BEYOND THE OBVIOUS:

MENTAL REPRESENTATIONS AND ELEMENTARY ARITHMETIC

DEMETRA PITTA, M.Sc.

Ph.D Thesis in Mathematics Education

The University of Warwick

Institute of Education

JUNE 1998

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ACKNOWLEDGEMENTS

Here are my thanks and love to all of you that made this study possible:

Firstly my supervisor, my "friend and foe" Dr Eddie Gray. Without your patience, support, hard work, guidance and many arguments this dissertation would truly have not been possible. We both know how true this is! I would also like to thank Mrs Mareea Gray, you have been so supportive and kind to me and I would also like to apologise for all the inconvenience I have caused especially during the last two weeks of the study.

Teta, Miltos, Alexia, Malcolm, Katia, Alistair, Jeremy, Anne, Natalie, Sonia, Orlando, Ioannis, Matti, Gabriel, David and all the other children for allowing me to 'see' and 'hear' what was in your mind. This study is as much yours as it is mine. You were the ones that provided me with all the material and ideas to write. I thank you for this and I hope I was true to your thoughts. I would also like to thank your teachers, especially Synthia and Sophie. You were so good to me in organising my interviews, giving me further information about the children and allowing me to talk to them for substantial amount of time.

My parents, who I can not thank enough for the innumerable ways and uncountable sacrifices that made this possible. You did not only put up with me and my big and costly dreams, but you were there for me, giving your love and support in every step of the way. I would also like to thank my granny, your daily prayers and love were giving me the courage and strength to carry on.

My fiancé, for your patience and impatience, your love, encouragement and belief in me and not least your frequent and long phone-calls that kept me sane during stressful times and the long periods that we had to be apart.

My parents-in-law, for your kind words and support which were always giving me strength.

And finally my friends in Britain and Ireland, you were like a family to me providing your support and understanding especially in moments of crisis.

DEMETRA PITTA, JUNE, 1998

DECLARATION

I declare that the material within this thesis has not been presented in any other thesis. Chapter 5 contains material which has formed the basis of several papers:

Gray, E. M. & Pitta, D. (1996). Number Processing: Qualitative differences in thinking and the role of imagery. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol.4, pp 155–162. Valencia: Spain.

Pitta, D., & Gray, E., (1996). Nouns, Adjectives and Images in Elementary Mathematics. In L. Puig and A Guitiérrez (Eds.), *Proceedings of XX International Conference for the Psychology of Mathematics Education*, Vol.3, pp 35–42. Valencia: Spain.

Pitta, D., & Gray, E., (1997). In the Mind...What can imagery tell us about success and failure in arithmetic? In G.A. Makrides (Ed.), *Proceedings of the First Mediterranean Conference on Mathematics*, pp 29–41. Nicosia: Cyprus.

I acknowledge Eddie Gray's guidance and assistance which was seminal to the publication of each article. The distribution of effort within these papers was approximately 50/50. Dr Gray provided comments and additions to the structure of the papers and valuable assistance in the use of descriptive statistics. The balance of the thesis is my own work but again Dr Gray provided help with statistical analysis and insights into structure and presentation.

SUMMARY

This study seeks to answer the question: "What kinds of mental representation do children project and how may these be associated with their level of achievement in elementary arithmetic?". Drawing upon theories offering some explanation for the way in which arithmetical activity is transformed into numerical concepts and those that hypothesise the form and quality of mental representations the study suggests that qualitatively different kinds of mental representation may be associated with qualitatively different kinds of arithmetical behaviour.

The evidence is drawn from the classification and categorisation of data from two series of semi-clinical interviews carried out with children aged eight to twelve who were at extremes of numerical achievement. The first, a pilot study, largely concentrated on mental representations associated with numerical concepts and skills. Its results suggest that mental representations projected by children may have a disposition towards different kinds of mental representation which transcends arithmetical and non-arithmetical boundaries.

Issues raised by this study, in conjunction with a re-appraisal of the psychological evidence, informed the development of the main study. With a similar sample of children this considered the relationship between children's projections, reports and descriptions of mental representations in numerical and non-numerical contexts and in elementary arithmetic. Words, pictures, icons and symbols stimulated the projection of these representations. The evidence suggests that there is indeed a disposition towards the formation of particular kinds of mental representation. 'Low achievers' projected mental representations which have descriptive emphasis. 'High achievers', whilst able to do the same, also project those with relational characteristics, the frequency of which increases as the stimulus becomes more 'language like'. This provides them with the flexibility to oscillate between descriptive and abstract levels of thought.

