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Estimates for multiple stochastic integrals and stochastic Hamilton-Jacobi equations

Vassili N. Kolokol'tsov, René L. Schilling and Alexei E. Tyukov

Abstract

We study stochastic Hamilton-Jacobi-Bellman equations and the corresponding Hamiltonian systems driven by jump-type Lévy processes. The main objective of the present paper is to show existence, uniqueness and a (locally in time) diffeomorphism property of the solution: the solution trajectory of the system is a diffeomorphism as a function of the initial momentum. This result enables us to implement a stochastic version of the classical method of characteristics for the Hamilton-Jacobi equations. An –in itself interesting– auxiliary result are pointwise a.s. estimates for iterated stochastic integrals driven by a vector of not necessarily independent jump-type semimartingales.

1. Introduction

Over the last few years interest in stochastic Hamilton-Jacobi-Bellman (HJB for short) equations has increased, see e.g. the papers [R], [So], [DaPDe] and references given there. The HJB equations are important as they describe the evolution of optimally controlled systems with random dynamics, but they are also useful tools when studying various classes of stochastic models in probability theory and mathematical physics. Presently, the notion of *stochastic HJB equation* is used in two different contexts: firstly, for classical differential equations with a random Hamiltonian and, secondly, for truly stochastic differential equations where the Hamiltonian includes a non-homogeneous semimartingale term which does not allow to write down the corresponding equation in classical form.

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In the sequel we will consider the second type of HJB equations, that is to say equations of the form

$$dS + H\left(x, \frac{\partial S}{\partial x}\right) dt + c(x)\, d\xi_t = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{1.1}$$

where $H : \mathbb{R}^{2d} \to \mathbb{R}$ and $c : \mathbb{R}^d \to \mathbb{R}^d$ are smooth functions and ξ_t is a stochastic process (driving noise) in $\mathbb{R}^d$. The corresponding Hamilton system is then

$$\begin{cases} dx = \frac{\partial H}{\partial p}\, dt \\ dp = -\frac{\partial H}{\partial x}\, dt - \frac{\partial c}{\partial x}\, d\xi_t. \end{cases} \tag{1.2}$$

The equation (1.1) driven by a Wiener process $\{\xi_t\}_{t\geqslant 0}$ was considered in [K1], [K2], and [TrZ1], [TrZ2] for various classes of real H and c. The corresponding case of complex valued H and c was taken up in [K3]. The main objective of the present paper is to study the equations (1.1), (1.2), where $\{\xi_t\}_{t\geqslant 0}$ is a Lévy noise without Brownian part and to develop a stochastic analogue of the theory of classical (i.e. smooth in x) solutions of the Cauchy problem for equation (1.1). Generalised solutions can then be constructed (see [KT2]) in the same way as they are constructed for the case of a Wiener process $\{\xi_t\}_{t\geqslant 0}$ in [K1], [K2] (see also [KMa]).

For this programme we need to be able to solve a boundary value problem for the stochastic Hamilton system (1.2). The analysis of the latter problem is the second main topic of our paper. Boundary value problems for Hamilton systems of type (1.2) with a Wiener process $\{\xi_t\}_{t\geqslant 0}$ and their connections with the calculus of variations were investigated in [K2]. However, the proof of the existence and uniqueness of the solution of the boundary value problem was only sketched in [K2]. In this paper we give complete proofs of the corresponding results for Hamilton systems driven by Lévy noise without a Brownian part.

An important tool for the analysis of the behaviour of the solutions for Hamilton systems is the study of their linearised approximations (equation in variations). These linearised approximations turn out to be linear nonhomogeneous Hamilton systems. Using perturbation theory we can derive a representation of the solutions of such linear systems as series of multiple stochastic integrals. In order to prove the convergence of these series, we are led to the third topic of the article —obtaining estimates for multiple stochastic integrals. In the present paper we use these estimates as auxiliary tools for the study of linear stochastic Hamilton systems. We believe, however, that they are of independent value. Let us mention here the paper [Ta], where a rather general linear system driven by Brownian motion

was considered, convergence of the series from perturbations theory proved, and necessary estimates for multiple integrals obtained. Multiple stochastic integrals with respect to general semimartingales or infinitely divisible processes were also considered, see e.g. [KwW], [Sz] and references given there.

Let us give a brief outline how our paper is organized. In Section 3 we obtain estimates for the symmetric sum of multiple integrals driven by semimartingales. Under some additional assumptions on H and c we prove in Section 4 well-posedness of certain boundary value problems for stochastic Hamilton systems (1.2) for times t not exceeding some stopping time. For proving this we first observe that well-posedness is equivalent to the statement that the map $p_0 \mapsto X(t, t_0, x_0, p_0)$ (where $X(t, t_0, x_0, p_0)$ is a solution of (1.2) with initial conditions (x_0, p_0) at time $t = t_0$) is a diffeomorphism. We call this statement *diffeomorphism theorem.* In Section 5 we develop the method of stochastic characteristics to solve stochastic Hamilton-Jacobi equations with Lévy noise.

Some applications of our results to the theory of stochastic heat equation (large deviation type asymptotics) are considered in the paper [KT2].

2. Preliminaries

Throughout this paper we consider Lévy processes as driving noise terms in the Hamiltonian system (1.2). Our standard references for Lévy processes are the monographs by Bertoin [Ber1] and Sato [Sa]. For Lévy processes and stochastic calculus with jumps we use the books by Jacod and Shiryaev [JSh] and Protter [Pro]. We will collect a few definitions and results from these books.

A *Lévy process* (on $\mathbb{R}^d$) is a stochastic process $\{\xi_t\}_{t\geqslant 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with stationary and independent increments which is also stochastically continuous. We will assume that $\xi_0 = 0$ a.s. The state space will always be $\mathbb{R}^d$. We can (and will) choose a version that has right-continuous sample paths with everywhere finite left-hand limits (càdlàg, for short); if not otherwise mentioned, we will use the augmented canonical filtration of $\{\xi_t\}_{t\geqslant 0}$. The process $\{\xi_t\}_{t\geqslant 0}$ is uniquely (up to stochastic equivalence) determined through its Fourier transform,

$$\mathbb{E}e^{i\eta\xi_t} = e^{-t\psi(\eta)}, \qquad t > 0,\ \eta \in \mathbb{R}^d,$$

where the characteristic exponent $\psi : \mathbb{R}^d \to \mathbb{C}$ is given by the Lévy-Khinchine representation

$$\psi(\eta) = i\ell \cdot \eta + \eta \cdot Q\eta + \int_{y\neq 0} \left(1 - e^{iy\cdot\eta} + \frac{iy \cdot \eta}{1 + |y|^2}\right) \nu(dy).$$

Here, ℓ is some vector in $\mathbb{R}^d$, $Q \in \mathbb{R}^{d\times d}$ is a positive semi-definite matrix and ν is the Lévy or jump measure with support in $\mathbb{R}^d \setminus \{0\}$ such that $\int_{y\neq 0} |y|^2 \wedge 1\, \nu(dy) < \infty$. The Lévy-Khinchine formula is actually a one-to-one correspondence between the function ψ and the *Lévy triplet* (ℓ, Q, ν).

Stochastically, the Lévy-Khinchine representation translates into a path decomposition of the process $\{\xi_t\}_{t\geqslant 0}$. Fix some Borel set $A \subset \mathbb{R}^d \setminus \{0\}$, and write $N_t(\omega, A)$ for the Poisson point process with intensity measure $\nu(A)$. It is known that $N_t(\omega, A)$ describes jumps of ξ_t with sizes contained in A and we get

$$\xi_t(\omega) = \alpha t + B_t(\omega) + M_t(\omega) + J_t(\omega), \tag{2.1}$$

where $\alpha = \mathbb{E}\Big(\xi_1 - \int_{|y|\geqslant 1} y\, N_1(\omega, dy)\Big)$ is the drift coefficient, B_t is a $d-$dimensional Wiener process with (possibly degenerate) covariance matrix Q,

$$M_t(\omega) = \int_{|y|<1} y\, (N_t(\omega, dy) - t\nu(dy))$$

is a martingale which is the compensated sum of all small jumps (modulus less than 1), and

$$J_t(\omega) = \sum_{0<s\leqslant t} \Delta\xi_s\, \mathbf{1}_{\{|\Delta\xi_s|\geqslant 1\}}$$

is the sum of all big jumps (modulus greater than 1). As usual, we write $\Delta\xi_s = \xi_s - \xi_{s-} = \xi_s - \lim_{r\uparrow s} \xi_r$ for the jump at time $s > 0$. Note that J_t is a process of bounded variation on compact time-intervals. This is the case since càdlàg paths can have only finitely many jumps of size $\geqslant 1$ on any finite time interval. The above decomposition of ξ_t shows that Lévy processes are semimartingales and, therefore, good stochastic integrators.

The following two formulae for point processes hold whenever the right-hand side is finite:

$$\mathbb{E}\Big(\int_A f(y)\, N_t(\bullet, dy)\Big) = t \int_A f(y)\, \nu(dy)$$

and

$$\mathbb{E}\Big(\Big\{\int_A f(y)(N_t(\bullet, dy) - t\nu(dy))\Big\}^2\Big) = t \int_A f(y)^2\, \nu(dy). \tag{2.2}$$

In particular, we get

$$\mathbb{E}\Big(\sum_{s\leqslant t} f(\Delta\xi_s)\Big) = t \int f(y)\, \nu(dy) \tag{2.3}$$

for finite right-hand sides.

It is not hard to see that

$$t \mapsto \xi_t \quad \text{has a.s. finite variation if and only if} \quad \int_{0<|y|<1} |y|\, \nu(dy) < \infty$$

and that

$$\mathbb{E}|\xi_t| < \infty \quad \text{if and only if} \quad \int_{|y|\geqslant 1} |y|\, \nu(dy) < \infty.$$

If ξ_t has a.s. bounded jumps, i.e., if the support of ν is a bounded set, ξ_t has absolute moments of any order.

We will also need the following simple Lemma. Since we could not find a precise reference for it, we include a short proof.

Lemma 2.1 *Let $\{\xi_t\}_{t\geqslant 0}$, $\xi = (\xi_1, \dots, \xi_d)$ be a Lévy process with $Q = 0$ and Lévy measure ν satisfying $\int_{|y|>1} |y|^2\, \nu(dy) < \infty$. For any $0 < \varepsilon < 1/2$ we find a stopping time $\mathcal{R}_\varepsilon(\omega) < 1$ such that*

$$\vartheta_t = 2\sum_{i=1}^{d} \left(\sup_{\tau\in[0,t]} |\xi_{i,\tau}| + [\xi_i, \xi_i]_t^{\frac{1}{2}} \right) < t^{\frac{1}{2}-\varepsilon}$$

holds for all $t < \mathcal{R}_\varepsilon$, where $\mathbb{P}(\mathcal{R}_\varepsilon > 0) = 1$. In particular, one can find a stopping time $\mathcal{R} > 0$ a.s. such that for all $t < \mathcal{R}$

$$\vartheta_t = 2\sum_{i=1}^{d} \left(\sup_{\tau\in[0,t]} |\xi_{i,\tau}| + [\xi_i, \xi_i]_t^{\frac{1}{2}} \right) < 1. \tag{2.4}$$

Remarks

1. In this paper we will use only the fact that $\vartheta_t < 1$ for $t < \mathcal{R}$. The stopping times $\mathcal{R}_\varepsilon$ will be needed in [KT2].
2. Lemma 2.1 remains valid if $Q \neq 0$. Since we do not need this result, we settle for the case $Q = 0$ and the somewhat simpler proof.

Proof. As usual we write $\xi_t^* = \sup_{\tau\in[0,t]} |\xi_\tau|$. Since $Q = 0$, we get from (2.1)

$$\mathbb{E}\left(\{\xi_t^*\}^2\right) \leqslant 3\left[|\alpha|^2 t^2 + \mathbb{E}\left(\{M_t^*\}^2\right) + \mathbb{E}\left(\{J_t^*\}^2\right)\right].$$

Using $(a+b)^2 \leqslant 2a^2 + 2b^2$ and (2.2) we get for $t < 1$

$$\begin{aligned}
\mathbb{E}\left(\{J_t^*\}^2\right) &\leqslant \mathbb{E}\left(\{\sum_{s\leqslant t} |\Delta\xi_s| \mathbf{1}_{\{|\Delta\xi_s|>1\}}\}^2\right) = \mathbb{E}\left(\{\int_{|x|>1} |x|\, N_t(\cdot, dx)\}^2\right) \\
&\leqslant 2\mathbb{E}\left(\{\int_{|x|>1} |x|\, (N_t(\cdot, dx) - t\nu(dx))\}^2\right) + 2t^2\Big(\int_{|x|>1} |x|\, \nu(dx)\Big)^2 \\
&\leqslant 2t(1 + \nu(B_1^c(0))) \int_{|x|>1} |x|^2\, \nu(dx),
\end{aligned}$$

where we used $t < 1$ and Jensen's inequality for the last term.

Doob's martingale inequality and (2.2) give

$$\mathbb{E}\left(\{M_t^*\}^2\right) \leqslant 4\mathbb{E}\left(M_t^2\right) = 4t \int_{|x|<1} |x|^2 \, \nu(dx).$$

Formula (2.3) implies that

$$\mathbb{E}([\xi,\xi]_t) = t \int_{|x|>0} |x|^2 \, \nu(dx).$$

Thus, the process

$$\zeta_t = \{\xi_t^*\}^2 + [\xi,\xi]_t$$

satisfies $\mathbb{E}\,\zeta_t < Ct$ for $t < 1$, where $C = C(\alpha,\nu) > 0$ is a constant. By Chebyshev's inequality

$$\mathbb{P}(\zeta_t > R) \leqslant \frac{\mathbb{E}(\zeta_t)}{R} \leqslant \frac{Ct}{R}, \qquad t < 1.$$

Choosing $t = 2^{-k}$ and $R = (8d)^{-1}2^{-(1-\varepsilon)k}$ we find

$$\sum_{k=1}^{\infty} \mathbb{P}\left\{\zeta_{2^{-k}} > (8d)^{-1}2^{-(1-\varepsilon)k}\right\} \leqslant 8dC \sum_{k=1}^{\infty} 2^{-\varepsilon k} < \infty.$$

The Borel-Cantelli Lemma implies that

$$\zeta_{2^{-k}} \leqslant (8d)^{-1}2^{-(1-\varepsilon)k} \quad \text{for} \quad k > k_0(\omega) \quad \text{for some} \quad k_0(\omega) \in \mathbb{N}.$$

Set $k_1(\omega) = k_0(\omega) \vee \left[\frac{(1-2\varepsilon)}{\varepsilon} + 1\right]$. Then $(1-2\varepsilon)\frac{k+1}{k} \leqslant 1-\varepsilon$ for $k > k_1(\omega)$. If $2^{-(k+1)} \leqslant t < 2^{-k}$ for some $k > k_1(\omega)$ we find, as $t \mapsto \zeta_t$ is an increasing function,

$$(8d)\zeta_t \leqslant (8d)\zeta_{2^{-k}} \leqslant 2^{-(1-\varepsilon)k} \leqslant \left(2^{-(k+1)}\right)^{\frac{(1-\varepsilon)k}{k+1}} \leqslant \left(2^{-(k+1)}\right)^{(1-2\varepsilon)} \leqslant t^{(1-2\varepsilon)}.$$

Using the elementary inequality $(a_1 + \cdots + a_{2d})^2 \leqslant (2d)(a_1^2 + \cdots + a_{2d}^2)$ we get with $\mathcal{R}_\varepsilon(\omega) = 2^{-k_1(\omega)}$

$$\vartheta_t^2 \leqslant (8d)\zeta_t \leqslant t^{1-2\varepsilon}, \qquad \forall\, t < \mathcal{R}_\varepsilon,$$

and the lemma follows. ■

Notation. Most of our notation should be standard or self-explanatory. All stochastic integrals are Itô-integrals and our main reference texts for stochastic integrals with jumps are Jacod and Shiryaev [JSh] and Protter [Pro]. In particular, we follow the conventions of Protter

$$\int_a^b X_s \, dY_s := \int_{(a,b]} X_s \, dY_s, \qquad [X,Y]_0 = X_0 Y_0 \quad \text{and} \quad X_{0-} = (X_-)_0 = 0.$$

3. Estimates for multiple stochastic integrals driven by semimartingales

Here we derive some estimates for multiple stochastic integrals which will be needed later on. We use the following notation. For any $A \in \mathbb{R}^{M\times N}$ we write

$$\|A\|_\infty = \max_{\substack{i=1,\dots,M\\ j=1,\dots,N}} |(A)_{i,j}|.$$

We will always consider k-fold stochastic integrals driven by (general) real-valued semimartingales $\{\eta_{j,t}\}_{t\geqslant 0}$, $j=1,\dots,d$, with càdlàg paths or by the deterministic process $\eta_{0,t}=t$. We assume that all semimartingales are on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ and are adapted to the same filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. The filtration is assumed to satisfy the usual conditions, i.e., it is right-continuous and augmented. Since the $d\eta_{j,t}$, $j=0,\dots,d$, may appear in any order we want to keep track when we deal with a Stieltjes differential $d\tau = d\eta_{0,\tau}$ and a (genuinely) stochastic differential $d\eta_{j,\tau}$, $j=1,\dots,d$. To do so we introduce and fix throughout this section a sequence $\ell_n \in \mathbb{N}$ such that

$$\ell_{2n-1} \leqslant \ell_{2n} \qquad \text{and} \qquad \ell_{2n}+1 < \ell_{2n+1}. \tag{3.1}$$

We set

$$\mathcal{M}_i = \begin{cases} \{1,\dots,d\} & \text{if} \quad \ell_{2n-1}\leqslant i \leqslant \ell_{2n} \\ \{0\} & \text{otherwise.} \end{cases}$$

This means that for $(j_1,\dots,j_k)\in \mathcal{M} = \mathcal{M}_1\times\cdots\times\mathcal{M}_k$ the first ℓ_1-1 integrals of the stochastic differential

$$d\eta_{j_1,\tau_1} d\eta_{j_2,\tau_2}\dots d\eta_{j_k,\tau_k} \tag{3.2}$$

are deterministic, the next up to label ℓ_2 are semimartingales, those up to ℓ_3-1 are again deterministic, etc.:

$$\begin{aligned} \ell_1 &= \min\{s\in[0,k] : j_s\neq 0\} \\ \ell_2 &= \min\{s\in(\ell_1,k] : j_s = 0\} - 1 \\ &\vdots \\ \ell_{2n-1} &= \min\{s\in(\ell_{2n-2},k] : j_s\neq 0\} \\ \ell_{2n} &= \min\{s\in(\ell_{2n-1},k] : j_s = 0\} - 1 \end{aligned}$$

($\min\emptyset = k+1$). Moreover,

$$\mathfrak{m} = \mathfrak{m}(k) = \sum_{n\in\mathbb{N},\,\ell_{2n}\leqslant k} (\ell_{2n}-\ell_{2n-1}+1) \tag{3.3}$$

is the number of non-trivial differentials in (3.2), i.e.

$$\mathfrak{m} = \#\{s : j_s\neq 0\}.$$

Let $W_{j,\tau} = W_{j,\tau}(\omega)$, $I_{0,\tau} = I_{0,\tau}(\omega)$ be $\mathbb{R}^{M\times M}$-valued continuous processes adapted to the filtration $\{\mathcal{F}_\tau\}_{\tau\geqslant 0}$ and such that

$$\sup_{s\geqslant t_0} \|W_{j,s}\|_\infty \leqslant 1, \quad j = 0,\dots,d, \tag{3.4}$$

and let v_τ be some real-valued $\mathcal{F}_t$-adapted increasing process such that for any $s,t \in \mathbb{R}_+$, $s < t$,

$$\|W_{j,t} - W_{j,s}\|_\infty \leqslant v_t - v_s, \quad j = 1,\dots,d. \tag{3.5}$$

We also assume that $[\eta_i,\eta_i]^c = 0$, $i = 1,\dots,d$, and

$$2 \sup_{t_0\leqslant\tau_1\leqslant\tau} |\eta_{i,\tau_1} - \eta_{i,t_0}| \leqslant 1, \qquad i = 1,\dots,d. \tag{3.6}$$

Notice that the assumptions (3.4), (3.6) can always be achieved by suitable (pre-)stopping arguments.

