INVERSE SYSTEMS OF SPECTRA AND GENERALIZATIONS OF A THEOREM OF W.H. LIN

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Abstract

In this thesis we generalize a theorem of W.H. Lin. Lin's results are concerned with the homotopy and cohomotopy of an inverse system of spectra $\{P_{-k}\}$. Using the quadratic construction we construct an inverse system of spectra $\{P_{-k}(E)\}$. We generalize Lin's results by studying the homotopy and cohomotopy of $\{P_{-k}(E)\}$.

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§1. Introduction

M. Mahowald investigated the homotopy groups of stunted real projective spaces and found that they exhibited interesting periodic properties. Mahowald made a conjecture based on these calculations which was described by J.F. Adams in [2]. W.H. Lin verified Mahowald's conjecture in [15] and also proved a 'dual' version in cohomotopy. Lin's theorem was the essential step in his verification of Segal's conjecture for the case $G = \mathbb{Z}/2$.

We begin by briefly describing Lin's theorem. It is possible to construct spectra P_k, $k \in \mathbb{Z}$, and maps of spectra $P_k \to P_{k+1}$ with the following properties:

(a) If $k \geq 1$ $P_k = \mathbb{R}P^\infty/\mathbb{R}P^{k-1}$ and $P_0 = \mathbb{R}P^\infty_+$ (+ means add a disjoint basepoint).

(b) If $\mathbb{F}_2[u,u^{-1}]$ is given the structure of a module over the mod 2 Steenrod algebra, A, by

$$Sq^j u^\ell = \binom{\ell}{j} u^{\ell+j} ,$$

then $H^*(P_{-k};\mathbb{Z}/2)$ is isomorphic to the A-submodule of $\mathbb{F}[u,u^{-1}]$ generated by u^ℓ, $\ell \geq -k$.

(c) $H^*(P_{-k};\mathbb{Z})$ has no odd torsion.

(d) The map of spectra $P_k \to P_{k+1}$ induces the obvious inclusion in mod 2 cohomology.

The maps $P_{-k} \to P_{-k+1}$ give us an inverse system of spectra

$$\cdots \to P_{k-1} \to P_{-k} \to P_{-k-1} \to \cdots \ .$$

Let $\hat{\pi}_*$ and $\hat{\pi}^*$ denote the homology and cohomology theories associated to $S(\hat{\mathbb{Z}_2})$, the Moore spectrum of the 2-adic integers. Then

$$\cdots \to \pi_*P_{-k-1} \to \pi_*P_{-k} \to \pi_*P_{-k+1} \to \cdots$$

is an inverse system of abelian groups (as $-k \to -\infty$) and

$$\cdots \to \hat{\pi}^*P_{-k+1} \to \hat{\pi}^*P_{-k} \to \hat{\pi}^*P_{-k-1} \to \cdots$$

is a direct system (as $-k \to -\infty$) of abelian groups. If $\lim_k$ denotes the inverse limit and colim_k denotes the direct limit, then the following result is a direct consequence of Lin's results:

<u>Lin's Theorem</u> ([15]).

There are isomorphisms:

(a) $\lim_k \pi_*P_{-k} \cong \hat{\pi}_*S^{-1}$;

(b) $\mathrm{colim}_k \hat{\pi}^*P_{-k} \cong \hat{\pi}^*S^0$.

In this thesis we shall generalize Lin's Theorem. The key observation is that the spectra $\{P_{-k}\}$ can be constructed by using the quadratic construction.

Let X be a CW-complex with basepoint x_0. Define a free involution on $S^\infty \times (X \wedge X)$ by $T(w, x \wedge y) = (-w, y \wedge x)$. The quadratic construction on X, written $D_2(X)$, is the complex $S^\infty \times_T X \wedge X / S^\infty \times_T x_0 \wedge x_0$. Let $\mathcal{S}p$ denote the homotopy category of CW-spectra. Then May et al ([13] and [18]) have extended this construction to a functor

$$D_2 : \mathcal{S}p \to \mathcal{S}p .$$

There is a natural transformation of functors

$$SD_2(E) \to D_2(SE) \ , \quad E \in \mathcal{S}p .$$

For each $k \in \mathbb{Z}$ we define functors ([12]) $P_{-k} : \mathcal{S}p \to \mathcal{S}p$ by

$$P_{-k}(E) = S^k D_2(S^{-k}E) .$$

If we take $E = S^{-k}X$, $X \in \mathcal{S}p$, then the above natural transformation gives us a map of spectra

$$SD_2(S^{-k}X) \to D_2(S^{-k+1}X) \quad ;$$

hence, after applying S^{k-1} to the above map we obtain a map of spectra

$$P_{-k}(X) = S^k D_2(S^{-k}X) \to S^{k-1} D_2(S^{-k+1}X) = P_{-k+1}(X) .$$

Thus we obtain an inverse system of spectra

$$\cdots \to P_{-k-1}(E) \to P_{-k}(E) \to P_{-k+1}(E) \to \cdots \quad .$$

If we let $P_{-k} = P_{-k}(S^0)$, then $\{P_{-k}\}$ is an inverse system of spectra satisfying properties (a) through (d) and Lin's Theorem is true for this inverse system. We are led to ask 'under what hypothesis on E will Lin's Theorem hold with the inverse system $\{P_{-k}\}$ replaced by $\{P_{-k}(E)\}$?'

In Sp we can form the homotopy inverse limit, written holim_k . If p is a prime and E is a spectrum, we can define the p-completion of E , written $\hat{E}_p$. We say that E is a spectrum of finite type if each skeleton of E is finite.

<u>Theorem A</u>

If E is a spectrum type, then there is a natural equivalence of spectra

$$S^{-1}\hat{E}_2 \to \mathrm{holim}_k P_{-k}(E) \ .$$

The hypothesis of Theorem A is sufficient to ensure that

$$\pi_* \mathrm{holim}_k P_{-k}(E) \cong \lim_k \pi_* P_{-k}(E) \ .$$

Thus we obtain the following corollary which is a generalization of part (a) of Lin's Theorem.

Corollary

If E is a spectrum of finite type, then there is a natural isomorphism

$$\hat{\pi}_* S^{-1}E \cong \lim_k \pi_* P_{-k}(E) .$$

The generalization of part (b) of Lin's Theorem is not quite so nice.

Theorem B ([12])

If E is a finite spectrum, then there is a natural isomorphism

$$\hat{\pi}^* E \cong \operatorname{colim}_k \hat{\pi}^* P_{-k}(E) .$$

In Theorem B we cannot replace 'finite spectrum' by even 'CW-complex of finite type' since we show that

$$\hat{\pi}^* P_0 \not\cong \operatorname{colim}_k \hat{\pi}^* P_{-k}(P_0) .$$

However, we are able to construct a particularly nice isomorphism for Theorem B. Note that $P_{-t}(S^t) = S^t D_2(S^0) = S^t \mathbb{R}P^\infty_+$. There is a canonical map $\mathbb{R}P^\infty_+ \to S^0$ defined by 'map the basepoint to the basepoint and $\mathbb{R}P^\infty$ to the other point'. This defines a map

$$\rho : P_{-t}(S^t) \to S^t .$$

If σ is the composite of maps from the inverse system $\{P_{-k}(X)\}$, $\ell \geq 0$,

$$P_{-t-\ell}(X) \to P_{-t}(X) ,$$

then define

$$\gamma_{t,\ell}:\pi^{t}X \to \pi^{t}P_{-t-\ell}(X)$$

by $\gamma_{t,\ell}(\alpha) = \sigma^{*}(\rho P_{-t}(\alpha))$, $\alpha \in \pi^{-t}X$. If $\ell \geq 1$, then $\gamma_{t,\ell}$ is a homomorphism, and $\gamma_{t,\ell}$ extends to a homomorphism

$$\gamma_{t,\ell}:\hat{\pi}^{t}X \to \hat{\pi}^{t}P_{-t-\ell}(X) .$$

Choose $\ell \geq 1$ and let Γ be the composite

$$\hat{\pi}^{t}X \xrightarrow{\gamma_{t,\ell}} \hat{\pi}^{t}P_{-t-\ell}(X) \to \mathrm{colim}_{k}\hat{\pi}^{t}P_{-k}(X)$$

where the last map is the usual homomorphism from a term in a directed system into the direct limit. By construction Γ is independent of $\gamma \geq 1$. Γ is essentially the total power operation in cohomotopy.

<u>Theorem C</u> ([12])

If E is a finite spectrum, then

$$\Gamma:\hat{\pi}^{*}E \to \mathrm{colim}_{k}\hat{\pi}^{*}P_{-k}(E)$$

is an isomorphism.

Next we briefly remark on the proofs of Theorems A, B and C. To prove Theorem B we show that $\mathrm{colim}_k \hat{\pi}^* P_{-k}(E)$ is a cohomology theory (not necessarily satisfying Milnor's wedge axiom). Then we construct a natural transformation of cohomology theories that is an isomorphism, using part (b) of Lin's Theorem, when $E = S^0$. The Eilenberg-Steenrod uniqueness theorem will complete the proof. To prove Theorem C we use Theorem B to regard Γ as a cohomology operation in $\hat{\pi}^*$. We show that this operation is the identity by evaluating it on $1 \in \hat{\pi}^* S^0$ and Theorem C follows.

The proof of Theorem A is slightly more technical. Fix a prime p and let $\hat{E} = \hat{E}_p$, for $E \in Sp$. Given a spectrum of finite type X and $\{Y_k\}$ an inverse system of spectra, each Y_k of finite type, we construct a spectral sequence converging in a strong sense to the group

$$[X, \mathrm{holim}_k \hat{Y}_k]_* .$$

This spectral sequence is constructed by taking inverse limits of Adams spectral sequences. To prove Theorem A we construct a map

$$S^{-1}E \to \mathrm{holim}_k P_{-k}(E) .$$

Then we use this map to construct a morphism of spectral sequences, namely, from the Adams spectral sequence converging to $\pi_* \hat{E}$ to the above spectral sequence converging to $\pi_* \mathrm{holim}_k P_{-k}(E)$. Using a theorem of Adams, Gunawardena and Miller ([4]) we show that this morphism is an isomorphism of spectral sequences. Convergence arguments complete the proof.

Another interesting inverse system of spectra to consider is

$$\cdots \to P_{-k-1}\wedge E \to P_{-k}\wedge E \to P_{-k+1}\wedge E \to \cdots$$

obtained by smashing the maps in the inverse system $\{P_{-k}\}$ with the identity map on E. If bo and $H(\mathbb{Z})$ denote the representing spectra for connective KO-theory and integral cohomology, then Davis and Mahowald prove:

Theorem D ([11])

There is an equivalence of spectra

$$\Big(\bigvee_{j\in\mathbb{Z}} S^{4j-1}H(\mathbb{Z})\Big)_2^\wedge \simeq \operatorname{holim}_k P_{-k}\wedge bo .$$

We use the spectral sequence mentioned above to prove Theorem D.

For the sake of clarity and completeness, the first three sections lay down the technical groundwork for the remaining sections and are distilled from the literature. Section two contains an account of inverse limits and completions, and is taken mainly from [5] and [6]. Section three is taken from [7] and discusses spectral sequences and convergences. Section four contains an account of the Adams spectral sequence and its convergence properties taken from [7].

In section five we construct the spectral sequence we need, paying full attention to its convergence properties. In section six we prove Theorem A. In section seven we prove Theorems B and C and provide the counter-example mentioned above. In section eight we give our proof of Theorem D. Lastly we provide a proof in the appendix of the Theorem of Adams, Gunawardena and Miller. This proof comes directly from notes taken in seminars given by Adams and Miller.

§2. Inverse limits and completions

This section has been distilled from [5] and [6].

An inverse system of abelian groups is a collection of abelian groups $\{A_n\}$, indexed by the natural numbers, and homomorphisms

$$\sigma_{n+1} : A_{n+1} \to A_n \quad .$$

Given two inverse systems $\{A_n\}$ and $\{B_n\}$ with homomorphisms $\{\sigma_n\}$ and $\{\theta_n\}$, a map of inverse systems is a collection of homomorphisms $\{f_n\}$

$$f_n : A_n \to B_n$$

that satisfy $\theta_n f_n = f_{n-1}\sigma_n$.

