Correction of the definition of mass flow parameter in dynamic inflow modelling

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Abstract: The dynamic inflow model, which has been extensively developed by Peters and his co-workers for the last two decades, is a powerful tool for predicting the induced velocity distribution over a rotor disc. Scant attention has, however, been paid so far in using the model to analyse an autorotative rotor. The authors identified a necessary change to the mass flow parameter of the original model of Peters for autorotating rotors. In response, questions were raised by readers about how the modified and original inflow parameters should consistently be explained, why the original model of Peters is insufficient for autorotation, and how the validity of the modified mass flow parameter can mathematically be enunciated. Indeed, the discussion in the article by Murakami and Houston (2008) was held in a manner that readers might have received an impression that the modified model was derived independently of the original model, and the connection to the original model was not clearly presented. This technical note is written with an aim to elucidate the mathematically consistent relationship between the original and modified mass flow parameters, and it is confirmed that the original definition of the mass flow parameter contains an error in its definition. It is concluded that the unified mass flow parameter, which is presented in this article, should always be used in the dynamic inflow model.

Keywords: dynamic inflow modelling, mass flow parameter, autorotation, autogyro, gyroplane

1 INTRODUCTION

The Pitt and Peters model [**1**] is arguably one of the most popular dynamic inflow models widely used for rotor application today. It is typically represented as

$$[\mathbf{M}]\begin{Bmatrix}\dot{\lambda}_0\\ \dot{\lambda}_{1s}\\ \dot{\lambda}_{1c}\end{Bmatrix} + [\mathbf{L}]^{-1}V\begin{Bmatrix}\lambda_0\\ \lambda_{1s}\\ \lambda_{1c}\end{Bmatrix} = \begin{Bmatrix}C_{\mathrm{T}}\\ C_{\mathrm{L}}\\ C_{\mathrm{M}}\end{Bmatrix} \tag{1}$$

where [**M**], [**L**], and V are the apparent mass matrix, gain matrix, and the mass flow parameter, respectively. In the Peters and He model [**2**, **3**], which is a more advanced dynamic inflow model, the inflow components, $\{\lambda_0, \lambda_{1s}, \lambda_{1c}\}$, and the rotor load components, $\{C_{\mathrm{T}}, C_{\mathrm{L}}, C_{\mathrm{M}}\}$, in equation (1) are expanded not only in the azimuth direction but also in the radial direction, but the basic matrix form holds the same. The mass flow parameter, V, may be the free stream speed, V_∞, but is recommended to be replaced with the mass flow parameter [**4**], which is defined as

$$V_{m+} = V_{\mathrm{T}} + \lambda_m \frac{\partial V_{\mathrm{T}}}{\partial \lambda_m} = \frac{\mu^2 + \lambda(\lambda + \lambda_m)}{\sqrt{\mu^2 + \lambda^2}} = \frac{\mu^2 + (\lambda_{\mathrm{f}} + \lambda_m)(\lambda_{\mathrm{f}} + 2\lambda_m)}{V_{\mathrm{T}}} \tag{2}$$

where μ, λ, λ_m, λ_{f}, and V_{T} are advance ratio, the total inflow, inflow due to the rotor thrust, free stream inflow, and total flow at rotor disc, respectively. Note that

$$\lambda = \lambda_{\mathrm{f}} + \lambda_m \tag{3}$$

$$V_{\mathrm{T}} = \sqrt{\mu^2 + \lambda^2} \tag{4}$$

See also Fig. 1 for their geometric relationships.

The mass flow parameter defined in equation (2) was first introduced in reference [**4**], and then has been always used in other variations of dynamic inflow

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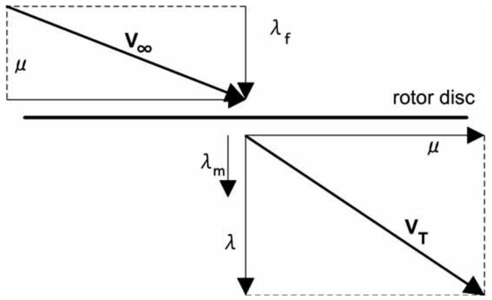

Fig. 1 Geometric relationships between inflow parameters

models of Peters and his co-workers [**1–3**, **5–9**] over 20 years. Reference [**7**] recommends that the mass flow parameter should be replaced with the mass flow matrix

$$V_{m+} \longrightarrow \begin{bmatrix} V_{\mathrm{T}} & 0 & 0 \\ 0 & V_{m+} & 0 \\ 0 & 0 & V_{m+} \end{bmatrix} \tag{5}$$

for the non-linear version of the dynamic inflow model, but the definition of V_{m+} itself remains the same. Few discussions were held about the definition of the mass flow parameter, V_{m+}, in part because the derivation of equation (2) has not clearly been disclosed in the past. References [**10**] and [**11**] are rare examples to try to extend the definition of V_{m+} so as to cover the vortex-ring and windmill-brake states.