For $0 \leqslant t_0 \leqslant \tau$ we set

$$I_{k,\tau} = \tag{3.7}$$

$$= \sum_{(j_1,\dots,j_k)\in\mathcal{M}} \int_{t_0}^{\tau} W_{j_k,\tau_k}\Big(\int_{t_0}^{\tau_k-}\cdots\Big(\int_{t_0}^{\tau_2-} W_{j_1,\tau_1} I_{0,\tau_1}\, d\eta_{j_1,\tau_1}\Big)\dots d\eta_{j_{k-1},\tau_{k-1}}\Big) d\eta_{j_k,\tau_k},$$

where $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_k$ as above, and

$$D_\tau = M\cdot d\left[v_\tau - v_{t_0} + 4\sum_{j=1}^{d}\Big(\sup_{t_0\leqslant\tau_1\leqslant\tau} |\eta_{j,\tau_1} - \eta_{j,t_0}| + ([\eta_j\,,\eta_j]_\tau - [\eta_j\,,\eta_j]_{t_0})^{\frac{1}{2}}\Big)\right]. \tag{3.8}$$

We will use the abbreviation *BV-process* for a process with (almost surely) paths of bounded variation on compact time-intervals.

We can now state the main result of this section.

Proposition 3.1 *Let $\eta_{i,\tau}$, $i = 1,\dots,d$, be $\mathcal{F}_\tau$-semimartingales, $W_{j,\tau}$ be $\mathcal{F}_\tau$-adapted continuous processes as set out above, $\ell_n \in \mathbb{N}$ be any fixed sequence such that* (3.1) *holds and $I_{k,\tau}$ be as defined above in* (3.7)*. Moreover we assume that*

$$W_{j,\tau}W_{i,\tau} = W_{i,\tau}W_{j,\tau} \qquad j,i = 1,\dots,d. \tag{3.9}$$

If $I_{0,\tau} = E_M \in \mathbb{R}^{M\times M}$ is the identity matrix or if $\ell_1 > 1$ (or both), we have

$$\|I_{k,\tau}\|_\infty \leqslant b_k\, D_\tau^{\mathfrak{m}}\{M(\tau - t_0)\}^{k-\mathfrak{m}} \|I^*_{0,\tau}\|_\infty, \tag{3.10}$$

where $\|I^*_{0,\tau}\|_\infty = \sup_{t_0\leqslant s\leqslant\tau} \|I_{0,s}\|_\infty$, with $\mathfrak{m}$ from (3.3) *and*

$$b_k = \frac{(2^7)^k}{(\ln\{\ln\{k+2\}\})^{\frac{k}{16}}}. \tag{3.11}$$

The proof of Proposition 3.1 is quite technical and a few words of guidance are in order. Basically, we prove Proposition 3.1 by induction in k, the number of iterated integrals in I_k. There are two different ways how I_{k+1} can arise from I_k: **(i)** the $(k+1)$st integrator is deterministic, or **(ii)** the $(k+1)$st integrator is a semimartingale. Case **(i)** is easily dealt with (Case 1 of Step 1 of the proof), while **(ii)** requires a few auxiliary results which we prove in Lemmas 3.1–3.6 below. The idea here is to consider the last two, i.e. kth and $(k+1)$st, integrations simultaneously and to look first at the more complicated (but symmetrised) object

$$z^{\widehat{m}}_{k+1,\tau} = \sum_{i=1}^{d} \sum_{J\in\mathbb{N}_0^d,\,|J|=\widehat{m}} \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^{\tau} U_{J,\tau_2} \left(\int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-} d\eta_{i,\tau_1} \right) d\eta^{J'}_{\tau_2}, \tag{3.12}$$

where $U_{J,\tau}$ is a product of W_j's and $\mathfrak{R}_{J',J,\tau}$ are the coefficients of a multinomial series (see (3.19) and (3.20)). Notice that $I_{k+1,\tau} = z^1_{k+1,\tau}$. The symmetrisation has the effect that we are effectively integrating against $d(\eta_\bullet - \eta_\tau)^J$—but this integrator would not be well-defined in Itô's sense.

The technique is to apply integration by parts to the two inner integrals in (3.12) (the general formula is given in Lemma 3.1, Corollary 3.1 contains the 'symmetrised' version which we are going to use later on) and then to estimate the three appearing terms (3.27)–(3.29) which is done in Lemma 3.3. Depending on the nature of the integrators in (3.12), the main estimates are done in Lemmas 3.5 and 3.6. Their proofs use mainly Lemmas 3.2–3.4 which are of technical nature and can be skipped on first reading. This induction gives the basic estimate of Proposition 3.1 without revealing the form of the coefficients b_k. The coefficients are obtained in a separate induction (Step 2 in the proof of Proposition 3.1). This part of the proof rests entirely on some tedious estimates and recurrence relations which we deferred to the appendix (Lemmas A.1–A.4).

Notation

(***i***) We will use throughout the paper matrix and vector notation for stochastic integrals. Since matrices are, in general, non-commutative the position of stochastic differentials etc. is important.

(***ii***) All stochastic integrals, where the integrand is a vector (or a matrix) and the integrator is an $\mathbb{R}$-valued semimartingale will be understood coordinate-wise. In a similar way, brackets of vectors and $\mathbb{R}$-semimartingales or matrices of $\mathbb{R}$-semimartingales are understood coordinatewise. The bracket of two matrices A, B is defined as a matrix

$$[A,B]_{ik} = \sum_{j=1}^{M} [a_{ij}, b_{jk}], \quad i,k = 1,\dots,M,$$

which is compatible with the rules of stochastic calculus and matrix algebra.

We start with rewriting the integration by parts formula in the form we need later on.

Lemma 3.1 *Let $\{U_\tau\}_{\tau\geqslant 0}$ be an $\mathcal{F}_\tau$-adapted continuous BV process with values in $\mathbb{R}^{M\times M}$, $\{\Phi_\tau\}_{\tau\geqslant 0}$ an $\mathcal{F}_\tau$-adapted càdlàg process with values in $\mathbb{R}^{M\times M}$ and let $\{\nu_\tau\}_{\tau\geqslant 0}$, $\{\kappa_\tau\}_{\tau\geqslant 0}$ be two real-valued $\mathcal{F}_\tau$-semimartingales. Then the iterated stochastic integral*

$$Q_\tau = \int_0^\tau U_{\tau_2}\bigg(\int_0^{\tau_2-} \Phi_{\tau_1-}\,d\nu_{\tau_1}\bigg)d\kappa_{\tau_2}$$

can be written in the form

$$\begin{aligned} Q_\tau = \kappa_\tau \int_0^\tau dU_{\tau_2} \times \int_0^{\tau_2-} \Phi_{\tau_1-}\,d\nu_{\tau_1} + \kappa_\tau \int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\,d\nu_{\tau_2} - \int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\,d[\nu,\kappa]_{\tau_2} \\ - \int_0^\tau dU_{\tau_2} \times \kappa_{\tau_2-} \int_0^{\tau_2-} \Phi_{\tau_1-}\,d\nu_{\tau_1} - \int_0^\tau \kappa_{\tau_2-}U_{\tau_2}\Phi_{\tau_2-}\,d\nu_{\tau_2}. \end{aligned} \tag{3.13}$$

Proof. We use the following integration by parts formula for $\mathbb{R}$-valued semimartingales:

$$\int_0^\tau Y_{\tau_2-}\,dZ_{\tau_2} = Y_\tau Z_\tau - Y_0 Z_0 - [Y,Z]_\tau - \int_0^\tau Z_{\tau_2-}\,dY_{\tau_2}.$$

With the coordinate conventions detailed in the above remark we may choose

$$Y_{\tau_2} = U_{\tau_2}\int_0^{\tau_2} \Phi_{\tau_1-}\,d\nu_{\tau_1} \qquad \text{and} \qquad Z_{\tau_2} = \kappa_{\tau_2}.$$

Clearly, $Y_0 = 0$ and therefore

$$\begin{aligned} Q_\tau &= \kappa_\tau U_\tau \int_0^\tau \Phi_{\tau_1-}\,d\nu_{\tau_1} - \bigg[U_\bullet \int_0^\bullet \Phi_{\tau_1-}\,d\nu_{\tau_1}, \kappa_\bullet\bigg]_\tau - \int_0^\tau \kappa_{\tau_2-}\,d\bigg(U_{\tau_2}\int_0^{\tau_2} \Phi_{\tau_1-}\,d\nu_{\tau_1}\bigg) \\ &= \mathbf{I} + \mathbf{II} + \mathbf{III}. \end{aligned}$$

An application of Itô's formula gives

$$\begin{aligned} d\bigg(U_{\tau_2}\int_0^{\tau_2} \Phi_{\tau_1-}\,d\nu_{\tau_1}\bigg) &= \\ = dU_{\tau_2} \times \int_0^{\tau_2-} \Phi_{\tau_1-}\,d\nu_{\tau_1} &+ U_{\tau_2}\Phi_{\tau_2-}\,d\nu_{\tau_2} + d\bigg[U_\bullet\,, \int_0^\bullet \Phi_{\tau_1-}\,d\nu_{\tau_1}\bigg]_{\tau_2}. \end{aligned} \tag{3.14}$$

Since U is a continuous BV-process, the square bracket vanishes and we get

$$\mathbf{I} = \kappa_\tau \int_0^\tau dU_{\tau_2} \times \int_0^{\tau_2-} \Phi_{\tau_1-}\,d\nu_{\tau_1} + \kappa_\tau \int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\,d\nu_{\tau_2}. \tag{3.15}$$

From (3.14) we also find

$$\begin{aligned}\mathbf{II} &= -\left[\left(U_\bullet \int_0^\bullet \Phi_{\tau_1-}\, d\nu_{\tau_1}\right), \kappa_\bullet\right]_\tau \\ &= -\left[\left(\int_0^\bullet U_{\tau_2}\Phi_{\tau_2-}\, d\nu_{\tau_2}\right), \kappa_\bullet\right]_\tau - \left[\int_0^\bullet dU_{\tau_1} \times \int_0^{\tau_1-} \Phi_{\tau_2-}\, d\nu_{\tau_2}, \kappa_\bullet\right]_\tau .\end{aligned}$$

Since $U_\bullet$ is a continuous BV-process, so is the stochastic integral driven by dU, and the last bracket above vanishes. So,

$$\mathbf{II} = -\int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\, d[\nu,\kappa]_{\tau_2}. \tag{3.16}$$

Finally, using (3.14) the third time gives

$$\mathbf{III} = -\int_0^\tau \kappa_{\tau_2-} dU_{\tau_2} \times \int_0^{\tau_2-} \Phi_{\tau_1-}\, d\nu_{\tau_1} - \int_0^\tau \kappa_{\tau_2-} U_{\tau_2}\Phi_{\tau_2-}\, d\nu_{\tau_2}.$$

Combining this with (3.15) and (3.16) completes the proof. ■

Later on, we will use Lemma 3.1 in the following form.

Corollary 3.1 *Let $\{U_\tau\}_{\tau\geqslant 0}$, $\{\Phi_\tau\}_{\tau\geqslant 0}$, $\{\nu_\tau\}_{\tau\geqslant 0}$ be as in Lemma 3.1 and let $\{\widetilde{\kappa}_{\alpha,\tau}\}_{\tau\geqslant 0}$ and $\{\kappa_{\alpha,\tau}\}_{\tau\geqslant 0}$ be finitely many ($\alpha \in \mathfrak{A}$) real-valued $\mathcal{F}_\tau$-semimartingales such that*

$$\sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau}\kappa_{\alpha,\tau} = 0. \tag{3.17}$$

Then the iterated stochastic integral

$$Q_\tau = \sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau} \int_0^\tau U_{\tau_2}\left(\int_0^{\tau_2-} \Phi_{\tau_1-}\, d\nu_{\tau_1}\right) d\kappa_{\alpha,\tau_2}$$

can be written in the form

$$\begin{aligned}Q_\tau = &-\sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau} \int_0^\tau U_{\tau_2}\Phi_{\tau_2-}\, d[\nu,\kappa_\alpha]_{\tau_2} - \sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau} \int_0^\tau dU_{\tau_2} \times \kappa_{\alpha,\tau_2-} \int_0^{\tau_2-} \Phi_{\tau_1-}\, d\nu_{\tau_1} \\ &- \sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau} \int_0^\tau \kappa_{\alpha,\tau_2-} U_{\tau_2}\Phi_{\tau_2-}\, d\nu_{\tau_2}.\end{aligned} \tag{3.18}$$

Proof. For a fixed $\alpha \in \mathfrak{A}$ we can apply Lemma 3.1 with $\kappa_\tau = \kappa_{\alpha,\tau}$ to get (3.13). Now multiply (3.13) with $\widetilde{\kappa}_{\alpha,\tau}$ and sum over $\alpha \in \mathfrak{A}$. Because of (3.17), the first two terms vanish and we get (3.18). ■

Recall the following common notations for multi-indices $J = (j_1, \dots, j_d)$, $J' = (j'_1, \dots, j'_d) \in \mathbb{N}_0^d$: we say $J' \leqslant J$ if $j'_1 \leqslant j_1, \dots, j'_d \leqslant j_d$, $|J| = j_1 + \dots + j_d$, $J \pm J' = (j_1 \pm j'_1, \dots, j_d \pm j'_d)$, and if $\eta_\tau = (\eta_{1,\tau}, \dots, \eta_{d,\tau})$,

$$\eta_\tau^J = \eta_{1,\tau}^{j_1} \dots \eta_{d,\tau}^{j_d}.$$

We also set

$$\mathfrak{R}_{J',J,\tau} = (-1)^{|J|-|J'|} \binom{j_1}{j'_1} \dots \binom{j_d}{j'_d} \eta_\tau^{J-J'} \qquad J' \leqslant J. \tag{3.19}$$

From the binomial formula we easily see

$$\sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \eta_{\tau_2}^{J'} = (\eta_{\tau_2} - \eta_\tau)^J = (\eta_{1,\tau_2} - \eta_{1,\tau})^{j_1} \dots (\eta_{d,\tau_2} - \eta_{d,\tau})^{j_d}. \tag{3.20}$$

Below is a technical lemma which we need in order to estimate the norm of some integrals driven by BV-processes.

Lemma 3.2 *Let $\{\eta_{i,\tau}\}_{\tau \geqslant 0}$, $i = 1, \dots, d$, be real-valued $\mathcal{F}_\tau$-semimartingales with $[\eta_i, \eta_i]^c = 0$, $\eta_{i,0} = 0$ and such that the inequalities* (3.6) *are satisfied. Then we have for all $J', J \in \mathbb{N}_0^d$ with $J' \leqslant J$ and $|J'| = m'$, $|J| = m$*

(***i***) $\sup\limits_{0 \leqslant s \leqslant \tau} |\eta_s^{J-J'}| \leqslant (4M \cdot d)^{-(m-m')+1} D_\tau^{m-m'-1}$,

(***ii***) $\sup\limits_{0 \leqslant s \leqslant \tau} |\eta_s^{J-J'}| \leqslant (4M \cdot d)^{-(m-m')} D_\tau^{m-m'}$,

(***iii***) $\sup\limits_{0 \leqslant s \leqslant \tau} |\eta_s^{J'}| \leqslant (4M \cdot d)^{-m'} D_\tau^{m'}$,

(***iv***) $|\Delta[\eta^{J'}, \eta_i]_\tau| \leqslant (2M \cdot d)^{-m'-1} D_\tau^{m'} \Delta D_\tau$,

(***v***) $\sum\limits_{J' \leqslant J} |\mathfrak{R}_{J',J,\tau}| \sup\limits_{0 \leqslant \tau_2 \leqslant \tau} |\Delta[\eta^{J'}, \eta_i]_{\tau_2}| \leqslant 2^{-1} (M \cdot d)^{-m-1} D_\tau^m \Delta D_\tau$,

(***vi***) $\sum\limits_{J' \leqslant J} |\mathfrak{R}_{J',J,\tau}| \sup\limits_{0 \leqslant \tau_2 \leqslant \tau} |\eta_{\tau_2}^{J'}| \leqslant 2(2M \cdot d)^{-m+1} D_\tau^{m-1}$,

(***vii***) $\sum\limits_{J' \leqslant J} |\mathfrak{R}_{J',J,\tau}| \, (2M \cdot d)^{-m'-1} D_\tau^{m'} \leqslant (M \cdot d)^{-m-1} D_\tau^m$.

Proof. By the very definition (3.8) of D_τ we get

$$\max_{i=1,\dots,d} \max\{|\eta_{i,\tau}|, |\Delta \eta_{i,\tau}|\} \leqslant (4M \cdot d)^{-1} D_\tau. \tag{3.21}$$

Hence, due to (3.6),

$$\begin{aligned} \sup_{0 \leqslant s \leqslant \tau} |\eta_s^{J-J'}| \leqslant \sup_{0 \leqslant s \leqslant \tau} \max_{i=1,\dots,d} |\eta_{i,s}|^{m-m'} &\leqslant \sup_{0 \leqslant s \leqslant \tau} \max_{i=1,\dots,d} |\eta_{i,s}|^{m-m'-1} \\ &\leqslant (4M \cdot d)^{-(m-m')+1} D_\tau^{m-m'-1} \end{aligned}$$

and (3.2) follows. Similarly we obtain (3.2) and (3.2).