Given an inverse system $\{A_n\}$, define a homomorphism

$$d : \prod_n A_n \to \prod_n A_n$$

by $d(a_n) = a_n - \sigma_{n+1} a_{n+1}$. Then we define the inverse limit of the inverse system $\{A_n\}$, written $\lim_n A_n$, by $\lim_n A_n = \ker d$. Clearly $\lim_n A_n$ is the subgroup of $\prod_n A_n$ consisting of sequences (a_n) with $a_n = \sigma_{n+1} a_{n+1}$. We also define $R\lim_n A_n = \operatorname{coker} d$. Then we have an exact sequence

$$0 \to \lim_n A_n \to \prod_n A_n \xrightarrow{d} \prod_n A_n \to R\lim_n A_n \to 0 \ .$$

We now record standard results about $\lim_n$, paying particular attention to its exactness properties.

Let $\{A_{s,t}\}$ be a collection of abelian groups with homomorphisms $\{\sigma_t\}$, $\{\theta_s\}$ making the following diagram commute:

$$\begin{array}{ccc} A_{s+1,t+1} & \xrightarrow{\sigma_{t+1}} & A_{s+1,t} \\ {\scriptstyle\theta_{s+1}}\downarrow & & \downarrow{\scriptstyle\theta_{s+1}} \\ A_{s,t+1} & \xrightarrow[\sigma_{t+1}]{} & A_{s,t} \end{array} .$$

<u>Lemma 2.1</u> ([5])

Let $\{A_{s,t}\}$ be as above. Then

$$\lim_s \lim_t A_{s,t} = \lim_t \lim_s A_{s,t} .$$

<u>Lemma 2.2</u> ([6])

If

$$0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$$

is a short exact sequence of inverse systems, then

$$0 \to \lim_n A_n \to \lim_n B_n \to \lim_n C_n \to R\lim_n A_n \to R\lim_n B_n \to R\lim_n C_n \to 0$$

is exact.

Let

$$\cdots \to \{C_{s+1}(k)\} \xrightarrow{\{d_{s+1}(k)\}} \{C_s(k)\} \to \cdots \to \{C_0(k)\}$$

be an inverse system (indexed by k) of chain complexes. We make the following definitions

$$Z_s(k) = \operatorname{Ker} d_s(k) \qquad B_s(k) = \operatorname{Im} d_{s+1}(k) \qquad H_s(k) = Z_s(k)/B_s(k)$$

$$C_s = \lim_k C_s(k) \qquad d_s = \lim_k d_s(k)$$

$$Z_s = \operatorname{Ker} d_s \qquad B_s = \operatorname{Im} d_{s+1} \qquad H_s = Z_s/B_s \quad .$$

<u>Lemma 2.3</u> ([5])

If $R1\lim_k H_s(k) = 0$ and $R1\lim_k B_s(k) = 0$, then $\lim_k H_s(k) = H_s$.

The next lemma gives sufficient conditions for the vanishing of Rlim.

<u>Lemma 2.4</u> ([5])

Let $\{A_n\}$ be an inverse system with homomorphisms $\{\sigma_n\}$. Then either of the following conditions implies that $R1\lim_n A_n = 0$:

(a) For each n there exists m with

$$\operatorname{Im}(\sigma_p \circ \sigma_{p-1} \circ \cdots \circ \sigma_n) = \operatorname{Im}(\sigma_m \circ \cdots \circ \sigma_n)$$

whenever $p \geq m$.

(b) Each group A_n is compact and Hausdorff under a topology that makes each σ_n continuous.

Condition (a) above is known as the Mittag-Leffler condition, but condition (b) will be more useful in this thesis. We shall use the p-adic topology to construct topologies that satisfy (b).

Given an abelian group A and a filtration of A by subgroups

$$\dots \subset F^{s-1} \subset F^{s} \subset F^{s+1} \subset \dots \bigcup_{s} F^{s} = A \ ,$$

let neighbourhoods about $a \in A$ be the cosets $a+A$. The topology that this defines is called the filtration topology. This topology is Hausdorff if $\bigcap_s F^s = 0$. The completion of A with respect to the filtration topology, written $\hat{A}$, is given by

$$\hat{A} = \lim_s A/F^s \ .$$

For a prime p , when we set $F^s = p^s A$ the filtration topology is called the p-adic topology; we write $\hat{A}_p$ for the p-adic completion of A . Note that any homomorphism $f:A \to B$ is continuous with respect to the p-adic topologies on A and B since

$$p^n A \subseteq f^{-1} p^n B \ .$$

Let $\hat{\mathbb{Z}}_p$ denote the p-adic completion of $\mathbb{Z}$, called the p-adic integers.

Lemma 2.5 ([6])

If A is a finitely generated abelian group, then

$$\hat{A}_p \cong A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p .$$

Give each A/p^sA the p-adic topology and $\prod_s A/p^sA$ the usual product topology. Then the subspace topology on $\hat{A}_p$,

$$\hat{A}_p = \lim_s A/p^sA \subseteq \prod_s A/p^sA ,$$

that this defines is the same as the p-adic topology on $\hat{A}_p$.

Lemma 2.6

If A is finitely generated, then the p-adic topology on $\hat{A}_p$ is compact and Hausdorff.

Pf Let $A(p)$ denote the p-torsion subgroup of A . Then $\hat{A}_p \cong \lim_s A(p)/p^sA(p)$. Each of the groups $A(p)/p^sA(p)$ has a compact, Hausdorff p-adic topology, hence $\prod_s A(p)/p^sA(p)$ is compact and Hausdorff. $\lim_s A(p)/p^sA(p)$ is a closed subspace of $\prod_s A(p)/p^sA(p)$, so $\hat{A}_p$ is compact and Hausdorff.

§3. Spectral sequences and convergence

In this section we discuss spectral sequences and convergence following [7] and [17]. A collection of chain complexes $\{E_r^*, d_r\}$ $1 \le r < \infty$ with differentials

$$d_r : E_r^s \to E_r^{s+r}$$

is said to be a spectral sequence, written (E_r, d_r), if for each r there are isomorphisms

$$H(E_r) = E_{r+1} \quad .$$

Construct subgroups $Z_r^s \subseteq E_1^s$ inductively as follows: Let $Z_1^s = E_1^s$. At stage 'r' we have constructed subgroups

$$Z_r^s \subset Z_{r-1}^s \subset \ldots \subset Z_1^s$$

and quotient maps $q_r : Z_r^s \to E_r^s$. Define $Z_{r+1}^s \subset Z_r^s$ to be the subgroup of elements $a \in Z_r^s$ with $d_r(q_r a) = 0$. Let $B_1^s = 0$. Then define B_r^s to be the subgroup of Z_r^s whose image under $q_r : Z_r^s \to E_r^s$ corresponds to the image of $d_r : E_r^{s-r} \to E_r^s$. This gives a sequence of subgroups

$$0 = B_1^s \subset B_2^s \subset \ldots \subset B_r^s \subset \ldots \subset Z_r^s \subset \ldots \subset Z_1^s = E_1^s \quad .$$

By construction

$$E_r^s \cong Z_r^s/B_r^s$$

$$d_r:Z_r^s/B_r^s \rightarrow Z_1^{s+r}/B_1^{s+r}$$

$$\text{Ker}d_r = Z_{r+1}^s/B_r^s \qquad \text{Im}d_r = B_{r+1}^{s+r}/B_r^{s+r} \quad .$$

We also define groups

$$Z_\infty^s = \bigcap_r Z_r^s \subseteq E_1^s$$

$$B_\infty^s = \bigcup_r B_r^s \subseteq E_1^s$$

$$E_\infty^s = Z_\infty^s/B_\infty^s \quad .$$

All of the above groups are meant to be graded groups, graded by codegree (the codegree of a graded group is minus the degree) since we are using 'cohomology notation'. The differential, d_r , is required to have codegree + 1 , that is d_r raises codegree by one. We shall rarely need to specify the codegree so generally we will supress the grading.

A morphism of spectral sequences

$$f:(E_r,d_r) \rightarrow (\bar{E}_r,\bar{d}_r)$$

is a collection of homomorphisms

$$f_r:E_r^s \rightarrow \bar{E}_r^s \quad .$$

each of which fits into the commutative diagram

$$\begin{array}{ccc} E_r^s & \xrightarrow{f_r} & \bar{E}_r^s \\ d_r \downarrow & & \downarrow \bar{d}_r \\ E_r^{s+r} & \xrightarrow[f_r]{} & \bar{E}_r^{s+r} \end{array} \quad .$$

We record the standard comparison theorem for spectral sequences (e.g. see [7]).

Theorem 3.1

Let $f:(E_r,d_r) \to (\bar{E}_r,\bar{d}_r)$ be a morphism of spectral sequences with

$$f_k:E_k^* \to \bar{E}_k^*$$

an isomorphism for some k . Then

$$f_r:E_r^* \to \bar{E}_r^* \qquad k \le r \le \infty$$

is an isomorphism.

Let (E_r,d_r) be a spectral sequence and let G be a group filtered by subgroups F^s , $s \in \mathbb{Z}$, with $\bigcup_s F^s = G$. We say that (E_r,d_r) :

(a) converges to G weakly if we are given isomorphisms

$$E_\infty^s \cong F^s/F^{s+1} \quad ;$$

(b) <u>converges to G</u> if (a) holds and the filtration topology on G is Hausdorff;

(c) <u>converges to G</u> strongly if the spectral sequence converges to G and

$$G \cong \lim_s G/F^s ,$$

i.e. the filtration topology on G is complete.

If (E_r, d_r) and $(\bar{E}_r, \bar{d}_r)$ converge weakly to G and $\bar{G}$, then we say that a morphism $f:(E_r, d_r) \to (\bar{E}_r, \bar{d}_r)$ is compatible with a homomorphism $g:G \to \bar{G}$ if, using the isomorphisms in (a) above,

$$\begin{array}{ccc} E_\infty^s & \xrightarrow{f_\infty} & \bar{E}_\infty^s \\ \wr\wr & & \wr\wr \\ F^s/F^{s+1} & \xrightarrow[\bar{g}]{} & \bar{F}^s/\bar{F}^{s+1} \end{array}$$

commutes.

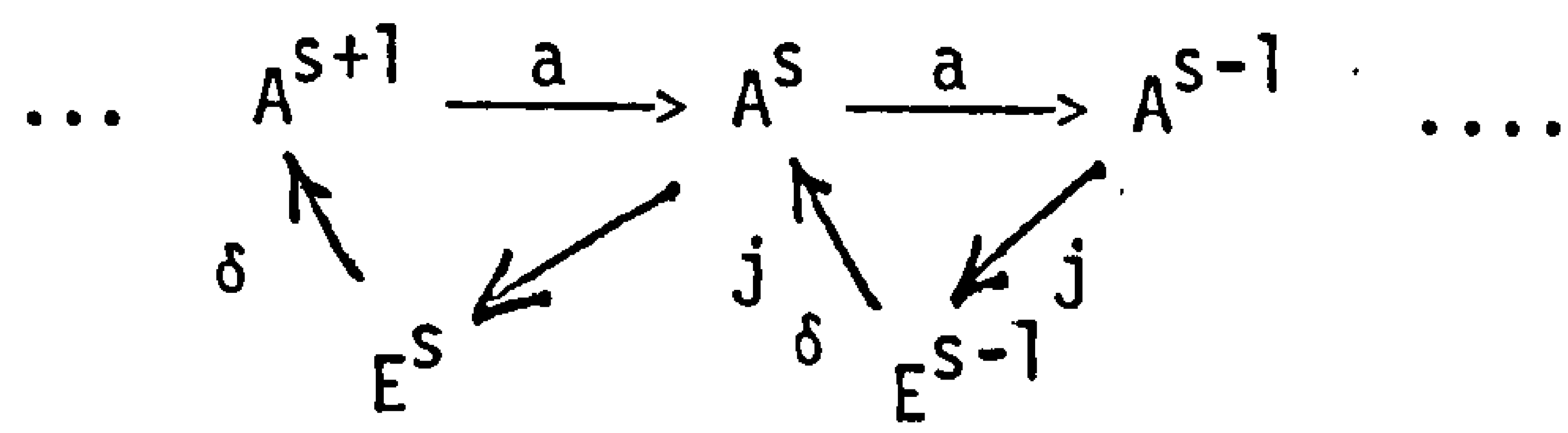
<u>Theorem 3.2</u> ([7])

Suppose that (E_r, d_r) and $(\bar{E}_r, \bar{d}_r)$ converge strongly to G and $\bar{G}$. Let $f:(E_r, d_r) \to (\bar{E}_r, \bar{d}_r)$ be a morphism compatible with $g:G \to \bar{G}$. If

$$f_k : E_k^* \to \bar{E}_k^*$$

is an isomorphism for some $k \leq \infty$, then $g:G \to \bar{G}$ is an isomorphism of filtered groups, i.e. $F^sG \cong \bar{F}^s\bar{G}$.

