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Measure-valued limits of interacting particle systems with k-nary interactions II. Finite-dimensional limits.

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Abstract. It is shown that Markov chains in $\mathbf{Z}_+^d$ describing k-nary interacting particles of d different types approximate (in the continuous state limit) Markov processes on $\mathbf{R}_+^d$ having pseudo-differential generators $p(x, i\frac{\partial}{\partial x})$ with symbols $p(x, \xi)$ depending polynomially (degree k) on x. This approximation can be used to prove existence and non-explosion results for the latter processes. Our general scheme of continuous state (or finite-dimensional measure-valued) limits to processes of k-nary interaction yields a unified description of these limits for a large variety of models that are intensively studied in different domains of natural science from interacting particles in statistical mechanics (e.g. coagulation-fragmentation processes) to evolutionary games and multidimensional birth and death processes from biology and social sciences.

Key words. Interacting particles, k-nary interaction, measure-valued limits, Markov processes with pseudo-differential generators, growing symbols, martingale problem, evolutionary games, population dynamics.

AMS 2000 subject classification 60J25, 60J75, 60J80, 60K35, 91A22.

1. Introduction.

This paper is the second in a series of papers devoted to k-nary interacting particles (see [18, 19]) and ideologically it is a development of [18]. However formally it does not depend on [18] that is devoted to one-dimensional processes, where quite special tools are available. On the contrary, the main results of this paper dwells on the theory developed in [17].

Let $\mathbf{Z}^d$ denote the integer lattice in $\mathbf{R}^d$ and let $\mathbf{Z}_+^d$ be its positive cone (which consists of vectors with non-negative coordinates). We equip $\mathbf{Z}^d$ with the usual partial order saying that $N \leq M$ iff $M - N \in \mathbf{Z}_+^d$. A state $N = \{n_1, ..., n_d\} \in \mathbf{Z}_+^d$ will designate a system consisting of n_1 particles of the first type, n_2 particles of the second type, etc. For such a state we shall denote by $supp(N) = \{j : n_j \neq 0\}$ the support of N (considered as a measure on $\{1, ..., d\}$). We shall say that N has a full support if $supp(N)$ coincides with the whole set $\{1, ..., d\}$. We shall write $|N|$ for $n_1 + ... + n_d$.

For a locally compact topological space, we denote by $B(X)$ (resp. $C(X)$) the Banach space of all measurable bounded functions (resp. continuous and bounded) equipped with the usual sup-norm. By $C_c(\mathbf{R}^d)$ (respectively $C^s(\mathbf{R}^d)$) we shall denote the space of continuous functions with a compact support (respectively having s continuous derivatives). We shall also use the standard notations of the theory of pseudo-differential operators. Namely, for a continuous function $p : \mathbf{R}^{2d} \mapsto \mathbf{C}$ we shall denote by $p(x, -i\nabla)$ the (pseudo-differential) operator with the symbol p defined as

$$p(x, -i\nabla)u(x) = (2\pi)^{-d/2} \int_{\mathbf{R}^d} e^{ix\xi} p(x, \xi)\hat{u}(\xi)\, d\xi, \quad u \in S(\mathbf{R}^d),$$

where $\hat{u}(\xi) = (2\pi)^{-d/2} \int e^{-ix\xi} u(x)\, d\xi$ is the Fourier transform of u.

Roughly speaking, k-nary interaction means that any group of k particles (chosen randomly from a given state N) are allowed to have an act of interaction with the effect that some of these particles (maybe all or none of them) may die producing a random number of offspring of different types. More precisely, each sort of k-nary interaction is specified by:

(i) a vector $\Psi = \{\psi_1, ..., \psi_d\} \in \mathbf{Z}_+^d$, which we shall call the profile of the interaction, with $|\Psi| = \psi_1 + ...\psi_d = k$, so that this sort of interaction is allowed to occur only if $N \geq \Psi$ (i.e. ψ_j denotes the number of particles of type j which take part in this act of interaction);

(ii) a family of non-negative numbers $g_\Psi(M)$ for $M \in \mathbf{Z}^d$, $M \neq 0$, vanishing whenever $M \geq -\Psi$ does not hold.

The generator of a Markov process (with the state space $\mathbf{Z}_+^d$) describing *k-nary interacting particles of types* $\{1, ..., d\}$ is then an operator on $B(\mathbf{Z}_+^d)$ defined as

$$(G_k f)(N) = \sum_{\Psi \leq N, |\Psi| = k} C_{n_1}^{\psi_1} ... C_{n_d}^{\psi_d} \sum_M g_\Psi(M)(f(N + M) - f(N)), \tag{1.1}$$

where C_n^k denote the usual binomial coefficients. Notice that each $C_{n_j}^{\psi_j}$ in (1.1) appears from the possibility to choose randomly (with the uniform distribution) any ψ_j particles of type j from a given group of n_j particles. Consequently, the generator $\sum_{k=0}^{|K|} G_k$ of k -nary interactions with profiles not exceeding a given profile K can be written as

$$(G_K f)(N) = \sum_{\Psi \leq K} C_{n_1}^{\psi_1} ... C_{n_d}^{\psi_d} \sum_{M \in \mathbf{Z}^d} g_\Psi(M)(f(N + M) - f(N)), \tag{1.2}$$

where we used the usual convention that $C_n^k = 0$ for $k > n$. The term with $\Psi = 0$ corresponds to the external input of particles.

