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Landau damping is described in relativistic electron-positron plasmas. Relativistic electron-positron plasma theory contains important new effects when compared with classical plasmas. For example, there are undamped superluminal wave modes arising from both a continuous and discrete mode structure, the former even in the classical limit. We present here a comprehensive analytical treatment of the general case resulting in a compact and useful form for the dispersion relation. The classical pair-plasma case is addressed, for completeness, in an appendix.

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I. INTRODUCTION

In this paper we describe the phenomenon of linear Landau damping in an unmagnetised, collisionless electron-positron plasma. In order to be relevant to the modelling of astrophysical electron-positron plasmas, in which the particle thermal energy may be a significant fraction of its rest energy, it is essential to adopt a fully relativistic description of the Vlasov equation.

Landau damping has been a key aspect of collective behaviour in plasmas since the original paper by Landau [1]. Subsequent detailed analysis by Buti [2] and Lerche [3] produced additional mathematical insight into the rich physics of the phenomenon. Collisionless damping has become important in a wider context (see the reviews by Ryutov [4] and Ivanov [5]) and other recent works [6–10] show that progress in understanding, and modelling, this topic is still being made, albeit in classical electron-ion plasmas. For example, experimental measurements of classical linear collisionless damping [11] show quantitative agreement with Landau’s treatment for low-energy electron-ion plasmas.

For our contribution to the discussion we present the full, relativistic treatment of Landau damping in electron-positron plasmas.

We show that Landau damping of a relativistic electron-positron plasma exhibits many novel features, including undamped superluminal waves, which we shall show are also relevant to the classical case. The key parameter is a , the ratio of rest to thermal energy of a plasma particle:

$$a = mc^2/(k_B T) \quad (1)$$

Large a corresponds to the classical limit; as $a \rightarrow 0$, the plasma is ultra-relativistic. The damping of wavemodes is a complicated function of a , but general points can be made. A key aspect of our analysis is the treatment of the singularity present in the momentum integration.

The conventional, purely classical approach, can be written entirely in terms of the plasma dispersion function. However, if the relativistic approach is used, the location of the singularity becomes a more complicated function of the momentum, and the subsequent integration yields a more structured wave response.

A striking feature is the existence of stable undamped propagating electrostatic waves, which is essentially a relativistic mechanism. These were proposed (in electron-ion plasmas) by Buti [2] and presented in more detail by Lerche [3]. Further exploration and modelling by Schlickeiser [6] and Schlickeiser and Kneller [7] revealed more properties, with a measure of numerical verification by Sartori and Coppa [9]. The particle-wave resonance condition essential for classical Landau damping is modified by the presence of the Lorentz factor $\gamma = [1 + \mathbf{p}^2/(mc)^2]^{1/2}$, where \mathbf{p} is the particle momentum. As a result, if the wave phase velocity exceeds c there exists a mode in which the singularity in the Landau description is suppressed, and the usual Landau treatment of the singularity which leads to damping is not relevant.

This article is structured in the following way. Section II describes the linear response of a relativistic electron-positron plasma to an electrostatic wave. Section III contains a summary of results, together with general remarks and conclusions. A table of notation used in this article is presented in Appendix A, and Appendices B and C present details of certain critical integrals. For completeness, Appendix D presents the theory of Landau damping for the classical electron-positron plasma.

II. RELATIVISTIC ELECTRON-POSITRON PLASMAS

The starting point is the Vlasov equations for positrons and electrons written for momentum-space, together

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with Poisson's equation:

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{r}} + e\mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \quad (2)$$

$$\frac{\partial g}{\partial t} + \mathbf{u} \cdot \frac{\partial g}{\partial \mathbf{r}} - e\mathbf{E} \cdot \frac{\partial g}{\partial \mathbf{p}} = 0 \quad (3)$$

$$\frac{en}{\epsilon_0} \int (f - g) d\mathbf{u} = \nabla \cdot \mathbf{E} \quad (4)$$

where f and g are the distribution functions for positrons and electrons respectively, and \mathbf{p} is the particle momentum. We consider linearised equations for small amplitude waves. Positrons and electrons will be taken to have the same equilibrium distribution function, which in this case is the relativistic counterpart of the Maxwellian:

$$f_0(\mathbf{p}) = g_0(\mathbf{p}) = (4\pi m^3 c^3)^{-1} \frac{a}{K_2(a)} e^{-a\gamma} \quad (5)$$

in which $\gamma = [1 + p^2/(m^2 c^2)]^{1/2}$ is the usual Lorentz factor; $a = mc^2/(k_B T)$ is the characteristic temperature parameter for a relativistic plasma, and K_2 is the modified Bessel function of the second kind, of order 2. Note that $\mathbf{p} = \gamma m \mathbf{u}$.

Following the usual steps we arrive at the dispersion relation

$$2 \frac{m\omega_p^2}{k^2} \int \frac{\mathbf{k} \cdot \partial f_0 / \partial \mathbf{p}}{\mathbf{k} \cdot \mathbf{u} - \omega} d\mathbf{p} = 1 \quad (6)$$

where

$$\mathbf{k} \cdot \frac{\partial f_0}{\partial \mathbf{p}} = -(4\pi m^3 c^3)^{-1} \frac{a^2}{K_2(a)} \frac{e^{-a\gamma}}{m^2 c^2 \gamma} \mathbf{k} \cdot \mathbf{p} \quad (7)$$

Define $\mathbf{q} = \mathbf{p}/(mc)$ to get Eq. (6) in the form

$$\frac{\omega_p^2 a^2}{2\pi c^2 k^2 K_2(a)} \int \frac{\mathbf{ck} \cdot \mathbf{q} e^{-a\gamma}}{\gamma\omega - c\mathbf{k} \cdot \mathbf{q}} d\mathbf{q} = 1 \quad (8)$$

The presence of γ in the integral above complicates its subsequent analytical development. Other authors (for example [12]) have made some analytical progress for a restricted range of problems by approximating to a one-dimensional distribution function in velocity space.

In this article we shall retain full generality by adopting a spherical polar co-ordinate system for \mathbf{q} , with \mathbf{k} along the polar axis. Defining M to be the integral in Eq. (8) we have

$$M = 2\pi \int_0^\infty dq q^2 \int_0^\pi \frac{kcq \cos \theta \sin \theta}{\gamma\omega - ckq \cos \theta} e^{-a\gamma} d\theta \quad (9)$$

$$= 2\pi \int_0^\infty q^2 e^{-a\gamma} dq \int_{-1}^1 \frac{q\mu}{\gamma\xi - q\mu} d\mu, \quad (10)$$

$$= -4\pi \int_0^\infty q^2 e^{-a\gamma} dq - 2\pi\xi \int_0^\infty \gamma q e^{-a\gamma} \log \left(\frac{\gamma\xi - q}{\gamma\xi + q} \right) dq \quad (11)$$

$$= -4\pi K_2(a)/a - 2\pi\xi \int_{-\infty}^\infty \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq \quad (12)$$

where $\xi = \omega/(kc)$, $\mu = \cos \theta$ and we have used the integral expression

$$\int_0^\infty q^2 e^{-a\gamma} dq = \frac{K_2(a)}{a} \quad (13)$$

Full details as to how this is established are given in Appendix B.