The study indicates that qualitative different thinking in number processing is closely associated to a disposition towards qualitatively different kinds of mental representation. Its concluding comments suggest that these differences may have some considerable implication for the received belief that active methods may supply all children with a basis for numerical understanding.

CHAPTER 1

OVERVIEW OF THE PROBLEM

"Ουδεποτε νοει ανευ φαντασματος η ψυχη"
(Αριστοτελης, Περι Ψυχης, 431α, σ.16)
"Nobody thinks without a mental image"
(Aristotle, About the Soul, 431a, p. 16)

1.1 INTRODUCTION

The study seeks to provide additional insight into some of the qualitative differences in thinking seen in elementary arithmetic. Gray and Tall (1994) suggest that one reason for the difference may be the way in which numerical symbolism is interpreted. It can represent a process to *do* and a concept to *know.* This ambiguity has the potential to create a proceptual divide whenever there is a bifurcation in approach between the use of procedural approaches (manifestations of processes to do) and flexible proceptual methods (arising from concepts to know).

It is a conjecture of this study that the root of a proceptual divide in elementary arithmetic may be associated with the different kinds of mental representation that children project. It is a belief that these will affect the nature of the cognitive shift associated with the encapsulation of numerical procedures. Thus the study suggests that there is another divide, that which is seen in the qualitatively different kinds of mental representations projected by children at extremes of numerical achievement. It is conjectured that qualitative differences in children's interpretations of numerical symbolism are founded in their disposition towards these different kinds of mental representations. Therefore it is the thesis of this study that:

> *Children at the extremes of arithmetical achievement project different kinds of mental representation.* It arises from this that:
>
> *The qualitatively different thinking that children display in elementary arithmetic is linked to their predisposition towards qualitatively different kinds of mental representation.*

Within this chapter it is the intention to do four things; set out a background for the study, give some notion of its theoretical perspective, provide a brief outline of the hypotheses and core research questions and to provide an outline to guide the reader through the study.

The background for the study draws upon three influences: experience in school, the theories of cognitive development in mathematics — particularly those associated with Warwick — and a wish to reflect my own interests in psychology. This inevitably means that the theoretical influences of the study have been drawn from mathematics education (for example Steffe, Richards and Cobb, 1983; Skemp, 1986; Gray & Tall, 1994) and psychology, (Kosslyn, 1994; Paivio, 1986; De Beni & Pazzaglia, 1995). It is by drawing these influences together that the hypotheses of the study are formed.

By its very nature the study seeks to identify common forms of behaviour exhibited by children at extremes of arithmetical achievement. This is not to be interpreted that all children at these extremes will behave in the ways that are suggested. In supporting the hypotheses in the study we do not wish to lose sight of the individuals and neither do we wish to loose sight of children with the wide range of achievements that have not been directly investigated. It is the hope that our understanding of the longer term prognosis for all children may gain something from the investigation of the extremes. It may mean that some of the children's difficulties in encapsulating numerical processes may need to be viewed from a perspective that is broader than the mathematical one. If we are to understand what it is that children are doing differently and sharpen our perspective upon the qualitatively different behaviour that is seen it may be that interdisciplinary approaches should become more of a regular feature than they are. The first Director of the Mathematics Education Centre became the second President of the International Group for the Psychology of Mathematics Education. It is his tradition that this study hopes to build upon.

1.2 BACKGROUND TO THE STUDY

The developmental increase in children's thinking in primary school years is self-evident, not just in terms of knowledge they possess, but in the style and capacity of their reasoning and thoughts. However, while the development itself is apparent, the psychological mechanisms and the components that cause it are not fully identified or understood. If we could recognise the components and understand better these mechanisms and their associations we could probably further our ability to describe the qualitative difference between those people who fail in mathematics and those who succeed. It was these sort of issues that guided the rationale for this study. The synthesis of experience within school with literature on children's cognitive development always lead towards the same concern "*What are they really doing in their heads?*". Would the answer say anything about deep seated differences between them which offers one child support in a mathematical context but is a hindrance for another?