The aim of the paper is to show that the measure-valued limits (which in the present finite-dimensional framework means just the continuous state limits) of the Markov chains with generators (1.2) are given by Markov processes on $\mathbf{R}_+^d$ having pseudo-differential generators with polynomially growing symbols. and to use this limiting procedure in order to prove the existence and non-explosion of such Markov processes on $\mathbf{R}_+^d$.

To this end, instead of Markov chains on $\mathbf{Z}_+^d$ we shall consider the corresponding scaled Markov chains on $h\mathbf{Z}_+^d$, h being a positive parameter, with generators of type

$$(G_K^h f)(hN) = \sum_{\Psi \le K} h^{|\Psi|} C_{n_1}^{\psi_1} ... C_{n_d}^{\psi_d} \sum_{M \in \mathbf{Z}^d} g_\Psi(M)(f(Nh + Mh) - f(Nh)), \quad (1.3)$$

which clearly can be considered as the restriction on $B(h\mathbf{Z}_+^d)$ of an operator on $B(\mathbf{R}_+^d)$ (which we shall again denote by G_K^h with some abuse of notations) defined as

$$(G_K^h f)(x) = \sum_{\Psi \le K} C_\Psi^h(x) \sum_{M \in \mathbf{Z}^d} g_\Psi(M)(f(x + Mh) - f(x)), \quad (1.4)$$

where we introduced a function C_Ψ^h on $\mathbf{R}_+^d$ defined as

$$C_\Psi^h(x) = \frac{x_1(x_1 - h)...(x_1 - (\psi_1 - 1)h)}{\psi_1!} ... \frac{x_d(x_d - h)...(x_d - (\psi_d - 1)h)}{\psi_d!}$$

in case $x_j \ge (\psi_j - 1)h$ for all j and $C_\Psi^h(x)$ vanishes otherwise.

As

$$\lim_{h \to 0} C_\Psi(x) = \frac{x^\Psi}{\Psi!} = \prod_{j=1}^d \frac{x_j^{\psi_j}}{\psi_j},$$

one can expect that (with an appropriate choice of $g_\Psi(M)$, possibly depending on h) the operators G_K^h will tend to the generator of a stochastic process on $\mathbf{R}_+^d$ which has the form of a polynomial in x with "coefficients" being generators of spatially homogeneous processes with i.i.d. increments (i.e. Lévy processes) on $\mathbf{R}_+^d$, which are given therefore by the Lévy-Khintchine formula with the Lévy measures having support in $\mathbf{R}_+^d$.

The paper is organized as follows. In Section 2 we formulate our main results: Theorems 1 -3. Theorems 2 and 3 are obtained as consequences of more general results from [17] (obtained by developing some ideas from [14-16] and [13]), and Theorem 1 is proved in Section 3. Section 4 is devoted to some examples of the processes with k-nary interaction taken from various domains of natural science.

Let us stress for conclusion that this paper describes a $\mathbf{R}_+^d$-valued limit of a re-scaled number of particles under k-nary interaction. As $\mathbf{R}_+^d$ is the space of measures on a finite set $\{1, ..., d\}$, we have got a measure-valued limit of the Markov chain initially defined on $\mathbf{Z}_+^d$. Alternatively, as is usual in the theory of superprocesses and interacting superprocesses (see e.g. [9, 10, 23] and references therein), one considers points on $\mathbf{Z}_+^d$ as integer -valued measures on $\{1, ..., d\}$ (empirical measures) and the limit $Nh \to x$, $h \to 0$, describes the limit of empirical measures as the number of particles tend to infinity but the "mass" of each particle is re-scaled in such a way that the whole mass tend to x. The finite-dimensionality of the limit is of course due to the fact that we have considered only a finite number of types of particles. In the next papers of this series (see [19, 20]), we shall consider the bona fide (infinite dimensional) measure-valued processes, which arise as the limits for general systems of k-nary interacting particles (which may be characterised by various discrete or continuous parameters like position in space, mass, or genotype for biological models, etc) and which can be described by generators that have the form of polynomials with coefficients given by Lévy-Khintchine formula or its infinite-dimensional analogues. (the case of linear polynomials corresponds to superprocesses). More precisely, if all jumps are scaled uniformly one obtains a deterministic limit described by a general kinetic equation (derived formally in [4] developing some ideas from [3]) that includes as particular cases the well known equations of Vlasov, Boltzman, Smoluchovskii and others. If one accelerates some short range interactions (say, with $|M| = 1$ in (1.3)), one gets a second order parabolic operator as part of a limiting generator, and if one slows down the long range interactions (large M in (1.3)), one gets non-local (Lévy-type) terms.

2. Results.

By $Z_t(G_K)$ (respectively $Z_t(G_K^h)$ we shall denote the minimal Markov chain on $\mathbf{Z}_+^d$ (respectively on $h\mathbf{Z}_+^d$) specified by the generator of type (1.2) (respectively (1.3)). For a given $L \in \mathbf{Z}_+^d$, we shall say that $Z_t(G_K)$ and the generators G_K, G_K^h are *L-subcritical* (respectively *L-critical*) if

$$\sum_{M \neq 0} g_\Psi(M)(L, M) \leq 0 \tag{2.1}$$

for all $\Psi \leq K$ (respectively, if the equality holds in (2.1)), where (L, M) denotes the usual scalar product in $\mathbf{R}^d$. Putting for convenience $g_\Psi(0) = -\sum_{M \neq 0} g_\Psi(M)$, we conclude from (2.1) that the Q-matrix Q^K of the chain $Z_t(G_K)$ defined as

$$Q^K_{NJ} = \sum_{\Psi \leq K} C_{n_1}^{\psi_1} ... C_{n_d}^{\psi_d} g_\Psi(J - N) \tag{2.2}$$

satisfies the condition $\sum_J Q^K_{NJ}(L, J - N) \leq 0$ for all $N = \{n_1, ..., n_d\}$.