In the integral in Eq. (12), the term $\log(\gamma\xi - q)$ has a branch point in the complex plane of q where $\gamma\xi = q$. For the classical case (Appendix D) it can be shown that the only solutions are damped waves, that is, waves having $\Im(\omega) < 0$. However there is the possibility here of a purely real ω .

A. General Treatment

Consider here the case of ξ complex, with $\Im(\xi) < 0$. Let $\xi = x - iy$, x, y real and positive. The integral possesses a singular point q_0 , defined by

$$\xi \sqrt{1 + q_0^2} = q_0 \quad (14)$$

that is,

$$q_0 = \frac{x - iy}{[1 - (x - iy)^2]^{1/2}} = q_{0r} - iq_{0i} \quad (15)$$

Note that we only have the positive q_{0r} solution, since there is only one singularity in the integral when written in the form Eq. (11). The consequence of the singularity is that there is a branch point at q_0 , as shown in Figure 1. The full expression for the dispersion relation is

$$\xi \int_\Gamma \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq + (S^2 + 2) \frac{K_2(a)}{a} = 0 \quad (16)$$

where $S = k\lambda_D$.

We must be clear as to what is meant by the integration contour. The full details of this integration are given in Appendix C; in the interests of brevity and clarity, we state here only the general method and the final result.

Integrating along the contour Γ as shown in Figure 1, note that the contribution from the circular contour CDE vanishes in the limit of its radius vanishing *except* that on following this route the logarithm switches to the upper branch because of the branch cut. Also, the linear segments BC and EF produce contributions which cancel exactly except for the extra part generated by the branch cut. The final form of the dispersion relation, taking into account the branch cut corrections, is

$$\begin{aligned} & \xi \int_{-\infty}^{+\infty} \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq \\ & + i \frac{2\pi\xi}{a^3} (a^2 \psi^2 + 2a\psi + 2) e^{-a\psi} \\ & + (S^2 + 2) \frac{K_2(a)}{a} = 0 \end{aligned} \quad (17)$$

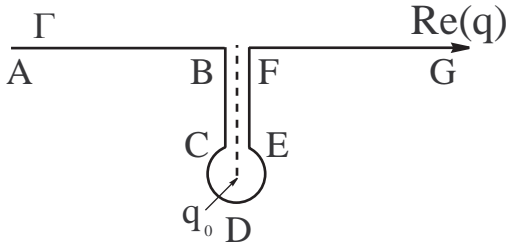


FIG. 1: The integration contour for the integral in Eq. (16). The branch cut is shown as the broken line.

where

$$\psi = \left[1 + (q_{0r} - iq_{0i})^2\right]^{1/2} \quad (18)$$

$$= (1 - \xi^2)^{-1/2} \quad (19)$$

and where in the integral in Eq. (17) the logarithm term is evaluated in the principal branch:

$$\log[\gamma(x - iy) - q] = \frac{1}{2} \log[(\gamma x - q)^2 + \gamma^2 y^2] - i \tan^{-1} \left(\frac{\gamma y}{\gamma x - q} \right) \quad (20)$$

taking the inverse tangent to lie in the range $[0, \pi]$. Hence the full dispersion relation can be written in the form

$$\begin{aligned} & \frac{1}{2} \xi \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \log[(\gamma x - q)^2 + \gamma^2 y^2] dq \\ & - i \xi \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \tan^{-1} \left(\frac{\gamma y}{\gamma x - q} \right) dq \\ & + i \frac{2\pi\xi}{a^3} (a^2 \psi^2 + 2a\psi + 2) e^{-a\psi} \\ & + (S^2 + 2) \frac{K_2(a)}{a} = 0 \end{aligned} \quad (21)$$

in which the treatment of the branch cut is explicitly given.

B. Relationship with classical Landau damping

In Eq. (D9) we state that the classical dispersion relation for electron-positron plasmas is given by

$$\zeta \mathcal{Z}(\zeta) + \frac{1}{2} S^2 + 1 = 0 \quad (22)$$

We now show that Eq. (17) reduces to Eq. (22) in the non-relativistic limit, that is, $a \rightarrow \infty$. In this limit, let

$$\gamma \approx 1 + \frac{1}{2} q^2 \quad (23)$$

$$\xi \ll 1 \quad (24)$$

Then to the lowest order, the integral in Eq. (17) can be written

$$\int_{-\infty}^{\infty} q e^{-\frac{1}{2} a q^2} \log(\xi - q) dq \quad (25)$$

which can be integrated by parts to yield

$$\frac{1}{a} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} a q^2}}{\xi - q} dq$$

The dispersion relation in the large a limit now takes the form

$$\begin{aligned} & -\frac{\xi}{a} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2} a q^2}}{\xi - q} dq \\ & + \frac{2\pi i}{a} \xi e^{-\frac{1}{2} a \xi^2} + \sqrt{\frac{\pi}{2}} a^{-3/2} (2 + S^2) \approx 0 \end{aligned} \quad (26)$$

Using the notation

$$\bar{q} = \sqrt{\frac{a}{2}} q$$

$$\bar{\xi} = \sqrt{\frac{a}{2}} \xi$$

Eq. (26) can be written eventually as

$$\frac{\bar{\xi}}{\sqrt{\pi}} \left(\int_{-\infty}^{\infty} \frac{e^{-\bar{q}^2}}{\bar{q} - \bar{\xi}} d\bar{q} + 2\pi i e^{-\bar{\xi}^2} \right) + 1 + \frac{1}{2} S^2 \approx 0 \quad (27)$$

which is indeed the classical limit given in Eq. (22), noting that the expression in parenthesis in the above equation is the integral along the real line plus $2\pi i$ times the residue.

C. Undamped Solutions

Undamped solutions are those for which the imaginary part of the frequency is identically zero. There are two cases in which this can be satisfied here: (i) let $y \rightarrow 0$ in the full expression; and (ii) demand that the argument to the logarithm is positive definite for all q in the general form of the dispersion relation. In case (i), the imaginary part of the whole dispersion relation has to vanish, including the branch cut contribution, in the limit that $\Im(\xi)$ vanishes, leading to a set of discrete x values. In case (ii), the branch cut analysis is not relevant, and so makes no contribution. This leads to a limited but continuous spectrum of undamped modes.