Spearman (1927) suggested that mathematical ability is a set of primary mental abilities which are required in certain combinations in any specific area. In this study this suggestion is transposed to the realm of mental representations which is regarded as one member of this set of abilities. The ability to use the ambiguity inherent in mathematical symbolism is another. Within this study the term mental representation, though closely tied to the notion of mental image, is used to avoid both the ordinary informal meaning of the word 'image' and psychological debate on the form of mental representation (see Section 3.3.3). A mental representation is identified as the product of imaging in any modality whether it be visual, verbal, olfactory, auditory and/or kinaesthetic. It is conjectured that the ability to form particular kinds of mental representation and to recognise the ambiguity of mathematical symbolism are linked; the quality of the cognitive shift which permits an individual to recognise the latter is dependent upon what it is the individual chooses to concentrate on in any numerical activity.

This individual selection process had been displayed during several encounters in school. James a 9-year-old identified as having difficulty in arithmetic ignored opportunities to use perceptual items to support his obvious reliance upon counting. He explained that he was trying to find the answer to 3+4 by "*seeing fingers*" in his head. There was some external evidences supporting his claim. The elongation of the words "*o-n-e, t-w-o, t-h-r-e-e* ..." and the exaggerated movement of his eyes, which did not appear to be seeing in the real sense, and the movement of his head from left to right all pointed to the fact that he was counting something. Observations of this sort are regular in the literature (Steffe, Richards, von Glasersfeld and Cobb, 1983; Gray and Tall, 1994, 1995) but for James the procedure was causing evident tension which is largely unreported in the literature. So why was he doing it this way? The answer lay in his perception of what others were doing. He "*wanted to be like the clever children* [who] *did things in their heads*". It appeared to James that counting fingers in his head was a natural transition from counting using real fingers. In his response to number words and symbols James was re-creating mentally a counting activity which involved using real fingers. He did not realise that the 'clever' children were doing things differently.

In a different school another child, the same age as James, faced different sorts of tensions:

> "I find it easier **not** to do it [simple addition]with my fingers because sometimes I get into a big muddle with them [and] I find it much harder to add up because I am not concentrating on the sum. I am concentrating on getting my fingers right...which takes a while. It can take longer to work out the sum than it does to work out the sum in my head. "If we don't [use our fingers] the teacher is going to think, 'why aren't they using their fingers... they are just sitting there thinking'... we are meant to be using our fingers because it is easier... which it is not".
>
> (Amanda, age 9)

This implicit recognition that there may be some conflict with an interpretation of numerical activity as the external manipulation of physical 'things' as opposed to the internal manipulation of mental 'things' placed some sort of perspective on James's comments. That 'things' that were mentally manipulated in an efficient way qualitatively different is implicit in the Gray & Tall study, they are also implicit in the notion of "concept image" (Tall & Vinner, 1981). More specifically, although Gray and Tall's theory did not account for mental representations, their role seemed to be implied. The fact that those who are successful in arithmetic make use of the numerical symbol whereas those who have difficulties seem to be using more perceptual and active counting procedures was taken as an indication of differences in their "internalisations". Investigating what kinds and uses of mental representations influence the quality of thinking and subsequently achievement in early number arithmetic seemed to be a question worth exploring. In the context of elementary arithmetic it leads to an important question:

> *What is it that, in the absence of perceptual items, children see in the mind's eye or hear in the mind's ear and what are the consequences of any differences for children's arithmetical behaviour?*

The importance of meaningful counting as a basis for arithmetical development appears to be beyond question. Research evidence now provides an integrated picture of this development. It ranges from the development of early counting skills (for example, Fuson, 1982; Gelman & Gallistel, 1986), the qualitative changes associated with counting units, (Steffe *et al*, 1983), children's ability to solve word problems in elementary arithmetic (for example, Carpenter & Moser, 1982), the results of instruction in the development of higher order strategies (for example, Thornton, 1990), the order in which children acquire number concepts (Denvir & Brown, 1986a, 1986b), and the role that symbolism plays in this development (Gray & Tall, 1994). A common feature that emerges from this literature is the observation that for some the act of counting is a stepping stone to higher order thinking but for others is becomes a hurdle which is hard to jump.

