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On Markov processes with decomposable pseudo-differential generators

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Abstract. The paper is devoted to the study of Markov processes in finite-dimensional convex cones (especially $\mathbf{R}^d$ and $\mathbf{R}^d_+$) with a decomposable generator, i.e. with a generator of the form $L = \sum_{n=1}^N A_n \psi_n$, where every A_n acts as a multiplication operator by a positive, not necessarily bounded, continuous function $a_n(x)$ and where every ψ_n generates a Lévy process, i.e. a process with i.i.d. increments in $\mathbf{R}^d$. The following problems are discussed: (i) existence and uniqueness of Markov or Feller processes with a given generator, (ii) continuous dependence of the process on the coefficients a_n and the starting points, (iii) well posedness of the corresponding martingale problem, (iv) generalized solutions to the Dirichlet problem, (v) regularity of boundary points.

Keywords. Markov processes, Feller processes, pseudo-differential non-local generators, martingale problem, exit time, Dirichlet problem, boundary points, coupling.

Mathematics Subject Classification. 60J25,60J50,60J75.

1. Introduction, main results and content of the paper.

1.1. Basic notations. For a subset $M \subset \mathbf{R}^d$, we shall denote by $C(M)$ (respectively $C_b(M)$, $C_c(M)$, $C_\infty(M)$) the space of continuous functions on M (respectively its subspace consisting of bounded functions, functions with a compact support, functions tending to zero as $x \in M$ tends to infinity). All these spaces are equipped with the usual sup-norm $\|.\|$. If M is an open set and Γ is a subset of the boundary ∂M of M, we denote by $C^s(M \cup \Gamma)$ (respectively $C^s_b(M \cup \Gamma)$) the space of functions having continuous (respectively continuous and bounded) derivatives in M up to and including the order s that have a continuous extension to $M \cup \Gamma$. If M is omitted, it will be tacitly assumed that $M = \mathbf{R}^d$, i.e. we shall write, say, C_∞ to denote $C_\infty(\mathbf{R}^d)$. We shall use all three standard notations $f'(x)$, $\nabla f(x)$, and $\frac{\partial f}{\partial x}(x)$ to denote the gradient field of a smooth function. Similarly, $f''(x)$ denotes the matrix of the second derivatives.

For a locally compact space M (usually $\mathbf{R}^d$, or its one-point compactification $\dot{\mathbf{R}}^d$, or its subdomains) we shall use the standard notation $D_M[0,\infty)$ to denote the Skorokhod space of càdlàg paths in M.

We shall usually denote by the capital letters E and P the expectation and respectively the probability defined by a process under consideration.

1.2. General description of results. Let ψ_n, $n = 1, ..., N$, be a finite family of generators of Lévy processes in $\mathbf{R}^d$, i.e. for each n

$$\psi_n f(x) = tr\,(G^n \frac{\partial^2}{\partial x^2}) f(x) + (\beta^n, \frac{\partial}{\partial x}) f(x)$$

$$+ \int (f(x+y) - f(x) - \nabla f(x) y) \nu^n(dy) + \int (f(x+y) - f(x)) \mu^n(dy), \tag{1.1}$$

where $G^n = (G^n_{ij})$ is a non-negative symmetric $d \times d$-matrix, $\beta^n \in \mathbf{R}^d$, ν^n and μ^n are Radon measures on the ball $\{|y| \leq 1\}$ and on $\mathbf{R}^d$ respectively (Lévy measures) such that

$$\int |y|^2 \nu^n(dy) < \infty, \quad \int \min(1, |y|) \mu^n(dy) < \infty, \quad \nu^n(\{0\}) = \mu^n(\{0\}) = 0 \tag{1.2}$$

(such a partition of the Lévy measure in two parts makes our further assumptions on this measure more transparent), and where

$$tr\,(G \frac{\partial^2}{\partial x^2}) f = \sum_{i,j=1}^d G_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

The function

$$p_n(\xi) = (G^n \xi, \xi) - i(\beta^n, \xi) + \int (1 - e^{i\xi y} + i\xi y) \nu^n(dy) + \int (1 - e^{i\xi y}) \mu^n(dy) \tag{1.3}$$

is called the symbol of the operator $-\psi_n$. This terminology reflects the observation that ψ_n is in fact a pseudo differential operator of the form

$$\psi_n = -p_n(-i\nabla), \quad \nabla = (\nabla_1, ..., \nabla_d) = (\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_d})$$

We shall denote by p_n^ν, p_n^μ the corresponding integral terms in (1.3), e.g. $p_n^\mu(\xi) = \int (1 - e^{i\xi y}) \mu^n(dy)$. We also denote $p_0 = \sum_{n=1}^N p_n$.

Let a_n be a family of positive continuous functions on $\mathbf{R}^d$. Denote by A_n the operator of multiplication by a_n. In the extensive literature on the Feller processes with pseudo-differential generators (see e.g. [14] for a recent review), special attention was given to the decomposable generators of the form $\sum_{n=1}^N A_n\psi_n$, because analytically they are simpler to deal with, but at the same time their properties capture the major qualitative features of the general case. On the other hand, the decomposable generators appear naturally in connection with the interacting particle systems (see [19]-[22]). In fact, the results of this paper (mainly the last Theorems 9, 10) supply the corner stones to the proof of the main result of [20]. In the context of interacting particle systems, the corresponding functions a_n are usually unbounded but smooth.

This paper addresses all fundamental issues of the theory of processes with decomposable generators (with possibly unbounded a_n), namely the problems of the existence and uniqueness of Markov process with a given generator (Theorem 1 and Theorem 3 (i)), the continuous dependence of the process on the coefficients a_n and the starting points (Theorems 2 -5), the restriction of such processes to a subdomain of $\mathbf{R}^d$ (Theorems 6 and 7) and the corresponding Dirichlet problem (Theorem 8), and the application of these results to the analysis of processes in $\mathbf{R}^d_+$ (Theorems 9 and 10). In Appendix we give some general results on the existence of a solution to the martingale problems with pseudo-differential generator (not necessarily decomposable) and on the classification of the boundary points.

We use a variety of techniques both analytic (perturbation theory, chronological or T-products, Sobolev spaces) and probabilistic (martingale problem characterization of Markov semigroups, stopping times, coupling, etc).

1.3. Existence and uniqueness of processes in $\mathbf{R}^d$ *(perturbation theory, the T-product method and the martingale problem approach).* After a large amount of work done by using different deep techniques, the results obtained on the existence of Markov processes with decomposable generators are still far from being complete. The two basic assumptions under which it was proved that to a decomposable operator there corresponds a unique Markov process (see [9]) are the following:

(a1) reality of symbols: all $p_n(\xi)$ are real

(a2) non-degeneracy: $\sum_{n=1}^N p_n(\xi) \geq c|\xi|^\alpha$ with some positive c, α.

Moreover, it was always supposed that $a_n \in C_b^s(\mathbf{R}^d)$ for all n and some s (depending on the dimension d). As indicated in [13], using the methods from [9,12] condition (a1) can be relaxed to the following one:

$(a1')$ $|Im\, p_n(\xi)| \leq c|Re\, p_n(\xi)|$ for all n with some $c > 0$.

Clearly these conditions are very restrictive. For example, they do not include even degenerate diffusions. Notice however that one-dimensional theory is fairly complete by now (see e.g. the pioneering paper [1] and also [19] for more recent developments). Some other related results can be found in [26].

In the present paper we start by proving the existence and uniqueness of the Markov process with generator $\sum_{n=1}^N A_n\psi_n$ under the following assumptions on the symbols p_n: there exists $c > 0$ and constants $\alpha_n > 0$, $\beta_n < \alpha_n$ such that for each $n = 1, ..., N$

(A1) $|Im\, p_n^\mu(\xi) + Im\, p_n^\nu(\xi)| \le c|p_0(\xi)|$,

(A2) $Re\, p_n^\nu(\xi) \ge c^{-1}|pr_{\nu^n}(\xi)|^{\alpha_n}$ and $|(p_n^\nu)'(\xi)| \le c|pr_{\nu^n}(\xi)|^{\beta_n}$, where pr_{ν^n} is the orthogonal projection on the minimal subspace containing the support of the measure ν^n.

Remarks. 1. Clearly the condition $|Im\, p_n| \le cRe\, p_n$ (of type (a1') above) implies $|Im\, p_n| \le c|p_0|$, but is not equivalent to it. 2. Condition (A2) is practically not very restrictive. It allows, in particular, any α-stable measures ν (whatever degenerate) with $\alpha \ge 1$ (the case $\alpha < 1$ can be included in μ_n). Moreover, if $\int |\xi|^{1+\beta_n}\nu_n(d\xi) < \infty$, then the second condition in (A2) holds, because $|e^{ixy} - 1| \le c|xy|^\beta$ for any $\beta \le 1$ and some $c > 0$. In particular, the second inequality in (A2) always holds with $\beta_n = 1$. Hence, in order that (A2) holds it is enough to have the first inequality in (A2) with $\alpha_n > 1$. 3. As no restrictions on the differential part of p_n are imposed, all (possibly degenerate) diffusion processes with symbols are covered by our assumptions.

To formulate our results on existence that include possibly unbounded coefficients we shall also use the following conditions:

(A3) $a_n(x) = O(|x|^2)$ as $x \to \infty$ for those n where $G^n \neq 0$ or $\nu^n \neq 0$, $a_n(x) = O(|x|)$ as $x \to \infty$ for those n where $\beta^n \neq 0$,

(A3′) there exists a positive function $f \in C^2(\mathbf{R}^d)$ with bounded first derivatives such that $f(x) \to \infty$ and $|f''(x)| = |\frac{\partial^2 f}{\partial x^2}| = O(1)(1 + |x|)^{-1}$ as $|x| \to \infty$, and $a_n(x)\psi_n f(x) \le c$ for some constant $c \ge 0$ and all n,

(A4) $a_n(x)$ is bounded whenever $\mu^n \neq 0$,

(A4′) $\int |y|\mu^n(dy) < \infty$ for all n,

(A4″) $a_n(x) = O(|x|)$ whenever $\mu^n \neq 0$.

Theorem 1. *Suppose (A1),(A2) hold for the family of operators ψ_n, and suppose that all a_n are positive functions taken from $C^s(\mathbf{R}^d)$ for $s > 2 + d/2$.*

(i) If (A3), (A4) hold, then there exists a unique extension of the operator $L = \sum_{n=1}^N A_n\psi_n$ (with the initial domain being $C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$) that generates a Feller semigroup in $C_\infty(\mathbf{R}^d)$.

(ii) If (A3′) and (A4′) hold, then there exists a unique strong Markov process whose generator coincides with the operator $L = \sum_{n=1}^N A_n\psi_n$ on $C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$. Moreover, its semigroup preserves the set $C_b(\mathbf{R}^d)$, the process $f(X_t^x) - \int_0^t Lf(X_s^x)\, ds$ is a martingale and

$$Ef(X_t^x) \le f(x) + Nct \tag{1.4}$$

for all t and x, where X_t^x denotes the process with the initial point x.

Remarks. 1. Some information on the domain of the generators of the Markov processes obtained is given in the corollary to Theorem A1 of Appendix 1 for case (A3) and at the end of Section 4 for case ($A3'$).

2. Clearly, condition ($A3'$) allows examples with coefficients increasing arbitrary fast (see Section 7).

3. Statement (ii) still holds if instead of condition $a_n(x)\psi_n f(x) \leq c$ for all n, one assumes the more cumbersome but more general condition that $\sum_{n=1}^N \tilde{a}_n(x)\psi_n f(x) \leq c$ for all $\tilde{a}_n$ such that $0 \leq \tilde{a}_n \leq a_n$.

4. Statement (i) of Theorem 1 is a natural generalization to processes with jumps of a well known criterion for non-explosion of diffusions that states that a diffusion process does not explode and defines a Feller semigroup whenever its diffusion coefficients grow at most quadratically and the drift grows at most linearly.

The proof of this theorem will be given in the next three sections (using also Appendix 1), each of which is based on different ideas and techniques, which seemingly can be used for more general Feller processes. In Section 2 we shall prove (see Proposition 2.1) the result of Theorem 1 subject to some additional bounds for coefficients a_n and under the additional assumption

($A1'$) $|Im\, p_n(\xi)| \leq c|p_0(\xi)|$

on the symbols p_n. Clearly ($A1'$) is a version of (A1) for the whole symbol, which thus combines (A1) and some restrictions on the drift. The proof will be based on the perturbation theory representation for semigroups in Sobolev spaces (as in [18], and not for resolvents as in, say [9,10], [12,13]), which shall give us other nice properties of the semigroup constructed, for example, that $C^2 \cap C_c$ is a core for the generator.

In Section 3 we shall use the methods of T-products and of the "interaction representation" to get rid of the additional assumption ($A1'$).

In Section 4, we shall get rid of the bounds on the norms $\|a_n\|$ and complete the proof of Theorem 1 using the martingale problem approach. This last part of the proof of Theorem 1 has three ingredients: a general existence result for the solution to a martingale problem proved in Appendix 1, standard localization arguments for proving the uniqueness of these solutions (see e.g. [9] in the similar context of Feller processes and [6] in general), and a simple argument to prove the Feller property in case (A3).

4. Continuity properties by the coupling method. Theorems 2 -5 formulated below are proved in Section 5. We are going to use the coupling method to relax the smoothness assumptions on the coefficients $a_n(x)$ and to prove the continuous dependence of the process on these coefficients. Unfortunately, we are able to do it only under very restrictive assumptions on the measures ν^n, namely, we shall assume that for all n

(A5) if $\nu^n \neq 0$, then $a_n(x) = a_n$ is a constant.

Remark. The following results and their proofs are still valid if instead of (A5) one assumes that $d = 1$ (one-dimensional case), $a_n(x)$ is an increasing function of x (respectively decreasing) and ν_n has a support on $(0, \infty)$ (respectively on $(-\infty, 0)$).

Let us recall the notion of coupling (for details, see e.g. [4]). For a probability measures P_1, P_2 on $\mathbf{R}^d$, a measure P on $\mathbf{R}^{2d}$ is called a coupling of P_1, P_2, if

$$P(B \times \mathbf{R}^d) = P_1(B), \quad P(\mathbf{R}^d \times B) = P_2(B)$$

for all measurable $B \subset \mathbf{R}^d$. The W_p-metric between P_1 and P_2 (sometimes called also Kantorovich or Wasserstein metric) is defined by the formula

$$W_p(P_1, P_2) = \inf_P \{ \int |x_1 - x_2|^p P(dx_1, dx_2) \}^{1/p}, \quad p \geq 1, \tag{1.5}$$

where inf is taken over all couplings P of P_1, P_2. We shall write simply W for W_1. For the application of coupling the most important fact is that the convergence of distributions in any of W_p-metric implies the weak convergence. For given $\mathbf{R}^d$-valued processes X_t, Y_t, $t \geq 0$, a process Z_t valued in $\mathbf{R}^{2d}$ is called a coupling of X_t and Y_t, if the distribution of Z_t is a coupling of the distributions of X_t and Y_t for all t. In other words, the coordinates of the process Z_t have the distributions of X_t and Y_t so that one can write $Z_t = (X_t, Y_t)$. With some abuse of notations, we shall denote by $W(X_t, Y_t)$ the W_1-distance between the distributions of X_t and Y_t. For X_t, Y_t, Z_t being Feller processes with generators L_X, L_Y, L_Z respectively, the condition of coupling can be written as $L_Z f_X(x, y) = L_X f(x)$ and $L_Z f_Y(x, y) = L_Y f(y)$ for all f from the domains of L_X and L_Y respectively, where $f_X(x, y) = f(x)$ and $f_Y(x, y) = f(y)$.

The following result reflects the continuous dependence of Feller processes with decomposable generators on their coefficients and initial conditions.

Theorem 2. *Let (A1), (A2), (A4′) hold and let a_n, $\tilde{a}_n$ be two families of positive functions from $C^s(\mathbf{R}^d)$ with $s > 2 + d/2$ such that (A3), (A4), (A5) hold for both of them (see also the Remark after (A5)), $\omega = \max_n \|a_n - \tilde{a}_n\| < \infty$,*

$$K = \max(\max_{n: G^n \neq 0} \|\nabla \sqrt{a_n}\|, \max_{n: \beta^n \neq 0 \, or \, \mu^n \neq 0} \|\nabla a_n\|) < \infty, \tag{1.6}$$

and $\tilde{a}_n = a_n$ if $\nu_n \neq 0$. Let $X_t^{x_0}$ be the Feller process with generator (1.1) starting from some point x_0 and let $Y_t^{y_0}$ be the Feller process with generator (1.1) where all a_n are replaced by $\tilde{a}_n$ and starting from y_0. Then for any $\epsilon > 0$ and $T > 0$ there exists a coupling $Z_t^\epsilon = (X_t^{x_0}, Y_y^{y_0})$ of $X_t^{x_0}$ and $Y_t^{y_0}$ which is a Feller process with a decomposable symbol starting from (x_0, y_0) such that for all $t \in [0, T]$

$$E^\epsilon |X_t^{x_0} - Y_t^{y_0}| \leq C(T, K)(|x_0 - y_0| + \epsilon + \max(\omega, \sqrt{\omega})) \tag{1.7}$$

with some constant $C(T,K)$ depending on T, K and the bound in (A4'). Here E^ϵ denotes the expectation with respect to the coupling process Z_t^ϵ. In particular, taking $\epsilon \to 0$ and using definition (1.5) yields

$$W(X_t^{x_0}, Y_t^{y_0}) \le C(|x_0 - y_0| + \max(\omega, \sqrt{\omega})). \tag{1.8}$$

If additionally all measures μ^n have a finite second moment, i.e. if

$$\sup_n \int |y|^2 \mu^n(dy) < \infty, \tag{1.9}$$

then

$$E^\epsilon |X_t^{x_0} - Y_t^{y_0}|^2 \le C(T,K)(|x_0 - y_0|^2 + |x_0 - y_0| + \epsilon + \omega + \omega^2). \tag{1.10}$$

It is not difficult now to get the following improvements of the results obtained.