Nearly all the spectral sequences that we shall use arise from the method of unravelled exact couples. We say that the diagram of graded (by codegree) abelian groups



is an unravelled exact couple if each of the sequences

$$\cdots \quad A^s \xrightarrow{j} E^s \xrightarrow{\delta} A^{s+1} \xrightarrow{a} A^s \quad \cdots$$

is long exact. The homomorphism δ is required to have a codegree + 1 while a and j are to have codegree zero.

From the above unravelled exact couple we obtain a spectral sequence (E_r, d_r) as follows: define subgroups

$$Z_r^s = \delta^{-1}\mathrm{Im}(a^{r-1}: A^{s+r} \to A^{s+1}) \subseteq E^s$$

$$B_r^s = j\,\mathrm{Ker}(a^{r-1}: A^s \to A^{s-r+1}) \subseteq E^s$$

with

$$B_1^s \subset \ldots \subset B_r^s \subset \ldots \subset Z_r^s \subset \ldots \subset Z_1^s \ ,$$

and define $E_r^s = Z_r^s/B_r^s \quad r \geq 2$, $E_1^s = E^s$. Define the differential $d_r : E_r^s \to E_r^{s+r}$ by lifting

$$\delta E_r \subset A^{s+1}$$

to A^{s+r} and then applying j . This is well defined and it is easy to verify that $H(E_r) = E_{r+1}$. Then, as before

$$Z_\infty^s = \bigcap_r Z_r^s \subseteq E_1^s$$

$$B_\infty^s = \bigcup_r B_r^s \subseteq E_1^s$$

$$E_\infty^s = Z_\infty^s/B_\infty^s \ .$$

A morphism of unravelled exact couples $F:\{A^s, E^s, a, j, \delta\} \to \{\bar{A}^s, \bar{E}^s, \bar{a}, \bar{j}, \bar{\delta}\}$ is a collection of homomorphisms $f^s : A^s \to \bar{A}^s$, $g^s : E^s \to E^{s-1}$ that fit into the commutative diagram

$$\begin{array}{ccccccc}
A^{s+1} & \xrightarrow{a} & A^s & \xrightarrow{j} & E^s & \xrightarrow{\delta} & A^{s+1} \\
f^{s+1}\downarrow & & f^s\downarrow & & g^s\downarrow & & \downarrow f^{s+1} \\
\bar{A}^{s+1} & \xrightarrow{\bar{a}} & \bar{A}^s & \xrightarrow{\bar{j}} & \bar{E}^s & \xrightarrow{\bar{\delta}} & \bar{A}^{s+1}
\end{array} \ .$$

Note that such a morphism induces a morphism of the corresponding spectral sequences.

Suppose that $E^s = 0 \quad s < 0$, then we say that the resulting spectral sequence is a right-half-plane spectral sequence. Given such a spectral sequences we take as its target group (for convergence) $A^0 = G$ and we filter G by the subgroups

$$F^sG = \mathrm{Im}(a^s:A^s \to A^0) .$$

Suppose that $f:\{A^s,E^s\} \to \{\bar{A}^s,\bar{E}^s\}$ is a morphism of unravelled exact couples and that $f:(E_r,d_r) \to (\bar{E}_r,\bar{d}_r)$ is the corresponding morphism of spectral sequences. Then, if (E_r,d_r) and $(\bar{E}_r,\bar{d}_r)$ converge weakly to A^0 and $\bar{A}^0$, $f:(E_r,d_r) \to (\bar{E}_r,\bar{d}_r)$ is compatible with $f:A^0 \to \bar{A}^0$.

The convergence properties of spectral sequences arising from unravelled exact couples takes on a particularly nice form. Following Boardman ([7]), we say that a right-half-plane spectral sequence <u>converges conditionally to $G = A^0$</u> if

$$\lim_s A^s = 0 \qquad R\lim_s A^s = 0 .$$

Note that without extra assumptions a conditionally convergent spectral sequence need not converge weakly to G .

Let $RE^s_\infty = R\lim_r Z^s_r$.

Theorem 3.3 ([7])

Let (E_r, d_r) be a right-half-plane spectral sequence, arising from an exact couple, that converges conditionally to $G = A^0$. Then

(a) the filtration topology on G is complete;

(b) (E_r, d_r) converges strongly to G if and only if $RE_\infty^* = 0$.

A left-half-plane spectral sequence is a spectral sequence arising from an unravelled exact couple $\{A^s, E^s\}$ that satisfies $E^s = 0$ for $s > 0$. Let $\operatorname{colim}_s A^s = 0$; then we filter the target group $H = A^1$ by

$$F^s H = \operatorname{Ker}(A^1 \xrightarrow{a^{s-1}} A^s) \quad .$$

Theorem 3.4 ([7])

If (E_r, d_r) is a left-half-plane spectral sequence as above, then (E_r, d_r) converges strongly to H.

§4. Convergence of the Adams spectral sequence

We begin this section by setting down some basic properties of $\mathcal{S}p$. (See [3], [7] and [8] for general references.) For X and $Y \in \mathcal{S}p$, let $[X,Y]$ denote homotopy classes of maps $X \to Y$, and let $[X,Y]_t = [S^t X, Y]$. If $\alpha \in [X,Y]_t$, then we say that α has degree t or codegree $-t$.

An inverse system of spectra is a collection of spectra $\{Y_k\}$, indexed by the natural numbers, with maps of spectra

$$\sigma_{k+1} : Y_{k+1} \to Y_k .$$

In $\mathcal{S}p$ we can form the spectrum $\prod_k Y_k$ constructed so that, for any spectrum W

$$[W, \prod_k Y_k]_* \cong \prod_k [W, Y_k]_* .$$

In particular, let $W = \prod_k Y_k$ and define the projections $p_n : \prod_k Y_k \to Y_n$ by

$$1_W = (p_n) .$$

Then we define a map $d : \prod_k Y_k \to \prod_k Y_k$ by

$$d = (p_n - \sigma_{n+1} p_{n+1}) \in \prod_n [\prod_k Y_k, Y_n] .$$

We define the homotopy inverse limit of the inverse system $\{Y_k\}$, written $\operatorname{holim}_k Y_k$, by $\operatorname{holim}_k Y_k = \text{fibre}(d)$. This definition is motivated by

$$\operatorname{Ker}(d_*:\prod_k \pi_* Y_k \to \prod_k \pi_* Y_k) = \lim_k \pi_* Y_k \quad .$$

<u>Proposition 4.1</u> ([3])

If W is a spectrum and $\{Y_k\}$ is an inverse system of spectra, then there is an exact sequence

$$0 \to R\lim_k [W,Y_k]_{*+1} \to [W,\operatorname{holim}_k Y_k]_* \to \lim_k [W,Y_k]_* \to 0 \; .$$

'Dual' to Proposition 4.1 we have

<u>Proposition 4.2</u> ([19]) (Milnor's exact sequence.)

Let X be a spectrum filtered by subspectra X_s with $\bigcup_s X_s = X$. Then for any spectrum E there is a short exact sequence

$$0 \to R\lim_s [X_s,E]_{*+1} \to [X,E]_* \to \lim_s [X_s,E]_* \to 0 \quad .$$

If G is an abelian group let SG denote the Moore spectrum of type G (see [3]). If E is a spectrum, then there is an inverse system of spectra

$$.. \to S\mathbb{Z}p^3 \wedge E \to S\,\mathbb{Z}/p^2 \wedge E \to S\,\mathbb{Z}/p \wedge E \quad .$$

The p-completion of E, written $\hat{E}_p$, is the homotopy inverse limit of this inverse system.

<u>Lemma 4.3</u>

If E is a spectrum of finite type, then

$$\hat{E}_p \simeq E \wedge S\,\hat{\mathbb{Z}}_p \ .$$

<u>Pf</u> Since E is of finite type

$$\prod_n E \wedge S\,\mathbb{Z}/p^n \simeq E \wedge \prod_n S\,\mathbb{Z}/p^n$$

(see [3]) and the result follows.

<u>Lemma 4.4</u>

If F is a finite spectrum and E is a spectrum of finite type, then

$$[F,\hat{E}_p]_* \cong [F,E]_* \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}_p \ .$$

<u>Pf</u> The previous lemma shows that

$$\hat{E}_p \simeq E \wedge S\,\hat{\mathbb{Z}}_p \ .$$

Since F is finite there is a short exact sequence

$$0 \to [F,E]_* \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}_p \to [F, E\wedge S\hat{\mathbb{Z}}_p]_* \to \mathrm{Tor}_1^{\mathbb{Z}}([F,E]_{*-1}, \hat{\mathbb{Z}}_p) \to 0$$

(see [3]). The group $\hat{\mathbb{Z}}_p$ is torsion free, so $\mathrm{Tor}_1^{\mathbb{Z}}$ vanishes and the result follows.

We now give an account of the classical modp Adams spectral sequence. For a prime p, let $H(p)$ denote the representing spectrum for modp cohomology. Let $i:S^0 \to H(p)$ be the inclusion. Let $\bar{H}(p)$ be the fibre of i, then we have the exact triangle

$$\bar{H}(p) \to S^0 \xrightarrow{\ i\ } H(p) \xrightarrow{\ q\ } S^1\bar{H}(p) .$$

For convenience define

$$I^s = \underbrace{S^1\bar{H}(p) \wedge \ldots \wedge S^1\bar{H}(p)}_{s \text{ times}} .$$

Then for any spectrum Y and each s we have an exact triangle

$$I^s \wedge Y \xrightarrow{1\wedge i\wedge 1} I^s \wedge H(p) \wedge Y \xrightarrow{1\wedge q\wedge 1} I^{s+1} \wedge Y \to SI^s \wedge Y .$$

If X is a spectrum, then by applying $[X,-]_*$ to the above exact triangles we obtain the unravelled exact couple

$$\cdots \to [X, I^{s+1} \wedge Y]_* \xrightarrow{\bar{a}} [X, I^s \wedge Y]_* \to \cdots \to [X,Y]_*$$
$$\bar{\delta} \nwarrow \qquad \swarrow \bar{j}$$
$$[X, I^s \wedge H(p) \wedge Y]_* \quad .$$

The spectral sequence (E_r, d_r) associated to this unravelled exact couple is the classical modp Adams spectral sequence.

We wish to show that this spectral sequence converges conditionally, i.e.

$$\lim_s [X, I^s \wedge Y]_* = 0 \quad R\lim_s [X, I^s \wedge Y]_* = 0 \ .$$

For general Y, however, it is not obvious that either of these conditions will hold. So we will construct a new unravelled exact couple that satisfies the convergence conditions and gives rise to the same spectral sequence (E_r, d_r).

To begin, define

$$Y^\infty_{(p)} = \operatorname{holim}_s I^s \wedge Y \quad .$$

Then form the exact triangle

$$Y^\infty_{(p)} \to I^s \wedge Y \to I^s \wedge Y / Y^\infty_{(p)} \to SY^\infty_{(p)} \ .$$

Since the homotopy inverse limit of a system of exact triangles is an exact triangle,

$$\operatorname{holim}_s(I^s \wedge Y/Y^\infty_{(p)}) \simeq * \, .$$

Hence Proposition 4.1 shows

$$\lim_s [X, I^s \wedge Y/Y^\infty_{(p)}]_* = 0 \quad R\lim_s [X, I^s \wedge Y/Y^\infty_{(p)}]_* = 0 \, .$$

The commutative diagram of exact triangles

$$\begin{array}{ccccccc}
Y^\infty_{(p)} & \to & Y^\infty_{(p)} & \to & * & \to & SY^\infty_{(p)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I^{s+1} \wedge Y & \to & I^s \wedge Y & \to & I^s \wedge H_{(p)} \wedge Y & \to & SI^{s+1} \wedge Y \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I^{s+1} \wedge Y/Y^\infty_{(p)} & \to & I^s \wedge Y/Y^\infty_{(p)} & \to & I^s \wedge H_{(p)} \wedge Y & \to & SI^{s+1} \wedge Y/Y^\infty_{(p)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
SY^\infty_{(p)} & \to & SY^\infty_{(p)} & \to & * & \to & S^2 Y^\infty_{(p)}
\end{array}$$

gives us, via the third row from the top, a new unravelled exact couple

$$\begin{array}{ccccc}
\cdots \to [X, I^{s+1} \wedge Y/Y^\infty_{(p)}]_* & \xrightarrow{a} & [X, I^s \wedge Y/Y^\infty_{(p)}]_* & \to \cdots \to & [X, Y/Y^\infty_{(p)}]_* \\
{\scriptstyle \delta} \nwarrow & & \swarrow {\scriptstyle j} & & \\
& [X, I^s \wedge H_{(p)} \wedge Y]_* & & &
\end{array}$$

and, using the middle row of vertical maps, a morphism of unravelled exact couples

$$f:\{[X,I^S\wedge Y]_*,[X,I^S\wedge H_{(p)}\wedge Y]_*\} \to \{[X,I^S\wedge Y/Y^\infty_{(p)}]_*,[X,I^S\wedge H_{(p)}\ Y]_*\} \quad .$$

Theorem 4.5 ([7])

For any spectra X and Y the classical modp Adams spectral sequence converges conditionally to the group $[X,Y/Y^\infty_{(p)}]_*$.