Encompassing the development of this meaning is the way in which actions on concrete objects are the source of the abstractions that are represented through symbolism. It is hypothesised that dynamic actions become conceptual entities through processes variously described as 'interiorisation' (Piaget, 1965), 'encapsulation' (Dubinsky, 1991) and 'reification' (Sfard, 1991). However, these theories share a common weakness. They identify cognitive processes associated with a cognitive shift but they have not been subjected to 'cognitive decomposition' (used after Dubinsky, 1992). We know that the process takes place and Piaget (1973) has indicated that the key which projects actions to thought or mental representation is 'reflective abstraction'. Intertwined with the notion of reflective abstraction is a sense of the cognitive shift associated with the mental activity which reconstructs at a higher level everything drawn from a co-ordination of the actions. This of course leads us to the second question associated with the study:

> *Are different kinds of mental representation better linked to the ability to encapsulate mathematical actions and do children indicate a predisposition towards those that support it?*

In the field of cognitive development, mathematics education may be suffering from its tendency to 'isolate' itself from other disciplines. Although in a research sense mathematics educators may have looked towards psychology and mathematics to define what may be seen as scholarly work, the accelerated trend since the 1960s has been for mathematics education to develop its own body of research literature. Links with other disciplines such as psychology are not as strong as they could have been. Kilpatrick (1992) draws our attention to the ambivalent way in which mathematics educators receive the discourse of psychologists. When sharing a common interest they are welcomed but when pursuing their own concerns they are ignored. This is a theme that has been expanded upon by Golding (1992) although he does extend it to indicate that even in mathematics education, "we... often do not learn sufficiently from each other, build sufficiently on each others perspectives, or give each others work sufficient acknowledgement" (p. 237).

It was an attempt to make these links, to build upon other perspectives, particularly in the field of psychology, that provided the third aspect of the rationale for this study. The consensus is that mental representations are a significant component of cognition but it is their relationship with children's achievement in elementary arithmetic which provides the third question that led to the study:

> *To what extent may an interdisciplinary view of mental representation provide insight into qualitatively different forms of numerical behaviour?*

1.3 DEFINING MENTAL REPRESENTATION

Eysenck & Keane (1995) noted that research on mental representations has been one of the most explored areas in psychology in the past two decades. Indeed, philosophers, have been considering the issue for centuries. Aristotle, claimed that:

> No soul thinks without a mental image (Aristotle)

The more recent exploration has caused problems. Much of the research has concentrated on the nature of these mental representations and whether or not they have been visual or language like. In summarising the current consensus they give two points of view. First, arguments to prove that one form of representation are redundant is a waste of time. Secondly that the richness of human cognition requires different representational constructs to describe it.

These issues are ones that this study will avoid. It takes a broad view of the nature of mental representation and as has been indicated earlier, though closely tied to the notion of mental image, within this study the term **mental representation** is used to describe:

> *a mental reference which, divorced from the objects that give it a place in the real world, is the product of imaging in any modality whether it be visual, verbal, olfactory, auditory or kinaesthetic.*

Referring to mental representations provides a major benefit for this study: it is inclusive of both imagery and propostional representations but due to the nature of the study it will be very difficult, if not impossible, to identify the difference between 'verbal images' and 'propositional representations'. Therefore, referring to verbal mental representations will allow us to argue our case without getting caught up in the ongoing debate about the nature of 'images' and 'propositional representations'.

1.4 PURPOSE OF THE STUDY

One may ask, since we were not intending to answer the question of the format of these mental representations how may these mental representations be discussed and what benefit would derive from limited considerations of format? Irrespective of this issue there are other aspects of the mental representation that have not been widely discussed which may prove important to our understanding of cognition in elementary mathematics.

Tall (1994) suggested that:

> "To be successful in mathematics, it is desirable to have a rich mental representation of concepts. A representation is rich if it contains many linked aspects of the concept. A representation is poor if it has too few elements to allow for flexibility in problem solving." (Tall, 1994, p. 32)

Such a notion suggests so many questions not least that which relates to the notion of 'rich mental representation' and its comparison with a poor one. In the context of elementary mathematics do we really know what it is to possess rich mental representations and how these are linked to objects?

On a theoretical basis, this study will address the extent within which different kinds of mental representations possessed by individuals may be linked to the qualitative differences that are associated with procedural and proceptual thinking. Understanding the role of mental representation may not only provide answers to where the problem of mathematical achievement lies but it may also provide a new dimension of what the source of the problem might be.

In doing this the study draws upon a theoretical background which is embedded in the psychology of mathematics education, with particular reference to cognitive development, and within psychology.

1.5 THEORETICAL BACKGROUND

The best chance of studying the development of mathematical knowledge may be best understood by considering children (Piaget, 1971). Drawing a distinction between this form of knowledge and physical knowledge Piaget indicated that mathematical knowledge is abstract whilst physical knowledge, "knowledge based on experience in general — is concrete" (p. 16). It was his belief that the abstraction that is mathematical knowledge stemmed not from abstraction from the objects themselves but from the abstraction of the actions carried out upon the objects.