Proposition 2.1. *If G_K is L-subcritical with some L having full support, then (i) $Z_t(G_K)$ is a unique Markov chain with the Q-matrix (2.2), (ii) $Z_t(G_K)$ is a* *regular jump process (i.e. it is non-explosive), (iii)* $(L, Z_t(G_K))$ *is a non-negative supermartingale, which is a martingale iff* G_K *is L-critical.*

Proof. This is a direct consequence of (2.1), (2.2) and the standard theory of continuous-time Markov chains. For example, statement (iii) follows either from Dynkin formula (see e.g. [6]) or from the Feller backward integral recursion formula (see e.g. [2]) for $Z_t(G_K)$.

Let us describe now precisely the generators of limiting processes on $\mathbf{R}_+^d$ and the approximating chains in $\mathbf{Z}_+^d$. Suppose that to each $\Psi \leq K$ there correspond

(i) a non-negative symmetric $d\times d$-matrix $G(\Psi) = G_{ij}(\Psi)$ such that $G_{ij}(\Psi) = 0$ whenever i or j does not belong to $supp(\Psi)$,

(ii) vectors $\beta(\Psi) \in \mathbf{R}_+^d$, $\gamma(\Psi) \in \mathbf{R}_+^d$ such that $\gamma_j(\Psi) = 0$ whenever $j \notin supp(\Psi)$,

(iii) Radon measures ν_Ψ and μ_Ψ on $\{|y| \leq 1\} \subset \mathbf{R}^d$ and on $\mathbf{R}_+^d \backslash \{0\}$ respectively (Lévy measures) such that

$$\int |\xi|^2 \nu_\Psi(d\xi) < \infty, \int |\xi| \mu_\Psi(d\xi) < \infty, \quad \mu(\{0\}) = \nu(\{0\}) = 0 \tag{2.3}$$

and $supp\, \nu_\Psi$ belongs to the subspace in $\mathbf{R}^d$ spanned by the unit vectors e_j with $j \in supp(\Psi)$.

These objects define an operator in $C(\mathbf{R}_+^d)$ by the formula

$$(\Lambda_K f)(x) = -\sum_{\Psi \leq K} \frac{x^\Psi}{\Psi!} p_\Psi(-i\nabla), \tag{2.4}$$

where

$$-p_\Psi(-i\nabla) = tr\,(G(\Psi)\frac{\partial^2}{\partial x^2})f + \sum_{j=1}^d (\beta_j(\Psi) - \gamma_j(\Psi))\frac{\partial f}{\partial x_j} + \int (f(x+y) - f(x) - f'(x)y)\nu_\Psi(dy)$$

$$+ \int (f(x+y) - f(x))\mu_\Psi(dy) \tag{2.5}$$

is the pseudo-differential operator with the symbol $-p_\Psi(\xi)$, where

$$p_\Psi(\xi) = (\xi, G(\Psi)\xi) - i(\beta - \gamma, \xi)$$

$$+ \int (1 - e^{iy\xi} + iy\xi)\nu_\Psi(dy) + \int (1 - e^{iy\xi})\mu_\Psi(dy) \tag{2.6}$$

and where as usual

$$tr\,(G(\Psi)\frac{\partial^2}{\partial x^2})f = \sum_{i,j=1}^d G_{ij}(\Psi)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Operator (2.5) is known to represent the generator of a Lévy process in $\mathbf{R}^d$, or a process with i.i.d. (independent identically distributed) increments. Hence one can say that operators (2.4) are polynomials in x with "coefficients" being the generators of Lévy processes.

Remark. Conditions in (i) and (iii) concerning the $supp(\Psi)$ mean simply that a particle of type i can not kill a particle of type j without an interaction. Condition (iii) highlights the fact that in the framework of interacting particles, it is natural to write the generators of a Lévy process in the form (2.5) with two measures ν and μ (the first one having bounded support but infinite first moment), because only ν is subject to an additional condition on its support.

We shall say that operator (2.5) is *L-subcritical* (respectively *L-critical*) for an $L \in \mathbf{Z}_+^d$, if

$$(\beta(\Psi) - \gamma(\Psi) + \int y\mu_\Psi(dy), L) \leq 0 \tag{2.7}$$

for all Ψ (respectively, if the equality holds in (2.7)).