We proceed with (3.2). Using (3.21), the elementary identity

$$\Delta(a_1 \cdot a_2 \cdot \ldots \cdot a_{m'}) = \sum_{r=1}^{m'} a_0 \ldots a_{r-1}(\Delta a_r)(a_{r+1} - \Delta a_{r+1}) \ldots (a_{m'+1} - \Delta a_{m'+1}),$$

where $a_0 = a_{m'+1} = 1$, $\Delta a_0 = \Delta a_{m'+1} = 0$, and $m' 2^{-m'+1} \leqslant 1$ we find

$$\begin{aligned} |\Delta \eta_\tau^{J'}| &\leqslant m' \left(\sup_{0 \leqslant s \leqslant \tau} \max_{i=1,\ldots,d} |\eta_{i,s}| \right)^{m'-1} \left(\max_{i=1,\ldots,d} |\Delta \eta_{i,\tau}| \right) \\ &\leqslant (2M \cdot d)^{-m'+1} D_\tau^{m'-1} \max_{i=1,\ldots,d} |\Delta \eta_{i,\tau}|. \end{aligned}$$

Since η_i and η^J are pure jump semimartingales, it follows

$$|\Delta[\eta^{J'}, \eta_i]_\tau| = |\Delta \eta_\tau^{J'} \Delta \eta_{i,\tau}| \leqslant (2M \cdot d)^{-m'+1} D_\tau^{m'-1} \max_{i=1,\ldots,d} |\Delta \eta_{i,\tau}|^2. \tag{3.22}$$

Due to (3.8)

$$\Delta[\eta_i, \eta_i]_\tau^{\frac{1}{2}} \leqslant (4M \cdot d)^{-1} \Delta D_\tau$$

for any $i = 1, \ldots, d$, where we used that v_τ and $\sup_{0 \leqslant s \leqslant \tau} |\eta_s|$ are increasing functions of τ. Since

$$\Delta([\eta_i, \eta_i]_\tau^{\frac{1}{2}}) = \frac{\Delta[\eta_i, \eta_i]_\tau}{[\eta_i, \eta_i]_\tau^{\frac{1}{2}} + [\eta_i, \eta_i]_{\tau-}^{\frac{1}{2}}} \geqslant \frac{\Delta[\eta_i, \eta_i]_\tau}{2[\eta_i, \eta_i]_\tau^{\frac{1}{2}}} \geqslant \frac{|\Delta \eta_{i,\tau}|^2}{(2M \cdot d)^{-1} D_\tau},$$

we conclude

$$|\Delta \eta_{i,\tau}|^2 \leqslant (2M \cdot d)^{-2} D_\tau \Delta D_\tau \tag{3.23}$$

and the combination of (3.22), (3.23) proves (3.2).

Using the fact that

$$\sum_{J' \leqslant J} \binom{j_1}{j_1'} \ldots \binom{j_d}{j_d'} = 2^m$$

we see the implications: (3.2), (3.2) $\Longrightarrow$ (3.2); (3.2), (3.2) $\Longrightarrow$ (3.2); and (3.2) $\Longrightarrow$ (3.2) . ■

Given $L' = (l_1', \ldots, l_d') \in \mathbb{N}_0^d$ and $J = (j_1, \ldots, j_d) \in \mathbb{N}_0^d$ we put

$$f_{L'}(x_1, \ldots, x_d) = x_1^{l_1'} \ldots x_d^{l_d'}, \tag{3.24}$$

$$\delta_{L',\tau_2} = \Delta[f_{L'}(\eta)]_{\tau_2} - \sum_{i=1}^{d} \frac{\partial f_{L'}(\eta_{\tau_{\tau_2}-})}{\partial x_i} \Delta \eta_{i,\tau_2} \tag{3.25}$$

and

$$U_{J,\tau_2} = \frac{1}{J!} (W_{1,\tau_2})^{j_1} \ldots (W_{d,\tau_2})^{j_d}, \quad J! = j_1! \ldots j_d!. \tag{3.26}$$

We are going to estimate the norms of the following integrals and sums

$$A_{J,i,\tau} = -\sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\, d[\eta^{J'}, \eta_i]_{\tau_1}, \tag{3.27}$$

$$B_{J,i,\tau} = -\sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau dU_{J,\tau_2} \times \eta^{J'}_{\tau_2-} \int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1}, \tag{3.28}$$

$$C_{L,\tau} = \sum_{L'\leqslant L} \sum_{0<\tau_2\leqslant\tau} \mathfrak{R}_{L',L,\tau} U_{L,\tau_2} I_{k-1,\tau_2-} \delta_{L',\tau_2} \tag{3.29}$$

driven by BV-processes. As we will see in Lemmas 3.5 and 3.6, these integrals arise from an application of the integration by parts formula to iterated integrals of a certain type.

Lemma 3.3 *For $m, k \in \mathbb{N}$ and $\mathcal{B}_m = \{J \in \mathbb{N}_0^d : |J| = m\}$ we have*

$$\Big\| \sum_{i=1}^d \sum_{J\in\mathcal{B}_m} A_{J,i,\tau} \Big\|_\infty \leqslant \frac{1}{2m!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1}, \tag{3.30}$$

$$\Big\| \sum_{i=1}^d \sum_{J\in\mathcal{B}_m} B_{J,i,\tau} \Big\|_\infty \leqslant \frac{1}{(m-1)!} D_\tau^{m-1} \int_0^\tau \|I_{k,\tau_2-}\|_\infty \, dD_{\tau_2}, \tag{3.31}$$

$$\Big\| \sum_{L\in\mathcal{B}_{m+1}} C_{L,\tau} \Big\|_\infty \leqslant \frac{1}{2m!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1}. \tag{3.32}$$

Proof. Applying (3.2) of Lemma 3.2 we have

$$\begin{aligned} \|A_{J,i,\tau}\|_\infty &= \Big\| \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\, d[\eta^{J'}, \eta_i]_{\tau_1} \Big\|_\infty \\ &\leqslant 2^{-1}(M\cdot d)^{-m-1} D_\tau^m \int_0^\tau \|U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1}. \end{aligned}$$

Because of

$$\|YZ\|_\infty \leqslant M\|Y\|_\infty \|Z\|_\infty, \qquad Y, Z \in \mathbb{R}^{M\times M}, \tag{3.33}$$

definition (3.26), and condition (3.4) we obtain for $J \in \mathcal{B}_m$

$$\|U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\|_\infty \leqslant \frac{M^m}{J!} \|W_{i,\tau_1} I_{k-1,\tau_1-}\|_\infty \leqslant \frac{M^{m+1}}{J!} \|I_{k-1,\tau_1-}\|_\infty,$$

and so

$$\|A_{J,i,\tau}\|_\infty \leqslant \frac{1}{2} \frac{d^{-m-1}}{J!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1}. \tag{3.34}$$

Summing (3.34) over all $J \in \mathcal{B}_m$, $i = 1, \dots, d$, and using the multinomial identity

$$\sum_{J \in \mathcal{B}_m} \frac{1}{J!} = \frac{d^m}{m!} \tag{3.35}$$

gives (3.30).

To estimate $B_{J,i,\tau}$ we observe that for the continuous BV-processes $W_{j,\tau}$ which satisfy (3.9) one has

$$dU_{J,\tau_2} = \frac{1}{J!} \sum_{r=1}^{d} (W_{1,\tau_2})^{j_1} \dots \big(d(W_{r,\tau_2})^{j_r}\big) \dots (W_{d,\tau_2})^{j_d} = \sum_{\substack{r=1 \\ j_r>0}}^{d} U_{J-e_r,\tau_2}\, dW_{r,\tau_2},$$

where

$$e_r = (\underbrace{0, \dots, 0, 1}_{r}, 0, \dots, 0) \in \mathbb{N}_0^d. \tag{3.36}$$

Therefore, using (3.5), (3.33) and

$$\sum_{\substack{r=1 \\ j_r>0}}^{d} \frac{1}{(J-e_r)!} = \frac{m}{J!}$$

we obtain for any càdlàg $\mathcal{F}_\tau$-adapted $\mathbb{R}^{M\times M}$-valued process Z_τ

$$\left\| \int_0^\tau dU_{J,\tau_2} \times Z_{\tau_2-} \right\|_\infty \leqslant \frac{m}{J!} M^m \int_0^\tau \|Z_{\tau_2-}\|_\infty \, dv_{\tau_2}. \tag{3.37}$$

Note that according to the assumption of the lemma

$$I_{k,\tau} = \sum_{i=1}^{d} \int_0^{\tau-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1}$$

(i.e. $\mathcal{M}_k = \{1, \dots, d\}$ in the set $\mathcal{M} = \mathcal{M}_1 \times \dots \times \mathcal{M}_k$) and hence, by the very definition (3.7) of $I_{k,\tau}$

$$\begin{aligned}
\left\| \sum_{i=1}^{d} B_{J,i,\tau} \right\|_\infty &= \left\| \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau dU_{J,\tau_2} \times \eta^{J'}_{\tau_2-} \Big(\sum_{i=1}^{d} \int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1} \Big) \right\|_\infty \\
&\leqslant \frac{m}{J!} M^m \sum_{J' \leqslant J} |\mathfrak{R}_{J',J,\tau}| \int_0^\tau \left\| \eta_{\tau_2-} \sum_{i=1}^{d} \int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1} \right\|_\infty dv_{\tau_2} \\
&\leqslant \frac{m}{J!} M^m \Big(\sum_{J' \leqslant J} |\mathfrak{R}_{J',J,\tau}| \sup_{0 \leqslant \tau_2 \leqslant \tau} |\eta^{J'}_{\tau_2}| \Big) \int_0^\tau \|I_{k,\tau_2-}\|_\infty \, dv_{\tau_2} \\
&\leqslant \frac{m}{J!} M (2d)^{-m+1} D_\tau^{m-1} \int_0^\tau \|I_{k,\tau_2-}\|_\infty \, dv_{\tau_2},
\end{aligned}$$

where we used (3.2) of Lemma 3.2. Summing over all $J \in \mathcal{B}_m$ and using the multinomial identity (3.35) we get

$$\left\| \sum_{i=1}^{d} \sum_{J \in \mathcal{B}_m} B_{J,i,\tau} \right\|_\infty \leqslant \frac{2^{-m+1}}{(m-1)!} (M \cdot d) D_\tau^{m-1} \int_0^\tau \|I_{k,\tau_2-}\|_\infty \, dv_{\tau_2}.$$

Since $\Delta v_\tau \leqslant (M \cdot d)^{-1} \Delta D_\tau$ we arrive at (3.31).

We proceed with $C_{L,\tau}$. An application of Taylor's formula and (3.25) imply

$$|\delta_{L',\tau_2}| \leqslant \frac{1}{2} \left(\sup_{0 \leqslant \theta \leqslant 1} \sum_{i,j=1}^{d} \left| \frac{\partial^2 f_{L'}(\eta_{\tau_2-} + \theta \Delta \eta_{\tau_2})}{\partial x_i \partial x_j} \right| \right) \max_{n=1,\dots,d} |\Delta \eta_{n,\tau_2}|^2.$$

Since, by (3.24), $f_{L'}(x_1, \dots, x_d) = x^{L'} = \prod_{j=1}^{d} x_j^{l'_j}$

$$\sum_{i,j=1}^{d} \left| \frac{\partial^2 f_{L'}(x)}{\partial x_i \partial x_j} \right| \leqslant (m'+1) m' \max_{n=1,\dots,d} |x_n|^{m'-1},$$

we get from (3.8) and (3.23)

$$\begin{aligned} |\delta_{L',\tau_2}| &\leqslant \frac{1}{2} (m'+1) m' \left(\max_{i=1,\dots,d} \sup_{0 \leqslant s \leqslant \tau_2} (|\eta_{i,s}| + |\Delta \eta_s|)^{m'-1} \right) \left(\max_{i=1,\dots,d} |\Delta \eta_{i,\tau_2}|^2 \right) \\ &\leqslant \frac{1}{2} (m'+1) m' ((4M \cdot d)^{-m'+1} D_{\tau_2}^{m'-1}) \left((2M \cdot d)^{-2} D_{\tau_2} \Delta D_{\tau_2} \right) \\ &\leqslant \frac{1}{2} (m'+1)(2M \cdot d)^{-m'-1} D_{\tau_2}^{m'} \Delta D_{\tau_2}. \end{aligned}$$

We apply (3.2) of Lemma 3.2 to find for $L \in \mathcal{B}_{m+1}$

$$\begin{aligned} \|C_{L,\tau}\|_\infty &\leqslant \frac{m+1}{2} \left(\sum_{L' \leqslant L} |\mathfrak{R}_{L',L,\tau}| (2M \cdot d)^{-m'-1} D_\tau^{m'} \right) \int_0^\tau \|U_{L,\tau_1} I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1} \\ &\leqslant \frac{m+1}{2} (M \cdot d)^{-m-1} D_\tau^m \int_0^\tau \|U_{L,\tau_1} I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1} \\ &\leqslant \frac{d^{-m-1}}{2} \frac{(m+1)}{L!} D_\tau^m \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1}, \end{aligned} \tag{3.38}$$

where we used the fact that $\sup_{0 \leqslant s \leqslant t} \|W_{j,s}\|_\infty \leqslant 1$, cf. (3.4). Since

$$\sum_{L \in \mathcal{B}_{m+1}} \frac{m+1}{L!} d^{-m-1} = \frac{1}{m!},$$

summing (3.38) over all $L \in \mathcal{B}_{m+1}$ completes the proof. ■

Lemma 3.4 *For any $m, k \in \mathbb{N}$ one has*

$$\sum_{i=1}^{d} \sum_{L \in \mathcal{B}_{m+1}} \sum_{L' \leqslant L} \mathfrak{R}_{L',L,\tau} \int_0^\tau l'_i \eta_{\tau_2-}^{L'-e_i} U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2} = \sum_{i=1}^{d} \sum_{J \in \mathcal{B}_m} \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau \eta_{\tau_2-}^{J'} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2-} d\eta_{i,\tau_2}, \tag{3.39}$$

where $e_i \in \mathbb{N}_0^d$ is the i-th unit multi-index.

Proof. Given $L', L \in \mathbb{N}_0^d$, $i \in \{1, \dots, d\}$ such that $L' \leqslant L$ and $l'_i > 0$ we take

$$J = L - e_i, \qquad J' = L' - e_i. \tag{3.40}$$

From (3.19) we find that

$$\mathfrak{R}_{J',J,\tau} = \frac{l'_i}{l_i} \mathfrak{R}_{L',L,\tau}. \tag{3.41}$$

Since

$$U_{J,\tau_2} W_{i,\tau_2} = (j_i + 1) U_{J+e_i,\tau_2} = l_i U_{L,\tau_2},$$

it follows

$$l'_i \int_0^\tau \eta_{\tau_2-}^{L'-e_i} U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2} = \frac{l'_i}{l_i} \int_0^\tau \eta_{\tau_2-}^{J'} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2}. \tag{3.42}$$

Multiplying both sides of (3.42) by $\mathfrak{R}_{L',L,\tau}$ and using (3.41) gives

$$\mathfrak{R}_{L',L,\tau} l'_i \int_0^\tau \eta_{\tau_2-}^{L'-e_i}\, U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2} = \mathfrak{R}_{J',J,\tau} \int_0^\tau \eta_{\tau_2-}^{J'} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2}. \tag{3.43}$$

We sum (3.43) over all triples (L, L', i) such that $i = 1, \dots, d$, $L' \leqslant L$, $L \in \mathcal{B}_{m+1}$ and $l'_i > 0$. By (3.40), $L = J + e_i$ and $L' = J' + e_i$ which means that

$$\{(L, L', i) : i = 1, \dots, d,\ L' \leqslant L \in \mathcal{B}_{m+1}, l'_i > 0\} = \{(J, J', i) : i = 1, \dots, d, J' \leqslant J \in \mathcal{B}_m\},$$

i.e. we can express the summation of the r.h.s. of (3.43) in terms of (J, J', i). On the left, the terms where $l'_i = 0$ do not contribute and we may, therefore, simply sum over all (L, L', i) with $L' \leqslant L \in \mathcal{B}_{m+1}$, $i = 1, \dots, d$. This proves (3.39). ■

Using integration by parts for $I_{k,\tau}$ will lead to expressions of the form

$$Q_{J,i,\tau} = \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_2} \bigg(\int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1} \bigg) d\eta^{J'}_{\tau_2}, \tag{3.44}$$

where $J \in \mathbb{N}_0^d$, $i \in \{0, \dots, d\}$ and U_{J,τ_2} is given by (3.26). We want to estimate the norm of the sum $\sum_{|J|=m} Q_{J,i,\tau}$ through the norm of $\sum_{|L|=m+1} Q_{L,i,\tau}$ plus some correction terms. This will be needed in the main induction step in the proof of Proposition 3.1.

Lemma 3.5 *For any $m \in \mathbb{N}$ the stochastic integrals $Q_{J,i,\tau}$ satisfy*

$$\begin{aligned} \Big\| \sum_{i=1}^d \sum_{J \in \mathcal{B}_m} Q_{J,i,\tau} \Big\|_\infty &\leqslant \\ &\leqslant \frac{D_\tau^m}{m!} \int_0^\tau \|I_{k-1,\tau_1-}\|_\infty \, dD_{\tau_1} + \frac{D_\tau^{m-1}}{(m-1)!} \int_0^\tau \|I_{k,\tau_2-}\|_\infty \, dD_{\tau_2} \\ &\quad + \Big\| \sum_{L \in \mathcal{B}_{m+1}} \sum_{L' \leqslant L} \mathfrak{R}_{L',L,\tau} \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta^{L'}_{\tau_2} \Big\|_\infty . \end{aligned} \tag{3.45}$$

Proof. We write for the right-hand side of (3.45) $\mathbf{I} + \mathbf{II} + \mathbf{III}$. For any $J \in \mathcal{B}_m$, $i \in \{1, \dots, d\}$ an application of Corollary 3.1 with $\mathfrak{A} = \{J' \in \mathbb{N}_0 : J' \leqslant J\}$, $\alpha = J'$ and

$$U_{\tau_2} = U_{J,\tau_2}, \quad \Phi_{\tau_1} = W_{i,\tau_1} I_{k-1,\tau_1}, \quad \nu_{\tau_1} = \eta_{i,\tau_1}, \quad \kappa_{\alpha,\tau} = \eta^{J'}_\tau, \quad \widetilde{\kappa}_{\alpha,\tau} = \mathfrak{R}_{J',J,\tau}$$

yields

$$\begin{aligned} Q_{J,i,\tau} = &- \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_1} W_{i,\tau_1} I_{k-1,\tau_1-}\, d[\eta^{J'}, \eta_i]_{\tau_1} \\ &- \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau dU_{J,\tau_2} \times \eta^{J'}_{\tau_2-} \int_0^{\tau_2-} W_{i,\tau_1} I_{k-1,\tau_1-}\, d\eta_{i,\tau_1}, \\ &- \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau \eta^{J'}_{\tau_2-} U_{J,\tau_2} W_{i,\tau_2} I_{k-1,\tau_2-} d\eta_{i,\tau_2}, \end{aligned} \tag{3.46}$$

where we used that, by (3.20),

$$\sum_{\alpha \in \mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau} \kappa_{\alpha,\tau} = \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \eta^{J'}_\tau = 0,$$

i.e. (3.17) is satisfied.