The Piagetian notions of cognitive development in mathematics are underpinned by the notion of 'reflective abstraction'. This notion, which was deemed to reflect the process through which an action was projected to thought or mental representation and the mental activity associated with the reconstruction of the co-ordinations of actions, was essentially seen as a self-referential system which involves the construction of relationships between and amongst objects.

The theme of reconstructing knowledge is implicit within this study. Though we do not directly investigate the way in which this is done, we do investigate the prediposition that children may bring to it. In this way it falls into a theoretical perspective that tries to account for the cognitive shift associated with abstractions from actions carried out on objects (Dubinsky 1992; Sfard, 1992). But its essence is qualitative differences in thinking that may arise from relational and non-relational constructions. Thus the work of Skemp (1977, 1986) is important in this context. Not only does he allude to qualitative differences in thinking, relational and instrumental forms, but he also provides a framework for considering the role of numerical symbolism in concept formation. This is a theme later carried forward by Gray and Tall (1994). Their notion of procept leads to a hypothesised proceptual divide to account for the qualitative differences that they observed emerging from interpretations of mathematical symbolism.

Elementary arithmetic remains a sound area of study in this context because abstraction from the use of physical objects forms the background for the conceived cognitive development of simple arithmetic. The literature associated with the development of number concepts amongst young children indicates that counting plays a sophisticated and central role in the procedural encapsulation of number (e.g. Steffe *et al*, 1982; Fuson & Hall, 1983). Through a co-ordinated series of actions associated with counting, an object of thought is created, for example '5', which is associated with verbal and written symbolism. But the full realisation of the function of the symbol is dependent upon the way in which it may be used independently of any exemplar or embodiment (Skemp, 1979).

The manner through which the child's understanding of this notion is mediated through their mental representations is pivotal in this study. Some theoretical perspectives offered by psychology suggest that children's difficulties with elementary arithmetic may be ascribed to limitations in working memory (Geary, 1990). Such a view does not account for the extraordinary feats of memory that may be seen in some of children (see for example, Gray & Tall, 1995). It is for this reason that the study looks closely at kinds of mental representation. A theoretical perspective is drawn from the work of Paivio (1990) and Kosslyn (1981, 1994) to examine the nature of mental representations, from De Beni & Pazzaglia (1995) to form a platform for analysis of different kinds of mental representation, and the work of Baddeley (1966, 1986) and Hitch (1978) to examine the role of memory and its links with arithmetic.

1.6 STATEMENT OF HYPOTHESES

The central thesis of this study is that:

Children at the extremes of arithmetical achievement project different kinds of mental representation.

Since children at different levels of numerical achievement have been shown to project qualitatively different forms of thinking this thesis may be reworded to state:

The qualitatively different thinking that children display in elementary arithmetic is linked to their predisposition towards qualitatively different kinds of mental representation.

In order to confirm this thesis the study proposes to address in a qualitative way the following hypotheses:

- children at extreme levels of numerical achievement project different kinds of mental representations,
- children project qualitatively different kinds of mental representation to support their thinking in elementary arithmetic and
- different kinds of mental representation are associated with qualitatively different kinds of arithmetical thinking.

It is from such hypotheses that it will be conjectured that a predisposition towards particular kinds of mental representations will inhibit the encapsulation of arithmetical procedures as numerical objects. It may be further conjectured that a predisposition towards other kinds of mental representation will facilitate it. Although this conjecture is not directly tested within the study it is one that should follow from the force of argument within it.

To respond to these hypotheses the study has three main components:

(i) First it considers the written and verbal projections, reports, descriptions and interpretations that children give for mental representations triggered by objects, words, pictures, icons and symbols.

(ii) Secondly it considers the 'explanation' given to the uninitiated for numerical and non-numerical words.

(iii) Thirdly it considers the approaches children use in elementary arithmetic and what it is they claim to "see" and "hear" in their mind when they are dealing with elementary number combinations.

1.7 METHODOLOGICAL CONSIDERATIONS

The study will concentrate on different kinds of mental representation and the different uses to which individuals put these mental representations in an arithmetical context. Within such a framework we will draw analogies and links between mental representations from an arithmetical context and mental representations associated with words, pictures, icons and mathematical symbols. The desire is to link numerical procedures and concepts with different kinds of mental representations that children possess and the uses to which they are put. It is an intention to provide some explanation for the interaction of these three elements and their effect on success and failure in arithmetic.