Next, let $\Delta_h(\Psi, G)$ be a finite-difference operator of the form

$$\begin{array}{c}(\Delta_h(\Psi, G)f)(x) = \frac{1}{h^2}\sum_{i \in supp(\Psi)} \omega_i(\Psi)(f(x + he_i) + f(x - he_i) - 2f(x)) \\ +\frac{1}{h^2}\sum_{i \neq j: i,j \in supp(\Psi)} \big[\omega_{ij}(\Psi)(f(x + he_i + he_j) + f(x - he_i - he_j) - 2f(x)) \\ +\tilde{\omega}_{ij}(\Psi)(f(x + he_i - he_j) + f(x - he_i + he_j) - 2f(x))\big]\end{array} \tag{2.8}$$

with some constants $\omega_i, \omega_{ij}, \tilde{\omega}_{ij}$ (where e_j are the vectors of the standard basis in $\mathbf{R}^d$) that approximate $tr\,(G(\Psi)\frac{\partial^2}{\partial x^2})$ in the sense that

$$\|\, tr\,(G(\Psi)\frac{\partial^2}{\partial x^2}) - \Delta_h(\Psi, G))f\| = O(h)\|f'''\|. \tag{2.9}$$

for $f \in C^3(\mathbf{R}^d)$. If $f \in C^4(\mathbf{R}^d)$, then the l.h.s. of (2.9) can be better estimated by $O(h^2)\|f^{(4)}\|$.

Remark. Such $\Delta_h(\Psi, G)$ is surely not unique, but its existence is clear, because (2.8) is just a standard finite difference approximation of the second order operator $tr\,(G(\Psi)\frac{\partial^2}{\partial x^2})$. Moreover, other finite difference approximations to $tr\,(G(\Psi)\frac{\partial^2}{\partial x^2})$ could be used.

Putting $B_h = \{x \in \mathbf{R}_+^d : 0 \leq x_j < h \,\forall j\}$ and choosing an arbitrary $\omega \in (0, 1)$ we can now define an operator of type (1.4) as

$$\Lambda_K^h = \sum_{\Psi \leq K} C_\Psi^h(x)\pi_\Psi^h$$

with

$$
\begin{aligned}
(\pi^h_\Psi f)(x) &= (\Delta_h(\Psi, G)f)(x) \\
&+ \frac{1}{h}\sum_j (\beta_j(\Psi)(f(x+he_j)-f(x)) + \gamma_j(\Psi)(f(x-he_j)-f(x))) \\
&+ \sum_{M: M_j \geq h^{-\omega} \forall j} (f(x+Mh)-f(x)+\sum_j M_j(f(x-he_j)-f(x)))v(M,h) \\
&+ \sum_{M: M_j \geq h^{-\omega} \forall j} (f(x+Mh)-f(x))\mu_\Psi(B_h+Mh),
\end{aligned}
\tag{2.10}
$$

where

$$v(M,h) = \frac{1}{h^2M^2}\tilde{\nu}(B_h+Mh), \quad \tilde{\nu}(dy) = y^2\nu(dy)$$

Proposition 2.2. *Operator (2.10) is L-subcritical, if and only if*

$$(\beta(\Psi) - \gamma(\Psi) + \sum_M M\mu_\Psi(B_h+Mh), L) \leq 0.$$

In particular, if Λ_K is L-subcritical or critical, then the same holds for its approximation Λ^h_K.

Proof. It follows from a simple observation that operator (2.8) and the operator given by the sum in (2.10) that depends on the measure ν are always L-critical for any L, i.e. they are e_j-critical for all j.

Let $Z^{x,h}_t$ denote the minimal (càdlàg) Markov chain in $x + h\mathbf{Z}^d_+ \subset \mathbf{R}^d$ generated by Λ^h_K.

We shall denote by $D_{\bar{\mathbf{R}}^d_+}[0,\infty)$ (respectively $D_{\mathbf{R}^d_+}[0,\infty)$) the space of càdlàg sample paths $[0,\infty) \mapsto \bar{\mathbf{R}}^d_+$ (respectively $[0,\infty) \mapsto D_{\mathbf{R}^d_+}[0,\infty)$) equipped with the canonical filtration $\mathcal{F}_t = \sigma(X_s : s \leq t)$, and by $X_t(\omega) = \omega(t)$, $\omega \in D_{\bar{\mathbf{R}}^d_+}[0,\infty)$, the corresponding canonical projections. We shall say that a probability measure P_x on $D_{\bar{\mathbf{R}}^d_+}[0,\infty)$ (respectively $D_{\mathbf{R}^d_+}[0,\infty)$) is a solution to the *martingale problem* with sample paths in $D_{\bar{\mathbf{R}}^d_+}[0,\infty)$ (respectively in $D_{\mathbf{R}^d_+}[0,\infty)$) and with the initial position $x \in \mathbf{R}^d_+$, if $X_0 = x$ P_x almost surely and for any function $\phi \in C^\infty(\mathbf{R}^d) \cap C_c(\mathbf{R}^d)$ the process

$$\phi(X_t) - \phi(x) - \int_0^t L\phi(X_s)\,ds \tag{2.11}$$

is a $\mathcal{F}_t$-martingale with respect to P_x. We say that the martingale problem is well-posed if for any $x \in \mathbf{R}^d_+$, a solution exists and is unique. Our first result is the following.

Theorem 1. *Suppose Λ_K of form (2.4), (2.5) is L-subcritical with some L having full support.*

(i) There exists a solution to the martingale problem for Λ_K^h from (2.7) with sample paths $D_{\bar{\mathbf{R}}_+^d}[0,\infty)$ for any $x \in \mathbf{R}_+^d$.

(ii) The family of processes $Z_t^{hN,h}$, $h \in (0,1]$, $N = x/h$, with any given $x \in \mathbf{R}_+^d$ is tight and it contains a subsequence that converges (in the sense of distribution) as $h \to 0$ to a solution of the martingale problem for Λ_K.

Part (i) is a consequence of (ii), and part (ii) is proved in Section 3.

