho_i(\hat{\boldsymbol{x}})\big)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leq \mathbf{0}$$
$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0(\sigma_i(\boldsymbol{x}) - \sigma_i(\hat{\boldsymbol{x}})) \geq \mathbf{0} \Rightarrow \nabla(\sigma_i(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \geq \mathbf{0}$$
$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

for all $\boldsymbol{x} \in \mathbb{X}$ and $i = 1,2,\dots,r$.

This function is also said to be weak quasistrictly pseudo type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\hat{\boldsymbol{x}} \in \mathbb{X}$ if there exist real-valued functions $b_0, b_1, \varphi_0, \varphi_1$ and η such that

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\big(\rho_i(\boldsymbol{x}) - \rho_i(\hat{\boldsymbol{x}})\big) \leq \mathbf{0} \Rightarrow \nabla\big(\rho_i(\hat{\boldsymbol{x}})\big)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$
$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0(\sigma_i(\boldsymbol{x}) - \sigma_i(\hat{\boldsymbol{x}})) \geqq \mathbf{0} \Rightarrow \nabla(\sigma_i(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \geqq \mathbf{0}$$
$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leq \mathbf{0}$$

for all $\boldsymbol{x} \in \mathbb{X}$ and $i = 1,2,\dots,r$.

4.2 *First-Order Optimality Conditions*

In this section, the author presents first-order optimality conditions for the multi-objective problems discussed in the previous section.

Theorem 4.2.1: Consider the following problem,

$$\begin{array}{ll} \text{Minimise} & f(\boldsymbol{x}) = \big(f_1(\tau(\boldsymbol{x})), f_2(\tau(\boldsymbol{x})), \dots, f_p(\tau(\boldsymbol{x}))\big)^T \\ \text{Subject to} & g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0} \\ & \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n \end{array} \tag{4.1}$$

and suppose that:

I. $\hat{\boldsymbol{x}} \in \mathbb{X}$
II. $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\hat{\boldsymbol{x}} \in \mathbb{X}$
III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\lambda} > \mathbf{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \mathbf{0}$ such that

$$
\begin{aligned}
&a)\ \boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}) = \mathbf{0}\\
&b)\ \boldsymbol{\mu} g(\hat{\boldsymbol{x}}) = \mathbf{0}\\
&c)\ \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^p
\end{aligned}
$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.1).

Proof: Suppose the contrary that $\hat{\boldsymbol{x}}$ is not a Pareto-optimal solution to (4.1). Then, there is another feasible solution $\boldsymbol{x}$ to (4.1) such that

$$f_i(\tau(\boldsymbol{x})) - f_i(\tau(\hat{\boldsymbol{x}})) \le \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\big(f_i(\tau(\boldsymbol{x})) - f_i(\tau(\hat{\boldsymbol{x}}))\big) \le \mathbf{0} \Longrightarrow \nabla\big(f_i(\tau(\hat{\boldsymbol{x}}))\big)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \le \mathbf{0} \tag{4.2}$$

As $\hat{\boldsymbol{x}}$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Longrightarrow \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \tag{4.3}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.2) and (4.3) give

$$(\boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Theorem 4.2.2: Consider the following problem,

$$
\begin{aligned}
\text{Minimise}\quad & f(\boldsymbol{x}) = \big(f_1(\tau_1(\boldsymbol{x})), f_2(\tau_2(\boldsymbol{x})), \dots, f_p(\tau_p(\boldsymbol{x}))\big)^T\\
\text{Subject to}\quad & g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0}\\
& \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n
\end{aligned} \tag{4.4}
$$

and suppose that:

I. $\hat{\boldsymbol{x}} \in \mathbb{X}$

II. $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\hat{\boldsymbol{x}} \in \mathbb{X}$

III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\lambda} > \mathbf{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \mathbf{0}$ such that

$$
\begin{aligned}
&a)\ \boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}) = \mathbf{0}\\
&b)\ \boldsymbol{\mu} g(\hat{\boldsymbol{x}}) = \mathbf{0}\\
&c)\ \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^p
\end{aligned}
$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.4).

Proof: Suppose the contrary that $\hat{\boldsymbol{x}}$ is not Pareto-optimal solutions to (4.4). Then, there is another feasible solution $\boldsymbol{x}$ to (4.4) such that

$$f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\hat{\boldsymbol{x}})) \le \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\big(f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\hat{\boldsymbol{x}}))\big) \le \mathbf{0} \Longrightarrow \nabla\big(f_i(\tau_i(\hat{\boldsymbol{x}}))\big)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \le \mathbf{0} \tag{4.5}$$

As $\hat{\boldsymbol{x}}$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \qquad (4.6)$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.5) and (4.6) give

$$(\boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Theorem 4.2.3: Consider the following problem,

$$\begin{array}{ll} \text{Minimise} & f(\boldsymbol{x}) = \Big(f_1\big((\tau_1 \circ \theta)(\boldsymbol{x})\big), f_2\big((\tau_2 \circ \theta)(\boldsymbol{x})\big), \dots, f_p\left((\tau_p \circ \theta)(\boldsymbol{x})\right)\Big)^T \\ \text{Subject to} & g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0} \\ & \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n \end{array} \qquad (4.7)$$

and suppose that:

I. $\hat{\boldsymbol{x}} \in \mathbb{X}$

II. $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\hat{\boldsymbol{x}} \in \mathbb{X}$

III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\lambda} > \mathbf{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \mathbf{0}$ such that

$$\begin{array}{l} a)\ \boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}) = \mathbf{0} \\ b)\ \boldsymbol{\mu} g(\hat{\boldsymbol{x}}) = \mathbf{0} \\ c)\ \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^p \end{array}$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.7).

Proof: Suppose the contrary that $\hat{\boldsymbol{x}}$ is not a Pareto-optimal solution to (4.7). Then, there is another feasible solution $\boldsymbol{x}$ to (4.7) such that

$$f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) - f_i\big((\tau_i \circ \theta)(\hat{\boldsymbol{x}})\big) \leq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\left(f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) - f_i\big((\tau_i \circ \theta)(\hat{\boldsymbol{x}})\big)\right) \leq \mathbf{0} \Rightarrow \nabla\left(f_i\big((\tau_i \circ \theta)(\hat{\boldsymbol{x}})\big)\right)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leq \mathbf{0} \quad (4.8)$$

As $\hat{\boldsymbol{x}}$ is feasible solution, we have

$$-\boldsymbol{\mu} g(\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \qquad (4.9)$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.8) and (4.9) give

$$(\boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Theorem 4.2.4: Consider the following problem,

$$\begin{array}{ll} \text{Minimise} & f(\boldsymbol{x}) = \left(\frac{(\rho_1(\boldsymbol{x}))^2}{\sigma_1(\boldsymbol{x})}, \dots, \frac{(\rho_r(\boldsymbol{x}))^2}{\sigma_r(\boldsymbol{x})}\right)^T \\ \text{Subject to} & g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0} \\ & \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n \end{array} \tag{4.10}$$

and suppose that:

I. $\rho(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are continuously differentiable at $\boldsymbol{x} = \hat{\boldsymbol{x}}$

II. $f(\boldsymbol{x})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\hat{\boldsymbol{x}} \in \mathbb{X}$

III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\lambda} > \mathbf{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \mathbf{0}$ such that

$$\begin{array}{l} a)\ \boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}) = \mathbf{0} \\ b)\ \boldsymbol{\mu} g(\hat{\boldsymbol{x}}) = \mathbf{0} \\ c)\ \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^T \in \mathbb{R}^p \end{array}$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.10).

Proof: Suppose the contrary that $\hat{\boldsymbol{x}}$ is a not Pareto-optimal solution to (4.10). Then, there is another feasible solution $\boldsymbol{x}$ to (4.10) such that

$$\left(\frac{(\rho(\boldsymbol{x}))^2}{\sigma(\boldsymbol{x})} - \frac{(\rho(\hat{\boldsymbol{x}}))^2}{\sigma(\hat{\boldsymbol{x}})}\right) \leq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_0\left(\frac{(\rho(\boldsymbol{x}))^2}{\sigma(\boldsymbol{x})} - \frac{(\rho(\hat{\boldsymbol{x}}))^2}{\sigma(\hat{\boldsymbol{x}})}\right) \leq \mathbf{0} \Rightarrow \nabla\left(\frac{(\rho(\hat{\boldsymbol{x}}))^2}{\sigma(\hat{\boldsymbol{x}})}\right)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leq \mathbf{0} \tag{4.11}$$

As $\hat{\boldsymbol{x}}$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) \leqq \mathbf{0} \tag{4.12}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.11) and (4.12) give

$$(\boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}}))\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Theorem 4.2.5: Consider the following problem,

$$\begin{array}{ll} \text{Minimise} & \left(\frac{\rho_1(\boldsymbol{x})}{\sigma_1(\boldsymbol{x})}, \dots, \frac{\rho_r(\boldsymbol{x})}{\sigma_r(\boldsymbol{x})}\right)^T \\ \text{Subject to} & g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0} \\ & \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n \end{array} \tag{4.13}$$

Suppose the following for problem (4.13),

I. $\rho(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are continuously differentiable at $\boldsymbol{x} = \widehat{\boldsymbol{x}}$

II. $\rho(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\widehat{\boldsymbol{x}} \in \mathbb{X}$

III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^r, \boldsymbol{\lambda} > \boldsymbol{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \boldsymbol{0}$ such that

$$a)\ \boldsymbol{\lambda}\nabla\left(\frac{\rho(\widehat{\boldsymbol{x}})}{\sigma(\widehat{\boldsymbol{x}})}\right) + \boldsymbol{\mu}\nabla g(\widehat{\boldsymbol{x}}) = \boldsymbol{0}$$
$$b)\ \boldsymbol{\mu} g(\widehat{\boldsymbol{x}}) = \boldsymbol{0}$$
$$c)\ \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^r$$

Then, $\widehat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.13).

Proof: Suppose the contrary that $\widehat{\boldsymbol{x}}$ is not a Pareto-optimal solution to (4.13). Then, there is another feasible solution $\boldsymbol{x}$ to (4.13) such that

$$\left(\frac{\rho(\boldsymbol{x})}{\sigma(\boldsymbol{x})} - \frac{\rho(\hat{\boldsymbol{x}})}{\sigma(\hat{\boldsymbol{x}})}\right) \leq \boldsymbol{0}$$

By condition (II) and the above inequality, one gets

$$b_0(x,\hat{x})\varphi_0\left(\frac{\rho(x)}{\sigma(x)} - \frac{\rho(\hat{x})}{\sigma(\hat{x})}\right) \leq \boldsymbol{0} \Longrightarrow \nabla\left(\frac{\rho(\hat{x})}{\sigma(\hat{x})}\right)\eta(x,\hat{x}) \leq \boldsymbol{0} \tag{4.14}$$

As $\widehat{\boldsymbol{x}}$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\widehat{\boldsymbol{x}}) \leqq \boldsymbol{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\widehat{\boldsymbol{x}})\varphi_1 g(\widehat{\boldsymbol{x}}) \leqq \boldsymbol{0} \ \Longrightarrow \nabla g(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \leqq \boldsymbol{0} \Longrightarrow \boldsymbol{\mu}\nabla g(\widehat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) \leqq \boldsymbol{0} \tag{4.15}$$

Since $\boldsymbol{\lambda} > \boldsymbol{0}$, (4.14) and (4.15) give

$$\left(\boldsymbol{\lambda}\nabla\left(\frac{\rho(\hat{x})}{\sigma(\hat{x})}\right) + \boldsymbol{\mu}\nabla g(\widehat{\boldsymbol{x}})\right)\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}}) < \boldsymbol{0},$$

This contradicts condition (III). So, the proof is completed.

Theorem 4.2.6: Suppose the following for problem (4.13),

I. $\rho(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are continuously differentiable at $\boldsymbol{x} = \widehat{\boldsymbol{x}}$

II. $\rho(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ are weak quasistrictly pseudo type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\widehat{\boldsymbol{x}} \in \mathbb{X}$

III. there exist $\boldsymbol{\lambda} \in \mathbb{R}^r, \boldsymbol{\lambda} > \boldsymbol{0}, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \boldsymbol{0}$ such that

$$a)\ \boldsymbol{\lambda}\nabla\left(\frac{\rho(\hat{\boldsymbol{x}})}{\sigma(\hat{\boldsymbol{x}})}\right)+\boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})=\mathbf{0}$$
$$b)\ \boldsymbol{\mu}g(\hat{\boldsymbol{x}})=\mathbf{0}$$
$$c)\ \boldsymbol{\lambda e}=1,\boldsymbol{e}=(1,1,\dots,1)^{\boldsymbol{T}}\in\mathbb{R}^r$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to (4.13).

Proof: Suppose the contrary that $\hat{\boldsymbol{x}}$ is not a Pareto-optimal solution to (4.13). Then, there is another feasible solution $\boldsymbol{x}$ to (4.13) such that

$$\left(\frac{\rho(\boldsymbol{x})}{\sigma(\boldsymbol{x})}-\frac{\rho(\hat{\boldsymbol{x}})}{\sigma(\hat{\boldsymbol{x}})}\right)\leq\mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(x,\hat{x})\varphi_0\left(\frac{\rho(x)}{\sigma(x)}-\frac{\rho(\hat{x})}{\sigma(\hat{x})}\right)\leq\mathbf{0}\Longrightarrow\nabla\left(\frac{\rho(\hat{x})}{\sigma(\hat{x})}\right)\eta(x,\hat{x})\leq\mathbf{0}\tag{4.16}$$

As $\hat{\boldsymbol{x}}$ is a feasible solution, we have

$$-\boldsymbol{\mu}g(\hat{\boldsymbol{x}})\leqq\mathbf{0}$$

By condition (II) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}})\leqq\mathbf{0}\ \Longrightarrow\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}})\leqq\mathbf{0}\Longrightarrow\boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}})\leqq\mathbf{0}\tag{4.17}$$

Since $\boldsymbol{\lambda}>\mathbf{0}$, (4.16) and (4.17) give

$$\left(\boldsymbol{\lambda}\nabla\left(\frac{\rho(\hat{x})}{\sigma(\hat{x})}\right)+\boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\right)\boldsymbol{\eta}(\boldsymbol{x},\hat{\boldsymbol{x}})<\mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

4.3 *Duality Results*

The duality results corresponding to the optimality conditions previously illustrated are derived in this section. Consider the following problem,

$$\begin{aligned}&\text{Maximise}\quad f(\boldsymbol{\zeta})=\left(f_1(\tau(\boldsymbol{\zeta})),f_2(\tau(\boldsymbol{\zeta})),\dots,f_p(\tau(\boldsymbol{\zeta}))\right)^T,\\&\text{Subject to}\quad \boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta})+\boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})=\mathbf{0},\\&\qquad\qquad\quad \boldsymbol{\mu}g(\boldsymbol{\zeta})\geqq\mathbf{0},\boldsymbol{\lambda}\geqq\mathbf{0},\boldsymbol{\mu}\geqq\mathbf{0},\\&\qquad\qquad\quad \boldsymbol{\lambda e}=1,\boldsymbol{e}=(1,1,\dots,1)^{\boldsymbol{T}}\in\mathbb{R}^p\end{aligned}\tag{4.18}$$

Let

$$\Lambda=\{(\boldsymbol{\zeta},\boldsymbol{\lambda},\boldsymbol{\mu}):\boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta})+\boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})=\mathbf{0},\boldsymbol{\mu}g(\boldsymbol{\zeta})\geqq\mathbf{0},\boldsymbol{\lambda}\in\mathbb{R}^p,\boldsymbol{\mu}\in\mathbb{R}^m,\boldsymbol{\mu}\geqq\mathbf{0}\}$$

denotes the set of feasible solutions of problem (4.18). The following theorem is suggested.

Theorem 4.3.1: Suppose the following for problem (4.18)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $f(\boldsymbol{\zeta})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

III. $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > \mathbf{0}$;

Then $f(\boldsymbol{x}) \nleq f(\boldsymbol{\zeta})$.

Proof: Suppose the contrary, i.e.,

$$f_i(\tau(\boldsymbol{x})) \leq f_i(\tau(\boldsymbol{\zeta}))$$

By condition (II) and the above inequality, one gets

$$b_0(x, \boldsymbol{\zeta})\varphi_0\big(f_i(\tau(\boldsymbol{x})) - f_i(\tau(\boldsymbol{\zeta}))\big) \leq \mathbf{0} \Rightarrow \nabla\big(f_i(\tau(\boldsymbol{\zeta}))\big)\eta(x, \boldsymbol{\zeta}) \leq \mathbf{0} \tag{4.19}$$

As $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\boldsymbol{\zeta}) \leqq \mathbf{0}$$

By condition (III) and the above inequality, one gets

$$-b_1(\boldsymbol{x}, \boldsymbol{\zeta})\varphi_1 g(\boldsymbol{\zeta}) \leqq \mathbf{0} \;\Rightarrow \nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.20}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.19) and (4.20) give

$$(\boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}))\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Consider the following problem,

$$\begin{aligned} \text{Maximise} \quad & f(\boldsymbol{\zeta}) = \big(f_1(\tau_1(\boldsymbol{\zeta})), f_2(\tau_2(\boldsymbol{\zeta})), \dots, f_p(\tau_p(\boldsymbol{\zeta}))\big)^T, \\ \text{Subject to} \quad & \boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}) = \mathbf{0}, \\ & \boldsymbol{\mu} g(\boldsymbol{\zeta}) \geqq \mathbf{0}, \boldsymbol{\lambda} \geqq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \\ & \boldsymbol{\lambda}\boldsymbol{e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^p \end{aligned} \tag{4.21}$$

The following theorem is suggested.

Theorem 4.3.2: Suppose the following for problem (4.21)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $f(\boldsymbol{\zeta})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

III. $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > \mathbf{0}$;

Then $f(\boldsymbol{x}) \nleq f(\boldsymbol{\zeta})$.

Proof: Suppose the contrary, i.e.,

$$f_i(\tau_i(\boldsymbol{x})) \leq f_i(\tau_i(\boldsymbol{\zeta})) \tag{4.22}$$

By condition (II) and the above inequality, one gets

$$b_0(x,\boldsymbol{\zeta})\varphi_0\big(f_i(\tau_i(\boldsymbol{x})) - f_i(\tau_i(\boldsymbol{\zeta}))\big) \le \mathbf{0} \Rightarrow \nabla\big(f_i(\tau_i(\boldsymbol{\zeta}))\big)\eta(x,\boldsymbol{\zeta}) \le \mathbf{0} \tag{4.23}$$

As $(\boldsymbol{\zeta},\boldsymbol{\lambda},\boldsymbol{\mu}) \in \Lambda$ is a feasible solution, we have

$$-\boldsymbol{\mu}g(\boldsymbol{\zeta}) \leqq \mathbf{0}$$

By condition (III) and the above inequality, one get

$$-b_1(\boldsymbol{x},\boldsymbol{\zeta})\varphi_1 g(\boldsymbol{\zeta}) \leqq \mathbf{0} \;\Rightarrow \nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.24}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.23) and (4.24) give

$$(\boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}))\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Consider the following problem,

$$\begin{aligned} &\text{Maximise} \quad f(\boldsymbol{\zeta}) = \Big(f_1\big((\tau_1 \circ \theta)(\boldsymbol{\zeta})\big), f_2\big((\tau_2 \circ \theta)(\boldsymbol{\zeta})\big), \dots, f_p\Big(\big((\tau_p \circ \theta)(\boldsymbol{\zeta})\big)\Big)\Big)^T, \\ &\text{Subject to} \quad \boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}) = \mathbf{0}, \\ &\qquad\qquad \boldsymbol{\mu}g(\boldsymbol{\zeta}) \geqq \mathbf{0}, \boldsymbol{\lambda} \geqq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \\ &\qquad\qquad \boldsymbol{\lambda}\boldsymbol{e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^p \end{aligned} \tag{4.25}$$

The following theorem is suggested.

Theorem 4.3.3: Suppose the following for problem (4.25)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $f(\boldsymbol{\zeta})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

III. $(\boldsymbol{\zeta},\boldsymbol{\lambda},\boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > \mathbf{0}$;

Then $f(\boldsymbol{x}) \nleq f(\boldsymbol{\zeta})$.

Proof: Suppose the contrary, i.e.,

$$f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) \le f_i\big((\tau_i \circ \theta)(\boldsymbol{\zeta})\big)$$

By condition (II) and the above inequality, one gets

$$b_0(x,\boldsymbol{\zeta})\varphi_0\Big(f_i\big((\tau_i \circ \theta)(\boldsymbol{x})\big) - f_i\big((\tau_i \circ \theta)(\boldsymbol{\zeta})\big)\Big) \le \mathbf{0} \Rightarrow \nabla\Big(f_i\big((\tau_i \circ \theta)(\boldsymbol{\zeta})\big)\Big)\eta(x,\boldsymbol{\zeta}) \le \mathbf{0} \tag{4.26}$$

As $(\boldsymbol{\zeta},\boldsymbol{\lambda},\boldsymbol{\mu}) \in \Lambda$ is a feasible solution, we have

$$-\boldsymbol{\mu}g(\boldsymbol{\zeta}) \leqq \mathbf{0}$$

By condition (III) and the above inequality, one get

$$-b_1(\boldsymbol{x},\boldsymbol{\zeta})\varphi_1 g(\boldsymbol{\zeta}) \leqq \mathbf{0} \;\Rightarrow \nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \Rightarrow \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.27}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.26) and (4.27) give

$$(\boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}))\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) < \mathbf{0},$$

This contradicts condition (III). So, the proof is completed.

Consider the following problem,

$$\begin{aligned} &\text{Maximise} \quad f(\boldsymbol{\zeta}) = \left(\frac{(\rho_1(\zeta))^2}{\sigma_1(\zeta)}, \dots, \frac{(\rho_r(\zeta))^2}{\sigma_r(\zeta)}\right)^T, \\ &\text{Subject to} \quad \boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}) = \mathbf{0}, \\ &\qquad \boldsymbol{\mu} g(\boldsymbol{\zeta}) \geqq \mathbf{0}, \boldsymbol{\lambda} \geqq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \\ &\qquad \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^r \end{aligned} \tag{4.28}$$

The following theorem is suggested.

Theorem 4.3.4: Suppose the following for problem (4.28)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $f(\boldsymbol{\zeta})$ is strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

III. $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > \mathbf{0}$;

Then $f(\boldsymbol{x}) \nleq f(\boldsymbol{\zeta})$.

Proof: Suppose the contrary, i.e.,

$$\left(\frac{(\rho(\boldsymbol{x}))^2}{\sigma(\boldsymbol{x})} - \frac{(\rho(\boldsymbol{\zeta}))^2}{\sigma(\boldsymbol{\zeta})}\right) \leq \mathbf{0}$$

By condition (II) and the above inequality, one gets

$$b_0(x,\boldsymbol{\zeta})\varphi_0\left(\frac{(\rho(\boldsymbol{x}))^2}{\sigma(\boldsymbol{x})} - \frac{(\rho(\boldsymbol{\zeta}))^2}{\sigma(\boldsymbol{\zeta})}\right) \leq \mathbf{0} \Longrightarrow \nabla\left(\frac{(\rho(\boldsymbol{\zeta}))^2}{\sigma(\boldsymbol{\zeta})}\right)\eta(x,\boldsymbol{\zeta}) \leq \mathbf{0} \tag{4.29}$$

As $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\boldsymbol{\zeta}) \leqq \mathbf{0}$$

By condition (III) and the above inequality, one gets

$$-b_1(\boldsymbol{x},\boldsymbol{\zeta})\varphi_1 g(\boldsymbol{\zeta}) \leqq \mathbf{0} \Longrightarrow \nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \Longrightarrow \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.30}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.29) and (4.30) give

$$(\boldsymbol{\lambda}\nabla f(\boldsymbol{\zeta}) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}))\boldsymbol{\eta}(\boldsymbol{x},\boldsymbol{\zeta}) < \mathbf{0}$$

This contradicts condition (III). So, the proof is completed.

Consider the following problem:

$$\begin{aligned} &\text{Maximise} \quad \left(\frac{\rho_1(\zeta)}{\sigma_1(\zeta)}, \dots, \frac{\rho_r(\zeta)}{\sigma_r(\zeta)}\right)^T, \\ &\text{Subject to} \quad \boldsymbol{\lambda}\nabla\left(\frac{\rho(\zeta)}{\sigma(\zeta)}\right) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}) = \mathbf{0}, \\ &\qquad \boldsymbol{\mu} g(\boldsymbol{\zeta}) \geqq \mathbf{0}, \boldsymbol{\lambda} \geqq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \\ &\qquad \boldsymbol{\lambda e} = 1, \boldsymbol{e} = (1,1,\dots,1)^{\boldsymbol{T}} \in \mathbb{R}^r \end{aligned} \tag{4.31}$$

Let

$$\Lambda = \left\{(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) : \boldsymbol{\lambda}\nabla\left(\frac{\rho(\boldsymbol{\zeta})}{\sigma(\boldsymbol{\zeta})}\right) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta}) = \mathbf{0}, \boldsymbol{\mu} g(\boldsymbol{\zeta}) \geqq \mathbf{0}, \boldsymbol{\lambda} \in \mathbb{R}^r, \boldsymbol{\mu} \in \mathbb{R}^m, \boldsymbol{\mu} \geqq \mathbf{0}\right\}$$

denotes the set of feasible solutions of problem (4.31). The following theorems are suggested.

Theorem 4.3.5: Suppose the following for problem (4.31)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $\rho(\boldsymbol{\zeta})$ and $\sigma(\boldsymbol{\zeta})$ are continuously differentiable at $\boldsymbol{\zeta}$,

III. $\rho(\boldsymbol{\zeta})$ and $\sigma(\boldsymbol{\zeta})$ are strong pseudoquasi type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

IV. $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > \mathbf{0}$;

Then $\frac{\rho(x)}{\sigma(x)} \nleq \frac{\rho(\zeta)}{\sigma(\zeta)}$

Proof: Suppose the contrary, i.e.,

$$\frac{\rho(\boldsymbol{x})}{\sigma(\boldsymbol{x})} \leq \frac{\rho(\boldsymbol{\zeta})}{\sigma(\boldsymbol{\zeta})}$$

By condition (III) and the above inequality, one gets

$$b_0(x, \boldsymbol{\zeta})\varphi_0\left(\frac{\rho(x)}{\sigma(x)} - \frac{\rho(\zeta)}{\sigma(\zeta)}\right) \leq \mathbf{0} \Longrightarrow \nabla\left(\frac{\rho(\zeta)}{\sigma(\zeta)}\right)\eta(x, \boldsymbol{\zeta}) \leq \mathbf{0} \tag{4.32}$$

As $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ is a feasible solution, we have

$$-\boldsymbol{\mu} g(\boldsymbol{\zeta}) \leqq \mathbf{0}$$

By condition (III) and the above inequality, one gets

$$-b_1(\boldsymbol{x}, \boldsymbol{\zeta})\varphi_1 g(\boldsymbol{\zeta}) \leqq \mathbf{0} \Longrightarrow \nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) \leqq \mathbf{0} \Longrightarrow \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.33}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.32) and (4.33) give

$$\left(\boldsymbol{\lambda}\nabla\left(\frac{\rho(\boldsymbol{\zeta})}{\sigma(\boldsymbol{\zeta})}\right) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\right)\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) < \mathbf{0}$$

This contradicts condition (IV). So, the proof is completed.