It cannot be claimed that any one methodology guided the development of this study from the outset. Indeed, it cannot be claimed that any one research question did. The final study grew out of issues arising from a preliminary study and a pilot study but there were some overriding factors that guided the work:

(i) how we account for differences in children's arithmetical behaviour. This may be seen to have direct links with actions interiorised as concepts (Piaget, 1965), different forms of mathematical understanding (Skemp, 1977) and qualitatively different forms of mathematical thinking (Gray & Tall, 1994), and

(ii) the way in which children use their knowledge. This inevitably falls within the information processing paradigm (see, for example, Davis, 1983). However, an investigation which seeks to discover how this knowledge is constructed and the way in which mental representations may contribute to this construction draws upon a constructivist methodology.

As a result of the preliminary investigations these initial considerations were supported by two more:

(iii) how different kinds of mental representation may be classified. This draws upon psychological influences (see, in particular, De Beni & Pazzaglia, 1995), and is supported by a phenomenological orientation and

(v) the way in which children deal with elementary arithmetic and therefore the methodology draws on approaches used by Carpenter and Moser (1981), Seigler and Jenkins (1989) and Gray (1991).

Of course the investigation of mental representations is fraught with difficulty. They belong to an environment which is confined to the individual. They are restricted to a world no one else can enter and efforts to consider them are fraught with difficulties. Our understanding relies upon words and/or pictures but because of their disguised nature it is only possible to make conjectures about them. Well wrapped possessions, they may be covered in many layers and sometimes found as discrete packages. We may believe it is possible to shake the package to find out what is inside, but by doing this we risk breaking it. The pitfalls, particularly in terms of operational definitions and interpretation, are clearly identified by Pylyshyn (1973).

In developing the method used within this study there were two main departures from the main psychological standpoints which has investigated themes associated with mental representations. The first was a decision not to embrace the notion of a clinical experiment. Drake's (1996) view that results from clinically designed experiments carried out in the laboratory have not been easily translated to the classroom was one which guided methodological deliberations. Giorgi (1987) suggested that the study of imagery needed a radical shift from the science-based or quantitative model indicating that a phenomenological method may be the best way to understand the meaning of human phenomena. The second related to the the scope of the study which wished to place a broader perspective upon the notion of mental representation. There was the sense that much of the psychological literature created its own artificial situation by requesting subjects to 'imagine', 'hear' or 'see' things in the mind. Although in these contexts the issues may have been more easily investigated it seemed that questions of this form would not be answering "what is really happening" in the children's mind. From the very beginning a major component in dealing with objects was excluded. To make this point clearer, by asking subjects to "visually imagine" they may have been asked to do something that they do not normally do or use.

By placing a qualitative perspective on the analysis of the results the study does run the risk of criticism from those who see only the benefits of the quantitative perspective. For example Ahsen (1989) argues that a qualitative methodology only deals with the surface details of images. Whereas scientific approaches, in contrast to the simplistic qualitative approaches, enable both surface details and the deeper layers to be considered at the same time.

The attempt to investigate both "visual and verbal mental representations" is not infallible. Since the method involves both immediate and more prolonged reflection on the items presented it is possible that children's responses may not be providing just the verbal or visual mental representations but descriptions, interpretations of mental representations, beliefs, or possibly descriptions of other mental processes of the mind that are not the actual mental representations. However, even these pieces of information have the potential of being informative. Mental representations mediated by description is an accepted methodology within psychology (Kosslyn, 1980; Paivio, 1986, De Beni & Pazzaglia, 1996). In a slightly different form they have formed the backbone for much of the research into children's strategies in elementary arithmetic (Steffe, *et al*, 1983; Carpenter and Moser, 1982; Seigler & Jenkins, 1989; Gray & Tall, 1994). Combining the two forms of investigation within the current study will, it is hoped, provide further insight into children's qualitatively different thinking.

1.7 OVERVIEW OF THE STUDY

Besides the current chapter there are nine others within this study.

The first section of Chapter 2 presents a framework from which the nature of a mathematical concept may be considered. It then examines theories which attempt to account for the creation of these concepts through the interiorisation of mathematical actions. After examining cognitive construction within elementary arithmetic particularly with reference to Steffe *et al* (1983) it considers the role of numerical symbolism and the contribution it may make to qualitatively different forms of numerical understanding.