Observe that the first two integrals in (3.46) are just $A_{J,i,\tau}$ and $B_{J,i,\tau}$ defined in (3.27) and (3.28). Therefore, Lemma 3.3 gives

$$\left\| \sum_{i=1}^{d} \sum_{J\in\mathcal{B}_m} Q_{J,i,\tau} \right\|_\infty \leqslant \frac{1}{2}\,\mathbf{I} + \mathbf{II} + \left\| \sum_{i=1}^{d} \sum_{J\in\mathcal{B}_m} \widetilde{C}_{J,i,\tau} \right\|_\infty, \tag{3.47}$$

with **I**, **II** as required for (3.45) and with $-\widetilde{C}_{J,i,\tau}$ given by the last term in (3.46). We still have to estimate the last sum.

Recall that $f_{L'}(x_1,\dots,x_d)$ and δ_{L',τ_2} are given by formulae (3.24) and (3.25) respectively. Since the $\eta_{j,\tau}$ are pure-jump semimartingales, an application of Itô's formula to $f_{L'}(x_1,\dots,x_d) = x^{L'}$ and the process $\eta_{\tau_2} = (\eta_{1,\tau_2},\dots,\eta_{d,\tau_2})$ yields

$$\begin{aligned}\int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{\tau_2}^{L'} &= \sum_{i=1}^{d} \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-} \frac{\partial f_{L'}(\eta_{\tau_2-})}{\partial x_i}\, d\eta_{i,\tau_2} \\ &\quad + \sum_{0<\tau_2\leqslant\tau} U_{L,\tau_2} I_{k-1,\tau_2-} \delta_{L',\tau_2}.\end{aligned} \tag{3.48}$$

We multiply (3.48) by $\mathfrak{R}_{L',L,\tau}$, sum it over all $L' \leqslant L$ and obtain using $\partial f_{L'}(\eta_{\tau_2-})/\partial x_i = l'_i \eta_{\tau_2-}^{L'-e_i}$

$$\begin{aligned}&\sum_{L'\leqslant L} \mathfrak{R}_{L',L,\tau} \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{\tau_2}^{L'} \\ &\qquad = \sum_{i=1}^{d} \sum_{L'\leqslant L} \mathfrak{R}_{L',L,\tau} l'_i \int_0^\tau \eta_{\tau_2-}^{L'-e_i} U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{i,\tau_2} + C_{L,\tau},\end{aligned} \tag{3.49}$$

with $C_{L,\tau}$ as in (3.29). Summing (3.49) over all $L \in \mathcal{B}_{m+1}$ and applying Lemma 3.4 we find

$$\sum_{L\in\mathcal{B}_{m+1}} \sum_{L'\leqslant L} \mathfrak{R}_{L',L,\tau} \int_0^\tau U_{L,\tau_2} I_{k-1,\tau_2-}\, d\eta_{\tau_2}^{L'} = \sum_{i=1}^{d} \sum_{J\in\mathcal{B}_m} \widetilde{C}_{J,i,\tau} + \sum_{L\in\mathcal{B}_{m+1}} C_{L,\tau}.$$

Therefore, by (3.32),

$$\left\| \sum_{i=1}^{d} \sum_{J\in\mathcal{B}_m} \widetilde{C}_{J,i,\tau} \right\|_\infty \leqslant \left\| \sum_{L\in\mathcal{B}_{m+1}} C_{L,\tau} \right\|_\infty + \mathbf{III} \leqslant \frac{1}{2}\,\mathbf{I} + \mathbf{III}.$$

Combining this and (3.47) we arrive at (3.45). ■

Here is the modification of Lemma 3.5 for the case when the inner integral in (3.44) is driven by $d\tau$.

Lemma 3.6 *If $\ell_{2n} < k' < \ell_{2n+1}$ for some $n \in \mathbb{N}$ or $1 \leqslant k' < \ell_1$ then*

$$\left\| \sum_{J\in\mathcal{B}_m} \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_2} \Big(\int_0^{\tau_2-} W_{0,\tau_1} I_{k'-1,\tau_1}\, d\tau_1 \Big) d\eta^{J'}_{\tau_2} \right\|_\infty \tag{3.50}$$
$$\leqslant \frac{M}{m!} D_\tau^m \int_0^\tau \|I_{k'-1,\tau_2}\|_\infty\, d\tau_2 + \frac{1}{(m-1)!} D_\tau^{m-1} \int_0^\tau \|I_{k',\tau_2-}\|_\infty\, dD_{\tau_2}$$

Proof. For any $J \in \mathcal{B}_m$, $i \in \{1,\dots,d\}$ we apply Corollary 3.1 with $\mathfrak{A} = \{J' \in \mathbb{N}_0 : J' \leqslant J\}$, $\alpha = J'$, $\nu_\tau = \eta_{0,\tau} = \tau$ and

$$U_{\tau_2} = U_{J,\tau_2}, \quad \Phi_{\tau_1} = W_{0,\tau_1} I_{k'-1,\tau_1}, \quad \kappa_{\alpha,\tau} = \eta^{J'}_\tau, \quad \widetilde{\kappa}_{\alpha,\tau} = \mathfrak{R}_{J',J,\tau}.$$

As before

$$\sum_{\alpha\in\mathfrak{A}} \widetilde{\kappa}_{\alpha,\tau}\kappa_{\alpha,\tau} = \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau}\eta^{J'}_\tau = 0.$$

Moreover, $[\nu, \kappa_\alpha] = 0$. Therefore we estimate the left-hand side of (3.50) by $\mathbf{I} + \mathbf{II}$, where

$$\mathbf{I} = \left\| \sum_{J\in\mathcal{B}_m} \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau \eta^{J'}_{\tau_2} U_{J,\tau_2} W_{0,\tau_2} I_{k'-1,\tau_2}\, d\tau_2 \right\|_\infty,$$
$$\mathbf{II} = \left\| \sum_{J\in\mathcal{B}_m} \sum_{J'\leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau dU_{J,\tau_2} \times \eta^{J'}_{\tau_2-} \int_0^{\tau_2-} W_{0,\tau_1} I_{k'-1,\tau_1}\, d\tau_1 \right\|_\infty.$$

Similar calculations to those in the proof of Lemma 3.3 give

$$\mathbf{I} \leqslant M^{m+1} \sum_{J\in\mathcal{B}_m} \frac{1}{J!} \Big(\sum_{J'\leqslant J} |\mathfrak{R}_{J',J,\tau}| \sup_{0\leqslant\tau_1\leqslant\tau} |\eta^{J'}_{\tau_1}| \Big) \int_0^\tau \|I_{k'-1,\tau_2}\|_\infty\, d\tau_2$$
$$\leqslant M \frac{D_\tau^m}{m!} \int_0^\tau \|I_{k'-1,\tau_2}\|_\infty\, d\tau_2.$$

From (3.37) and arguments similar to those of Lemma 3.3 we deduce

$$\mathbf{II} \leqslant M^m \sum_{J\in\mathcal{B}_m} \frac{m}{J!} \Big(\sum_{J'\leqslant J} |\mathfrak{R}_{J',J,\tau}| \sup_{0\leqslant\tau_1\leqslant\tau} |\eta^{J'}_{\tau_1}| \Big) \int_0^\tau \|I_{k',\tau_1-}\|_\infty\, dv_{\tau_1}$$
$$\leqslant (Md) \frac{D_\tau^{m-1}}{(m-1)!} \int_0^\tau \|I_{k',\tau_1-}\|_\infty\, dv_{\tau_1}$$
$$\leqslant \frac{D_\tau^{m-1}}{(m-1)!} \int_0^\tau \|I_{k',\tau_1-}\|_\infty\, dD_{\tau_1}.$$

Piecing together the above estimates completes the proof. ■

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality we may assume that $t_0 = 0$. We define p_l, $l \in \mathbb{N}_0$, recursively by $p_0 = 1$ and for

$$l \in (\gamma_{n-1}, \gamma_n] := \left(\sum_{i=1}^{n-1} (\ell_{2i} - \ell_{2i-1} + 1), \sum_{i=1}^{n} (\ell_{2i} - \ell_{2i-1} + 1) \right], \quad \gamma_0 = 0$$

by

$$p_l = p_{\gamma_{n-1}} q_{l-\gamma_{n-1}}, \tag{3.51}$$

where the q_k are defined by formula (A.3) in the appendix.

Recall that $\mathfrak{m} = \mathfrak{m}(k)$, D_τ were defined at the beginning of this section. Throughout this proof we suppress the argument in $\mathfrak{m}(\cdot)$ if the argument is k, i.e. $\mathfrak{m} = \mathfrak{m}(k)$. We split the proof into two steps.

Step 1. We show by induction that

$$\|I_{k,\tau}\|_\infty \leqslant p_{\mathfrak{m}} D_\tau^{\mathfrak{m}} \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \|I^*_{0,\tau}\|_\infty, \qquad k \in \mathbb{N}. \tag{3.52}$$

Clearly, (3.52) is true for $k = 0$. Assume that (3.52) holds for $0, \dots, k-1$.

Case 1. $\ell_{2n} < k < \ell_{2n+1}$ for some $n \in \mathbb{N}$ or $1 \leqslant k < \ell_1$. In this case $\eta_{j_k,\tau} = \tau$ in the definition of $I_{k,\tau}$ and $\mathfrak{m}(k) = \mathfrak{m}(k-1)$. Therefore,

$$\begin{aligned} \|I_{k,\tau}\|_\infty \leqslant M \int_0^\tau \|I_{k-1,s}\|_\infty \, ds &\leqslant M \int_0^\tau p_{\mathfrak{m}} D_s^{\mathfrak{m}} \frac{(Ms)^{k-1-\mathfrak{m}}}{(k-1-\mathfrak{m})!} \, ds \, \|I^*_{0,\tau}\|_\infty \\ &\leqslant p_{\mathfrak{m}} D_\tau^{\mathfrak{m}} \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!} \|I^*_{0,\tau}\|_\infty, \end{aligned}$$

which is just (3.52).

Case 2. $\ell_{2n-1} \leqslant k \leqslant \ell_{2n}$ for some $n \in \mathbb{N}$; then $\eta_{j_k,\tau}$ is a semimartingale. For $\widehat{m} \in \mathbb{N}$ and $\hat{r} = \ell_{2n-1}, \dots, k$, we denote by

$$z^{\widehat{m}}_{\hat{r},\tau} = \sum_{J \in \mathcal{B}_{\widehat{m}}} \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,s} I_{\hat{r}-1,s-} \, d\eta^{J'}_s.$$

Note that $z^1_{\hat{r},\tau} = I_{\hat{r},\tau}$ and

$$z^{\widehat{m}}_{\hat{r}+1,\tau} = \sum_{i=1}^{d} \sum_{J \in \mathcal{B}_{\widehat{m}}} Q_{J,i,\tau}, \quad z^{\widehat{m}+1}_{\hat{r},\tau} = \sum_{L \in \mathcal{B}_{\widehat{m}+1}} \sum_{L' \leqslant L} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{L,\tau_2} I_{\hat{r}-1,\tau_2-} \, d\eta^{L'}_{\tau_2},$$

where

$$Q_{J,i,\tau} = \sum_{J' \leqslant J} \mathfrak{R}_{J',J,\tau} \int_0^\tau U_{J,\tau_2} \left(\int_0^{\tau_2-} W_{i,\tau_1} I_{\hat{r}-1,\tau_1-} \, d\eta_{i,\tau_1} \right) d\eta^{J'}_{\tau_2}.$$

Applying Lemma 3.5 with the above $Q_{J,i,\tau}$ we get

$$\|z^{\widehat{m}}_{\widehat{r}+1,\tau}\|_\infty \leqslant \tag{3.53}$$
$$\leqslant \frac{D^{\widehat{m}}_\tau}{\widehat{m}!}\int_0^\tau \|I_{\widehat{r}-1,s-}\|_\infty\, dD_s + \frac{D^{\widehat{m}-1}_\tau}{(\widehat{m}-1)!}\int_0^\tau \|I_{\widehat{r},s-}\|_\infty\, dD_s + \|z^{\widehat{m}+1}_{\widehat{r},\tau}\|_\infty.$$

If $k > \ell_{2n-1}$ we obtain after the change of indices $\widehat{m} = m+1$, $\widehat{r}+1 = k-m$, $m = 0, 1, \dots, k-\ell_{2n-1}-1$,

$$\|z^{m+1}_{k-m,\tau}\|_\infty - \|z^{m+2}_{k-(m+1),\tau}\|_\infty \leqslant \beta_{m+1,\tau} + \beta_{m,\tau}, \tag{3.54}$$

where

$$\beta_{m,\tau} = \frac{D^m_\tau}{m!}\int_0^\tau \|I_{k-m-1,s}\|_\infty\, dD_s. \tag{3.55}$$

Summing (3.54) over $m = 0, \dots, (k-\ell_{2n-1}-1)$, we get

$$\|z^1_{k,\tau}\|_\infty - \|z^{k-\ell_{2n-1}+1}_{\ell_{2n-1},\tau}\|_\infty \leqslant \beta_{0,\tau} + \beta_{k-\ell_{2n-1},\tau} + 2\sum_{m=1}^{k-\ell_{2n-1}-1} \beta_{m,\tau}. \tag{3.56}$$

Set $k' = \ell_{2n-1}-1$ and observe that $\ell_{2n-2} < k' < \ell_{2n-1}$. An application of Lemma 3.6 gives

$$\|z^{k-\ell_{2n-1}+1}_{\ell_{2n-1},\tau}\|_\infty \leqslant \frac{D^{k-\ell_{2n-1}+1}_\tau}{(k-\ell_{2n-1}+1)!}\, M\int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty\, d\tau_2 + \beta_{k-\ell_{2n-1},\tau}. \tag{3.57}$$

If $k = \ell_{2n-1}$, the inequality (3.57) is obviously true by Lemma 3.6. Combining (3.56), (3.57) we arrive at

$$\|z^1_{k,\tau}\|_\infty \leqslant \frac{D^{k-\ell_{2n-1}+1}_\tau}{(k-\ell_{2n-1}+1)!}\, M\int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty\, d\tau_2 + 2\sum_{m=0}^{k-\ell_{2n-1}} \beta_{m,\tau}. \tag{3.58}$$

Since

$$\mathfrak{m}(k-m-1) = \mathfrak{m}(k) - m - 1, \qquad m = 0, \dots, k-\ell_{2n-1}, \tag{3.59}$$

we can use the induction hypothesis (3.52) and deduce from (3.55) for $m = 0, 1, \dots, k-\ell_{2n-1}$,

$$\begin{aligned} \beta_{m,\tau} &\leqslant \frac{D^m_\tau}{m!}\, p_{\mathfrak{m}-m-1}\int_0^\tau D^{\mathfrak{m}-m-1}_s \frac{(Ms)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\, dD_s\, \|I^*_{0,\tau}\|_\infty \\ &\leqslant \frac{p_{\mathfrak{m}-m-1}}{(\mathfrak{m}-m)\, m!}\, D^{\mathfrak{m}}_\tau \frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\, \|I^*_{0,\tau}\|_\infty\,. \end{aligned} \tag{3.60}$$

Since $\mathfrak{m}(\ell_{2n-1}-2)=\mathfrak{m}(\ell_{2n-1}-1)=\mathfrak{m}(k)-(k-\ell_{2n-1}+1)$ we may use again the induction hypothesis (3.52) to get

$$M\int_0^\tau \|I_{\ell_{2n-1}-2,\tau_2}\|_\infty\,d\tau_2 \leqslant p_{\mathfrak{m}-(k-\ell_{2n-1}+1)}D_\tau^{\mathfrak{m}-(k-\ell_{2n-1}+1)}\,M\int_0^\tau \frac{(Ms)^{k-\mathfrak{m}-1}}{(k-\mathfrak{m}-1)!}\,ds\,\|I^*_{0,\tau}\|_\infty\,. \tag{3.61}$$

Together (3.58), (3.60) and (3.61) show

$$\|z^1_{k,\tau}\|_\infty \leqslant p_{\mathfrak{m}-(k-\ell_{2n-1}+1)}\,\frac{D_\tau^{\mathfrak{m}}}{(k-\ell_{2n-1}+1)!}\,\frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\,\|I^*_{0,\tau}\|_\infty + 2\sum_{m=0}^{k-\ell_{2n-1}}\frac{p_{\mathfrak{m}-m-1}D_\tau^{\mathfrak{m}}}{(\mathfrak{m}-m)\,m!}\,\frac{(M\tau)^{k-\mathfrak{m}}}{(k-\mathfrak{m})!}\,\|I^*_{0,\tau}\|_\infty\,. \tag{3.62}$$

Since $\mathfrak{m}-(k-\ell_{2n-1}+1)=\gamma_{n-1}$ it follows from (3.51), (3.59) with $l=\mathfrak{m}-m-1$ that

$$p_{\mathfrak{m}-m-1}=p_{\gamma_{n-1}}q_{(k-\ell_{2n-1}+1)-m-1}, \qquad m=0,\dots,k-\ell_{2n-1}.$$

From this and definition (A.3) in the appendix we find

$$\frac{p_{\mathfrak{m}-(k-\ell_{2n-1}+1)}}{(k-\ell_{2n-1}+1)!}+\sum_{m=0}^{k-\ell_{2n-1}}\frac{2p_{\mathfrak{m}-m-1}}{(\mathfrak{m}-m)\,m!}\leqslant p_{\gamma_{n-1}}\sum_{m=0}^{k-\ell_{2n-1}}\frac{3q_{(k-\ell_{2n-1}+1)-m-1}}{((k-\ell_{2n-1}+1)-m)\,m!}$$

$$= p_{\gamma_{n-1}}q_{k-\ell_{2n-1}+1}=p_{\mathfrak{m}}, \tag{3.63}$$

where we used $\mathfrak{m}\geqslant k-\ell_{2n-1}+1$. Combining (3.62) and (3.63) we arrive at (3.52).

