Wälde, K. (2008) Corrigendum to 'optimal saving under Poisson uncertainty' [J. Econ. Theory 87 (1999) 194–217]. *Journal of Economic Theory* 140(1):e5-e9.

Optimal Saving under Poisson Uncertainty: Corrigendum

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January 2002

The derivation of the Keynes-Ramsey rule in Wälde (1999) contains two errors. Correcting them affects the expression for the Keynes-Ramsey rule in (11) but leaves all other findings of the paper unaffected. It especially does not affect the major finding of the paper, i.e. the dichotomy in general equilibrium between a stochastic and a deterministic regime. As a consequence, the analysis of endogenous business cycles in very tractable models remains entirely valid.

1 Introduction

The paper "Optimal Saving under Poisson Uncertainty" by Wälde (1999) established several findings. After presenting the economic environment where uncertainty results from a Poisson process (section 2.1), it derived a household's budget constraint (section 2.2) and formulated the Bellman equation appropriate for the maximization problem of the household (section 2.3). Section 3 derived first order conditions that determine optimal consumption levels and an optimal allocation of savings across the risky and riskless investment form. Section 3 also derived the Keynes-Ramsey rule that determines optimal evolution of consumption over time. Section 4 put the household in a general equilibrium setup and showed that corner solution will be observed under standard assumptions: Households either allocate all savings to the riskless or the risky investment form, depending on relative (expected) returns.

The present note corrects two errors that were made in deriving the Keynes-Ramsey rule. These errors occured when using rules for computing differentials of functions that are not appropriate in a context where the arguments of the functions follow stochastic processes. The present note therefore corrects part of section 3 of Wälde (1999). All other findings of the paper remain valid (for example the steps made to derive the budget constraint, first order conditions and general equilibrium properties), as these other findings do not depend on the Keynes-Ramsey rule or on how this rule was derived. The main contribution of the paper, the simple analysis of endogenous business cycles in a general equilibrium context as in e.g. Wälde (2002) therefore remains valid and useful for future research.

The next section will compute the Keynes-Ramsey rule using the appropriate rules for computing differentials. The main difference lies in the fact that the evolution of consumption is no longer explicitly computed but implicitly through a rule that determines the evolution of marginal utility.

2 The Keynes-Ramsey rule

Wälde (1999) studied a household whose wealth a evolves according to

$$da = (ra + w - i - e)\,dt + \left(\varpi \frac{i}{I} - sa\right) dq. \tag{5*}$$

This is equation (5) in the original paper (which also explains the notation) and therefore marked by an asterisk (*). This household maximizes expected utility at each point in time t, given by

$$U(t) = \varepsilon \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\,d\tau. \tag{6*}$$

The marginal value of a unit of wealth $V_a(a, \gamma)$ is a function of both assets a and of the technological level γ. As assets a and the technological level γ are stochastic, computing the differential of $V_a(a, \gamma)$ requires an appropriate version of Ito's Lemma. Hence, one can not write

$$dV_a = V_{aa} da = V_{aa}[ra + w - i - e]\,dt + V_{aa}\left[\varpi \frac{i}{I} - sa\right] dq, \tag{25*}$$

where rules for computing differentials were used that apply only if all arguments of $V_a(.)$ follow deterministic paths.

Applying the modified, appropriate version of Ito's Lemma (Wälde, 1999, appendix 1), the differential of the marginal value reads

$$dV_a(a, \gamma) = V_{aa}(a, \gamma)[ra + w - i - e]\,dt + [V_{\tilde{a}}(\tilde{a}, \gamma + 1) - V_a(a, \gamma)]\,dq \tag{1}$$

where

$$\tilde{a} = \varpi \frac{i}{I} + (1 - s)\,a.$$

This was the first error in the derivation of the Keynes-Ramsey rule. It is important to note that Ito's Lemma is applied to the partial derivative of the function $V(.)$ with respect to the first argument. This means that the jump-term $V_{\tilde{a}}(\tilde{a}, \gamma+1) - V_a(a, \gamma)$ is a difference between partial derivatives with respect to first arguments and not a difference between partial derivatives with respect to a.

With $V_a \equiv V_a(a, \gamma)$, $V_{aa} \equiv V_{aa}(a, \gamma)$ and $V_{\tilde{a}} \equiv V_{\tilde{a}}(\tilde{a}, \gamma+1)$ and replacing $V_{aa}[ra + w - i - e]$ by

$$[\rho - r + \lambda] V_a(a) - [1 - s] \lambda V_{\tilde{a}}(\tilde{a}) = V_{aa}(a) [ra + w - i - e], \tag{10*}$$

we obtain

$$\begin{aligned} dV_a &= [(\rho - r + \lambda) V_a - (1 - s) \lambda V_{\tilde{a}}] dt + [V_{\tilde{a}} - V_a] dq \Leftrightarrow \\ \frac{dV_a}{V_a} &= \left[\rho - r + \lambda \left[1 - (1 - s) \frac{V_{\tilde{a}}}{V_a}\right]\right] dt + \left[\frac{V_{\tilde{a}}}{V_a} - 1\right] dq. \end{aligned} \tag{2}$$

Using the first order condition for consumption

$$u'(c) p^{-1} = V_a(a), \tag{8*}$$

we can express the differential of V_a in (2) as

$$dV_a = d\left(u'(c) p^{-1}\right). \tag{3}$$

Computing the differential on the right hand side requires assumptions concerning the evolution of consumption c and prices p. Again, we can not use simple rules for differentials that are valid only for deterministic paths of c and p, as was done on p. 213 in Wälde (1999), where it implicitly says $d(u'(c) p^{-1}) = u''(c) p^{-1} dc - u'(c) p^{-2} dp$. This was the second error.