Theorem 3. *(i) The statement of Theorem 1 still holds under assumptions (A1), (A2), (A3), (A4), $(A4')$,(A5) if the positive functions a_n are not necessarily smooth but such that $\sqrt{a_n}$ (respectively a_n) are Lipschitz continuous whenever $G^n \neq 0$ (respectively whenever β^n or μ^n do not vanish). (ii) The statement of Theorem 2 still holds if a_n and $\tilde{a}_n$ are not necessarily smooth and instead of (1.6) the functions a_n satisfy condition (i). Moreover, in (1.7) one can take $\epsilon = 0$, i.e. there exists a coupling $Z_t = (X_t^{x_0}, Y_t^{y_0})$ obtained as the limit $\epsilon \to 0$ from the couplings Z_t^ϵ such that*

$$E^0 |X_t^{x_0} - Y_t^{y_0}| \le C(T,K)(|x_0 - y_0| + \max(\omega, \sqrt{\omega})) \tag{1.11}$$

holds, and analogously (1.10) holds with $\epsilon = 0$.

In the following theorem we collect some useful estimates describing in various ways the continuous dependence of the process under consideration on their starting points.

Theorem 4. *Let P^0 and E^0 denote the probability and the expectation given by the coupling $Z_t^0 = (X_t^x, X_t^y)$ described in Theorem 3. Under the assumptions of Theorem 3 (i)*

$$\lim_{|x-y| \to 0} P^0(\sup_{0 \le s \le t} |X_s^x - X_s^y| > r) = 0 \tag{1.12}$$

for all $r > 0$,

$$\lim_{|x-y| \to 0} E^0(|u(X_t^x) - u(X_t^y)|) = 0 \tag{1.13}$$

for any bounded continuous function u and

$$\lim_{r \to \infty} P(\sup_{0 \le s \le t} |X_s^x - x| > r) = 0, \quad \lim_{t \to 0} P(\sup_{0 \le s \le t} |X_s^x - x| > r) = 0, \tag{1.14}$$

the first limit being uniform for all x from any compact set and $0 \le t \le T$ and the second limit being uniform for all x from any compact set and $r \ge r_0$ with any $r_0 > 0$. If all coefficients of the generator L are bounded, all limits above are uniform with respect to all x.

We are going to generalize the main results obtained under condition (A3) to a more general case of condition (*A*3′).

Theorem 5. *Let $a_n \in C^s(\mathbf{R}^d)$ for $s > 2 + d/2$ and let conditions (A1), (A2), (A3′), (A4), (A5) hold. Then for any $\epsilon > 0$, there exists a coupling $Z_t^\epsilon = (X_t^x, X_t^y)$ such that*

$$\lim_{\epsilon \to 0} \lim_{|x-y| \to 0} P^0 (\sup_{0 \le s \le t} |X_s^x - X_s^y| > r) = 0 \tag{1.15}$$

for all $r > 0$, and

$$\lim_{\epsilon \to 0} \lim_{|x-y| \to 0} E^0 (|u(X_t^x) - u(X_t^y)|) = 0 \tag{1.16}$$

for any bounded continuous function u. Moreover, (1.14) holds.

1.5. Processes in cones and the Dirichlet problem. We shall turn now to the study of the processes reduced to an open convex cone $U \subset \mathbf{R}^d$ (with the vertex at the origin). We shall denote by $\bar{U}$ and ∂U the closure and the boundary of U respectively. The dual cone $\{v : (v, w) > 0$ for all non-vanishing $w \in \bar{U}\}$ will be denoted by $U^\star$.

Remark. More general domains could be considered, but for decomposable generators defined in cones all results are much more transparent, the main example being surely $\mathbf{R}_+^d$ considered below in more detail.

To further simplify the formulation of the results, we shall assume that the cone U is proper, i.e. $U^\star \cap U$ is also an open convex cone. Let $\mathbf{e}$ denote some (arbitrary chosen) unit vector in $U \cap U^\star$. Let L denote a decomposable operator in U, i.e. $L = \sum_{n=1}^N A_n \psi_n$ with ψ_n of type (1.1) and with A_n being the operators of multiplications by the real functions a_n on U. We shall widely use the following notion that has its origin in the theory of branching process.

Definition. If $l \in U^\star$, we shall say that L is l-subcritical (respectively, l-critical), if $\psi_n f_l \le 0$ (respectively, $\psi_n f_l = 0$) for all n, where $f_l(x) = (l, x)$. (Notice that $\psi_n f_l$ is a constant.) We say that l-subcritical L is strictly subcritical, if there is n such that $\psi_n f_l < 0$.

From now on, we shall use the classification of the boundary points, the definition of exit times and stopped processes together with the general characterization of the stopped processes in terms of the martingale problem formulation, which are given in Appendix 2. Here we shall study the continuity property (Feller property) of the corresponding semigroups under the following conditions:

(B1) $a_n \in C_b(\bar{U})$ for all n and they are (strictly) positive and smooth (of class $C^s(U)$ with $s > 2 + d/2$ in case of a non-vanishing ν_n and of class $C^1(U)$ for vanishing ν_n) in U;

(B2) the support of the measure $\mu^n + \nu^n$ is contained in U for all n (this condition ensures that U is transmission admissible as discussed in Appendix 2);

(B3) there exists $l \in U^\star$ such that L is l-subcritical.

Occasionally we shall use the following additional assumptions:

(B4) all a_n are extendable as smooth (strictly) positive functions to the whole $\mathbf{R}^d$; in this case we shall assume that this extension is made in such a way that a_n are uniformly bounded outside $U - \mathbf{e}$.

Example. The operator $x\frac{d^2}{dx^2}$ on $\mathbf{R}_+$ can not be extended to $\mathbf{R}_-$ as a diffusion operator with a (positive) smooth coefficient.

The following result is simple.

Proposition 1.1. *(i) Suppose (A1), (A2), (A4′), (B1)-(B4) hold for L. Then there exists a function $f \in C^2(\mathbf{R}^d)$ that coincides with f_l inside U up to an additive constant and such that condition $(A3')$ of Theorem 1 holds, and hence the martingale problem is well posed for L and its solution uniquely defines a strong Markov process X_t in $\mathbf{R}^d$. In particular, condition (U1) of Appendix 2 holds. Moreover, $L\phi \in C_\infty$ whenever $\phi \in C^2 \cap C_c$.*

(ii) If (A1), (A2), (A4′), (B1)-(B3) hold, then the operator L and the domain U satisfy the condition (U2) of Appendix 2 with $U_m = U + \frac{1}{m}\mathbf{e}$. Moreover, $L\phi \in C_\infty(U)$ whenever $\phi \in C^2 \cap C_c$.

Proof. (i) Choose a positive constant K such that $f_l + K$ is strictly positive in $U - \mathbf{e}$. Then let us extend the restriction of this function to $U - \mathbf{e}$ as a smooth positive function ϕ on $\mathbf{R}^d$ such that ϕ' is bounded and $\phi'' = O(1 + |x|^{-1})$. Then $L\phi \leq 0$ in $U - \mathbf{e}$ by subcriticallity, and $L\phi \leq c$ everywhere with some $c > 0$ because all a_n are bounded outside $U - \mathbf{e}$. (ii) Similarly one can extend the restrictions of a_n on U_m to the whole $\mathbf{R}^d$ in such a way that they are bounded outside U and Theorem 1 can be applied. The last statements in both (i) and (ii) are obvious.

Hence Proposition A1 from Appendix 2 holds under assumptions of Proposition 1.1, so that the stopped process X_t^{stop} in U is correctly defined and is uniquely specified as a solution to the corresponding martingale problem.

The semigroup T_t^{stop} of the process stopped on the boundary and the semigroup of the corresponding process killed on the boundary are defined as

$$(T_t^{stop}u)(x) = E_x u(X_{\min(t,\tau_U)}), \quad (T_t^{kil}u)(x) = E_x(u(X_t)\chi_{t<\tau_U}) \tag{1.17}$$

on the space of bounded measurable functions on $\bar{U}$.

An important question is whether the semigroups (1.17) are Feller or not (whether they preserve the class of continuous functions and the class of functions vanishing at infinity). Clearly the second semigroup preserves the set of functions vanishing on the boundary ∂U and actually coincides with the restriction of the first semigroup to this set of functions. Hence the Feller property of the first semigroup would imply the Feller property for the second one.

Some criteria for boundary points to be t-regular, inaccessible or an entrance boundary (that can be used to verify the assumptions in the following results) are given in Appendix 3. The estimates for the exit times are discussed at the end of Section 7 (Propositions 7.2 - 7.4).

Theorem 6. *Under assumptions of Proposition 1.1 (ii), suppose that all ν_n vanish, that X_t leaves U almost surely, and $\partial U \setminus \partial U_{treg}$ is an inaccessible set. Then*

(i) the set $C_b(U \cup \partial U_{treg})$ of bounded continuous functions on $U \cup \partial U_{treg}$ is preserved by the semigroup T_t^{stop}; in particular, if $\partial U = \partial U_{treg}$ and (A3), (A4″) hold, the semigroup T_t^{stop} is a Feller semigroup in $\bar{U}$;

(ii) the subset of $C_b(U \cup \partial U_{treg})$ consisting of functions vanishing at ∂U_{treg} is preserved by T_t^{kil},

(iii) for any continuous bounded function h on ∂U_{treg}, the function $E_x h(X_{\tau_U})$ is continuous in $U \cup \partial U_{treg}$ and for any $u \in C_b(U \cup \partial U_{treg})$ and $x \in U$ there exists a limit

$$\lim_{t \to \infty} T_t^{stop} u(x) = E_x u(X_{\tau_U}); \tag{1.18}$$

(iv) if $P_x(\tau_U > t) \to 0$ uniformly in x (in particular, if $\sup_x E_x \tau_U < \infty$), then the limit in (1.18) is uniform (i.e. it is a limit in the topology of $C_b(U \cup \partial U_{treg})$), and moreover, the function $E_x h(X_{\tau_U})$ is invariant under the action of T_t^{stop} for any $h \in C_b(\partial U_{treg})$.

It is not difficult to give an example when T_t^{stop} does not preserve the whole space $C_b(U \cup \partial U)$. However, if (B4) holds and the inaccessible set $\partial U \setminus \partial U_{treg}$ consists of the entrance boundary points only, one can consider a natural modification of T_t^{stop}, where the process is supposed to stop only on ∂U_{treg}, i.e. one can define a stopping time

$$\tilde{\tau}_U = \inf\{t : X_t^x \in \partial U_{treg}\} \tag{1.19}$$

and the corresponding semigroups

$$(\tilde{T}_t^{stop} u)(x) = E_x u(X_{\min(t, \tilde{\tau}_U)}), \quad (\tilde{T}_t^{kil} u)(x) = E_x(u(X_t) \chi_{t < \tilde{\tau}_U}) \tag{1.20}$$

on the space of bounded measurable functions on $\bar{U}$.

A simple example that illustrates the difference between T^{stop} and $\tilde{T}^{stop}$ is given by the process in $U = \mathbf{R}_+^2 = \{(x, y) : x > 0, y > 0\}$ with the generator $-\partial/\partial x$. Here $\partial U_{treg} = \{(x, y) \in \partial U : x = 0\}$. One sees by inspection that T_t^{stop} is not Feller in $\bar{U}$, but $\tilde{T}_t^{stop}$ is. This example makes the following result not surprising.

Theorem 7. *Let the assumptions of Theorem 6 and condition (B4) hold, and let the inaccessible set $\partial U \setminus \partial U_{treg}$ consist of the entrance boundary points only. Then (i) the space $C_b(\bar{U})$ is preserved by $\tilde{T}_t^{stop}$; in particular, if (A3), (A4″) hold, the semigroup $\tilde{T}_t^{stop}$ and the corresponding process on $\bar{U}$ are Feller; (ii) for any* *continuous bounded function h on ∂U_{treg}, the function $E_x h(X_{\tilde{\tau}_U})$ is continuous in $\bar{U}$, coincides with $E_x h(X_{\tau_U})$ for $x \in U$, and for any $u \in C_b(\bar{U})$ there exists a limit*

$$\lim_{t \to \infty} \tilde{T}_t^{stop} u(x) = E_x u(X_{\tilde{\tau}_U}).$$

A natural application of Theorems 6 and 7 is in the study of the Dirichlet problem.

Definition. Let $h \in C_b(U_{treg})$. A function $u \in C_b(U \cup \partial U_{treg})$ is called *a generalized solution of the Dirichlet problem* for L in U if (i) u coincides with h on ∂U_{treg}, (ii) u belongs to the domain $D(L^{stop})$ of the generator L^{stop} of the semigroup T^{stop} and $L^{stop} u = 0$.

To show that this definition is reasonable, one should prove that any classical solution (i.e. a function $u \in C_b(U \cup \partial U_{treg})$ which satisfies the boundary condition, is two times continuously differentiable and satisfies $Lu = 0$ in U), is also a generalized solution. This question as well as the well posedness of the problem are addressed in the following theorem.

Theorem 8. *Suppose the assumptions of Theorem 6 hold. Then*

(i) a generalized solution exists, is unique, and is given by the formula

$$u(x) = E_x h(X_{\tau_U})$$

for any $h \in C_b(U_{treg})$;

(ii) any classical solution is a generalized solution;

(iii) if, in addition, the conditions of Theorem 7 hold, the generalized solution u is continuous (or can be extended continuously) on $\bar{U}$, belongs to the domain of $\tilde{L}^{stop}$ and $\tilde{L}^{stop} u = 0$.

Some bibliographical comments on the Dirichlet problem for the generators of Markov processes seem to be in order here. For degenerate diffusions the essential progress was begun with the papers [15] and [7]. In particular, in [7] the Fichera function was introduced giving the partition of a smooth boundary into subsets $\Sigma_0, \Sigma_1, \Sigma_3, \Sigma_4$ which in one-dimensional case correspond to natural boundary, entrance boundary, exit boundary and regular boundary respectively studied by Feller (see e.g. [24] for one-dimensional theory). A hard analytic work was done afterwards on degenerate diffusions (see e.g. [27], [16,17], or more recent development in [29], [32]). However, most of the results obtained by analytic methods require very strong assumptions on the boundary, namely that it is smooth and the four basic parts $\Sigma_0, \Sigma_1, \Sigma_3, \Sigma_4$ are disjoint smooth manifolds. Probability theory suggests very natural notions of generalized solutions to the Dirichlet problem that can be defined and to be proved to exist in rather general situations (see [28] for a definition based on the martingale problem approach, [2] for the approach based on the general Balayage space technique, [11] for comparison of different approaches and the generalized Dirichlet space approach), however the interpretation of the general regularity conditions in terms of the given concrete generators and domains becomes a non-trivial problem. Usually it is supposed, in particular, that the process can be extended beyond the boundary. For degenerate diffusions some deep results on the regularity of solutions can be found e.g. in [28] and [8]. But for non-local generators of Feller processes with jumps, the results obtained so far seem to be dealing only with the situations when the boundary is infinitely smooth and there is a dominating non-degenerate diffusion term in the generator (see e.g. [30,31]). Theorem 8 above (in combination with criteria from Appendix 3) clearly includes the situations without a dominating diffusion term and also the situations when the process is not extendable beyond the boundary. The most important example with $U = \mathbf{R}_+^d$ is considered in more detail below. Our definition of the generalized solution to the Dirichlet problem is the same as used in [8] for degenerate diffusions (the only difference is that we included the continuity of the solution in the definition) . Similar results can be obtained by generalizing to jump processes the martingale problem definition from [28].

6. Processes on $\mathbf{R}_+^d$. There is a variety of situations when the state space of a stochastic model is parametrized by positive numbers only. This happens, for instance, if one is interested in the evolution of the number (or the density) of particles or species of different kinds. In this case, the state space of a system is $\mathbf{R}_+^d$. Consequently, one of the most natural application of the results discussed above concerns the situation when $D = \mathbf{R}_+^d$. We shall discuss this situation in more detail. Theorems 9 and 10 formulated below are proved in Section 7.

From now on, let a co-ordinate system $\{x^1, ..., x^d\}$ be fixed in $\mathbf{R}^d$ and let $U = \mathbf{R}_+^d$ be the set of points with all co-ordinates being strictly positive. Then $U^\star = U$ and one can take as a unit vector $\mathbf{e}$ used above the vector $\mathbf{e} = (1, ..., 1)$. We shall suppose that the assumptions (and consequently the conclusions) of Proposition 1.1 (i) or (ii) hold. We shall denote by U_j the subset of the boundary of U where $x^j = 0$ and all other x^k are strictly positive.

As $\mathbf{R}_+^d$ is a proper cone, Theorems 6 - 8 in combination with the criteria established in Appendix 3 (in particular see Remark 2 following Proposition A6) can be applied to construct processes in that cone. In the next Theorem we are going to single out some important particular situations which ensure also that the corresponding semigroup is a Feller one.