Theorem 4.3.6: Suppose the following for problem (4.31)

I. $\boldsymbol{x} \in \mathbb{X}$;

II. $\rho(\boldsymbol{\zeta})$ and $\sigma(\boldsymbol{\zeta})$ are continuously differentiable at $\boldsymbol{\zeta}$;

III. $\rho(\boldsymbol{\zeta})$ and $\sigma(\boldsymbol{\zeta})$ are weak quasistrictly pseudo type I univex with respect to $b_0, b_1, \varphi_0, \varphi_1$ and η at $\boldsymbol{\zeta}$;

IV. $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu}) \in \Lambda$ and $\boldsymbol{\lambda} > 0$;

Then $\frac{\rho(x)}{\sigma(x)} \nleq \frac{\rho(\zeta)}{\sigma(\zeta)}$

Proof: Suppose the contrary, i.e.,

$$\frac{\rho(\boldsymbol{x})}{\sigma(\boldsymbol{x})} \leq \frac{\rho(\boldsymbol{\zeta})}{\sigma(\boldsymbol{\zeta})}$$

By conditions (III) and the above inequality, we get (4.32). By the feasibility of $(\boldsymbol{\zeta}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ and condition (III), we get (4.33). By inequalities (4.32) and (4.33) with condition (III), we have

$$\nabla\left(\frac{\rho(\zeta)}{\sigma(\zeta)}\right)\eta(x, \boldsymbol{\zeta}) \leq \mathbf{0} \tag{4.34}$$

and

$$\boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) \leqq \mathbf{0} \tag{4.35}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, (4.34) and (4.35) give

$$\left(\boldsymbol{\lambda}\nabla\left(\frac{\rho(\boldsymbol{\zeta})}{\sigma(\boldsymbol{\zeta})}\right) + \boldsymbol{\mu}\nabla g(\boldsymbol{\zeta})\right)\boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{\zeta}) < \mathbf{0}$$

This contradicts condition (IV). So, the proof is completed.

4.4 Proposed Second-order Optimality Conditions

In this chapter, second order optimality conditions for multi-objective optimisation problems are proposed under a generalised form of univexity concept. Before we discuss these results, let us recall Equation (1.1):

$$\begin{aligned} &\text{Minimise} \quad f(\boldsymbol{x}) = \left(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \dots, f_p(\boldsymbol{x})\right)^T \\ &\text{Subject to} \ \ g(\boldsymbol{x}) = (g_1(\boldsymbol{x}), g_2(\boldsymbol{x}), \dots, g_m(\boldsymbol{x}))^T \leqq \mathbf{0} \\ &\qquad\qquad\quad \boldsymbol{x} \in \mathbb{X} \subseteq \mathbb{R}^n \end{aligned} \tag{4.36}$$

Where, $f: \mathbb{X} \longrightarrow \mathbb{R}^p$ and $g: \mathbb{X} \longrightarrow \mathbb{R}^m$ are continuously differentiable real-valued functions and $\mathbb{X}$ is an open set.

In this section, some sufficient second-order optimality conditions for multi-objective optimisation problems defined in (4.36) are established. The objective functions and inequality constraints involved in these conditions should satisfy certain features of type I univex functions previously defined.

Theorem 4.4.1: Suppose that f, g are strong pseudoquasi type I univex at $\hat{\boldsymbol{x}} \in \mathbb{X}$ with respect to some $b_0, b_1, \varphi_0, \varphi_1$ and η (critical direction) for all feasible $\boldsymbol{x}$. Assuming that f, g are twice continuously differentiable at $\hat{\boldsymbol{x}} \in \mathbb{X}$. If for each critical direction $\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \neq \mathbf{0}$, there exist $\boldsymbol{\lambda} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^m$ such that:

$$I. \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \mathbf{0},$$

$$II. \quad \left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}}) \right) (\boldsymbol{\eta}, \boldsymbol{\eta}) \geqq \mathbf{0},$$

$$III. \quad \begin{cases} \boldsymbol{\lambda} > \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \lambda_i = 0 \ \forall \ i \notin \Xi(\boldsymbol{\eta}) = \{i \in \{1,2,\dots,p\}: \nabla f_i(\hat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}\} \\ \mu_j = 0 \ \forall \ j \notin \Psi(\boldsymbol{\eta}) = \{j \in \{1,2,\dots,m\}: g_j(\hat{\boldsymbol{x}}) = 0, \nabla g_j(\hat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}\} \end{cases}$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to problem (4.36).

Proof: Assume that, for each critical direction $\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \neq \mathbf{0}$, there exist $\boldsymbol{\lambda} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^m$ such that (I)-(III) hold and that $\hat{\boldsymbol{x}}$ is not a Pareto-optimal solution for (4.36). Then, there exists a feasible $\boldsymbol{x}$ to (4.36) such that

$$f(\boldsymbol{x}) \leq f(\hat{\boldsymbol{x}})$$

By the strong pseudoquasi type I univex and the above inequality, one gets

$$b_0(\boldsymbol{x}, \hat{\boldsymbol{x}})\varphi_0[f(\boldsymbol{x}) - f(\hat{\boldsymbol{x}})] \leq \mathbf{0} \tag{4.37}$$

Since $\boldsymbol{x}$ is a feasible solution, one has

$$-\mu\, g(\hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

Using the above inequality with the strong pseudoquasi type I univex, we have

$$-b_1(\boldsymbol{x}, \hat{\boldsymbol{x}})\varphi_1 g(\hat{\boldsymbol{x}}) \leqq \mathbf{0} \tag{4.38}$$

From (4.37) and (4.38), we have

$$\nabla f(\hat{\boldsymbol{x}})\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leq \mathbf{0},$$
$$\mu \nabla g(\hat{\boldsymbol{x}})\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leqq \mathbf{0}$$

Since $\boldsymbol{\lambda} > \mathbf{0}$, the above inequalities with Motzkin's theorem of alternative (Dax and Sreedharan, 1997) imply that the following system is inconsistent:

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \mathbf{0},$$

thereby contradicting condition (I). Therefore, $\hat{\boldsymbol{x}}$ is a Pareto optimal solution to problem (4.36). As $\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \neq \mathbf{0}$ is a critical direction, then the system

$$\begin{aligned} \nabla f_{\Xi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 f_{\Xi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta}) < \boldsymbol{0} \\ \nabla g_{\Psi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 g_{\Psi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta}) \leqq \boldsymbol{0} \end{aligned} \tag{4.39}$$

has no solution $\boldsymbol{z}$. This is equivalent to

$$\begin{aligned} \nabla f_{\Xi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 f_{\Xi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta})t < \boldsymbol{0} \\ \nabla g_{\Psi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 g_{\Psi(\boldsymbol{\eta})}(\hat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta})t \leqq \boldsymbol{0} \\ -t < 0 \end{aligned}$$

which has no solution $\boldsymbol{z} \in \mathbb{R}^n, t \in \mathbb{R}$. By Motzkin's theorem of alternative, there exist multipliers $\boldsymbol{\xi} \in \mathbb{R}, \boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\mu} \in \mathbb{R}^m$ such that

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \boldsymbol{0}, \\ &\left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) - \boldsymbol{\xi} = \boldsymbol{0}, \\ &(\boldsymbol{\lambda},\boldsymbol{\xi}) \geq \boldsymbol{0}, \boldsymbol{\mu} \geqq \boldsymbol{0}, \lambda_i = 0 \; \forall \; i \notin \Xi(\boldsymbol{\eta}), \mu_j = 0 \; \forall \; j \notin \Psi(\boldsymbol{\eta}) \end{aligned}$$

Since $(\boldsymbol{\lambda},\boldsymbol{\xi}) \geq \boldsymbol{0}$ implies

$$(\boldsymbol{\lambda} \geq \boldsymbol{0} \text{ and } \boldsymbol{\xi} \geqq \boldsymbol{0}) \text{ or } (\boldsymbol{\lambda} \geqq \boldsymbol{0} \text{ and } \boldsymbol{\xi} > \boldsymbol{0}),$$

there exist multipliers $\boldsymbol{\lambda} \in \mathbb{R}^p, \boldsymbol{\mu} \in \mathbb{R}^m$ such that either (4.40) or (4.41) below holds:

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \boldsymbol{0}, \\ &\left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) > \boldsymbol{0}, \\ &\boldsymbol{\lambda} \geqq \boldsymbol{0}, \boldsymbol{\mu} \geqq \boldsymbol{0}, \lambda_i = 0 \; \forall \; i \notin \Xi(\boldsymbol{\eta}), \mu_j = 0 \; \forall \; j \notin \Psi(\boldsymbol{\eta}) \end{aligned} \tag{4.40}$$

Or,

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \boldsymbol{0}, \\ &\left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) > \boldsymbol{0}, \\ &\boldsymbol{\lambda} \geq \boldsymbol{0}, \boldsymbol{\mu} \geqq \boldsymbol{0}, \lambda_i = 0 \; \forall \; i \notin \Xi(\boldsymbol{\eta}), \mu_j = 0 \; \forall \; j \notin \Psi(\boldsymbol{\eta}) \end{aligned} \tag{4.41}$$

Assuming that (4.40) does not hold. This is equivalent to the inconsistency of the system:

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\widehat{\boldsymbol{x}}) = \mathbf{0},$$
$$\left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\widehat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) > \mathbf{0},$$
$$\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \boldsymbol{\xi} \geqq \mathbf{0}, \lambda_i = 0 \;\forall\, i \notin \Xi(\boldsymbol{\eta}), \mu_j = 0 \;\forall\, j \notin \Psi(\boldsymbol{\eta})$$

By Motzkin's theorem of the alternative, there exist $\boldsymbol{z}$ and $t \geqq 0$ satisfying

$$\nabla f_{\Xi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 f_{\Xi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta})t < \mathbf{0}$$
$$\nabla g_{\Psi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{z} + \nabla^2 g_{\Psi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta})t \leqq \mathbf{0}$$

Since (4.39) has no solution, we have $t = 0$; hence,

$$\nabla f_{\Xi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{z} < \mathbf{0}$$
$$\nabla g_{\Psi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{z} \leqq \mathbf{0}$$

However,

$$\nabla f_{\Xi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}, \nabla f_{L\backslash\Xi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{\eta} < \mathbf{0}, L = \{1,2,\dots,p\}$$
$$\nabla g_{\Psi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}, \nabla g_{K\backslash\Psi(\boldsymbol{\eta})}(\widehat{\boldsymbol{x}})\boldsymbol{\eta} < \mathbf{0}, K = \{1,2,\dots,m\}$$

As $\boldsymbol{\eta}(\boldsymbol{x},\widehat{\boldsymbol{x}})$ is critical, this implies

$$\nabla f(\widehat{\boldsymbol{x}})(\boldsymbol{\eta} + \boldsymbol{\varepsilon z}) < 0,$$
$$\nabla g(\widehat{\boldsymbol{x}})(\boldsymbol{\eta} + \boldsymbol{\varepsilon z}) \leqq \mathbf{0}$$

For any sufficiently small $\boldsymbol{\varepsilon} > 0$, this contradicts the first-order optimality conditions. This completes the proof.

The proof of theorem 4.4.2 is similar to theorem 4.4.1 proof and is omitted here.

Theorem 4.4.2: Suppose that f, g are weak strictly pseudoquasi type I univex at $\widehat{\boldsymbol{x}} \in \mathbb{X}$ with respect to some $b_0, b_1, \varphi_0, \varphi_1$ and η for all feasible $\boldsymbol{x}$. Assuming that f, g are twice continuously differentiable at $\widehat{\boldsymbol{x}} \in \mathbb{X}$. If for each critical direction $\eta(\boldsymbol{x},\widehat{\boldsymbol{x}}) \neq \mathbf{0}$, there exist $\boldsymbol{\lambda} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^m$ such that:

$$I. \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\widehat{\boldsymbol{x}}) = \mathbf{0},$$
$$II. \quad \left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\widehat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\widehat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) \geqq \mathbf{0},$$
$$III. \quad \begin{cases} \boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\mu} \geqq \mathbf{0}, \lambda_i = 0 \;\forall\, i \notin \Xi(\boldsymbol{\eta}) = \{i \in \{1,2,\dots,p\}: \nabla f_i(\widehat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}\} \\ \mu_j = 0 \;\forall\, j \notin \Psi(\boldsymbol{\eta}) = \{j \in \{1,2,\dots,m\}: g_j(\widehat{\boldsymbol{x}}) = 0, \nabla g_j(\widehat{\boldsymbol{x}})\boldsymbol{\eta} = \mathbf{0}\} \end{cases}$$

Then, $\hat{\boldsymbol{x}}$ is a Pareto-optimal solution to problem (4.36).

Theorem 4.3.3: Suppose that f, g are weak quasistrictly pseudo type I univex at $\hat{\boldsymbol{x}} \in \mathbb{X}$ with respect to some $b_0, b_1, \varphi_0, \varphi_1$ and η for all feasible $\boldsymbol{x}$. Assuming that f, g are twice continuously differentiable at $\hat{\boldsymbol{x}} \in \mathbb{X}$. If for each critical direction $\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \neq \boldsymbol{0}$, there exist $\boldsymbol{\lambda} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^m$such that:

$$I. \quad \sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla g_j(\hat{\boldsymbol{x}}) = \boldsymbol{0},$$

$$II. \quad \left(\sum_{i=1}^{p} \lambda_i \nabla^2 f_i(\hat{\boldsymbol{x}}) + \sum_{j=1}^{m} \mu_j \nabla^2 g_j(\hat{\boldsymbol{x}}) \right) (\boldsymbol{\eta}, \boldsymbol{\eta}) > \boldsymbol{0},$$

$$III. \quad \begin{cases} \boldsymbol{\lambda} \geq \boldsymbol{0}, \boldsymbol{\mu} \geqq \boldsymbol{0}, \lambda_i = 0 \ \forall \ i \notin \Xi(\boldsymbol{\eta}) = \{i \in \{1,2,\dots,p\}: \nabla f_i(\hat{\boldsymbol{x}})\boldsymbol{\eta} = \boldsymbol{0}\} \\ \mu_j = 0 \ \forall \ j \notin \Psi(\boldsymbol{\eta}) = \{j \in \{1,2,\dots,m\}: g_j(\hat{\boldsymbol{x}}) = 0, \nabla g_j(\hat{\boldsymbol{x}})\boldsymbol{\eta} = \boldsymbol{0}\} \end{cases}$$

Then, $\hat{\boldsymbol{x}}$ is Pareto-optimal solution to problem (4.36).

Proof: Assume that, for each critical direction $\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \neq \boldsymbol{0}$, there exist $\boldsymbol{\lambda} \in \mathbb{R}^p$, $\boldsymbol{\mu} \in \mathbb{R}^m$ such that (I)-(III) hold and that $\hat{\boldsymbol{x}}$ is not a Pareto-optimal solution for (4.36). Then, there exists a feasible $\boldsymbol{x}$ to (4.36) such that

$$f(\boldsymbol{x}) \leq f(\hat{\boldsymbol{x}})$$

By the weak quasistrictly pseudo type I univex and the above inequality, one gets

$$\nabla f(\hat{\boldsymbol{x}})\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leqq \boldsymbol{0},$$
$$\nabla g(\hat{\boldsymbol{x}})\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leq \boldsymbol{0}$$

Two cases are to be considered:

Case 1: If $\nabla f(\hat{\boldsymbol{x}})\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) < 0$, $\eta(\boldsymbol{x}, \hat{\boldsymbol{x}}) = \boldsymbol{x} - \hat{\boldsymbol{x}} \neq \boldsymbol{0}$ and from the feasibility of $\hat{\boldsymbol{x}}$, one gets

$$\nabla f(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) < \boldsymbol{0},$$
$$\boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) \leq \boldsymbol{0}$$

Since $\boldsymbol{\lambda} \geq \boldsymbol{0}$, the above inequalities give

$$\left(\boldsymbol{\lambda}\nabla f(\hat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\hat{\boldsymbol{x}})\right)\boldsymbol{\eta}(\boldsymbol{x}, \hat{\boldsymbol{x}}) < \boldsymbol{0}$$

This contradicts the condition (I).

Case 2: If $\nabla f(\hat{x})\boldsymbol{\eta}(x, \hat{x}) = \boldsymbol{0}$ and let $\bar{\boldsymbol{x}}(\boldsymbol{t}) = \hat{\boldsymbol{x}} + t\,\boldsymbol{\eta}, t \in (\boldsymbol{0}, \boldsymbol{1}]$, one can get the following using the differentiability and the weak quasistrictly pseudo type I univex conditions:

$$f(\hat{\boldsymbol{x}} + t\,\boldsymbol{\eta}) - f(\hat{\boldsymbol{x}}) = t\nabla f(\hat{\boldsymbol{x}})\boldsymbol{\eta} + \frac{1}{2}\,t^2\nabla^2 f(\hat{\boldsymbol{x}})(\boldsymbol{\eta}, \boldsymbol{\eta}) + \boldsymbol{o}(t^2) \leq \boldsymbol{0}$$

Therefore,

$$\nabla f(\widehat{\boldsymbol{x}})\boldsymbol{\eta} + \frac{1}{2}\, t\nabla^2 f(\widehat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta}) + \frac{o(t^2)}{t} \leq \mathbf{0} \tag{4.42}$$

Similarly,

$$\nabla g(\widehat{\boldsymbol{x}})\boldsymbol{\eta} + \frac{1}{2}\, t\nabla^2 g(\widehat{\boldsymbol{x}})(\boldsymbol{\eta},\boldsymbol{\eta}) + \frac{o(t^2)}{t} \leqq \mathbf{0} \tag{4.43}$$

Using (III) in (4.42) and (4.43) we have

$$\left(\boldsymbol{\lambda}\nabla f(\widehat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla g(\widehat{\boldsymbol{x}})\right)\boldsymbol{\eta} + \frac{1}{2}\, t^2\left(\left(\boldsymbol{\lambda}\nabla^2 f(\widehat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla^2 g(\widehat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) + \frac{o(t^2)}{t}\right) \leq \mathbf{0}$$

From (I) and since $t > 0$ we obtain

$$\left(\boldsymbol{\lambda}\nabla^2 f(\widehat{\boldsymbol{x}}) + \boldsymbol{\mu}\nabla^2 g(\widehat{\boldsymbol{x}})\right)(\boldsymbol{\eta},\boldsymbol{\eta}) \leq \mathbf{0}$$

This contradicts (II). So the proof is completed.

4.5 Summary

This chapter has proposed first and second-order optimality conditions for dealing with multi-objective optimisation problems. The first-order optimality conditions for some specified classes of multi-objective optimisation problems have been illustrated and proved. Furthermore, the duality results corresponding to the first-order optimality conditions have been illustrated. These classes are general classes of multi-objective optimisation problems that cover the test and real-life cases studies tackled in the next chapters. For example, all the test problems that are solved in the next two chapters are examples of the class represented by the equation 4.1. The other classes are more general classes that cover problems different from those used in this thesis or may be used to cover problems in the future.

This chapter has achieved the following.

- It has proposed some classes of multi-objective optimisation problems.
- It has proposed first-order optimality conditions under univexity concept for these classes.
- It has identified the duality results corresponding to the first-order optimality conditions proposed above.
- It has proposed second-order optimality conditions for multi-objective optimisation problems under univexity concept.

This chapter has proposed first and second-order optimality conditions for handling the challenges of multi-objective optimisation problems where the objectives and the constraints should satisfy a generalisation type of the univexity aspect. However, there is a need to develop symbolic algorithms based on these theoretical results that can be used to deal with these challenges in a systematic way. The next chapter aims to propose a multi-objective optimisation symbolic algorithm based on the first-order optimality conditions. It is used to get an analytical formula for the Pareto-optimal front along with analysing the relationships among the decision variables used for constructing this formula in an analytical way.

5 PROPOSED FIRST-ORDER MULTI-OBJECTIVE OPTIMISATION SYMBOLIC ALGORITHM (MOSA-I)

This chapter aims to present a multi-objective symbolic algorithm (MOSA-I) for solving multi-objective optimisation problems with particular aspects. These aspects include the class of all objective functions and inequality constraints that are continuous, differentiable, and convex. The algorithm is used to identify the relationship between the decision variables that leads to the analytical formula of the Pareto-optimal front between the conflicting objective functions. This relationship is provided by the symbolic algorithm as functions of Lagrange's multipliers. It is demonstrated that in some cases better solutions are obtained compared to those obtained by state-of-the-art optimisation algorithms. The algorithm's steps have been coded using Mathematica toolbox© and the core of it is based on Karush-Kuhn-Tucker (KKT) conditions. The algorithm is tested by solving some multi-objective optimisation problems that have been carefully selected from literature. A state-of-the art optimisation algorithm, NSGA-II, has been used for comparison as it has given better solutions for these problems than the current optimisation algorithms. This chapter attempts to achieve the following.

- *To propose a multi-objective symbolic algorithm (MOSA-I).*
- *To present some test problems used to test the proposed algorithm.*
- *To discuss and compare the results from the algorithm with those from NSGA-II.*
- *To measure the performance of NSGA-II.*
- *To analysis the advantages and disadvantages of the proposed algorithm.*

5.1 *Multi-objective Symbolic Algorithm (MOSA-I)*

In this section, a multi-objective symbolic algorithm, called MOSA-I, is presented for solving multi-objective optimisation problems. The algorithm's steps are entirely described below. Each step of the algorithm is coded by Mathematica Toolbox©. Details on the performance of the algorithm on some well-known test problems are shown later. Figure 5.1 shows the flow chart of the proposed symbolic algorithm. The following steps are given:

Step 1: Define the objective functions $f_i, i = 1, \dots, M$ to be minimised.

Step 2: Define the inequality constraints $g_j, j = 1, \dots, J$.

Step 1 and Step2 of the MOSA-I algorithm are given. They present the mathematical formulae of the objective functions and the constraints. They only need to be typed under Mathematica Toolbox©.

The algorithm's main steps:

Step 3: Check the continuity and differentiability of the objective functions and constraints. If yes, go to step 4, otherwise terminate.

Step 4: Calculate the second-order partial derivatives of each objective function separately.

Step 5: Form the Hessian matrix $\mathbb{H}$ for each objective function. For example, the Hessian matrix for f_1 can be written as follows:

$$\mathbb{H} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_1}{\partial x_2 \partial x_1} & \frac{\partial^2 f_1}{\partial x_2^2} & \cdots & \frac{\partial^2 f_1}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_1}{\partial x_n \partial x_1} & \frac{\partial^2 f_1}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_n^2} \end{pmatrix}$$

Step 6: Check if the Hessian matrix $\mathbb{H}$ is positive semi-definite, i.e., $\boldsymbol{x}^{t}\mathbb{H}\boldsymbol{x} \geq \boldsymbol{0}$ for all solution vector $\boldsymbol{x} \neq \boldsymbol{0}$. If yes, then, in this case, $f_1$is convex, otherwise go to step 7.

Step 7: Check if the condition $\nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \leq \boldsymbol{0}$ is satisfied for one $\boldsymbol{y}$ in the feasible space. If yes, go to step 8, otherwise terminate.

Step 8: Solve the system of equations $\sum_{i=1}^{M} \lambda_i \nabla f_i(\boldsymbol{x}) + \sum_{j=1}^{J} \mu_j \nabla g_j(\boldsymbol{x}) = \boldsymbol{0}$ to find $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Step 9: Solve the system of equations $\mu_j g_j(\boldsymbol{x}) = 0$, for all $j \notin \{j \in \{1,2,\dots,J\} |\, g_j(\boldsymbol{x}) = 0\}$ to get $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\lambda})$.

Step 10: Substitute by the results obtained from step 9 in step 8 to obtain $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\lambda})$.

Step 11: Substitute the results from step 10 in the objective functions (Step 1) to formulate the analytical formula of the Pareto-optimal front, as for example, f_2 is a function of f_1, i.e., $f_2 = \varphi(f_1)$. Note that, in some cases, the multiplier $\boldsymbol{\lambda}$ can not be removed and the thus the relationship between the conflicting functions will depend on that multiplier. To overcome this, different values for this multiplier are given by trial and error and consequently, the Pareto-optimal front is found.

Step 12: End

Clarification of the algorithm's steps:

Step1 - Step2: Both steps are given. Their mathematical description should be firstly coded under Mathematica so that the other steps can be executed.

Step3: As MOSA-I algorithm is created to solve multi-objective optimisation problems that have continuous and differentiable objectives and constraints; this step is used to make sure that the objectives and constraints are continuous and differentiable.

Step4-Step6: To the check the convexity of the objectives, the second-order partial derivatives are used. Using step 1, setp2 and the second-order partial derivatives for each objective function, the matrix of the Hessian is now formed. After that the positive semi-definite of the Hessian is checked.

Step7: This step is an alternative one to step 4. It is used to identify the convexity of the objectives using the first-order partial derivatives when the second-order partial derivatives are not exist or the Hessian matrix is not positive semi-definite.

Step8: In this step, all the gradient vectors are known and only unknown vectors are λ, μ and x. This step can be rewritten as follows:

$$[\lambda_1 \quad \cdots \quad \lambda_M]_{1\times M}\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_n} \end{bmatrix}_{M\times n} + [\mu_1 \quad \cdots \quad \mu_J]_{1\times J}\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_J}{\partial x_1} & \frac{\partial g_J}{\partial x_2} & \cdots & \frac{\partial g_J}{\partial x_n} \end{bmatrix}_{J\times n} = 0$$

So the number of unknown variables in the above systems is now $M + n + J$. The above system is solved to get $x = x(\lambda, \mu)$.

Step 9: This step is used to reduce the number of unknown variables in step 8. The system is this step is solved to get $\boldsymbol{\mu}$ as functions of $\boldsymbol{\lambda}$. Here, we solve the system of equations $\mu_j g_j(\boldsymbol{x}) = 0$, for all $j \notin \{j \in \{1,2,\dots,J\} \big| \, g_j(\boldsymbol{x}) = 0\}$ to get $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\lambda})$.

Step 10: This step is used to form the Pareto set as functions of $\boldsymbol{\lambda}$ multiplier. This Pareto set is used later to form the Pareto-optimal front. Note that, in some cases, the multiplier $\boldsymbol{\lambda}$ can not be removed and the thus the relationship between the conflicting functions will depend on that multiplier. To overcome this, different values for this multiplier are given by trial and error and consequently, the Pareto-optimal front is found.