Reference [**12**] is arguably the first study in which a necessary change to equation (2) for the windmill-brake state has been discussed. Reference [**12**] elucidated the modification of equation (2) to

$$\begin{aligned} V_{m-} &= V_{\mathrm{T}} - \lambda_m \frac{\partial V_{\mathrm{T}}}{\partial \lambda_m} = \frac{\mu^2 + \lambda(\lambda - \lambda_m)}{\sqrt{\mu^2 + \lambda^2}} \\ &= \frac{\mu^2 + (\lambda_{\mathrm{f}} + \lambda_m)\lambda_{\mathrm{f}}}{V_{\mathrm{T}}} \end{aligned} \tag{6}$$

where the minus sign in the subscript means the windmill-brake state, following the denomination used in reference [**12**]. In reference [**12**], the necessity of the sign change is discussed from the purely physical point of view, considering the fact that while in the normal working state the inflow is given energy and accelerated, in the wind-mill brake state the inflow is taken energy and decelerated. Also, regarding the definition of wake skew angle, χ, reference [**12**] confirmed that χ should be defined in Peters' manner as

$$\chi = \tan^{-1}\left|\frac{\mu}{\lambda}\right| \tag{7}$$

with modulus sign, unlike Chen's definition [**13**] which has no modulus sign for μ/λ. Since the normal working and windmill-brake states are somewhat separately treated in reference [**12**], questions naturally arising from this situation should include how to mathematically consistently explain the relationship between equations (2) and (6), and if it is possible to define the unified form of equations (2) and (6) so as to consistently describe both states.

2 CORRECTION OF THE CONVENTIONAL DEFINITION

In reference [**7**], in relation with the definition of the disc angle of attack, $\alpha \simeq \pi/2 - \chi$, Peters and HaQuang wrote, 'α is always positive, whether the flow is from above or below'. The latter case in which the flow is from below the rotor may indicate the windmill-brake state, thus it is believed that those authors designed equation (2) for both the normal working and windmill-brake states.

Differentiating equation (4) yields

$$\frac{\partial V_{\mathrm{T}}}{\partial \lambda_m} = \frac{\lambda}{V_{\mathrm{T}}} = \frac{\lambda_{\mathrm{f}} + \lambda_m}{V_{\mathrm{T}}} = \cos\chi \tag{8}$$

Since $\cos\chi \simeq \sin\alpha$ in high-speed forward flight, the right-hand side of equation (8) is required to be positive due to the above-mentioned requirement that $\tan\alpha$ should be positive. Note that λ_m is always positive by definition, but $\lambda_{\mathrm{f}} > 0$ in the normal working state and $\lambda_{\mathrm{f}} < 0$ in the windmill-brake state. Since $V_{\mathrm{T}} > 0$ and $\lambda_{\mathrm{f}} + \lambda_m < 0$ in the windmill-brake state, $\tan\alpha$ will be negative following equation (8), and hence becomes inconsistent with equation (7). However, the minus sign appearing in equation (6) before $\lambda_m(\partial V_{\mathrm{T}}/\partial\lambda_m)$ will make the term of $\partial V_{\mathrm{T}}/\partial\lambda_m$ positive, resulting in the same value of $V_{\mathrm{T}} + \lambda_m(\partial V_{\mathrm{T}}/\partial\lambda_m)$ as in the normal working state. Therefore, it can be said that equation (2) alone becomes inconsistent with equation (7) in the windmill-brake state, despite reference [**7**] mentioning the state in which the inflow is coming from below as well as the normal working state. This suggests a flaw in the definition of equation (2) in reference [**7**]. Separating the mass flow parameter into equations (2) and (6) for the normal working and windmill-brake states, respectively, makes the definition consistent with Peters' definition of the rotor angle of attack. This can be confirmed in axial flight too. In axial flight ($\mu = 0$)

$$V_{\mathrm{T}} = \sqrt{\lambda^2} = |\lambda| \tag{9}$$

Then, equation (2) yields

$$V_{m+} = 2\lambda_m + \lambda_{\mathrm{f}} \ (\text{for } \lambda_{\mathrm{f}} > 0) \tag{10}$$

$$V_{m+} = -2\lambda_m - \lambda_{\mathrm{f}} \ (\text{for } \lambda_{\mathrm{f}} < 0) \tag{11}$$