Theorem 9. *(i) Suppose (A1),(A2),(A4′),(B1)-(B3) hold for a decomposable pseudo-differential operator L in U. For any $j = 1, ..., d$ and $n = 1, ..., N$, let $a_n(x) = O((x^j)^2)$ in a neighborhood of $\bar{U}_j$ uniformly on compact sets whenever $G^n_{jj} \neq 0$ or $\int (x^j)^2 \nu^n(dx) \neq 0$, and $a_n(x) = O(x^j)$ uniformly on compact sets whenever $\beta^n_j < 0$. Then the whole boundary ∂U is inaccessible, and Proposition A5 is valid that ensures that there exists a unique solution to the martingale problem for L in U, which is a Markov process whose semigroup T_t preserves the space $C_b(U)$.*

(ii) Suppose additionally that $a_n(x) = O(x^j)$ uniformly on compact sets whenever either $\beta_j^n \neq 0$ or $\int x^j \mu^n(dx) \neq 0$. Then T_t preserves the subspace of $C_b(\bar{U})$ of functions vanishing on the boundary. If additionally conditions (A3), (A4$''$) on the growth of a_n hold, then T_t is a strongly continuous Feller semigroup on the Banach space of continuous function on U vanishing when x approaches infinity or the boundary of U.

Our last purpose is to study a natural class of processes which have possibly accessible boundary but which do not stop on the boundary but stick to it as soon as they reach it. For any subset I of the set of indices $\{1, ..., d\}$, let $U_I = \cap_{j \in I} U_j$.

Definition. Let us say that the boundary subspace U_I is *gluing* if for all $j \in I$, $x \in U_I$ and all ξ

$$\frac{\partial}{\partial \xi_j} \sum_{n=1}^{N} a_n(x) p_n(\xi) = 0.$$

Clearly if the boundary U_j, say, is gluing, the values $Lf(x)$ for $x \in U_j$ do not depend on the behavior of f outside U_j. This is the key property of the gluing boundary that allows the process (with generator L) to live on it without leaving it. In the Theorem below, we shall call U_j accessible if it is not inaccessible.

Our main result on gluing boundaries is the following.

Theorem 10. *Let (A1), (A2), (B1)-(B3) hold.*

(i) Suppose that for any j, the boundary U_j is inaccessible or gluing and the same hold for the restrictions of L to any accessible U_j, i.e. for the process on U_j defined by the restriction of L to U_j (well defined due to the gluing property) each of its boundaries U_{ji}, $i \neq j$ is either inaccessible or gluing, and the same holds for the restriction of L to each accessible U_{ji} and so on. Then there exists a unique Markov process Y_t in $\bar{U}$ with sample paths in $D_{\bar{U}}[0, \infty)$ such that

$$\phi(Y_t) - \phi(x) - \int_0^t L\phi(Y_s)\, ds$$

is a P_x-martingale for any $x \in U$ and any $\phi \in C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$ and moreover such that $Y_t \in U_j$ for all $t \geq s$ almost surely whenever $Y_s \in D_j$. Moreover, this process coincides with the process X_t which is uniquely defined as follows: for any $x \in U$, the process X_t is defined as the (unique) solution to the stopped martingale problem in U up to the time τ_1 when it reaches the boundary at some point $y \in U_{j_1}$ with some j_1 such that U_{j_1} is not inaccessible and hence gluing. Starting from y it evolves like a unique solution to the stopped martingale problem in U_{j_1} (with the same generator L) till it reaches a boundary point at $U_{j_1} \cap U_{j_2}$ with some j_2, hence it evolves as the unique solution of the stopped martingale problem in $U_{j_1} \cap U_{j_2}$ and so on, so that it either stops at the origin or ends at some U_I with an inaccessible boundary.

(ii) If additionally all ν_n vanish and $\partial U \setminus \partial U_{treg}$ is an inaccessible set (for all restrictions of L to all accessible boundary spaces), then the corresponding semigroup preserves the set of functions $C_b(U \cup \partial U_{treg})$. In particular, if either $\partial U = \partial U_{treg}$ or $\partial U \setminus \partial U_{treg}$ consists of entrance boundaries only, then the space $C_b(\bar{U})$ is preserved, and if condition (A3), $(A4'')$ hold, then the corresponding semigroup is Feller in $\bar{U}$.

(iii) In order that condition (i) holds it is sufficient that $a_n(x) = 0$ whenever $x \in U_j$ and either $G^n_{jj} \neq 0$, or $\beta^n_j \neq 0$, or $\int (x^j)^2 \nu^n(dx) \neq 0$, or $\int x^j \mu^n(dx) \neq 0$. Then all U_I are gluing.

Remark. Surely the condition in (iii) is just a simplest reasonable criterion for (i) to hold. Other conditions for (i), as well as various conditions for (ii) follow from Propositions A6-A10 of Appendix 3.

The end of the Section 7 is devoted to some simple estimates for exit times from U.

2. Perturbation theory in Sobolev spaces.

Recall first that a Sobolev space H^s is defined as the completion of the Schwarz space $S(\mathbf{R}^d)$ with respect to the norm

$$\|f\|_s^2 = \int (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi,$$

where $\hat{f}(\xi) = (2\pi)^{-d/2} \int e^{-ix\xi} f(x) \, dx$ is the Fourier transform of f. In particular, H^0 (with the norm $\|.\|_0$) is the usual L^2-space.

Let a_n and ψ_n be as in Theorem 1. Let $L_0 = \sum_{n=1}^N \psi_n$ and

$$L = L_0 + \sum_{n=1}^N A_n \psi_n \tag{2.1}$$

(the pseudo-differential operator with the symbol $-\sum_{n=1}^N (1 + a_n(x)) p_n(\xi)$). In this section we shall prove the following result.

Proposition 2.1. *Suppose $(A1')$ and (A2) hold for the family of operators ψ_n, all $a_n \in C^s_b(\mathbf{R}^d)$ for $s > 2 + d/2$ and*

$$2(c+1) \sum_{n=1}^N \|a_n\| < 1, \tag{2.2}$$

where the constant c is taken from condition $(A1')$ (let us stress that $\|.\|$ always denotes the usual sup-norm of a function). Then the closure of $\sum_{n=1}^N A_n \psi_n$ (with the *initial domain $C_c \cap C^2$) generates a Feller semigroup in $C_\infty(\mathbf{R}^d)$ and the (strongly) continuous semigroups in all Sobolev spaces $H^{s'}$, $s' \leq s$, including $H^0 = L^2$.*

From now on, we shall suppose that the assumptions of Proposition 2.1 are satisfied.

We shall start with defining an equivalent family of norms on H^s. Namely, let $b = \{b_I\}$ be any family of (strictly) positive numbers parametrized by multi-indices $I = \{i_1, ..., i_d\}$ such that $0 < |I| = i_1 + ... + i_d \leq s$ and $i_j \geq 0$ for all j. Then the norm $\|.\|_{s,b}$ defined by

$$\|f\|_{s,b} = \|f\|_0 + \sum_{0<|I|\leq s} b_I \|\frac{\partial^{|I|}}{\partial x_I} f\|_0 = \sqrt{\int |\hat{f}(\xi)|^2 \, d\xi} + \sum_{0<|I|\leq s} b_I \sqrt{\int |\xi|^{2I} |\hat{f}(\xi)|^2 \, d\xi},$$

where $|\xi|^{2I} = |\xi_1|^{2i_1} ... |\xi_d|^{2i_d}$ for $I = \{i_1, ..., i_d\}$, is a norm in $S(\mathbf{R}^d)$ which is obviously equivalent to norm $\|.\|_s$. We shall denote by $H^{s,b}$ the corresponding completion of $S(\mathbf{R}^d)$ which coincides with H^s as a topological vector space.

Lemma 2.1. *Let $a(x) \in C_b^s(\mathbf{R}^d)$. Then for an arbitrary $\epsilon > 0$ there exists a collection $b = \{b_I\}$, $0 < |I| \leq s$, of positive numbers such that the operator A of multiplication by $a(x)$ is bounded in $H^{s,b}$ with the norm not exceeding $\|a\| + \epsilon$ (i.e. the bounds on the derivatives of $a(x)$ are essentially irrelevant for the norm of A).*

Proof. To simplify the formulas, we shall give a proof for the case $s = 2, d = 1$. In this case we have

$$\|f\|_{2,b} = \|f\|_0 + b_1 \|f'\|_0 + b_2 \|f''\|_0$$

and

$$\|Af\|_{2,b} \leq (\|a\| + b_1 \|a'\| + b_2 \|a''\|) \|f\|_0$$
$$+ (b_1 \|a\| + 2b_2 \|a'\|) \|f'\|_0 + b_2 \|a\| \, \|f''\|_0.$$

Clearly by choosing b_1, b_2 small enough we can ensure that the coefficient of $\|f\|_0$ is arbitrary close to $\|a\|$ and then by decreasing (if necessary) b_2 we can make the coefficient at $\|f'\|_0$ arbitrary close to $b_1 \|a\|$. The proof is complete.

We are going to construct a semigroup in L^2 and H^s with generator L which is considered as a perturbation of L_0. To this end, for a family of functions ϕ_s, $s \in [0, t]$, on $\mathbf{R}^d$ let us define a family of functions $\mathcal{F}_s(\phi)$, $s \in [0, t]$, on $\mathbf{R}^d$ as

$$\mathcal{F}_s(\phi) = \int_0^s e^{\tau L_0} \sum_{n=1}^N (L - L_0) \phi_\tau \, d\tau. \tag{2.3}$$

From the perturbation theory one knows that formally the solution to the Cauchy problem

$$\dot{\phi} = L\phi, \quad \phi(0) = f \tag{2.4}$$

is given by the series of the perturbation theory

$$\phi = (1 + \mathcal{F} + \mathcal{F}^2 + ...)\phi^0, \quad \phi_s^0 = e^{-sL_0} f. \tag{2.5}$$

In order to carry out a rigorous proof on the basis of this formula, we shall study carefully the properties of the operator $\mathcal{F}$. We shall start with the family of operators F_t on the Schwarz space $S(\mathcal{R}^d)$ defined as

$$F_t(\phi) = \int_0^t e^{sL_0}(L - L_0)\phi \, ds.$$

Lemma 2.2. *F_t is a bounded operator in $L^2(\mathcal{R}^d)$ for all $t > 0$. Moreover, for an arbitrary $\epsilon > 0$, there exists $t_0 > 0$ such that for all $t \leq t_0$*

$$\|F_t\|_0 \leq 2(c+1)\sum_n \|a_n\| + \epsilon$$

and hence $\|F_t\|_0 < 1$ for small enough ϵ.

Proof. As

$$F_t = \sum_{n=1}^N \int_0^t e^{-sL_0}\psi_n A_n \, ds - \sum_{n=1}^N \int_0^t e^{-sL_0}[\psi_n, A_n] \, ds$$

one has for $f \in S(\mathbf{R}^d)$:

$$[\psi_n, A_n]f(x) = (\psi_n(a_n))(x)f(x) + 2(G^n \nabla a_n, \nabla f)(x)$$

$$+ \int (a_n(x+y) - a_n(x))(f(x+y) - f(x))(\nu^n(dy) + \mu^n(dy))$$

$$= 2\sum_{k,l} \nabla_k (G^n_{kl}(\nabla_l a_n)f)(x) + \int \sum_k y_k((\nabla_k a_n f)(x+y) - (\nabla_k a_n f)(x))\nu^n(dy)$$

$$+ (\psi_n(a_n) - 2\sum_{k,l} \nabla_k(G^n_{kl}\nabla_l a_n))(x)f(x)$$

$$+ \int (a_n(x+y) - a_n(x) - (\nabla a_n(x), y))(f(x+y) - f(x))\nu^n(dy)$$

$$- \int \sum_k (\nabla_k a_n(x+y) - \nabla_k a_n(x))f(x+y)y_k \nu^n(dy)$$

$$+ \int (a_n(x+y) - a_n(x))(f(x+y) - f(x))\mu^n(dy).$$

Apart from the first two terms, all other terms in the last expression define bounded operators of f in L^2. Hence

$$F_t(f) = \sum_{n=1}^N \int_0^t e^{sL_0}\psi_n A_n(f)\,ds + 2\sum_{n=1}^N \int_0^t e^{sL_0} \sum_{k,l} \nabla_k (G^n_{kl}(\nabla_l a_n) f)\,ds$$

$$+\sum_{n=1}^N \int_0^t e^{sL_0}\,ds \sum_k \int (e^{i(y,\nabla)}-1)y_k\nu^n(dy)(\nabla_k a_n f) + O(t)\|f\|_0.$$

We can estimate the first term using

$$\| \int_0^t e^{sL_0}\psi_n A_n\,ds\|_0 \leq \|a_n\| \| \int_0^t e^{-sp_0(\xi)} p_n(\xi)\,ds\|$$

$$\leq \|a_n\|\, \|\frac{p_n}{p_0}(1-e^{-tp_0})\| \leq 2(1+c)\|a_n\|$$

(due to $(A1')$, the second term as

$$\| \int_0^t e^{sL_0}\nabla_k (G^n_{kl}(\nabla_l a_n) f)\,ds\| = O(t^{1/2})\|\nabla a_n\|\,\|G\|$$

(to get the latter estimate one should decompose $\mathbf{R}^d$ in the orthogonal sum of the two sub-spaces such that G^n is non-degenerate on the first subspace and vanishes on the other one), and the last term using

$$\| \int_0^t e^{sL_0}\,ds \int (e^{i(y,\nabla)}-1)y_k\nu^n(dy)\|_0 \leq \| \int_0^t e^{-s\,Re\,p_0}\,|\nabla_k p^\nu_n| ds\|$$

$$= O(1)\int_0^t s^{-\beta/\alpha}\,ds = O(t^{1-\beta/\alpha})$$

(which holds due to (A2)). These estimates prove the Lemma.

It turns out that the same holds in H^s.

Lemma 2.3. *For an arbitrary $\epsilon > 0$ there exists $t_0 > 0$ and a family of positive numbers $b = \{b_I\}$, $0 < |I| \leq s$ such that for all $t \leq t_0$ and $s' \leq s$*

$$\|F_t\|_{s',b} \leq 2(c+1)\sum_n \|a^n\| + \epsilon$$

Proof. Follows by the same arguments as the proof of Lemma 2.2 with the use of Lemma 2.1 and the definition of the norm $\|.\|_{s,b}$.

Lemma 2.4. *The family F_t is strongly continuous in $H^{s',b}$ for all $s' \leq s$, i.e. $F_{t+\tau}f - F_t f \to 0$ in $H^{s',b}$ for any $f \in H^{s',b}$.*

Proof. From the estimates on F_t obtained in the proof of Lemma 2.2, we conclude that we only need to prove that $\int_0^t e^{sL_0}\psi_n f\, ds \to 0$ as $t \to 0$ (because the other terms in F_t tends to 0 uniformly). By $(A1')$, it is sufficient to show that $(1 - e^{tL_0})f \to 0$ as $t \to 0$, i.e that the family of operators of multiplication the Fourier image $\hat{f}$ of f by the function $1 - e^{-tp_0(\xi)}$ is strongly continuous, but this is obvious (in a bounded region of ξ the function $1 - e^{-tp_0(\xi)}$ tends to 0 uniformly, and we can always choose a bounded domain such that outside of it the function $\hat{f}$ is small).

We can now deduce the necessary properties of the operator $\mathcal{F}$.

For a Banach space B of functions on $\mathbf{R}^d$ let us denote by $C([0,t], B)$ the Banach space of continuous functions ϕ_s from $[0,t]$ to B with the usual sup-norm $\sup_{s\in[0,t]} \|\phi_s\|_B$. We shall identify B with a closed subspace of functions from $C([0,t], B)$ which do not depend on $s \in [0,t]$.

Lemma 2.5. *Under conditions of Lemma 2.3, the operator $\mathcal{F}$ defined by (2.3) is a continuous operator in $C([0,t], H^{s,b})$ and $\|\mathcal{F}\| < 1$ for small enough t.*

Proof. The statement about the norm follows from Lemma 2.3. Let us show that $\mathcal{F}(\phi) \in C([0,t], H^{s,b})$ whenever $\phi \in C([0,t], H^{s,b})$. One has

$$\mathcal{F}_{t+\tau}(\phi) - \mathcal{F}_t(\phi)$$

$$= \int_0^t (e^{(t+\tau-s)L_0} - e^{(t-s)L_0})(L - L_0)\phi_s\, ds + \int_t^{t+\tau} e^{(t+\tau-s)L_0})(L - L_0)\phi_s\, ds \quad (2.6)$$

The first integral in this expression tends to zero as $\tau \to 0$, because $(1 - e^{\tau L_0})$ converges to zero strongly as $\tau \to 0$ (see proof of Lemma 2.4). Next, writing $\phi_s = \phi_t + (\phi_s - \phi_t)$ in the second integral and again using Lemma 2.4, we conclude that the second integral also tends to zero as $\tau \to 0$.

As a consequence of Lemma 2.5 (and the assumptions of Proposition 2.1) we get the following result.

Lemma 2.6. *Under the conditions of Lemma 2.3, there exists t_0 such that the series (2.5) converges in $C([0,t], H^{s',b})$ for all $s' \leq s$ and $t \leq t_0$. Moreover, the r.h.s. of (2.5) defines a strongly continuous family of bounded operators $f \mapsto T_t f$ in all $H^{s'}$, $s' \leq s$.*

Prove of Proposition 2.1. By the Sobolev lemma, H^s can be continuously imbedded in $C_\infty \cap C^l$ whenever $s > l + d/2$. Hence, T_t defines also a strongly continuous family of bounded operators in C_∞. Next, as $s > 2 + d/2$, $\mathcal{F}_t(\phi)$ is differentiable in t for any $\phi \in C([0,t], H^{s,b})$ and

$$\frac{d}{dt}\mathcal{F}_t(\phi) = \lim_{\tau\to 0} \frac{1}{\tau}(\mathcal{F}_{t+\tau}(\phi) - \mathcal{F}_t(\phi)) = L_0\mathcal{F}_t + (L - L_0)\phi_t, \quad (2.7)$$

where the limit is understood in the norm of H^{s-2}. Therefore, one can differentiate the series (2.5) to show that for $f \in H^s$, the function $T_t f$ gives a (classical) solution to the Cauchy problem (2.4). Since a classical solution in C_∞ for such a Cauchy problem is always positivity preserving and unique, because L is an operator with the PMP (positive maximum principle) property (see e.g. [18], sect.8), we conclude that T_t defines a positivity preserving semigroup in each $H^{s'}$, $s' \leq s$, and in C_∞ for all t (using the semigroup property one can prolong T_t to all finite $t > 0$ thus taking away the restriction $t \leq t_0$). By the standard arguments one can now deduce that T_t defines a contraction semigroup (and thus a Feller semigroup) in C_∞, for example using Hille-Yosida theorem and the fact that the resolvent $R_\lambda f = \int_0^\infty e^{-t\lambda} T_t f \, dt$ is defined on the whole C_∞ for all sufficiently large $\lambda > 0$.