Pf If we let $(\bar{E}_r,\bar{d}_r)$ denote the spectral sequence associated with the unravelled exact couple

$$\{[X,I^S\wedge Y/Y^\infty_{(p)}]_*,[X,I^S\wedge H_{(p)}\wedge Y]_*\} \quad ,$$

then the morphism of unravelled exact couples f induces a morphism of spectral sequences $f:(E_r,d_r) \to (\bar{E}_r,\bar{d}_r)$. The homomorphism $f_1:E_1 \to \bar{E}_1$ is an isomorphism so Theorem 3.1 applies to show that f is an isomorphism of spectral sequences. Since

$$R\lim_s[X,I^S\wedge Y/Y^\infty_{(p)}]_* = 0 \quad \lim_s[X,I^S\wedge Y/Y^\infty_{(p)}]_* = 0 \quad ,$$

$(\bar{E}_r,\bar{d}_r)$ converges conditionally to $[X,Y/Y^\infty_{(p)}]_*$.

This completes the proof.

We would of course like to know what Y/Y^∞ is and Bousfield [9] provides the answer.

Theorem 4.6 ([9])

If Y is a spectrum of finite type, then $Y/Y^{\infty}_{(p)} \approx \hat{Y}_p$.

If Y is of finite type, define

$$A^s = [X, I^s \wedge \hat{Y}_p]_* \quad E^s = [X, I^s \wedge H_{(p)} \wedge Y]_* \quad .$$

Recall that a, j, δ are the homomorphisms in the unravelled exact couple $\{A^s, E^s\}$.

Lemma 4.7

If X and Y are spectra of finite type, then there are compact Hausdorff topologies on A^s and E^s which make a, j and δ continuous.

Pf Filter X by its finite skeleta $\{X_n\}$. Then we have Milnor's exact sequence

$$0 \to R\lim_n [X_n, I^s \wedge \hat{Y}_p]_{*+1} \to [X, I^s \wedge \hat{Y}_p]_* \to \lim_n [X_n, I^s \wedge \hat{Y}_p]_* \to 0 \quad .$$

Lemma 4.4, Lemma 2.5 and Lemma 2.6 show that each group $[X_n, I^s \wedge \hat{Y}_p]_*$ has a compact, Hausdorff p-adic topology. Thus by Lemma 2.4 (b)

$$R\lim_n [X_n, I^s \wedge \hat{Y}_p]_* = 0 \quad ,$$

hence

$$[X,I^S\wedge\hat{Y}_p]_* = \lim_n[X_n,I^S\wedge\hat{Y}_p]_* \quad .$$

If each factor has the p-adic topology, then

$$\prod_n[X_n,I^S\wedge\hat{Y}_p]_*$$

is compact and Hausdorff under the product topology. The subspace

$$A^S = \lim_n[X_n,I^S\wedge\hat{Y}_p]_* \subseteq \prod_n [X_n,I^S\wedge\hat{Y}_p]_*$$

is closed, hence compact and Hausdorff.

Since the group $[X_n,I^S\wedge H_{(p)}\wedge Y]_*$ is a finite p-group it is compact and Hausdorff under the p-adic topology. Thus the above procedure shows that

$$E^S = \lim_n[X_n,I^S\wedge H_{(p)}\wedge Y]_* \subseteq \prod_n[X_n,I^S\wedge H_{(p)}\wedge Y]_*$$

is a compact, Hausdorff subspace. The homomorphisms a, j and δ extend to continuous homomorphisms of the product spaces, hence they are continuous.

Theorem 4.8

If X and Y are spectra of finite type, then the classical modp

Adams spectral sequence converges strongly to $[X,\hat{Y}_p]_*$.

<u>Pf</u> Recall the definition

$$Z_r^s = \delta^{-1} \operatorname{Im}(a^{r-1}:A^{s+r} \to A^{s+1}) \subseteq E^s \ .$$

Then the previous lemma shows that each Z_r^s is a compact, Hausdorff subspace of E^s . Each of the inclusions

$$\ldots \subset Z_r^s \subset \ldots \subset Z_2^s \subset Z_1^s$$

is automatically continuous, so the inverse system $\{Z_r^s\}$ satisfies the hypothesis of Lemma 2.4 (b). Hence $R\lim_r Z_r^s = 0$. Theorem 4.5, Theorem 4.6 and Theorem 3.3 complete the proof.

§5. The construction of the spectral sequence

Throughout this section let X be a spectrum of finite type and let $\{Y_k\}$ be an inverse system of spectra, each Y_k a spectrum of finite type. For convenience fix a prime p, and write $H = H_{(p)}$ and $\hat{E} = \hat{E}_p$. Then we will construct in this section a spectral sequence converging strongly to

$$[X, \operatorname{holim}_k \hat{Y}_k]_* .$$

In the previous section we showed that the unravelled exact couple

$$\begin{array}{ccccc} \cdots \to [X, I^{s+1} \wedge \hat{Y}_k]_* & \xrightarrow{a(k)} & [X, I^s \wedge \hat{Y}_k]_* & \to \cdots \to & [X, \hat{Y}_k]_* \\ \delta(k) \nwarrow & & \swarrow j(k) & & \\ & [X, I^s \wedge H \wedge Y_k]_* & & & \end{array}$$

gives rise to the classical modp Adams spectral sequence converging strongly to the group $[X, \hat{Y}_k]_*$. We will construct the spectral sequence by taking the inverse limit, over k, of the above unravelled exact couples.

To begin, define

$$A^s(k) = [X, I^s \wedge \hat{Y}_k] \qquad A^s = \lim_k A^s(k)$$

$$E^s(k) = [X, I^s \wedge \hat{Y}_k] \qquad E^s = \lim_k E^s(k)$$

$$a = \lim_k a(k) \qquad \delta = \lim_k \delta(k) \qquad j = \lim_k j(k) \quad .$$

Proposition 5.1

The following is an unravelled exact couple:

$$\begin{array}{ccccccc}
\cdots\; A^{s+1} & \xrightarrow{a} & A^{s} & & \xrightarrow{a} & A^{s-1} & \to \cdots \to A^0 \\
{\scriptstyle\delta}\nwarrow & \swarrow & & {\scriptstyle j}\;\nwarrow & \swarrow{\scriptstyle j} & & \\
& E^s & & {\scriptstyle\delta} & E^{s-1} & &
\end{array} \quad .$$

Pf We are required to show that each of the sequences

$$\begin{array}{ccc}
A^{s+1} & \xrightarrow{a} & A^{s} \\
{\scriptstyle\delta}\nwarrow & & \swarrow{\scriptstyle j} \\
& E^s &
\end{array}$$

is exact. Lemma 4.7 shows that each of the inverse systems $\{\mathrm{Im}\,\delta(k)\}$, $\{\mathrm{Im}\,a(k)\}$ and $\{\mathrm{Im}\,j(k)\}$ satisfies condition (b) of Lemma 2.4. Lemma 2.3 proves the result.

Let (E_r, d_r) denote the spectral sequence determined by Proposition 5.1. Recall the definitions

$$Z_r^s(k) = \delta_{(k)}^{-1}\, \mathrm{Im}(a(k)^{r-1} : A^{s+r}(k) \to A^{s+1}(k)) \subseteq E^s(k)$$

$$B_r^s(k) = j(k)\mathrm{Ker}(a(k)^{r-1} : A^s(k) \to A^{s-r-1}(k)) \subseteq E^s(k)$$

$$Z_r^s = \delta^{-1}\,\mathrm{Im}(a^{r-1}:A^{s+r} \to A^{s+1}) \subseteq E^s\ .$$

Proposition 5.2

The sequence of chain complexes $(\lim_k E_r(k), \lim_k d_r(k))$ is a spectral sequence.

Pf Using Lemma 4.7, the groups $Z_r^s(k)$ and $B_r^s(k)$ are compact, Hausdorff subspaces of $E^s(k)$. Thus, under the quotient topology, the inverse system $\{E_r^s(k)\}_k$ satisfies condition (b) of Lemma 2.4, hence

$$R\lim_k E_r^s(k) = 0\ .$$

Likewise, using $\mathrm{Im} d_r(k) = B_{r+1}^{s+r}(k)/B_r^{s+r}(k)$, we see that

$$R\lim_k \mathrm{Im} d_r(k) = 0\ .$$

Lemma 2.3 now proves the result.

Proposition 5.3

There is an isomorphism of spectral sequences

$$(E_r, d_r) \cong (\lim_k E_r(k),\ \lim_k d_r(k))\ .$$

Pf The obvious morphism of unravelled exact couples

$$\{A^s, E^s\} \to \{A^s(k), E^s(k)\}$$

induces a morphism of spectral sequences

$$(E_r, d_r) \to (E_r(k), d_r(k)) .$$

This gives a morphism of spectral sequences

$$(E_r, d_r) \to (\lim_k E_r(k), \lim_k d_r(k))$$

which is an isomorphism when $r = 1$. Theorem 3.1 completes the proof.

We turn now to the question of convergence:

Theorem 5.4

The spectral sequence (E_r, d_r) converges strongly to $[X, \mathrm{holim}_k \hat{Y}_k]_*$.

Pf First we note that by Lemma 4.7, Lemma 2.4(b) and Proposition 4.1,

$$[X, \mathrm{holim}_k \hat{Y}_k]_* = \lim_k [X, \hat{Y}_k]_* .$$

So to prove the theorem, by Theorem 3.3, it suffices to show that

$$\lim_s A^s = 0 \qquad R\lim_s A^s = 0 \qquad R\lim_r Z_r^s = 0 .$$

As a subspace of $\prod_k A^s(k)$, A^s is compact and Hausdorff by Lemma 4.7. The maps in the inverse system $\{A^s\}$ are continuous under this topology so Lemma 2.4(b) applies to show that $R\lim_s A^s = 0$.

Recall that for each k $\lim_s A^s(k) = 0$ by Theorem 4.5. Thus, using Lemma 2.1,

$$\lim_s A^s = \lim_s \lim_k A^s(k) = \lim_k \lim_s A^s(k) = 0 .$$

Lemma 4.7 shows that as a subspace of $\prod_k E^s(k)$, E^s is compact and Hausdorff. Likewise A^s is a compact, Hausdorff subspace of $\prod_k A^s(k)$ and the homomorphisms a,j and δ are continuous. Thus Z_r^s is a compact, Hausdorff subspace of E^s . Each of the inclusions

$$\cdots \subset Z_r^s \subset \cdots \subset Z_2^s \subset Z_1^s$$

is automatically continuous, hence Lemma 2.4(b) shows that $R\lim_r Z_r^s = 0$. This completes the proof.

Let $A(p)$ denote the mod p Steenrod algebra, $B = H^*X$ and $M_k = H^*Y_k$. Since $\{Y_k\}$ is an inverse system of spectra, the $A(p)$-modules M_k form a directed system. Let $M = \operatorname{colim}_k M_k$. The standard identification of the E_2-term of $(E_r(k),d_r(k))$ (see [3]) is

$$E_2^{s,*}(k) = \operatorname{Ext}_{A(p)}^{s,*}(M_k,B) .$$

Then, using Proposition 5.3, the E_2-term of (E_r, d_r) is given by

$$E_2^{s,*} = \lim_k \operatorname{Ext}^{s,*}_{A(p)}(M_k, B) \ .$$

We note that the codegree of $E_2^{s,t}$ is $s-t$ ([7]).

Theorem 5.5

The homomorphism

$$\operatorname{Ext}^{s,*}_{A(p)}(M,B) \to \lim_k \operatorname{Ext}^{s,*}_{A(p)}(M_k,B)$$

is an isomorphism.