The interdisciplinary nature of the study is brought to the fore within Chapter 3. The chapter develops three main themes; contemporary psychological standpoints on the structure and process of memory, the nature of mental representations and finally a context for both of these within elementary arithmetic, in order to provide a unified whole as a background for the investigation that is to follow.

Chapter 4 considers the methodology and is to be seen in conjunction with Chapter 6 which considers in more detail the method applicable to the main study. The purpose of Chapter 4 is to consider the several influences which have guided the development of the method used in the study. It sets out the research questions that arise from the considerations within this current chapter, examines issues of reliability and validity within a study of this sort, and outlines the way in which the samples of 'high' and 'low achievers' in arithmetic were selected and the way in which data was collected. Central to the issue of data collection is the way in which mental representations may be evoked. Thus the chapter outlines the way in which verbal and visual stimuli were used to do this before presenting issues associated with the numerical component of the study. Because of the complexities associated with data collection and analysis the chapter largely confines itself to general methodological issues and those that related to the preliminary and the pilot studies. It is these that form the focus for the next chapter.

Chapter 5 first examines the small preliminary study and considers the way in which this was used to inform the development of a pilot study. The major component of the chapter is devoted to the collection and analysis of data in this study. First the chapter examines the children's approaches to elementary arithmetic. It then considers children's mental representations associated with the arithmetical phase and indicates the differences that emerge between the perceptual and figural representations that seem to dominate the 'low achievers'' arithmetical activity and the symbolic representations that support that of the 'high achievers'. After considering the way in which responses to free context numerical and non-numerical items verbal and visual were classified the descriptive analysis of results is placed within a context which takes account of the children's qualitatively different approaches to the arithmetic. The analysis suggests that children who display qualitatively different forms of mathematical thinking called upon different kinds of mental representation to support this thinking. The conclusions to the chapter suggest that children may be predisposed to project different kinds of mental representation that transcend arithmetical and non-arithmetical boundaries.

Initially Chapter 6 re-examines some of the aspects considered in chapter 4 and how these were blended from the experience of the pilot study into an appropriate method for the main study. An issue of central importance within this chapter is the way in which the work of De Beni and Pazzagalia (1996) was used to form a core for the classification of different kinds of mental representation. The chapter argues that the kinds of mental representation identified by De Beni and Pazzaglia do not account for those mental representations associated with more abstract ideas. Consequently the chapter introduces the notions of 'generic' and 'proceptual' mental representations.

Chapters 7, 8 and 9 report the findings of the main study. Whist the analysis associated with the first two presents an overall model for linking qualitatively different kinds of mathematical behaviour with qualitatively different kinds of mental representation, Chapter 9 examines this model and the way it may be considered with individual children.

Chapter 7 reports the responses given by two groups of children — 'high' and 'low achievers' in mathematics — to the visual and verbal free-context phases of the main study. The classification and analysis of the children's responses is considered in sections which separately consider the visual phase, the verbal phase, the similarities and differences between these two phases, visual imagery, the children's explanations of different items and finally how the quality of mental representation may change with age. The conclusions to the chapter suggest that there is a qualitative difference in the kinds of mental representations projected by children at extremes in numerical achievement. Those identified as 'low achievers' give 'general', 'episodic' and 'specific' mental representations which are 'descriptive' in nature. 'High achievers' appear to have greater choice available to them. The mental representations that they project are related to the nature of the stimulus and have the potential to provide descriptive or relational mental representations which have 'generic' and proceptual' qualities.

With Chapter 8 the arithmetical component of the main study is considered. It reports on responses given by the two groups to visually and verbally presented arithmetical combinations at three levels of difficulty. The chapter considers children's strategies, the representations used and their modality and the nature of the object used to support thinking. The chapter indicates that the responses of the 'high achievers' are associated with 'abstract' mental representations which utilise the flexibility of numerical symbolism. Those of the 'low achievers' suggest that they rely on both external and mental representations (perceptual, figural, verbal counting) which are evolved from the general number sequence. The main components of these representations are the counting act and countable objects (physical or mental).