3. T-products for Feller generators.

Let $B_1 \subset B_2$ be two Banach spaces with norms $\|.\|_{B_1} \geq \|.\|_{B_2}$, such that B_1 is dense in B_2. Let $L_t : B_1 \mapsto B_2$, $t \geq 0$, be a family of uniformly (in t) bounded operators such that the closure in B_2 of each L_t is the generator of a strongly continuous semigroups of bounded operators in B_2. For a partition $\Delta = \{0 = t_0 < t_1 < ... < t_N = t\}$ of an interval $[0, t]$ let us define a family of operators $U_\Delta(\tau, s)$, $0 \leq s \leq \tau \leq t$, by the rules

$$U_\Delta(\tau, s) = \exp\{(\tau - s) L_{t_j}\}, \quad t_j \leq s \leq \tau \leq t_{j+1},$$

$$U_\Delta(\tau, r) = U_\Delta(\tau, s) U_\Delta(s, r), \quad 0 \leq r \leq s \leq \tau \leq t.$$

Let $\Delta t_j = t_{j+1} - t_j$ and $\delta(\Delta) = \max_j \Delta t_j$. If the limit

$$U(s, r) f = \lim_{\delta(\Delta) \to 0} U_\Delta(s, r) f \tag{3.1}$$

exists for some f and all $0 \leq r \leq s \leq t$ (in the norm of B_2), it is called the T-product (or chronological exponent of L_t) and is denoted by $T \exp\{\int_r^s L_\tau \, d\tau\} f$. Intuitively, one expects the T-product to give a solution to the Cauchy problem

$$\frac{d}{dt} \phi = L_t \phi, \quad \phi_0 = f, \tag{3.2}$$

in B_2 with the initial conditions f from B_1. In particular, the following (not very hard) statement is proved in [25] (Lemma 1.1). If the T-product exists for $f \in B_1$ and the following basic assumption holds:

(C) the limit

$$\lim_{\tau \to 0} \| \frac{\exp\{\tau L_t\} f - f}{\tau} - L_t f \|_{B_2} = 0 \tag{3.3}$$

is uniform on the bounded sets of B_1,

then $T\exp\{\int_r^s L_\tau\, d\tau\}f$ is a solution of the problem (3.2).

From this fact, we shall deduce now the following simple statement.

Lemma 3.1. *If (i) $L_t f$ is continuous in t locally uniformly in f (i.e. for f from bounded domains of B_1), (ii) all $\exp\{sL_t\}$ preserve B_1 and define a strongly continuous in s,t and a uniformly bounded family of operators in B_1, (iii) condition (C) holds, then*

(i) the T-product (3.1) exists for all $f \in B_2$,

(ii) the convergence in (3.1) is uniform for f from any bounded set of B_1,

(iii) the obtained T-product defines a strongly continuous (in t,s) family of uniformly bounded operators both in B_1 and B_2,

(iv) $T\exp\{\int_0^s L_\tau\, d\tau\}f$ is a solution of the problem (3.2) for any $f \in B_1$.

Proof. Due to the above stated result from [25], the statement (iv) follows from (i)-(iii). Next, since B_1 is dense in B_2, it is enough to prove only the claims from (i)-(iii) concerning B_2. But they follow from the formula

$$U_\Delta(s,r) - U_{\Delta'}(s,r) = U_{\Delta'}(s,\tau)U_\Delta(\tau,r)|_{\tau=r}^{\tau=s} = \int_r^s \frac{d}{d\tau} U_\Delta(s,\tau)U_{\Delta'}(\tau,r)\, d\tau$$

$$= \int_r^s U_\Delta(s,\tau)(L_{[\tau]_{\Delta'}} - L_{[\tau]_\Delta})U_{\Delta'}(\tau,r)\, ds$$

(where we denoted $[s]_\Delta = t_j$ for $t_j \le s < t_{j+1}$) and the uniform continuity of L_t.

The aim of this section is to apply Lemma 3.1 to a particular example of Feller generators and to prove the following result.

Proposition 3.1. *The statement of Proposition 2.1 still holds if we assume (A1) instead of* (A1$'$).

Proof. The difference between (A1) and (A1') concerns only the drift terms of L. So, our statement will be proved, if we will be able to show, that if L is as in Proposition 2.1 and γ be an arbitrary vector field of the class $C_b^s(\mathbf{R}^d)$, then the statements of Proposition still holds for the generator $L + (\gamma(x), \nabla)$. Let S_t be the family of diffeomorphisms of $\mathbf{R}^d$ defined by the equation $\dot{x} = -\gamma(x)$ in $\mathbf{R}^d$. With some abuse of notation we shall denote by S_t also the corresponding action on function, i.e. $S_t f(x) = f(S_t(x))$. In the interaction representation (with respect to the group S_t), the equation

$$\dot{\phi} = (L + (\gamma(x), \nabla))\phi, \quad \phi(0) = f, \tag{3.4}$$

has the form

$$\dot{g} = L_t g = (S_t^{-1} L S_t) g, \quad g(0) = f, \tag{3.5}$$

i.e. equations (3.4) and (3.5) are equivalent for g and $\phi = S_t g$. We shall now apply Lemma 3.1 to the operators L_t from (3.5) using the pair of Banach spaces

$B_1 = H^s$, $s > 2 + d/2$, and $B_2 = H^{s-2}$. The only non-obvious condition to be checked is (C). For this we observe that (i) the convergence in (2.7) is uniform on the "localized" subsets $M \subset H^{s-2}$, i.e. on such subsets M that for any ϵ there exists a compact set K such that $\int_{\mathbf{R}^d \setminus K}(1+|\xi|^{s-2})|\hat{f}(\xi)|\,d\xi < \epsilon$ for all $f \in M$, and (ii) the bounded subsets of H^s are localized subsets in H^{s-2}. Hence the validity of (C) follows. Consequently the T-product yields the classical solutions of (3.5) (and hence of (3.4)) for any f from H^s. But as we mentioned before, the uniqueness of the classical solution follows directly from the PMP property. Hence we obtain a semigroup in both H^s and C_∞.

4. Martingale problem approach.

Proof of Theorem 1. Let us first prove the well posedness of the martingale problem for the operator $L = \sum A_n \psi_n$ under the assumptions of Theorem 1 (see Appendix 1 for the definition of the martingale problem). It follows from Theorem A1 (given in Appendix 1) that under conditions (A3),(A4) the martingale problem for the operator $L = \sum A_n \psi_n$ with sample paths in $D_{\mathbf{R}^d}[0,\infty)$ has a solution. Moreover, in a neighborhood of any point in $\mathbf{R}^d$ one can represent the operator $\sum A_n \psi_n$ in the form $\sum a_n(x_0)\psi_n + \sum (a_n(x) - a_n(x_0))\psi_n$ in such a way that Proposition 3.1 can be applied, and hence in this neighborhood L coincides with an operator for which the martingale problem is well posed (because for generators of the Feller processes the martingale problem is known to be well posed, see [6]). Consequently, assuming (A3), (A4) the uniqueness of the solution of the martingale problem with sample paths in $D_{\mathbf{R}^d}[0,\infty)$ (and hence the well-posedness) follows from the standard localization procedure (see Theorem 7.1 in [9] or Theorems 6.1, 6.2 in Chapter 4 of [6]).

Assume now that $(A3')$, $(A4')$ hold. For each n, let us choose an increasing sequence of continuous positive bounded functions $a_n^m(x)$ converging to $a_n(x)$ and let L_m denote the operator $\sum A_n^m \psi_n$, where A_n^m denote the multiplication by a_n^m. Due to Theorem A1 (ii) from Appendix, the processes

$$f(X_t^m) - \int_0^t L_m f(X_s^m)\,ds \tag{4.1}$$

is a martingale for all m. Moreover, from our assumptions it follows that $a_n^m(x)\psi_n f(x) \leq c$ for all m and n. Hence

$$0 \leq E f(X_t^m) \leq f(x) + tNc.$$

Moreover, since the negative part of the martingale (4.1) is uniformly bounded by tNc, we conclude that the expectation of its magnitude is bounded by $f(x) + 2Nct$ and hence by Doob's inequality

$$\lim_{r\to\infty} P_x\left(\sup_{0\leq s\leq t} f(X_s^m) > r\right) = 0 \tag{4.2}$$

uniformly for x from any compact set and $t \leq T$ with arbitrary T. This clearly implies the compact containment condition and the relative compactness of the family X_t^m (similar arguments are given in more detail in the proof of Theorem A1 of Appendix). Hence, taking a converging subsequence we obtain as a limit a solution to the martingale problem for the operator L which satisfies (1.4). Uniqueness again follows by localization as above. Moreover, as the limit in (4.2) is uniform on x from compact sets, it follows that for arbitrary $r > 0$ and $\epsilon > 0$ there exists $R > 0$ such that for the solution P_η of the martingale problem with an arbitrary initial probability measure η

$$P_\eta(\sup_{0\leq s\leq t} |X_s| \geq R, |X_0| \leq r) \geq (1-\epsilon)\eta(\{|X_0| \leq r\}). \tag{4.3}$$

Due to this estimate one can apply Theorem 5.11 (b), (c) from Chapter 4 of [6] to deduce that the family P_x of the solutions to the martingale problem is a family of measures on $D_{\mathbf{R}^d}[0,\infty)$ that depends weakly continuous on x and that the corresponding semigroup preserves the space $C_b(\mathbf{R}^d)$.

Since it is well known (Theorem 4.2 from Chapter 4 of [6]) that the well posedness of the martingale problem implies that its solution is a strong Markov process, to prove Theorem 1 it remains to show that under condition (A3) the set of functions vanishing at infinity is preserved by the corresponding semigroup. But this follows from a more general Corollary to Theorem A1 from Appendix.

Let us give now some information on the domain of the generator of the (generally speaking not a Feller) contraction semigroup of the Markov process given by Theorem 1 with condition $(A3')$.

Proposition 4.1. *Let X_t be a Markov process given by Theorem 1 under conditions (A1), (A2), $(A3')$, $(A4')$, let T_t denote the corresponding contraction semigroup on C_b, and let $C(L)$ denote the "classical domain" of L, i.e. the space of functions $f \in C^2 \cap C_b$ such that $L\phi \in C_b$. Then*

(i) if $\phi \in C(L)$, then the pair $(\phi, L\phi)$ belongs to the domain of the full generator of T_t, i.e.

$$T_t\phi - f = \int_0^t T_s L\phi \, ds; \tag{4.4}$$

(ii) the mapping $t \mapsto T_t\phi$ is strongly continuous for any ϕ from the closure $\bar{C}(L)$ of $C(L)$ in C_b;

(iii) if $\phi \in C(L)$ and $L\phi \in \bar{C}(L)$, then $T_t\phi$ is differentiable with respect to t and $\frac{d}{dt}T_t\phi = T_tL\phi$ for all t; in particular, such ϕ belongs to the domain $D(L)$ of the generator L in the sense that $\lim_{t\to 0}(T_t\phi - \phi)/t$ exists in the uniform topology of C_b and equals $L\phi$.

Proof. (i) Let $\phi \in C(L)$, and let $\phi_m = \phi\chi_m$, $m = 1, 2, ...$, where χ_m is a smooth function $\mathbf{R}^d \mapsto [0,1]$ such that $\chi_m(y) = 1$ (respectively 0) for $|y \ \leq m$ (respectively

$|y|\geq m+1)$. As $\phi_m\in C^2\cap C_c$, it follows that $\phi_m(X_t)-\phi_m(x)-\int_0^t L\phi_m(X_s)\,ds$ is a martingale with respect to any P_x. To get the same property for ϕ itself from the dominated convergence theorem we need the uniform boundedness of $L\phi_m$, which does not seem to be obvious. To circumvent this difficulty let us apply Doob's option sampling theorem to conclude that

$$\phi_{m+K(m)}(X_{\min(t,\tau_m)})-\phi_{m+K(m)}(x)-\int_0^{\min(t,\tau_m)} L\phi_{m+K(m)}(X_s)\,ds$$

is a P_x-martingale, where $K(m)$ is chosen in such a way that

$$\mu^n(|y|\ \geq K(m))\leq \frac{1}{n}(\sup_{j,|x|\leq m}|a_j(x)|)^{-1},$$

and τ_m is the exit time from the ball $|y|\leq m$. Hence

$$E_x\int_0^{\min(t,\tau_m)} L\phi_{m+K(m)}(X_s)\,ds=E_x\int_0^{\min(t,\tau_m)} L\phi(X_s)\,ds+O(t/n)$$

and a random variable under the integral is uniformly bounded. Hence by the dominated convergence theorem as $m\to\infty$ we get that

$$E_x\phi(X_t)-\phi(x)=\int_0^t E_x\phi(X_s)\,ds.$$

This yields (4.4).

(ii) From (4.4) and due to the contraction property of T_t (which ensures that $T_tL\phi$ is uniformly bounded) it follows that $t\mapsto T_t\phi$ is continuous for $\phi\in C(L)$. One extends this property to all $\phi\in\bar{C}(L)$ by a standard $\epsilon/3$ trick.

(iii) If $L\phi\in\bar{C}(L)$, then the function under the integral in (4.4) is continuous by (ii). Hence the statement follows from (4.4).

Remark. One can find examples where $C(L)$ is rather poor. However, in many reasonable situations it is pretty obvious that $C(L)$ contains $C^2\cap C_c$ and hence $\bar{C}(L)$ contains C_∞. For instance, this is the case if all μ^n have a finite support, or in the case of processes on cones considered in Sections 6 and 7.

5. Coupling for processes with decomposable generators.

Here we shall prove Theorems 2 - 5 essentially by the coupling method.

Proof of Theorem 2. We shall omit here for brevity the upper subscripts in the notations of the process $X_t^{x_0}$ and $Y_t^{y_0}$. Moreover, also for brevity we shall assume that all $\nu_n=0$ noting that if there exists a $\tilde{n}$ such that $\nu_{\tilde{n}}\neq 0$ (and then $a_{\tilde{n}}$ is a constant), one only needs to include in the coupling operator L_ϵ given below the extra term

$$a_{\tilde{n}} \int (f(x+v,y+v) - f(x,y) - \frac{\partial f}{\partial x}(x,y)v - \frac{\partial f}{\partial y}(x,y)v)\nu^{\tilde{n}}(dv)$$

to get the same result.

For an arbitrary $\epsilon > 0$ let M_ϵ denote a regularized function of minimum, i.e. M_ϵ is an infinitely smooth function on $\mathbf{R}^2$ such that $M_\epsilon(b,c) = \min(b,c)$ for $|b-c| \geq \epsilon$ and $\min(b,c) - \epsilon \leq M_\epsilon \leq \min(b,c)$ for all b,c. As a coupling Z_t^ϵ let us take the Feller process in $\mathbf{R}^{2d}$ with the generator

$$L_\epsilon f(x,y) = \sum_{n=1}^N \big(a_n(x)\, tr\,(G^n \frac{\partial^2}{\partial x^2}) f(x,y) + \tilde{a}_n(y)\, tr\,(G^n \frac{\partial^2}{\partial y^2}) f(x,y)$$

$$+2\sqrt{a_n(x)\tilde{a}_n(y)}\, tr\,(G^n \frac{\partial^2}{\partial x \partial y}) f(x,y) + a_n(x)(\beta^n, \frac{\partial}{\partial x}) f(x,y) + \tilde{a}_n(y)(\beta^n, \frac{\partial}{\partial y}) f(x,y)$$

$$+M_\epsilon(a_n(x), \tilde{a}_n(y)) \int (f(x+v,y+v) - f(x,y))\mu^n(dv)$$

$$+(a_n(x) - M_\epsilon(a_n(x), \tilde{a}_n(y))) \int (f(x+v,y) - f(x,y))\mu^n(dv)$$

$$+(a_n(y) - M_\epsilon(a_n(x), \tilde{a}_n(y))) \int (f(x,y+v) - f(x,y))\mu^n(dv)\big). \qquad (5.1)$$

This is a sort of the combination of a regularized marching coupling for the jump part of the generator with the standard coupling of the diffusion processes coming from their representations as solutions to the Ito stochastic equations. The existence of the Feller process with the generator L_ϵ follows from Theorem 1. The key property of the generator L_ϵ is the following: if a function $f(x,y)$ depends only on the difference $(x-y)$, then $L_\epsilon f(x,y)$ equals

$$\sum_{n=1}^N \big((\sqrt{a_n(x)} - \sqrt{\tilde{a}_n(y)})^2\, tr\,(G^n \frac{\partial^2}{\partial x^2}) f(x,y) + (a_n(x) - \tilde{a}_n(y))(\beta_n, \frac{\partial}{\partial x}) f(x,y)$$

$$+(a_n(x) - M_\epsilon(a_n(x), \tilde{a}_n(y))) \int (f(x+v,y) - f(x,y))\mu^n(dv)$$

$$+(\tilde{a}_n(y) - M_\epsilon(a_n(x), \tilde{a}_n(y))) \int (f(x,y+v) - f(x,y))\mu^n(dv). \qquad (5.2)$$

Now let ρ_δ denote a regularized amplitude function on $\mathbf{R}$, i.e.

$$\rho_\delta(b) = \begin{cases} |b|, & |b| \geq \delta \\ \frac{1}{2\delta} b^2 + \frac{\delta}{2}, & |b| < \delta \end{cases} \tag{5.3}$$

and let $f_\delta(x,y) = \rho_\delta(|x-y|)$ denote the corresponding regularized distance on $\mathbf{R}^d$.