1. case(i)

Here we have to ensure that the imaginary part of the dispersion relation vanishes, and simultaneously $\Im(\xi) = 0$. The full dispersion relation Eq. (21) can be written explicitly as real and imaginary parts as follows:

$$\tau_1 + \tau_2 + \tau_3 = 0 \quad (28)$$

where

$$\tau_1 = (x - iy)(A - iB) \quad (29)$$

$$A = \frac{1}{2} \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \log[(\gamma x - q)^2 + \gamma^2 y^2] dq \quad (30)$$

$$B = \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \tan^{-1}\left(\frac{\gamma y}{\gamma x - q}\right) dq \quad (31)$$

$$\tau_2 = i \frac{2\pi\xi}{a^3} (a^2 \psi^2 + 2a\psi + 2) e^{-a\psi} \quad (32)$$

and

$$\tau_3 = (S^2 + 2) \frac{K_2(a)}{a} \quad (33)$$

Note that τ_3 is always real. In the limit $y \rightarrow 0$, $\Im(\tau_1) \rightarrow 0$ leaving only the imaginary part of τ_2 as the restriction on x , if the dispersion relation is to be satisfied for a discrete undamped mode. This yields, for $x > 1$,

$$\Im(\tau_2) = \frac{2\pi}{a^3} x [2\Phi \sin \Phi + (2 - \Phi^2) \cos \Phi] \quad (34)$$

$$\Phi = a(x^2 - 1)^{-1/2} \quad (35)$$

and if $x < 1$,

$$\Im(\tau_2) = \frac{2\pi}{a^3} x \left[\frac{2a}{(1 - x^2)^{1/2}} + \frac{a^2}{1 - x^2} + 2 \right] \times \exp\left[-\frac{a}{(1 - x^2)^{1/2}}\right] \quad (36)$$

There is no value of x other than 0 for which Eq. (36) can be made to vanish. Therefore we conclude that for $x < 1$ all wave solutions have non-zero damping. However, for $x > 1$, $\Im(\tau_2)$ vanishes if

$$\tan \Phi = \frac{\Phi^2 - 2}{2\Phi} \quad (37)$$

As an example, Eq. (37) is satisfied if $\Phi = 4.222$. For the case $a = 2$ this yields a value of $x = 1.107$, and numerically it can be shown that there is zero damping for $\Omega_r = 12.30$ and $S = 11.11$. This is to be compared with the continuum case (ii) (to be discussed below), in which $x = 1.107$ leads uniquely to $\Omega_r = 0.870$, $S = 0.786$ (see Fig. 7). Clearly these are two quite different modes of wave propagation.

Similarly, the case $a = 5$ yields $x = 1.55$, with an associated $\Omega_r = 88.9$, and $S = 36.3$ for the discrete mode (see Fig. 6), and $\Omega_r = 0.877$, $S = 0.358$ for the continuum case (see Fig. 7).

Note that the discrete modes are very high frequency, short wavelength waves; while this is consistent with the relativistic Vlasov theory upon which this article is based, a comprehensive kinetic analysis embracing additional physical effects would involve a consideration of collisional terms appropriate for these scales. We hope such a study will be presented in the near future.

As $x \rightarrow 1$ for fixed a , or equivalently, as $a \rightarrow \infty$ for a given x , $\Phi \rightarrow \infty$, and so solutions that satisfy Eq. (37) for large Φ take the form

$$\Phi = (2n + 1) \frac{\pi}{2} - \epsilon_n \quad (38)$$

where $n \gg 1$ is integer, and $\epsilon_n > 0$, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. However, not all of these solutions will produce real values of S , since

$$\sin\left[(2n + 1) \frac{\pi}{2}\right] = (-1)^n$$

and the contribution from τ_2 to the real part of the dispersion relation can be approximated, in the limit $x \rightarrow 1$, by

$$\Re(\tau_2) \approx (-1)^n 2\pi \frac{\Phi^2}{a^2} \quad (39)$$

Given that τ_1 and τ_3 are both real and positive, then only odd values of n will yield physically acceptable solutions.

This analysis for the discrete modes shows that for $x > 1$, that is for such waves with superluminal phase velocity, there is no damping term, irrespective of whether a is small or large. Hence we confirm that there are classical and relativistic superluminal modes which are not damped, as deduced by [2] in the ultra-relativistic limit $a \ll 1$, and derived by [6] using the argument principle for contours in the complex frequency plane.

2. case(ii)

Consider the case of ξ purely real, $\xi = x$, say. The condition for a singular point is that $x\sqrt{1 + q^2} - q = 0$; clearly if $x > 1$ then this can never be satisfied. Hence for waves with phase velocity greater than c there is no singular point, and no damping term arising from the branch point. The dispersion relation then takes on the simple form

$$x \int_{-\infty}^{+\infty} \gamma q e^{-a\gamma} \log(\gamma x - q) dq + (S^2 + 2) \frac{K_2(a)}{a} = 0 \quad (40)$$

The case $x = 1$:

Consider the case of $x = 1$, which is a limiting case for zero damping. The integral in Eq. (40) takes the form

$$- \int_{-\infty}^{+\infty} t \cosh^2(t) \sinh(t) e^{-a \cosh(t)} dt = I_0 \quad (41)$$

on changing the integration variable from q to t , where $q = \sinh(t)$. Noting that $d/dt(\exp(-a \cosh(t))) =$

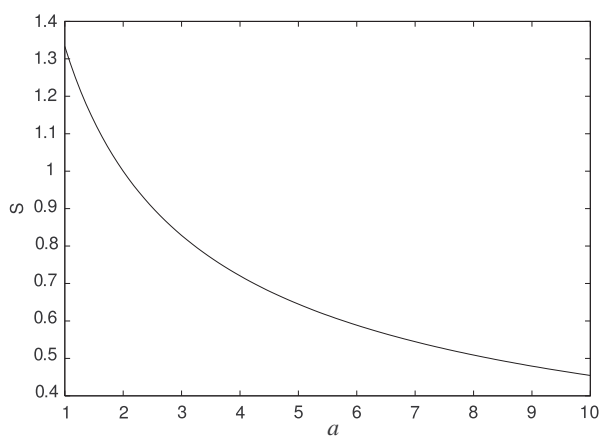


FIG. 2: Graph of S (y-axis) as a function of a (x-axis) for the limiting case of $\omega = kc$, using Eq. (46).