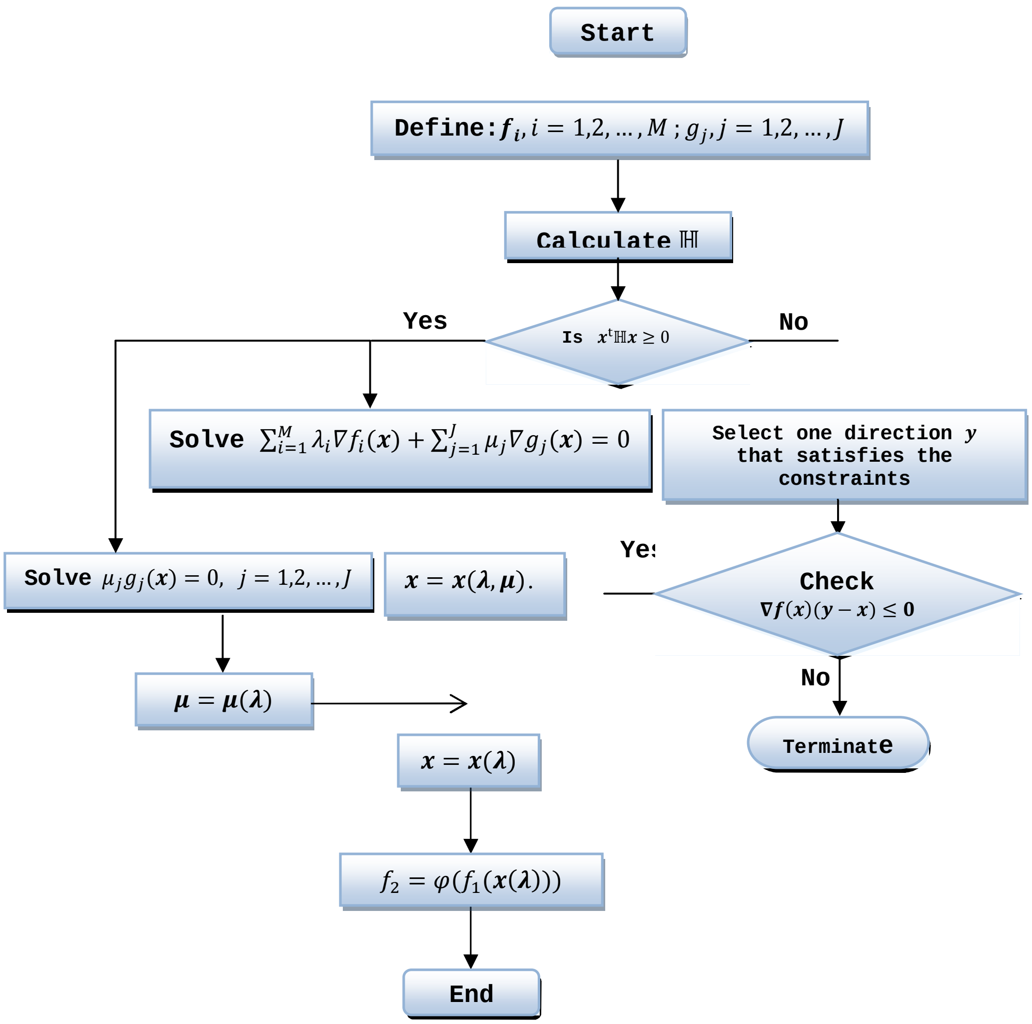

Figure 5.1: The flow chart of the proposed MOSA-I algorithm.

5.2 Test Problem Formulation

In order to test and validate MOSA-I, some problems that have been solved using the state-of-the-art optimisation algorithms were selected from the literature (Mehnen, 2006). The test problems ZDT1 to ZDT6 are known as complex benchmark problems and have not been analytically solved yet using KKT conditions. What we mean here by analytically solved is that the analytical relationship between the decision variables that is used to construct the analytical formula of the Pareto-optimal front has not found yet as functions of KKT multipliers. As can be seen below, these problems have been framed such that the function g has the effect of producing difficulty in progressing towards the true Pareto-optimal front and the function h has the effect of causing difficulty for spread along the Pareto-optimal front. Here, three of these problems ZDT1, ZDT2 and ZDT4 along with other problems carefully selected from literature are used to check the algorithm. These problems have been chosen because they have different characteristics such as, for example, the shape of the Pareto-optimal front, convexity and concavity of the Pareto-optimal front. A complete description about the features of these problems is given below (Table 5.1):

Table 5.1: Features of test problems

Problems	Problem features						
	constraints	No. of objectives	Pareto-optimal front features				
			Geometry	Connected	Disconnected	Convex	Concave
ZDT1 (Deb,2001)	Box	2	Curve	✓		✓	
ZDT2 (Deb,2001)	Box	2	Curve	✓			✓
ZDT4 (Deb,2001)	Box	2	Curve	✓		✓	
Deb 2 (Deb,2001)	Box	2	Curve		✓	✓	
FON (Deb,2001)	Box	2	Curve	✓			✓
VNT1 (Coello,2002)	Box	3	Surface	✓		✓	
Constrained-problem (Deb,2001)	Inequalities	2	Curve		✓	✓	

Problem formulation (ZDT1)

This problem can have any number of variables and the Pareto-optimal front for any number of variables is convex. It has two objective functions which are to be minimised (Deb, 2001). The analytical formulation of this problem is as follows.

$$
\begin{aligned}
&f_1(\boldsymbol{x}) = x_1, \\
&f_2(\boldsymbol{x}) = g(x) \times h(f_1, g) \\
&g(\boldsymbol{x}) = 1 + \tfrac{9}{n-1}\textstyle\sum_{i=2}^{n} x_i \\
&h(f_1, g) = 1 - \sqrt{\tfrac{f_1}{g}} \\
&0 \le x_i \le 1,\ \ i = 1,2, \dots, n
\end{aligned}
\qquad (5.1)
$$

The Pareto-optimal set for this problem is $0 \le x_1 \le 1$ and $x_i = 0, i = 2,3, \dots, n$ (Deb, 2001).

Problem formulation (ZDT2)

This problem can have any number of variables and the Pareto-optimal front for any number of variables is non-convex. It has two objective functions which are to be minimised (Deb, 2001). The analytical formulation of this problem is as follows.

$$\begin{aligned} &f_1(\boldsymbol{x}) = x_1, \\ &f_2(\boldsymbol{x}) = g(x) \times h(f_1, g) \\ &g(\boldsymbol{x}) = 1 + \frac{9}{n-1} \textstyle\sum_{i=2}^{n} x_i \\ &h(f_1, g) = 1 - \left(\frac{f_1}{g}\right)^2 \\ &0 \leq x_i \leq 1,\ i = 1,2, \dots, n \end{aligned} \tag{5.2}$$

The Pareto-optimal set for this problem is $0 \leq x_1 \leq 1$ and $x_i = 0, i = 2,3, \dots, n$ (Deb, 2001). The only difficulty with this problem is that the Pareto-optimal front is non-convex. Thus, the weighted approaches will have difficulty in finding a good spread of solutions on the Pareto-optimal front.

Problem formulation (ZDT 4)

This problem can have any number of variables and the Pareto-optimal front for any number of variables is convex. It has two objective functions which are to be minimised (Deb, 2001). The analytical formulation of this problem is as follows.

$$\begin{aligned} &f_1(\boldsymbol{x}) = x_1, \\ &f_2(\boldsymbol{x}) = g(x) \times h(f_1, g) \\ &g(\boldsymbol{x}) = 1 + 10\,(n-1) + \textstyle\sum_{i=2}^{n}\left(x_i^2 - 10\cos(4\pi \cdot x_i)\right) \\ &h(f_1, g) = 1 - \sqrt{\frac{f_1}{g}} \\ &x_1 \in [0,1] \\ &x_i \in [-5,5], i = 2, \dots, n \end{aligned} \tag{5.3}$$

The Pareto-optimal set for this problem is $0 \leq x_1 \leq 1$ and $x_i = 0, i = 2,3, \dots, n$ (Deb, 2001). This problem is more difficult than ZDT1. It has a large number of local Pareto-optimal fronts which produce hurdles for optimisation algorithms to converge to the global Pareto-optimal front (Deb, 2001).

Problem formulation (FON)

This problem is a typical multi-objective optimisation benchmark problem. It consists of two objective functions to be minimised and n decision variables. It has the following mathematical description:

$$f_1 = 1 - e^{-\sum_{i=1}^{n}\left(x_i - \frac{1}{\sqrt{n}}\right)^2}$$
$$f_2 = 1 - e^{-\sum_{i=1}^{n}\left(x_i + \frac{1}{\sqrt{n}}\right)^2} \quad (5.4)$$
$$-4 \le x_i \le 4$$

Problem formulation (Deb 2)

This problem is a two-variable problem. It consists of two objective functions to be minimised. It has the following mathematical description:

$$f_1(\boldsymbol{x}) = f(x) = x,$$
$$f_2(\boldsymbol{x}) = g(\boldsymbol{y}) \times h(f,g),$$
$$g(\boldsymbol{y}) = 1 + 10\,y$$
$$h(f,g) = 1 - \left(\frac{f}{g}\right)^{\alpha} - \left(\frac{f}{g}\right) \cdot \sin(2\pi \cdot q \cdot f), \alpha = 2 \quad (5.5)$$
with q defined as the number of legs in [0,1]
$$0 \le x, y \le 1$$

The parameter q allows us to define the number of discontinuous areas in the interval [0,1]. The parameter α is often chosen to be equal to 2.

Problem formulation (VNT1)

This problem is a two-variable problem. It consists of three objective functions that have the following form:

$$f_1(x,y) = x^2 + (y-1)^2,$$
$$f_2(x,y) = x^2 + (y+1)^2 + 1, \quad (5.6)$$
$$f_3(x,y) = (x-1)^2 + y^2 + 2$$
$$-2 \le x, y \le 2$$

Problem formulation (Constrained problem)

This problem is a two-variable problem. It consists of two objective functions and two inequality constraints. It has the following mathematical description:

$$f_1(\boldsymbol{x}) = x_1,$$
$$f_2(\boldsymbol{x}) = \frac{1+x_2}{x_1}$$
$$g_1(\boldsymbol{x}) = x_2 + 9x_1 \ge 6 \quad (5.7)$$
$$g_2(\boldsymbol{x}) = -x_2 + 9x_1 \ge 1$$
$$0.1 \le x_1 \le 1, 0 \le x_2 \le 5$$

5.3 Comparison of Results

The following results and observations have been found after executing MOSA-I on the test problems defined in the previous section. The results provided by NSGA-II have been obtained using a standard code of NSGA-II.

The following results are obtained:

Results analysis of Problem 1 (ZDT1): step 1 to step 4 are very simple to implement. They need one to be familiar with Mathematica to define the objectives and constraints and to check their continuity and differentiability.

Step 5: This step gives the following Hessian matrices for f_1 and f_2:

$$\mathbb{H}_{f_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{H}_{f_2} = \begin{bmatrix} \frac{1}{4\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)} & -\frac{9x_1}{8\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & -\frac{9x_1}{8\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^2} \\ -\frac{9x_1}{8\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & \frac{81x_1^2}{16\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^3} & \frac{81x_1^2}{16\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^3} \\ -\frac{9x_1}{8\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & \frac{81x_1^2}{16\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^3} & \frac{81x_1^2}{16\left(\frac{x_1}{1+\frac{9}{2}(x_2+x_3)}\right)^{\frac{3}{2}}\left(1+\frac{9}{2}(x_2+x_3)\right)^3} \end{bmatrix}$$

Step 6: For the matrix $\mathbb{H}_{f_1}$, $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_1} \boldsymbol{x} = 0$ for all non-zero $\boldsymbol{x} \in [\mathbf{0}, \mathbf{1}]$. This means that f_1 is convex. For the matrix $\mathbb{H}_{f_2}$, we have the following:

$$\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2} \boldsymbol{x} = \frac{(-9(b+c)x_1 + a(2+9x_2+9x_3))^2}{4\sqrt{2}\left(\frac{x_1}{2+9x_2+9x_3}\right)^{\frac{3}{2}}(2+9x_2+9x_3)^3}$$

This means that $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2} \boldsymbol{x} > 0$ for all non-zero $x = (a, b, c)$. Thus, f_2 is strictly convex function.

Step 8-10: By solving the two systems in step 8 and 9, one can see that the global Pareto-optimal set is $\left(\frac{\lambda_2^2}{4\lambda_1^2}, 0, 0\right)$, where λ_1 and λ_2 are KKT multipliers. This means that the exact Pareto-optimal front occurs at $g(x) = 1$. Some other dominated solutions are found by Mathematica at $\left(\frac{\lambda_2^2}{4\lambda_1^2}\left(1 + \frac{9}{2}(\alpha+\beta)\right), \alpha, \beta\right)$ where $0 < \alpha, \beta \le 1$ and $\alpha = 0\ or\ \beta = 0$. For example, if $\alpha = 0\ and\ \beta = 1$, then the Pareto-optimal front in this case is with . This means that the Pareto-optimal set is .

Step 11: Substituting by the global Pareto-optimal set calculated in the previous step, the exact Pareto-optimal front for ZDT1 at takes the form:

(5.8)

This exact Pareto-optimal front is constructed by the Pareto-optimal set . It is clear that this Pareto-optimal front is convex (see Figure 5.2). Figure 5.2 shows the exact and approximated Pareto-optimal front found by MOSA-I and NSGA-II at The tests reported here are carried out using population size, generations, crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II. Here, we have used the NSGA-II algorithm because it gives better solutions for these problems.

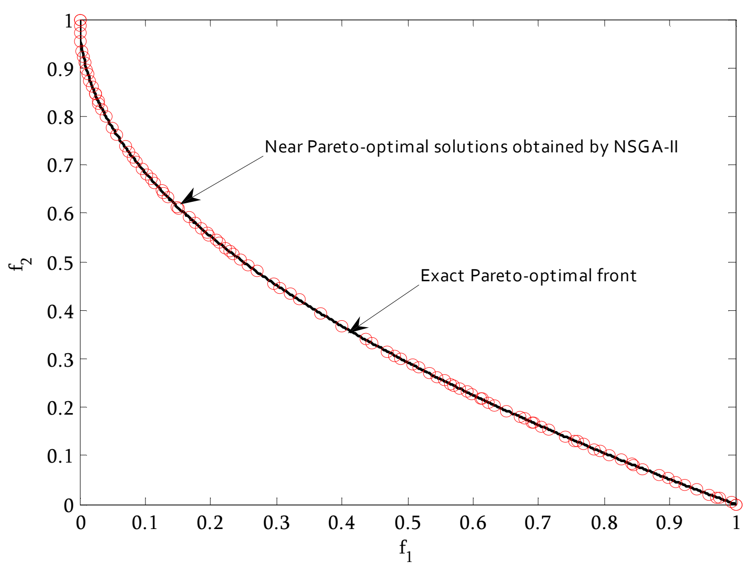

Figure 5.2: The exact and approximated Pareto-optimal front for ZDT1 at

It has been mentioned in Deb (2001) that evolutionary algorithms face difficulty in tackling a large number of decision variables in this problem. MOSA-I has not found any difficulty in solving this problem with high dimensions in the decision space. The exact Pareto-optimal front for this problem with using MOSA-I takes the same form as Equation (5.8) with at Pareto-optimal set . Some other dominated solutions are found by Mathematica at are in the form with Pareto-optimal set . For example, if and then the Pareto-optimal front in this case is . In general, when solving this problem at , MOSA-I yielded the following equations:

(5.9)

The solution of (5.9) takes the following form:

(5.10)

If then the Pareto-optimal set becomes and the corresponding Pareto-optimal front is the same equation as equation (5.8). Other dominated Pareto-optimal front can be obtained at for at least one . Figure 5.3 shows the exact and approximated Pareto-optimal front using MOSA-I and NSGA-II. The tests reported here are carried out using population size, generations, crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II.

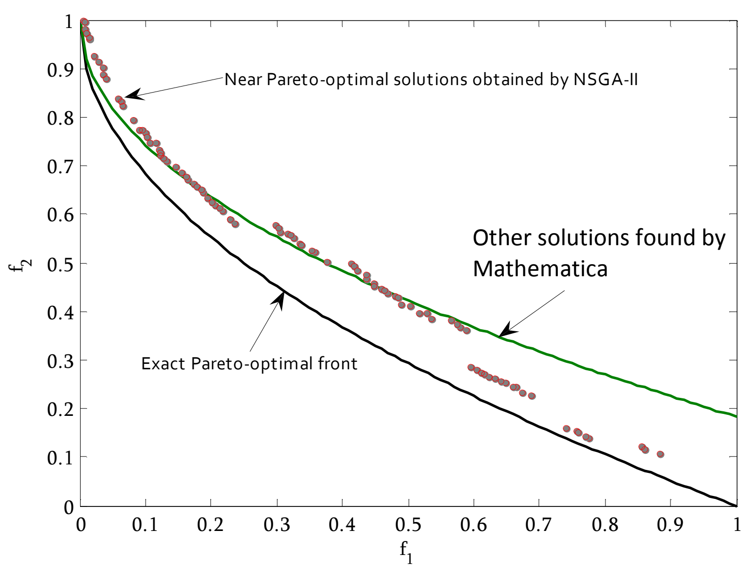

Figure 5.3: The exact and approximated Pareto-optimal front for ZDT1 at

Results analysis of Problem 2: step 1 to step 4 are very simple to implement.

Step 5: This step gives the following Hessian matrices for f_1 and f_2:

$$\mathbb{H}_{f_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{H}_{f_2} = \begin{bmatrix} \frac{-2}{\left(1+\frac{9}{2}(x_2+x_3)\right)} & \frac{9x_1}{\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & \frac{9x_1}{\left(1+\frac{9}{2}(x_2+x_3)\right)^2} \\ \frac{9x_1}{\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & -\frac{81x_1^2}{2\left(1+\frac{9}{2}(x_2+x_3)\right)^3} & -\frac{81x_1^2}{2\left(1+\frac{9}{2}(x_2+x_3)\right)^3} \\ \frac{9x_1}{\left(1+\frac{9}{2}(x_2+x_3)\right)^2} & -\frac{81x_1^2}{2\left(1+\frac{9}{2}(x_2+x_3)\right)^3} & -\frac{81x_1^2}{2\left(1+\frac{9}{2}(x_2+x_3)\right)^3} \end{bmatrix}$$

Step 6: For the matrix $\mathbb{H}_{f_1}$, $\boldsymbol{x}^{\mathrm{t}}\,\mathbb{H}_{f_1}\boldsymbol{x} = 0$ for all non-zero $\boldsymbol{x} \in [\mathbf{0}, \mathbf{1}]$. This means that f_1 is convex. For the matrix $\mathbb{H}_{f_2}$, we have the following:

$$\boldsymbol{x}^{\mathrm{t}}\,\mathbb{H}_{f_2}\boldsymbol{x} = -\frac{4(-9(b+c)x_1 + a(2+9x_2+9x_3))^2}{(2+9x_2+9x_3)^3}$$

This means that $\boldsymbol{x}^{\mathrm{t}}\,\mathbb{H}_{f_2}\boldsymbol{x} < 0$ for all non-zero $\boldsymbol{x} = (a, b, c)$. Then, we go to step 7.

step 7: For one $y = (0,1,1)$ one can easy find that $\nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \leq \mathbf{0}$. This means that f_2 is pseudoconvex function.

Step 8-10: By solving the two systems in step 8 and 9, one can see that the global Pareto-optimal set is $\left(\frac{\lambda_1}{2\lambda_2}, 0{,}0\right)$, where λ_1 and λ_2 are KKT multipliers. This means that the exact Pareto-optimal front occurs at $g(x) = 1$. Some other dominated solutions are found by Mathematica at $\left(\frac{\lambda_1}{2\lambda_2}\right(1 +$ 92α+β,α,β where *0<α,β≤1* and *α=0 or β=0*. For example, if *α=0 and β=1,* then the Pareto-optimal front in this case is $f_2 = 1 - \frac{2}{11}f_1^2, 0 \leq f_1 \leq 1$ with $g(\boldsymbol{x}) = \frac{11}{2}$. This means that the Pareto-optimal set in this case is $\left(\frac{11\lambda_1}{4\lambda_2}, 0{,}1\right)$.

Step 11: Substituting by the global Pareto-optimal set calculated in the previous steps, the exact Pareto-optimal front for ZDT2 at $n = 3$ takes the form:

$$f_2 = 1 - f_1^2, 0 \leq f_1 \leq 1 \tag{5.11}$$

This exact Pareto-optimal front is constructed by the Pareto-optimal set $\left(\frac{\lambda_1}{2\lambda_2}, 0{,}0\right)$. It is clear that Equation 5.11 of the Pareto-optimal front is non-convex (see Figure 5.6). Figure 5.4 shows the exact and approximated Pareto-optimal front found by MOSA-I and NSGA-II at $n = 3$. The tests reported here are carried out using 100 population size, 16000 generations, 0.8 crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II.

When solving this problem at , MOSA-I yielded the following equations:

(5.12)

The solution of (5.12) takes the following form:

(5.13)

If then the Pareto-optimal set in this case becomes and the corresponding Pareto-optimal front is the same equation as equation (5.11). Some other dominated solutions are found by Mathematica at

for at least one . It is worthy to mention here that since the exact Pareto-optimal front for this problem is non-convex, the weighted-sum approach has difficulty in finding a good spread of solutions on the Pareto-optimal front.

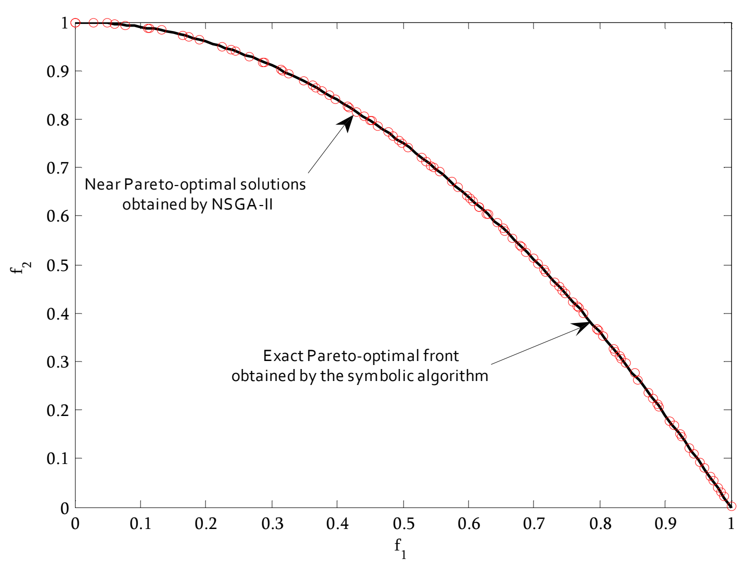

Figure 5.4: Pareto-optimal front obtained by the symbolic algorithm and NSGA-II for ZDT2 at

Results analysis of Problem 3: step 1 to step 6 are very simple to implement such as ZDT1 and ZDT2.

Step 8-10: By solving the two systems in step 8 and 9, one can see that the global Pareto-optimal set is , where and are KKT multipliers. This means that the exact Pareto-optimal front occurs at . This means that the exact Pareto-optimal front takes the form:

(5.14)

This exact Pareto-optimal front is constructed by the Pareto-optimal set . It is clear that this Pareto-optimal front is convex (see Figure 5.5). In general, the Pareto-optimal set for this problem at is with the same Pareto-optimal front equation as equation (5.14). Figure 5.5 shows the exact and approximated Pareto-optimal front obtained by MOSA-I and NSGA-II.

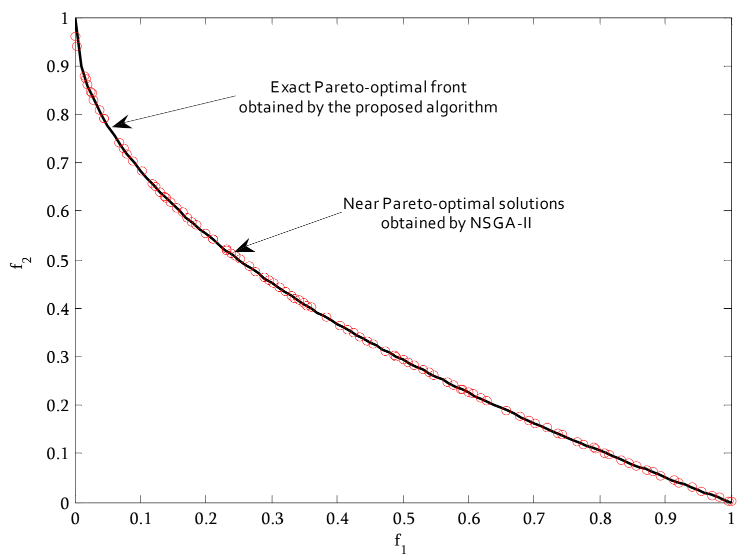

Figure 5.5: Pareto-optimal front obtained by the symbolic algorithm and NSGA-II for ZDT4 at

For this problem at , the following equations for the decision variables are obtained by the MOSA-I algorithm:

The solution of above system is:

This means that and hence the exact Pareto-optimal front is the same equation as equation (5.14). The Pareto-optimal set corresponding to this case is — . This problem can be solved by MOSA-I for any number of decision variables. In addition, the equations provided by the MOSA-I algorithm will be the same as in equation (5.14). However, NSGA-II faces difficulty in converging to the exact Pareto-optimal front as this problem has huge number of local Pareto-optimal sets.

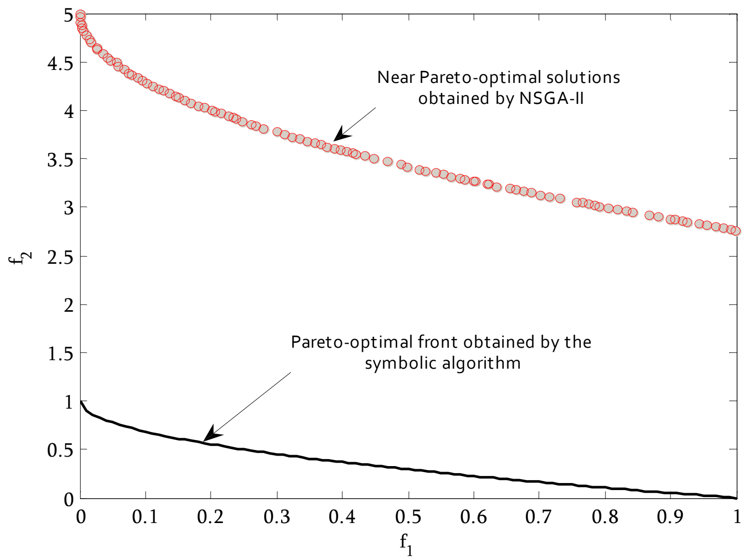

Figure 5.6: Exact and approximated Pareto-optimal front found by the MOSA-I and NSGA-II algorithms for problem ZDT 4 at .

The tests reported here are carried out using population size, generations, crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II. Figure 5.6 shows the results obtained by NSGA-II and the MOSA-I algorithm.

Results analysis of Problem 4: step 1 to step 6 are very simple to implement.

Step 8-10: By solving the two systems in step 8 and 9 for Fonseca and Fleming problem (Deb, 2001) at , one can see that the global Pareto-optimal set is taken the following form:

(5.15)

where and are KKT multipliers. After separating KKT multipliers from equation (5.15), one gets

(5.16)

Using equation (5.16), the analytical formula of the exact Pareto-optimal front is given by step 11 of MOSA-I algorithm as follows:

(5.17)

Although both the objective functions are convex functions, the exact Pareto-optimal front is a non-convex curve as can be seen in Figure 5.7.