By Theorem A1 (ii) of the Appendix, the process

$$f_\delta(Z_t^\epsilon) - \int_0^t L_\epsilon f_\delta(Z_s^\epsilon)\, ds \tag{5.4}$$

is a martingale. Consequently, using (1.6), (5.2) and the trivial formula

$$\max_{a,b\geq 0, |a-b|\leq \omega} |\sqrt{a} - \sqrt{b}| = \max_{x\geq 0}(\sqrt{x+\omega} - \sqrt{x}) = \sqrt{\omega}$$

yields

$$E^\epsilon f_\delta(X_t, Y_t) \leq f_\delta(x_0, y_0)$$

$$+C(T,K) \int_0^t E^\epsilon \big((|X_s - Y_s| + \sqrt{\omega})^2 \min(\frac{1}{\delta}, \frac{1}{|X_s - Y_s|}) + (|X_s - Y_s| + \omega + \epsilon)\big)\, ds.$$

Remark. Notice that one seemingly can not obtain a similar estimate for non-trivial measure ν^n (and $a_n(x)$ being not a constant).

Choosing $\delta = \sqrt{\omega}$ yields

$$E_\epsilon^Z(|X_t - Y_t|) \leq E_\epsilon^Z f_\delta(X_t, Y_t)$$

$$\leq |x_0 - y_0| + \sqrt{\omega} + C \int_0^t \big(E_\epsilon^Z(|X_s - Y_s|) + \max(\sqrt{\omega}, \omega) + \epsilon\big)\, ds,$$

and one gets (1.7) by the standard application of the Gronwall lemma. The proof of (1.10) is quite analogous. Namely, one applies the martingale property of the process (5.4) with $f(x,y) = |x-y|^2$ (which is possible due to (1.9) and Theorem A1 (iii) of the Appendix) to get the estimate

$$E^\epsilon |X_t - Y_t|^2 \leq |x_0 - y_0|^2$$

$$+C(T,K) \int_0^t E^\epsilon(|X_s - Y_s|^2 + |X_s - Y_s|(\omega + 1) + \omega^2 + \epsilon)\, ds.$$

Estimating here $E^\epsilon |X_s - Y_s|$ by (1.7) and then using Gronwall's lemma yields (1.10).

Proof of Theorem 3. (i) Approximating Lipshitz continuous functions a_n by smooth functions $\tilde{a}_n^\omega$ such that (1.6) holds and noticing that (due to Theorem 2) the family of process Y_t^ω (constructed from the family $\tilde{a}_n^\omega$) is fundamental in W-metric as $\omega \to 0$ one concludes that there exists a limiting process Y_t (in W-metric and hence in the sense of the weak convergence) that does not depend on the approximating family $\tilde{a}_n^\omega$. (ii) The first part of this statement is now obvious. To get the coupling with $\epsilon = 0$ one needs only to notice that the only reason to have $\epsilon > 0$ in the proof of Theorem 2 is the necessity to have smooth coefficients in the coupling in order to get the existence of the coupling process from Theorem 1. Since the Lipschitz continuity of the coefficients $a_n(x)$ and their square roots $\sqrt{a}_n(x)$ is now proved to be sufficient for the existence of the process, everything works fine with $\epsilon = 0$ and with the (non-smooth) function min instead of its regularized version M_ϵ used in the proof of Theorem 2.

Proof of Theorem 4. This proof borrows some ideas from the proofs of the analogous results on degenerate diffusions from [8], the essential difference being the use of Dynkin's formula and the coupling Z^0 instead of the use of stochastic equations and Ito's formula in [8]. Let F_t^δ denote the martingale (4.4) where $f = f_\delta(x,y)$ as in the proof of (1.7) above, where $\epsilon = 0$ (which is possible due to Theorem 3) and where $Y_t^y = X_t^y$. From the estimates obtained when proving (1.7) it follows that $E^0(|F_s|) \leq C(|x-y|+\delta)$ for all $s \leq t$ and $E^0(\int_0^t |L_0 f_\delta(Z_s^0)|\, ds \leq C(|x-y|+\delta)$ with some constant $C = C(t)$. Hence by a standard martingale inequality

$$P^0(\sup_{0\leq s\leq t} |F_s| > r) \leq C(r)(|x-y|+\delta)$$

and by the Chebyshev inequality

$$P^0(\int_0^t |L_0 f_\delta(Z_s^0)|\, ds > r) \leq C(r)(|x-y|+\delta)$$

with some constant $C(r)$. This implies that

$$P^0(\sup_{0\leq s\leq t} |X_t^x - X_t^y| > r) \leq P^0(\sup_{0\leq s\leq t} \rho_\delta(|X_t^x - X_t^y|) > r) \leq C(|x-y|+\delta)$$

with some (may be different) constant C. This implies (1.12) since δ can be chosen arbitrary small.

Next, due to the continuity and boundedness of the function u, for any $\epsilon > 0$ one can find a $r > 0$ such that $|u(x_1) - u(x_2)| \leq \epsilon$ whenever $|x_1 - x_2| \leq r$ and $\max(|x_1|,|x_2|) \leq \epsilon^{-1}$. Hence $E^0|u(X_t^x) - u(X_t^y)|$ does not exceed

$$\epsilon + 2\|u\|(P^0(|X_t^x - X_t^y| > r) + P^0(|X_t^x| > \epsilon^{-1}) + P^0(|X_t^y| > \epsilon^{-1}))$$

$$\leq \epsilon + 2\|u\|(r^{-1}E^0(|X_t^x - X_t^y|) + \epsilon E(|X_t^x|) + \epsilon E(|X_t^y|)).$$

This expression can be made arbitrary small by choosing first small ϵ and then small $|x-y|$ (because of (1.11) and the boundedness of $E|X_t^x|$ that follows from Theorem A1 (ii) of the Appendix). This proves (1.13).

First limit in (1.14) is obvious, because it just expresses the non-explosion property of the process. For the case of bounded coefficients a_n, the second limit in (1.14) follows from a more precise and a more general formula (Ap7) of the Appendix. For general situation one observes that changing the generator L outside a domain does not change the behavior of the process inside this domain (see e.g. Theorem 6.1 of Chapter 4 from [6] for a precise formulation of this result). Hence using the first limit in (1.14) one can first reduce the situation to the set of trajectories living in a ball, then change L to $\tilde{L}$ having bounded coefficients $\tilde{a}_n$ that coincide with a_n inside this ball and then again apply (Ap7).

Proof of Theorem 5. We define the coupling by the same operator (5.1). The corresponding process is well defined as a strong Markov process due to Theorem 1 (ii) (see also Remark 3 after this theorem), where as a function f in condition $(A3')$ one can take $f(x)+f(y)$. From (1.4) and the martingale property of $f(X_t)-\int_0^t f(X_s)\,ds$ it follows (by Doob's inequality) that

$$P^\epsilon(\sup_{0\le s\le t}(|X_t|+|Y_t)>r)=o(1) \tag{5.5}$$

as $r\to\infty$. Again as in the proof of Theorem 4, one can change the generator L_ϵ to $\tilde{L}_\epsilon$ having bounded coefficients $\tilde{a}_n$ that coincide with a_n inside the ball of radius r centered at the origin without changing the behavior of the process inside the ball. Hence, (1.14)-(1.16) are obtained by first choosing r to make the r.h.s. of (5.5) arbitrary small and then using (1.12)-(1.14) for a suitable modification of L_ϵ outside the ball of radius r.

6. Processes in cones and the Dirichlet problem.

Proof of Theorem 6. (i) Let $u\in C_b(U\cup\partial U_{treg})$. Then

$$|E_x u(X_{t\wedge\tau_U})-E_y u(X_{t\wedge\tau_U})|\le E^\epsilon|u(X^x_{t\wedge\tau_U^x})-u(X^y_{t\wedge\tau_U^y})|.$$

We need to show that this can be made arbitrary small by choosing $|x-y|$ small enough. Taking into account that the process leaves U almost surely, and hence $\lim_{T\to\infty}P(\tau_U^x>T)=0$, one concludes that one can assume additionally that both τ_U^x and τ_U^y do not exceed some large (but fixed) T and that their trajectories lie in some fixed compact set, because one can ensure that these properties hold with probability arbitrary close to one.

By the definition of t-regularity, for arbitrary positive t and ϵ, and any $z\in\partial U_{treg}$, there exists a ball V_z centered at z such that $P(\tau_U^v>t)<\epsilon$ for all $v\in V_z$.

Choosing a dense denumerable subset of ∂U_{treg}, we can get a denumerable covering of ∂U_{treg} by these V_z. Now by the countable additivity of probability measures, we can choose a finite number of these subsets V_j, $j = 1, ..., q$, such that $P(X_{\tau_U^x} \notin V)$ is arbitrary small for $V = \cup_{j=1}^q V_j$. Next, by Proposition A1 and again by the countable additivity of probability measures one concludes that by choosing m large enough one can ensure that $X_s \in V$ for all $s \in [\tau_m, \tau_U]$ with probability arbitrary close to one. Since for any fixed m, Theorem 5 is applicable, one deduces that for arbitrary positive t and ϵ, there exists $\delta > 0$, an integer m, and an open subset $V \in U \cup \partial U_{treg}$ such that $P(\tau_U^v > t) < \epsilon$ for all $v \in V$ and for any $y : |y - x| \leq \delta$, $X^y_{\tau_m^y} \in V$ with probability not less than ϵ. As τ_U^x and τ_U^y are both less than some large (but fixed T) for y near x, the trajectories X_t^x and X_t^y are uniformly close to each other till the time $\tau_m^x \wedge \tau_m^y$ (again with probability arbitrary close to one). Hence by the above, one can ensure that τ_U^x and τ_U^y are arbitrary close to each other with probability arbitrary close to one and hence by (1.14), $X_{\tau_U^x}$ and $X_{\tau_U^y}$ are also arbitrary close (notice that we use the fact that the estimates (1.14) are uniform for all L_m, because their coefficients are uniformly bounded in any compact domain). Hence $E^\epsilon |u(X^x_{t \wedge \tau_U^x}) - u(X^y_{t \wedge \tau_U^y})|$ tends to zero as y tends to x for any continuous u.

Thus we have proved that $T_t^{stop} u(x)$ is continuous inside U, but exactly the same argument shows that it is continuous for $x \in \partial U_{treg}$. At last, the Feller property (i.e. that the set of functions vanishing at infinity is preserved by the semigroup) in case (A3), (A4) follows directly from Theorem 1. In case $(A4'')$, we notice that condition $(Ap4')$ from Appendix holds in U and the corresponding result follows from Theorem A1, if one observes that its proof works also in the situation when $(Ap4')$ holds on a cone and (Ap4) holds outside it.

Statement (ii) is obvious.

(iii) and (iv). The continuity of $E_x h(X_{\tau_U})$ is proved in exactly the same way as above. To prove that the limit (1.18) exists, we write

$$T_t^{stop} u(x) = E_x \left(u(X_t) \mathbf{1}_{\tau_U \geq t} \right) + E_x \left(u(X_{\tau_U}) \mathbf{1}_{\tau_U < t} \right), \tag{6.1}$$

where as usual $\mathbf{1}_M$ for an event M means the indicator of M that equals 1 if M holds and vanishes otherwise. The first term here tends to zero, because we assumed that the process leaves the domain almost surely in a finite time, and second term tends to the r.h.s. of (1.18) by the dominated convergence theorem. If $P_x(\tau_U \geq t) \to 0$ as $t \to \infty$ uniformly in x, then the first term tends to zero uniformly and

$$E_x \left(u(X_{\tau_U}) \mathbf{1}_{\tau_U < t} \right) - E_x (u(X_{\tau_U})) = E_x \left(u(X_{\tau_U}) \mathbf{1}_{\tau_U \geq t} \right)$$

also tends to zero uniformly in x. The invariance of $E_x(u(X_{\tau_U}))$ is now obtained by the application of T_t^{stop} to both sides of (1.18).

Proof of Theorem 7. One just observes that all arguments given in the proof of Theorem 6 hold for $\tilde{\tau}_U^x$ for all $x \in \bar{U}$.

We shall give now some information on the generator of the stopped process similar to the one given for T_t in Proposition 4.1. Similarly to the space $C(L)$ used in Proposition 4.1, we define now the space $C_U(L)$ (the classical domain of L in U) as the space of functions $\phi \in C^2(U) \cap C_b(U \cup \partial U_{treg})$ such that $L\phi \in C_b(U \cup \partial U_{treg})$.

Proposition 6.1. *Let the assumptions of Theorem 6 hold. Then*

(i) if $\phi \in \bar{C}_U(L)$, then $T_t^{stop}\phi$ is a continuous function $t \mapsto C_b(U \cup \partial U_{treg})$;

(ii) if $\phi \in C_U(L)$, $L\phi \in \bar{C}_U(L)$, and $L\phi(x) = 0$ for all $x \in \partial U_{treg}$, then ϕ belongs to the domain of the generator of the semigroup T_t^{stop} in the sense that $\lim_{t\to 0}(T_t^{stop}\phi - \phi)/t$ exists in the uniform topology of $C_b(U \cup \partial U_{treg})$ and equals $L\phi$.

Proof.

(i) If $\phi \in C_U(L)$, then

$$T_t^{stop}\phi(x) - \phi(x) = E_x \int_0^{\min(t,t_U)} L\phi(X_s)\, ds. \tag{6.2}$$

Since $L\phi \in C_b(U)$, it implies that $T_t^{stop}\phi$ depends continuously on t. For general $\phi \in \bar{C}_U(L)$, one gets the assertion by the usual $\epsilon/3$-trick.

(ii) Since $L\phi$ vanishes on the boundary, one concludes that the r.h.s. of equation (6.2) can be written as

$$E_x \int_0^t L\phi(X_{\min(s,\tau_U)})\, ds = \int_0^t T_s^{stop}(L\phi)(x)\, ds.$$

By (i), the function $T_t^{stop}(L\phi)$ depends continuously on t. Consequently, one gets (ii) by differentiating (6.2).

Remark. Similar result can be obtained under assumptions of Theorem 7.

Proof of Theorem 8. (i) From Theorem 6,

$$u(x) = E_x h(X_{\tau_U}) \in C_b(U \cup \partial U_{treg}).$$

Next, $T_t^{stop}u = u$. In case $\sup_x E_x(\tau_U) < \infty$, it is already proved in Theorem 6 (iv). In general case, this is a consequence of the strong Markov property of X_t quite similar to the case of diffusions (see [8], p.224). Consequently $u \in D(L^{stop})$ and $L^{stop}u = 0$. Hence u is a generalized solution. To show uniqueness, suppose u is a solution vanishing at ∂D_{treg}. Hence $T_t^{stop}u = u$ and from (1.21) it follows that $u = \lim_{t\to\infty} T_t^{stop}u = 0$.

(ii) If $u \in C^2(\mathbf{R}^d) \cap C_b(\mathbf{R}^d)$, $Lu = 0$, then $u \in D(L^{stop})$ and $L^{stop}u = 0$ by Proposition 6.1 (ii) (or simply because $T_t^{stop}u = u$). If $u \in C^2(U)$ only, consider a sequence of functions $u_m \in C^2 \cap C_b$, $Lu \in C_\infty$ such that u_m coincide with u in U_m and vanishes outside U_{m+1}. Hence

$$E_x u(X_{\min(t,\tau_m)}) - u(x) = E_x \int_0^{\min(t,\tau_m)} Lu(X_s)\, ds = 0,$$

where τ_m denote the exit times from U_m, and by the dominated convergence theorem $T_t^{stop} u = u$, which again implies $u \in D(L^{stop})$ and $L^{stop} u = 0$.

(iii) Is the same as (i).

7. Processes in $\mathbf{R}_+^d$.

Proof of Theorem 9. (i) This follows directly from Theorem 1 and Propositions A4, A5 (ii).

(ii) The last statement follows from Theorem A1 (see proof of Theorem 6 above). Let us prove that T_t preserves the space of bounded functions vanishing on the boundary. To this end, let us choose an arbitrary $j \in \{1, ..., d\}$ and let us show that

$$\lim_{R \to \infty, x \to x_0, x \in U} P_x(\tau_\epsilon^R < t) = 0, \tag{7.1}$$

$t > 0$, $\epsilon > 0$, uniformly for x_0 from an arbitrary compact subset K of U_j, where $B_R = \{x_0 \in U_j : |x_0| \leq R\}$ and τ_ϵ^R denotes the exit time from $V_\epsilon^R = \{x : d(x, B_R) < \epsilon\}$ (d denotes the usual distance, of course). By (1.14), by taking R large enough one can ensure that the process does not leave the domain $\{x \in U : |x| \leq R\}$ with the probability arbitrary close to 1 when started in $x_0 \in K$. Let $f(x)$ be a function from $C^2(\{x : x^j > 0\})$ such that $f(x) = (x^j)^\gamma$ with some $\gamma \in (0,1)$ in a neighborhood of V_ϵ^R and vanishes outside some compact set. Then $|Lf(x)| \leq cf$ in a neighborhood of V_ϵ^R with some constant c and as in the proof of Proposition A6 from Appendix 3 (and taking into account that the whole boundary ∂U is inaccessible) one shows that

$$E_x f(X_{\min(t, \tau_\epsilon^R)}) \leq f(x) e^{ct}$$

for x from V_ϵ^R. As (up to an arbitrary small probability which allows the process to leave the domain $\{x : |x| \leq R\}$) the l.h.s. of this inequality can be estimated from below by

$$P_x(\tau_\epsilon < t) \min\{f(x) : x^j = \epsilon\} = P_x(\tau_\epsilon < t)\epsilon^\gamma,$$

the limiting formula (7.1) follows. This formula implies that for any given time t, if the initial point of the process tends to a boundary point, the process is obliged to stay near the boundary the whole time t. This clearly implies that $T_t f(x) = E_x f(X_t)$ tends to zero as x tends to a boundary point whenever $f(x)$ vanishes on the boundary. Consequently the proof of Theorem 9 is completed.