$-a \sinh(t) \exp(-a \cosh(t))$, integration by parts yields

$$I_0 = -\frac{1}{a} \int_{-\infty}^{+\infty} (\cosh^2 t + 2t \cosh t \sinh t) e^{-a \cosh t} dt \quad (42)$$

$$= -\frac{2}{a} [K_0(a) + K_1(a)/a] - \frac{2}{a^2} \int_{-\infty}^{+\infty} [t \cosh t \sinh t] e^{-a \cosh t} dt \quad (43)$$

$$= -\frac{2}{a} \left(1 + \frac{2}{a^2}\right) K_0(a) - \frac{6}{a^2} K_1(a) \quad (44)$$

The relation between S and a for the case $x = 1$ (that is, for the case $\omega = kc$) can now be written in the form

$$S^2 = -2 + 2 \left(1 + \frac{2}{a^2}\right) \frac{K_0(a)}{K_2(a)} + \frac{6}{a} \frac{K_1(a)}{K_2(a)} \quad (45)$$

$$= \frac{2}{a} \frac{2K_0(a)/a + K_1(a)}{K_2(a)} \quad (46)$$

and is presented as a plot in Figure 2. This agrees with the analysis of [6] and the numerical simulations of [9]. Notice that there is always a physically real solution for S , though as $a \rightarrow \infty$, $S \rightarrow 0$ from the asymptotic expansion of Eq. (46), showing that there is strictly no solution for such wave modes in the traditional classical treatment. However, the asymptotic behaviour for large a does give some insight:

$$S^2 + 2 \sim 2 + \frac{2}{a} \left(1 + \frac{1}{2a}\right), \quad x \gg 1 \quad (47)$$

which can be written in the following form, using Eqs. (D14, 1)

$$\omega^2 = k^2 c^2 \sim 2\omega_p^2 \left(1 + \frac{1}{2a}\right), \quad x \gg 1 \quad (48)$$

It is clear that the frequency is always above the plasma frequency, but recall that this is a limiting case for undamped wave solutions with $\xi > 1$.

The ultra-relativistic case $a \rightarrow 0$ yields the following dispersion relation, noting that $K_0(a)/K_2(a) \approx -\frac{1}{2}a^2 \log(a)$, and $K_1(a)/K_2(a) \approx \frac{1}{2}a$:

$$\omega^2 \approx 2 \frac{c^2}{\lambda_D^2} \log(1/a) \quad (49)$$

which agrees with Eq. (59) of [6].

The case $x \gg 1$:

The case of $x \gg 1$ is also tractable. In this limit, the integral in Eq. (40) is most usefully expressed as

$$\int_0^\infty \gamma q e^{-a\gamma} \log\left(\frac{1 - q/(\gamma x)}{1 + q/(\gamma x)}\right) dq \quad (50)$$

The logarithm can now be Taylor expanded, given that $q/(\gamma x) \ll 1$. To order x^{-4} the integral in Eq. (50) becomes

$$-2 \int_0^\infty q^2 \left(1 + \frac{q^2}{3x^2\gamma^2} + \frac{q^4}{5x^4\gamma^4}\right) e^{-a\gamma} dq \quad (51)$$

The dispersion relation Eq. (40) reduces to

$$\Omega^4 - \frac{1}{3}\Omega^2 a^2 \frac{I(a)}{K_2(a)} - \frac{1}{10}S^2 a^3 \frac{J(a)}{K_2(a)} = 0 \quad (52)$$

where $\Omega = \omega/(2\omega_p^2)^{1/2}$ and

$$I(a) = \int_0^\infty \frac{q^4}{\gamma^2} e^{-a\gamma} dq \quad (53)$$

$$J(a) = \int_0^\infty \frac{q^6}{\gamma^4} e^{-a\gamma} dq \quad (54)$$

As $a \rightarrow \infty$, $K_2(a) \rightarrow \sqrt{\pi/2} e^{-a} a^{-1/2}$. Since this is a classical, non-relativistic limit, we can approximate $I(a)$ as follows:

$$I(a) \rightarrow \int_0^\infty q^4 e^{-a(1+q^2/2)} dq = 3\sqrt{\pi/2} a^{-5/2} e^{-a} \quad (55)$$

Similarly,

$$J(a) \rightarrow 15\sqrt{\pi/2} a^{-7/2} e^{-a} \quad (56)$$

Hence, as $a \rightarrow \infty$, Eq. (52) becomes

$$\Omega^4 - \Omega^2 - \frac{3}{2}S^2 = 0 \quad (57)$$

which is the classical limit given in Eq. (D18). However, note here that we are discussing specifically waves which are superluminal and undamped. The fact that the classical dispersion relation is recovered in an asymptotic limit tells us that the classical description of plasma oscillations contains both the traditional damped mode of Landau [1] represented by Eq. (D18), or more correctly,

by Eq. (D17), and also an undamped solution given by Eq. (52). This agrees with the analysis of Schlickeiser [6], who notes the presence of undamped superluminal modes in his treatment.

The ultra-relativistic limit $a \rightarrow 0$ can also be treated here. Note that $I(a)$ and $J(a)$ have the same limiting behaviour as $a \rightarrow 0$:

$$I(a) \sim 2/a^3, \quad J(a) \sim 2/a^3, \quad a \rightarrow 0 \quad (58)$$

so that the dispersion relation takes the form

$$\Omega^4 - a\Omega^2/3 - a^2S^2/10 \approx 0. \quad (59)$$

Given that $x \gg 1$, we can recast the dispersion relation Eq. (59) in terms of x and S to get the solutions

$$x^2 \approx \frac{1}{3S^2} \left(1 \pm \sqrt{1 + 18S^2/5} \right) \quad (60)$$

Since $x \gg 1$, S must be small, leaving the only physically consistent solution as

$$x^2 \approx \frac{2}{3S^2} + \frac{3}{5} + \dots \quad (61)$$

which can be expressed as

$$\omega^2 \approx \frac{2c^2}{3\lambda_D^2} + \frac{3}{5}k^2c^2 + \dots \quad (62)$$

in agreement with Eq. (58) of [6].

D. Calculating the dispersion relation

In the case of classical Landau damping we outline in Appendix D the procedure for obtaining solutions independently of the density and temperature of the plasma, via Ω - S plots, in which Ω is normalised with respect to temperature, and S scales with density. An Ω - S plot is non-dimensional, and valid for all values of temperature and number density.

The relativistic counterpart follows the same process, but the temperature behaviour this time is properly associated with a . The natural parameters to choose here are $\bar{\xi}$ and S , where

$$\bar{\xi} = \sqrt{a/2}\xi = \Omega/S = (\Omega_r - i\Omega_i)/S \quad (63)$$

where $\Omega = \omega/(\sqrt{2}\omega_p)$

The dispersion relation is split into real and imaginary parts, noting that the real part involves the normalised wavenumber S , as in the classical case.