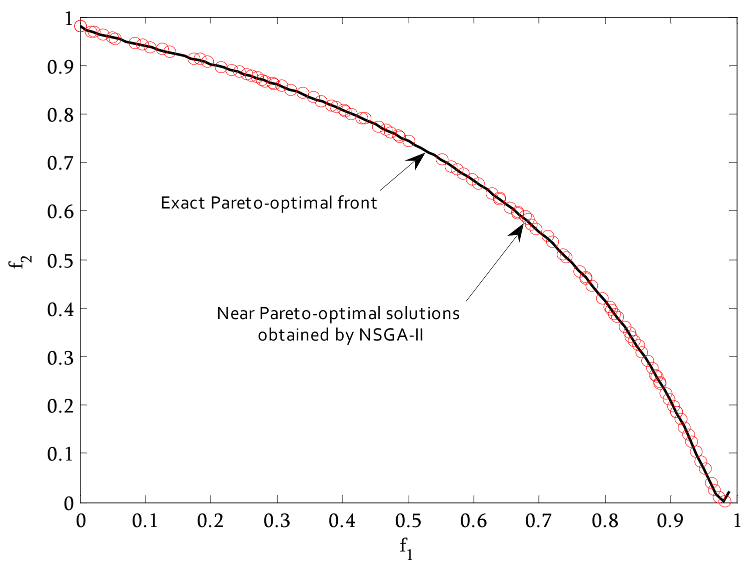

Figure 5.7: Objective space of Fonseca and Fleming problem

The above curve of the exact Pareto-optimal front is constructed by the linear relationship (equation 5.16) between the decision variables (red colour, Figure 5.8).

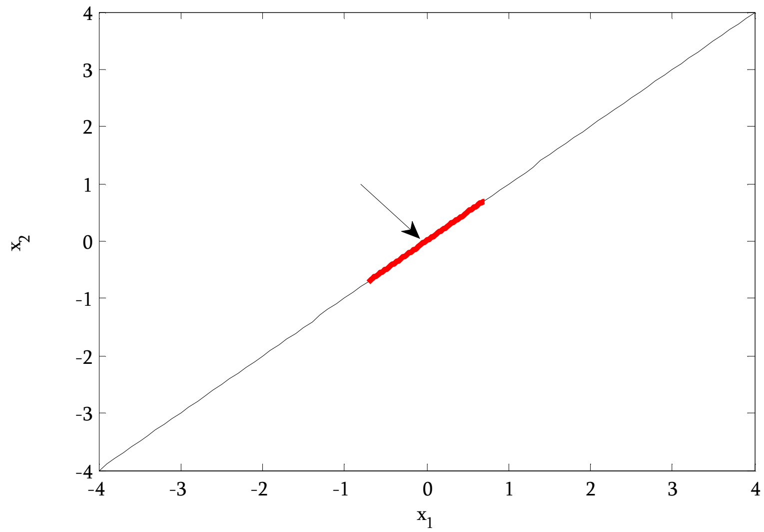

Figure 5.8: Decision space of Fonseca and Fleming problem

As can be shown from Figure 5.8, not all the linear relationships between the decision variables are used to construct the exact Pareto-optimal front.

This problem has been solved using the state-of-the-art evolutionary algorithm, NSGA-II, with population size 100 and 100 generations using standard parameters. The result is plotted in Figure 5.6 (red circles). It is shown that NSGA-II is robust in finding uniform solutions near the exact curve of the Pareto-optimal front. MOSA-I algorithm provides a connected curve of the Pareto-optimal front. Furthermore, it is guided to the relationship between the decision variables responsible for constructing that curve. In addition, this problem has been solved using the MOSA-I algorithm at and has yielded the following:

(5.18)

This means that the dimension of the search space has no direct impact on the shape of the Pareto-optimal front; only the range of the objective will be changed. The algorithm shows that the Pareto-optimal solutions for this problem satisfy the equation:

(5.19)

Results analysis of Problem 5: step 1 to step 6 are very simple to implement.

Step 8-10: By solving the two systems in step 8 and 9 for Deb 2 at , the following solutions are found:

Solution 1: , this is a dominated Pareto-optimal point (see figure 5.8).

Solution 2: , this is also a dominated Pareto-optimal point (see Figure 5.8).

Solution 3: and satisfies the equation:

(5.20)

where and are KKT multipliers. Using equation (5.20) with step 11 of MOSA-I algorithm, the exact curve of the Pareto-optimal front is constructed as follows:

(5.21)

This formula is plotted in Figure 5.8 (black bold curve). Figure 5.9 shows a disconnected Pareto-optimal front. Cases 1 and 2 are dominated by points on this curve. This problem has been solved using NSGA-II with population size 100 and 100 generations using standard parameters. The results are plotted in Figure 5.9 (red circles).

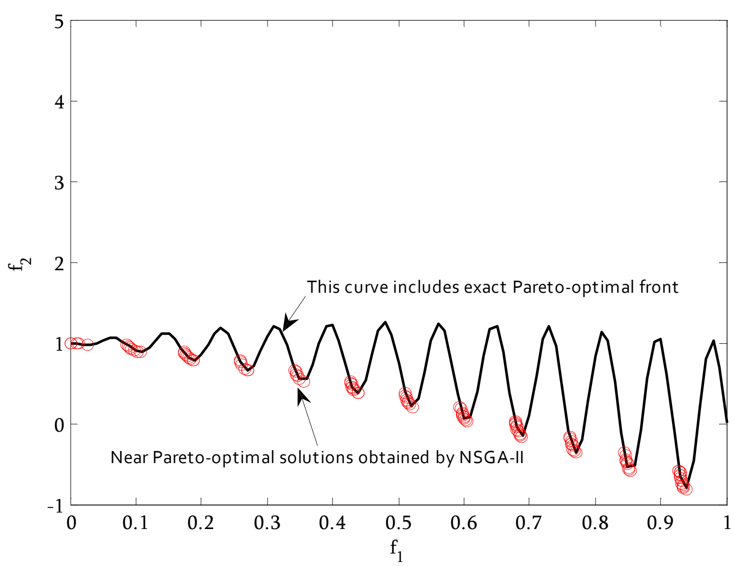

Figure 5.9: Exact and approximated Pareto-optimal front for Deb2 problem

Results analysis of Problem 6: step 1 to step 6 are very simple to implement.

Step 8-10: By solving the two systems in step 8 and 9 for VNT1, one can see that the global Pareto-optimal set is taken the following form:

(5.22)

where and are KKT multipliers. Using equation (5.22), the analytical formula of the exact Pareto-optimal front is given by step 11 of MOSA-I algorithm as follows:

(5.23)

This equation represents the hyper-surface involving the exact Pareto-optimal front. Figure 5.10 shows the surface of this equation including the Pareto-optimal solutions (cross points) obtained by NSGA-II. The interesting observation here is that this curve is constructed using the linear relationship between the decision variables (equation 5.22).

The tests reported here are carried out using population size, generations, crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II.

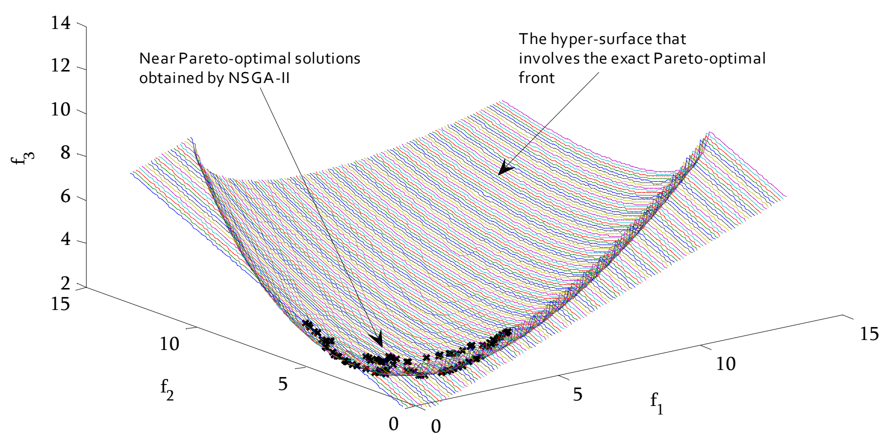

Figure 5.10: Exact and approximated Pareto-optimal front for VNT1 problem

Results analysis of Problem 7: step 1 to step 6 are very simple to implement.

Step 8-10: By solving the two systems in step 8 and 9 for the constrained problem, step 10 of MOSA-I algorithm has given the mathematical formulae for the relationship among the decision variables as functions of Lagrange multipliers. These formulae are as follows.

(5.24)

To eliminate from Equation (5.24) step 9 in the algorithm gives 19 cases that represent the relationship between and . Not all these case are accepted. Only 8 cases are accepted and the rest is rejected. The rejected cases contain complex relationship between and and therefore are out of the feasible search space since it is real space. The 19 cases are as follows:

Case 1: — — — .

This case gives the point on the border of the feasible decision space (bold line, Figure 5.11). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is on the region A (bold curve, Figure 5.12).

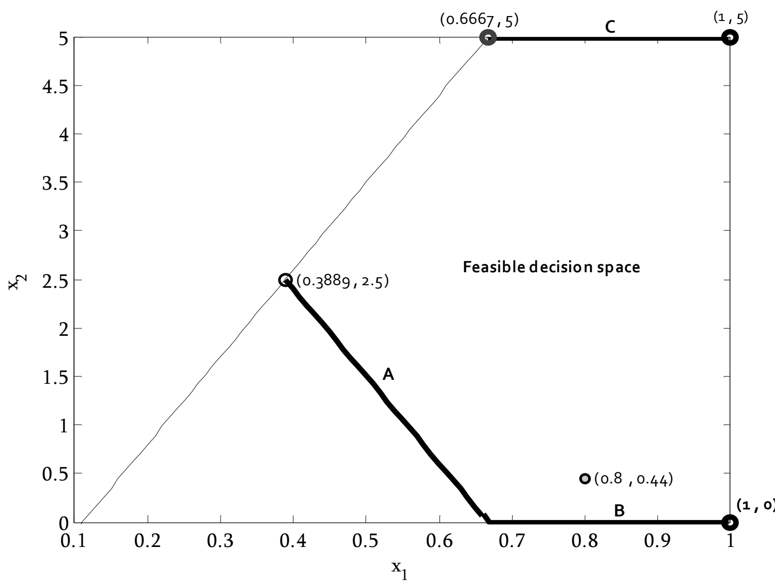

Figure 5.11: Decision space of the constrained problem

Case2: .

This case yields the point in the feasible decision space (bold point, Figure 5.11). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is (bold point, Figure 5.12). This point is better than some points in region and is dominated by some points from regions and .

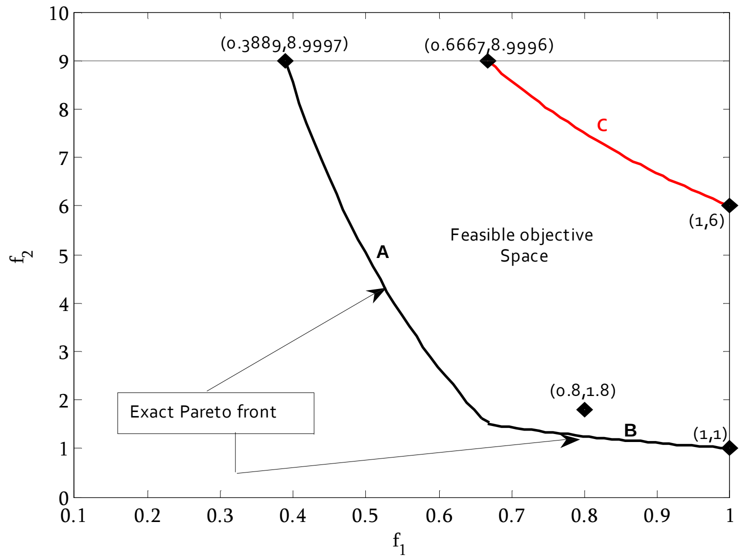

Figure 5.12: Objective space of the constrained problem obtained by MOSA-I algorithm

Case 3: .

This case gives the following relationship between the decision variables:

$$ \tag{5.25}$$

This relationship between and represents the bold line in the decision space (Figure 5.11). The points satisfying this line are used to construct the formula:

$$ \tag{5.26}$$

This formula is the first part of the exact Pareto-optimal front (bold curve , Figure 5.12).

Case 4: .

This case gives the following relationship between the decision variables:

$$ \tag{5.27}$$

The relationship (5.27) between and is represented by the bold line in the decision space (Figure 5.11). The points satisfying this line are used to construct the formula:

$$ \tag{5.28}$$

This formula is the second part of the exact Pareto-optimal front (bold curve B, Figure 5.12).

Case 5: $\mu_1 = 0, \mu_2 = 0.111111\lambda_1 - 1.5\lambda_2,\ \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = -0.111111\lambda_1$.

This case yields the point $(0.6667,5)$ in the feasible decision space (bold point on the bold line C, Figure 5.11). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(0.6667,8.9996)$ (bold point on the curve C, Figure 5.12). This point is dominated by points on the curve A.

Case 6: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \mu_6 = -\sqrt{\frac{\lambda_1\lambda_2}{6}}$.

This case gives the following relationship between the decision variables:

$$x_1 = \sqrt{\frac{6\lambda_2}{\lambda_1}}, x_2 = 5 \qquad (5.29)$$
$$0.6667 \le x_1 \le 1$$

The relationship (5.29) between x_1 and x_2 is represented by the bold line C in the decision space (Figure 5.11). The points satisfying this line are used to construct the formula:

$$f_2 = \frac{6}{f_1}, \qquad (5.30)$$
$$0.6667 \le f_1 \le 1$$

This formula creates the part C (red curve in Figure 5.12). This Pareto-optimal front is dominated by both curves A and B (Figure 5.12).

Case 7: $\mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -\lambda_1 + \lambda_2,\ \mu_5 = \lambda_2,\ \mu_6 = 0$.

This case yields the point $(1,0)$ on the border of the feasible decision space (bold line B, Figure 5.11). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(1,1)$ on the border of the region B (bold curve, Figure 5.12).

Case 8: $\mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -\lambda_1 + 6\lambda_2,\ \mu_5 = 0,\ \mu_6 = -\lambda_2$.

This case yields the point $(1,5)$ on the border of the feasible decision space (bold line C, Figure 5.11). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(1,6)$ on the border of the region C (red bold curve, Figure 5.12). This point is dominated by the point $(1,1)$.

Case 9: $\mu_1 = \mu_4 = \mu_5 = \mu_6 = 0, \mu_2 = -10\lambda_2,\ \mu_3 = \lambda_1$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 10: $\mu_1 = \mu_3 = \mu_5 = \mu_6 = 0, \mu_2 = -\lambda_2,\ \mu_4 = -\lambda_1$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 11: $\mu_3 = \mu_4 = \mu_5 = \mu_6 = 0, \mu_1 = 0.0555556\lambda_1,\ \mu_2 = 0.0555556\lambda_1 - 2.57143\lambda_2$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 12: $\mu_1 = 0.111111\lambda_1 - 54\lambda_2,\ \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \mu_6 = 0.111111\lambda_1 - 63\lambda_2$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 13: $\mu_1 = 0.111111\lambda_1 - 0.25\lambda_2,\ \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = -0.111111\lambda_1 - 1.5\lambda_2$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 14: $\mu_1 = -\sqrt{\frac{\lambda_1\lambda_2}{7}}, \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 15: $\mu_1 = 0, \mu_2 = 0.111111\lambda_1 - 9\lambda_2,\ \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = 0.111111\lambda_1$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 16: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = -\sqrt{\lambda_1\lambda_2}$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 17: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \mu_6 = \sqrt{\frac{\lambda_1\lambda_2}{6}}$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 18: $\mu_1 = 10\lambda_2\ \mu_2 = \mu_4 = \mu_5 = \mu_6 = 0, \mu_3 = \lambda_1 - 700$.

This case is rejected since it gives values for the decision variables which are not included in their range.

Case 19: $\mu_1 = \mu_2 = \mu_4 = \mu_5 = 0,\ \mu_3 = \lambda_1 - 600,\ \ \mu_6 = -10\lambda_2$.

This case is rejected since it gives values for the decision variables which are not included in their range.

The tests reported here are carried out using 100 population size, 500 generations, 0.8 crossover probability, 0.05 mutation probability, and simulated binary crossover with 10 crossover distribution and 50 mutation distribution indexes for NSGA-II. Figure 5.13 shows the results obtained by NSGA-II for this problem.

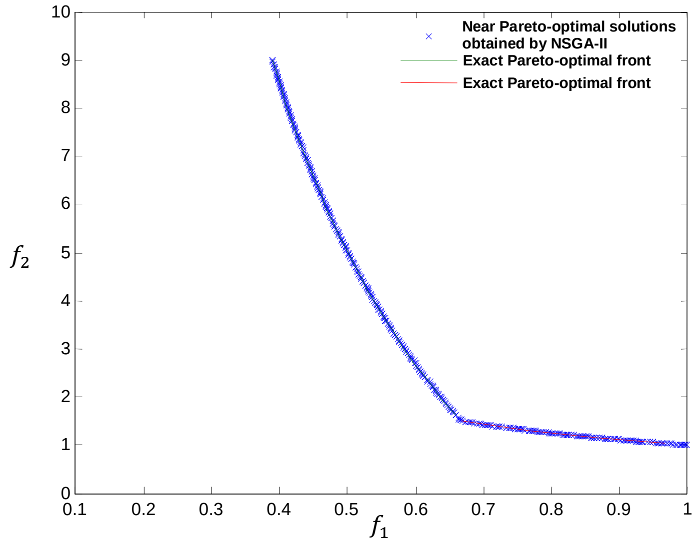

Figure 5.13: The approximated Pareto-optimal front for the constrained problem using NSGA-II

5.4 Performance Comparison Between MOSA-I and NSGA-II

The term performance is always involved when comparing different optimisation techniques experimentally. In the case of multi-objective optimisation, the definition of quality is substantially complex because the optimisation goal itself consists of multiple objectives (Zizler et al., 2002):

- The distance of the resulting non-dominated set to the Pareto-optimal front should be minimised.
- A good (in most cases) uniform distribution of the solutions found is desirable. The assessment of this criterion might be based on a certain distance metric.
- The span of the obtained non-dominated front should be maximised, i.e., for each objective, a wide range of values should be covered by the non-dominated solutions.

In the literature, some attempts can be found to formalise the above definition (or parts of it) by means of quantitative metrics (Coello et al., 2002). Within this chapter, the generational distance $\boldsymbol{GD}$ metric is used. This metric is the average distance from the obtained Pareto-optimal front $\boldsymbol{PF_{Known}}$ to the true Pareto-optimal front $\boldsymbol{PF_{true}}$ and is defined as follows (Coello et al., 2002).

$$\boldsymbol{GD} = \frac{(\sum_{i=1}^{n} \boldsymbol{d}_i^p)^{\frac{1}{p}}}{n} \tag{5.31}$$

where, $\boldsymbol{n}$ is the number of vectors in $\boldsymbol{PF_{Known}}$, $p = 2$ and $\boldsymbol{d_i}$ is the Euclidean distance (in objective space) between each vector and the nearest vector of $\boldsymbol{PF_{true}}$. The result $\boldsymbol{GD = 0}$ indicates $\boldsymbol{PF_{Known} = PF_{true}}$; any other value indicates $\boldsymbol{PF_{Known}}$ deviates from $\boldsymbol{PF_{true}}$.

The performance of MOSA-I and NSGA-II has been illustrated on the FON problem that has been randomly chosen from the set discussed in the previous sections. The performance of MOSA-I and NSGA-II can be easily measured for the rest of the problems as the FON problem. Here, the NSGA-II with 100 generations and 100 individuals in each generation has been executed on FON problem. The exact Pareto-optimal front for this problem is given by MOSA-I (Eq. 5.17). This means that $\boldsymbol{GD = 0}$ for MOSA-I.

By choosing 100 values on the true Pareto-optimal front (provided by the MOSA-I algorithm) next to 100 individuals obtained by NSGA-II, the generational distance metric has been calculated for each experiment separately using Equation (5.31). The experiments show that with a minimum $\boldsymbol{GD} = 0.00082983$, NSGA-II can approximate the Pareto-optimal front in some runs. However, in other runs, the approximated Pareto-optimal front obtained by NSGA-II is worse since maximum $\boldsymbol{GD} = 0.001065864$. The mean and standard deviation of all the 10 experiments are $\mu = 0.982912\,E - 3$ and $\sigma = 7.23078\,E - 5$, respectively. The small standard deviation shows that the $\boldsymbol{GD}$ values after 100 generations of the NSGA-II are already close to the mean. However, a $\boldsymbol{GD}$ different from zero indicates an approximation. As expected from a stochastic technique, NSGA-II is able to find some points but not all the points on the true Pareto-optimal front in its final generation. There are also some minor deviations from the exact Pareto-optimal front as shown by the value of the $\boldsymbol{GD}$.

From the above results, one can conclude the following:

- The proposed MOSA-I algorithm outperforms NSGA-II algorithm in providing an exact curve/hyper surface representing the conflicting between the objectives which is not obtained by NSGA-II.
- The proposed MOSA-I algorithm outperforms NSGA-II in providing an analytical equation of the exact Pareto-optimal front. This is also not provided by NSGA-II.
- The proposed MOSA-I algorithm outperforms NSGA-II in providing the relationship between the design variables in an analytical form that is responsible for forming the exact Pareto-optimal front. This is also not provided by NSGA-II.
- The proposed MOSA-I algorithm, by providing a formula of the Pareto-optimal front, allows a precise statistical analysis of the performance of stochastic multi-objective optimisation techniques such as NSGA-II using an absolute performance measure such as the $\boldsymbol{GD}$. Figure

5.14 and Figure 5.15 illustrate the residuals between the approximated Pareto-optimal front and the exact Pareto-optimal front.

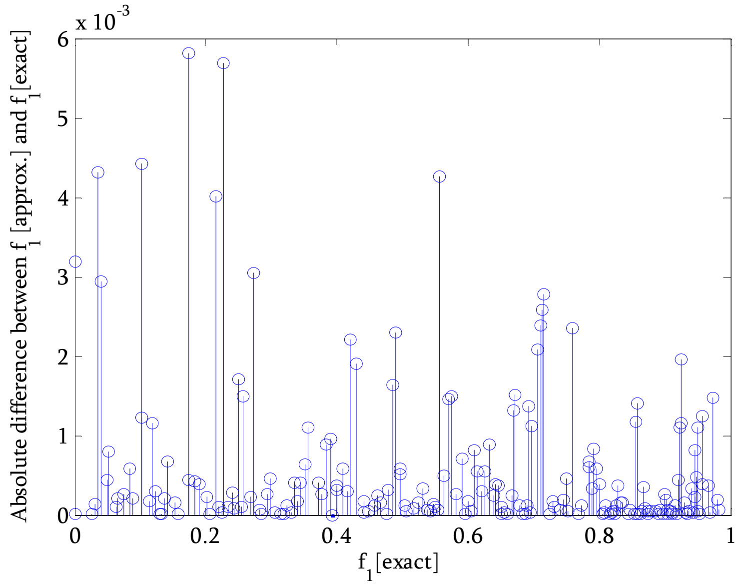

Figure 5.14: The residual plot between the approximated Pareto-optimal front and the exact Pareto-optimal front for FON problem

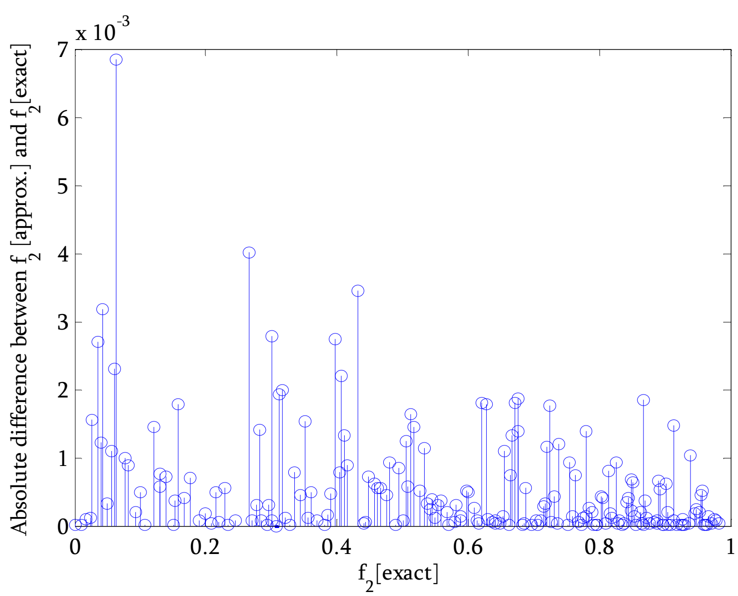

Figure 5.15: The residual plot between the approximated Pareto-optimal front and the exact Pareto-optimal front for FON problem

5.5 Discussion

The most important contribution in this chapter is the development of a multi-objective symbolic algorithm (MOSA-I) which is able to solve a certain class of multi-objective optimisation problems. The main advantage of the MOSA-I algorithm is its ability to find exact solutions as opposed to the approximated solutions provided by evolutionary optimisation algorithms. The proposed MOSA-I algorithm provides the exact curve/hyper surface of the Pareto-optimal front which contains all the exact compromised solutions for the multi-objective problems at hand. In addition, this algorithm provides important relationship between the decision variables (innovative design principles). This relationship may be a linear or nonlinear relationship between the decision variables and between the objectives and can be used to form the true Pareto-optimal front analytically. This relationship is also used to discover which decision variables have effects on the relationship between the conflicting objective functions. This relationship is not provided by evolutionary optimisation algorithms. It can be found numerically by optimisation algorithms using for instance regression analysis. The results obtained by the proposed MOSA-I algorithm have demonstrated the importance of this relationship not only in constructing the analytical formula of the true Pareto-optimal front but also in supporting the process of innovation. As can be seen from the four-bar plane truss results found by the MOSA-I algorithm (chapter 7), if the engineering designer wants to get the best trade-off for this problem, he/she should create bars with cross sections that satisfy the relationship found by the proposed algorithm.

One of the most important difficulties that stochastic algorithms face at the current time is the problem of high dimension in the objective space. MOSA-I algorithm can be applied to multi-objective problems with high dimensions in both the decision space and the objective space. This because step 8 in MOSA-I algorithm tackles the high dimensions in both spaces. Moreover, MOSA-I can be used to evaluate the performance of a stochastic algorithm and can help in supporting stochastic algorithms in identifying a termination criterion. This is because many performance metrics (Coello et al., 2002) depend on knowing the exact formula of the Pareto-optimal front. So, they can not be applied to evaluate the performance of the stochastic algorithms unless this exact formula is shown. So, knowing the exact Pareto-optimal front in an analytical form will help to measure the robustness of stochastic algorithms using quantitative metrics. In addition, obtaining the analytical equation that represents the Pareto-optimal set will help researchers to find out unknown information about the problem at hand.

The disadvantage of MOSA-I algorithm is that in some cases, e.g., the algorithm cannot find the relationship between the decision variables responsible for constructing the Pareto-optimal front, and therefore the equation of the Pareto-optimal front may not be found. However, better solutions than the solutions identified by stochastic algorithms can be found by trying different values for Lagrange multipliers (this is shown later on in chapter 7).

Another disadvantage of this algorithm is that it fails to solve multi-objective optimisation problems that have Pareto-optimal front constructed by non-linear relationship between the decision variables.

5.6 Summary

This chapter has proposed a novel analytical algorithm capable of handling some classes of multi-objective optimisation problems. As illustrated below, the proposed MOSA-I algorithm meets all its proposed characteristics as mentioned at the beginning of this chapter.