Pf Let J^s be an injective $A(p)$-coresolution of B . Then

$$\lim_k \operatorname{Hom}^*_{A(p)}(M_k, J^s) \cong \operatorname{Hom}^*_{A(p)}(M, J^s) \ .$$

Let $C^{s,*}(k)$ and $D^{s,*}(k)$ denote the cycles and boundaries of the cochain complex $\operatorname{Hom}^*_{A(p)}(M_k,B)$. The cohomology of this cochain complex is

$$\operatorname{Ext}^{s,*}_{A(p)}(M_k,B)$$

and the cohomology of the cochain complex $\operatorname{Hom}^*_{A(p)}(M,B)$ is $\operatorname{Ext}^{s,*}_{A(p)}(M,B)$ (see [17]).

Using the identification from the Adams spectral sequence

$$E_2^{s,*}(k) = \mathrm{Ext}_{A(p)}^{s,*}(M_k,B) \quad ,$$

in the proof of Proposition 5.2 we showed that

$$R\lim_k \mathrm{Ext}_{A(p)}^{s,*}(M_k,B) = 0 \; .$$

For each K let Q_k denote the kernel of the homomorphism $M_k \to M_{k+1}$. Let $\bar{M}_k$ be the $A(p)$-module defined by the short exact sequence

$$0 \to Q_k \to M_k \to \bar{M}_k \to 0 \quad .$$

Then we have a short exact sequence of inverse systems

$$0 \to \{Q_k\} \to \{M_k\} \to \{\bar{M}_k\} \to 0 \quad ,$$

with the homomorphisms in $\{Q_k\}$ being zero, and the homomorphisms in $\{\bar{M}_k\}$ being injective. Because J^s is injective, $\mathrm{Hom}_{A(p)}^{*}(-,J^s)$ is exact (see [17]), so

$$0 \to \{\mathrm{Hom}_{A(p)}^{*}(\bar{M}_k,J^s)\} \to \{\mathrm{Hom}_{A(p)}^{*}(M_k,J^s)\} \to \{\mathrm{Hom}_{A(p)}^{*}(Q_k,J^s)\} \to 0$$

is a short exact sequence of inverse systems. The homomorphisms in

$\{\mathrm{Hom}^*_{A(p)}(\bar{M}_k, J^s)\}$ are onto and the homomorphisms in $\{\mathrm{Hom}^*_{A(p)}(Q_k, J^s)\}$ are zero. Hence these systems satisfy the Mittag-Leffler condition and

$$R1\mathrm{im}_k\ \mathrm{Hom}^*_{A(p)}(\bar{M}_k, J^s) = 0 = R1\mathrm{im}_k \mathrm{Hom}^*_{A(p)}(Q_k, J^s)$$

by Lemma 2.4(a). Lemma 2.2 now shows that

$$R1\mathrm{im}_k\ \mathrm{Hom}^*_{A(p)}(M_k, J^s) = 0 \ .$$

The short exact sequence of inverse systems

$$0 \to \{C^{s,*}(k)\}_k \to \{\mathrm{Hom}^*_{A(p)}\ (M_k, J^s)\}_k \to \{D^{s,*}(k)\}_k \to 0$$

and Lemma 2.2 show that

$$R1\mathrm{im}_k D^{s,*}(k) = 0 \ .$$

Lemma 2.3 now completes the proof.

We summarize the results of this section with the following theorem:

<u>Theorem 5.6</u>

If X is a spectrum of finite type and $\{Y_k\}$ is an inverse system of spectra, each of finite type, then there is a spectral sequence with E_2-term

$$E_2^{s,*} = \mathrm{Ext}^{s,*}_{A(p)}(\mathrm{colim}_k\ H^*Y_k, H^*X)$$

converging strongly to

$$[X, \mathrm{holim}_k \hat{Y}_k]_* \ .$$

§6. The proof of Theorem A

Given a spectrum E we define

$$P_{-k}(E) = S^k D_2(S^{-k}E)$$

using the quadratic construction on spectra, D_2 (see [12], [13]). Let $\mathcal{C}$ denote the homotopy category of CW-complexes. Let $\mathcal{S}$ denote the full subcategory of $\mathcal{S}p$ consisting of objects $S^\ell X$, $\ell \in \mathbb{Z}$ and $X \in \mathcal{C}$. We refer to $\mathcal{S}$ as the stable category. We recall five facts about the functors P_{-k} which we will need in this section and section seven (see [13] or [12]).

Fact 1

Given a CW-complex Z and a spectrum E, there is a map of spectra

$$\phi(Z,E): Z \wedge P_{-k}(E) \to P_{-k}(Z \wedge E)$$

which is a natural transformation of functors $\mathcal{C} \times \mathcal{S}p \to \mathcal{S}p$.

If $W, E \in \mathcal{S}p$ let $T: E \wedge W \to W \wedge E$ be the map which switches factors.

Fact 2

Given CW-complexes Z and X, and a spectrum E, the following diagram commutes:

$$\begin{array}{ccccc}
Z\wedge X\wedge P_{-k}(E) & \xrightarrow{1\wedge\phi(X,E)} & Z\wedge P_{-k}(X\wedge E) & \xrightarrow{\phi(Z,X\wedge E)} & P_{-k}(Z\wedge X\wedge E) \\
\downarrow T\wedge 1 & & & & \downarrow P_{-k}(T\wedge 1) \\
X\wedge Z\wedge P_{-k}(E) & \xrightarrow[1\wedge\phi(Z,E)]{} & X\wedge P_{-k}(Z\wedge E) & \xrightarrow[\phi(X,Z\wedge E)]{} & P_{-k}(X\wedge Z\wedge E)
\end{array}$$

For $E \in \mathcal{S}p$ let $\sigma_{-k}:P_{-k}(E) \to P_{-k+1}(E)$ be the natural map defined by

$$\sigma_{-k} = S^{-1}\phi(S^1,E) .$$

<u>Fact 3</u>

There are natural maps of spectra

$$i_{-k}:S^{-k}\wedge E\wedge E \to P_{-k}(E)$$

$$p_{-k}:P_{-k}(E) \to S^{-k}\wedge E\wedge E$$

which fit into the exact triangle

$$S^{-k}\wedge E\wedge E \xrightarrow{i_{-k}} P_{-k}(E) \xrightarrow{\sigma_{-k}} P_{-k+1}(E) \xrightarrow{p_{-k+1}} S^{-k+1}\wedge E\wedge E .$$

Moreover, $p_{-k}i_{-k} = 1 + (-1)^k T$.

The maps σ_{-k} give us an inverse system $\{P_{-k}(E)\}$.

Fact 4

If we define an inverse system of spectra $\{P_{-k}\}$ by $P_{-k} = P_{-k}(S^0)$, then $\{P_{-k}\}$ satisfies properties (a) through (d) in the introduction and Lin's Theorem holds for this inverse system.

Fact 5

Given an exact triangle in C

$$A \to X \to Y \to SA \ ,$$

there is a spectrum $P_{-k}(X;A)$ and an exact triangle in $\mathcal{S}p$

$$P_{-k}(A) \to P_{-k}(X) \to P_{-k}(X;A) \to SP_{-k}(A) \ .$$

If $\Delta:S^1 \to S^2$ is the diagonal map, then there are maps $P_{-k}(X;A) \to P_{-k+1}(X;A)$ which fit into the commutative diagram of exact triangles

$$\begin{array}{ccccccc} S^{-k}A\wedge Y & \to & P_{-k}(X;A) & \to & P_{-k}(Y) & \to & S^{-k+1}A\wedge Y \\ S^{-k-1}\Delta\wedge 1 & & & & \sigma_{-k} & & S^{-k}\Delta\wedge 1 \\ S^{-k+1}A\wedge Y & \to & P_{k+1}(X;A) & \to & P_{-k}(Y) & \to & S^{-k+2}A\wedge Y \end{array} \ .$$

Note that the vertical maps to the far right and the far left are zero.

Our first task is to construct a map of inverse system of spectra

$$S^{-1}E \to \{P_{-k}(E)\} \quad .$$

Recall that Lin's Theorem shows that

$$\hat{\mathbb{Z}}_2 \cong \lim_k \pi_{-1} P_{-k} \quad .$$

Choose a map of inverse systems

$$S^{-1} \xrightarrow{\{\psi_{-k}\}} \{P_{-k}\}$$

that passes to $1 \in \mathbb{Z}_2$ under the above isomorphism. Given a spectrum of finite type E, let $\{E_n\}$ denote the set of all finite subspectra of E. For each m there exists $\ell_m \geq 0$ with

$$S^{\ell_m} E_m \in C \quad .$$

Using Fact 1 we form the composite

$$S^{-1} \wedge S^{\ell_m} E_m \xrightarrow{\psi_{-k-\ell_m} \wedge 1} P_{-k-\ell_m} \wedge S^{\ell_m} E_m \xrightarrow{\phi(S^{\ell_m} E_m, S^0)} P_{-k-\ell_m}(S^{\ell_m} E_m)$$

which we desuspend to form the composite

$$S^{-1} E_m \to P_{-k-\ell_m} \wedge E_m \to S^{-\ell_m} P_{-k-\ell_m}(S^{\ell_m} E_m) \quad .$$

Using $P_{-k}(E_m) = S^{-\ell_m}P_{-k-\ell_m}(S^{\ell_m}E_m)$ and the map of spectra $P_{-k}(E_m) \to P_{-k}(E)$, induced by the inclusion, we obtain the composite

$$S^{-1}E_m \to P_{-k-\ell_m} \wedge E_m \to P_{-k}(E_m) \to P_{-k}(E) .$$

Given an inclusion $E_m \subseteq E_n$, $\ell_n \geq \ell_m$, using Fact 2 we obtain the commutative diagram

$$\begin{array}{ccccc} S^{-1}E_m & \to & P_{-k-\ell_m} \wedge E_m & \to & P_{-k}(E_m) & & \\ \downarrow & & & & \downarrow & & P_{-k}(E) \\ S^{-1}E_n & \to & P_{-k-\ell_n} \wedge E_n & \to & P_{-k}(E_n) & & \end{array} .$$

Since $E = \underset{n}{\cup} E_n$ we have constructed a map $\Phi_k : S^{-1}E \to P_{-k}(E)$. Fact 2 and the construction show that these maps form a map of inverse systems $S^{-1}E \to \{P_{-k}(E)\}$.

We wish to identify this map in mod 2 cohomology, H^*. Let A denote the mod 2 Steenrod algebra. Following Singer ([20]), given a left A-module M, define the $\mathbb{Z}/2$-vector space $\Delta(M)$ by

$$\Delta(M) = \mathbb{F}_2[u,u^{-1}] \otimes M .$$

Give $\Delta(M)$ the structure of a left A-module via

$$Sq^a(u^\ell \otimes m) = \sum_j \binom{\ell-j}{a-2j} u^{\ell+a-j} \otimes Sq^j m, \quad m \in M .$$

Note that as a functor from the category of left A-modules to itself, Δ is exact.

There is an A-homomorphism (see [20]) $\varepsilon:\Delta(M) \to \Sigma^{-1}M$ defined by

$$\varepsilon(u^{\ell}\otimes m) = \begin{cases} Sq^{\ell+1}m & \ell \geq -1 \\ 0 & \text{otherwise.} \end{cases}$$

After applying mod 2 cohomology to the inverse system $\{P_{-k}(E)\}$ we obtain a direct system of A-modules. The following observation is due to H. Miller and a proof may be found in [12].

Proposition 6.1

There is a natural isomorphism defined on $\mathcal{S}p$

$$\Delta(H^*E) \cong \operatorname{colim}_k H^*P_{-k}(E) .$$

Let $\Phi^* = \operatorname{colim}_k \Phi_k^*:\operatorname{colim}_k H^*(P_{-k}(E)) \to \Sigma^{-1}H^*E$.

Proposition 6.2 ([12])

If F is a finite spectrum, then $\Phi^* = \varepsilon$.

Corollary 6.3

If E is a spectrum of finite type, then $\Phi^* = \varepsilon$.

Pf Let E_n denote the collection of all finite subspectra of E. Then we have the commutative diagram

$$\begin{array}{ccc} \Delta(H^*E) & \xrightarrow{\Phi^*} & \Sigma^{-1}H^*E \\ \downarrow & & \downarrow \\ \Delta(H^*E_n) & \xrightarrow[\Phi^*]{} & \Sigma^{-1}H^*E_n \end{array} .$$

Fixing $*$ choose m so that $H^*E \cong H^*E_m$. Since Δ is exact, if $x \in H^*E$ then under the homomorphism $\Delta(H^*E) \to \Delta(H^*E_m)$

$$u^\ell \otimes X \to u^\ell \otimes X \quad .$$

Proposition 6.2 now completes the proof.