The results of the full dispersion relation calculations are presented in Figures 3 to 6, which show combined contour plots similar in style to the classical case of Figure 9, but this time for finite values of a . The case $a = 50$, in Figure 3, differs from the strictly classical case in that the ‘fingers’, along which allowable values of S are contoured, are twisted clockwise as a result of the relativistic

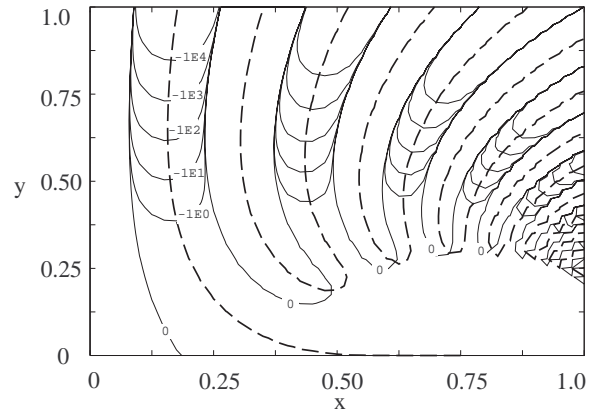


FIG. 3: Combined contour plot of the real part of the integral in Eq. (16) (solid lines) and the zero contour of the imaginary part of Eq. (16) (dashed lines), for the case $a = 50$. The horizontal axis is $x = \Re(\xi)$; vertical axis is $y = \Im(\xi)$. Allowable wave modes have ω and k calculated from the intersection of the dashed line with those solid line contours that are negative definite, corresponding to positive definite values of $(S^2 + 2)K_2(a)/a$. Sample values of such contours are given on the graph.

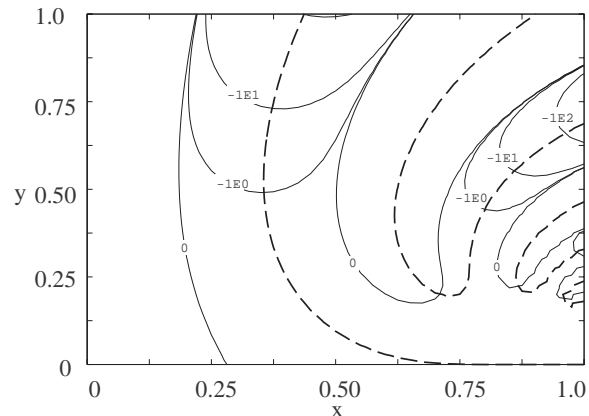


FIG. 4: Caption is the same as for Figure 3, but for the case $a = 20$.

influence. This twist becomes progressively more pronounced as a decreases. Physically relevant modes trace those broken lines that follow paths between the ‘fingers’, where contours of $-(S^2 + 2)K_2(a)/a$ are shown. The relative damping for a given solution can be determined by comparing the x and y values for a given point. Undamped modes for $x \geq 1$ are revealed by the intersections of the broken lines with the $y = 0$ axis. Note that these intersections correspond to the discrete modes that are the solutions to Eq. (35, 37). The continuum undamped modes derived from Eq. (40) are not represented in these contour plots, but instead two examples of the dispersion relations for such waves (for the cases $a = 2$ and $a = 5$) are given in Fig. 7.

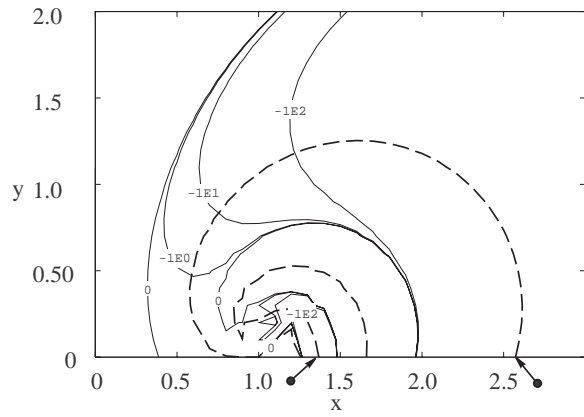


FIG. 5: Caption is the same as for Figure 3, but for the case $a = 10$. Discrete undamped solutions are indicated by the arrows below the x-axis.

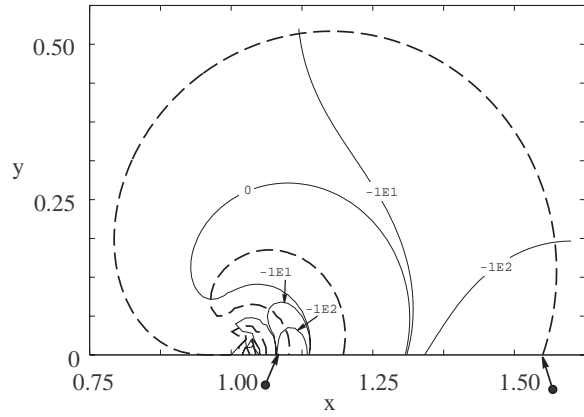


FIG. 6: Caption is the same as for Figure 3, but for the case $a = 5$. Discrete undamped solutions are indicated by the arrows below the x-axis.

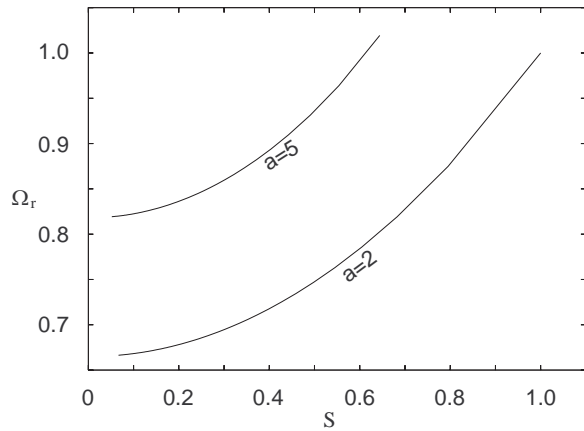


FIG. 7: The dispersion relation for the superluminal continuum mode for the cases $a = 2$ and $a = 5$, showing Ω_r as a function of S .

The conventional classical treatment of Landau damping in plasmas, based essentially entirely on the plasma dispersion function, $\mathcal{Z}(\zeta)$, is the usual formulation for describing collisionless damping in the long wavelength limit. In this article (Appendix D) we have revisited the classical case in detail for a pair plasma, and presented the higher mode behaviour that comes from a meticulous exposition of the characteristics of $\mathcal{Z}(\zeta)$ for the case where each plasma species has equal mobility.

Where the energetics of the plasma (and mathematical rigour) preclude the classical approach, we have generalised the modelling to calculate Landau damping for relativistic plasmas, and in so doing, we can embrace the traditional classical approach as a special, limiting case. As a result we have been able to present a comprehensive analytical formulation of the general case, with selected examples evaluated for clarity and detailed interpretation. The results can be summarized in the following bullet points.