Proof of Theorem 10. (i) Notice first that the process X_t described in the Theorem is well (and uniquely) defined due to Theorem 1, Proposition A1 and Proposition 1.1. If Y_t solves the martingale problem and does not leave U_j after reaching it, then applying the option sampling theorem we conclude that

$$E\big[\phi(Y_{\min(\tau_1+t_2,\tau_2)}) - \phi(Y_{\min(\tau_1+t_1,\tau_2)}) - \int_{\min(\tau_1+t_1,\tau_2)}^{\min(\tau_1+t_2,\tau_2)} L\phi(Y_s)\, ds | Y_{\min(\tau_1+t_1,\tau_2)}\big] = 0$$

for any $t_1 < t_2$, which is precisely the condition that Y_t coincides with X_t between stopping times τ_1 and τ_2. Similar formulas are valid for τ_k, $k = 1, ..., d$ and hence Y_t coincides with X_t. Similar arguments show that conversely, X_t the solves the global martingale problem, which completes the proof of (i).

(ii) This is obtained by the same argument as in the proof of Theorem 6 by analyzing separately the cases with $t \in [\tau_k, \tau_{k+1}]$.

(iii) Follows from the definition of a gluing boundary.

We shall conclude this Section with three Propositions that give some criteria for the exit times from U that can be used to verify the corresponding assumptions from Theorem 6 - 8. Let us assume for the rest of this Section that the assumptions of Proposition 1.1 (i) or (ii) hold and $U = \mathbf{R}^d_+$.

The following statement shows that the deterministic case (when the generator has only drift terms) is quite special for the treatment of the exit from $\mathbf{R}^d_+$.

Proposition 7.2. *The process X_t leaves the domain U almost surely, if there exists n such that $a_n(x)$ is (strictly) positive up to the boundary ∂U and either $G^n \neq 0$ or $\mu_n \neq 0$.*

Proof. A key argument in the proof is based on the observation that (due to subcriticallity), the process $Z_t = f_l(X_{\min(t,\tau_U)})$ is a positive supermartingale, and hence it has a finite limit as $t \to \infty$ almost surely. Hence almost surely, there exists a compact subset $U^{b_1,b_2} = \{x : b_1 \leq (l, x) \leq b_2\}$ in U where the process lives all the time starting from some time t_0. Consequently, to prove the statement, one only need to show that the process leaves any subset U^{b_1,b_2} almost surely using Proposition A2 from Appendix 2. If there exist n and j such that $G^n_{jj} \neq 0$, one takes as a barrier function a positive function that equals $f(x) = \lambda^{-1}(e^{\lambda R} - e^{\lambda x})$ in U^b, where R and λ are large enough. Next, suppose $G^n = 0$ for all n. As $\mu_n \neq 0$, it follows that there exists j such that $\int (l, x)\mu_n(dx) > 0$. Hence, by subcriticallity $\sum_{n=1}^N (\beta^n, l) < 0$. Consequently, one proves that X_t leaves U^{b_1,b_2} almost surely as in Proposition A8 of Appendix 3.

Let us give now some estimates on the expectation of the exit time considering separately the subcritical and critical cases.

Proposition 7.3. *Let L be strictly l-subcritical and let n be such that $\psi_n f_l = -c < 0$. Then (i) if $a_n(x) \geq a > 0$ for all x, then $E_x \tau_U \leq (l, x)/(ac)$; (ii) if $a_n(x) \geq a(1 + |x|^\alpha)$ with some $a > 0$, $\alpha > 1$, then*

$$E_x \tau_U \leq K \min(1, (l, x)) \tag{7.2}$$

for some $K > 0$; in particular, the expectation of the exit time is uniformly bounded in U.

Proof. (i) Follows from Proposition A2 of Appendix 2 with the barrier function $f_l(x) = (l, x)$.

(ii) As a barrier function let us take the function $f(x)=g_\alpha(f_l(x))$, where g_α is a real function such that $g_\alpha(0)=0$ and $g'_\alpha(y)=(1+y)^{-\alpha}$. Since f is positive increasing and does exceed f_l, it is easy to show that it satisfies the martingale condition and hence Proposition A2 from Appendix is applicable. Next, as $g''_\alpha(y)<0$ for all y, it follows that the results of the application of the diffusion part of L and the integral part depending on ν^n to f is always non-positive. As $g'_\alpha(y)$ is positive decreasing it follows that the result of the application of the integral part of ψ_n depending on μ^n to f is positive and does not exceed $g'(f_l(x))\int(l,y)\mu^n(dy)$ at the point x, and hence, as $g'_\alpha(y)$ is of order $y^{-\alpha}$, it follows that $Lf(x)\le -b<0$ uniformly for all x. Hence, the statement follows from Proposition A2 and the observation that $f(x)\le C\min(1,f_l(x))$ for some C.

Proposition 7.4. *Let L be l-critical and let there exists n such that $G^n\neq 0$ and $a_n(x)>a(1+|x|^{1+\alpha})$ with some $\alpha>0$. (i) If $\alpha>1$, then again (7.2) holds. (ii) If $\alpha\in(0,1)$, then $E_x\tau_U\le K\min((l,x)^\alpha,(l,x))$.*

Proof. Is the same as for Proposition 7.5 above: one uses Proposition A2 from Appendix 2 and the barrier $f=g_\alpha(f_l(x))$ taking into account that

$$tr\,(G^n\frac{\partial^2 f(x)}{\partial x^2})=(Gl,l)g''_\alpha(f_l(x))$$

is negative and of order $|x|^{-(1+\alpha)}$.

Appendix 1. On the existence of solutions to martingale problems.

Here we prove a rather general existence result for the martingale problem corresponding to a pseudo-differential (or integro-differential) operator of the form

$$Lu(x)=tr\,(G(x)\frac{\partial^2}{\partial x^2})u(x)+(\beta(x),\nabla)u(x)$$

$$+\int(u(x+y)-u(x)-\mathbf{1}_{|y|\le 1}(y,\nabla)u(x))\nu(x,dy), \qquad (Ap1)$$

where $\mathbf{1}_M(y)$ is an indicator function for a set M (that equals one or zero respectively for $y\in M$ and $y\notin M$), $\nu(x,.)$ is a Lévy measure for all x (i.e. it is a Borel measure on $\mathbf{R}^d$ such that $\nu(x,\{0\})=0$ and $\int\min(1,y^2)\nu(x,dy)<\infty$), and $G(x)=(G_{ij}(x))$, $\beta(x)=(\beta_j(x))$, $i,j=1,...,d$, are respectively non-negative matrix and vector valued functions on $\mathbf{R}^d$.

We shall denote by $X_t(\omega)=\omega(t)$, $\omega\in D_{\dot{\mathbf{R}}^d}[0,\infty)$, the canonical projections of the Skorokhod space and by $\mathcal{F}_t=\sigma(X_s:s\le t)$ the corresponding canonical filtration. For a given probability measure η on $\mathbf{R}^d$, a probability measure $\mathbf{P}_\eta$ on $D_{\mathbf{R}^d}[0,\infty)$ (respectively on $D_{\dot{\mathbf{R}}^d}[0,\infty)$) is called a solution to the martingale problem for L and the initial measure η with sample paths in $D_{\mathbf{R}^d}[0,\infty)$ (respectively in $D_{\dot{\mathbf{R}}^d}[0,\infty)$) if the distribution of X_0 under $\mathbf{P}_\eta$ coincides with η and if for all $\phi \in C_c \cap C^2$ the process

$$\phi(X_t) - \int_0^t L\phi(X_s)\, ds \qquad (Ap2)$$

is an $\mathcal{F}_t$-martingale with respect to $\mathbf{P}_\eta$. If for all initial distributions η there is a unique solution, then the martingale problem is called well-posed. We some abuse of notations we shall write shortly P_x for P_{δ_x} for any $x \in \mathbf{R}^d$, and we shall denote by E_x the corresponding expectation.

The following result (and its proof) generalizes Theorem 3.2 from [Ho2], where the case of bounded real symbols was considered.

Theorem A1. *(i) Suppose that the symbol*

$$p(x,\xi) = (G(x)\xi,\xi) - i(\beta(x),\xi) + \int (1 - e^{i\xi y} + i\mathbf{1}_{|y|\leq 1}(y)(\xi,y))\nu(x,dy) \qquad (Ap3)$$

of the operator $(-L)$ *is continuous,* $|G(x)| = O(x^2)$, $\int_{|y|\leq 1} y^2 \nu(x,dy) = O(x^2)$, $|\beta(x)| = O(|x|)$ *as* $x \to \infty$ *and either*

$$\sup_x \int_{|y|>1} \nu(x,dy) < \infty, \qquad (Ap4)$$

or

$$\int_{|y|>1} |y|\nu(x,dy) = O(|x|), \quad \mathrm{supp}\, \nu(x,.) \cap \{|y|>1\} \subset \{y : |y+x| > |x|\}. \quad (Ap4')$$

Then the martingale problem corresponding to L *has a solution* P_η *with sample paths in* $D_{\mathbf{R}^d}[0,\infty)$ *for any initial probability distribution* η.

(ii) If (i) with (Ap4') holds, or (i) with (Ap4) holds together with a stronger condition

$$\sup_x \int_{|y|>1} |y|\nu(x,dy) < \infty, \qquad (Ap5)$$

then

$$E_x|X_t| \leq (1+|x|)e^{Ct} \qquad (Ap6)$$

for all x *and* $t > 0$ *with some constant* C*, and for any (strictly) positive* $\phi \in C^2$ *such that* $|\phi'(x)|$ *is bounded and* $|\phi''(x)| = O(1)(1+|x|)^{-1}$*, the process (Ap2) is a* P_x*-martingale.*

(iii) If (i) holds and moreover

$$\sup_x \int_{|y|>1} |y|^2 \nu(x,dy) < \infty,$$

then

$$E_x(|X_t - x|^2) \le (1+|x|^2)(e^{Ct}-1) \qquad (Ap7)$$

for all t and process of type (Ap2) is a martingale for any $\phi \in C^2$ with uniformly bounded second derivative. Moreover, if all coefficients are bounded, i.e. $G(x)$, $\beta(x)$, $\int_{|y|\le 1} y^2\nu(x,dy)$ are bounded, then for any $T>0$ and a compact set $K \subset \mathbf{R}^d$

$$P(\sup_{s\le t} |X_s^x - x| > r) \le \frac{t}{r} C(T,K) \qquad (Ap8)$$

for all $t \le T$ and $x \in K$ with some constant $C(T,K)$.

Proof. (i) Writing $L = L_0 + L_1$ with

$$L_1 u(x) = \int_{\{|y|\ge 1\}} (u(x+y)-u(x))\nu(x,dy),$$

and using a perturbation theory result (Proposition 10.2 from Chapter 4 of [6]) one concludes that if (Ap4) holds, the existence of the solutions to the martingale problem for the operator L_0 with sample paths in $D_{\mathbf{R}^d}[0,\infty)$ implies the existence for the same martingale problem for the operator L. This reduces the proof of Theorem A1 to the case when either the support of all measures $\nu(x,.)$ is contained in a unit ball or $(Ap4')$ holds. But in this case it is known (see Theorem 5.4 from Chapter 4 of [6]) that there exists a solution to the martingale problem with the sample paths in $D_{\dot{\mathbf{R}}^d}[0,\infty)$. So, one needs to prove only that the paths of this solution lie in $D_{\mathbf{R}^d}[0,\infty)$ almost surely. Notice also that it is enough to prove this for initial measures of type δ_x only.

Suppose first that the coefficients of the generator L are bounded, i.e. (A5) holds, and the functions $G(x)$, $\beta(x)$ and $\int_{|y|\le 1} |y|^2\nu(x,dy)$ are bounded. Choose a positive increasing smooth function $f_{\ln}$ on $\mathbf{R}_+$ such that $f_{\ln}(y) = \ln y$ for $y \ge 2$. We claim that the process (Ap2) is a martingale for $\phi(y) = f_{\ln}(|y|)$ under any P_x. Here one needs to be a bit cautious because the function $f_{\ln}(|y|)$ does not belong to $C_c \cap C^2$. However, approximating it by the increasing sequence of positive functions $g_n(y) = f_{\ln}(|y|)\chi(|y|/n)$, $n = 1,2,...$, where χ is a smooth function $[0,\infty) \mapsto [0,1)$ which has a compact support and equals 1 in a neighborhood of the origin, noticing that $|Lg_n(x)|$ is a uniformly bounded function of x and n (because g_n and g_n' are uniformly bounded and the coefficients of L are uniformly bounded) and using the dominated convergence theorem we justify the martingale property of (Ap2) with $\phi = f_{\ln}(|y|)$. Hence $E_x f_{\ln}(X_t) \le f_{\ln}(x) + ct$ with some constant $c > 0$. From Doob's martingale inequality we conclude that

$$P_x\left(\sup_{0\le s\le t} f(|X_s|) \ge r\right) \le \frac{C(t+f_{\ln}(x))}{r} \qquad (Ap9)$$

for all $r > 0$ and some $C > 0$. This clearly implies that, almost surely, the paths do not reach infinity in finite time, which completes the proof for the case of bounded coefficients.

In general case, we approximate $G(x), \beta(x)$ and $\nu(x, .)$ by a (uniformly on compact sets) converging sequence of bounded G_m, β_m, ν_m such that all estimates required in (i) are uniform for all m and all operators L_m obtained from L by changing G, β, ν by G_m, β_m, ν_m respectively have bounded coefficients. It follows that $|L_m \phi(x)|$ is a uniformly bounded function of m and x for $\phi(y) = f_{\ln}(|y|)$. But the processes (Ap2) with this ϕ and with L_m instead of L is a martingale (as was shown above). Hence we conclude that $E_x^m f_{\ln}(X_t) \leq f_{\ln}(x) + ct$ uniformly for all m. Again by Doob's inequality it implies that (Ap9) holds uniformly for all processes X_t^m defined by L_m. In turn this implies the so called compact containment condition for the family of processes X_t^m (or the corresponding measures P_x^m), i.e. that for every $\epsilon > 0$ and every $T > 0$ there exists a compact set $\Gamma_{\epsilon,T} \subset \mathbf{R}^d$ such that

$$\inf_m P_x^m \{X_t \in \Gamma_{\epsilon,T} \text{ for all}\, t \in [0, T]\} \geq 1 - \epsilon. \qquad (Ap10)$$

Hence by a well known criterion (see Lemma 5.1 and Remark 5.2 from Chapter 4 of [6]) the family of measures P_x^m on $D_{\mathbf{R}^d}[0, \infty)$ or the corresponding processes $X_t^{x,m}$ is relatively compact, and its limit will solve the martingale problem for the operator L.

(ii) Process (Ap2) is surely a martingale for $\phi \in C^2 \cap C_c$ and for their shifts on constants. Let

$$\rho(x) = \begin{cases} |x|, & |x| \geq 1 \\ (1 + |x|^2)/2 & |x| \leq 1 \end{cases} \qquad (Ap11)$$

and for $n > 1$

$$\rho_n(x) = \begin{cases} \rho(x), & |x| \leq n \\ 2n, & |x| \geq 2n \\ 2n - (|x| - 2n)^2/n, & n \leq |x| \leq 2n \end{cases} \qquad (Ap12)$$

Then each ρ_n can be obtained by shifting a function with a compact support by a constant and hence (Ap2) is a martingale for each ρ_n. (Strictly speaking, the second derivative of ρ_n is not continuous everywhere, but one can approximate it by infinitely smooth functions having the same estimates on its first and second derivatives.) Hence

$$E_x \rho_n(X_t) = \rho_n(x) + \int_0^t E_x L \rho_n(X_s)\, ds \qquad (Ap13)$$

As $|\rho_n'(x)|$ are uniformly bounded and $|\rho_n''(x)| \leq C(1 + |x|)^{-1}$ with some C for all n and x, one concludes that

$$L\rho_n(x) \leq K\rho_n(x)$$

with some $K > 0$ uniformly for all n and x (by inspection, considering separately the cases when $|x| \leq n$, $|x| \geq 2n$ and $n \leq |x| \leq 2n$ and the three terms in the expression for L). Hence from (Ap13) and Gronwall's lemma one gets

$$E_x \rho_n(X_t) \leq \rho_n(x) e^{Kt} \tag{$Ap14$}$$

with some $K > 0$ uniformly for all n and x. As $\rho_n(x)$ is an increasing sequence of functions converging to $\rho(x)$ this implies by the monotone convergence theorem that

$$E_x |X_t| \leq E_x \rho(X_t) \leq \rho(x) e^{Kt} \leq (1 + |x|) e^{Kt}.$$

Next, for any ϕ from the condition (ii) of the theorem, let us take a sequence $g_n(x) = \phi(x)\chi(|x|/n)$ where χ is an infinitely differentiable non-increasing function on $\mathbf{R}_+$ with a compact support taking value in $[0, 1]$ and equal to 1 in a neighborhood of the origin. Then all g_n have compact support and the process (Ap2) with g_n for ϕ is a martingale. As one sees by inspection $|Lg_n(x)| \leq K(1+|x|)$ with some constant $K > 0$ uniformly for all n. As $E_x(1 + |X_t|)$ is already proved to be bounded, one can apply the dominated convergence theorem to the sequence g_n to obtain the required result for its limit ϕ.