- This algorithm can handle the interactions between the objectives and between the variables for a certain class of multi-objective optimisation problems.
- The proposed MOSA-I algorithm exhibits better performance that the high-performing NSGA-II on a variety of multi-objective optimisation problems.
    - A complete curve/surface of the exact Pareto-optimal front can be obtained.
    - Better distribution of the Pareto-optimal solutions can be obtained.

This chapter has achieved the following.

- It has identified the challenges that the interactions between variables and between objectives pose for multi-objective optimisation problems.
- It has detected the interactions between variables and between objectives in an analytical form.
- It has proposed a new multi-objective optimisation algorithm (MOSA-I).
- It has analysed the performance of the proposed algorithm using existing test problems and engineering applications.

This chapter has proposed the MOSA-I algorithm for handling multi-objective optimisation problems. The aim of the next chapter is to propose a second-order multi-objective symbolic algorithm. It modifies MOSA-I so that it can deal with a wider class of multi-objective optimisation problems.

6 PROPOSED SECOND-ORDER MULTI-OBJECTIVE OPTIMISATION SYMBOLIC ALGORITHM (MOSA-II)

This chapter aims to modify MOSA-I presented in the previous chapter. To carry out this modification, the second-order optimality conditions of Karush-Kuhn-Tuck are relaxed. This relaxation involves the pseudoconvex[1] and quasiconvex concepts. Based on these conditions the modified version, called MOSA-II, is used to solve multi-objective optimisation problems with more general aspects. These aspects include the class of the objective functions and inequality constraints that are continuous, differentiable, convex, pseudoconvex, quasiconvex, and satisfy the univexity concept. The only difference between the MOSA-II algorithm and previous algorithm mentioned in the previous chapter is that this one includes second-order optimality conditions that are used to reduce solutions provided by the first-order optimality conditions. In this way, the class of the problems tackled by MOSA-II is extended. The MOSA-II algorithm is used to identify the relationship between the decision variables that leads to the analytical formula of the Pareto-optimal front between the conflicting objective functions. This relationship is provided by the MOSA-II algorithm as functions of Lagrange's multipliers. As the previous algorithm, the algorithm's steps are coded using Mathematica toolbox©. The algorithm is checked by solving some multi-objective optimisation problems that are carefully selected from literature. Comparison is carried out with the state-of-the art optimisation algorithm, NSGA-II, since it has given better solutions for these problems than the other optimisation algorithms. This chapter attempts to achieve the following.

- *To propose the MOSA-II algorithm.*
- *To illustrate the purpose of using second-order optimality conditions.*
- *To illustrate MOSA-II on a test problem.*
- *To analysis the advantages and disadvantages of the proposed MOSA-II algorithm.*

---

[1] Let f be a differentiable function defined on the open convex set $\mathbb{X}$. It turns out that f is pseudoconvex (quasiconvex) on $\mathbb{X}$ if and only if for all $x, y \in \mathbb{X}$ $(x \neq y)$, $f(x) < f(y) (f(x) \leq f(y)) \Rightarrow (x - y)\nabla f(y) \leq 0$.

6.1 Multi-objective Optimisation Symbolic Algorithm (MOSA-II)

Considering the optimality conditions derived in the previous chapters, a symbolic algorithm, called MOSA-II, based on both the first and second-optimality conditions for multi-objective optimisation is proposed here. The purpose of this algorithm is to reduce the solutions provided by the first-order optimality conditions and also detect the relationship between the decision variables that is used to create the analytical curve (or hyper-surface) of Pareto-optimal front. The algorithm's steps are entirely described below. The MOSA-II algorithm is coded by Mathematica Toolbox©. Details on the performance of the algorithm are shown later. Figure 6.1 shows the flow chart of the proposed symbolic algorithm. The following steps are given:

Step 1: Define the objective functions $f_i, i = 1, \dots, M$ to be minimised.

Step 2: Define the inequality constraints $g_j, j = 1, \dots, J$.

Step 1 and Step2 of the MOSA-II algorithm are given. They present the mathematical formulae of the objective functions and the constraints. They only need to be typed under Mathematica Toolbox©.

The algorithm's main steps:

Step 3: Check the continuity and differentiability of the objective functions and constraints. If yes, go to step 4, otherwise terminate.

Step 4: Calculate the second-order partial derivatives of each objective function separately.

Step 5: Form the Hessian matrix $\mathbb{H}$ for each objective function. For example, the Hessian matrix for f_1 can be written as follows:

$$\mathbb{H} = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2} & \frac{\partial^2 f_1}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f_1}{\partial x_2 \partial x_1} & \frac{\partial^2 f_1}{\partial x_2^2} & \cdots & \frac{\partial^2 f_1}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f_1}{\partial x_n \partial x_1} & \frac{\partial^2 f_1}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f_1}{\partial x_n^2} \end{pmatrix}$$

Step 6: Check if the Hessian matrix $\mathbb{H}$ is positive semi-definite, i.e., $\boldsymbol{x}^{t}\mathbb{H}\boldsymbol{x} \geq \boldsymbol{0}$ for all solution vector $\boldsymbol{x} \neq \boldsymbol{0}$. If yes, then, in this case, $f_1$is convex, otherwise go to step 7.

Step 7: Check if the condition $\nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \leq \boldsymbol{0}$ is satisfied for one $\boldsymbol{y}$ in the feasible space. If yes, go to step 8, otherwise terminate.

Step 8: Solve the system of equations $\sum_{i=1}^{M} \lambda_i \nabla f_i(\boldsymbol{x}) + \sum_{j=1}^{J} \mu_j \nabla g_j(\boldsymbol{x}) = \boldsymbol{0}$ to find $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Step 9: Solve the system of equations $\mu_j g_j(\boldsymbol{x}) = 0$, for all $j \notin \{j \in \{1,2,\ldots,J\} | \, g_j(\boldsymbol{x}) = 0\}$ to get $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\lambda})$.

Step 10: Substitute by the results obtained from step 9 in step 8 to obtain $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\lambda})$.

Step 11: Check if $\Lambda = \boldsymbol{d}^t \left(\sum_{i=1}^{M} \lambda_i \nabla^2 f_i(\boldsymbol{x}) + \sum_{j=1}^{J} \mu_i \nabla^2 g_j(\boldsymbol{x}) \right) \boldsymbol{d} \geq \boldsymbol{0}$, where, $\boldsymbol{d} = \boldsymbol{y} - \boldsymbol{x} \neq \boldsymbol{0}$. If yes, go to step 12 and, in this case, the solutions provided by step 8 are reduced. If no, we will only have the solutions provided by step 8 which include solutions from the infeasible space or may have complex solutions that do not belong to the feasible space.

Step 12: Substitute the results from step 10 in the objective functions (Step 1) to formulate the analytical formula of the Pareto-optimal front, as for example, f_2 is a function of f_1, i.e., $f_2 = \varphi(f_1)$. Note that, in some cases, the multiplier $\boldsymbol{\lambda}$ can not be removed and the thus the relationship between the conflicting functions will depend on that multiplier. To overcome this, different values for this multiplier are given by trial and error and consequently, the Pareto-optimal front is found.

Step 13: End

Clarification of the algorithm's steps:

Step1 - Step2: Both steps are given. Their mathematical description should be firstly coded under Mathematica so that the other steps can be executed.

Step3: This step is used to make sure that the objectives and constraints are continuous and differentiable.

Step4-Step6: To the check the convexity, pseudoconvexity and quasiconvexity of the objectives, the second-order partial derivatives are used. Using step 1, setp2 and the second-order partial derivatives for each objective function, the matrix of the Hessian is now formed. After that the positive semi-definite of the Hessian is checked.

Step7: This step is an alternative one to step 4. It is used to identify the convexity, pseudoconvexity and quasiconvexity of the objectives using the first-order partial derivatives when the second-order partial derivatives are not exist or the Hessian matrix is not positive semi-definite.

Step8: In this step, all the gradient vectors are known and only unknown vectors are $\boldsymbol{\lambda}, \boldsymbol{\mu}$ and $\boldsymbol{x}$. This step can be rewritten as follows:

$$[\lambda_1 \quad \cdots \quad \lambda_M]_{1\times M} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_M}{\partial x_1} & \frac{\partial f_M}{\partial x_2} & \cdots & \frac{\partial f_M}{\partial x_n} \end{bmatrix}_{M\times n} + [\mu_1 \quad \cdots \quad \mu_J]_{1\times J} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial g_J}{\partial x_1} & \frac{\partial g_J}{\partial x_2} & \cdots & \frac{\partial g_J}{\partial x_n} \end{bmatrix}_{J\times n} = 0$$

So the number of unknown variables in the above systems is now $M + n + J$. The above system is solved to get $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\lambda}, \boldsymbol{\mu})$.

Step 9: This step is used to reduce the number of unknown variables in step 8. The system is this step is solved to get $\boldsymbol{\mu}$ as functions of $\boldsymbol{\lambda}$. Here, we solve the system of equations $\mu_j g_j(\boldsymbol{x}) = 0$, for all $j \notin \{j \in \{1,2,\dots,J\} | \, g_j(\boldsymbol{x}) = 0\}$ to get $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{\lambda})$.

Step 10: This step is used to form the Pareto set as functions of $\boldsymbol{\lambda}$ multiplier. This Pareto set is used later to form the Pareto-optimal front.

Step 11: This step shows the difference between MOSA-I and MOSA-II. In this step, one direction vector $\boldsymbol{y}$ satisfying the condition $\nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \leq \boldsymbol{0}$ is selected from the feasible space. After that, substituting the solutions obtained by steps 9 and 10 (solutions obtained by the first-order conditions) in step 11. Solutions that do not satisfy the conditions in step 11 should be rejected and therefore, the solutions provided by the first-order conditions are reduced.

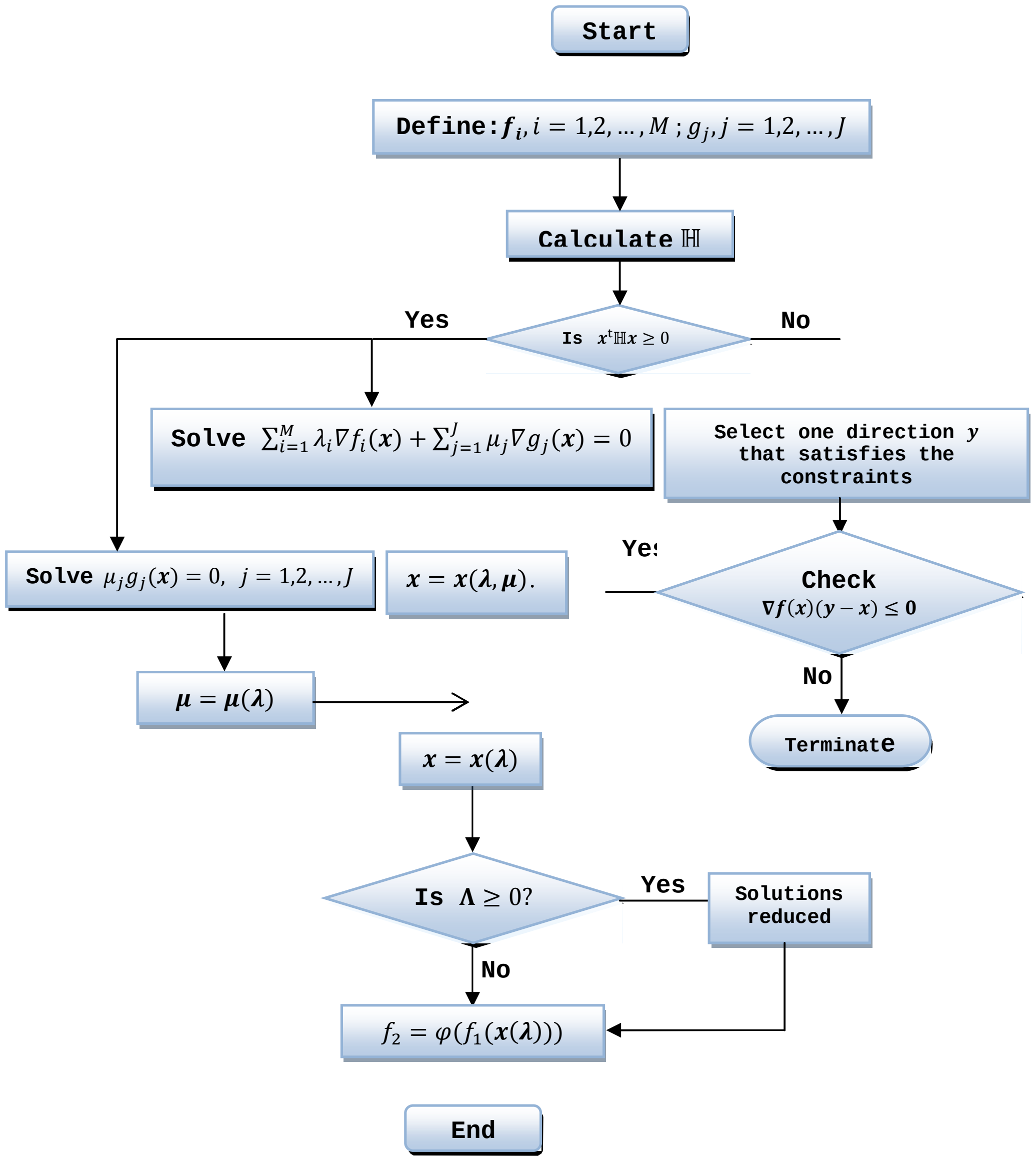

Figure 6.1: Flow chart of the proposed MOSA-II algorithm

6.2 Illustration of the effectiveness of second-order optimality conditions

Here, MOSA-II is applied to the constrained problem used in the previous chapter to illustrate the impact of the second-order optimality conditions in reducing the solutions obtained by the first-order optimality conditions. Because the only difference between this algorithm and MOSA-I is the second-order optimality conditions involved in MOSA-II, there is no need to show the application of MOSA-II to all the test problems tackled by MOSA-I if the results obtained by MOSA-I are similar to those obtained by MOSA-II. We have only selected the constrained problem since it shows the importance of using second-order optimality conditions to reduce the solutions provided by first-order conditions. The following results have been found after executing the MOSA-II algorithm on the constrained problem.

Results analysis of constrained problem: For this problem, the MOSA-II algorithm has given the same results as in chapter 5 but the only difference is that some previously accepted cases are rejected as they do not satisfy step 11 in the algorithm presented in this chapter. To show that, let us implement MOSA-II on this problem.

Step 1-4: step 1 to step 4 are very simple to implement. They need one to be familiar with Mathematica to define the objectives and constraints and to check their continuity and differentiability.

Step 5: This step gives the following Hessian matrices for f_1 and f_2:

$$\mathbb{H}_{f_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbb{H}_{f_2} = \begin{bmatrix} \frac{2(1+x_2)}{x_1^3} & -\frac{1}{x_1^2} \\ -\frac{1}{x_1^2} & 0 \end{bmatrix}$$

Step 6: : For the matrix $\mathbb{H}_{f_1}$, $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_1}\boldsymbol{x} = 0$ for all non-zero $\boldsymbol{x}$. This means that f_1 is convex. For the matrix $\mathbb{H}_{f_2}$, we have the following:

$$\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2}\boldsymbol{x} = \frac{2a\,(a - bx_1 + ax_2)}{x_1^3}$$

This means that $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2}\boldsymbol{x} < 0$ for all non-zero $x = (a, b)$. Then, we go to step 7.

Step 7: Let $y = (1,5)$ and $x = (1,0)$ then the condition $\nabla f_2(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \leq \boldsymbol{0}$ is satisfied. This means that f_2 is pseudoconvex function.

Step 8-10: After executing these steps the following cases are obtained:

Case 1: $\mu_1 = \frac{\lambda_1}{18}, \mu_2 = \frac{\lambda_1}{18} - \frac{18\lambda_2}{7}, \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0.$

Substituting by this case in Eq. 5.24, one gets the point $(0.3889, 2.5)$ on the border of the feasible decision space (bold line, Figure 6.2). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space using step 12 is $(0.3889, 8.9997)$ in the region A (bold curve, Figure 6.3). It is easy to check that this point satisfies step 11 of the MOSA-II algorithm using an arbitrary direction $y = (1,0)$. Therefore, this case is accepted.

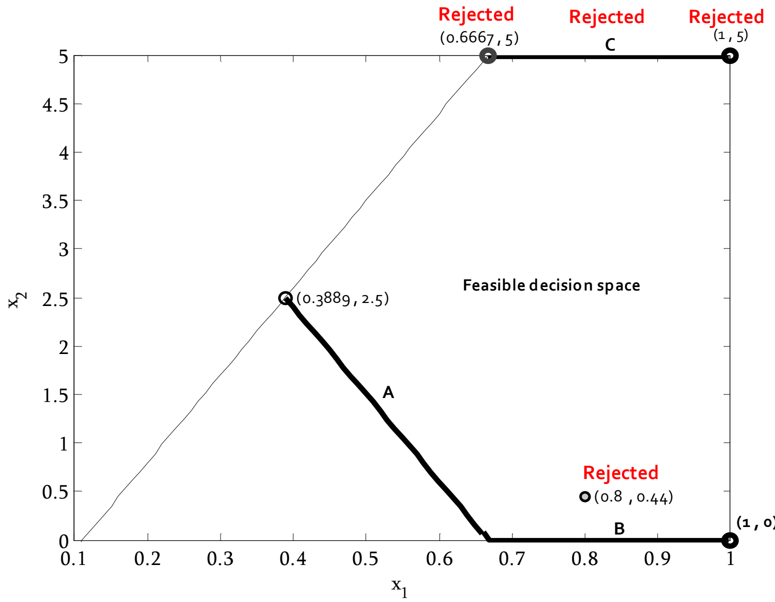

Figure 6.2: Feasible search region for the constrained problem in the decision variable space

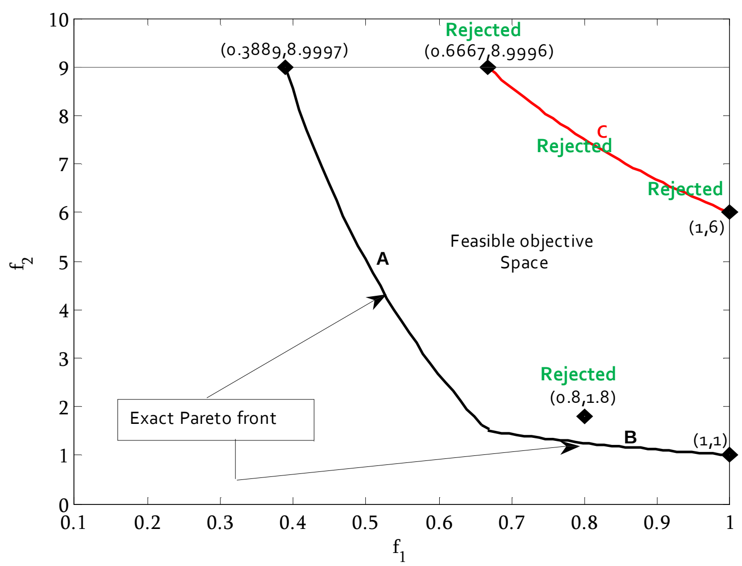

Figure 6.3: Exact and local Pareto front for the constrained problem in the objective space

Case 2: .

Substituting by this case in Eq. 5.24, one gets the point in the feasible decision space (bold point, Figure 6.2). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space using step 12 is (bold point, Figure 6.3). This point is excluded as it does not satisfy the second order optimality condition of the MOSA-II algorithm at . Therefore, this case is rejected.

Case 3: .

Substituting by this case in Eq. 5.24, one gets the following relationship between the decision variables:

$$x_1 = \sqrt{\frac{7\lambda_2}{\lambda_1}}, x_2 = 6 - 9x_1 \qquad (6.1)$$
$$0.3889 \leq x_1 \leq 0.6667$$

This relationship between x_1 and x_2 represents the bold line A in the decision space (Figure 6.2). The points satisfying this line are used to construct the following formula using step 12:

$$f_2 = \frac{7}{f_1} - 9, \qquad (6.2)$$
$$0.3889 \leq f_1 \leq 0.6667$$

This formula is the first part of the exact Pareto-optimal front (bold curve A, Figure 6.3). Therefore, this case is accepted.

Case 4: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = \sqrt{\lambda_1\lambda_2}$.

Substituting by this case in Eq. 5.24, one gets the following relationship between the decision variables:

$$x_1 = \sqrt{\frac{\lambda_2}{\lambda_1}}, x_2 = 0 \qquad (6.3)$$
$$0.6667 \leq x_1 \leq 1$$

The relationship (6.3) between x_1 and x_2 is represented by the bold line B in the decision space (Figure 6.2). The points satisfying this line are used to construct the following formula using step 12:

$$f_2 = \frac{1}{f_1}, \qquad (6.4)$$
$$0.6667 \leq f_1 \leq 1$$

This formula is the second part of the exact Pareto-optimal front (bold curve B, Figure 6.3). Therefore, this case is accepted.

Case 5: $\mu_1 = 0, \mu_2 = 0.111111\lambda_1 - 1.5\lambda_2, \ \mu_3 = \mu_4 = \mu_6 = 0, \mu_5 = -0.111111\lambda_1$.

Substituting by this case in Eq. 5.24, one gets the point $(0.6667,5)$ in the feasible decision space (bold point on the bold line C, Figure 6.2). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space is $(0.6667,8.9996)$ (bold point on the curve C, Figure 6.3). This point is dominated by points on the curve A and does not satisfy step 11 of the MOSA-II algorithm at $y = (1,0)$. So, it is rejected.

Case 6: $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = 0, \mu_6 = -\sqrt{\frac{\lambda_1\lambda_2}{6}}$.

Substituting by this case in Eq. 5.24, one gets the following relationship between the decision variables:

$$x_1 = \sqrt{\frac{6\lambda_2}{\lambda_1}}, x_2 = 5 \qquad (6.5)$$
$$0.6667 \le x_1 \le 1$$

The relationship (6.5) between x_1 and x_2 is represented by the bold line C in the decision space (Figure 6.2). The points satisfying this line are used to construct the following formula using step 12:

$$f_2 = \frac{6}{f_1}, \qquad (6.6)$$
$$0.6667 \le f_1 \le 1$$

This formula creates the part C (red curve in Figure 6.3). This is rejected as it does not satisfy step 11 of the MOSA-II algorithm at $y = (1,0)$.

Case 7: $\mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -\lambda_1 + \lambda_2,\ \mu_5 = \lambda_2,\ \mu_6 = 0.$

Substituting by this case in Eq. 5.24, one gets the point $(1,0)$ on the border of the feasible decision space (bold line B, Figure 6.2). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space using step 12 is $(1,1)$ on the border of the region B (bold curve, Figure 6.3).

It is easy to check that this point satisfies step 11 of the algorithm using the direction $y = (1,0)$. Therefore, this case is accepted.

Case 8: $\mu_1 = \mu_2 = \mu_3 = 0,\ \mu_4 = -\lambda_1 + 6\lambda_2,\ \mu_5 = 0,\ \mu_6 = -\lambda_2.$

Substituting by this case in Eq. 5.24, one gets the point $(1,5)$ on the border of the feasible decision space (bold line C, Figure 6.2). It satisfies the inequality constraints imposed on the problem. The corresponding point in the feasible objective space using step 12 is $(1,6)$ on the border of the region C (red bold curve, Figure 6.3). This point is dominated by the point $(1,1)$ and does not satisfy step 11 of the MOSA-II algorithm at $y = (1,0)$. So, it is rejected.

With these results discussed above, it has been shown how the proposed second-order optimality conditions are helpful in reducing the solutions provided by the first-order optimality conditions.

Now, we have selected the ZDT3 problem from literature as a box-constrained problem to be solved by MOSA-II. The reason for choosing this problem is that the Pareto-optimal front of it is discontinuous. The mathematical description of the ZDT3 problem takes the following form:

$$
\begin{aligned}
&f_1(\boldsymbol{x}) = x_1 \\
&f_2(\boldsymbol{x}) = g(x) \cdot h(f_1, g) \\
&g(\boldsymbol{x}) = 1 + \frac{9}{n-1} \times \sum_{i=2}^{n} x_i \;, n = 30 \\
&h(f_1, g) = 1 - \sqrt{\frac{f_1}{g}} - \left(\frac{f_1}{g}\right) \times \sin(10\pi f_1) \\
&0 \le x_i \le 1, i = 1, \dots, n
\end{aligned}
$$

Let us implement MOSA-II on this problem. As previously shown, the steps from 1 to 7 are easy to implement:

Step 8: After executing this step the following two systems are obtained:

$$\lambda_1 = \frac{\lambda_2}{2}\left(2\sin(10\pi x_1) + 20\pi x_1 \cos(10\pi x_1) + \frac{1}{\sqrt{\frac{x_1}{1+9x_2}}}\right)$$

$$\lambda_2 \sqrt{\frac{x_1}{1+9x_2}} = 2\lambda_2$$

The above two systems have no solutions. This does not mean that MOSA-II faces a difficulty in solving this problem because of the insolvability of the first-order optimality system. The difficulty is in the problem itself. The ZDT3 problem has been artificially framed such that no analytical solutions can be found for it. To explain this, let us solve ZDT3 using MOSA-II at $n = 2, M = 2$ (for simplicity). Step 8 of MOSA-II gives the following equation:

$$\lambda_1 + \lambda_2\left(\frac{1}{2\sqrt{x_1}} - \sin(10\pi x_1) - 10\pi x_1^2 \cos(10\pi x_1)\right) = 0$$

The above equation can not be solved for any values of $\lambda_1$and λ_2. This means that Pareto-optimal solutions based on KKT conditions can not be found. This does not mean that NSGA-II or any other evolutionary algorithms outperform MOSA-II in solving this problem. For simple reason which is, NSGA-II still faces criticism about the theoretical optimality of the obtained solutions.

Another way to get solutions for this problem is by looking in the decision space of ZDT3. It takes the following form ($n = 3$):

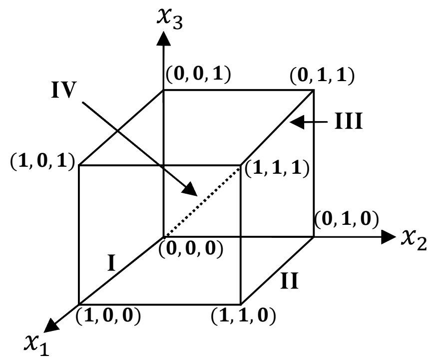

Figure 6.4: Decision space for ZDT3 at $n = 3$

As shown in Figure 6.4, the research space of the ZDT3 problem is divided into five regions as follows:

***Region* I**: This region is represented by the following relationship:

$$0 \leq x_1 \leq 1, x_2 = x_3 = 0 \tag{6.7}$$

By substituting (6.7) in step 1 of MOSA-II, one gets

$$f_2 = 1 - \sqrt{f_1} - f_1 \times sin(10\pi f_1), \quad 0 \leq f_1 \leq 1 \tag{6.8}$$

Equation (6.8) represents the Pareto-optimal front between the two objective functions based on the relationship (6.7). It is easy to check that this Pareto-optimal front is constructed at $\boldsymbol{g(x) = 1}$. This gives us confidence about the solutions represented by the relationship (6.7) because the ZDT problems have been artificially framed in such a way the exact Pareto-optimal front is constructed at $\boldsymbol{g(x) = 1}$. Figure 6.5 shows this Pareto-optimal front. Interestingly, one can clearly see that the near Pareto-optimal solutions provided by NSGA-II (red circles) matches this one.