- For subluminal waves, that is for $x < 1$, all wave modes, for all values of a , have non-zero damping. The concept of mode limitation ([8]) is also confirmed: the relativistic treatment permits only a finite number of subluminal damped modes.
- For superluminal waves, that is for $x > 1$, there is no damping term for solutions that arise as a result of the continuum analysis irrespective of whether a is small or large. In particular, classical superluminal modes are undamped. This is in complete agreement with Lerche [3] and Schlikeiser and Kneller [6–8], albeit that the analysis in this article is for pair plasmas.
- There are also relativistic superluminal modes which generally exhibit damping, but there is a discrete spectrum of these modes for which the damping is precisely zero. These modes are not presented in Lerche [3] and Schlikeiser and Kneller [7]; the formal statement of the dispersion relation is identical to that in [8], for example: Eq. 21 and Eq. (A8) in [8] are the same. However, the choice of variable in our treatment allows a further analytical integration step that is not accessible in [8], affording additional insight into the wave character. For these particular modes, there is continuity in the damping term in the transition from subluminal to superluminal (see for example Figs. 5 and 6). To this extent, we do not see the discontinuous damping term inferred in the mode completion treatment of [8].

The real utility of the analysis presented here is that the general dispersion relation is expressed in compact form in Eq. (21); it is simply a matter of quadrature to evaluate the solutions for parameters of interest. The

special case of continuum superluminal undamped modes is also a simple expression, Eq. (40), which once again yields results on numerical integration.

Relativistic undamped electrostatic oscillations in a pair plasma may be relevant to the pulsar environment, in which electrostatic oscillations are a significant collective effect in the particle dynamics of trapped pair-plasmas [15]; these waves can couple to electromagnetic modes, and therefore act as moving sources of coherent radiation. Such a mechanism may offer a new interpretation of drifting subpulses in pulsars, which have been the subject of renewed interest recently [16]. We hope to report on such an application in a subsequent article.

IV. ACKNOWLEDGMENTS

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APPENDIX A: NOTATION

In the interests of clarity, some of the key notation used in this paper is detailed in the table below.

APPENDIX B: INTEGRALS INVOLVING MODIFIED BESSEL FUNCTION

Several results involving the modified Bessel function of the second kind, $K_n(a)$, have been quoted in this article. They all stem from the general relation

$$K_n(a) = \left(\frac{a}{2}\right)^n \frac{\Gamma(1/2)}{\Gamma(n+1/2)} \int_1^\infty e^{-at} (t^2 - 1)^{n-1/2} dt \quad (\text{B1})$$

In Eq. (12) we quote the result

$$\int_0^\infty q^2 e^{-a\gamma} dq = K_2(a)/a \quad (\text{B2})$$

This arises from setting $n = 1$ in Eq. (B1), and then changing the variable of integration from q to $t = \gamma = (1 + q^2)^{1/2}$. Then

$$\int_0^\infty q^2 e^{-a\gamma} dq = \int_1^\infty e^{-at} (t^2 - 1)^{1/2} t dt \quad (\text{B3})$$

$$= -\frac{\partial}{\partial a} \int_1^\infty e^{-at} (t^2 - 1)^{1/2} dt \quad (\text{B4})$$

$$= -\frac{d}{da} \left(\frac{K_1(a)}{a} \right) \quad (\text{B5})$$

$$= \frac{K_2(a)}{a} \quad (\text{B6})$$

where we have used the recurrence relation for derivatives with respect to argument of $K_n(z)$ [17, 18].

TABLE I: Selected notation used in this paper

Symbol	Definition
a	$= mc^2/(k_B T)$
\mathbf{E}	electric field
c_s	plasma sound speed
f	positron distribution function
f_0	equilibrium distribution for positrons
g	electron distribution function
g_0	equilibrium distribution for electrons
k	wavenumber
m	e^\pm mass
\mathbf{p}	particle momentum
q	$\Im(\zeta) = -qx$
\bar{q}	$= (a/2)^{1/2} q$
\mathbf{q}	$= \mathbf{p}/(mc)$
S	$= k\lambda_D$
T	common temperature of e^\pm
\mathbf{u}	particle velocity
x	generic real part: $= \Re(\xi), \Re(\zeta)$
y	generic imaginary part: $= -\Im(\xi), -\Im(\zeta)$
z	$= u/\alpha$
\mathcal{Z}	Plasma Dispersion Function
α	$= (2k_B T/m)^{1/2}$
γ	$= [1 + (\mathbf{p}/(mc))^2]^{1/2}$
ζ	$= \omega/(k\alpha) = \Omega/S$
λ_D	Debye length for e^\pm
ξ	$= \omega/(kc)$
$\bar{\xi}$	$= (a/2)^{1/2} \xi$
ϕ	electric potential
Φ	$= a(x^2 - 1)^{-1/2}$
ω	(complex) wave frequency
ω_p	plasma frequency for e^\pm
Ω	$= \omega/(2\omega_p^2)^{1/2}$

APPENDIX C: INTEGRATION ALONG THE CONTOUR Γ

Using Fig. 1, it is clear that the integral defined in Eq. (16) is made up of the following contributions.

$$\int_{AB} = \int_{-\infty}^{q_{0r}} \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq, \quad \xi = x - iy \quad (\text{C1})$$

$$\int_{BC} = -i \int_0^{q_{0i}} \gamma_0 (q_{0r} - iq) e^{-a\gamma_0} \log(\gamma_0\xi - q_{0r} + iq) dq \quad (\text{C2})$$

where $\gamma_0 = [1 + (q_{0r} - iq)^2]^{1/2}$.

$$\int_{CDE} = 0, \quad (\text{C3})$$

a zero contribution in the limit that the radius of the circle $\rightarrow 0$, but the logarithmic term switches to the upper branch, increasing by $2\pi i$ because of the branch cut.

$$\int_{EF} = -i \int_{q_{0i}}^0 \gamma_0 (q_{0r} - iq) e^{-a\gamma_0} \times [\log(\gamma_0\xi - q_{0r} + iq) + 2\pi i] dq \quad (\text{C4})$$

$$\int_{FG} = \int_{q_{0r}}^{\infty} \gamma q e^{-a\gamma} [\log(\gamma\xi - q) + 2\pi i] dq \quad (C5)$$

noting that the upper branch still applies.