Surely this result is not quite satisfactory, because it does not include the uniqueness of the limiting point for $Z_t^{Nh,h}$. And without uniqueness one even can not be sure that the solution to the martingale problem defines a Markov process. The uniqueness of a solution to a martingale problem is known to be usually much harder to get than the existence. In case without interaction ($|K| = 1$ in our setting), i.e. for superprocesses, the uniqueness is usually obtained via duality, which seems to be not available in the general case. We shall get the uniqueness under some additional assumptions using results from [17].

First we shall need some assumptions on the measures μ and ν. Let

$$p_0(\xi) = \sum_{\Psi \le K} p_\Psi(\xi).$$

We shall suppose that there exists $c > 0$ and constants $\alpha_\Psi > 0$, $\beta_\Psi < \alpha_\Psi$ such that for each Ψ

(A1) $|Im\, p_\Psi^\mu(\xi) + Im\, p_\Psi^\nu(\xi)| \le c|p_0(\xi)|$,

(A2) $Re\, p_\Psi^\nu(\xi) \ge c^{-1}|pr_{\nu_\Psi}(\xi)|^{\alpha_\Psi}$ and $|(p_\Psi^\nu)'(\xi)| \le c|pr_{\nu_\Psi}(\xi)|^{\beta_\Psi}$, where pr_{ν_Ψ} is the orthogonal projection on the minimal subspace containing the support of the measure ν_Ψ.

Remarks. These conditions are not very restrictive. It allows, in particular, any stable Lévy measures of any degree of degeneracy. Moreover, if $\int |\xi|^{1+\beta_\Psi}\nu_\Psi(d\xi) < \infty$, then the second condition in (A2) holds, because $|e^{ixy} - 1| \le c|xy|^\beta$ for any $\beta \le 1$ and some $c > 0$. In particular, the second inequality in (A2) always holds with $\beta_\Psi = 1$. Hence, in order that (A2) holds it is enough to have the first inequality in (A2) with $\alpha_\Psi > 1$.

Let us say that a type j of particles is *immortal*, if for any solution of the martingale problem for Λ_K, the j-th co-ordinate of the process X_t^x will be positive for all times almost surely whenever the j-th co-ordinate of x was positive. In other words this means that the boundary $\bar{U}_j = \{x \in \bar{\mathbf{R}}^d : x_j = 0\}$ is inaccessible. Various criteria for immortality can be found in Appendix 3 of [17], for instance, as a simple sufficient condition one can assume that $\psi_j \ge 2$ whenever either $G_{jj}(\Psi) \ne 0$ or $\int (x_j)^2 \nu_\Psi(dx) \ne 0$.

Now we can formulate our first result on uniqueness.

Theorem 2. *(i) Let the conditions of Theorem 1 together with (A1), (A2) be satisfied. If, in addition, all types of particles are immortal, then the martingale problem of Λ_K is well-posed and has sample paths in $D_{\mathbf{R}^d_+}[0,\infty)$; i.e. the boundary is almost surely inaccessible. Hence this solution defines a strong Markov process in $\mathbf{R}^d_+$, which is a limit (in the sense of distributions) of the Markov chains $Z_t^{Nh,h}$, as $h \to 0$ with Nh tending to a constant.*

(ii) If, in addition to the hypotheses in (i), $\psi_j \geq 2$ whenever either $G_{jj}(\Psi) \neq 0$ or $\int (x_j)^2 \nu_\Psi(dx) \neq 0$, and $\psi_j \geq 1$ whenever either $\beta_j(\Psi) \neq 0$ or $\int x_j \mu_\Psi(dx) \neq 0$, the semigroup of the corresponding Markov process preserves the space of bounded continuous functions on $\bar{\mathbf{R}}^d_+$ vanishing on the boundary. If, moreover, $|K| \leq 2$ (i.e. only binary interactions are allowed) and for $|K| = 2$ the drift term and the integral term depending on μ_Ψ vanish, the corresponding semigroup is Feller, i.e. it preserves the space of continuous functions on $\mathbf{R}^d$ that tend to zero when the argument approaches either the boundary or infinity.

Proof. This is a consequence of a more general Theorem 9 in [17].

Our second result on uniqueness will be more general. Let us say that a type j of particles is *not revivable* if $\beta_j(\Psi) = 0$ whenever j is not contained in the support of Ψ, and *supp* μ_Ψ belongs to the subspace spanned by the vectors e_j with $j \in$ *supp* Ψ. In more general terminology from [17] this means that the boundary hyperspace $\bar{U}_j = \{x \in \bar{\mathbf{R}}^d : x_j = 0\}$ is gluing. The meaning of the term revivable is revealed in the following result.

Theorem 3. *Under the conditions of Theorem 1 and conditions (A1), (A2), suppose that all types of particles are either immortal or are not revivable. Then for any $x \in \mathbf{R}^d_+$ there exists a unique solution to the martingale problem for Λ_K under the additional assumption that, for any j, if at some (random) time τ the j-th coordinate of X_t vanishes, then it remains zero for all future times almost surely (i.e. once dead, the particles of type j are never revived). Moreover, the family of Markov chains $Z_t^{Nh,h}$ converges in distribution to this solution to the martingale problem.*

Proof. The uniqueness follow from more general Theorem 10 from [17]. Since the family of processes $Z_t^{Nh,h}$ converges in distributional sense to the martingale solution X_t, Theorem 1 applies to the effect that X_t inherits the non-revivability property, because each process $Z_t^{Nh,h}$ is non-revivable.