Step 2. We are going to prove that

$$\frac{p_{\mathfrak{m}}}{(k-\mathfrak{m})!}\leqslant\frac{(2^7)^k}{(\ln\ln(k+1))^{\frac{k}{16}}}\,. \tag{3.64}$$

If $1\leqslant k<\ell_1$ then $\mathfrak{m}(k)=0$, $p_{\mathfrak{m}}=p_0=1$, and estimate (3.64) is clear. From definition (3.51) we deduce

$$p_{\mathfrak{m}}=q_{k-\ell_{2n-1}+1}\prod_{j=1}^{n-1}q_{\ell_{2j}-\ell_{2j-1}+1} \qquad \text{for} \quad \ell_{2n-1}\leqslant k\leqslant\ell_{2n}$$

and

$$p_{\mathfrak{m}}=\prod_{j=1}^{n}q_{\ell_{2j}-\ell_{2j-1}+1} \qquad \text{for} \quad \ell_{2n}<k<\ell_{2n+1}.$$

From (A.4) we know

$$\frac{p_{\mathfrak{m}}}{(k-\mathfrak{m})!} \leqslant (2^6)^k Z^{-1}, \tag{3.65}$$

where

$$Z = (k-\mathfrak{m})! \prod_{j=1}^{n} (\ln\{\alpha_j + 1\})^{\frac{\alpha_j}{2}}.$$

Here $\alpha_j = \ell_{2j} - \ell_{2j-1} + 1$, $j = 1, \dots, n-1$, and $\alpha_n = k - \ell_{2n-1} + 1$ for $\ell_{2n-1} \leqslant k \leqslant \ell_{2n}$, $\alpha_n = \ell_{2n} - \ell_{2n-1} + 1$ for $\ell_{2n} < k < \ell_{2n+1}$. Clearly $k - \mathfrak{m} \geqslant n - 1$. Using the estimate from Lemma $A.3$ with $m = k - \mathfrak{m}$ gives

$$Z \geqslant \frac{1}{2^k} (\ln\ln(k+1))^{\frac{k}{16}} \tag{3.66}$$

and (3.65), (3.66) show (3.64). The proposition now follows from (3.52) and (3.64). ■

4. Boundary value problems for stochastic Hamilton systems (diffeomorphism theorem)

We consider the following Hamiltonian system

$$\begin{cases} dx = p\,dt \\ dp = \dfrac{\partial V(x)}{\partial x}\,dt - \dfrac{\partial c(x)}{\partial x} d\xi_t, \end{cases} \tag{4.1}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = t_0$. We write

$$(X, P) = (X(t, t_0, x_0, p_0), P(t, t_0, x_0, p_0)) \in \mathbb{R}^{2d}$$

for its solution. The coefficients

$$\frac{\partial V}{\partial x} \in \mathbb{R}^d, \quad \frac{\partial c}{\partial x} \in \mathbb{R}^{d\times d}$$

are derivatives of functions $V : \mathbb{R}^d \to \mathbb{R}^1$ and $c = (c_1, \dots, c_d) : \mathbb{R}^d \to \mathbb{R}^d$ which admit (at least) continuous partial derivatives up to order 3 such that

$$\left|\frac{\partial^{|L|} V(x)}{\partial x^L}\right|, \left|\frac{\partial^{|L|} c(x)}{\partial x^L}\right| \leqslant K \quad |L| = 2, 3 \tag{4.2}$$

and

$$\frac{\partial c(x)}{\partial x} = 0, \qquad \forall \quad |x| > K \tag{4.3}$$

for some constant $K > 1$.

The driving noise $\xi_t = (\xi_{1,t}, \dots, \xi_{d,t})$ is a d-dimensional Lévy process such that

$$[\xi_j, \xi_j]^c = 0\,, \qquad j = 1, \dots, d, \tag{4.4}$$

i.e. it contains no Brownian component. The main result of this section is the following.

Theorem 4.1 *Under the assumptions* (4.2)-(4.4)*, there exists a stopping time T such that $\mathbb{P}(T > 0) = 1$ and for $0 \leqslant t_0 < t < T(\omega)$, $x_0 \in \mathbb{R}^d$,*
(***i***) *the system* (4.1) *has a solution (X,P),*

$$\frac{\partial X}{\partial x_0} = E_d + O(t - t_0), \quad \frac{\partial P}{\partial p_0} = E_d + O(t - t_0), \tag{4.5}$$

$$\frac{\partial X}{\partial p_0} = (t - t_0)E_d + O((t - t_0)^2), \tag{4.6}$$

where $O(\cdot)$ is uniform with respect to x_0, p_0,
(***ii***) *the map*

$$\mathfrak{D} : \mathbb{R}^d \to \mathbb{R}^d, \quad p_0 \mapsto X(t, t_0, x_0, p_0)$$

is a diffeomorphism.

Remark. We can rewrite the system (4.1) in the following form

$$\begin{pmatrix} X(t) \\ P(t) \end{pmatrix} = \int_{t_0}^{t} \mathcal{V}(X(s), P(s))\, ds - \int_{t_0}^{t} \gamma(X(s), P(s))\, d\zeta_s$$

with coefficients

$$\mathcal{V}(x, p) = \begin{pmatrix} p \\ \partial V(x)/\partial x \end{pmatrix} \in \mathbb{R}^{2d}, \qquad \gamma(x, p) = \begin{pmatrix} 0 & 0 \\ 0 & \partial c(x)/\partial x \end{pmatrix} \in \mathbb{R}^{2d \times 2d}$$

and the (degenerate) Lévy noise

$$\zeta_t = \begin{pmatrix} 0 \\ \xi_t \end{pmatrix} \in \mathbb{R}^{2d}.$$

Notice that $\mathcal{V}$ and γ are globally Lipschitz continuous; Theorem 7 of [Pro, pp. 197-198] guarantees existence and uniqueness of a solution

$$\left\{ \begin{pmatrix} X(t) \\ P(t) \end{pmatrix} \right\}_{t \geqslant t_0}$$

.

Moreover, if the coefficients have globally Lipschitz continuous partial derivatives up to order $N + 2$, then we may differentiate

$$\left\{ \begin{pmatrix} X(t) \\ P(t) \end{pmatrix} \right\}_{t \geqslant t_0}$$

w.r.t. the initial conditions up to order N, cf. [Pro, p. 254, Theorem 40].

For the proof of Theorem 4.1 we need the following auxiliary result.

Lemma 4.1 *There exists a constant $K_1 = K_1(K, d)$ such that for $t_0 \leqslant a \leqslant b < \mathcal{R} \wedge K_1^{-1}$ ($\mathcal{R}$ is the stopping time from Lemma 2.1 and K is the constant from (4.2))*

$$\int_a^b |P(\tau)|\, d\tau \leqslant 3|X(a) - X(b)| + K_1(b-a),$$

where

$$X(\tau) = X(\tau, t_0, x_0, p_0), \qquad P(\tau) = P(\tau, t_0, x_0, p_0). \tag{4.7}$$

Proof. *Step 1.* From the system (4.1) we find

$$P(\tau) - P(a) = \int_a^\tau \frac{\partial V(X(s))}{\partial x}\, ds - \int_a^\tau \frac{\partial c(X(s))}{\partial x}\, d\xi_s\,.$$

Since $X(t)$, $\partial V(X(t))/\partial x$ and $\partial c(X(t))/\partial x$ are continuous BV-processes, we find by integration by parts

$$\begin{aligned} P(\tau) - P(a) &= \frac{\partial V(X(a))}{\partial x}(\tau - a) - \frac{\partial c(X(a))}{\partial x}\,(\xi_\tau - \xi_a) \\ &\quad - \int_a^\tau (s-\tau)\frac{\partial^2 V(X(s))}{\partial x^2} P(s)\, ds + \int_a^\tau \frac{\partial^2 c(X(s))(\xi_s - \xi_a)}{\partial x^2} P(s)\, ds. \end{aligned} \tag{4.8}$$

Here

$$\frac{\partial^2 c(X(s))(\xi_s - \xi_a)}{\partial x^2} = \sum_{i=1}^d \frac{\partial^2 c_i(X(s))(\xi_{i,s} - \xi_{i,a})}{\partial x^2} \in \mathbb{R}^{d\times d}.$$

We know from Lemma 2.1 that

$$|\xi_\tau - \xi_a| \leqslant 2 \sup_{0\leqslant s\leqslant \tau} |\xi_s| \leqslant 2\sum_{i=1}^d \sup_{0\leqslant s\leqslant\tau} |\xi_{i,s}| \leqslant \vartheta_t < 1$$

for $\tau < \mathcal{R}$ and so

$$|P(\tau)| \leqslant |P(a)| + C_1 + C_1 \int_a^\tau |P(s)|\, ds, \tag{4.9}$$

where

$$C_1 = \max_{i=1,\dots,d} \sup_{x\in\mathbb{R}^d} \left(d\left\|\frac{\partial^2 V}{\partial x^2}\right\|_\infty + d\left\|\frac{\partial^2 c_i}{\partial x^2}\right\|_\infty\right) \vee \left(\sqrt{d}\left\|\frac{\partial V}{\partial x}\right\|_\infty + \sqrt{d}\left\|\frac{\partial c}{\partial x}\right\|_\infty\right).$$

Integrating (4.9) we have for $b < \mathcal{R}$

$$\int_a^b |P(\tau)|\, d\tau \leqslant |P(a)|(b-a) + C_1(b-a) + C_1 \int_a^b \int_a^\tau |P(s)|\, ds d\tau. \tag{4.10}$$

Since for $b-a<(3C_1)^{-1}$

$$\begin{aligned} C_1\int_a^b\int_a^\tau |P(s)|\,dsd\tau &= C_1(b-a)\int_a^b |P(s)|\,ds - C_1\int_a^b(\tau-a)|P(\tau)|\,d\tau \\ &\leqslant \frac{1}{3}\int_a^b |P(s)|\,ds, \end{aligned} \tag{4.11}$$

we deduce from (4.10) that

$$\int_a^b |P(\tau)|\,d\tau \leqslant \frac{3}{2}\,(|P(a)|+C_1)(b-a). \tag{4.12}$$

Step 2. Similarly, we find from (4.8)

$$|P(\tau)-P(a)| \leqslant C_1 + C_1\int_a^\tau |P(s)|\,ds. \tag{4.13}$$

We integrate (4.13) to get

$$\begin{aligned} \int_a^b |P(\tau)-P(a)|\,d\tau &\leqslant C_1(b-a)+C_1\int_a^b\int_a^\tau |P(s)|\,dsd\tau \\ &\leqslant C_1(b-a)+\frac{1}{3}\int_a^b |P(s)|\,ds, \end{aligned}$$

where we used (4.11) and so, by (4.12)

$$\int_a^b |P(\tau)-P(a)|\,d\tau \leqslant \frac{1}{2}(|P(a)|+3C_1)(b-a).$$

Thus

$$\begin{aligned} \left|\int_a^b P(\tau)\,d\tau\right| &\geqslant |P(a)|(b-a)-\int_a^b |P(\tau)-P(a)|\;d\tau \\ &\geqslant \frac{1}{2}(|P(a)|-3C_1)(b-a). \end{aligned} \tag{4.14}$$

Combining (4.12) and (4.14) we arrive at

$$\int_a^b |P(\tau)|\,d\tau \leqslant 6C_1(b-a)+3\left|\int_a^b P(\tau)\,d\tau\right|.$$

The assertion follows with $K_1=6C_1$. ■

Corollary 4.1 *Let $f : \mathbb{R}^d \to \mathbb{R}$, $f \in C^1$ and*

$$f^{(1)}(x) = 0 \quad \text{if} \quad |x| > K$$

(K is the constant from (4.2)). Then for $0 \leqslant t_0 \leqslant t < \mathcal{R} \wedge K_2^{-1}$

$$\operatorname{Var}_{[t_0,t]} f(X(\cdot)) \leqslant K_2, \tag{4.15}$$

where $K_2 = K_2(K, d, f)$ is some constant.

Proof. Let $\mathcal{B} = \{\tau \in [t_0, t] : |X(\tau)| \leqslant K\}$. If $\mathcal{B} = \emptyset$ then the left-hand side of (4.15) vanishes and the assertion of the corollary is clear. Otherwise we set $a = \inf\{\tau : \tau \in \mathcal{B}\}$, $b = \sup\{\tau : \tau \in \mathcal{B}\}$.
Since the first derivative $f^{(1)}$ has compact support, $\operatorname{supp} f^{(1)} \subset \{x : |x| \leqslant K\}$,

$$\begin{aligned}\operatorname{Var}_{[t_0,t]} f(X(\cdot)) &= \int_a^b |f^{(1)}(X(s))P(s)|\, ds \\ &\leqslant \sup_{x\in\mathbb{R}^d} |f^{(1)}(x)| \int_a^b |P(\tau)|\, d\tau\end{aligned}$$

and (4.15) follows from Lemma 4.1. ■

We introduce a new stopping time

$$T = \mathcal{R} \wedge \widetilde{K}_2^{-1},$$

where $\widetilde{K}_2 = \max\limits_{i,j=1,\dots,d} K_2(K, d, \partial c_i/\partial x_j)$ and K_2 is defined in Corollary 4.1.

Proof of Theorem 4.1. *Step 1.* Since (X, P) can be differentiated with respect to the initial data (x_0, p_0), we find that the matrix-valued process

$$G = \frac{\partial(X,P)}{\partial(x_0,p_0)} = \begin{pmatrix} \partial X/\partial x_0 & \partial X/\partial p_0 \\ \partial P/\partial x_0 & \partial P/\partial p_0 \end{pmatrix}$$

satisfies the formally differentiated system (4.1) (cf. also [Pro], proof of Theorem 39, p. 250):

$$dG = W_{0,t} G\, dt + \sum_{j=1}^d W_{j,t} G\, d\xi_{j,t}, \qquad G\big|_{t=t_0} = G_0 = \begin{pmatrix} E_d & 0 \\ 0 & E_d \end{pmatrix}, \tag{4.16}$$

where $E_d \in \mathbb{R}^{d\times d}$ is the identity matrix and

$$W_{0,t} = \begin{pmatrix} 0 & E_d \\ \partial^2 V(X(t))/\partial x^2 & 0 \end{pmatrix}, \quad W_{j,t} = \begin{pmatrix} 0 & 0 \\ -\partial^2 c_j(X(t))/\partial x^2 & 0 \end{pmatrix}. \tag{4.17}$$

A solution of the system (4.16) can be given by the following (formal) series expansion,

$$G = \sum_{k=0}^{\infty} G_k \tag{4.18}$$

with $G_0 = G\big|_{t=t_0}$,

$$G_1 = \sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau} G_0 \, d\eta_{j,\tau} \quad \text{and} \quad G_k = \sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau} G_{k-1} \, d\eta_{j,\tau} \,, \quad (k \in \mathbb{N})$$

where $\eta_\tau = (\eta_{0,\tau}, \eta_{1,\tau}, \dots, \eta_{d,\tau}) = (\tau, \xi_{1,\tau}, \dots, \xi_{d,\tau})$ is a $(d+1)$-dimensional semimartingale.

Indeed, it is immediate that

$$\sum_{j=0}^{d} \int_{t_0}^{t} W_{j,\tau}(G_0 + G_1 + \cdots + G_k) \, d\eta_{j,\tau} = G_1 + \cdots + G_{k+1},$$

so (4.18) will give a solution of (4.16) whenever it converges uniformly (on compact intervals) in t.

Since the terms of series (4.18) are k-fold integrals, we get

$$G = E_{2d} + \sum_{k=1}^{\infty} \sum_{j_1,\dots,j_k=0}^{d} \widetilde{I}_{j_1,\dots,j_k,t} = E_{2d} + \widetilde{I}_{0,t} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{4.19}$$

where $A_{ij} \in \mathbb{R}^{d\times d}$ are suitable (series of) block-matrices and

$$\widetilde{I}_{j_1,\dots,j_k,t} = \int_{t_0}^{t} W_{j_k,\tau_k} \Big(\int_{t_0}^{\tau_k -} \cdots \Big(\int_{t_0}^{\tau_2 -} W_{j_1,\tau_1} \, d\eta_{j_1,\tau_1} \Big) \dots d\eta_{j_{k-1},\tau_{k-1}} \Big) d\eta_{j_k,\tau_k}. \tag{4.20}$$

Because of the particular form of the $W_{j,\tau}$'s in (4.17), we know more about the structure of A_{ij} in (4.19). Let

$$\mathcal{J}_1 = \{(j_1, \dots, j_k) : k \in \mathbb{N}, \ \text{ none of } \ j_1, \dots, j_k \text{ equals to } \ 0\}$$

(i.e. all integrators in (4.20) are Lévy processes) and

$$\mathcal{J}_2 = \{(j_1, \dots, j_k) : \ k \geqslant 2 \text{ at most one } j_1, \dots, j_k \text{ equals to } \ 0\}$$

(i.e. at most one $d\tau$ integration happens). If $(j_1, \dots, j_k) \in \mathcal{J}_1$, then the iterated integrals have the form

$$\widetilde{I}_{j_1,\dots,j_k,t} = \begin{pmatrix} 0 & 0 \\ r & 0 \end{pmatrix}, \quad r \in \mathbb{R}^{d\times d}$$

and if $(j_1, \dots, j_k) \in \mathcal{J}_2$, they are of the form

$$\widetilde{I}_{j_1,\dots,j_k,t} = \begin{pmatrix} r_{11} & 0 \\ r_{21} & r_{22} \end{pmatrix}, \quad r_{11}, r_{21}, r_{22} \in \mathbb{R}^{d\times d}.$$

Thus

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty \leqslant \Big\| \sum_{(j_1,\dots,j_k)\in\mathcal{J}\setminus\mathcal{J}_1} \widetilde{I}_{j_1,\dots,j_k,t} \Big\|_\infty, \tag{4.21}$$

$$\|A_{12}\|_\infty \leqslant \Big\| \sum_{(j_1,\dots,j_k)\in\mathcal{J}\setminus\mathcal{J}_2} \widetilde{I}_{j_1,\dots,j_k,t} \Big\|_\infty, \tag{4.22}$$

where $\mathcal{J} = \cup_{k=1}^\infty \{0,\dots,d\}^k$.