Contributions from BC and EF cancel except for the additional term $2\pi i$, so that the net contribution from BC and EF is

$$-2\pi i \int_{q_{0r}}^{q_{0r}-iq_{0i}} (1 + \beta^2)^{1/2} \beta e^{-a(1+\beta^2)^{1/2}} d\beta \quad (C6)$$

where $\beta = q_{0r} - iq$ is a convenient new integration variable. The full contribution from AB, BC, EF and FG can then be written as

$$\begin{aligned} & \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq \\ & + 2\pi i \left(\int_{q_{0r}}^{\infty} \gamma q e^{-a\gamma} dq + \int_{q_{0r}-iq_{0i}}^{q_{0r}} \gamma q e^{-a\gamma} dq \right) \\ = & \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq + 2\pi i \int_{q_{0r}-iq_{0i}}^{\infty} \gamma q e^{-a\gamma} dq \end{aligned} \quad (C7)$$

where we have reset β to q , since they are each dummy integration variables. Now, given that [19]

$$\begin{aligned} & \int_h^{\infty} \sqrt{1+x^2} e^{-a\sqrt{1+x^2}} x dx = \\ & a^{-3} \left[2 + 2a(1+h^2)^{1/2} + a^2(1+h^2) \right] e^{-a\sqrt{1+h^2}} \end{aligned} \quad (C8)$$

we can write the final expression for the integration around the contour as

$$\begin{aligned} & \int_{-\infty}^{\infty} \gamma q e^{-a\gamma} \log(\gamma\xi - q) dq \\ & + \frac{2\pi i}{a^3} (a^2\psi^2 + 2a\psi + 2) e^{-a\psi} \end{aligned} \quad (C9)$$

where $\psi = [1 + (q_{0r} - iq_{0i})^2]^{1/2} = (1 - \xi^2)^{-1/2}$. This leads to the result quoted in Eq. (17).

APPENDIX D: CLASSICAL ELECTRON-POSITRON PLASMA

For completeness, this section contains a concise treatment of the classical pair-plasma case. The starting point is the Vlasov equations for positrons and electrons, together with Poisson's equation:

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{e}{m} \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{u}} = 0 \quad (D1)$$

$$\frac{\partial g}{\partial t} + \mathbf{u} \cdot \frac{\partial g}{\partial \mathbf{r}} - \frac{e}{m} \mathbf{E} \cdot \frac{\partial g}{\partial \mathbf{u}} = 0 \quad (D2)$$

$$\frac{en}{\epsilon_0} \int (f - g) d\mathbf{u} = \nabla \cdot \mathbf{E} \quad (D3)$$

where f is the distribution function for positrons, g is that for electrons, e is the electronic charge, m is the electron (rest) mass, n is the particle number density, and \mathbf{E} is the electric field. We suppose that the electrons and positrons are initially in a steady state, at temperature T , described by the Maxwellians

$$f_0(\mathbf{u}) = g_0(\mathbf{u}) = \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp\left(-\frac{m\mathbf{u}^2}{2k_B T} \right) \quad (D4)$$

Defining

$$\alpha^2 = 2k_B T/m \quad (D5)$$

$$z = u/\alpha \quad (D6)$$

$$\zeta = \omega/(k\alpha) \quad (D7)$$

the dispersion relation can be written in the form

$$\frac{4}{\pi^{1/2}} \frac{\omega_p^2}{k^2 \alpha^2} \int_C \frac{ze^{-z^2}}{z - \zeta} dz + 1 = 0 \quad (D8)$$

where C is the usual Landau contour. A more compact version of the full dispersion relation for an electron-positron plasma is

$$\Delta = \zeta \mathcal{Z}(\zeta) + \frac{1}{2} k^2 \lambda_D^2 + 1 = 0 \quad (D9)$$

in which \mathcal{Z} is the plasma dispersion function [14], and $\lambda_D = [\epsilon_0 k_B T / (ne^2)]^{1/2}$ is the electron or positron Debye length. The dispersion relation given in Eq. (D9) should be compared with the usual case of electrons with stationary ions:

$$\zeta \mathcal{Z}(\zeta) + k^2 \lambda_D^2 + 1 = 0 \quad (D10)$$

Note that the plasma dispersion function can be written in several forms, but the one most useful in the context of this article is the following:

$$\mathcal{Z}(\zeta) = i\pi^{1/2} e^{-\zeta^2} [\text{erf}(i\zeta) + 2] \quad (D11)$$

We know that ζ is a complex variable with negative imaginary part, since this is the basis of our investigation. Hence let $\zeta = x - iy$. To determine the behaviour of the plasma, we must find values of x and y for a given k such that the dispersion relation Eq. (D9) is satisfied; specifically, we must ensure the simultaneous solution of

$$\Re(\Delta) = 0 \quad (D12)$$

$$\Im(\Delta) = 0 \quad (D13)$$

where the following notation has been used:

$$S = k\lambda_D \quad (D14)$$

$$\zeta = x - iy = \Omega/S \quad (D15)$$

$$\Omega = \omega/(\sqrt{2}\omega_p) \quad (D16)$$

When a solution set (x, y, S) is obtained, it is straightforward to obtain the real frequency $\Re(\Omega) = \Omega_r = xS$ and

the damping term $\Im(\Omega) = \Omega_i = -yS$. In solving for the dispersion relation this way, the results are independent of the plasma density and temperature.

Previous descriptions of classical Landau damping have concentrated on a full theoretical treatment, especially in all the major texts which describe plasma kinetic theory. Exceptions are Landau's original paper [1], which discusses the limiting cases of long ($S \rightarrow 0$) and short ($S \rightarrow \infty$) wavelengths, and the work of Fried and Conte [14], which contains extensive tables of the plasma dispersion function \mathcal{Z} .

1. Long Wavelength Approximation

The asymptotic expansion of \mathcal{Z} is required for the long wavelength approximation:

$$\zeta \mathcal{Z}(\zeta) \sim -1 - \frac{1}{2\zeta^2} - \frac{3}{4\zeta^4} + i2\sqrt{\pi}\zeta e^{-\zeta^2} \quad (\text{D17})$$

Ignoring the small imaginary part in comparison with the real part, we have the approximate long wavelength dispersion relation

$$\Omega_r^2 \approx 1 + \frac{3}{2}S^2, \quad |\Omega_i| \ll 1 \quad (\text{D18})$$

which becomes, in dimensional variables, the familiar dispersion relation

$$\omega^2 \approx 2\omega_p^2 + 3k^2 c_s^2 \quad (\text{D19})$$

where c_s is the plasma sound speed.