Remark. Various criteria for the semigroup of the process from Theorem 3 to be Feller can be found in [17].

3. Proof of Theorem 1.

Step 1. *The family of processes $Z_t^{Nh,h}$, $h \in (0,1]$, $Nh = x$ is tight.*

Proof. First we observe that the compact containment condition holds, i.e. for every $\epsilon > 0$ and every $T > 0$ there exists a compact set $\Gamma_{\epsilon,T} \subset \mathbf{R}^d_+$ such that

$$\inf_h P\{Z_t^{Nh,h} \in \Gamma_{\epsilon,T} \,\forall t \in [0,T]\} \geq 1 - \epsilon$$

uniformly for all starting points x from any compact subset of $\mathbf{R}^d_+$. In fact, the compact containment condition for $(L, Z_t^{Nh,h})$ follows directly from maximal inequalities for positive supermartingales and Proposition 2.1. It implies the compact containment condition for X_t^x, because L is assumed to have full support. The tightness can now be deduced by standard methods. First, as the Dynkin formula for $f(Z_t^{Nh,h})$ with any $f \in S(\mathbf{R}^d)$ gives explicit expressions for predictable projection and the quadratic variation of supermartingale $f(Z_t^{Nh,h})$, the tightness follows from Aldous-Rebolledo criterion in precisely the same manner as in e.g. [10] for the case of superprocesses. Alternatively, even simpler, one deduces the tightness directly from Remark 5.2 in Chapter 4 of [11] and Step 2 below.

Step 2. *The operators Λ_K^h approximate Λ_K on the space $C^3(\mathbf{R}^d_+) \cap C_c(\mathbf{R}^d)$, i.e. for an arbitrary function f in this space*

$$\|(\Lambda_K^h - \Lambda_K)f\| = o(1) \sup_x (1 + |x|^{|K|}) \max_{|y| \geq |x| - h} (|f'(y)| + |f''(y)| + |f'''(y)|), \quad (3.1)$$

with $o(1)$ as $h \to 0$ not depending on f (but only on the family of measures μ_Ψ, ν_Ψ, see (3.2), (3.4) below for a precise dependence of $o(1)$ on h).

Proof. Estimate (2.9) shows that the diffusion part of Λ_K is approximated by finite sums of the form

$$\sum C_\Psi^h(x) \Delta_h(\Psi, G)$$

in the required sense. It is obvious that the drift part of λ_K is approximated by the sum in (2.10) depending on β and γ. Let us prove that the integral part of $-p_\Psi(-i\nabla)$ depending on ν_Ψ is approximated by the corresponding sum from (2.10) (similar fact for the integral part depending on μ is simpler and is omitted).

Since

$$(f(x - he_j) - f(x)) = -hf'(x) + \frac{1}{2} h^2 f''(x - \theta e_j), \quad \theta \in [0, h],$$

and

$$\sum_{M: M_j \geq h^{-\omega}} \sum_{j=1}^d M_j h^2 v(M, h) \leq 2 \sum_{M: M_j \geq h^{-\omega}} \sum_{j=1}^d M_j h^2 \nu(Bh + Mh)$$

$$\leq 2h \sum_{j=1}^d \int_{y: y_j \geq h^{1-\omega} \,\forall j} y_j \nu(dy) \leq 2h^\omega \int |y|^2 \nu(dy),$$

the sum in (2.10) depending on ν can be written in the form

$$\sum_{M:M_j\geq h^{-\omega}} (f(x+Mh)-f(x)-h(f'(x),M))v(M,h)$$

$$+O(h^\omega)\sup_{|y|\geq|x|-h}|f''(y)|\int|y|^2\nu(dy)$$

and hence the difference between this sum and the corresponding integral from (2.5) has the form

$$\begin{aligned}&\sum_{M:M_j\geq h^{-\omega}} (f(x+Mh)-f(x)-h(f'(x),M))v(M,h)\\ &-\int_{y:y_j\geq h^{-\omega}\,\forall\, j}(f(x+y)-f(x)-f'(x)y)\nu(dy))\\ &+\sup_{|y|\geq|x|-h}|f''(y)|\left(O(1)\int_0^{h^{1-\omega}}|y|^2\nu(dy)+O(h^\omega)\int|y|^2\nu(dy)\right).\end{aligned} \tag{3.2}$$

To estimate the difference between the sum and the integral here, we shall use the following simple general estimate

$$|\sum_M g(Mh)\tilde\nu(Mh+B_h)-\int g(x)\tilde\nu(dx)|\leq h\|g'(x)\|\int\tilde\nu(dx), \tag{3.3}$$

which is valid for any continuously differentiable function g in the cube $\bar{B}_1$. Estimating the difference between the sum and the integral in (3.2) by means of (3.3) with $g(y)=|y|^{-2}(f(x+y)-f(x)-f'(x)y)$ (that clearly satisfies the estimate $\|g'\|\leq\sup_{|y|\geq|x|}|f'''(y)|$) yields for this difference the estimate

$$\sup_{|y|\geq|x|}|f'''(y)|O(h)\int|y|^2\nu(dy). \tag{3.4}$$

Clearly (3.1) follows from (3.2), (3.4) and the observation that $C_\Psi^h(x)=O(1+|x|^{|K|})$ for $\Psi\leq K$.