(iii) This is quite similar to (ii). Namely, one first gets the result for the function $\phi(x) = 1 + |x|^2$ approximating it by the sequence

$$\phi_n(x) = \begin{cases} 1 + x^2, & |x| \leq n \\ 1 - n^2 + 2nx & |x| \geq n \end{cases}$$

and then for general $\phi(x)$ approximating it by $\phi(x)\chi(|x|/n)$. One gets (Ap7) picking up $\phi(y) = (y - x)^2$ and then using Gronwall's lemma.

At last, to prove (Ap8), picking up $\phi(y) = (y - x)^2$ and using the martingale property yields now $E_x(|X_t - x|^2) = O(t)$ uniformly for all x and then applying Doob's inequality yields the estimate

$$rP_x(\sup_{s \leq t}(|X_s - x|^2 + O(t)|X_t - x|^2 + O(t)) > r)$$

$$\leq 3 \sup_{s \leq t} E_x(|X_t - x|^2 + O(t)|X_t - x|^2 + O(t)),$$

which implies (Ap8) for small enough t. For finite t the result is straightforward.

Corollary. *Under conditions of Theorem A1 (i) suppose additionally that the solution to the martingale problem is unique, and hence this problem is well defined. Then*

(i) the corresponding process is a Feller process, i.e. its semigroup T_t preserves the space $C_\infty(\mathbf{R}^d)$;

(ii) if $\phi \in C^2 \cap C_\infty$ is such that $x^2\phi'' \in C_\infty$ and $|x|\phi' \in C_\infty$, then $L\phi \in C_\infty$ and ϕ belongs to the domain $D(L)$ of the generator L, i.e. $\lim_{t\to 0}(T_t\phi - \phi)/t = L\phi$ *in the uniform topology of the space C_∞.*

Proof. (i) Suppose first that the function $(Lg_y)(x)$ is uniformly bounded as a function of two variables, where $g_y(x) = \rho(x-y)$ with ρ from (Ap11) (this holds, say, if the coefficients of L are bounded). Then, applying the statement of Theorem A1 (ii) to the function g_y yields the estimate $E_y|X_t - y| \le Ke^{Kt}$ with some $K > 0$ uniformity with respect to all y. Hence $P(\sup_{0\le s\le t}|X_s - y| > r)$ tends to zero as $r \to \infty$ uniformly for all y. Consequently, for a $f \in C_\infty(\mathbf{R}^d)$, one has $E_y f(X_t) \to 0$ as $y \to \infty$.

Returning to the general case, first observe that due to the standard perturbation theory result (if A generates a Feller semigroup and B is bounded and satisfies the positive maximum principle, then $A+B$ generates a Feller semigroup), it is enough to prove the statement under additional assumption that either all measures $\nu(x,.)$ have a support in the unit ball or (Ap4') holds. In this case, changing the variable $x \mapsto \tilde{x}$ where $\tilde{x}(x)$ is a diffeomorphism of $\mathbf{R}^d$ such that $\tilde{x} = x$ for $|x| \le 1$, $\tilde{x}/|\tilde{x}| = x/|x|$ for all x, and $|\tilde{x}| = \ln|x|$ for $|x| \ge 3$ allows to reduce the problem to the case of an operator $\tilde{L}$ defined as $(\tilde{L}g)(\tilde{x}) = (Lf)(x(\tilde{x}))$ that has the same structure as L but has bounded drift and diffusion coefficients. Moreover, one observes that $\tilde{L}g_y(\tilde{x})$ is uniformly bounded as a function of two variables y and $\tilde{x}$. In fact, this is equivalent to the statement that $Lf_z(x)$ is uniformly bounded, where $f_z(x) = \rho(\tilde{x} - z)$, and the latter follows from the fact $f_z'(x) = O(|x|^{-1})$ uniformly for all z. Hence the previous arguments work for $\tilde{L}$, which completes the proof.

(ii) If, say, (Ap5) holds, it follows that

$$\sup_x \nu(x, \{\mathbf{R}^d \setminus \{y : |y| > r\}\}) \le \frac{1}{r} \sup_x \int_{\mathbf{R}^d \setminus \{y:|y|>r\}} |y|\nu(x, dy) \to 0$$

as $r \to \infty$, which implies that

$$\int_{\mathbf{R}^d \setminus \{y:|y|>1\}} \phi(x+y)\nu(x,dy) \to 0$$

as $x \to \infty$ for any $\phi \in C_\infty$. This implies that $L\phi \in C_\infty$ for any ϕ from the conditions (ii). Next, using the martingale property (statement (ii) of Theorem A1), we find that

$$E_x\phi(X_t) - \phi(x) = \int_0^t E_x L\phi(X_s)\, ds$$

and hence

$$T_t\phi(x) - \phi(x) = \int_0^t T_s(L\phi)(x)\, ds.$$

As $L\phi \in C_\infty$, $T_s(L\phi)$ is a continuous function $s \mapsto C_\infty$, which implies that $\lim_{t\to 0}(T_t\phi - \phi)/t = L\phi$. The proof is complete.

Remarks. 1. The statement (i) of this corollary is a particular case of a result claimed in [3]. However, the general result from [3] seems to be erroneous as can be seen already on a simple deterministic process with generator $-x^3(\partial/\partial x)$ on the line, whose martingale problem is well posed but the corresponding group is not Feller in the sense that it does not preserve the set of functions vanishing at infinity. 2. Clearly statement (ii) still holds if instead of (Ap5) one assumes only that $\sup_x \nu(x, \{\mathbf{R}^d \setminus \{y : |y| > r\}\}) \to 0$ as $r \to 0$.

Appendix 2. Exit from domain and classification of boundary points.

Let U be an open subset of $\mathbf{R}^d$ and let L be given by (Ap1) in U, i.e. $G(x)$, $\beta(x)$, $\nu(x, .)$ are well defined continuous functions on $\bar{U}$. Let U_{ext} be define as

$$U_{ext} = \{\cup_{x\in U} \operatorname{supp} \nu(x, .)\} \cup U$$

We shall say that U is transmission admissible (with respect to L), if $U = U_{ext}$.

Remark. This terminology stems from the observation that L satisfies the so called transmission property (see e.g [11] and references therein) in U whenever U is transmission admissible by our definition.

From now on, we shall fix some U and L assuming that at least one of the following two conditions holds.

(U1) The domain U is transmission admissible and the operator L can be extended to an operator on the whole $\mathbf{R}^d$ of form (Ap1) (which we shall again denoted by L) in such a way that its symbol is continuous and the corresponding martingale problem is well-posed (for instance, Theorem 1 or the results from [18,19] or[13] are applicable). As above, we shall denote by X_t the corresponding strong Markov process with sample paths in $D_{\mathbf{R}^d}[0, \infty)$ and by P_x the corresponding distribution on the path space when the process starts at x.

(U2) There is a sequence of transmission admissible subdomains U_m of U such that $\bar{U}_m \subset U_{m+1}$ for all m, $\cup_{m=1}^\infty U_m = U$ and the operator L_m obtained from L by the restriction to U_m can be extended to an operator on $\mathbf{R}^d$ of form (Ap1) (which we again denote by L_m) in such a way that (i) its symbol is continuous, (ii) the corresponding martingale problems are well posed, (iii) for any compact set $K \subset \mathbf{R}^d$

$$\sup_{x\in K\cap U} \int \min(y, y^2)\nu(x, dy) < \infty$$

(iv) for the corresponding Markov processes $X_{t,m}$ the following uniform compact containment condition holds: for any $\epsilon > 0$, $T > 0$ and a compact set $K \subset \mathbf{R}^d$, there exists a compact set $\Gamma_{\epsilon,T,K} \subset \mathbf{R}^d$ such that

$$P_x\left(X_{t,m} \in \Gamma_{\epsilon,T,K}\, \forall t \in [0, T]\right) \geq 1 - \epsilon$$

holds for all m and all $x \in K$.

There is a large variety of notions of regularity for boundary points of U. This Appendix is devoted to a discussion of the basic notions of regularity, where we are going to be more general than in usual texts on diffusions (see e.g. [8, 28]), but at the same time more concrete than in general potential theory (see [2]). In particular, we shall propose some generalization of the notion of the entrance boundary from the theory of one-dimensional diffusions (see e.g. [24]) to the case of processes with general pseudo-differential generators. Notice also that (U1) is a usual simplifying assumption for dealing with subdomains, for example, the results of [8] and [11] are formulated subject to this assumption. For our purposes, a generalization to (U2) is of vital importance (see [19, 20]).

For an open $D \subset U$ (including $D = U$) the exist time τ_D from D is defined as

$$\tau_D = \tau_D^x = \inf\{t \geq 0 : X_t^x \notin D \quad \text{or} \quad X_{t_-} \notin D\} \tag{$Ap15$}$$

if (U1) holds or as $\tau_D = \lim_{m \to \infty} \tau_{D \cap U_m}$ if (U2) holds. Clearly, if D itself is transmission admissible and (U1) holds, then

$$\tau_D = \inf\{t : X_t^x \in \partial D\} \tag{$Ap16$}$$

(where ∂D is the boundary of D), and the trajectories of X_t^x are almost surely continuous at $t = \tau_D$. We need a similar characterization of τ_D for the case (U2).

Proposition A1. *(i) Under condition (U2), if $\tau_U = \lim_{m \to \infty} \tau_{U_m} < \infty$, then almost surely there exists a limit $\lim_{m \to \infty} X_{m, \tau_{U_m}}$ and it belongs to ∂U.*

(ii) The stopped process X_t^{stop} in $\bar{U}$ is correctly defined by
(1) $X_t^{stop} = X_{\min(t, \tau_U)}$ in case (U1),
(2) $X_t^{stop} = X_t$ for $t \leq \tau_{U_m}$ for some m and $X_t^{stop} = \lim_{m \to \infty} X_{m, \tau_m}$ for $t \geq \lim_{m \to \infty} \tau_{U_m}$ in case (U2).

(iii) In case (U2) suppose additionally that $L\phi \in C_b$ whenever $\phi \in C^2 \cap C_c$. The stopped process in (ii) is the unique solution of the stopped martingale problem in U, i.e. for any initial probability measure η on U it defines a unique measure P_η^{stop} on $D_{\bar{U}}[0, \infty)$ such that X_0 is distributed according to η, $X_t = X_{\min(t, \tau_U)}$ almost surely and

$$\phi(X_t) - \phi(X_0) = \int_0^{\min(t, \tau_U)} L\phi(X_s) \, ds$$

is a P_η^{stop}-martingale for any $\phi \in C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$.

Proof. (i) The compact containment condition reduces the problem to the case when the coefficients of all L_m are uniformly bounded. In fact, since changing the generator outside a domain does not change the behavior of the process inside this domain (see e.g. Theorem 6.1 from Chapter 4 of [6] for a precise formulation of this result), one can change L to some $\tilde{L}$ by multiplying by an appropriate smooth function $a(x)$ outside a given compact set to get an operator with bounded coefficients with all other conditions preserved. Next, again by the compact containment condition there exists, almost surely, a finite limit point of the sequence $X_{m,\tau_{U_m}}$ which clearly belongs to ∂U. At last, it follows from (Ap8) that this limit point is unique, because if one supposes that there are two different limit points, y_1 and y_2, say, the process must perform infinitely many transitions from any fixed neighborhood of y_1 to any fixed neighborhood of y_2 and back in a finite time, which is impossible by (Ap8) and condition (iii) of (U2) that ensures that (Ap8) holds uniformly for all processes $X_{t,m}$. In fact, the probability of at least n jumps is of order $t^n/n!$.

(ii) This is a direct consequence of (i).

(iii) In case (U1) this is a consequence of a general Theorem 6.1 from Chapter 4 of [6]. In case (U2), from the same general result, it follows that the stopped (at U_m) processes $X_{t,m}^{stop}$ give unique solutions to the corresponding stopped martingale problem in U_m, and by the dominated convergence theorem we get from (i) that X_t^{stop} is a solution to the stopped martingale problem in U. Uniqueness is clear, because the (uniquely defined) stopped processes $X_{t,m}^{stop}$ defines X_t^{stop} uniquely for $t < \tau_U$, and hence up to τ_U inclusive (due to (i)). After τ_U the behavior of the process is fixed by the definition. Proposition is proved.

We shall say that the process X_t leaves a domain $D \subset U$ almost surely (respectively with a finite expectation) if $P_x(\tau_D < \infty) = 1$ for all x (respectively if $E_x \tau_D < \infty$ for all $x \in D$).

Definition. We shall say that

(i) a point $x_0 \in \partial U$ is *t-regular* if for all t

$$\lim_{x \to x_0} P_x(\tau_U > t) = 0; \qquad (Ap17)$$

(ii) a point $x_0 \in \partial U$ is *normally regular*, if there exists a neighborhood V of x_0 such that

$$\limsup_{x \to x_0, x \in U} \frac{E_x \tau_{U \cap V}}{|x - x_0|} < \infty;$$

(iii) a subset $\Gamma \subset \partial U$ is *inaccessible* if $P_x(X_{\tau_U} \in \Gamma, \tau_U < \infty) = 0$ for all $x \in U$;

(iv) a point $x_0 \in \partial U$ is called an *entrance boundary* if for any positive t and ϵ there exist an integer m and a neighborhood V of x_0 such that $P(\tau^x_{U \setminus U_m} > t) < \epsilon$ and $P(\tau^x_{U \setminus U_m} = \tau^x_U) < \epsilon$ for all $x \in V \cap U$, where U_m are the domains from condition (U2), if (U2) holds, or $U_m = \{x \in U : \rho(x, \partial U) > 1/m\}$ in case (U1) holds (ρ denotes the usual distance).

We shall denote by ∂U_{treg} the set of t-regular points of U (with respect to some given process).

Remarks. Notice that a point from an inaccessible set can be nevertheless normally regular. The notion of t-regularity is the key notion for the analysis of the continuity of stopped semigroups (see Theorems 1.6, 1.7) and the corresponding boundary value problems. The normal regularity of a point is required if one is interested in the regularity of the solutions to a boundary value problem beyond the simple continuity (see e.g. [8] for the case of degenerate diffusions under condition (U1)).

For the analysis of the exit times from a domain and for the classification of the boundary points, the major role is played by the method of barrier (or Lyapunov) functions, which is essentially contained in the following simple statement.

We shall say that $\phi \in C^2$ satisfies the martingale condition, if (Ap2) is a martingale for all measures P_x in case (U1) or the same holds for all L_m in case (U2). By definition, all $\phi \in C^2 \cap C_c$ satisfy the martingale condition, but not vice versa.

Proposition A2. Method of barrier function. *Let $f \in C^2$ satisfy the martingale condition and be non-negative in D_{ext} for some $D \subset U$.*

(i) If $Lf(x) \leq 0$ for $x \in D$, then for all $t > 0$ and $x \in D$

$$f(x) \geq E_x(f(X_{\min(t,\tau_D)})).$$

If, moreover, f is bounded and the process leaves D almost surely, then also $f(x) \geq E_x(f(X_{\tau_D}))$.

(ii) If $Lf(x) \leq -c$ for $x \in D$ with some $c > 0$, then

$$f(x) \geq cE_x(\min(t, \tau_D)).$$

In particular, the process leaves the domain D almost surely and $E_x(\tau_D) \leq f(x)/c$.

Proof. Consider the case (U2) only (the case (U1) being clearly simpler). Let $D_m = D \cap U_m$. As f satisfies the martingale condition,

$$E_x f(X_{\min(t,\tau_{D_m})}) = f(x) + E_x \int_0^{\min(t,\tau_{D_m})} Lf(X_s)\, ds,$$

which implies the statements of the Proposition concerning $\min(t, \tau_D)$ (using also Fatou's lemma for the statement (i)). To get the corresponding results for τ_D one takes a limit as $t \to \infty$ and uses the dominated convergence theorem in (i), and the monotone convergence theorem in (ii).

From Proposition A2, one can deduce some criteria for transience and recurrence for processes with pseudo-differential generators (see e.g. [23]). We shall use it now to deduce some criteria of t-regularity and inaccessibility generalizing the corresponding results from [8] devoted to diffusion processes under condition (U1).

Proposition A3. *Suppose $x_0 \in \partial U$, $f \in C^2(U)$ satisfies the martingale condition (or, in case (U2), the restrictions of f to U_m can be extended to $C^2(\mathbf{R}^d)$*

functions that satisfy the martingale conditions for L_m), $f(x_0) = 0$, and $f(x) > 0$ for all $x \in \bar{U} \setminus \{x_0\}$. Suppose there exists a neighborhood V of x_0 such that $Lf(x) \leq -c$ for $x \in U \cap V$ with some $c > 0$. Then x_0 is a t-regular point and

$$E_x(\tau_{V\cap U}) \leq f(x)/c \tag{Ap18}$$

for $x \in V \cap U$. In particular, if $f(x) \leq |x - x_0|$ for $x \in V \cap U$, then x_0 is normally regular.