2. Heavily Damped Modes

In Eq. (D18) we made the approximation of large, real argument in \mathcal{Z} . Allowing a complex argument in the asymptotic form reveals a rich structure. Let $\zeta = x(1 - iq)$, where x and q are real and positive, and consider the asymptotic form of the plasma dispersion function. The dispersion relation Eq. (D9) requires

$$\Im(\zeta \mathcal{Z}(\zeta)) = 0 \quad (\text{D20})$$

Hence we have, for $\zeta = x(1 - iq)$,

$$\Im(\zeta \mathcal{Z}(\zeta)) \sim 2\sqrt{\pi}x e^{-(1-q^2)x^2} [q \sin(2qx^2) + \cos(2qx^2)] - \frac{q}{(1+q^2)^2 x^2} \quad (\text{D21})$$

so that for a valid solution of the dispersion relation, the following necessary condition applies:

$$q \sin(2qx^2) + \cos(2qx^2) \approx \frac{q e^{(1-q^2)x^2}}{2\sqrt{\pi}x^3(1+q^2)} \quad (\text{D22})$$

A particularly simple case is that of $q = 1$, that is, heavily damped modes, for which approximate solutions to Eq. (D22) are given by

$$x_n \approx \left[\frac{\pi}{2} \left(n - \frac{1}{4} \right) \right]^{1/2}, \quad n = 1, 2, \dots \quad (\text{D23})$$

assuming $x \gg 1$, corresponding to modes with phase speeds much larger than the sound speed. For $q \gtrsim 1$ this approximation may be generalised to

$$x \approx \left[\frac{\pi}{2q} \left(n - \frac{1}{4} \right) \right]^{1/2} \quad (\text{D24})$$

However, Eq. (D23) is not sufficient to define a solution, since the corresponding real part of $\zeta \mathcal{Z}(\zeta)$ must be negative definite, in order to satisfy Eq. (D12). For $\zeta = x(1 - iq)$,

$$\Re(\zeta \mathcal{Z}(\zeta)) \sim 2\sqrt{\pi}x e^{-(1-q^2)x^2} [q \cos(2qx^2) - \sin(2qx^2)] - \frac{q}{(1+q^2)^2 x^2} \quad (\text{D25})$$

Concentrating on the case $q = 1$, we see that solutions to the full dispersion relation satisfy

$$S^2 \approx 4\sqrt{2\pi}x_n, \quad n \text{ odd} \quad (\text{D26})$$

that is,

$$\frac{\Omega_r}{k} \approx c_s \frac{S^2}{4\sqrt{2\pi}}, \quad n \text{ odd} \quad (\text{D27})$$

3. Moderately Damped Modes

The analysis for moderately damped waves is rather more involved, and only a limited insight is possible. The difficulty here is that for q small, the exponential in the term on the right-hand side of Eq. (D22) becomes problematic. Note also that although q may be small, in the sense $q \ll 1$, the quantity qx^2 needn't necessarily be negligible, in which case the asymptotic expansion of the plasma dispersion function can still be a useful approximation.

Solutions to the imaginary part of the dispersion relation have to satisfy

$$q \sin(2qx^2) + \cos(2qx^2) \approx qF(x) \quad (\text{D28})$$

$$F(x) = \frac{e^{x^2}}{2\sqrt{\pi}x^3} \quad (\text{D29})$$

and the real part of the dispersion relation demands

$$q \cos(2qx^2) - \sin(2qx^2) + x^2(1 + \frac{1}{2}S^2)F(x) \approx qF(x) \quad (\text{D30})$$

It is clear that there is no short wavelength solution that has precisely zero damping, since as $q \rightarrow 0$, the right-hand side of Eq. (D28) tends to zero, whilst the left-hand side tends to unity.

This result is best illustrated graphically. Contours of the imaginary part of the dispersion relation, Eq. (D28), for which only the zero contour is relevant, and of S^2 given by Eq. (D30), for which only the positive contours are relevant, are combined in Figure 8, where the arrowed section of line shows the necessary zero imaginary part intersecting the positive-definite S^2 contours.

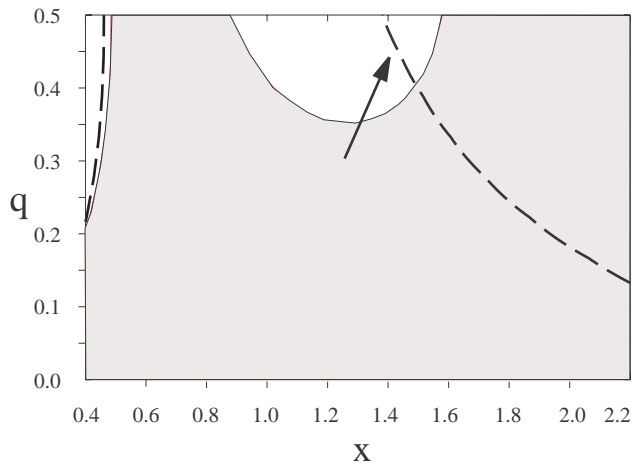


FIG. 8: Combined plot of the zero contour of the imaginary part of the dispersion relation (line) with contours of S^2 , using the asymptotic formulae appropriate for the moderately damped case. Only those sections of the zero contour of the imaginary part that lie in the unshaded region represent points that are damped wave solutions (as arrowed).

The general solution for wider ranges of x and y is best treated graphically in a similar manner, by combining the zero contour of the imaginary part with a contour plot of S^2 . The results are shown in Figure 9. Note that each value of S corresponds to a unique equilibrium plasma, so that for a given plasma, there is a discrete spectrum of possible Landau-damped wave modes. Indeed, Landau quoted two extreme solutions [1]: the approximate long-wavelength, undamped solution given in Eq. (D18), and the pathological extremely damped solution for which $\Omega_r \ll \Omega_i$ which can be shown to be given by $xy = \pi/2$, $y \exp(y^2) = S^2/(4\sqrt{\pi})$ in the case of electron-positron plasmas under the approximation $q \gg 1$.

The analysis here is a comprehensive treatment of the classical case, for electron-positron plasmas. .

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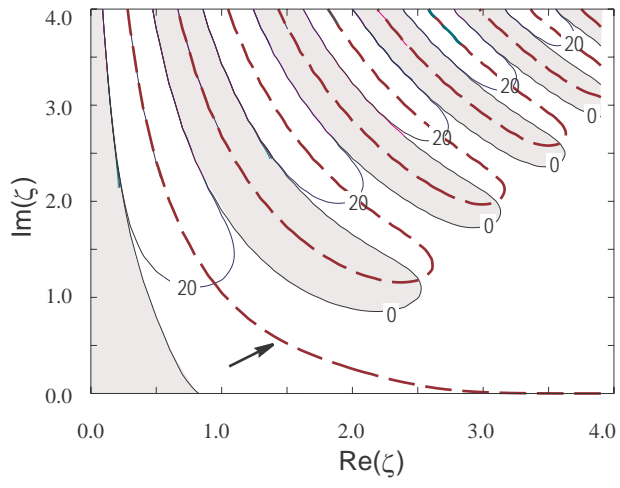


FIG. 9: Combined plot of the zero contour of the imaginary part of the full dispersion relation Eq. (D9) (broken line) with contours of S^2 . Only outside the shaded regions is S^2 positive definite; contours at 0 and 20 are shown for clarity. The arrowed section shows where the moderately damped solutions overlap with this more general approach.