Step 3. *End of the proof.* As the coefficients of Λ_K grow at most polynomially as $x\to\infty$, similarly to (3.1) one shows that the operators Λ_K^h, $h>0$, approximate Λ_K on the Schwarz space $S(\mathbf{R}^d)$, i.e. for an arbitrary $f\in S(\mathbf{R}^d)$ the estimate $\|(\Lambda_K^h-\Lambda_K)f\|=o(1)$ as $h\to 0$ holds uniformly for all f from the ball $\sup_x(1+|x|)^{|K|+4}|f'''(x)|<R$ with any R. Since $S(\mathbf{R}^d)$ is an algebra that separates points and vanishes nowhere, one uses Remark 5.2 from Chapter 4 of [11] to complete the

proof of tightness from Step 1 and Lemma 5.1 from Chapter 4 of [11] to conclude that the distribution of the limit of a converging subsequence of the family $Z_t^{Nh,h}$ solves the martingale problem for Λ_K.

4. Examples.

We discuss here shortly some examples of k-nary interactions from statistical mechanics and population biology giving some preference to the models, where Theorems 2 or 3 are applicable. For general background on interacting particles we refer to monographs [7] or [21].

1. Branching processes and finite-dimensional superprocesses. Branching without interaction in our model corresponds clearly to the cases with $K = 1$ and hence represents the simplest possible example. In this case the limiting processes in $\mathbf{R}^d$ have pseudo-differential generators with symbols $p(x, \xi)$ depending linearly on the position x. The corresponding processes are called (finite-dimensional) superprocesses and are well studied, see e.g. [9, 10].

2. Coagulation-fragmentation and general mass preserving interactions. These are natural models for the applications of our results in statistical mechanics. For these models, the function L from Theorem 1 usually represents the mass of a particle. We do not discuss this here, because the next issues of this series (see [19, 20]) deal with these models in detail; the discussion includes also infinite-dimensional measure-valued limits. Notice only that in the present finite-dimensional situation we always get an inaccessible boundary so that Theorem 2 applies.

3. Local interactions (birth and death processes). Generalizing the notion of local branching widely used in the theory of superprocesses (see e.g. [9, 10]), let us say that the interaction of particles of d types is *local*, if a group of particles specified by a profile Ψ can produce particles only of type $j \in supp\, \Psi$. Processes subject to this restriction include a variety of the so called birth and death processes from the theory of multidimensional population processes (see, e.g. [2] and references therein) such as competition processes, predator-prey processes, general stochastic epidemics and their natural generalizations (seemingly not much studied yet) that take into account the possibility of birth from groups of not only two (male, female) but also of more large number of species (say, for animals, living in groups containing a male and several females, as by gorillas). Excluded by the assumption of locality are clearly migration processes. In the framework of our general model, the assumption of locality gives the following additional restrictions to the generators (2.4), (2.5): $\beta_j(\Psi) = 0$ whenever j is not contained in the support of Ψ, and $supp\, \mu_\Psi$ belongs to the subspace spanned by the vectors e_j with $j \in supp\, \Psi$. This clearly implies that the whole boundary of the corresponding process in $\mathbf{R}_+^d$ is gluing and Theorem 3 is valid giving uniqueness and convergence.

4. Evolutionary games. A popular way of modelling the evolution of behavioral patterns in populations is given by the replicator dynamics (see [12, 22] for an extensive account of the theory), which is usually deduced by the following arguments. Suppose a population consists of individuals with d different types of behavior specified by their strategies in a symmetric two-player game given by the matrix A whose elements A_{ij} designate the payoffs to a player with the strategy i whenever the players apply the strategies i and j. Suppose the number of individuals playing strategy i at time t is $x_i = x_i(t)$ with the whole size of the population being $\mu(x) = \sum_{j=1}^d x_j$. If the payoff represents an individual's fitness measured as the number of offsprings per time unit, the average fitness $A_{ij}x_j/\mu(x)$ of a player with the strategy i coincides with the payoff of the pure strategy i playing against the mixed strategy $x/\mu(x) = \{x_1/\mu(x), ..., x_d/\mu(x)\}$. Assuming additionally that the background fitness and death rate of individuals (independent of outcomes in the game) are given by some constants B and C yields the following dynamics

$$\dot{x}_i = \left(B - C + \sum_{j=1}^d A_{ij} \frac{x_j}{\mu(x)} \right) x_i, \tag{4.4}$$

called the standard replicator dynamics (which is usually written in terms of the normalized vector $x/\mu(x)$). A rigorous deduction of this system of equation in $\mathbf{R}_+^d$ from the corresponding Markov chain on $\mathbf{Z}_+^d$ is given in [5]. Of course, it follows from our general Theorem 3.

Having in mind the recent increase in the interest to stochastic versions of replicator dynamics (see [8] and references therein), let us consider now a general model of this kind and analyze the possible stochastic processes that may arise as continuous state (or measure-valued) limits. Denoting by N_j the number of individuals playing the strategy j and by $N = \sum_{j=1}^d N_j$ the whole size of the population, assuming that the outcome of a game between players with strategies i and j is a probability distribution $A_{ij} = \{A_{ij}^m\}$ of the number of offsprings $m \geq -1$ of the players ($\sum_{m=-1}^\infty A_{ij}^m = 1$) and the intensity a_{ij} of the reproduction per time unit ($m = -1$ means the death of the individual) yields the Markov chain on $\mathbf{Z}_+^d$ with the generator

$$Gf(N) = \sum_{j=1}^d N_j \sum_{m=-1}^\infty \left(B_j^m + \sum_{k=1}^d a_{jk} A_{jk}^m \frac{N_k}{|N|} \right) (f(N + me_j) - f(N)) \tag{4.5}$$