Proof. Proposition A2 implies (Ap18). Hence $E_x(\tau_{V\cap U}) \to 0$ as $x \to 0$. And consequently $P_x(\tau_{V\cap U} > t) \to 0$ as $x \to x_0$ for any $t > 0$. Next, by Proposition A2 (i), for $x \in U \cap V$

$$f(x) \geq E_x f(X_{\tau_{V\cap U}}) = E_x(f(X_{\tau_U})\mathbf{1}_{\tau_{V\cap U}=\tau_U})$$

$$+E_x(f(X_{\tau_{V\cap U}})\mathbf{1}_{\tau_{V\cap U}<\tau_U}) \geq \min_{y\in U\setminus V} f(y)P_x(\tau_{V\cap U} < \tau_U),$$

where $\mathbf{1}_M$ for an event M denotes the indicator function of M. Hence $P_x(\tau_{V\cap U} < \tau_U) \to 0$ as $x \to x_0$. Consequently,

$$P_x(\tau_{V\cap U} < t \quad \text{and} \quad \tau_{V\cap U} = \tau_U) \to 1$$

as $x \to x_0$, $x \in V \cap U$, and so does

$$P_x(\tau_U < t) > P_x(\tau_U < t \quad \text{and} \quad \tau_{V\cap U} = \tau_U) = P_x(\tau_{V\cap U} < t \quad \text{and} \quad \tau_{V\cap U} = \tau_U).$$

Proposition A4. *(i) Let Γ be a subset of the boundary ∂U. Suppose there is a neighborhood V of Γ and a twice continuously differentiable non-negative function f on U such that f vanishes outside a compact subset of $\mathbf{R}^d$, $Lf(x) \leq 0$ for $x \in V \cap U$, and $f(x) \to \infty$ as $x \to \Gamma$, $x \in V \cap U$. Then Γ is inaccessible.*

(ii) Suppose $x_0 \in \partial U$ and V is a neighborhood of x_0 such that the set $V \cap \partial U$ is inaccessible. Suppose for any $\delta > 0$, there exist a positive integer m and a non-negative function $f \in C^2 \cap C_c$ such that $f(x) \in [0, \delta]$ and $Lf(x) \leq -1$ for $x \in V \cap (U \setminus U_m)$. Then x_0 is an entrance boundary.

Proof. (i) As U is transmission admissible, it is enough to prove that Γ is inaccessible for the domain $V \cap U$. Let us give a proof in case of the condition (U2) only (the other case being similar). For any m let us choose a function $f_m \in C^2 \cap C_c$ that coincides with f in U_m. For any $r > 0$ there exists a neighborhood V_r of Γ such that $\bar{V}_r \subset V$ and $\inf\{f(y) : y \in V_r \cap U\} \geq r$. By Proposition A2, for $x \in U_m \cap V$,

$$f(x) = f_m(x) \geq E_x f_m(X_{\min(t,\tau_{U_m\cap V})})$$

$$\geq \min\{f_m(y) : y \in V_r \cap \partial U_m\} P_x(\tau_{(U_m\cap V)} \leq t,\, X_{\tau_{U_m\cap V}} \in V_r)$$

Hence

$$P_x(\tau_{(U_m\cap V)} \le t,\, X_{\tau_{U_m\cap V}} \in V_r) \le f(x)/r$$

for all t, and consequently

$$P_x(\tau_{(U\cap V)} \le t,\, X_{\tau_{U\cap V}} \in V_r \cap \partial U) \le f(x)/r.$$

Hence

$$P_x(\tau_{(U\cap V)} \le \infty,\, X_{\tau_{U\cap V}} \in V_r \cap \partial U) \le f(x)/r.$$

Since $\cap_{r=1}^{\infty} V_r \supset \Gamma$, the proof of (i) is complete.

(ii) First by reducing t if necessary one can ensure that the probability of leaving $V \cap U$ in time t is arbitrary small (because $V \cap \partial U$ is inaccessible and because the coefficients of L are uniformly bounded which implies (1.14)). Next, by the Chebyshev inequality and Proposition A2 we conclude that

$$P(\tau^x_{V\cap(U\setminus U_m)} > t) \le \frac{1}{t} E_x(\tau_{V\cap(U\setminus U_m)}) \le \frac{1}{t} f(x) \le \frac{\delta}{t}$$

for $x \in V \cap (U \setminus U_m)$, which can be made arbitrary small because δ is arbitrary small.

In Appendix 3, we shall show how one can use the general results obtained above in order to obtain more concrete criteria (in terms of the coefficients of L). Now we shall give only the following simple (but important) consequences to Proposition A1.

Proposition A5. *(i) Under conditions of Proposition A1 (iii) suppose that $\Gamma \in \partial U$ is inaccessible and $L\phi(x) = 0$ for any $x \in \partial U \setminus \Gamma$ and any $\phi \in C^2 \cap C_c$. Then for any $x \in U$, the stopped process X_t^{stop} defines the unique distribution P_x on $D_{\bar{U}}[0,\infty)$ such that $X_0 = x$ and $X_t = X_{\min(t,\tau_U)}$ almost surely, and (Ap2) is a martingale for any $\phi \in C^2 \cap C_c$.*

(ii) If

$$\lim_{m\to\infty} P(\tau_m \le t) = 0 \qquad (Ap19)$$

almost surely for any t and any initial probability measure on U (in particular, (Ap19) is satisfied, if for any $x_0 \in \partial U$ there exists a neighborhood Γ of x_0 in ∂U such that Proposition A4 holds), then for any measure η on U there exists a unique measure P_η on the Skhorohod space $D_U[0,\infty)$ such that (Ap2) is a P_η-martingale for any $\phi \in C^2(U) \cap C_b(U)$ vanishing outside a bounded domain of $\mathbf{R}^d$. Moreover, this measure defines a strong Markov process whose the semigroup T_t preserves the space $C_b(U)$.

(iii) Under condition (ii), if both (U1) and (U2) hold, then the semigroup T_t of the corresponding Markov process in U preserves the subspace $C_b(\bar{U})$.

Proof. (i) It follows from Proposition A1 (iii) and the observation that

$$\int_0^{\min(t,\tau_u)} L\phi(X_s)\,ds = \int_0^t L\phi(X_s)\,ds$$

under conditions from (i).

(ii) For $\phi \in C^2 \cap C_c$ the required martingale property follows again from Proposition A1, or in this particular case it is in fact a direct consequence of Theorem 6.3 from Chapter 4 of [6]. For $\phi \in C^2(U) \cap C_b(U) \cap C_c(\mathbf{R}^d)$, it follows by first considering the stopped martingale problems in U_m and then as usual by the dominated convergence theorem. The last statement follows from Theorem 5.11 (b), (c) from Chapter 4 of [6],

(iii) By the same theorem 5.11 (b), (c) from Chapter 4 of [6], under (U1), (U2), the semigroup T_t of the process defined by L preserves the space $C_b(\mathbf{R}^d)$. Since $T_t\phi = T_t^{stop}\phi$ for all $x \in U$ and all $\phi \in C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$, it follows that $T_t^{stop}\phi \in C_b(\bar{D})$ for these ϕ. As $C^2(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$ is dense in $C_b(\bar{D})$, the required statement follows.

Appendix 3. Examples of barrier functions.

In this Appendix, we shall show how one can choose barrier functions in Propositions A2-A4 above in order to obtain the corresponding criteria in terms of the coefficients of the generator L of form (Ap1). More precisely, we shall consider the operator $\tilde{L}$ given by

$$\tilde{L}f(x) = Lf(x) + \int (f(x+y) - f(x))\,\mu(x,dy) \qquad (Ap20)$$

where μ is a Borel measure on $\mathbf{R}^d \setminus \{0\}$ such that $\int |y|\mu(x,dy)$ is finite for all x and L is of form (Ap1). Surely $\tilde{L}$ can be written in form (Ap1), but it is convenient to have special criteria for integral terms written in a form with μ above (which is possible to do when a Lévy measure has a finite first moment).

We are going to give local criteria for points lying on smooth parts of the boundary (however, they can be used also for piecewise smooth boundaries, see Remark 2 after Proposition A6). Since locally all these parts look like hyper-spaces (can be reduced to them by an appropriate change of the variables), we shall take U here to be the half-space

$$U = \mathbf{R}_+ \times \mathbf{R}^{d-1} = \{(z,v) \in \mathbf{R}^d : z > 0, v \in \mathbf{R}^{d-1}\}$$

and we shall denote by β_z and β_v the corresponding components of the vector field β and by $G_{zz}(x)$ the first entry of the matrix $G(x)$. We shall assume that the supports of all $\nu(x,.)$ and $\mu(x,.)$ belong to U and that condition (U2) holds with $U_m = \{z > 1/m\}$. Let us pick up positive numbers a and r, and for any $\epsilon > 0$ let

$$V_\epsilon = \{z \in (0,a), |v| \le r + \epsilon\}. \qquad (Ap21)$$

Proposition A6. *If*

$$\beta_z(x) = O(z), \quad G_{zz}(x) = O(z^2), \quad \int \tilde{z}^2 \nu(x, d\tilde{x}) = O(z^2), \qquad (Ap22)$$

in V_ϵ, then the ball $\{(0,v) : |v| \le r\}$ belongs to the inaccessible part of the boundary ∂U.

Remark 1. As one could expect, the measure μ does not enter this condition at all.

Remark 2. This criterion can be used also for piecewise smooth boundaries. For example, let $\tilde{U} = U \cap \{v : v^1 > 0\}$ and condition (Ap22) holds in $V_\epsilon \cap \tilde{U}$. Then the same proof as below shows that $\{|v| \le r\} \cap \{v : v^1 > 0\}$ is inaccessible. The same remark concerns other Propositions below.

Proof. A direct application of Proposition A4 is not enough here, but a proof given below is in the same spirit. Let a non-negative $f \in C^2(U)$ be such that it is decreasing in z, equals $1/z$ in V_ϵ and vanishes for large v or z. By f_m we denote a function $f \in C_c \cap C^2$ that coincides with f in U_m. Let τ_m denote the exit time from $V \cap U_m$. Condition (Ap22) implies that $\tilde{L}f(x) \le cf(x)$ for all $x \in V$ and some constant $c \ge 0$. Hence, considering the stopped martingale problem in $V \cap U_m$ and taking as a test function $f_m \in C^2 \cap C_c$ one obtains that

$$E_x f(X_{\min(t,\tau_m)}) - f(x) = E_x \int_0^{\min(t,\tau_m)} \tilde{L}f(X_s)\, ds$$

$$\le cE_x \int_0^{\min(t,\tau_m)} f(X_s)\, ds \le cE_x \int_0^t f(X_{\min(s,\tau_m)})\, ds.$$

Consequently, applying Gronwall's lemma yields the estimate

$$E_x f(X_{\min(t,\tau_m)}) \le f(x)e^{ct}.$$

Hence

$$P_x(\tau_m \le t,\, X_{\tau_m} \in \partial U_m \cap V_\epsilon) \le \frac{1}{m} f(x) e^{ct},$$

which implies that (a neighborhood of) Γ is inaccessible by taking the limit as $m \to \infty$.

Proposition A7. *Suppose there exist constants* $0 \le \delta_1 < \delta_2 \le 1$ *such that*

$$G_{zz}(x) = O(z^{1+\delta_2}), \qquad \int z^2 \nu(x, d\tilde{x}) = O(z^{1+\delta_2})$$

in V_ϵ *and also either*

$$\beta_j(x) \ge \omega z^{\delta_1}$$

or $\beta_j(x) \ge 0$ *and*

$$\int_{V_\epsilon \cap \{z \le \tilde{z} \le 3z/2\}} \tilde{z} \mu(x, d\tilde{x}) \ge \omega z^{\delta_1}$$

in $V_{\epsilon/2}$ *with some* $\omega > 0$*. Then the ball* $\{(0, v) : |v| \le r\}$ *belongs to the inaccessible part of the boundary.*

Proof. This is a direct consequence of Proposition A4, if one takes the same function f from Proposition A6 above to be a barrier function and observes that under the given conditions the diffusion term and the integral term depending on ν in $\tilde{L}$ are both of order $O(z^{\delta_2 - 2})$ and either the drift term is negative of order $z^{\delta_1 - 2}$ and the integral term depending on μ is negative (because f is decreasing in z) or the drift term is negative and the integral term depending on μ is negative of order $z^{\delta_1 - 2}$.

Proposition A8. *Suppose that for* $|v| \le r + \epsilon$ *either* $G_{zz}(0, v)$ *does not vanish, or*

$$\int_{V_{\epsilon/2} \cap \{\tilde{z} \le b\}} \tilde{z}^2 \nu((0, v), d\tilde{x}) \ge \omega b^\delta$$

with some $\delta > 0$*,* $\omega > 0$ *and all sufficiently small* b*, or* $\beta_z(0, v) < 0$*. Then the origin* 0 *belongs to* ∂U_{treg}*.*

Proof. Let us prove that $0 \in \partial U_{treg}$. Let f be defined as

$$f(x) = \begin{cases} cv^2 + z - bz^2, & z < 1/2b \\ cv^2 - \frac{1}{4b}, & z \ge \frac{1}{2b} \end{cases}$$

in V_ϵ and belongs to C^2 and is bounded from below and above by some positive constants. Then

$$\tilde{L} f(0, v) = \beta_z(0, v) - 2b G_{zz}(0, v) + \int \min(\tilde{z} - b\tilde{z}^2, \frac{1}{2b}) \mu((0, v), d\tilde{x})$$

$$+ \int (\min(\tilde{z} - b\tilde{z}^2, \frac{1}{2b} - \tilde{z}) \nu((0, v), d\tilde{x}) + O(c).$$

Clearly the integral term depending on μ tends to zero as $b \to \infty$, the integral depending on ν over the subset $\tilde{z} \ge 1/2b$ is negative. Hence

$$\tilde{L} f(0, v) \le \beta_z(0, v) - 2b G_{zz}(0, v) - b \int_{V_\epsilon \cap \{\tilde{z} \le 1/2b\}} \tilde{z}^2 \nu((0, v), d\tilde{x}) + o(1),$$

where $o(1)$ tends to zero if $c \to 0$, $b \to \infty$, and $z \to 0$. Clearly, if $\beta_z < 0$, this expression becomes negative (for small c and large b), and otherwise, other two assumptions of the Proposition ensure that this expression becomes negative for large enough $b > 0$. By continuity, it will be negative also for (z, v) with small enough z. Consequently, the application of Proposition A3 completes the proof.

Proposition A9. *Suppose $G_{zz}(z, v) = \kappa(v)z(1 + o(1))$ as $z \to 0$ in V_ϵ. (i) If $\kappa(v) > \beta_1(v)$ for $|v| \le r + \epsilon$, then the ball $\{|v| \le r\}$ belongs to ∂U_{treg}. (ii) If $\kappa(v) < \beta_1(v)$ for $|v| \le r + \epsilon$, then the ball $\{|v| \le r\}$ belongs to the inaccessible part of the boundary.*

Proof. (i) As a barrier function, let us take f that equals $z^\gamma + cv^2$ with a $\gamma \in (0, 1)$ in V_ϵ and is smooth and bounded from below and above by positive constants outside. Then near the boundary the sum of the drift and the diffusion terms of Lf is

$$\gamma z^{\gamma-1}(\beta_1 - (1 - \gamma)\kappa + o(1)),$$

which can be made negative by choosing small enough γ. The integral term depending on ν is negative and the integral term depending on μ can be made small by changing f outside an arbitrary small neighborhood of the boundary. Then the origin belongs to ∂U_{treg} by Proposition A3, and similarly one deals with other points of $\{|v| \le r\}$. (ii) This follows from Proposition A4 if one uses the same barrier function as in Propositions 6 and 7 above.

The set where $\kappa(v) = \beta_1(v)$ is known to be a nasty set for the classification even in the case of diffusions (see e.g. [8], [28], [32]). The following simple result is intended to show what kind of barrier function can be used to deal with this situation.

Proposition A10. *Let the boundary of the open set $\Gamma = \{v : \beta_z(0, v) > 0\}$ in ∂U is smooth, the vector field $\beta(x)$ on $\partial\Gamma$ has a positive component in the direction of outer normal η to $\partial\Gamma$, and diffusion term and the integral terms vanish in a neighborhood of Γ in $\bar{U}$. Then the closed subset $\bar{\Gamma} = \Gamma \cup \partial\Gamma$ of the boundary ∂U is inaccessible.*

Proof. Consider the barrier function $f = (z^2 + \rho(v)^2)^{-1}$, where ρ denotes the distance to Γ. Then

$$Lf \le -2\frac{z\beta_z(x)}{z^2 + (\rho(v))^2} - 2\rho(v)\frac{(\beta_v(x), \eta)}{z^2 + (\rho(v))^2},$$

and the second term dominates in a neighborhood of the boundary of Γ, because $\beta_z(x)$ is of order $\rho(v)$. Hence the proof is completed by the application of Proposition A4.

We conclude with a criterion for a point to be an entrance boundary.

Proposition A11. *Let V_ϵ be defined by (Ap21) and let the ball $\{(0, v) : |v| \le r + \epsilon\}$ is inaccessible. Suppose $\beta_z(x) \ge c > 0$ and $\int z^2\nu(x, d\tilde{x}) = O(z)$ in V_ϵ. Then all points from the ball $\{(0, v) : |v| \le r\}$ are entrance boundaries.*

Proof. Clearly it is enough to prove the statement for the origin. Suppose for brevity that $\nu(x,.)$ vanishes in V_ϵ (the modifications required in the general case are as above). Then our claim is a consequence of Proposition A4 (ii), if as a barrier function one takes a function $f(x)$ that equals $\delta - z/c$ for $z < c\delta/2$ and which is non-negative and decreasing in z. Then $f(x) \in [\delta/2, \delta]$ for $z \leq c\delta/2$ and $Lf(x) \leq -1$ for these z, because the contributions from the diffusion part of L and the integral part depending on μ are clearly negative.

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