We quote a theorem of Adams, Gunawardena and Miller ([4]) which will be proved in the appendix.

Theorem 6.4 ([4])

If M is a left A-module, then the homomorphism

$$\mathrm{Ext}_A^{s,*}(\Sigma^{-1}M, \mathbb{Z}/2) \xrightarrow{\varepsilon^*} \mathrm{Ext}_A^{s,*}(\Delta(M), \mathbb{Z}/2)$$

is an isomorphism.

Proposition 6.5

If E is a spectrum of finite type, then the spectrum $\mathrm{holim}_k P_{-k}(E)$ is two-complete.

Pf First we will show that for k odd $\pi_* P_{-k}(E)$ is finite. Let $H_*(Q)$ denote rational homology, then

$$\pi_* P_{-k}(E) \underset{\mathbb{Z}}{\otimes} Q \cong H_*(Q)(P_{-k}(E)) .$$

Using Fact 3 we construct an unravelled exact couple

$$\begin{array}{ccccccc} H_*(Q)(P_{-k}(E)) & \to & H_*(Q)(P_{-k+1}(E)) & \to \cdots \to & H_*(Q)(P_{-k+s}(E)) & \to & \cdots \\ {\scriptstyle i_*} \nwarrow & & \swarrow {\scriptstyle p_*} & & & & \\ & H_*(Q)(E \wedge E) & & & & & \end{array}$$

which determines a left-half-plane spectral sequence converging, by Theorem 3.4, strongly to $H_*(Q)(P_{-k}(E))$. When k is odd Fact 3 shows that

$$d_1^s = p_* i_* = 1 + (-1)^{k+s} ,$$

so $E_2^* = 0$. Thus when k is odd

$$\pi_* P_{-k}(E) \underset{\mathbb{Z}}{\otimes} Q = 0 ,$$

hence $\pi_* P_{-k}(E)$ is finite since $P_{-k}(E)$ is of finite type.

Next we will show that $\pi_* P_{-k}(E)$ is a two-group when k is odd. If p is prime and $H_*(p)$ denotes mod p homology, then using Fact 3 we obtain just as before a left-half-plane spectral sequence converging strongly to $H_*(p)(P_{-k}(E))$. The differential d_1 is determined by

$$d_1^s = p_* i_* = 1 + (-1)^{k+s} .$$

Hence if $p \neq 2$, p is prime and k is odd, $E_2^* = 0$. Thus $H_*(p)(P_{-k}(E)) = 0$. This shows that $\pi_* P_{-k}(E)$ is a two-group when k is odd.

If $\wedge$ denotes the two-completion of spectra, then $[\mathrm{holim}_k P_{-k}(E)]^\wedge = \mathrm{holim}_k P^\wedge_{-k}(E)$. Since each spectrum $P_{-k}(E)$ is of finite type $P^\wedge_{-k}(E) \simeq P_{-k}(E) \wedge S(\mathbb{Z}_2^\wedge)$ by Lemma 4.3. Hence the inverse system $\{\pi_* P^\wedge_{-k}(E)\}$ satisfies condition (b) of Lemma 2.4 by Lemma 4.4, Lemma 2.5 and Lemma 2.6. Proposition 4.1 now shows that

$$\pi_* \mathrm{holim}_k P^\wedge_{-k}(E) \cong \lim_k \pi_* P_{-k}(E) .$$

Since the groups $\pi_* P_{-k}(E)$ are finite for each odd k the inverse system $\{\pi_* P_{-k}(E)\}$ satisfies the Mittag-Leffler condition. Lemma 2.4 and Proposition 4.1 now show that

$$\pi_* \mathrm{holim}_k P_{-k}(E) \cong \lim_k \pi_* P_{-k}(E) .$$

We have the commutative diagram

$$\begin{array}{ccc} \pi_* \mathrm{holim}_k P_{-k}(E) & \to & \pi_* \mathrm{holim}_k P^\wedge_{-k}(E) \\ \| 2 & & \| 2 \\ \lim_k \pi_* P_{-k}(E) & \to & \lim_k \pi_* P^\wedge_{-k}(E) \end{array} .$$

Since for each odd k $\pi_* P_{-k}(E)$ is a two-group and using Lemma 4.4.

$$\lim_k \pi_* P_{-k}(E) \cong \lim_k \pi_* P^\wedge_{-k}(E) .$$

This completes the proof.

Proof of Theorem A.

Let $(\bar{E}_r, \bar{d}_r)$ denote the Adams spectral sequence converging to $\pi_* S^{-1} E^\wedge$ and let (E_r, d_r) denote the spectral sequence of Theorem 5.6 converging to $\pi_* \mathrm{holim}_k P_{-k}(E)$ by Proposition 6.5. Then the map of inverse systems

$$S^{-1}E \xrightarrow{\{\phi_k\}} \{P_{-k}(E)\}$$

induces a morphism of spectral sequences $f:(\bar{E}_r, \bar{d}_r) \to (E_r, d_r)$. On the E_2-terms, $f_2 = \phi^*$ which is an isomorphism by Corollary 6.3 and Theorem 6.4. Theorem 4.8, Theorem 5.6 and Theorem 3.2 complete the proof.

§7. The proofs of Theorem B and C

This section comes from [12]. If X is a CW-complex, then Fact 1 gives a natural map of spectra

$$\phi(X,S^0):P_{-k}\wedge X \to P_{-k}(X) .$$

We write Φ for the map of inverse systems of spectra $\{P_{-k}\wedge X\} \to \{P_{-k}(X)\}$ induces by the maps $\phi(X,S^0)$. In this thesis we do not require a cohomology theory to satisfy Milnor's wedge axiom.

Theorem 7.1 ([12])

Let E^* be a cohomology theory defined on Sp . Then

(a) $\operatorname{colim}_k E^* P_{-k}(E)$ is a cohomology theory defined on S ;

(b) the natural transformation of cohomology theories, defined on C ,

$$\Phi^*:\operatorname{colim}_k E^* P_{-k}(X) \to \operatorname{colim}_k E^* P_{-k}\wedge X$$

is an isomorphism when $X \in C$ is finite.

Pf (a) The only difficult point in the proof is showing that $\operatorname{colim}_k E^* P_{-k}(-)$ takes exact triangles in S to long exact sequences. Recall that colim_k is an exact functor. Then given an exact triangle in S

$$A \to X \to Y \to SA ,$$

Fact 5 shows that $\operatorname{colim}_k E^* P_{-k}(Y) \cong \operatorname{colim}_k E^* P_{-k}(X,A)$. Thus, using Fact 5, we obtain the required long exact sequence

$$\ldots \to \operatorname{colim}_k E^* P_{-k}(Y) \to \operatorname{colim}_k E^* P_{-k}(X) \to \operatorname{colim}_k E^* P_{-k}(A) \to \ldots .$$

(b) It is clear that $\operatorname{colim}_k E^* P_{-k} \wedge X$ is a cohomology theory defined on Sp . Φ^* , defined on C , is a natural transformation of cohomology theories. When $X = S^0$ Fact 4 shows that Φ^* is an isomorphism. The Eilenberg-Steenrod uniqueness theorem shows that Φ^* is an isomorphism for each finite $X \in C$.

<u>Corollary 7.2</u>

If X is a finite spectrum then there is a natural isomorphism $B: \operatorname{colim}_k E^* P_{-k}(X) \cong \operatorname{colim}_k E^* P_{-k} \wedge X$.

<u>Pf</u> Since X is a finite spectrum, for some $\ell \geq 0$ $S^\ell X \in C$. Now use Theorem 7.1.

<u>Proof of Theorem B</u>

Fact 4 shows that $P_0(P_0) \simeq \mathbb{R}P_+^\infty$, hence we have the canonical map

$$\rho : P_0(S^0) = \mathbb{R}P_+^\infty \to S^0$$

described in the introduction. Recall that $\{\sigma_{-k}\}$ is the collection of maps in the inverse system of spectra $\{P_{-k}\}$. Then we define maps $g_{-k}:P_{-k} \to S^0$ by $g_{-k} = \rho\circ\sigma_{-1}\circ\ldots\circ\sigma_{-k}$. The maps $g_{-k}\wedge 1$ determine a homomorphism

$$G:\hat{\pi}^*X \to \mathrm{colim}_k \hat{\pi}^*P_{-k}\wedge X$$

which is a natural transformation of coholomolgy theories, defined on Sp . When $X = S^0$ Lin's Theorem shows that G is an isomorphism (see [15]), hence G is an isomorphism, by the Eilenberg-Steenrod uniqueness theorem, when X is a finite spectrum. Now use corollary 7.2 to complete the proof.

We turn now to the proof of Theorem C. Recall that in the introduction we defined a map

$$\gamma_{t,\ell}:\pi^t X \to \mathrm{colim}_k \pi^t P_{-t-\ell}(X) .$$

Let q_1 and q_2 be the projections of $X\vee Y$ onto X and Y . Define $f:P_{-k}(X\vee Y) \to S^{-k}X\wedge Y$ to be the composite

$$P_{-k}(X\vee Y) \xrightarrow{p_{-k}} S^{-k}(X\vee Y)\wedge(X\vee Y) \xrightarrow{S^{-k}q_1\wedge q_2} S^{-k}X\wedge Y .$$

<u>Lemma 7.3</u> ([12])

The map of spectra

$$H:P_{-k}(X\vee Y) \to P_{-k}(X) \vee P_{-k}(Y) \vee S^{-k}X\wedge Y$$

with components $P_{-k}(q_1)$, $P_{-k}(q_2)$ and f is an equivalence.

Pf Let j_1 and j_2 be the inclusions of X and Y into $X\vee Y$. Define $g:S^{-k}X\wedge Y \to P_{-k}(X\vee Y)$ to be the composite

$$S^{-k}X\vee Y \xrightarrow{S^{-k} j_1\wedge j_2} S^{-k}(X\vee Y)\wedge(X\vee Y) \xrightarrow{i_k} P_{-k}(X\vee Y) .$$

Then the map $P_{-k}(X)\vee P_{-k}(Y)\vee S^{-k}X\wedge Y \to P_{-k}(X\vee Y)$, with components $P_{-k}(j_1)$, $P_{-k}(j_2)$ and g , is an inverse for H .

Lemma 7.4 ([12])

For $\alpha,\beta \in \pi^t X$, $X \in Sp$

$$\gamma_{t,0}(\alpha+\beta) = \gamma_{t,0}(\alpha) + \gamma_{t,0}(\beta) + S^{-t}\alpha\wedge\beta p_{-t} .$$

Pf Let $W:X \to X\vee X$ be the map with both components the identity. Then the composite

$$P_{-t}(X) \xrightarrow{P_{-t}(W)} P_{-t}(X\vee X) \xrightarrow{H} P_{-t}(X)\vee P_{-t}(X)\vee S^{-t}X\wedge X$$

has as its first two components the identity and p_{-t} as its last. The proof now follows.

Lemma 7.5 ([12])

The map $\gamma_{t,\ell}$ is a homomorphism if $\gamma \geq 1$.

Pf Use the previous lemma and the fact that

$$P_{-t-1}(X) \xrightarrow{\sigma_{-t-1}} P_{-t}(X) \xrightarrow{p_{-k}} S^{-t} \wedge X \wedge X$$

is part of an exact triangle to show that

$$\gamma_{t,}\ (\alpha+\beta) = \gamma_{t,\ell}(\alpha) + \gamma_{t,\ell}(\beta)$$

when $\ell \geq 1$.