***Region* II**: This region is represented by the following relationship:

$$0 \leq x_1 \leq 1,\ x_2 = 1, x_3 = 0 \tag{6.9}$$

By substituting (6.9) in step 1 of MOSA-II, one gets

$$f_2 = \frac{11}{2}(1 - \sqrt{\frac{2f_1}{11}} - \frac{2f_1}{11} \times sin(10\pi f_1), \quad 0 \leq f_1 \leq 1 \tag{6.10}$$

Equation (6.10) represents the Pareto-optimal front between the two objective functions based on the relationship (6.9). It is easy to check that this Pareto-optimal front is constructed at $\boldsymbol{g(x)} = \frac{\mathbf{11}}{\mathbf{2}}$. This means that solutions on this Pareto-optimal front are dominated by solutions on the exact Pareto-optimal front represented by the equation (6.8).This case is shown in Figure 6.5.

***Region* III**: This region is represented by the following relationship:

$$0 \leq x_1 \leq 1,\ x_2 = 1,\ x_3 = 1 \tag{6.11}$$

By substituting (6.11) in step 1 of MOSA-II, one gets

$$f_2 = 10(1 - \sqrt{\frac{f_1}{10}} - \frac{f_1}{10} \times sin(10\pi f_1), \quad 0 \leq f_1 \leq 1 \tag{6.12}$$

Equation (6.12) represents the Pareto-optimal front between the two objective functions based on the relationship (6.11). It is easy to check that this Pareto-optimal front is constructed at $\boldsymbol{g(x) = 10}$. This means that solutions on this Pareto-optimal front are dominated by solutions on the exact Pareto-optimal front represented by the equation (6.8).This case is shown in Figure 6.5.

***Region* IV**: This region is represented by the following relationship:

$$0 \leq \theta \leq 1, x_1 = x_2 = x_3 = \theta \tag{6.13}$$

By substituting (6.11) in step 1 of MOSA-II, one gets

$$f_2 = (1 + 9f_1) \times (1 - \sqrt{\frac{f_1}{1+9f_1}} - \frac{f_1}{1+9f_1} \times sin(10\pi f_1), \quad 0 \leq f_1 \leq 1 \tag{6.14}$$

Equation (6.14) represents the Pareto-optimal front between the two objective functions based on the relationship (6.13). It is easy to check that this Pareto-optimal front is constructed at $\boldsymbol{g(x) = 1 + 9\,\theta}$. This means that solutions on this Pareto-optimal front are dominated by solutions on the exact Pareto-optimal front represented by the equation (6.8).This case is shown in Figure 6.5.

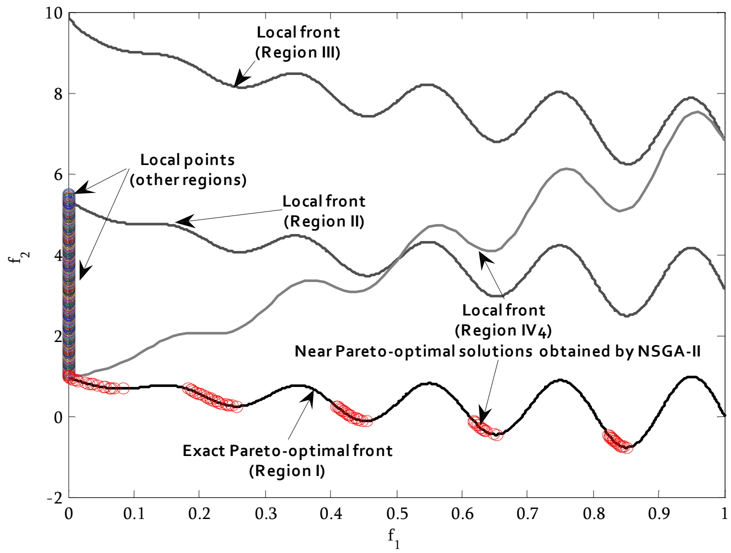

Figure 6.5: Pareto-optimal front obtained by the MOSA-II algorithm and NSGA-II for ZDT3

As can be seen from the first example, the results provided by the MOSA-II algorithm make the algorithm competent because these solutions are KKT based solutions. The algorithm provides an analytical equation for the Pareto-optimal front. This Pareto-optimal front is formed based on KKT relationship(s) between the decision variables and therefore, the Pareto-optimal solutions found have theoretical meanings. It is really important to note that these relationship(s) can be used to show which decision variable has influences on the objectives and the constraints.

It has been seen from ZDT3 problem that the linearity between the decision variables within the box constraints is responsible for constructing the curve of the exact Pareto-optimal front however, in other examples; the borders are responsible for this. The main difference between this algorithm and evolutionary algorithms is that this algorithm provides us with the analytical formula representing the Pareto-optimal front that is not provided by the evolutionary algorithms. Furthermore, the relationship between the decision variables is another important benefit that is not obtained by the evolutionary algorithms.

6.3 Discussion

This chapter proposes MOSA-II algorithm, which is based on second-order optimality conditions in order to solve multi-objective optimisation problems. The main contribution of this chapter is the capability of the proposed MOSA-II algorithm to deal with a broader class of multi-objective optimisation problems. Another important advantage of the proposed MOSA-II algorithm is its ability to find exact solutions compared to the approximated solutions provided by optimisation algorithms. It provides the exact curve/hyper surface of the Pareto-optimal front which inevitably contains all the exact compromised solutions for the multi-objective problem in hand. In addition, this algorithm provides the important relationship between the decision variables; this relationship may be linear or nonlinear relationship(s) between the decision variables and can be used to form the true Pareto-optimal front analytically. This relationship can also be used to discover which design variable (decision variable) has an effect on the relationship between the conflicting objective functions. This relationship is not provided by optimisation algorithms.

Furthermore, the proposed MOSA-II algorithm can be used to measure the performance of NSGA-II or any other stochastic optimisation algorithm. This has already been discussed in the previous chapter. The proposed MOSA-II algorithm can also help to support stochastic algorithms in identifying a termination criterion. In addition, MOSA-II can be used to reduce the optimal solutions obtained by the first-order optimality conditions as has been shown in the constrained problem.

The disadvantage of this algorithm is that in some cases (cases where Lagrange multipliers can not be separated from the relationship between the decision variables) the algorithm can not find the relationship between the decision variables responsible for constructing the Pareto-optimal front and therefore the equation of the Pareto-optimal front can not be found. However, better solutions than those identified by stochastic algorithms can be found by trying different values for Lagrange multipliers.

6.4 Summary

This chapter has modified the MOSA-I algorithm (presented in the previous chapter) based on the second-order optimality conditions of Karush-Kuhn-Tucker. The new algorithm is called MOSA-II. This algorithm is capable of handling the variable interaction in multi-objective optimisation problems. As illustrated below, the proposed MOSA-II algorithm meets all its objectives as mentioned at the beginning of this chapter.

- Although this algorithm can handle a broader class of multi-objective optimisation problems, it is expected that the algorithm would perform better than the existing ones in finding Pareto-optimal solutions based on theoretical optimality conditions.
- The proposed MOSA-II algorithm exhibits better performance than the high-performing NSGA-II on a variety of multi-objective optimisation problems.
  - A complete curve/surface of the exact Pareto-optimal front can be found by MOSA-II.

  - Better distribution of the Pareto-optimal solutions can be found using MOSA-II.
- Such as MOSA-I, MOSA-II can be used for measuring the performance of stochastic algorithms.

This chapter has achieved the following.

- It has extended the MOSA-I algorithm by proposing the new MOSA-II algorithm.
- It has illustrated the purpose of the second-order optimality conditions for multi-objective optimisation problems.
- It has illustrated the MOSA-II on a test problem.
- It has analysed the advantages and disadvantages of the proposed MOSA-II algorithm.

The next chapter validates the observations made in this chapter using a set of real-life case studies in the area of engineering optimisation. A brief analysis of the features of a selection of real-life problems is also presented in the next chapter.

7 REAL-LIFE CASE STUDIES

In the previous chapters, two symbolic algorithms (MOSA-I and MOSA-II) were proposed for tackling multi-objectives, constraints and variable interaction in optimisation problems. A number of test problems that have been commonly used in the literature were selected to analyse the performance of these proposed algorithms. The results found so far show that these algorithms are competent in finding good solutions for multi-objective optimisation problems in comparison with numerical solutions provided by stochastic algorithms. The aim of this chapter is to validate the observations made for the MOSA-II algorithm in the previous chapter using real-life case studies. MOSA-II is more general than MOSA-I because it involves the second-order optimality conditions that are used in reducing the optimality solutions obtained by the first-order optimality conditions involved in MOSA-I. For this reason MOSA-II is used for the validation process.

The aim of this chapter is realised through the achievement of the following.

- *To apply the MOSA-II to a number of the real-life case studies and therefore report the experimental results obtained by this algorithm for these applications.*
- *To apply the NSGA-II algorithm to these applications.*
- *To analyse the results obtained by the MOSA-II algorithm with the results provided by NSGA-II in order to validate the performance of the proposed MOSA-II algorithm.*

7.1 Real-Life Case Studies

This section introduces some real-life case studies from real-life engineering optimisation and analyses the features of these case studies. It has been reported in the literature that evolutionary algorithms have been successfully used in solving these case studies. In this chapter, three applications are chosen for analyse. The features of the chosen applications are tabulated in table 7.1. This table presents the features of these applications in terms of the number and nature of the variables and the number of the objective functions and constraints. It further looks at the application features that influence interaction between the decision variables. Another important feature is that, as the inseparable objective functions have interaction which is caused by the conflict among them, this table analyses the complexity of the objectives and the constraints. The complexity of the shape of the Pareto-optimal front together with the relationship involving design variables (innovative design principles) that lead to the Pareto-optimal front are also reported in this table. This relationship reveals the features among the design variables themselves. These features are related to the presence and nature of variables interactions in the applications. The table categorises these features for each application separately. Finally, some published results obtained from the application of stochastic optimisation algorithms on these applications are presented in the table. The following gives a brief description of these case studies.

1) **Pressure Vessel Design (Knowles, Corne and Deb, 2008)**: This application involves the design of a cylindrical vessel capped at both ends by hemispherical heads. The objective of the pressure vessel design is to minimise the total cost, including the cost of the material, forming and welding. There are four design variables for this problem: $\boldsymbol{x_1 = T_s}$ (thickness of the shell), $\boldsymbol{x_2 = T_h}$ (thickness of the head), $\boldsymbol{x_3 = \mathcal{R}}$ (inner radius) and $\boldsymbol{x_4 = \ell}$ (length of the cylindrical section of the vessel, not including the head). $\boldsymbol{T_s}$ and $\boldsymbol{T_h}$ are integer multiples of 0.0625 inch, which is the available thickness of the rolled steel plates, and $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{L}$ are continuous. Although this application is a single objective function with four inequality constraints and has been solved by many evolutionary algorithms as a single objective problem, it has been recently reported in the literature that the average percentage of feasible solutions found by CHNPGA (CH stands for constraint-handling, NPGA stands for niched Pareto genetic algorithm) during a single run is 33% with respect to the full population (Knowles, Corne and Deb, 2008). In this research, this problem is solved as two-objective functions and the second objective function to be minimised is the third inequality constraint. The reason for choosing this constraint to be the second objective function is that it has more non-linear terms in comparison with the other constraints.

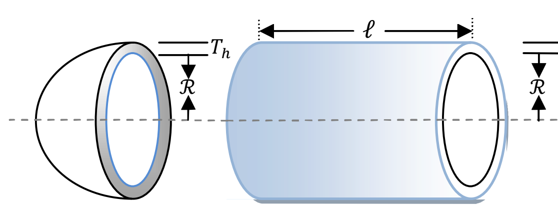

Figure 7.1: Pressure vessel (Source Knowles, Corne and Deb, 2008)

2) **Four-Member Truss Design (Ray and Liew, 2002)**: This mechanical engineering problem is a typical real-life problem. It was first introduced by Stadler and Dauer (1992). The aim of this application is to determine the cross-sectional areas of four joined bars (design variables) and the following objectives are to be minimised: [i] the volume of the structure [ii] the

displacement at the joint points. In this application, only physical restrictions (or design variables intervals) regarding the feasible cross-sectional areas of the bars are given. It has non-linear, convex and continuous Pareto-optimal front. In literature, most of the evolutionary optimisation algorithms perform well when dealing with this application. Complete information about the problem is as follows:

Design variables: $\begin{cases} x_1 = \text{the cross} - \text{sectional of the first bar} \\ x_2 = \text{the cross} - \text{sectional of the second bar} \\ x_3 = \text{the cross} - \text{sectional of the third bar} \\ x_4 = \text{the cross} - \text{sectional of the fourth bar} \end{cases}$

Objective functions: The problem is formulated as two conflicting objective functions of minimising both the volume V of the truss (or f_1) and the displacement Δ at the joint (or f_2) subject to given physical restrictions regarding the feasible cross-sectional areas x_1, x_2, x_3 and x_4 of the four bars. The stress on the truss is caused by three forces of magnitude F and $2F$ as depicted in Figure 7.2 below (Mezura-Montes and Coello, 2003).

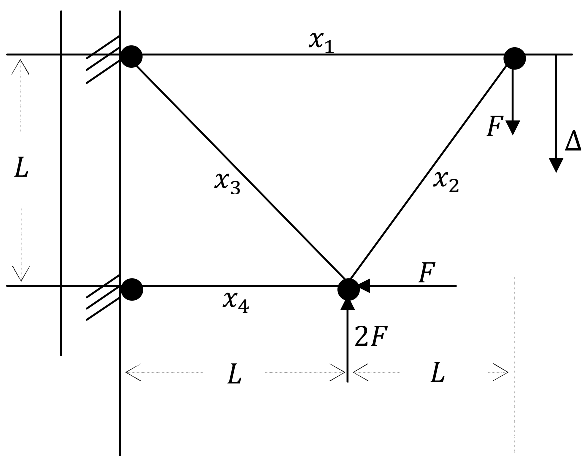

Figure 7.2: Four-bar plane truss (Source: Mezura and Coello, 2003)

3) Welded Beam Design (Knowles, Corne and Deb, 2008): The welded beam design application is considered as an important non-linear constraint problem for both single and multi-objective optimisation. As a single objective problem it has been well studied by Reklaitis et al., (1983). After that, it has been used by several authors and has become a significant case study to demonstrate the robustness and drawback of the stochastic algorithms. Here, we refer to two important papers (Coello, 2000; Mezura-Montes and Coello, 2008) which provide up-to-date results to the single objective version of this application. Later this problem was modified to be a bi-objective problem(Knowles, Corne and Deb, 2008). In this chapter, we use the bi-objective version to validate our techniques. Recently, new results for this version have been discovered (Deb, 1999; Deb and Srinivasan, 2008).

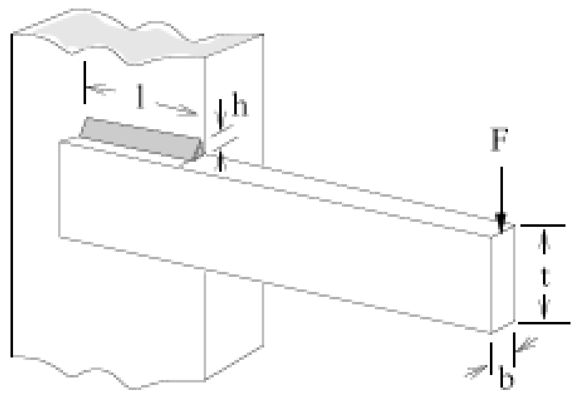

Figure 7.3: Welded beam design (Source: Knowles, Corne and Deb, 2008)

In this application, a beam needs to be welded on another beam and must carry a certain load (Figure 7.3). The objective of the design is to minimise the cost of beam fabrication (first objective function) and simultaneously minimise the vertical end deflection (second objective function) due to the load at the end of the beam. Here, the overhang portion of the beam and the applied force are specified, making the cross-sectional dimensions of the beam and the weld dimensions as the design variables. There are four non-linear inequality constraints in this application. The first constraint involves the shear stress at the support location of the beam. The purpose of this constraint is to make sure that this shear stress is smaller than the allowable shear strength of the material which in this case is . The second constraint includes the normal stress at the support location of the beam. The purpose of this constraint is to make sure that this normal stress is smaller than the allowable yield stress of the material which in this case is 30 . The third constraint makes sure that the thickness of the beam is not smaller than the weld thickness from a practical standpoint. Finally, the fourth constraint makes sure that the allowable buckling load (along direction) of the beam is more than the applied load which in this case is . It is worth mentioning here that any violation of the above four constraints will make the design unacceptable. Thus, in terms of discussion in Ignizio (1986), satisfaction of these constraints is the first priority. This application has non-linear convex Pareto-optimal front. NSGA-II with simulated binary crossover (SBX) and polynomial mutation operator has been shown to converge to the Pareto-optimal front and to distribute the solutions uniformly across the front.

The purpose of these case studies is to use them in this chapter for validating the performance of the MOSA-II algorithm. For this reason, we attempt in this section to present a set of applications that together represent some aspects, for example, multi-objectives, constraints and variable interactions in multi-objective optimisation problems. The following list identifies key aspects of real-life applications:

- Multiple variables.
- Integer, discrete and real variables.
- Multiple constraints.
- Polynomial, rational or complex objective functions.
- Implicit and multi-layered objective functions.
- Implicit and multi-layered constraints.
- Polynomial, rational or complex constraints.
- Unknown Pareto front.
- Multi-dimensional Pareto front.
- Non-linear (convex/concave) Pareto front.
- Continuous or discontinuous Pareto front.
- Biased search space.
- Multi-front (multiple local Pareto fronts).
- Variable interactions.

This list is used to select the presented applications so that all the above reported aspects are represented. This led to the selection of three applications listed below (shown as shaded regions in Table 7.1).

- ☑ Design of pressure vessel.
- ☑ Design of two-member truss.
- ☑ Design of welded beam.

Table 7.1: Case studies from real-life engineering optimisation

Real-life Case Studies			Application - 1	Application - 2
			Pressure Vessel	Four-member Truss
General Problem Features	No. of Variables		4	4
	Nature of Variables		4 real	4 real
	No. of Objectives		1	2
	No. of Inequality Constraints (+ Var. Bounds)		4	0
Variable Interaction	Inseparable Function Interaction	Complexity of Objectives	Non-linear function (O (3))	1. Linear function 2. Rational function (O(0)/O(1))
		Nature of Constraints	3 linear 1 non-linear	Only variable bound
		Reported Pareto-optimal Front	Only one point	Unknown
		Reported Complexity of Search Space	• The best average percentage of feasible solutions found during a single run is 33% • Biased search space	• Non-linear, convex & continuous Pareto-optimal front • Biased search space
Published Results from Optimisation Algorithms	Single Objective Optimisers		Order of performance: CHNPGA> CHVEGA> CHMOGA> COMOGA	Most algorithms give solutions on estimated Pareto-optimal front
	Multi-objective Optimisers		• Order of performance for convergence: NSGA-II>SPEA2 • Order of performance for distribution: NSGA-II>SPEA2	• Order of performance for convergence: NSGA-II>SPEA2 • Order of performance for distribution: NSGA-II>SPEA2

Table 7.1: Case studies from real-life engineering optimisation (contd.)

Real-life Case Studies			Application - 3 Welded Beam
General Problem Features	No. of Variables		4
	Nature of Variables		4 real
	No. of Objectives		2
	No. of Inequality Constraints (+ Var. Bounds)		4
Variable Interaction	Inseparable Function Interaction	Complexity of Objectives	1. Polynomial function (O(3)) 2. Rational function [O(0)/O(4)]
		Nature of Constraints	1 linear, 1 polynomial (O(5)) & 2 rational
		Reported Pareto-optimal Front	Corresponds to fixed values of 3 variables
		Reported Complexity of Search Space	• Non-linear, convex & disconnected Pareto-optimal front • Biased search space
Published Results from Optimisation Algorithms	Single Objective Optimisers		• Simple GA • Solution lies on estimated Pareto-optimal front
	Multi-objective Optimisers		• Order of performance for convergence: NSGA-II > SPEA2 • Order of performance for distribution: NSGA-II > SPEA2

7.2 *Design of Pressure Vessel*

This design is shown in Figure 7.1, and is briefly described in section 7.1. The mathematical model of this design is given in Equation 7.1 as follows:

$$
\begin{aligned}
&f_1(\boldsymbol{x}) \ = 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3,\\
&g_1(\boldsymbol{x}) = -x_1 + 0.0193x_3 \le 0,\\
&g_2(\boldsymbol{x}) = -x_2 + 0.00954x_3 \le 0,\\
&f_2(\boldsymbol{x}) \ = g_3(x) = -\pi x_3^2x_4 - \tfrac{4}{3}\pi x_3^3 + 1{,}296{,}000 \le 0,\\
&g_4(\boldsymbol{x}) = x_4 - 240 \le 0,\\
&1 \le x_1 \le 99, 1 \le x_2 \le 99, 10 \le x_3 \le 200 \text{ and } 10 \le x_4 \le 200
\end{aligned} \tag{7.1}
$$

where, f_1 is the total cost including the cost of material, forming and welding. The $g_i, i = 1,2,3,4$ is the constraints imposed on the design variables.

7.2.1 *Experimental Results*

Let us apply the MOSA-II steps on this problem as follows:

Steps 1-4: Step 1 to step 4 are simple to implement. They need one to be familiar with Mathematica to define the objectives and constraints and to check their continuity and differentiability. The design variables are denoted as x_1, x_2, x_3 and x_4.

Step 5: This step gives the following Hessian matrices for f_1 and f_2:

$$
\mathbb{H}_{f_1} = \begin{bmatrix} 39.68x_3 + 6.3322x_4 & 0 & 39.68x_1 + 0.6224x_4 & 6.3322x_1 + 0.6224x_4 \\ 0 & 0 & 3.5562x_3 & 0 \\ 39.68x_1 + 0.6224x_4 & 3.5562x_3 & 3.5562x_2 & 0.6224x_1 \\ 6.3322x_1 + 0.6224x_3 & 0 & 0.6224x_1 & 0 \end{bmatrix}
$$

$$
\mathbb{H}_{f_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8\pi x_3 - 2\pi x_4 & -2\pi x_3 \\ 0 & 0 & -2\pi x_3 & 0 \end{bmatrix}
$$

Step 6: For the matrix $\mathbb{H}_{f_1}$, we have the following:

$$
\begin{aligned}
\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_1}\boldsymbol{x} = {} & (79.36\, ac + 12.6644\, ad + 1.2448\, cd)x_1 + 3.5562\, c^2x_2 + 39.68\, a^2x_3 + 7.1124\, bcx_3 \\
& + 1.2448\, adx_3 + 6.3322\, a^2x_4 + 1.2448\, ac\, x_4 > 0
\end{aligned}
$$

for all non-zero $\boldsymbol{x} = (\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$. This means that f_1 is strictly convex.

For the matrix $\mathbb{H}_{f_2}$, we have the following:

$$
\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2}\boldsymbol{x} = -2c\pi[2(2c + d)x_3 + cx_4]
$$

This means that $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_2}\boldsymbol{x} < 0$ for all non-zero $x = (a, b, c, d)$. Then, we go to step 7.

Step 7: For one $y = (99{,}99{,}200{,}200)$, one can easy find that $\nabla f(\boldsymbol{x})^T(\boldsymbol{y} - \boldsymbol{x}) \le \boldsymbol{0}$. This means that f_2 is pseudoconvex function.

Step 8: For the pressure vessel problem, a set of trade-off solutions for this problem obtained by this step is presented in tables 7.2 and 7.3. Four solution vectors (as functions of Lagrange multipliers) have been found by MOSA-II algorithm. Three of them have been rejected because some of the decision variables were out of the range. The fourth vector of solutions gives better solutions than the solutions found in the literature (see Table 7.4).

Table 7.2: Trade-off Solutions Obtained by MOSA-II Algorithm

x_1	x_2	x_3	x_4	f_1	g_1	g_2	$f_2 = g_3$	g_4
0.8123	0.3999	42.0898	176.877	**5944.06**	0.00003314	0.00163669	-1263.49	-63.123
0.8092	0.3982	41.9182	180.819	**5981**	-0.00017874	0.00169963	-11213	-59.181
0.8099	0.3986	41.9560	180.579	**5987**	-0.0001492	0.00166024	-12522.7	-59.421
0.8105	0.3989	41.9867	180.383	**5993.36**	-0.00015669	0.00165312	-13579.2	-59.617
0.8107	0.3992	42.0163	180.233	**5997.05**	0.00021459	0.0016355	-14813.1	-59.767
0.8112	0.3992	42.0230	180.191	**6000.66**	-0.0001561	0.00169942	-15047.7	-59.809
0.8124	0.3998	42.0831	179.809	**6011.87**	-0.00019617	0.00167277	-17120.2	-60.191
0.8125	0.3999	42.0898	179.73	**6012.13**	-0.00016686	0.00163669	-17148.2	-60.27
0.8126	0.3999	42.0965	179.688	**6012.95**	-0.000013755	0.00170061	-17382.2	-60.312
0.8125	0.3999	42.0898	179.767	**6013**	-0.00016686	0.00163669	-17354.2	-60.233
0.8125	0.3999	42.0898	179.826	**6014.37**	-0.000016686	0.00163669	-18381.3	-60.174
0.8123	0.3999	42.0898	180.068	**6018.63**	0.00003314	0.00163669	-19030.1	-59.932
0.8131	0.4002	42.1232	179.556	**6018.67**	-0.00012224	0.00165533	-18511.2	-60.444
0.8138	0.4005	42.1566	179.345	**6025.05**	-0.00017762	0.00167762	-19666.8	-60.655
0.299997	0.357384	29.762	151.576	**1501.54**	0.2756	-0.0746448	763775	-88.4239

Table 7.3: Lagrange Multipliers Corresponding to Trade-off Solutions in Table 7.2

λ_1	λ_2	μ_1	μ_2	μ_3
0.00001	4.19876 x 10^{-8}	0.0689999	0.0315	0
0.00001	4.19999 x 10^{-8}	0.0699	0.0312437	0
0.00001	4.19999 x 10^{-8}	0.0699	0.0313	0
0.00001	4.19999 x 10^{-8}	0.0699	0.0313459	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.03139	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.0314	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.03149	0
0.00001	4.19999 x 10^{-8}	0.0699	0.0315	0
0.00001	4.19999 x 10^{-8}	0.0699	0.03151	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.0315	0
0.00001	4.19999 x 10^{-8}	0.06993	0.0315	0
0.00001	4.19876 x 10^{-8}	0.07	0.0315	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.03155	0
0.00001	4.19999 x 10^{-8}	0.0699116	0.0316	0
0.00002	4.19876 x 10^{-8}	0.0689999	0.0315	0

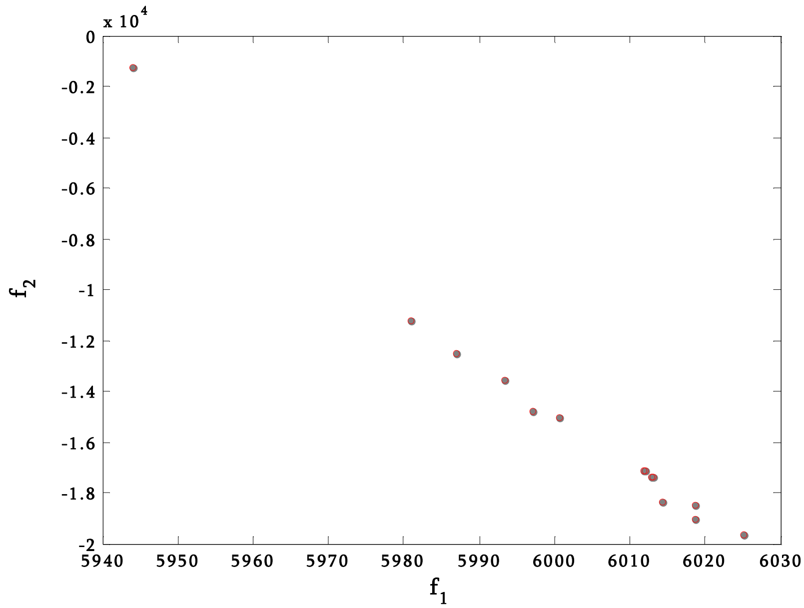

Figure 7.4: The relationship between the objectives for the pressure vessel

7.2.2 Discussion of Results

The salient observations from the above mentioned results are as follows. Tables 7.2 and 7.3 present Pareto-optimal solutions obtained by MOSA-II algorithm. Table 7.2 shows some Pareto-optimal solutions and the values of the objective functions corresponding to these solutions. Table 7.3 presents the KKT multipliers corresponding to these solutions. It is interesting to note how parameter affects the first objective function. It shows that any increase to this parameter gives better values for and, at the same time, gives worst values for and and, therefore, the constraints are unsatisfied.