Step 2. Let us now verify the conditions needed in Proposition 3.1. Lemma 2.1 and condition (4.4) imply that $\eta_t = (t, \xi_t)$ satisfies (3.6) for $0 \leqslant t_0 \leqslant t < \mathcal{R}$.

Condition (4.2) implies that for some constant $K > 1$

$$\|W_{j,t}\|_\infty \leqslant K, \quad j = 0,\dots,d$$

and, by (4.17), we find

$$W_{j,\tau}W_{i,\tau} = 0 = W_{i,\tau}W_{j,\tau} \quad i,j = 1,\dots,d.$$

Definition (3.8) (with M=2d) and formula (4.15) with $f = (\partial c_i/\partial x_j)$, $i,j = 1,\dots,d$, give

$$D_t \leqslant 2d^2(\widetilde{K}_2 + 2\vartheta_t + 2\vartheta_{t_0}) \leqslant 2d^2(\widetilde{K}_2 + 4\vartheta_t),$$

where ϑ_t, ϑ_{t_0} are given by formula (2.4), and Lemma 2.1 shows

$$D_t \leqslant 2d^2(\widetilde{K}_2 + 4) = O(1), \qquad 0 \leqslant t_0 \leqslant t < T. \tag{4.23}$$

Step 3. For any

$$\mathcal{M}^k = \mathcal{M}_1^k \times \dots \times \mathcal{M}_k^k, \quad \mathcal{M}_i^k = \{1,\dots,d\} \quad \text{or} \quad \mathcal{M}_i^k = \{0\} \tag{4.24}$$

an application of Proposition 3.1 to the matrices $K^{-1}W_{j,\tau} \in \mathbb{R}^{2d\times 2d}$, $j = 1,\dots,d$, shows (note that $I_{0,\tau} = E_{2d}$ is the identity!)

$$\Big\| \sum_{(j_1,\dots,j_k)\in\mathcal{M}^k} \widetilde{I}_{j_1,\dots,j_k,t} \Big\|_\infty \leqslant K^k b_k D_t^{\mathfrak{m}} \{2d(t-t_0)\}^{k-\mathfrak{m}},$$

where b_k are given by (3.11). One readily sees

$$\mathcal{M}^k \cap \mathcal{J}_1 = \emptyset \Longrightarrow \mathfrak{m} \leqslant k-1 \quad \text{and} \quad \mathcal{M}^k \cap \mathcal{J}_2 = \emptyset \Longrightarrow \mathfrak{m} \leqslant k-2. \tag{4.25}$$

Consequently, using the fact that the set $\{0,\dots,d\}^k$ is the disjoint union of 2^k subsets of type (4.24), we find from (4.21)

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty \leqslant \sum_{k=2}^\infty 2^k \max \Big\| \sum_{(j_1,\dots,j_k)\in\mathcal{M}^k} \widetilde{I}_{j_1,\dots,j_k,t} \Big\|_\infty,$$

where the maximum is taken over all $\mathcal{M}^k$ such that $\mathfrak{m} \leqslant k-1$.

Hence

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty \leqslant \sum_{k=2}^{\infty} 2^k K^k b_k (D_t + 2d(t-t_0))^{k-1}(2d)(t-t_0), \tag{4.26}$$

where we used that, by (4.25), $\mathfrak{m} \leqslant k-1$, and so

$$D_t^{\mathfrak{m}}\{2d(t-t_0)\}^{k-\mathfrak{m}} \leqslant (D_t + 2d(t-t_0))^{k-1}(2d)(t-t_0).$$

Since the coefficients b_k from (3.11) are rapidly decreasing, it is clear that the series $\Phi_1(x) = \sum_{k=2}^{\infty} 2^k K^k b_k x^{k-1}$ converges for all $x \in \mathbb{R}$; by (4.23), $\Phi_1(D_t + 2d(t-t_0)) = O(1)$. Then we deduce from (4.26) that

$$\|A_{11}\|_\infty, \|A_{22}\|_\infty = O(t-t_0), \quad 0 \leqslant t_0 \leqslant t < T. \tag{4.27}$$

Similarly, using (4.22) and (4.25) we have

$$\begin{aligned} \|A_{12}\|_\infty &\leqslant \sum_{k=2}^{\infty} 2^k K^k b_k (D_t + 2d(t-t_0))^{k-2}(2d)^2(t-t_0)^2 \\ &\leqslant (2d)^2 \Phi_2(D_t + 2d(t-t_0))(t-t_0)^2 \\ &= O((t-t_0)^2), \end{aligned} \tag{4.28}$$

where $\Phi_2(x) = \sum_{k=2}^{\infty} 2^k K^k b_k x^{k-2}$. Substituting estimates (4.27), (4.28) into (4.19) we arrive at (4.5), (4.6).

Step 4. From (4.6), we conclude (using the implicit function theorem) that the map $\mathfrak{D} : p_0 \mapsto X(t,p_0) = X(t,t_0,x_0,p_0)$ is a local diffeomorphism. Let us prove that it is injective. Since

$$X(t,p_2) - X(t,p_1) = \int_0^1 \frac{\partial X}{\partial p_0}(t, p_1 + \tau(p_2 - p_1))\,(p_2 - p_1)\,d\tau,$$

we have

$$\begin{aligned} |X(t,p_2) - X(t,p_1)|^2 &= \int_0^1 \int_0^1 (p_2 - p_1)^T \left(\frac{\partial X}{\partial p_0}(t, p_1 + s(p_2 - p_1))\right)^T \times \\ &\quad \times \left(\frac{\partial X}{\partial p_0}(t, p_1 + \tau(p_2 - p_1))\right)(p_2 - p_1)\; d\tau ds \\ &\geqslant C\|p_2 - p_1\|^2, \end{aligned} \tag{4.29}$$

for some constant $C = C(t,t_0) > 0$. The last inequality comes from (4.6). This shows that $\mathfrak{D}$ is injective and so $\mathfrak{D} : \mathbb{R}^d \to \mathfrak{D}(\mathbb{R}^d) \subset \mathbb{R}^d$ is a global diffeomorphism. Since, by (4.29), $\mathfrak{D}$ is open and closed, then $\mathfrak{D}(\mathbb{R}^d) \subset \mathbb{R}^d$ is open and closed. As $\mathfrak{D}(\mathbb{R}^d) \neq \emptyset$ we have $\mathfrak{D}(\mathbb{R}^d) = \mathbb{R}^d$. This finishes the proof of Theorem 4.1. ■

We assume that for all multi-indices $I \in \mathbb{N}_0^d$, $2 \leqslant |I| \leqslant q+2$ the partial derivatives

$$\left|\frac{\partial^{|I|}V}{\partial x^I}\right| \leqslant K, \qquad \left|\frac{\partial^{|I|}c_i}{\partial x^I}\right| \leqslant K, \quad i = 1, \dots, d, \tag{4.30}$$

are bounded and continuous.

Lemma 4.2 *If the coefficients V, c_i satisfy the above mentioned assumptions, we have for all $0 \leqslant t_0 \leqslant t < T$*

$$\frac{\partial^{|I|}X(t, t_0, x_0, p_0)}{\partial p_0^I} = O((t-t_0)^{|I|+1}), \tag{4.31}$$

$$\frac{\partial^{|I|}P(t, t_0, x_0, p_0)}{\partial p_0^I} = O((t-t_0)^{|I|}), \tag{4.32}$$

where $I \in \mathbb{N}_0^d$, $2 \leqslant |I| \leqslant q$ and $O(\cdot)$ is uniform with respect to x_0 and p_0.

For notational convenience we set $c_0(X) = -V(X)$. Let us first prove a technical lemma.

Lemma 4.3 *For $r \in \mathbb{N}$, $i_1, \dots, i_r \in \{1, \dots, d\}$, $i \in \{0. \dots, d\}$ and $\rho, \mu, \dots, \lambda, L \in \mathbb{N}_0^d$, $|\rho|, |\mu|, \dots, |L| \leqslant q$, we define $Z_i(\tau) = Z_{i,i_1,\dots,i_r,\rho,\mu,\dots,\lambda,L}(\tau)$ by*

$$Z_i(\tau) = \frac{\partial^{|\mu|}X_{i_1}(\tau)}{\partial p_0^\mu} \dots \frac{\partial^{|\lambda|}X_{i_r}(\tau)}{\partial p_0^\lambda} \frac{\partial^{|\rho|}c_i^{(2)}(X(\tau))}{\partial x^\rho} \left(\frac{\partial^{|L|}}{\partial p_0^L}\frac{\partial X}{\partial p_0}\right), \tag{4.33}$$

where $c_i^{(2)}(x) = (\partial^2 c_i(x)/\partial x^2) \in \mathbb{R}^{d\times d}$. Then

$$\begin{aligned}\Big\| \sum_{i=0}^d \int_{t_0}^s Z_i(\tau)\, d\eta_{i,\tau} \Big\|_\infty \\ \leqslant\ C \int_{t_0}^s \left|\frac{\partial^{|\mu|}P(\tau)}{\partial p_0^\mu}\right| d\tau \dots \int_{t_0}^s \left|\frac{\partial^{|\lambda|}P(\tau)}{\partial p_0^\lambda}\right| d\tau \int_{t_0}^s \left\|\frac{\partial^{|L|}}{\partial x^L}\frac{\partial P}{\partial p_0}\right\|_\infty d\tau\end{aligned}$$

for $t_0 \leqslant s < T$ and some constant $C > 0$.

Remark. By definition, $Z_i(\tau)$ is a continuous process, so we can write $Z_i(\tau)$ instead of $Z_i(\tau-)$.

Proof. The first equation in the system (4.1) implies

$$\left(\sup_{t_0 \leqslant \tau \leqslant s} \left|\frac{\partial^{|J|}X(\tau)}{\partial p_0^J}\right|\right) \vee \left(\mathrm{Var}_{[t_0,s]} \frac{\partial^{|J|}X}{\partial p_0^J}\right) \leqslant \int_{t_0}^s \left|\frac{\partial^{|J|}P(\tau)}{\partial p_0^J}\right| d\tau$$

for any $J \in \mathbb{N}_0^d$.

By (4.3), $(\partial c/\partial x) = 0$ for $|x| > K$. An application of Corollary 4.1 with $f = (\partial^{|\rho|} c_i^{(2)}(x)/\partial x^\rho)_{n,j}$, $n, j = 1, \ldots, d$, and condition (4.30) imply

$$\sup_{t_0 \leqslant \tau \leqslant s} \left\| \frac{\partial^{|\rho|} c_i^{(2)}(X(\tau))}{\partial x^\rho} \right\|_\infty \vee \left\| \mathrm{Var}_{[t_0,s]} \frac{\partial^{|\rho|} c_i^{(2)}(X)}{\partial x^\rho} \right\|_\infty \leqslant C_1$$

for some constant $C_1 > 0$ and all $|\rho| \leqslant K$. Using the formula

$$\sup_{t_0 \leqslant \tau \leqslant s} \Big| \prod_{l=1}^{r+2} a_l(\tau) \Big| \vee \mathrm{Var}_{[t_0,s]} \Big(\prod_{l=1}^{r+2} a_l \Big) \leqslant (r+2) \prod_{l=1}^{r+2} \Big(\sup_{t_0 \leqslant \tau \leqslant s} |a_l(\tau)| \vee \mathrm{Var}_{[t_0,s]} a_l \Big),$$

where $a_l : \mathbb{R} \to \mathbb{R}$, $l = 1, \ldots, r+2$, we find

$$\begin{aligned} (4.34) \quad & \sup_{t_0 \leqslant \tau \leqslant s} |Z_i(\tau)| \vee \|\mathrm{Var}_{[t_0,s]} Z_i\|_\infty \\ & \leqslant (r+2) C_1 \int_{t_0}^s \left| \frac{\partial^{|\mu|} P(\tau)}{\partial p_0^\mu} \right| d\tau \ldots \int_{t_0}^s \left| \frac{\partial^{|\lambda|} P(\tau)}{\partial p_0^\lambda} \right| d\tau \int_{t_0}^s \left\| \frac{\partial^{|L|}}{\partial x^L} \frac{\partial P}{\partial p_0} \right\|_\infty d\tau. \end{aligned}$$

Since $X(\tau)$ is a continuous process, so is $Z(\tau)$ and (4.34) shows that $Z(\tau)$ is a BV-process. Using integration by parts gives

$$\int_{t_0}^s Z_i(\tau)\, d\eta_{i,\tau} = Z_i(s)\eta_{i,s} - Z_i(t_0)\eta_{i,t_0} - \int_{t_0}^s \eta_{i,\tau-}\, dZ_i(\tau),$$

hence for $s < T$

$$\begin{aligned} (4.35) \quad & \left\| \int_{t_0}^s Z_i(\tau)\, d\eta_{i,\tau} \right\|_\infty \leqslant \\ & \leqslant 2 \sup_{t_0 \leqslant \tau \leqslant s} |\eta_{i,\tau}| \sup_{t_0 \leqslant \tau \leqslant s} \|Z_i(\tau)\|_\infty + \sup_{t_0 \leqslant \tau \leqslant s} |\eta_{i,\tau}|\, \|\mathrm{Var}_{[t_0,s]} Z_i\|_\infty. \end{aligned}$$

Recall that $\eta_{i,s}$ is a Lévy process. Therefore by Lemma 2.1 we have

$$2 \sum_{i=0}^d \big(\sup_{t_0 \leqslant \tau \leqslant s} |\eta_{i,s}| \big) \leqslant \theta_t + t \leqslant 2$$

for $0 \leqslant t_0 \leqslant s < T \leqslant \mathcal{R}$. Summing (4.35) over $i = 0, \ldots, d$, and using (4.34) completes the proof. ■

Proof of Lemma 4.2.

Step 1. Let us choose and (in this step) fix a sequence $(j_1, j_2, \ldots) \in \{1, \ldots, d\}^{\mathbb{N}}$. Write

$$\mathcal{A}_0(\tau) = \frac{\partial X(\tau)}{\partial p_0},\ \mathcal{B}_0(\tau) = \frac{\partial P(\tau)}{\partial p_0},\ \mathcal{A}_m(\tau) = \frac{\partial^{|J|} \mathcal{A}_0(\tau)}{\partial p_0^J},\ \mathcal{B}_m(\tau) = \frac{\partial^{|J|} \mathcal{B}_0(\tau)}{\partial p_0^J},$$

$\mathcal{A}_0(\tau), \mathcal{B}_0(\tau), \mathcal{A}_m(\tau), \mathcal{B}_m(\tau) \in \mathbb{R}^{d \times d}$, where

$$J = (j_1, \ldots, j_m) \quad \text{and} \quad p_0 = (p_{0,1}, \ldots, p_{0,d}) \in \mathbb{R}^d.$$

From equation (4.16) we get

$$d_\tau \begin{pmatrix} \mathcal{A}_0(\tau) \\ \mathcal{B}_0(\tau) \end{pmatrix} = \sum_{i=0}^{d} W_{i,\tau} \begin{pmatrix} \mathcal{A}_0(\tau) \\ \mathcal{B}_0(\tau) \end{pmatrix} d\eta_{i,\tau}. \tag{4.36}$$

Differentiating (4.36) with respect to $p_{0,j_1}, \dots, p_{0,j_m}$ we get the following system of SDE's

$$\begin{aligned} d_\tau \begin{pmatrix} \mathcal{A}_m(\tau) \\ \mathcal{B}_m(\tau) \end{pmatrix} &= \sum_{i=0}^{d} W_{i,\tau} \begin{pmatrix} \mathcal{A}_m(\tau) \\ \mathcal{B}_m(\tau) \end{pmatrix} d\eta_{i,\tau} - \sum_{i=0}^{d} \widetilde{A}_{m,i,\tau}\, d\eta_{i,\tau} \\ \mathcal{A}_m(t_0) &= \mathcal{B}_m(t_0) = 0, \end{aligned} \tag{4.37}$$

where $\widetilde{A}_{m,i,\tau} \in \mathbb{R}^{2d\times d}$ is given by the following recurrence relation

$$\widetilde{A}_{0,i,\tau} = 0, \quad \widetilde{A}_{m,i,\tau} = -\frac{\partial W_{i,\tau}}{\partial p_{0,j_m}} \begin{pmatrix} \mathcal{A}_{m-1}(\tau) \\ \mathcal{B}_{m-1}(\tau) \end{pmatrix} + \frac{\partial \widetilde{A}_{m-1,i,\tau}}{\partial p_{0,j_m}} \quad m > 0.$$

The interchange of stochastic and ordinary differentials (with respect to the initial conditions) is possible since the coefficients of the system (4.36) are smooth enough cf. Protter [Pro, p. 245, Theorem 40]. Using (4.17) gives

$$\widetilde{A}_{m,i,\tau} = \begin{pmatrix} 0 \\ A_{m,i,\tau} \end{pmatrix}, \qquad i = 0, \dots, d, \quad A_{m,i,\tau} \in \mathbb{R}^{d\times d}, \tag{4.38}$$

where

$$A_{0,i,\tau} = 0, \quad A_{m,i,\tau} = \frac{\partial c_i^{(2)}(X(\tau))}{\partial p_{0,j_m}} \mathcal{A}_{m-1}(\tau) + \frac{\partial A_{m-1,i,\tau}}{\partial p_{0,j_m}} \quad m > 0. \tag{4.39}$$

Recall that $c_i^{(2)}(x) = (\partial^2 c_i(x)/\partial x^2) \in \mathbb{R}^{d\times d}$. From (4.39) we find by induction

$$A_{m,i,\tau} = \sum_{k=2}^{m+1} \frac{\partial^{m+1-k}}{\partial p_{0,j_m} \dots \partial p_{0,j_k}} \left(\frac{\partial c_i^{(2)}(X(\tau))}{\partial p_{0,j_{k-1}}} \frac{\partial^{k-2} \mathcal{A}_0(\tau)}{\partial p_{0,j_{k-2}} \dots \partial p_{0,j_1}} \right). \tag{4.40}$$

A solution of (4.37) is given by the following (formal) series expansion

$$G = \sum_{k=1}^{\infty} G_k \tag{4.41}$$

with

$$G_1 = -\sum_{i=0}^{d} \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0}, \qquad G_k = \sum_{i=0}^{d} \int_{t_0}^{t} W_{i,\tau} G_{k-1}\, d\eta_{i,\tau} \quad k \geqslant 2.$$

This can be seen as in the proof of Theorem 4.1 and (4.41) is a solution of (4.37) whenever it converges uniformly (on compact intervals) in t.