Lemma 7.6 ([12])

If X is a finite spectrum, and $\ell \geq 1$, then $\gamma_{t,\ell}$ extends to a homomorphism

$$\gamma_{t,\ell}: \hat{\pi}^t X \to \hat{\pi}^t P_{-t-\ell}(X) .$$

Pf If $X \in C$ and $n \geq -k$, define the spectrum $P^n_{-k}(X)$ by replacing S^∞ with S^{n+k} throughout the definition of $P_{-k}(X)$. This works when $X \in Sp$ as well (see [12]). We define maps

$$\gamma^n_{t,\ell}:\pi^t X \to \pi^t P^n_{-t-\ell}(X)$$

just as before which are homomorphisms when $\ell \geq 1$. If X is a finite spectrum, then $P^n_{-k}(X)$ is finite and by Lemma 4.4 $\gamma^n_{t,\ell}$ extends to a homomorphism

$$\gamma^n_{t,\ell}:\hat{\pi}^t X \to \hat{\pi}^t P^n_{-t-\ell}(X) .$$

Following the proof of Lemma 4.7 we see that $\lim_n \hat{\pi}^t P^n_{t,\ell}(X) = \hat{\pi}^t P_{-t-\ell}(X)$, hence the homomorphisms $\{\gamma^n_{t,\ell}\}_n$ define a homomorphism

$$\gamma_{t,\ell}:\hat{\pi}^t X \to \hat{\pi}^t P_{-t-\ell}(X) .$$

Proof of Theorem C

If $X \in S$ and $\alpha \in \hat{\pi}^t X$, then $\gamma_{t+1,\ell}(S\alpha) = S\gamma_{t,\ell}(\alpha)$, hence Γ commutes with the suspension isomorphism in the two cohomology theories. If $1 \in \hat{\pi}^0 S^0$ is the unit, B is the isomorphism in Corollary 7.2 and G is the isomorphism in the proof of Theorem B, then $\Gamma(1) = B^{-1} \circ G(1)$. Since Γ and $B^{-1} \circ G$ are natural and commute with suspensions they agree on finite spectra. Thus Γ is an isomorphism on finite spectra since $B^{-1} \circ G$ is.

Lastly we show that Theorem B is false when $X = P_0$. Given a finite abelian group G, let BG denote the classifying space of G. Let $\hat{A}(G)$ denote the Burnside ring of G completed at its augmentation ideal. G. Segal made the conjecture for finite groups G

$$\pi^t BG = \begin{cases} \hat{A}(G) & t = 0 \\ 0 & t > 0 . \end{cases}$$

Let D_4 be the dihedral group with eight elements. Then ([1])

$$P_0 = B\mathbb{Z}/2_+ , \quad P_0 \wedge P_0 = B(\mathbb{Z}/2 \times \mathbb{Z}/2)_+ , \quad P_0(P_0) = BD_{4+} ,$$

$$[\hat{A}(\mathbb{Z}/2)]_2^{\wedge} = \bigoplus_2 \mathbb{Z}_2 , \quad [\hat{A}(\mathbb{Z}/2 \times \mathbb{Z}/2)]_2^{\wedge} = \bigoplus_5 \mathbb{Z}_2^{\wedge} ,$$

$$[\hat{A}(D_4)]_2^{\wedge} = \bigoplus_8 \mathbb{Z}_2^{\wedge} .$$

Adams, Gunawardena and Miller ([4]) verified Segal's conjecture when $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. Carlsson ([10]), using their results, proved the conjecture. Hence the cohomotopy exact sequence of the exact triangle

$$S^{-k} P_0 \wedge P_0 \to P_{-k}(P_0) \to P_{-k+1}(P_0) \to S^{-k+1} P_0 \wedge P_0$$

shows that

$$\operatorname{colim}_k \hat{\pi}^0 P_{-k}(P_0) \cong \hat{\pi}^0(P_{-1}(P_0)) .$$

Using the same exact triangle with $k = 1$, we see that $\operatorname{rank}_{\mathbb{Z}_2^{\wedge}} \hat{\pi}^0 P_{-1}(P_0) \geq 3$. However, $\hat{\pi}^0 P_0 = \bigoplus_2 \mathbb{Z}_2^{\wedge}$ ([15]) so $\hat{\pi}^0(P_0)$ cannot be isomorphic to $\operatorname{colim}_k \hat{\pi}^0 P_{-k}(P_0)$.

§8. The proof of Theorem D

In this section we will prove Theorem D using the spectral sequence constructed in section five.

For each $r \geq 1$ let A_r denote the subalgebra of A generated by Sq^{2^j} with $0 \leq j \leq r$. Recall from section six that

$$\Delta = \mathbb{F}_2[u, u^{-1}]$$

is a left A-module with its A-action defined by

$$Sq^j u^\ell = \binom{\ell}{j} u^{\ell+j} \quad .$$

Let $F_{\ell,r}$ be the A_r-submodule of Δ generated by u^j with $j < \ell$.

Lemma 8.1 ([16])

Given $\ell \in \mathbb{Z}$ define $I_\ell = \{i \in \mathbb{Z} : i \geq \ell \ , \ i \equiv -1(4)\}$. Then there is an isomorphism of A-modules

$$A \otimes_{A_1} \Delta/F_{\ell_1} \cong \bigoplus_{j \in I_\ell} \Sigma^j (A \otimes_{A_0} \mathbb{Z}/2) \quad .$$

Lemma 8.2

If $s \geq 1$ and $m \in \mathbb{Z}$, then

$$\mathrm{Ext}_{A_0}^{s,s-1}(\mathbb{Z}/2\ ,\ \Sigma^{4m}A\otimes_{A_0}\mathbb{Z}/2) = 0\ .$$

<u>Pf</u> For some free A_0-module F there is an isomorphism of A_0-modules

$$A\otimes_{A_0}\mathbb{Z}/2 \cong \mathbb{Z}/2 \oplus F\ .$$

Hence for $s \geq 1$

$$\mathrm{Ext}_{A_0}^{s,s-1}(\mathbb{Z}/2,\Sigma^{4m}A\otimes_{A_0}\mathbb{Z}/2) \cong \mathrm{Ext}_{A_0}^{s,s-1}(\mathbb{Z}/2,\Sigma^{4m}\mathbb{Z}/2) = 0\ .$$

<u>Lemma 8.3</u>

If $s \geq 1$, $m \in \mathbb{Z}$, then

$$\mathrm{Ext}_{A}^{s,s-1}(A\otimes_{A_1}\Delta,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) = 0\ .$$

<u>Pf</u> Each $F_{\ell,1}$ is bounded above. In fact $F_{\ell,1}$ is zero in dimensions $> \ell + 6$. Given $m,t \in \mathbb{Z}$ $s > 0$, we may choose ℓ so that

$$\mathrm{Ext}_{A_1}^{s,t}(F_{\ell,1},\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) = 0\ .$$

The change of rings isomorphism (see [3]) shows that

$$\mathrm{Ext}_{A_1}^{s,t}(F_{\ell,1},\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \cong \mathrm{Ext}_A^{s,t}(A\otimes_{A_1}F_{\ell,1},\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \quad .$$

Fix $m \in \mathbb{Z}$ and $s \geq 1$, then after apply Ext to the short exact sequence

$$0 \to A\otimes_{A_1}F_{\ell,1} \to A\otimes_{A_1}\Delta \to A\otimes_{A_1}\Delta/F_{\ell,1} \to 0$$

we see that for some ℓ

$$\mathrm{Ext}_A^{s,s-1}(A\otimes_{A_1}\Delta,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \cong \mathrm{Ext}_A^{s,t}(A\otimes_{A_1}\Delta/F_{\ell,1},\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \quad .$$

Lemma 8.1 shows that

$$\mathrm{Ext}_A^{s,s-1}(A\otimes_{A_1}\Delta/F_{\ell,1},\Sigma^{4m-1}A\otimes_{0}\mathbb{Z}/2)$$

$$\cong \prod_{j\in I_\ell} \mathrm{Ext}_A^{s,s-1}(\Sigma^j A\otimes_{A_0}\mathbb{Z}2,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \quad .$$

The change of rings isomorphism

$$\mathrm{Ext}_A^{s,s-1}(\Sigma^j A\otimes_{A_0}\mathbb{Z}/2,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \cong$$

$$\cong \mathrm{Ext}_{A_0}^{s,s-1}(\Sigma^j\,\mathbb{Z}/2,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2)$$

and Lemma 8.2 complete the proof.

Proof of Theorem D

First note that since the inverse system $\{P_{-4k-3}\wedge bo\}$ is cofinal in $\{P_{-k}\wedge bo\}$,

$$\text{holim}_k\, P_{-4k-3}\wedge bo \simeq \text{holim}_k P_{-k}\wedge bo\ .$$

Each of the spectra $P_{-4k-3}\wedge bo$ is two-complete which shows that $\text{holim}_k P_{-4k-3}\wedge bo$ is two-complete. Theorem 5.6 gives us a spectral sequence with E_2-term

$$\text{Ext}_A^{s,*}(\text{colim}_k H^* P_{-k}\wedge bo, \Sigma^{4m-1} H^* H(\mathbb{Z}))$$

converging strongly to

$$[S^{4m-1}H(\mathbb{Z}), \text{holim}_k P_{-4k-3}\wedge bo]_*\ .$$

Recall that $\text{colim}_k H^* P_{-k} \cong \Delta$, $H^* bo \cong A\otimes_{A_1} \mathbb{Z}/2$ and $H^* H(\mathbb{Z}) \cong A\otimes_{A_0} \mathbb{Z}/2$ ([3]), hence we may rewrite the E_2-term:

$$E_2^{s,*} \cong \text{Ext}_A^{s,*}(A\otimes_{A_1}\Delta, \Sigma^{4m-1} A\otimes_{A_0} \mathbb{Z}/2)\ .$$

Using the change of rings isomorphism,

$$\mathrm{Ext}_A^{0,0}(A\otimes_{A_1}\Delta,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) \cong \mathrm{Ext}_{A_1}^{0,0}(\Delta,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2) .$$

The map $\phi_m \in \mathrm{Hom}^0(\Delta,\Sigma^{4m-1}A\otimes_{A_0}\mathbb{Z}/2)$ defined by

$$\phi_m(u^\ell) = \begin{cases} Sq^{\ell+1-4m} & \ell \geq 4m-1 \\ 0 & \text{otherwise} \end{cases}$$

is an A_1-homomorphism, hence defines an element $\phi_m \in E_2^{0,0}$.
Lemma 8.3 shows that $\phi_m \in E_\infty$, thus defines a map of spectra

$$f_m : S^{4m-1}H(\mathbb{Z}) \to \mathrm{holim}_k P_{-4k-3}\wedge bo .$$

For each K let $\alpha_{-k}:\mathrm{holim}_k P_{-4k-3}\wedge bo \to P_{-4k-3}\wedge bo$ be the projection.
Then from the construction of f_m

$$f_m^* \circ \alpha_m^* \;: H^{4m-1}P_{4m-3}\wedge bo \;\to\; H^{4m-1}S^{4m-1}H(\mathbb{Z})$$

is an isomorphism. Thus

$$\alpha_{m*}\circ f_{m*}:\pi_{4m-1}S^{4m-1}H(\mathbb{Z}) \to \pi_{4m-1}P_{4m-3}\wedge bo$$

is onto. The homotopy of $\mathrm{holim}_k P_{-k}\wedge bo$ is well known (see [11]). We have

$$\pi_{4n+k}(P_{4n+1}{}^{\wedge}bo) \cong \begin{cases} \mathbb{Z}/2^{(k+3)/2} & K \equiv 3(8)\ k > 0 \\ \mathbb{Z}/2^{(k+1)/2} & K \equiv 7(8)\ k > 0 \\ \mathbb{Z}/2 & K \equiv 1,2(8)\ k > 0 \\ 0 & \text{otherwise;} \end{cases}$$

for each k and $n \in \mathbb{Z}$, the homomorphisms

$$\pi_{4n-1}P_{-4k-3}{}^{\wedge}bo \to \pi_{4n-1}P_{-4k+1}{}^{\wedge}bo$$

are onto. Hence

$$\pi_i\ \mathrm{holim}_k P_{-4k-3}{}^{\wedge}bo \cong \begin{cases} \mathbb{Z}_2^{\wedge} & i \equiv 3(4) \\ 0 & \text{otherwise.} \end{cases}$$

The homomorphism

$$f_{m*}:\pi_{4m-1}S^{4m-1}H(\mathbb{Z}) \to \pi_{4m-1}\mathrm{holim}_k P_{-k}{}^{\wedge}bo$$

is the completion homomorphism $\mathbb{Z} \to \mathbb{Z}_2^{\wedge}$. Thus the map

$$f = \prod_m f_m \in \prod_m [S^{4m-1}H(\mathbb{Z}),\ \mathrm{holim}_k P_{-4k-3}{}^{\wedge}bo]$$

induces the equivalence

$$(\bigvee_m S^{4m-1}H(\mathbb{Z}))_2^{\wedge} \simeq \mathrm{holim}_k P_{-4k-3}{}^{\wedge}bo\ .$$

Appendix

We shall prove Theorem 6.4 here. The proof is taken from notes of seminars given by Adams and Miller and is entirely due to Adams, Gunawardena and Miller. We include the proof only for the sake of completeness as their paper has not yet been circulated.