Table 7.4: Best Solutions found in Literature for the Pressure Vessel Design Problem

Design Variables	Details of the best solutions found				
	Author's results	Mezura-Montes and Coello (2003)	Deb (1997)	Kannan and Kramer (1994)	Sandgren (1988)
x_1	0.8123	0. 812500	0. 9375	1. 125	1. 125
x_2	0.3999	0. 437500	0. 5000	0. 625	0. 625
x_3	42.0898	42. 098370	48. 3290	58. 291	47. 700
x_4	176.877	176. 637146	112. 6790	43. 690	117. 701
f_1	**5944.06**	**6059. 714355**	**6410. 3811**	**7198. 0428**	**8129. 1036**
g_1	0.00003314	-0. 000001	-0. 004750	0. 000016	-0. 204390
g_2	0.00163669	-0. 035882	-0. 038941	-0. 068904	-0. 169942
$f_2 = g_3$	-1263.49	-0. 835772	-3652. 876838	-21. 220104	54. 226012
g_4	-63.123	-63. 362858	-127. 321000	-196. 310000	-122. 299000

7.3 Design of Four-Member Truss

This design is shown in Figure 7.2, and is briefly described in section 7.1. The mathematical model of this design is given in Equation 7.2 as follows:

$$
\begin{aligned}
&f_1(x) = L\left(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4\right)\\
&f_2(x) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} + \frac{2\sqrt{2}}{x_3} + \frac{1}{x_4}\right)\\
&\frac{F}{\sigma} \le x_1, x_4 \le 3\left(\frac{F}{\sigma}\right),\ \sqrt{2}\left(\frac{F}{\sigma}\right) \le x_2, x_3 \le 3\left(\frac{F}{\sigma}\right)\\
&F = 10\ KN, E = 2 \times 10^5\ (KN/cm^2), L = 200\ cm, \sigma = 10\ (KN/cm^2)
\end{aligned}
\tag{7.2}
$$

7.3.1 Experimental Results

Figure 7.5 depicts the results obtained by applying NSGA-II and MOSA-II to the optimisation of a four-member truss design. The tests reported here are carried out using 100 population size, 500 generations, 0.8 crossover probability, 0.05 mutation probability, and simulated binary crossover with 10 crossover distribution and 50 mutation distribution indexes for NSGA-II.

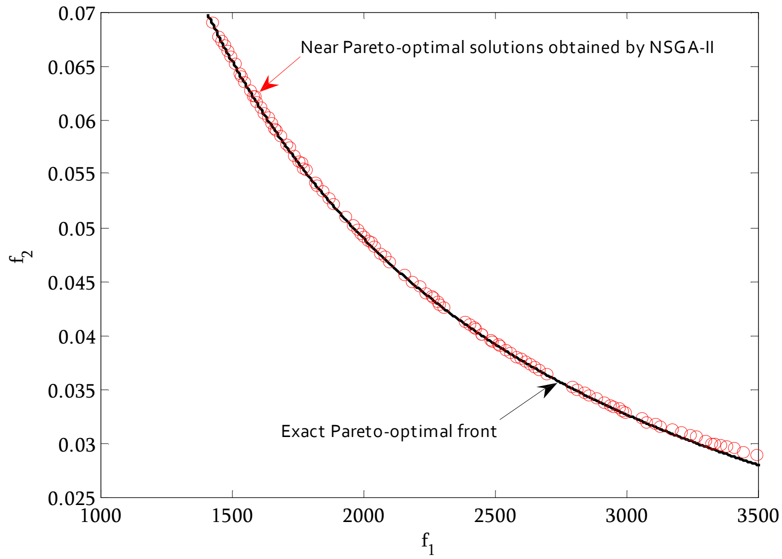

Figure 7.5: The Exact and Approximated Pareto-optimal Front for the Four-bar Plane Truss Problem Using MOSA-II and NSGA-II Algorithm.

7.3.2 Discussion of Results

Let us apply the MOSA-II steps on this problem as follows:

Steps 1-4: Step 1 to step 4 are simple to implement. They need one to be familiar with Mathematica to define the objectives and constraints and to check their continuity and differentiability. The design variables are denoted as and .

Step 5: This step gives the following Hessian matrices for and :

Step 6: For the matrix , for all non-zero . This means that is convex. For the matrix , we have the following:

This means that for all non-zero . Then, is strictly convex.

Step 8: For the four-bar plane truss problem, this step gives the following mathematical formula for the exact Pareto-optimal front:

$$f_2 = \left(\frac{49FL^2}{E}\right)\frac{1}{f_1}$$
$$\frac{7LF}{\sigma} \le f_1 \le \frac{21FL}{\sigma} \qquad (7.3)$$

As can be seen, Equation (7.3) is a non-linear convex relationship (see Figure 7.5) between the two conflicting objective functions. Figure 7.5 shows the Pareto-optimal front for this problem. The following observations for this problem are found by the MOSA-II algorithm:

- Equation (7.3) is constructed based on linear relationship between the decision variables that takes the following form:

$$x_1 = \sqrt{\frac{F\lambda_2}{E\lambda_1}} = x_4,\ \ x_2 = \sqrt{\frac{2F\lambda_2}{E\lambda_1}} = x_3\ , \boldsymbol{\lambda} = (\lambda_1, \lambda_2) > \mathbf{0}$$
$$\text{or} \qquad (7.4)$$
$$\sqrt{2}x_1 = x_2 = x_3 = \sqrt{2}x_4$$

  This means that the designer has to use bars with cross-sectional areas satisfying the relationship (7.4) so as to get the minimum values for both the volume V of the truss and the displacement Δ of the joint. Knowing the exact relationship between the design variables (which is used for formulating the Pareto-optimal front) helps the designers to find the exact optimal solutions for this problem.

- Interestingly, there are further relationship(s) involving the design variables. For example, the design variable x_1 has opposite effects on the two objectives. Thus, the optimal solutions reflect a similar pattern of variation for variable x_1 : a reduction in x_1 causes a reduction in the volume of the truss ($f_1 \propto x_1$) and an increase in the displacement of the joint $\left(f_2 \propto \frac{1}{x_1}\right)$.

7.4 *Design of Welded Beam*

This design is shown in Figure 7.3 and is briefly described in section 7.1. The mathematical model of this design is given in Equation 7.5. This equation assumes the following values.

- Overhang portion of the beam = 14 Inch.
- F = 6000 lb Force.
- Allowable shear strength of the material = 13,600 psi.
- Allowable yield strength of the material = 30,000 psi

$$\text{Minimise} \begin{cases} \text{Cost} = f_1(\boldsymbol{x}) = 1.10471\ h^2 l + 0.04811\ t\ b\ (14.0 + l), \\ \text{End of Deflection} = f_2(\boldsymbol{x}) = \frac{2.1952}{t^3 b} \end{cases} \qquad (7.5)$$

Subject to constraints

$$g_1(\boldsymbol{x}) = \tau(\boldsymbol{x}) - 13{,}600 \leq 0,$$
$$g_2(\boldsymbol{x}) = \sigma(\boldsymbol{x}) - 30{,}000 \leq 0,$$
$$g_3(\boldsymbol{x}) = h - b \leq 0,$$
$$g_4(\boldsymbol{x}) = 6000 - P_c(\boldsymbol{x}) \leq 0$$
$$0.125 \leq h, b \leq 5.0 \quad \text{and} \quad 0.1 \leq l, t \leq 10.0$$

Where,

$$\tau(\boldsymbol{x}) = \sqrt{\acute{\tau}^2 + \acute{\acute{\tau}}^2 + l\acute{\tau}\acute{\acute{\tau}}/\sqrt{0.25(l^2 + (h+t)^2)}},$$
$$\acute{\tau}(\boldsymbol{x}) = \frac{6000}{\sqrt{2}hl},$$
$$\acute{\acute{\tau}}(\boldsymbol{x}) = \frac{6000(14.0 + 0.5l)\sqrt{0.25(l^2 + (h+t)^2)}}{2\left\{0.707\, h\, l\, (\frac{l^2}{12} + 0.25(h+t)^2)\right\}},$$
$$\sigma(\boldsymbol{x}) = \frac{504000}{t^2 b},$$
$$P_c(\boldsymbol{x}) = 64746.022(1 - 0.0282346\, t)tb^3.$$

7.4.1 Experimental Results

Let us apply the MOSA-II steps on this problem as follows:

Steps 1-4: Step 1 to step 4 are simple to implement. They need one to be familiar with Mathematica to define the objectives and constraints and to check their continuity and differentiability. The design variables are denoted as x_1, x_2, x_3 and x_4 (for h, ℓ, t and b).

Step 5: This step gives the following Hessian matrices for f_1 and f_2:

$$\mathbb{H}_{f_1} = \begin{bmatrix} 2.20942x_2 & 2.20942x_1 & 0 & 0 \\ 2.20942x_1 & 0 & 0.04811x_4 & 0.04811x_3 \\ 0 & 0.04811x_4 & 0 & 0.04811(14.0 + x_2) \\ 0 & 0.04811x_3 & 0.04811(14.0 + x_2) & 0 \end{bmatrix}$$

$$\mathbb{H}_{f_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{26.3424}{x_3^5 x_4} & \frac{6.5856}{x_3^4 x_4^2} \\ 0 & 0 & \frac{6.5856}{x_3^4 x_4^2} & \frac{4.3904}{x_3^3 x_4^3} \end{bmatrix}$$

Step 6: For the matrix $\mathbb{H}_{f_1}$, we have the following:

$$\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_1} \boldsymbol{x} = 1.34708\, cd + 4.41884\, abx_1 + (2.20942\, a^2 + 0.09622\, cd)x_2 + 0.09622bdx_3 + 0.09622\, bcx_4$$

This means $\boldsymbol{x}^{\mathrm{t}}\, \mathbb{H}_{f_1} \boldsymbol{x} > 0$ for all non-zero $\boldsymbol{x} = (a, b, c, d)$. Then, f_1 is strictly convex.

For the matrix , we have the following:

This means that for all non-zero . Then, is strictly convex.

Step 8: In this step, the KKT systems give a complicated form for the design variables as functions of KKT multipliers. Therefore, these multipliers can not be separated from the variables to detect the relationship between them. So, the procedure used to solve ZDT3 in the previous chapter is performed here. Figure 7.6 and Figure 7.8 respectively depict the results obtained by applying NSGA-II, SPEA2 and the proposed procedure. The tests reported here are carried out using population size, generations, crossover probability, mutation probability, and simulated binary crossover with crossover distribution and mutation distribution indexes for NSGA-II and SPAE2. Figure 7.7 and Figure 7.9 show the impact of the decision variables on the two objective functions.

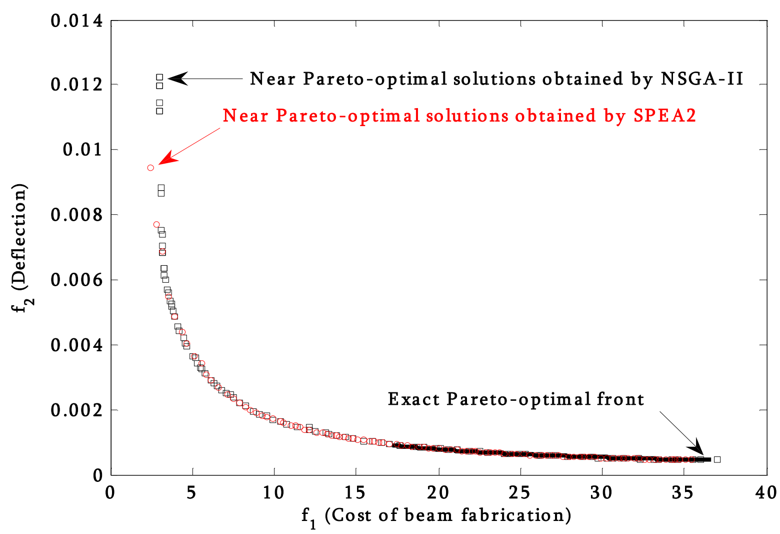

Figure 7.6: Results from NSGA-II, SPEA2 and the proposed procedure (part one)

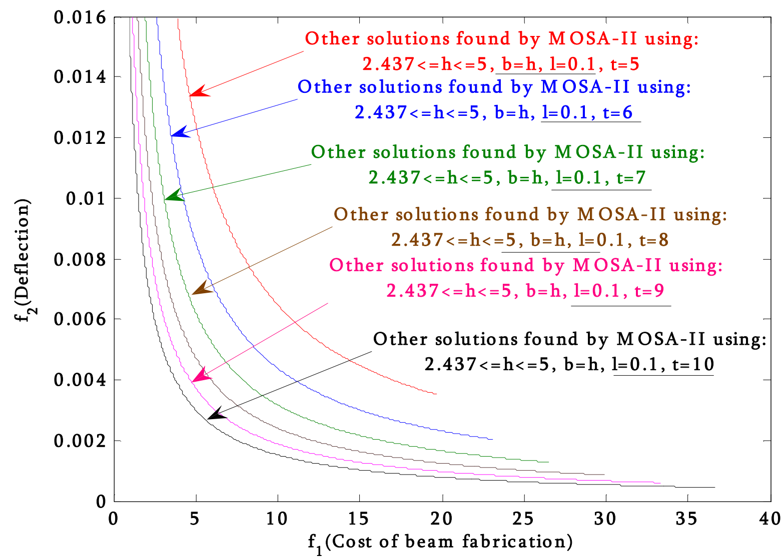

Figure 7.7: Impact of on the relationship between cost and deflection

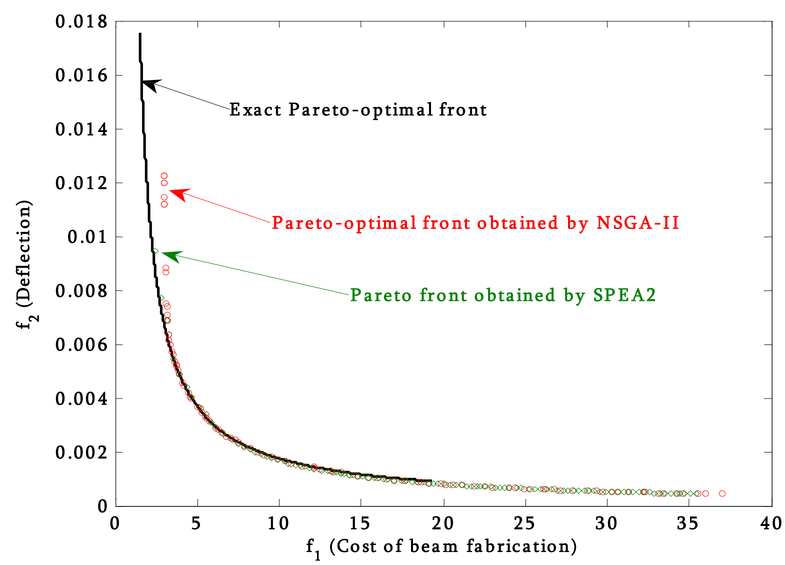

Figure 7.8: Results from NSGA-II, SPEA2 and the proposed procedure (part two)

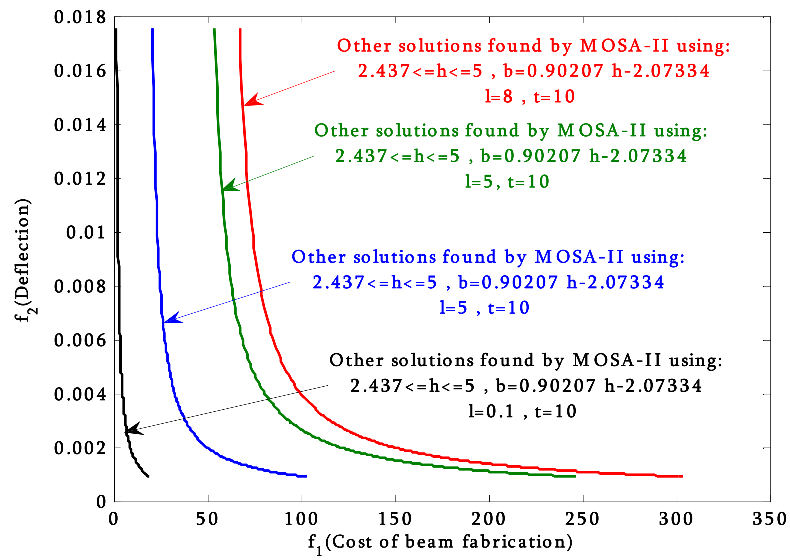

Figure 7.9: Impact of on the relationship between cost and deflection

7.4.2 Discussion of Results

The salient observations from the above mentioned results are as follows.

- The problem has a search space that is biased towards high values of cost and low values of deflection. Furthermore, the Pareto-optimal front of this problem is convex and discontinuous in nature.
- Although step 8 in MOSA-II could not be solved to separate the KKT multipliers from the design variables, the procedure performed for ZDT3 in the previous chapter has still given the exact Pareto-optimal front. Using this procedure, we first divide the relation between and into straight lines and fix the values for and until we get the best relationship between the conflicting objective functions. This can be clearly seen from the results plotted in Figure 7.7 and Figure 7.9. Moreover, the proposed procedure has provided the relationship between the design variables responsible for the conflicting relationship between the objectives.
- It is clear that NSGA-II and SPEA2 converge to the Pareto-optimal front. Figure 7.6 and Figure 7.8 show two parts of the exact Pareto-optimal front.
- Figure 7.6 shows a part of the relationship between the two conflicting objectives obtained by the proposed procedure; this relationship is marked by a solid curve. Interestingly, this relationship reduces the dimensionality of the two objectives as follows:

$$f_1 = 0.110471\, b^2 + 6.78351\, b,$$
$$f_2 = \frac{0.0021952}{b} \qquad (7.6)$$

Hence, the analytical equation between the two objective functions can be written in the following form:

$$f_2 = \frac{0.0021952}{\sqrt{\frac{f_1}{0.110471}+942.654}-30.6792} \qquad (7.7)$$
$$17.1877 \le f_1 \le 36.6792$$

- For the above part of the Pareto-optimal front, an important relationship between the design variables is found. This relationship is used to construct equation 7.6 and takes the following form:

$$2.437 \le h \le 5, b = h,$$
$$l = 0.1, t = 10 \qquad (7.8)$$

This means that for this part of the Pareto-optimal front all the design variables satisfy the relationship 7.8. This indicates that for this part the width of the beam t must be set to its upper limit.

- Figure 7.7 reveals that for optimal design the width of the beam needs to be fixed to its upper limit. Therefore, if solutions close to smaller deflection are desirable a beam with width $t = 10$ needs to be procured.
- Figure 7.8 shows the other part of the relationship between the two conflicting objectives obtained by the propose procedure; this relationship is marked by a solid curve. Interestingly, this relationship reduces the dimensionality of the two objectives as follows:

$$f_1 = 0.110471\,(1.10856\, b + 2.29843)^2 + 6.78351\, b,$$
$$f_2 = \frac{0.0021952}{b} \qquad (7.9)$$

Hence, the analytical equation between the two objective functions can be written in the following form:

$$f_2 = \frac{0.0021952}{\sqrt{\frac{f_1}{0.122464}+895.602}-30.0079} \qquad (7.10)$$
$$1.51733 \le f_1 \le 19.2352$$

- In equations 7.6 and 7.9, the end of deflection (f_2) has the same analytical equation. This equation reveals an interesting innovisation principle involving the design variables. It is observed that the thickness of the beam b needs to be inversely reduced ($b \propto \frac{1}{f_2}$) with deflection f_2 to retain optimality.

- For the second part of the Pareto-optimal front, an important relationship between the design variables is found. This relationship is used to construct equation 7.10 and takes the following form:

$$b = 0.90207h - 2.07334, 2.437 \leq h \leq 5, \quad l = 0.1, t = 10 \tag{7.11}$$

This indicates that for low-cost optimal solutions all the design variables satisfy the relationship 7.11. This means that for this part of the Pareto front the width of the beam t must be set to its upper limit.

- Other innovisation principles can be obtained when the designer tries to minimise cost and deflection separately. If low-cost solutions are desirable, t must be fixed to its upper limit, while increasing l and setting $b = h, 0.125 \leq b \leq 5$.
- Figure 7.9 shows that for optimal design the length of the weld needs to be fixed to its lower limit. Therefore, if low-cost solutions are desired the length of the weld $\ell = 0.1$ needs to be procured. In this way, this gives another innovisation principle between the design variables of the objective functions.

7.5 Validation of Results

The results obtained from the MOSA-II algorithm are validated here through comparison with results published in literature.

7.5.1 Design of Pressure Vessel

The results obtained by the MOSA-II algorithm when compared with results provided by NSGA-II clearly depict that the MOSA-II algorithm has been able to construct Pareto-optimal solutions for this problem (Table 7.2). Furthermore, the MOSA-II algorithm has provided values for the KKT multipliers used to construct these solutions (Table 7.3). Knowing the multipliers that lead to these solutions is a key contribution of the MOSA-II algorithm and is not provided by NSGA-II or any other stochastic algorithm. Literature reveals that in solving this problem the stochastic algorithms could not obtain better solutions than the Pareto-optimal solutions obtained by MOSA-II (Table 7.4).

7.5.2 Design of Four-Member Truss

The results obtained by the MOSA-II algorithm when compared with results provided by NSGA-II clearly depict that the MOSA-II algorithm has been able to construct a complete curve for the Pareto-optimal front (Figure 7.5). Furthermore, the MOSA-II algorithm has provided the mathematical equation that presents the Pareto-optimal front between the two objective functions. This equation can be used to create well distributed solutions across the Pareto-optimal front. Many innovative design principles for this problem have been detected by the MOSA-II algorithm.

Literature reveals that this problem has been solved by many stochastic algorithms (Coello et al., 2002); NSGA-II gives better solutions for this problem so far. Figure 7.5 depicts the NSGA-II results on this problem. Even though NSGA-II has been able to converge to the Pareto-optimal front, it does not give a mathematical formulation for the relationship between the design variables of the Pareto-optimal front. This is because NSGA-II is a stochastic algorithm. Moreover, the Pareto-optimal solutions obtained by MOSA-II are KKT based solutions, giving confidence about this Pareto-optimality.

7.5.3 Design of Welded Beam

The Pareto-optimal front for this problem is made up of two discontinuous parts. The results obtained by the proposed procedure (as also used for ZDT3) and the results provided by NSGA-II and SPEA2 clearly depict that the procedure has been able to construct a complete curve for the two parts of the Pareto-optimal front (Figures 7.8 and 7.9). Furthermore, the procedure has provided the mathematical equation that represents these two parts. This equation can be used to generate well distributed solutions across the two parts of the Pareto-optimal front. Knowing the relationship (innovative design principles) among the design variables, that leads and helps to construct this equation, helps the designer to move from one Pareto-optimal solution to another. It may as well contain information which can assist the designer make a decision and thus arrive at better designs. It is worth mentioning here that this relationship is a key contribution and is not provided by NSGA-II or any other stochastic algorithm.

It has been reported in literature that in solving this problem NSGA-II and SPEA2 give better performance than all other algorithms (Deb, Pratap and Moitra, 2000b; Deb and Srinivasan, 2008). Figures 7.6 and 7.8 depict the results obtained by NSGA-II and SPEA2. It can be seen from these Figures that NSGA-II is better than SPAE2. It is evident that even though NSGA-II has been able to converge to the Pareto-optimal front and it does exhibit uniform distribution of solutions, it does not give a mathematical formulation for the relationship between the design variables or between the objective functions for the solutions on the Pareto-optimal front. This is because NSGA-II is a stochastic algorithm.

7.6 Summary

In this chapter, it has been demonstrated how the MOSA-II algorithm is competent in solving three real-life engineering optimisation problems: design of pressure vessel, design of four-member truss and design of welded beam. Since these three problems constitute a representative set, it can be said that the application of the MOSA-II algorithm on these problems ensures its success in solving other problems. In this way, this chapter has used real-life problems to validate the observations made previously regarding the capability of the MOSA-II algorithm in tackling multiple objectives, constraints and variable interactions in multi-objective optimisation problems. This chapter has also demonstrated that the MOSA-II algorithm outperforms a state-of-the-art optimisation algorithm, NSGA-II, on a variety of multi-objective optimisation problems. In conclusion, this chapter has achieved the following.

- It has analysed a number of case studies from real-life engineering optimisation.
- It has reported the experimental results obtained by the MOSA-II algorithm and NSGA-II on these case studies.
- It has finally analysed these results in order to validate the performance of the MOSA-II algorithm.

8 DISCUSSION AND CONCLUSIONS

This chapter is devoted to concluding this thesis with a discussion on the deliverables of this research. It also identifies the limitations of this work and highlights the corresponding future research activities. This chapter aims to achieve the following.

- *To summarise key observations from this research.*
- *To identify the main contributions of this research.*
- *To discuss the limitations of this research.*
- *To frame future research activities based on this work.*

8.1 Discussion

This section discusses the key observations from this research. The generality of this research is discussed in this section as well.

8.1.1 Key Observations from this Research

This research aims to develop new multi-objective optimisation techniques based on the extension of the current theoretical results to handle the challenges posed by three features of multi-objective optimisation problems: multiple objective functions, constraints and interaction among decision variables. This is a part of board initiative to make symbolic optimisation algorithms popular in handling real-life engineering optimisation problems and help improve the current optimisation algorithms. The key observations of this research are summarised here.