(where B_j^m describe the background reproduction process), which is similar to the generator of binary interaction G_2 of form (1.1), but has an additional multiplier $1/|N|$ on the intensity of binary interaction that implies that in the corresponding scaled version of type (1.3) one has to put a simple common multiplier h instead of $h^{|\Psi|}$. Apart from this modification, the same procedure as for (1.1)-(1.3) applies leading to the limiting process on $\mathbf{R}^d_+$ with the generator of type

$$\Lambda_{EG} = \sum_{j=1}^d x_j \left(\phi_j + \sum_{k=1}^d \frac{x_k}{\mu(x)} \phi_{jk} \right), \tag{4.6}$$

where all ϕ_j and ϕ_{jk} are the generators of one-dimensional Lévy processes, more precisely

$$\phi_{jk} f(x) = g_{jk} \frac{\partial^2 f}{\partial x_j^2}(x) + \beta_{jk} \frac{\partial f}{\partial x_j}(x)$$

$$+ \int \left(f(x + ye_j) - f(x) - \mathbf{1}_{y \le 1}(y) \frac{\partial f}{\partial x_j}(x) y_j \right) \nu_{jk}(dy), \tag{4.7}$$

$$\phi_j f(x) = g_j \frac{\partial^2 f}{\partial x_j^2}(x) + \beta_j \frac{\partial f}{\partial x_j}(x) + \int \left(f(x + ye_j) - f(x) - \mathbf{1}_{y \le 1}(y) \frac{\partial f}{\partial x_j}(x) y \right) \nu_j(dy), \tag{4.8}$$

where $\mathbf{1}_M$ denotes as usual the indicator function of the set M and all ν_{jk}, ν_j are Borel measures on $(0, \infty)$ such that the function $\min(y, y^2)$ is integrable with respect to these measures, g_j and g_{jk} are non-negative. Let $\tilde{\nu}_{jk}(dy) = y^2 \nu_{jk}(dy)$, $\tilde{\nu}_j(dy) = y^2 \nu_j(dy)$ and $v_{jk} = (hl)^{-2} \tilde{\nu}_{jk}([lh, lh + 1))$, $v_j = (hl)^{-2} \tilde{\nu}_j([lh, lh + 1))$. Then the corresponding approximation to (4.6) of type (4.5) after scaling can be written in the form

$$\Lambda^h_{EG} f(Nh) = h \sum_{j=1}^d N_j \left(\phi^h_j + \sum_{k=1}^d \frac{N_k}{\mu(N)} \phi^h_{jk} \right) f(Nh), \tag{4.9}$$

where $\mu(N) = \sum_{j=1}^d N_j$ and $\phi^h_{jk} f(Nh)$ equals

$$\frac{1}{h^2} g_{jk} (f(Nh + he_j) + f(Nh - he_j) - 2f(Nh)) + \frac{1}{h} |\beta_{jk}| (f(Nh + he_j \, sgn\,(\beta_{jk})) - f(Nh))$$

$$+ \sum_{l \ge h^{-\omega}} [f(Nh + lhe_j) - f(Nh) + l(f(Nh - he_j) - f(Nh))] v_{jk}(l, h)$$

$$+ \sum_{l=1}^\infty [f(Nh + (1 + lh)e_j) - f(Nh)] \nu_{jk}([1 + lh, 1 + lh + h)), \tag{4.10}$$

and similarly ϕ^h_j are defined. The terms in (4.10) that approximate diffusion, drift and integral part of (4.7) have different scaling and have different interpretation in terms of population dynamics. Clearly the first term (approximation for diffusion) stands for a game that can be called "death or birth" game, which describes some sort of fighting for reproduction, whose outcome is that an individual either dies or produces an offspring. The second term (approximating drift) describes games for death or for life depending on the sign of β_{jk}. Other terms describe games for a large number of offsprings and are analogues of usual branching but with game-theoretic interaction. The same arguments as given for the proof of Theorem 1 and 3 yield the following result (observe only that no particles are revivable in this model, and no additional assumption of subcriticallity is required due to Theorem 1 in [17], since the coefficients grow at most linearly):

Proposition 4.1 *Suppose conditions (A1), (A2) hold for all measures ν_{jk}, ν_j. Then for any $x \in \mathbf{R}_+^d$ there exists a unique solution to the martingale problem for Λ_{EG} under the additional assumption that, for any j, if at some (random) time τ the j-th coordinate of X_t vanishes, then it remains zero for all future times almost surely. Moreover, the family of Markov chains $Z_t^{Nh,h}$ defined by Λ_{EG}^h converges in distribution to this solution to the martingale problem as $h \to 0$ and $Nh \to x$.*

If the limiting operator is chosen to be deterministic (i.e. the diffusion and non-local term vanish and only a drift term is left), we get the standard replicator dynamics (4.4).

Similarly one obtains the corresponding generalization to the case of non-binary (k-nary) evolutionary games (see e.g. [8] for biological and social science examples of such games), the corresponding limiting generator having the form $\sum_{j=1}^d x_j \Phi_j$, where Φ_j are polynomials of the frequencies $y_j = x_j / \sum_{i=1}^d x_i$ with coefficients being again generators of one-dimensional Lévy processes.

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