Thus

$$\begin{pmatrix} \mathcal{A}_m(t) \\ \mathcal{B}_m(t) \end{pmatrix} = -\sum_{i=0}^{d} \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} - \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k=0}^{d} \widetilde{I}^{m}_{i_1,\dots,i_k,t},$$

where

$$\widetilde{I}^{m}_{i_1,\dots,i_k,t} = \int_{t_0}^{t} W_{i_k,\tau_k} \dots \int_{t_0}^{\tau_2-} W_{i_1,\tau_1} \Big(\sum_{i=0}^{d} \int_{t_0}^{\tau_1-} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \Big) d\eta_{i_1,\tau_1} \dots d\eta_{i_k,\tau_k}.$$

Since $W_{i_1,\tau}\widetilde{A}_{m,i,\tau} = 0$ for $i_1 = 1,\dots,d$, it follows that

$$\widetilde{I}^{m}_{i_1,\dots,i_k,t} = 0 \qquad \text{for} \quad i_1 > 0,$$

i.e. the first integration in $\widetilde{I}^{m}_{i_1,\dots,i_k,t}$ is trivial, which means $\ell_1 > 0$. Therefore

$$\begin{aligned} \Big\| \sum_{k=1}^{\infty} \sum_{i_1,\dots,i_k=0}^{d} \widetilde{I}^{m}_{i_1,\dots,i_k,t} \Big\|_{\infty} &= \Big\| \sum_{k=1}^{\infty} \sum_{i_2,\dots,i_k=0}^{d} \sum_{i_1=1}^{d} \widetilde{I}^{m}_{i_1,\dots,i_k,t} \Big\|_{\infty} \\ &\leqslant \sum_{k=1}^{\infty} 2^k \max \Big\| \sum_{(i_1,\dots,i_k)\in\mathcal{M}^k} \widetilde{I}^{m}_{i_1,\dots,i_k,t} \Big\|_{\infty}, \end{aligned}$$

where $\mathcal{M}^k = \mathcal{M}^k_1 \times \dots \times \mathcal{M}^k_k$, $\mathcal{M}^k_j = \{1,\dots,d\}$ or $\mathcal{M}^k_j = \{0\}$ for $j > 1$ and the maximum is taken over all $\mathcal{M}^k$ such that $\mathcal{M}^k_1 = \{0\}$. Also observe that formula (3.10) still holds if $I_{0,t}$ is not a square $M \times M$ matrix, but rectangular of the form $I_{0,t} \in \mathbb{R}^{M\times N}$ and any $N \in \mathbb{N}$. An application of Proposition 3.1 to the matrices $K^{-1}W_{j,\tau} \in \mathbb{R}^{2d\times 2d}$, $j = 1,\dots,d$, with $I_{0,t} = \sum_{i=0}^{d} \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \in \mathbb{R}^{2d\times d}$ shows

$$\Big\| \sum_{(i_1,\dots,i_k)\in\mathcal{M}^k} \widetilde{I}^{m}_{i_1,\dots,i_k,t} \Big\|_{\infty} \leqslant K^k b_k D_t^{\mathfrak{m}} \{2d(t-t_0)\}^{k-\mathfrak{m}} \|I^*_{0,t}\|_{\infty},$$

where the b_k's are given by (3.11). Thus for $t < T$

$$\begin{aligned} &\|\mathcal{A}_m(t)\|_{\infty} \vee \|\mathcal{B}_m(t)\|_{\infty} \\ &\quad\leqslant \sum_{i=0}^{d} \Big\| \int_{t_0}^{t} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \Big\|_{\infty} \\ &\qquad + \sum_{k=1}^{\infty} 2^k K^k b_k D_t^{\mathfrak{m}} \{2d(t-t_0)\}^{k-\mathfrak{m}} \sup_{t_0\leqslant s\leqslant t} \sum_{i=0}^{d} \Big\| \int_{t_0}^{s} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \Big\|_{\infty} \\ &\quad\leqslant \sup_{t_0\leqslant s\leqslant t} \sum_{i=0}^{d} \Big\| \int_{t_0}^{s} \widetilde{A}_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \Big\|_{\infty} \widehat{\Phi}(D_t + 2d(t-t_0)), \end{aligned}$$

where $\widehat{\Phi}(x) = 1 + \sum_{k=1}^{\infty} 2^k K^k b_k x^k$, D_τ and b_k are given by (3.8) and (3.11) respectively.

By (4.23), $\widehat{\Phi}(D_t + 2d(t-t_0)) \leqslant \widehat{\Phi}(2d^2(\widetilde{K}_2+4)+2d) = O(1)$ for $t<T$, so we get from (4.38) arrive at

$$\left\| \frac{\partial^{|J|}\mathcal{B}_0(t)}{\partial p_0^J} \right\|_\infty = \|\mathcal{B}_m(t)\|_\infty = O(1) \sup_{t_0 \leqslant s \leqslant t} \sum_{i=0}^d \left\| \int_{t_0}^s A_{m,i,\tau_0-}\, d\eta_{i,\tau_0} \right\|_\infty . \tag{4.42}$$

Step 2. Now we apply induction in m to prove

$$\mathcal{A}_m = O((t-t_0)^{m+2}), \qquad \mathcal{B}_m = O((t-t_0)^{m+1}) \tag{4.43}$$

for any $J \in \mathbb{N}_0^m$, which is equivalent the assertion of the lemma.

Let us first check the claim for $m=1$. From (4.40) we find

$$A_{1,i,\tau} = \sum_{j=1}^d \frac{\partial c_i^{(2)}(X(\tau))}{\partial x_j} \frac{\partial X_j(\tau)}{\partial p_{0,j_1}} \frac{\partial X(\tau)}{\partial p_0}.$$

An application of Lemma 4.3 with $r=1$, $i_1=j$, $\rho=e_j$, $\mu=e_{j_1}$ (see definition (3.36)) and $L=0$ shows

$$\left\| \int_{t_0}^s \frac{\partial c_i^{(2)}(X(\tau))}{\partial x_j} \frac{\partial X_j(\tau)}{\partial p_{0,j_1}} \frac{\partial X(\tau)}{\partial p_0}\, d\eta_{i,\tau} \right\|_\infty = O(1) \left(\int_{t_0}^s \left\| \frac{\partial P(\tau)}{\partial p_0} \right\|_\infty d\tau \right)^2 .$$

By (4.5) we know $\|\partial P/\partial p_0\|_\infty = O(1)$. Hence

$$\sup_{t_0 \leqslant s \leqslant t} \sum_{i=0}^d \left\| \int_{t_0}^s A_{1,i,\tau-}\, d\eta_{i,\tau} \right\|_\infty = O((t-t_0)^2).$$

Combining this and (4.42) gives $\mathcal{B}_1 = O((t-t_0)^2)$. Using (4.1) we obtain $\mathcal{A}_1 = O((t-t_0)^3)$.

We now assume that (4.43) holds for $1,\dots,m-1$. From (4.40) we find that $A_{m,i,\tau}$ is a linear combination of the $Z_i(\tau) = Z_{i,i_1,\dots,i_r,\rho,\mu,\dots,\lambda,L}(\tau)$ given by (4.33) with

$$|\rho| = r \leqslant m, \qquad |\mu| + \ldots + |\lambda| + |L| = m. \tag{4.44}$$

By the induction assumption and by (4.5) we have $\left\|\partial^{|L|}\mathcal{B}_0(\tau)/\partial x^L\right\|_\infty = O((t-t_0)^{|L|})$ and

$$\left| \frac{\partial^{|\mu|}P(\tau)}{\partial p_0^\mu} \right| = O((t-t_0)^{\phi(|\mu|)}), \quad \ldots, \quad \left| \frac{\partial^{|\lambda|}P(\tau)}{\partial p_0^\lambda} \right| = O((t-t_0)^{\phi(|\lambda|)}),$$

where $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(1)=0$, $\phi(n)=n$ for all $n>1$. (Note that the choice of ϕ allows us to combine the formulae (4.5) and (4.32), which show

different behaviour for $|\mu| = 1$ resp. $|\mu| > 1$.) An application of Lemma 4.3 shows

$$\sum_{i=0}^{d} \left\| \int_{t_0}^{s} A_{m,i,\tau-}\, d\eta_{i,\tau} \right\|_{\infty} = O((t-t_0)^{\gamma}), \tag{4.45}$$

where

$$\gamma = \phi(|\mu|) + 1 + \ldots + \phi(|\lambda|) + 1 + |L| + 1.$$

Since $1 + \phi(n) \geqslant n$ we conclude from (4.44) that

$$\gamma \geqslant |\mu| + \ldots + |\lambda| + |L| + 1 = m + 1.$$

Combining this, (4.45) and (4.42) we have $\mathcal{B}_m = O((t-t_0)^{m+1})$. The assertion for $\mathcal{A}_m$ follows again from the first equation in (4.1). ■

5. The method of stochastic characteristics

As before we denote by $(X, P) = (X(t, t_0, x_0, p_0), P(t, t_0, x_0, p_0))$ the solution of the Hamilton system

$$\begin{cases} dx = \dfrac{\partial H}{\partial p}\, dt \\[2ex] dp = -\dfrac{\partial H}{\partial x}\, dt - \dfrac{\partial c}{\partial x}\, d\xi_t, \end{cases} \tag{5.1}$$

with initial condition $(x_0, p_0) \in \mathbb{R}^{2d}$ at $t = t_0$, where

$$H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}, \qquad c : \mathbb{R}^d \to \mathbb{R}^d.$$

We shall say that H and c *satisfy property* (D1) if

(D1) There exists a stopping time $T > 0$ a.s. such that for any $0 \leqslant t_0 < t < T$, $\forall x_0 \in \mathbb{R}^d$ the map $\mathfrak{D}_1 : \mathbb{R}^d \to \mathbb{R}^d, \quad p_0 \mapsto X(t, t_0, x_0, p_0)$ is a diffeomorphism.

Next we shall say that H, c and $S_0 : \mathbb{R}^d \to \mathbb{R}$ satisfy property (D2) if

(D2) There exists a stopping time $T > 0$ a.s. such that for any $0 \leqslant t_0 < t < T$ the map $\mathfrak{D}_2 : \mathbb{R}^d \to \mathbb{R}^d, \quad x_0 \mapsto X(t, t_0, x_0, \nabla S_0(x_0))$ is a diffeomorphism.

Remark. In the literature on Burgers turbulence, the map $\mathfrak{D}_2$ is called *Lagrangian function*, and its inverse $\mathfrak{D}_2^{-1}$ is called the *inverse Lagrangian function* [Ber2].

In the following statement we summarise the main results of the previous section.

Theorem 5.1 *Let $S_0 : \mathbb{R}^d \to \mathbb{R}$ be a twice differentiable function such that*

$$\frac{\partial^2 S_0(x)}{\partial x^2} \geqslant \Lambda \quad \textit{for some symmetric} \quad \Lambda \in \mathbb{R}^{d\times d}, \forall x \in \mathbb{R}^d, \tag{5.2}$$

$H(x,p) = (1/2)p^2 - V(x)$ and the conditions of the Theorem 4.1 *hold. Then* (D1) *and* (D2) *are satisfied.*

Proof. Theorem 4.1 immediately implies (D1).

Using formulae (4.5), (4.6) we deduce from

$$\begin{aligned}\frac{\partial X(t,t_0,x_0,\nabla S_0(x_0))}{\partial x_0} \\ = \frac{\partial X(t,t_0,x_0,p_0)}{\partial x_0}\Big|_{p_0=\nabla S_0(x_0)} + \frac{\partial X(t,t_0,x_0,\nabla S_0(x_0))}{\partial p_0}\frac{\partial^2 S_0(x_0)}{\partial x_0^2}\end{aligned}$$

that

$$\frac{\partial X(t,t_0,x_0,\nabla S_0(x_0))}{\partial x_0} = E_d + O(t-t_0) + [(t-t_0) + O(t-t_0)]\,\frac{\partial^2 S_0(x_0)}{\partial x_0^2}$$

and so there exist a constant $C_1 > 0$ such that

$$\frac{\partial X(t,t_0,x_0,\nabla S_0(x_0))}{\partial x_0} \geqslant \frac{1}{2}\,E_d \quad \text{for} \quad 0 \leqslant t_0 \leqslant t < T \wedge C_1.$$

Therefore the map $\mathfrak{D}_2 : x_0 \mapsto X(t,t_0,x_0,\nabla S_0(x_0))$ is a local diffeomorphism. Along the same lines as in the proof of Theorem 4.1 we conclude that $\mathfrak{D}_2$ is a global diffeomorphism. ■

Recall that the notations $X(\tau)$, $P(\tau)$ were introduced in (4.7). To each pair $(X(\tau), P(\tau))$ of solutions of (5.1) there corresponds the *action function* defined by the formula

$$\sigma(t,t_0,x_0,p_0) = \int_{t_0}^{t}\left[P(\tau)\frac{\partial X(\tau)}{\partial \tau} - H(X(\tau),P(\tau))\right]d\tau - \int_{t_0}^{t} c(X(\tau))\,d\xi_\tau. \tag{5.3}$$

If (D1) holds, the inverse map $p_0 = p_0(t,t_0,x,x_0)$ to $\mathfrak{D}_1$ exists, i.e.

$$X(t,t_0,x_0,p_0(t,t_0,x,x_0)) = x, \quad t > t_0, \quad x \in \mathbb{R}^n, \tag{5.4}$$

and we can define locally (for $0 \leqslant t_0 < t < T$) the *two-point function*

$$S(t,t_0,x,x_0) = \sigma(t,t_0,x_0,p_0(t,t_0,x,x_0)). \tag{5.5}$$

Notation. We set

$$p(t,t_0,x,x_0) = P(t,t_0,x_0,p_0(t,t_0,x,x_0)) \tag{5.6}$$

and

$$x(\tau) = X(\tau,t_0,x_0,p_0(t,t_0,x,x_0)), \quad p(\tau) = P(\tau,t_0,x_0,p_0(t,t_0,x,x_0)). \tag{5.7}$$

Finally, we put $X(\tau,x_0,p_0) = X(\tau,0,x_0,p_0)$, $P(\tau,x_0,p_0) = P(\tau,0,x_0,p_0)$, $p_0(\tau,0,x_0,p_0) = p_0(\tau,x_0,p_0)$, $\sigma(t,x_0,p_0) = \sigma(t,0,x_0,p_0)$ and $S(t,x,x_0) = S(t,0,x,x_0)$. We will use this notation throughout the rest of this paper.

The following results (and their proofs) are stochastic versions of the well known method of characteristics for solving the Hamilton-Jacobi equation (see e.g. [K3]).

Theorem 5.2 *Let $H(x,p)$ and $c(x)$ satisfy* (D1). *The function $(t,x) \mapsto S(t,t_0,x,x_0)$, as a function of the variables (t,x), satisfies the Hamilton-Jacobi equation*

$$dS + H\left(x, \frac{\partial S}{\partial x}\right) dt + c(x)\, d\xi_t = 0 \tag{5.8}$$

in the domain $(t_0,T) \times \mathbb{R}^d$ for the stopping time T with $\mathbb{P}(T > 0) = 1$. Moreover, we have

$$\frac{\partial S}{\partial x}(t,t_0,x,x_0) = p(t,t_0,x,x_0), \quad \frac{\partial S}{\partial x_0}(t,t_0,x,x_0) = -p_0(t,t_0,x,x_0). \tag{5.9}$$

Proof. Without loss of generality we may assume that $t_0 = 0$.

Step 1. We start with the proof of the first relation in (5.9). This equality can be rewritten as

$$\frac{\partial S}{\partial x}(t, X(t,x_0,p_0), x_0) = P(t,x_0,p_0)$$

which is, by (5.5),

$$\frac{\partial \sigma}{\partial p_0}(t,x_0,p_0)\frac{\partial p_0}{\partial x}(t, X(t,x_0,p_0), x_0) = P(t,x_0,p_0). \tag{5.10}$$

Due to (5.4),

$$\left(\frac{\partial p_0}{\partial x}(t, X(t,x_0,p_0), x_0)\right)^{-1} = \frac{\partial X}{\partial p_0}(t,x_0,p_0). \tag{5.11}$$

It follows that the first equation in (5.9), using (5.10), has the form

$$\frac{\partial \sigma}{\partial p_0}(t,x_0,p_0) = P(t,x_0,p_0)\,\frac{\partial X}{\partial p_0}(t,x_0,p_0). \tag{5.12}$$

Since $X(t)$, $\partial X(t)/\partial p_0$ are continuous and of bounded variation, we get from Itô's formula that

$$dP\,\frac{\partial X}{\partial p_0} = P\,d\frac{\partial X}{\partial p_0} + \frac{\partial X}{\partial p_0}\,dP. \tag{5.13}$$

The left-hand side of (5.12) can be expressed using (5.3). Together with (5.13) we calculate that its Itô differential gives

$$\frac{\partial}{\partial p_0}\left(P\,\frac{\partial X}{\partial t} - H\right)dt - \frac{\partial c(X)}{\partial p_0}\,d\xi_t = P\,d\frac{\partial X}{\partial p_0} + \frac{\partial X}{\partial p_0}\,dP.$$

Notice that we need the fact that

$$\frac{\partial}{\partial p_0}\int_0^t \frac{\partial c(X(\tau))}{\partial x}\,d\xi_\tau = \sum_{i=1}^d \int_0^t \frac{\partial^2 c_i(X(\tau))}{\partial p_0 \partial x}\,d\xi_{i,\tau},$$

which is justified by a special case of Theorem 36.9 [M], p. 258. Since by (4.1) $dP = (\partial H/\partial x)\,dt + (\partial c/\partial x)\,d\xi_t$, we find

$$\begin{aligned} &\frac{\partial P}{\partial p_0}\frac{\partial X}{\partial t}\,dt + P\,\frac{\partial^2 X}{\partial p_0 \partial t}\,dt - \frac{\partial H}{\partial x}\frac{\partial X}{\partial p_0}\,dt - \frac{\partial H}{\partial p}\frac{\partial P}{\partial p_0}\,dt - \frac{\partial X}{\partial p_0}\,\frac{\partial c}{\partial x}\,d\xi_t \\ &\quad = P\,\frac{\partial^2 X}{\partial p_0 \partial t}\,dt - \frac{\partial X}{\partial p_0}\left(\frac{\partial H}{\partial x}\,dt + \frac{\partial c}{\partial x}\,d\xi_t\right). \end{aligned} \tag{5.14}$$

As $\partial X/\partial t = P$ we find that (5.14) holds for all $t < T(\omega)$, and the first part of (5.9) is established.