Let M be a left A-module. Recall from section six that Singer defines a left A-module

$$\Delta(M) = \mathbb{F}_2[u,u^{-1}] \otimes M$$

with left A-action given by

$$Sq^a(u^\ell \otimes m) = \sum_j \binom{\ell-j}{a-2j} u^{\ell+a-j} \otimes Sq^j m \; .$$

We give $\Delta(A)$ a right A-action by $(u^\ell \otimes b)Sq^a = u^\ell \otimes bSq^a$.

Lemma A.1

Let M be a left A-module. Then as left A-modules $\Delta(M) \cong \Delta(A) \otimes_A M$.

Pf Define a map $G:\Delta(A) \otimes_A M \to \Delta(M)$ by

$$G(u^\ell \otimes Sq^a \otimes m) = u^\ell \otimes Sq^a m \; .$$

Then G is well defined and an A-homomorphism. G is the required isomorphism.

If N is a left A_r-module, then $\Delta(N)$ is a left A_{r+1}-module. $\Delta(A_r)$ is a right A_r-module.

<u>Lemma A.2</u>

Let M be a left A-module. Then as left A_r-modules $\Delta(M) \cong \Delta(A_{r-1}) \otimes_{A_{r-1}} M$.

<u>Pf</u> The A_r-homomorphism $G:\Delta(A_{r-1}) \otimes_{A_{r-1}} M \to \Delta(M)$ defined as above is the required homomorphism.

<u>Lemma A.3</u>

As a left A_r-module, $\Delta(A_{r-1})$ is free on generators $u^{k.2^{r+1}-1} \otimes 1$, $k \in \mathbb{Z}$.

<u>Pf</u> First we will show that for all $k \in \mathbb{Z}, u^{k.2^{r+1}-1} \otimes 1$ is non-zero in $\mathbb{Z}/2 \otimes_{A_r} \Delta(A_{r-1})$. Let $0 \leq a < 2^{r+1}$ and $\ell+a = k+2^{r+1}-1$. Then

$$Sq^a(u^\ell \otimes 1) = \sum_j \binom{\ell-j}{a-2j} u^{\ell+a-j} \otimes Sq^j .$$

However

$$\binom{\ell}{a} = \binom{\ell-k.2^{r+1}}{a} = \binom{-a-1}{a} = \binom{2a}{a} = 0 .$$

Let $\underline{A_r}$ denote the augmentation ideal of A_r. We have shown that $u^{k.2^{r+1}-1} \otimes 1 \notin \underline{A_r}\ \Delta(A_{r-1})$, hence $u^{k.2^{r+1}-1} \otimes 1$ is non-zero in $\mathbb{Z}/2 \otimes_{A_r} \Delta(A_{r-1})$.

Given ℓ, let $u^{\ell} \otimes A_{r-1}$ denote the $\mathbb{Z}/2$-vector subspace of $\Delta(A_{r-1})$ generated by elements of the form $u^{\ell} \otimes b$, $b \in A_{r-1}$.

Since $\quad Sq^1(u^{k.2^{r+1}-1} \otimes c) = u^{k.2^{r+1}} \otimes c$

$$u^{k.2^{r+1}} \otimes A_{r-1} \subseteq \underline{A_r}\ \Delta(A_{r-1}) .$$

Suppose by induction we have shown that for some $m < (k+1)2^{r+1}-1$

$$u^{\ell} \otimes A_{r-1} \subseteq \underline{A_r}\ \Delta(A_{r-1})$$

whenever $k.2^{r+1} \leq \ell < m$. If $c \in A_{r-1}$, then the A_r-action on $\Delta(A_{r-1})$ shows that there are $y_j \in A_{r-1}$ and a $n \geq 0$ with

$$Sq^{m-k.2^{r+1}+1}(u^{k.2^{r+1}-1} \otimes c) = u^m \otimes c + \sum_{j=1}^{n} u^{m-j} \otimes y_j .$$

Thus $u^m \otimes A_{r-1} \subseteq \underline{A}_r\, \Delta(A_{r-1})$. Hence by induction

$$u^{\ell} \otimes A_{r-1} \subseteq \underline{A_r}\, \Delta(A_{r-1})$$

whenever $\ell \neq k.2^{r+1} - 1$, $k \in \mathbb{Z}$. If $0 < m < 2^r$ then $Sq^m \in A_{r-1}$. Then given $c \in A_{r-1}$ and $k \in \mathbb{Z}$

$$Sq^{2m}(u^{k.2^{r+1}-m-1} \otimes c) = u^{k.2^{r+1}-1} \otimes Sq^m c + $$
$$+ \sum_{j=0}^{m-1} \binom{k.2^{r+1}-m-1-j}{2m-2j} u^{k.2^{r+1}+m-1-j} \otimes Sq^j m \; .$$

Thus $u^{k.2^{r+1}-1} \otimes \underline{A_{r-1}} \subseteq \underline{A_r}\, \Delta(A_{r-1})$. We have shown that $\mathbb{Z}/2 \otimes_{A_r} \Delta(A_{r-1})$ is precisely the set of elements $u^{k.2^{r+1}-1} \otimes 1$, $k \in \mathbb{Z}$. An easy 'counting ranks' argument shows that $\Delta(A_{r-1})$ is freely generated over A_r by these elements. This completes the proof.

Recall that there is an A-homomorphism

$$\varepsilon : \Delta(M) \to \Sigma^{-1} M$$

defined by

$$\varepsilon(u \otimes m) = \begin{cases} Sq^{\ell+1} m & \ell \geq -1 \\ 0 & \text{otherwise.} \end{cases}$$

Proposition A.4

Let M be a left A-module. Then $1\otimes\varepsilon: \mathbb{Z}/2 \otimes_A \Delta(M) \to \mathbb{Z}/2 \otimes_A \Sigma^{-1}M$ is an isomorphism of $\mathbb{Z}/2$-vector spaces.

Pf First we note that

$$\mathbb{Z}/2 \otimes_A \Delta(M) \cong \operatorname{colim}_r \mathbb{Z}/2 \otimes_{A_r} \Delta(M) .$$

Lemma A.2 shows that

$$\mathbb{Z}/2 \otimes_{A_r} \Delta(M) = \mathbb{Z}/2 \otimes_{A_r} \Delta(A_{r-1}) \otimes_{A_{r-1}} M .$$

We write $\oplus_k u^{k.2^{r+1}-1} \otimes 1$ for the $\mathbb{Z}/2$-vector space generated by $u^{k.2^{k+1}-1} \otimes 1$, $k \in \mathbb{Z}$. Then by Lemma A.3

$$\mathbb{Z}/2 \otimes_{A_r} \Delta(A_{r-1}) \otimes_{A_{r-1}} M \cong \oplus_k u^{k.2^{r+1}-1} \otimes 1 \otimes_{A_{r-1}} M ,$$

hence

$$\mathbb{Z}/2 \otimes_{A_r} \Delta(M) \cong \oplus_k u^{k.2^{r+1}-1} \otimes \underline{A_{r-1}}\, M .$$

After taking direct limits

$$\operatorname{colim}_r \mathbb{Z}/2 \otimes_{A_r} \Delta(M) \cong u^{-1} \otimes \underline{A}\, M .$$

The map $1 \otimes \varepsilon$ is seen to be an isomorphism as claimed.

<u>Theorem A.5</u>

Let M be a left A-module. Then the induced homomorphism

$$\varepsilon_*:\mathrm{Tor}^A_{s,t}(\mathbb{Z}/2,\Delta(M)) \to \mathrm{Tor}^A_{s,t}(\mathbb{Z}/2,\Sigma^{-1}M) \quad .$$

is an isomorphism.

<u>Pf</u> Let C_S be a projective A-resolution of M. Then as A_{r-1}-modules each C_S is projective, (e.g. see [14]) hence flat. Given a short exact sequence of A-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0 \ ,$$

$$0 \to M_1 \otimes_{A_r} \Delta(A_{r-1}) \otimes_{A_{r-1}} C_S \to M_2 \otimes_{A_r} \Delta(A_{r-1}) \otimes_{A_{r-1}} C_S \to$$
$$\to M_3 \otimes_{A_r} \Delta(A_{r-1}) \otimes_{A_{r-1}} C_S \to 0$$

is short exact by Lemma A.3. Since colim_k is an exact functor

$$0 \to M_1 \otimes_A \Delta(A) \otimes_A C_S \to M_2 \otimes_A \Delta(A) \otimes_A C_S \to M_3 \otimes_A \Delta(A) \otimes_A C_S \to 0$$

is short exact. Thus Lemma A.1 shows that the A-modules $\Delta(C_S)$ are flat. Hence the homology of the chain complex $\{\mathbb{Z}/2 \otimes_A \Delta(C_S)\}$ is

$\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}/2,\Delta(M))$ (see [17]). The homomorphism ε induces an isomorphism of chain complexes

$$\{\mathbb{Z}/2 \otimes_A \Delta(C_s)\} \xrightarrow{1\otimes\varepsilon} \{\mathbb{Z}/2 \otimes_A \Sigma^{-1} C_s\} \ .$$

Since the homology of $\{\mathbb{Z}/2 \otimes_A \Sigma^{-1} C_s\}$ is $\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}/2,\Sigma^{-1}C_s)$, this proves the theorem.

<u>Theorem A.6</u>

Let M be a left A-module. Then the induced homomorphism

$$\varepsilon^{*}:\mathrm{Ext}^{s,t}_{A}(\Sigma^{-1}M, \mathbb{Z}/2) \to \mathrm{Ext}^{s,t}_{A}(\Delta(M), \mathbb{Z}/2)$$

is an isomorphism.

<u>Pf</u> Given a $\mathbb{Z}/2$-vector space V , let V^{*} denote its dual space. Then for any left A-module N ,

$$\mathrm{Ext}^{s,t}_{A}(N, \mathbb{Z}/2) \cong (\mathrm{Tor}^{A}_{s,t}(\mathbb{Z}/2,N))^{*}$$

(see [16]). Now use the previous theorem.

References

[1] Adams, J.F. Graeme Segal's Burnside ring conjecture. Bull. A.M.S. 6 (1982), pp.201-210.

[2] Adams, J.F. Operations of the n^{th} kind in K-theory, and what we don't know about $\mathbb{R}P^{\infty}$. L.M.S. Lecture Notes No. 11, Cambridge Univ. Press (1974), pp.1-9.

[3] Adams, J.F. Stable Homotopy and Generalised Homology. The Univ. of Chicago Press, 1971.

[4] Adams, J.F., Gunawardena, J.H.C., Miller, H. To appear.

[5] Atiyah, M.F. Characters and cohomology of finite groups. Inst. Hautes Études Sci. Pub. Math. No. 9 (1961), pp.23-64.

[6] Atiyah, M.F., MacDonald, I.G. Introduction to Commutative Algebra. Addison-Wesley 1969.

[7] Boardman, J.M. Conditionally convergent spectral sequences. Preprint.

[8] Boardman, J.M. Stable homotopy theory. Warwick Lecture Notes (1965).

[9] Bousfield, A.K. The localization of spectra with respect to homology. Topology 18 (1979) pp.257-281.

[10] Carlsson, G. Equivariant stable homotopy theory and Segal's Burnside ring conjecture. Preprint.

[11] Davis, D.M., Mahowald, M.E. The spectrum $P_{-\infty} \wedge bo$. Preprint.

[12] Jones, J.D.S., Wegmann, S.A. Limits of stable homotopy and cohomotopy groups. Preprint.

[13] Lewis, G., May, J.P., McClure, J.E., Steinberger, M. Equivariant stable homotopy theory. Springer Lecture Notes. To appear.

[14] Lin, T.Y., Margolis, H.R. Homological aspects of modules over the Steenrod algebra. J. Pure and Applied Algebra 9 (1977) pp.121-129.

[15] Lin, W.H. On conjectures of Mahowald, Segal and Sullivan. Math. Proc. Camb. Phil. Soc. 87 (1980) pp.449-458.

[16] Lin, W.H., Davis, D.M., Mahowald, M.E., Adams, J.F. Calculation of Lin's Ext groups. Math. Proc. Camb. Phil. Soc. 87 (1980) pp.459-469.

[17] MacLane, S. Homology. Springer-Verlag 1963.

[18] May, J.P. H^∞-ring spectra and their applications. Proc. Symp. Pure Math. Vol. XXXII part 2. A.M.S. (1978) pp.229-243.

[19] Milnor, J. On axiomatic homology theory. Pacific J. Math. 12 (1962) pp.337-341.

[20] Singer, W.M. A new chain complex for the homology of the Steenrod algebra. Math. Proc. Camb. Phil. Soc. 90 (1981) pp.279-293.