Equation (11) will give a negative $\tan\alpha$, which is inconsistent with equation (7), but this is not the case with the definition of equation (6).

Equations (2) and (6) can be unified as

$$V_{m\pm} = V_{\mathrm{T}} + \lambda_m \left| \frac{\partial V_{\mathrm{T}}}{\partial \lambda_m} \right| = \frac{\mu^2 + \lambda^2 + \lambda_m |\lambda|}{\sqrt{\mu^2 + \lambda^2}} \qquad (12)$$

for both $\lambda > 0$ in the normal working state and $\lambda < 0$ in the windmill-brake state.

Equation (12) captures the physical phenomena described by equations (2) and (6), and since it is always consistent with equation (7) (either in the normal working or windmill-brake states), the authors believe that equations (2) and (6) can be replaced with equation (12).

3 DISCUSSION

Regarding the definition of mass flow parameter, the discussion held in reference [**12**] is expatiated in this article so that the necessity of the modification to Peters' original definition for the windmill-brake state should be both more mathematically and consistently enunciated. In so doing, it is pointed out that Peters' definition of the mass flow parameter is not consistent with the definition of the disc angle of attack in the windmill-brake state [**7**]. Therefore, the conventional definition of the mass flow parameter should be replaced with the unified form, equation (12), so as to be consistently used both in the normal working and windmill-brake states. In fact, Glauert [**14**] pointed out that the mass flow should be defined with modulus sign so as to be positive either in the normal working or windmill-brake state as early as in 1927. The modulus sign in equation (12) can be also explained in the same manner as in reference [**14**], and Glauert's argument underpins the discussion in this article. Unification of these two approaches is therefore considered a timely and significant result.

4 CONCLUSIONS

Conclusions drawn from this article can be summarized as follows.

1. The original definition of the mass flow parameter by Peters is inconsistent with the original definition of the disc angle of attack in the windmill-brake state.
2. The relationship between the original definition of the mass flow parameter and its modified version for the windmill-brake state [**12**] is both mathematically and consistently enunciated and hence the necessity of the modification for the windmill-brake state is clearly confirmed than by the physical argument in reference [**12**].
3. The unified form of the mass flow parameters is proposed so as to consistently include Peters' original definition for the normal working state and modified for the windmill-brake state. It is suggested that Peter's definition should be replaced with the unified form so as to be consistent with the definition of the disc angle of attack.

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APPENDIX

Notation

$\{C_{\mathrm{T}}, C_{\mathrm{L}}, C_{\mathrm{M}}\}$	rotor load components, non-dimensionalized on $\rho\Omega_0^2R^5$
L	gain matrix

M	apparent mass matrix
R	rotor radius (m)
V	mass flow parameter, non-dimensionalized on $\Omega_0 R$
V_{m+}	non-dimensionalized mass flow parameter by Peters
V_{m-}	non-dimensionalized mass flow parameter for windmill-brake state
$V_{m\pm}$	unified mass flow parameter, non-dimensionalized
V_{T}	total flow at rotor disc, non-dimensionalized
V_∞	free stream speed, non-dimensionalized on $\Omega_0 R$
α	angle between free stream and rotor disc (rad)
λ	total inflow, non-dimensionalized on $\Omega_0 R$
λ_{f}	non-dimensionaized free stream inflow, $V \sin\alpha$
λ_m	non-dimensionalized inflow due to the rotor thrust
$\{\lambda_0, \lambda_{1s}, \lambda_{1c}\}$	inflow components, non-dimensionalized on $\Omega_0 R$
μ	advance ratio in rotor co-ordinates, $V \cos\alpha$
ρ	density of air (kg/m^3)
χ	wake skew angle (rad)