By the first order condition (8*), consumption became a function of stochastic assets a and of the price p. Assume that the price level p follows a deterministic process

$$dp = g(.) p dt,$$

where $g(.)$ is some exogenously given function. This would be the case e.g. if p was chosen as numeraire.[1] The appendix then shows that applying the appropriate version of Ito's Lemma for (3) yields

$$d\left(u'(c) p^{-1}\right) = u'(c) dp^{-1} + p^{-1} du'(c) = -u'(c) p^{-2} dp + p^{-1} du'(c). \tag{4}$$

[1] The function $g(.)$ would of course be endogenously determined in general equilibrium.

Inserting (4) into (3) and dividing by (8*) yields

$$\frac{dV_a}{V_a} = \frac{-u'(c)\frac{dp}{p^2}}{u'(c)\,p^{-1}} + \frac{p^{-1}du'(c)}{u'(c)\,p^{-1}} = -\frac{dp}{p} + \frac{du'(c)}{u'(c)} \tag{5}$$

The correct version of the Keynes-Ramsey rule therefore reads with (2)

$$\begin{aligned} -\frac{du'(c)}{u'(c)} &= \left[r - \rho - \lambda\left[1 - \frac{V_{\tilde{a}}}{V_a}\right]\right] dt \\ &\quad - \left[\frac{V_{\tilde{a}}}{V_a} - 1\right] dq - \frac{dp}{p} \\ &= \left[r - \frac{dp/dt}{p} - \lambda - \rho\right] dt + [1 - s]\,\lambda\Omega dt + [1 - \Omega]\,dq, \end{aligned} \tag{6}$$

where dp/dt exists as we assumed p to be deterministic and where

$$\Omega = \frac{u'(c(\tilde{a}))\,p}{u'(c(a))\,\tilde{p}} = \frac{u'(c(\tilde{a}))}{u'(c(a))},$$

as $\tilde{p} = p$ also because of $f_4(.) = 0$ and *not*

$$\begin{aligned} -\frac{u''(c)}{u'(c)}dc &= \left[r - \frac{dp/dt}{p} - \lambda - \rho\right] dt + [1 - s]\,\lambda\Omega dt \\ &\quad - \frac{u''(c)}{u'(c)} c_a \left[\varpi\frac{i}{I} - sa\right] dq. \end{aligned} \tag{11*}$$

as was written in the original paper.

Interestingly, the Keynes-Ramsey rule (6) describes only the evolution of marginal utility and not the evolution of consumption. This is also true for the Keynes-Ramsey rule under uncertainty that is caused by Brownian motion (Turnovsky, 2000, ch.15.3).[2]

3 Appendix

We first proof a more general

Theorem 1 *Let* $F = F(x, y) = xy$ *and* $dx = f^x(x, y)\,dt + g^x(x, y)\,dq$ *and* $dy = f^y(x, y)\,dt + g^y(x, y)\,dq$. *Then* $dF = xdy + ydx + g^x(x, y)\,g^y(x, y)\,dq$.

[2] One might be tempted to apply Ito's Lemma to $du'(c)$ and compute this differential. This would give an expression that contains dc indeed, further terms would however only complicate matters and reduce insight.

Proof. RHS: Inserting dx and dy into the right hand side $xdy + ydx + g^x(.)\, g^y(.)\, dq$ gives

$$\begin{aligned} & x\left[f^y(.)\,dt + g^y(.)\,dq\right] + y\left[f^x(.)\,dt + g^x(.)\,dq\right] + g^x(.)\,g^y(.)\,dq \\ = & \left[xf^y(.) + yf^x(.)\right]dt + \left[xg^y(.) + yg^x(.) + g^x(.)\,g^y(.)\right]dq. \end{aligned}$$

LHS: By the appropriate version of Ito's Lemma (Wälde, 1999, appendix 1), the differential on the left-hand side is given by

$$dF = \left[F_x f^x(.) + F_y f^y(.)\right]dt + \left[F\left(x + g^x(.)\,, y + g^y(.)\right) - F(x,y)\right]dq.$$

Inserting $F(x,y) = xy$ yields

$$\begin{aligned} dF &= \left[yf^x(.) + xf^y(.)\right]dt + \left[\left(x + g^x(.)\right)\left(y + g^y(.)\right) - xy\right]dq \\ &= \left[xf^y(.) + yf^x(.)\right]dt + \left[xg^y(.) + yg^x(.) + g^x(.)\,g^y(.)\right]dq. \end{aligned}$$

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For our purposes in (4), we use the following

Corollary 2 *Let* $F = F(x,y) = xy$ *and* $dx = f^x(x,y)\,dt + g^x(x,y)\,dq$ *and* $dy = f^y(x,y)\,dt$*, i.e.* $g^y(x,y) = 0$*. Then* $dF = xdy + ydx$*.*

Replacing x by $u'(c)$ and y by p^{-1}, we obtain (4).

References

[1] Turnovsky, S. J., 2000. Methods of Macroeconomic Dynamics. 2nd edition. MIT Press, Cambridge, MA.

[2] Wälde, K., 1999, Optimal Saving under Poisson Uncertainty. Journal of Economic Theory 87: 194-217.

[3] Wälde, K., 2002, The Economic Determinants of Technology Shocks in a Real Business Cycle Model. Journal of Economic Dynamics and Control 27: 1 - 28.