Step 2. Using (5.5) we get

$$\frac{\partial S}{\partial x_0} = \frac{\partial \sigma}{\partial x_0} + \frac{\partial \sigma}{\partial p_0}\,\frac{\partial p_0(t,x,x_0)}{\partial x_0}$$

and so, by (5.4), we rewrite the second formula in (5.9) as

$$\frac{\partial \sigma}{\partial x_0} - \frac{\partial \sigma}{\partial p_0}\left(\frac{\partial X}{\partial p_0}\right)^{-1}\frac{\partial X}{\partial x_0} = -p_0. \tag{5.15}$$

The relation

$$\frac{\partial}{\partial \alpha}\left(\left[P(\tau)\,\frac{\partial X(\tau)}{\partial \tau} - H(X(\tau),P(\tau))\right]d\tau - c(X(\tau))\,d\xi_\tau\right) = d_\tau\left(P(\tau)\frac{\partial X(\tau)}{\partial \alpha}\right)$$

where $\alpha = x_0$ or $\alpha = p_0$, and definition (5.3) imply

$$\frac{\partial \sigma}{\partial x_0} = \left(P(\tau)\frac{\partial X(\tau)}{\partial x_0}\right)\Bigg|_0^t, \qquad \frac{\partial \sigma}{\partial p_0} = \left(P(\tau)\frac{\partial X(\tau)}{\partial p_0}\right)\Bigg|_0^t. \tag{5.16}$$

Using (5.16) and the fact that

$$\left.\frac{\partial X(\tau)}{\partial x_0}\right|_{\tau=0} = 1, \qquad \left.\frac{\partial X(\tau)}{\partial p_0}\right|_{\tau=0} = 0$$

gives (5.15).

Step 3. To prove (5.8), let us first rewrite it as

$$d\sigma(t, x_0, p_0) + \frac{\partial \sigma}{\partial p_0}\, dp_0 + H(x, p(t, x_0, x))dt + c(x)\, d\xi_t = 0.$$

Because of (5.3) we find

$$\begin{aligned} & P(t, x_0, p_0)\frac{\partial X}{\partial t}(t, x_0, p_0)\, dt - H(X(t, x_0, p_0), P(t, x_0, p_0))\, dt \\ - \quad & c(X(t, x_0, p_0))\, d\xi_t + \frac{\partial \sigma}{\partial p_0}\, dp_0 + H(x, p(t, x, x_0))\, dt + c(x)\, d\xi_t = 0. \end{aligned}$$

By construction, $X(\tau, x_0, p_0) = x$, $P(t, x_0, p_0) = p$ and expressing $\partial\sigma/\partial p_0$ by (5.12) gives

$$P(t, x_0, p_0)\frac{\partial X}{\partial t}(t, x_0, p_0)\, dt + P(t, x_0, p_0)\frac{\partial X}{\partial p_0}(t, x_0, p_0)\, dp_0 = 0. \tag{5.17}$$

Differentiating (5.4) with respect to t we get

$$dX(t, x_0, p_0(t, x_0, x)) = \frac{\partial X}{\partial t}(t, x_0, p_0)\, dt + \frac{\partial X}{\partial p_0}(t, x_0, p_0)\, dp_0 = 0.$$

Thus (5.17) is always satisfied and (5.8) follows. ■

Corollary 5.1 *Under the assumption of the Theorem* 5.2 *we have for* $0 \leqslant t_0 < t < T$

$$\frac{\partial^2 S(t, t_0, x, x_0)}{\partial x^2} = \frac{1}{t - t_0}(E_d + O(t - t_0)), \tag{5.18}$$

$$\frac{\partial^2 S(t, t_0, x, x_0)}{\partial x_0^2} = \frac{1}{t - t_0}(E_d + O(t - t_0)), \tag{5.19}$$

$$\frac{\partial^2 S(t, t_0, x, x_0)}{\partial x \partial x_0} = -\frac{1}{t - t_0}(E_d + O(t - t_0)), \tag{5.20}$$

where $O(\cdot)$ *is uniform with respect to* x_0, x.

Proof. Assume again that $t_0 = 0$. From (5.9) and (5.11) we deduce the equality

$$\frac{\partial^2 S(t,x,x_0)}{\partial x^2} = \frac{\partial P}{\partial p_0}(t,x_0,p_0)\left(\frac{\partial X}{\partial p_0}(t,x_0,p_0)\right)^{-1}.$$

Now (4.5), (4.6) imply the first formula in Corollary 5.1. The same argument can be used to prove the remaining formulae. ■

Theorem 5.3 *We assume that $H(x,p)$, $c(x)$ and $S_0(x)$ satisfy conditions* (D1), (D2). *Then for $0 \leqslant t_0 < t < T(\omega)$ the formula*

$$S(t,t_0,x) = S_0(x_0) + \int_{t_0}^{t}\left(\widetilde{p}(\tau)\,d\widetilde{x}(\tau) - H(\widetilde{x}(\tau),\widetilde{p}(\tau))\,d\tau - c(\widetilde{x}(\tau))\,d\xi_\tau\right) \tag{5.21}$$

(where the integral is taken along the trajectory $\widetilde{x}(\tau) = X(\tau,t_0,x_0,\nabla S_0(x_0))$, $\widetilde{p}(\tau) = P(\tau,t_0,x_0,\nabla S_0(x_0))$ and $x_0 = x_0(t,t_0,x)$ is the inverse map of $\mathfrak{D}_2$) gives the unique classical solution of the Cauchy problem for the equation

$$dS + H\left(x, \frac{\partial S}{\partial x}\right)\,dt + c(x)\,d\xi_t = 0 \tag{5.22}$$

with initial function $S_0(x)$. One can rewrite formula (5.21) *in the equivalent form*

$$S(t,t_0,x) = (S_0(x_0) + S(t,t_0,x,x_0))\,\big|_{x_0=x_0(t,t_0,x)}. \tag{5.23}$$

Proof. The definition of the two-point function (5.5) implies the equivalence of (5.21) and (5.23). From the system (5.1) we see that $X(t,t_0,x_0,p_0)$ is continuous in t and, using the implicit function theorem, we obtain from (D2) that $x_0 = x_0(t,t_0,x)$ is continuous in t. So, $[x_0,x_0] = 0$ and Itô's formula for this equation gives

$$\begin{aligned} &d_tS(t,t_0,x) \\ &\quad= \nabla S_0(x_0)d_tx_0(t,t_0,x) + \frac{\partial S(t,t_0,x,x_0)}{\partial t}\,dt + \frac{\partial S(t,t_0,x,x_0)}{\partial x_0}\,d_tx_0(t,t_0,x) \\ &\quad= \frac{\partial S(t,t_0,x,x_0)}{\partial t}dt. \end{aligned}$$

In the last equality we used $\nabla S_0(x_0) = p_0$ in conjunction with (5.9). From Theorem 5.2 we know that

$$\frac{\partial S(t,t_0,x,x_0)}{\partial t}\,dt = d_tS(t,t_0,x,x_0) = -H\left(x, \frac{\partial S(t,t_0,x,x_0)}{\partial x}\right)\,dt - c(x)\,d\xi_t,$$

and the theorem follows. ■

Appendix

In the proof of Proposition 3.1 we used some technical estimates which are not directly related to the arguments of Section 3.

Lemma A.1 *For $y_1, \dots, y_n \geqslant 1$ one has*

$$\prod_{l=1}^{n}(y_1 + \cdots + y_l)\Gamma(y_l) > \frac{1}{2^n}\,\Gamma(n+1)\left(\Gamma\left(\frac{y_1 + \cdots + y_n}{n} + 1\right)\right)^n.$$

Proof. Since $y_1 + \cdots + y_l > l - 1$ we easily see

$$\prod_{l=1}^{n}(y_1 + \cdots + y_l) > \frac{n!}{2^n}\prod_{l=1}^{n}\frac{y_1 + \cdots + y_l + l - 1}{l}\,. \tag{A.1}$$

We are going to prove by induction that

$$\prod_{l=1}^{n}\frac{y_1 + \cdots + y_l + l - 1}{l} \geqslant \prod_{\substack{1\leqslant l\leqslant n\\ i_l>0}}\prod_{s=1}^{i_l}(y_l + s - 1) \tag{A.2}$$

for some $i_1, \dots, i_n \geqslant 0$ such that $i_1 + \cdots + i_n = n$. Indeed, for $n = 1$ we take $i_1 = 1$. Assume that (A.2) is true for $n - 1$ and let $B = \min\{y_1 + i_1, \dots, y_{n-1} + i_{n-1}, y_n\}$. Then

$$\frac{y_1 + \cdots + y_n + n - 1}{n} = \frac{(y_1 + i_1) + \cdots + (y_{n-1} + i_{n-1}) + y_n}{n} \geqslant B.$$

If $B = y_r + i_r$ for some $1 \leqslant r \leqslant n - 1$ we take $j_r = i_r + 1$, $j_s = i_s$ for $s \neq r$ and $j_n = 0$. Otherwise we put $j_n = 1$, $j_s = i_s$, $s = 1, \dots, n - 1$. Then the inequality (A.2) holds for $(j_1, \dots, j_n)$, and all n.

Combining (A.1) and (A.2) we get

$$\prod_{l=1}^{n}(y_1 + \cdots + y_l)\Gamma(y_l) > \frac{n!}{2^n}\prod_{l=1}^{n}\Gamma(y_l + i_l).$$

The Lemma follows since the Gamma-function is log-convex, cf. [A]. ■

Lemma A.2 *We define q_k by the following recursion formula*

$$q_0 = 1, \qquad q_k = \sum_{m=0}^{k-1}\frac{3}{(k-m)\,m!}\,q_{k-m-1}, \qquad k \in \mathbb{N}. \tag{A.3}$$

Then

$$q_k \leqslant \frac{(2^6)^k}{(\ln(k+1))^{\frac{k}{2}}}, \qquad k \in \mathbb{N}. \tag{A.4}$$

Proof. With $a_{k,l} = 3/[(l+1)(k-l-1)!]$ formula (A.3) reads

$$q_k = \sum_{l=0}^{k-1} a_{k,l} q_l, \quad k \geqslant 1.$$

Clearly,

$$q_k = \sum_{\substack{i_1,\dots,i_n,\\ k=i_0>\dots>i_n=0}} a_{i_0,i_1} \dots a_{i_{n-1},i_n}. \tag{A.5}$$

Notice that

$$a_{i_0,i_1} \dots a_{i_{n-1},i_n} = \prod_{s=0}^{n-1} \frac{3}{(i_{s+1}+1)\Gamma(i_s - i_{s+1})}. \tag{A.6}$$

Lemma $A.1$ implies that for $b_1, \dots, b_r \geqslant 1$

$$\Gamma(b_n) \prod_{l=1}^{n-1} (b_1 + \dots + b_l)\Gamma(b_l) \geqslant \frac{1}{2^n} \frac{\Gamma(n+1)}{b_1 + \dots + b_n} \left(\Gamma\left(\frac{b_1 + \dots + b_n}{n} + 1 \right) \right)^n.$$

If $b_l = i_{n-l} - i_{n-l+1}$, $l = 2, \dots, n$, $b_1 = i_{n-1} - i_n + 1$, then

$$b_1 + \dots + b_n = i_0 - i_n + 1 = k + 1,$$

and it follows that

$$B := \Gamma(i_0 - i_1) \prod_{s=1}^{n-1} (i_s + 1)\Gamma(i_s - i_{s+1}) \geqslant \frac{1}{2^n} \frac{\Gamma(n+1)}{k+1} \left(\Gamma\left(\frac{k+1}{n} + 1 \right) \right)^n.$$

Recall that $n \leqslant k-1$. Because of the inequalities $\Gamma(b+1) \geqslant b^{\frac{b}{2}}$ for $b \geqslant 2$, $\Gamma(b+1) \geqslant 2^{-1} b^{\frac{b}{2}}$ for $1 \leqslant b \leqslant 2$ and $n^{n-k} k^k \geqslant 2^{-k} \left(\ln(k+1)\right)^k$ we have

$$B \geqslant \frac{1}{4^n} \frac{n^{\frac{n}{2}}}{k+1} \left(\frac{k}{n} \right)^{\frac{k}{2}} \geqslant \frac{1}{k 4^k} \left(n^{n-k} k^k \right)^{\frac{1}{2}} \geqslant \frac{1}{k 8^k} \left(\ln(k+1) \right)^{\frac{k}{2}}.$$

Consequently, the right-hand side of (A.6) does not exceed $3^n B^{-1}$ and so

$$q_k \leqslant 2^k \max_{k=i_0>\dots>i_n=0} a_{i_0,i_1} \dots a_{i_{n-1},i_n} \leqslant \frac{k(48)^k}{(\ln(k+1))^{\frac{k}{2}}},$$

where we used that the number of terms in (A.5) is equal to

$$\sum_{n=1}^{k} \binom{k-1}{n-1} = 2^{k-1}.$$

■

Lemma A.3 *For any* $m, k, \alpha_1, \dots, \alpha_n \in \mathbb{N}$ *with*

$$\alpha_1 + \dots + \alpha_n + m = k, \qquad m \geqslant n-1 \tag{A.7}$$

we have

$$\{\ln(\alpha_1+1)\}^{\frac{\alpha_1}{2}} \dots \{\ln(\alpha_n+1)\}^{\frac{\alpha_n}{2}} m! \geqslant 2^{-k} \{\ln(\ln(k+2))\}^{\frac{k}{16}}. \tag{A.8}$$

Proof. Denote the left-hand side of (A.8) by $\mathbf{I}$ and observe that

$$\mathbf{I} \geqslant (\sqrt{\ln 2})^n \geqslant 2^{-k}.$$

Since

$$\{\ln(\ln(k+2))\}^{\frac{k}{8}} \leqslant 1 \qquad \text{for } k \leqslant 10,$$

(A.8) holds for $k \leqslant 10$. If $k > 10$ we get from (A.7)

$$\frac{\alpha_1 + \cdots + \alpha_n}{n} \geqslant \frac{k-m}{m+1}$$

and so, using the log-convexity of $f(x) = \{\ln(x+1)\}^{\frac{x}{2}}$,

$$\mathbf{I} \geqslant \left\{ \ln\left(\frac{\alpha_1 + \cdots + \alpha_n}{n} + 1\right) \right\}^{\frac{\alpha_1+\cdots+\alpha_n}{2}} m! \geqslant \left\{ \ln\left(\frac{k-m}{m+1} + 1\right) \right\}^{\frac{k-m}{2}} m!$$

Applying Lemma $A.4$, completes the proof. ■

Lemma A.4 *For $1 \leqslant m < k$, $k \geqslant 10$, $m, k \in \mathbb{N}$ one has*

$$\left\{ \ln\left(\frac{k+1}{m+1}\right) \right\}^{\frac{k-m}{2}} \Gamma(m+1) \geqslant 2^{-k} \left\{ \ln\left(\ln(k+2)\right) \right\}^{\frac{k}{16}}. \tag{A.9}$$

Proof. We write $\mathbf{I}$ for the left-hand side of (A.9) and set $x_0 = (\ln k)^{-1} k$.

Case 1. $x_0 < m < k$. We split the proof into three steps. Take $x_0 < x < k$, $x \in \mathbb{R}$.

Step 1. Since $(k+1)/(x+1) < k/x < \ln k$ we get

$$\ln\left(\frac{k+1}{x+1}\right) < \ln\left(\frac{k}{x}\right) < \ln\ln k < \frac{k}{\ln k} < x,$$

and so

$$\frac{1}{8}\ln x > \frac{1}{8}\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right). \tag{A.10}$$

Step 2. Using the elementary inequality

$$\ln(1+a) \geqslant ab \qquad \text{for} \quad 0 < b < 1 \quad \text{and} \quad 0 < a < \frac{1}{b} - 1$$

with $a = (k-x)/(x+1)$, $b = 1/(2\ln x)$, and

$$0 < a = \frac{k-x}{x+1} \leqslant \frac{k-x_0}{x_0+1} < \ln k - 1 < 2\ln x_0 - 1 < 2\ln x - 1 = \frac{1}{b} - 1.$$

we find that

$$2\ln x\,\ln\left(\frac{k+1}{x+1}\right) = 2\ln x\,\ln\left(1+\frac{k-x}{x+1}\right) \geqslant \frac{k-x}{x+1},$$

and therefore,

$$\frac{1}{4}\ln x \geqslant \frac{k-x}{8}\,\frac{1}{\ln(\frac{k+1}{x+1})}\,\frac{1}{x+1}. \tag{A.11}$$

Step 3. Set

$$f(x) = \frac{k-x}{8}\,\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right) + \frac{1}{2}x\ln x.$$

Clearly,

$$f'(x) = -\frac{1}{8}\,\ln\left(\ln\left(\frac{k+1}{x+1}\right)\right) - \frac{k-x}{8}\,\frac{1}{\ln(\frac{k+1}{x+1})}\,\frac{1}{x+1} + \frac{1}{2}(\ln x+1).$$

Adding (A.10) and (A.11) we see $f'(x) \geqslant 0$ and so

$$\mathbf{I} \geqslant \exp\{f(x)\} \geqslant \exp\{f(x_0)\}. \tag{A.12}$$

Since $(k+1)(x_0+1)^{-1} \geqslant (3/4)\ln k$, we find

$$\ln\left(\frac{k+1}{x_0+1}\right) \geqslant \ln\left(\frac{3\ln k}{4}\right) \geqslant \frac{1}{2}\,(\ln\ln(k+2))^{\frac{1}{2}}, \qquad k \geqslant 10, \tag{A.13}$$

and so

$$\begin{aligned} f(x_0) &\geqslant \frac{1}{8}\,(k-x_0)\left(\frac{1}{2}\,\ln\ln\ln(k+2)-1\right) + \frac{1}{2}\,x_0\ln x_0 \\ &= \frac{k}{16}\ln\ln\ln(k+2) + Z, \end{aligned} \tag{A.14}$$

where

$$\frac{2}{k}\,Z = -\frac{1}{8}\,\frac{\ln\ln\ln(k+2)}{\ln k} + \frac{1}{4\ln k} + \frac{3}{4} - \frac{\ln\ln k}{\ln k} \geqslant \frac{3}{4} - \frac{9}{8}\,\frac{\ln\ln k}{\ln k} \geqslant 0, \quad k \geqslant 10.$$

Combining (A.12) and (A.14) gives (A.9).

Case 2. Let $1 < x \leqslant (k/\ln k)$. Using (A.13) we have

$$\mathbf{I} \geqslant \left(\ln\left(\frac{k+1}{x+1}\right)\right)^{\frac{k-x}{2}} \geqslant \left(\frac{\ln\ln(k+2)}{2}\right)^{\frac{k}{4}(1-\frac{1}{\ln k})} \geqslant 2^{-k}\,(\ln\ln(k+2))^{\frac{k}{8}}. \qquad \blacksquare$$

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