8.1.1.1 Literature Survey

This research has investigated the popular theoretical and evolutionary computing approaches in literature. It has surveyed the optimisation approaches for handling multi-objective optimisation problems. It has analysed the drawbacks of current theoretical approaches as well as evolutionary computing algorithms. Even though evolutionary computing algorithms are growing very quickly and many developments on these algorithms have been used to deal with complicated real-life multi-objective optimisation problems, it has been surveyed that evolutionary researchers still face criticism about the theoretical optimality of the obtained Pareto-optimal solutions found by their evolutionary techniques. The research in this thesis carried out a detailed survey of these techniques and the theoretical techniques with respect to the key features of the engineering optimisation problems such as: number of objective functions (multiple objectives), nature of constraints (linear, nonlinear), and design variable interactions.

When tackling multi-objective optimisation problems, the two main goals that researchers seek are convergence to the Pareto-optimal front and at the same time maintenance of diversity across the Pareto-optimal front. It is observed that the elitist multi-objective optimisation techniques that use Pareto domination and diversity-preserving operators perform better than other multi-objective optimisation techniques. Nevertheless, it is observed that there are a few evolutionary computing techniques that specialise in handling constraints in multi-objective optimisation problems. Recently, some constraint-handling techniques have been introduced (Deb, 2000; Coello, 1997, 2000; Coello et al, 2002). These constraint-handling techniques incorporate constraint violations in the definition of Pareto domination and have shown success in handling a variety of constrained multi-objective optimisation problems.

This research has further developed the current theoretical techniques to detect the analytical relationship among the design variables that consequently lead to constructing the Pareto-optimal front. Most of the evolutionary computing techniques that are discussed above fail to provide a complete curve of the Pareto-optimal front or give the analytical form of the relationship between the design variables that lead to this complete curve. The few evolutionary-based techniques that are used for handling variable interactions have limited usefulness for real-life optimisation problems since in most cases they cannot handle multiple objective functions and their performance is strongly dependent on the nature of the decision space. The presence of variable interactions introduces an additional level of complexity to the constrained, inseparable multi-objective optimisation problems. This reveals two important aspects that are not incorporated in evolutionary computing techniques. These two aspects are: identification of relationship between decision variables of Pareto-optimal solutions (aspect 1) and the theoretical optimality of variables (aspect 2). Although some techniques that can individually deal with these two aspects have been recently reported, there is a complete lack of dedicated theoretical and evolutionary computing frameworks to deal with variable interactions in optimisation problems.

The literature survey also analysed the existing theoretical approaches that have been applied to tackle multi-objective optimisation problems in many practical disciplines. It is observed that Karush-Kuhn-Tucker and Fritz-John conditions are the core of all the current researches reported in the literature. Recently, Mishra and Giorgi (2005) extended the Karush-Kuhn-Tucker first-order optimality conditions for a specific class of multi-objective optimisation problems. In this study, convex-concave fractional multi-objective optimisation problems were not included. This research further develops the current Karush-Kuhn-Tucker optimality conditions. The first-order and second-order optimality conditions of Karush-Kuhn-Tucker for convex-concave fractional multi-objective optimisation problems are proposed and discussed. Furthermore, weak and strong duality conditions for convex-concave fractional multi-objective optimisation problems are shown. An observation was made regarding these optimality conditions, where it was observed that there is a complete lack of research in relaxing these conditions. Relaxation of these conditions helps designing new algorithms and is useful for sensitivity analysis.

8.1.1.2 Gap Analysis: Theoretical Approaches

This research analyses the existing multi-objective optimality conditions along with the characteristics of the problems solved such as convexity, concavity, pseudo-convexity and quasi-convexity. This analysis reveals that the classical forms of sufficient optimality conditions involving Karush-Kuhn-Tucker multipliers often require the Lagrange function to be convex. However, there is a research gap in relaxing these conditions for handling various types of characteristics relating to the objectives and constraints. This gap identifies a focus of this research, which is to relax the first and second-order optimality conditions of Karush-Kuhn-Tucker to involve additional characteristics of the objectives and constraints such as pseudo-convexity, quasi-convexity and concavity that are important for solving real-life multi-objective optimisation problems.

8.1.1.3 Gap Analysis: Symbolic Algorithms

The solutions provided by the existing optimisation techniques still face criticism about their theoretical optimality. Moreover, by using these algorithms one cannot analyse the characteristics of the constructed shape of the Pareto-optimal front. This gap identifies the main focus of this research, which is to develop multi-objective symbolic algorithms that can effectively handle the above challenges in constrained multi-objective optimisation problems and give a complete description of the Pareto-optimal front.

8.1.1.4 Development of Optimality Conditions

The development of the first and second-order optimality conditions of Karush-Kuhn-Tucker require relaxation of the current conditions. This research has handled the class of multi-objective optimisation problems that satisfy the univexity concept. Firstly, different theoretical classes of multi-objective optimisation problems are presented. The first-order optimality conditions for the problems in each class are proposed and proved. Furthermore, the duality results for each class are separately suggested and some first-order duality theorems are illustrated and proved. Secondly, the research proposes second-order optimality conditions for Karush-Kuhn-Tucker under a generalised type of univexity functions. Finally, three theorems regarding the second-order optimality of Karush-Kuhn-Tucker conditions are proposed and proved.

8.1.1.5 Development of MOSA-I

This research identifies the challenges that objective functions and constraints pose for multi-objective optimisation problems. These challenges include multi-modality, deception and isolated optimum solutions that impede the current optimisation algorithms from convergence to the Pareto-optimal front, and discontinuity, non-uniformity and shape complexity of Pareto-optimal front that impede the optimisation algorithms from sustaining the diversity of Pareto-optimal solutions. This research proposes a symbolic solution strategy to deal with these challenges. It applies this solution strategy to develop a symbolic algorithm, the MOSA-I algorithm, which is capable of tackling constrained multi-objective optimisation problems having complex inseparable objective functions. The MOSA-I algorithm is based on the existence of relationship(s) – which are functions of Karush-Kuhn-Tucker multipliers – among the design variables of the solutions belonging to the exact Pareto-optimal front. It discovers this relationship using the first-order optimality conditions of Karush-Kuhn-Tucker. The relationship thus obtained is used to construct the analytical formula of the Pareto-optimal front. Two important steps are included in the MOSA-I algorithm for enabling it to cope with a broader class of multi-objective optimisation problems that have inseparable objectives and constraints characterised by pseudo-convex and quasi-convex features. The MOSA-I algorithm has been shown to outperform the existing state-of-the-art multi-objective optimisation algorithm, NSGA-II. It has also been proved to be effective in tackling a variety of multi-objective optimisation problems. This algorithm can be used to measure the performance of multi-objective optimisation algorithms as long as the analytical formula of the Pareto-optimal front exists. However, it is still dependent on the first-order optimality conditions of Karush-Kuhn-Tucker.

8.1.1.6 Development of MOSA-II

This research also proposes a second-order multi-objective symbolic algorithm, MOSA-II, to deal with the above challenges. It is a generalisation of MOSA-I. It is used to solve constrained multi-objective optimisation problems having complex inseparable objective functions. The MOSA-II algorithm is based on the existence of relationship(s) – which are functions of Karush-Kuhn-Tucker multipliers – among the design variables (same as the previous algorithm, MOSA-I). But the difference between this algorithm and the previous one is that the idea of this algorithm is based on the second-order optimality conditions of Karush-Kuhn-Tucker (presuming twice continuously differentiable objective and constraint functions). The reason for using the second-order optimality conditions is their ability to reduce some of the Pareto-optimal solutions obtained by the first-order optimality conditions (see the constrained problem in chapter 5). The MOSA-II algorithm has been shown to outperform the existing state-of-the-art multi-objective optimisation algorithm, NSGA-II. It has also been proved to be effective in tackling a variety of multi-objective optimisation problems. It has been observed that the first-order optimality conditions involved in this algorithm may cause difficulty when solving them to get the design variables separated from the KKT multipliers. Therefore, to overcome this drawback, a simple procedure to solve the first-order optimality conditions has been proposed. The use of this procedure commences when the first-order systems cannot be solved. This procedure depends on dividing the decision spaces using border and straight lines inside the decision space with different slopes, then substituting these lines in the objective functions. Once the borders or straight lines leading to the desired Pareto-optimal front have been identified, they consequently form the corresponding Pareto-optimal set for the problem at hand. They also give some insights about which design variable has impacted the conflicting relationship between the objectives.

8.1.1.7 Performance Analysis of MOSA-I and MOSA-II Algorithms

This research applies the MOSA-I and MOSA-II algorithms to a set of multi-objective test problems that have the required characteristics (multiple objectives, constraints and variable interactions) and enable controlled testing of the proposed symbolic algorithms. This research applies these algorithms to some real-life case studies that have been recently tackled by evolutionary algorithms and are available in literature. The choice of NSGA-II for comparison is based on the observation that it has been shown in literature to outperform most of the existing evolutionary computing techniques in dealing with multi-objective optimisation problems in real domains. The multi-objective test problems that are chosen for analysis have complex inseparable function interactions leading to multi-modality in the search space, discontinuity in Pareto-optimal front and bias in the search space. It is shown that in all cases these symbolic algorithms identify exact Pareto-optimal solutions. This is because of many reasons: (1) NSGA-II (and other stochastic algorithms) uses Pareto-optimal/elitism strategy that ceases to produce the driving force towards the global Pareto-optimal front once most of the solutions of the population share the same non-dominated level, and (2) the proposed symbolic algorithms use relaxation of Karush-Kuhn-Tucker conditions which guarantee Pareto-optimal solutions. These optimality conditions help finding out the relationship between the design variables for solutions on the Pareto-optimal front and therefore a complete curve/surface of the corresponding Pareto-optimal front can be formed. It is also observed that in most test problems the symbolic algorithms provide a mathematical formula for the Pareto-optimal front. With this mathematical formula of the Pareto-optimal front, better distribution of solutions can be obtained as compared to NSGA-II. As reported in literature, NSGA-II or any other stochastic optimisation algorithm still faces difficulties in providing uniform distribution on the Pareto-optimal front. The reason for this is that the crowded comparison operators used in these algorithms to attain solution diversity use external means without addressing the inherent characteristics that lead to diversity problems. The proposed symbolic algorithms address the core issue of this problem by determining the relationship(s) among the design variables of the Pareto-optimal solutions, and then using it to generate well distributed solutions over the Pareto-optimal front. These algorithms are able to identify variable interactions, and hence converge to the exact Pareto-optimal front. Furthermore, it is observed that since these symbolic algorithms use first and second-order optimality conditions as their optimisation engine, they have all the characteristics for effectively deal with inseparable function interactions in multi-objective optimisation problems.

8.1.1.8 Validation Using Real-life Problems

The performance of MOSA-II algorithm is also tested on a set of real-life engineering optimisation problems that together represent a variety of challenges that multiple objectives, constraints and variable interactions pose for optimisation algorithms. Here, the generalised form (MOSA-II algorithm) is used in the validation process rather than the MOSA-I algorithm. The set of real-life problems includes the design of a pressure vessel, a four-member truss and a welded beam. The validation of results is performed here based on the published results obtained for these problems using stochastic algorithms. The results from the testing of the MOSA-II algorithm are validated here. Particularly, the application of the MOSA-II algorithm on these real-life problems demonstrates its capability in dealing with multiple objectives, constraints and inseparable function interactions in multi-objective optimisation problems.

8.1.2 Main Contributions

In this research, a significant contribution to understanding about the handling of variable interactions in multi-objective engineering optimisation problems has been illustrated. The research has proposed two multi-objective symbolic algorithms for handling variable interactions in the presence of multiple objectives and constraints. In addition, some extensions of the first and second-order optimality conditions have been proposed.

The following points clearly identify the main contribution to knowledge of this work. There was a research gap in all these areas that has subsequently been filled by this research.

- ***Critical analysis of the existing evolutionary computing techniques***: The literature survey carried out in this research compares the capabilities of the existing evolutionary computing techniques against the challenges posed by multiple objective functions, constraints and variable interactions in finding a complete curve for the Pareto-optimal front. Based on the above analysis, this research has identified that there is a lack of techniques for handling variable interaction to form a complete curve for the Pareto-optimal front in multi-objective optimisation problems. This has led to the development of optimisation algorithms based on the development of the first and second-order optimality conditions of Karush-Kuhn-Tucker.
- ***Variable interactions***: This research has defined the relationship(s) between the design variables that lead to constructing the curve/surface of the Pareto-optimal front. It has further developed innovative design principles for each of these categories.
- ***Relaxation of optimality conditions***: This research has analysed optimality conditions, especially Karush-Kuhn-Tucker and Fritz-John conditions. It has analysed different theoretical classes of multi-objective optimisation problems under univexity concept. For these classes, extra conditions are imposed on the objectives and constraints to achieve univexity. Based on these additional conditions, the first-order optimality conditions for the problems in each class are separately proposed and proved. Furthermore, the duality results for each class are presented and some first-order duality theorems are illustrated and proved. In addition, this research has proposed second-order optimality conditions of Karush-Kuhn-Tucker under a generalised type of univexity functions. A set of sufficient second-order optimality conditions for multi-objective optimisation problems is established. Three important theorems regarding the second-order optimality conditions of Karush-Kuhn-Tucker are proposed and proved.

- ***Development of MOSA-I and MOSA-II for handling variable interactions***: This research has proposed MOSA-I and MOSA-II algorithms for tackling inseparable function interactions in multi-objective optimisation problems. The two algorithms need the objective functions and constraints to be continuous and differentiable. Furthermore, the research has compared the performance of these algorithms with NSGA-II based on a set of test problems carefully selected from literature. The algorithms are also validated using three real-life case studies. Moreover, these algorithms can be used to measure the performance of stochastic algorithms and can also be used to design stopping criteria for stochastic algorithms.

8.1.3 Research Limitations

An attempt has been made in this research to keep it as general as possible. However, as with any other research, this work also has some limitations. Here, some of these limitations are identified.

8.1.3.1 Limitations of Research Methodology

The following are the main limitations of the methodology used in this research.

- Although the literature review has systematically reported and analysed many articles related to theoretical and evolutionary approaches in the area of multi-objective optimisation, the analysis provided little insight into the statistical aspects of the information gathered.
- The algorithm used for comparison in this research is only NSGA-II. The choice of this algorithm is justified since it outperforms most of the existing optimisation algorithms in dealing with multi-objective optimisation problems.
- The real-life case studies that have been used in this research for the validation process are borrowed from literature. This has provided a limited insight into the process of model development for optimisation. Furthermore, the nature of these case studies could have been better if they were designed in consultation with the industrial designers.
- The literature review carried out in this research has shown that Karush-Kuhn-Tucker optimality conditions are the core of all extensions applied to optimality conditions. Therefore, this has focused this research only on Karush-Kuhn-Tucker optimality conditions.

8.1.3.2 Limitations of MOSA-I

The limitations of the MOSA-I algorithm are as follows.

- The performance of the MOSA-I algorithm is dependent on some characteristics of the objectives and constraints. It requires that the objectives and constraints should have a well-defined mathematical formula. In addition, the objectives and constraints should be continuous, differentiable and convex. Hence, use of multi-objective optimisations problems that do not involve these aspects make the MOSA-I algorithm to fail. So to overcome this limitation some modifications should be added to the MOSA-I algorithm. These modifications are incorporated in the MOSA-II algorithm.
- The MOSA-I algorithm depends on the first-order optimality conditions of Karush-Kuhn-Tucker. These conditions are used to get the analytical relationship(s) between the design variables and in this way the Pareto-optimal front is constructed. Hence, some complicated non-linear multi-objective optimisation problems could make these conditions unsolvable and therefore the MOSA-I algorithm may fail to find the analytical relationship in some cases.
- For complex relationship(s) (high order, non-linear) among the design variables of the Pareto-optimal solutions, the performance of the MOSA-I algorithm has not been tested yet.
- As the number of dimensions and objectives in the problem is increased, the MOSA-I algorithm's performance is not directly affected but the first-order optimality conditions in these cases become difficult to be solved.
- In certain classes of multi-objective optimisation problems, the MOSA-I algorithm finds out the relationship between the design variable as functions of Karsh-Kuhn-Tucker multipliers. In these problems, the analytical formula of the Pareto-optimal front cannot be mathematically formulated in some cases.
- Solutions provided by the MOSA-I algorithm can be categorised into two parts: the first part satisfies the constraints imposed on the problem and the other part contains solutions that do not satisfy the constraints. This is an inherent limitation of the first-order optimality conditions used in this algorithm.

8.1.3.3 Limitations of MOSA-II

The limitations of the MOSA-II algorithm are as follows.

- Although the MOSA-II algorithm tackles the drawbacks of the MOSA-I algorithm, its performance is still dependent on some characteristics of the objectives and constraints.

MOSA-II algorithm can deal with concave problems, but the use of multi-objective optimisations problems that have discontinuous objectives make it fail.

- The MOSA-II algorithm depends on the second-order optimality conditions of Karsh-Kuhn-Tucker. Although these conditions reduce the solutions provided by the first-order optimality conditions, the solutions still may contain those that are in the infeasible decision space.
- For very complex relationship(s) (high order, non-linear) among the design variables of the Pareto-optimal solutions, the performance of the MOSA-II algorithm has not been tested yet.
- As the number of dimensions and objectives in the problem is increased, the MOSA-II algorithm's performance is not directly affected but the first-order optimality conditions in these cases become difficult to be solved.
- For combinatorial multi-objective optimisation problem, the performance of the MOSA-II algorithm has not been tested yet.
- When the first-order optimality systems cannot be solved, the decision space is divided. So, the performance of the MOSA-II algorithm is dependent on dividing the decision space into straight lines (liner relationship(s) between the design variables) and checking if these lines guide to the Pareto-optimal front. Hence, multi-objective optimisation problems that have Pareto-optimal front constructed by non-linear relationship(s) make the MOSA-II algorithm fail and consequently the proposed procedure needs some modifications.

8.2 Future Research

This research concentrates on multi-objective optimisation problems. There is a need to extend this research to incorporate other areas of optimisation, such as combinatorial problems (e.g. knapsack problem). The extension of optimality conditions and optimisation strategies could enhance the capabilities of the proposed symbolic algorithms and might help evolutionary algorithms indentifying the theoretical optimality of the solutions obtained by them. More experiments are required to compare the performance of the proposed algorithms against other optimisation algorithms.

The future research activities corresponding to the optimality conditions are as follows.

- Further research needs to be carried out to relax optimality conditions based on the optimisation theory of Karsh-Kuhn-Tucker and Fritz-John.

- Sensitivity analysis and mathematical stability are not universally defined in the literature and have been widely ignored in the area of multi-criteria optimisation. Further research needs to be carried out in this area.
- Non-differentiable optimality conditions under constraints qualifications based on directional derivatives need to be relaxed.
- Multi-objective optimality condition using fractional derivatives is another area of future research.

The future research activities for further development of the MOSA-I algorithm can be summarised as follows.

- Further research needs to be carried out to enhance the MOSA-I algorithm for tackling discontinuous, non-differentiable and black-box multi-objective optimisation problems.
- The first-order optimality conditions need to be relaxed so that the MOSA-I algorithm can deal with a wider class of multi-objective optimisation problems.
- Further research needs to be carried out to enhance the MOSA-I algorithm for incorporating designers' preferences in certain classes of problems at an intermediate stage of the optimisation process.
- An estimation tool should be included in the MOSA-I algorithm to get estimated values for Karsh-Kuhn-Tucker multipliers in case the MOSA-I algorithm is not able to give solutions of the first-order optimality conditions.
- The MOSA-I algorithm needs more enhancements to deal with multi-objective optimisation problems that have noisy objective functions that prevent the algorithm from reaching the exact Pareto-optimal front.
- Although the first-order optimality conditions in the MOSA-I algorithm have been developed to cover a wider class of multi-objective optimisation problems, the algorithm still needs a relaxation in these conditions to handle fractional non-convex multi-objective optimisation problems.
- A systematic procedure to show how an arbitrary direction is selected to check pseudo-convexity and quasi-convexity is needed to be included in the MOSA-I algorithm.

The future research activities for further development of the MOSA-II algorithm are as follows.

- Although the MOSA-II algorithm covers a broader class of multi-objective optimisation problems than the MOSA-I algorithm, further research needs to be carried out to enhance it for incorporating optimality condition for discontinuous and non-differentiable problems.
- A systematic procedure to show how an arbitrary direction is selected to check the second-order optimality conditions is needed to be included in the MOSA-II algorithm.
- An estimation tool should be included in the MOSA-II algorithm to get estimated values for Karsh-Kuhn-Tucker multipliers in case the algorithm is not able to check the second-order optimality conditions.
- The enhancement of this algorithm to deal with noisy objective functions is another area of future research.
- The performance of this algorithm needs to be studied for other multi-objective optimisation problems which have non-linear relationship(s) among design variables or more complicated Pareto set (Li and Zhang, 2010).
- A neural network strategy could be incorporated into the algorithm to enable it to get the relationship(s) among design variables.
- A systematic procedure should be added to the algorithm to enable it to calculate points of intersection among the borders used to construct the feasible space.
- Further research needs to be carried out to enhance this algorithm for incorporating designers' preferences in certain classes of problems at an intermediate stage of the optimisation process.
- Further research needs to be carried out to combine the MOSA-II algorithm with any other algorithms (either stochastic or analytical).

8.3 Conclusions

This section compares the achievements of this research with the objectives stated in chapter 3. The following discussion analyses each research objective and compares it with what is achieved in this research.

- This thesis provides a literature review of the theoretical and evolutionary computing approaches used for solving multi-objective optimisation problems. It analyses theoretical techniques and evolutionary computing techniques with respect to the key features of engineering optimisation problems such as: number of objective functions (multiple objectives), nature of constraints (linear, nonlinear), and design variables interactions. This survey reveals that there are effective techniques available in literature for handling the above features. However, there is a research gap in these techniques for handling variable interactions. This gap defines the main focus of this research which is to develop new multi-objective optimisation techniques (based on the extension of the current theoretical approaches) that can effectively construct the relationship(s) among the design variables and therefore generate the Pareto-optimal front.

- This research also proposes extension of the optimality conditions of Karsh-Kuhn-Tucker. The convexity assumption in first and second-order optimality conditions is relaxed. Some classes of multi-objective optimisation problems are proposed. The first-order optimality conditions for these classes are established under the univexity assumption. Furthermore, the duality results corresponding to these classes are established.
- This research proposes two symbolic algorithms, MOSA-I and MOSA-II, for identifying interactions among design variables that are used to construct Pareto-optimal front for multi-objective optimisation problems. The algorithms are dependent on the first and second-order optimality conditions of Karush-Kuhn-Tucker and deal with specified classes of multi-objective optimisations problems. The second algorithm is more general than the first one. It provides a complete framework for handling multiple objectives in the presence of constraints and inseparable function interactions.
- This research shows how the two symbolic algorithms can be used for measuring the performance of stochastic algorithms. It is observed that these symbolic algorithms give a measurement for the performance of stochastic algorithms and can be used to design termination criteria for them.
- This research selects a representative set from these problems, and analyses the performance of the MOSA-II algorithm on this representative set of real-life engineering optimisation. The reason for choosing the MOSA-II algorithm is that it is a generalisation of MOSA-I one and can deal with a wider class of multi-objective optimisation problems. The performance of the MOSA-II algorithm is also compared with NSGA-II on these problems. The validation is performed here based on the results published in literature.

The achievements of this research can be briefly and precisely stated as follows.

- Critical analysis of the existing theoretical and evolutionary computing techniques.
- Extension of the optimality conditions of Karush-Kuhn-Tucker.
- Establishment of the duality results under the univexity assumption.
- Development of two symbolic algorithms for multi-objective optimisation problems (MOSA-I and MOSA-II).

In this way, the research aim which is dedicated to develop a new multi-objective optimisation technique capable of detecting relationship(s) among decision variables that can be used for constructing the analytical formula of Pareto-optimal front based on the extension of the current theoretical research, has been achieved.

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APPENDIX

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\\ MOSA-I Mathematica code for the constrained problem
\\ Define the objective functions and the constraints
f1[x1_,x_2]:=x1;
f2[x1_,x2_]:=(1+x2)/x1;
g1[x1_,x2_]:=6-x2-9*x1;
g2[x1_,x2_]:=1+x2-9*x1;
g3[x1_,x2_]:=0.1-x1;
g4[x1_,x2_]:=x1-1;
g5[x1_,x2_]:=x2;
g6[x1_,x2_]:=x2-5;
```

\\ Calculate the Hessian matrix for each objective function
\\ Hessian matrix for f1
m1={{$\partial_{x1,x1}$f1, $\partial_{x1,x2}$f1},{$\partial_{x2,x1}$f1, $\partial_{x2,x2}$f1}};
\\ Form m1 as a matrix
MatrixForm[m1];

\\Hessian matrix for f2
m2={{$\partial_{x1,x1}$f2, $\partial_{x1,x2}$f2},{$\partial_{x2,x1}$f2, $\partial_{x2,x2}$f2}}
\\ Form m2 as a matrix
MatrixForm[m2]

\\ Check the positive semi-definite for the matrix m1
\\ Define a symbolic matrix m3 that represents the vector x
m3={{a},{b}};
m4=Transpose[m3]*m1*m3
Simplify[m4]

\\ Check the positive semi-definite for the matrix m2
m5=Transpose[m3]*m2*m3
Simplify[m5]

\\ If m4>=0 and m5>=0 then solve the following first-order system
Eqs={$\lambda 1 * \partial_{x1}$f1+ $\lambda 2 * \partial_{x1}$f2+$\mu 1 * \partial_{x1} g1$+ $\mu 2 * \partial_{x1} g2$+ $\mu 3 * \partial_{x1} g3$+ $\mu 4 *$
$\partial_{x1} g4$+ $\mu 5 * \partial_{x1} g5$+ $\mu 6 * \partial_{x1} g6$==0, $\lambda 1 * \partial_{x2}$f1+ $\lambda 2 * \partial_{x2}$f2+$\mu 1 * \partial_{x2} g1$+ $\mu 2 *$
$\partial_{x2} g2$+ $\mu 3 * \partial_{x2} g3$+ $\mu 4 * \partial_{x2} g4$+ $\mu 5 * \partial_{x2} g5$+ $\mu 6 * \partial_{x2} g6$==0}
Solve [Eqs, {x1,x2}]

\\ Removing $\mu 1$, $\mu 2$, $\mu 3$, $\mu 4$, $\mu 5$ and $\mu 6$
Eq={ $\mu 1$g1==0, $\mu 2$g2==0, $\mu 3$g3==0, $\mu 4$g4==0, $\mu 5$g5==0, $\mu 6$g6==0}
Solve[Eq, { $\mu 1$, $\mu 2$, $\mu 3$, $\mu 4$, $\mu 5$, $\mu 6$}]

\\If m4<0 or m5<0 or both do the following
\\ Define one y
y={{a},{b},{c}}
\\ Calculate the gradient at any solution x

grade f1={{∂_{x1}f1},{∂_{x2}f1}}
grade f2={{∂_{x1}f2},{∂_{x2}f2}}
m6=Transpose[grade f1]*(y-x)
m7=Transpose[grade f2]*(y-x)
\\ If m6<=0 then f1 is pseudoconvex and then go to solve the first-order system
\\ If m7>=0 then f2 is quasiconvex and then go to solve the first-order system