Chapter 9 gives four indicative case studies of four children. Two are at the extreme of ages and achievement, two are of the same age but extremes of achievement. The purpose of this chapter is to give in a holistic and uninterrupted fashion the behaviour and responses of real children across all the items used in this study. This chapter shows that 'high' and 'low achievers' have different qualities of mental representation and thinking. The bifurcation is apparent in both numerical and non-numerical tasks and it seems to increase as the children get older.

Chapter 10 draws together the findings of the study which concludes that qualitatively different thinking in arithmetic is associated with qualitatively different kinds of mental representation. Four themes are developed from this conclusion. The first considers the implications of the results for theories associated with cognitive development in mathematics, the second considers the relationship between the different kinds of metal representation and memory whilst the third reconsiders the limitations of the study before making recommendations for further study.

A Glossary of common terms is included before the references. The Appendix, subdivided into three parts, presents the data obtained from classification of children's reponses. These appendices are used for the construction of figures and tables within the study.

*

CHAPTER 2

CONSTRUCTING ARITHMETICAL KNOWLEDGE: PROCESSES AND CONCEPTS

"Το γαρ αυτο νοειν εστιν τε και ειναι" (Παρμενιδης ο Ελεατης)
"What 'exists' is a 'concept'" (Parmenidis Eleatis)

2.1 INTRODUCTION

Over 2000 years ago Parmenidis stated that "what 'exists' is a 'concept'". In mathematics we often refer to numerical concepts and their development but these are not like any other concepts that exist "out there" and are readily available for the individual to grasp. Numerical concepts cannot be touched nor can they be seen. They do not exist with concrete features that can be perceived through any sensory modality and yet they "do exist". We talk about them, manipulate them, write about them, symbolise them. They serve the purpose of compressing multiple ideas into one, of forming a platform for identifying relationships, building new ideas, communicating and yet they remain intangible, a figment of the mind yet powerful and real. How is this possible? How may an individual construct a concept of something that does not exist? Perhaps more importantly, why do some do it with relative ease and yet others seem to have extraordinary difficulty?

The construction of arithmetical knowledge is the central platform upon which this study is based. However, it takes the view that hypothesised notions of development may not fit reality particularly if children at extremes of the achievement spectrum are considered. Nevertheless, whilst it may be trying to provide further insights into qualitative differences in thinking that may provide a two pronged view of children's mental representations, it takes as a starting point contemporary theories on the formation of mathematical concepts, memory and mental representations. Two issues are central to the themes that will be discussed as a result of the empirical evidence:

- How may we account for cognitive development in arithmetic?

- What role may memory and mental representation play in this development?

The first of these themes is considered largely through reference to literature within the field of mathematics education. The second draws mainly from the field of psychology. In the field of mental representations, it became apparent that although workers within mathematics education use many notions that are taken from psychology, both the contexts and the insights from that field of study were, until very recently, largely ignored. It is from this integrated platform that the empirical work of this study will be examined.

By focusing on processes and objects the first of the two issues is addressed within the current chapter. After examining a framework to establish the nature of a mathematical concepts the chapter considers the way in which the objects which give rise to these concepts are formed. Building upon the notions of 'interiorised actions' (Beth & Piaget, 1966; Sfard, 1991; Dubinsky, 1992), forms of mathematical knowledge (Skemp, 1976; Hiebert & Lefevre, 1986) and the consequences of qualitatively different forms of mathematical behaviour (Gray, 1991; Gray & Tall, 1994), the discussion within the chapter is completed by considering the nature of divergent thinking ascribed to children's qualitatively different thinking in elementary arithmetic. Thus a context is prepared for the second issue, memory and mental representations which, together with associated literature from within mathematics education, forms the focus of Chapter 3. It is in such a way that an integrated approach to the theoretical perspectives is prepared prior to the later empirical evidence which indicates that children who display qualitatively different thinking in elementary mathematics project qualitatively different kinds of mental representation.

2.2. MATHEMATICAL CONCEPTS AND BEHAVIOUR

Tall (1995) has indicated that the construction of mathematical knowledge may be seen as a two-pronged affair grounded in the learner's interaction with the environment. One, linked with perception and manipulation, is hypothesised to be a visuo-spatial to verbal deductive transformation. The other builds upon successive process-to-concept encapsulations using manipulable symbols. Very different types of mathematics follow from the use of visual clues on the one hand, and the representation of processes by symbols on the other. It has been more usual to associate mental representations, particularly in the field of visualisation with the first of these two strands, the geometric (see for example, Lean & Clements, 1981; Hershkowitz, 1989; Clements & Battista, 1990). However, Krutetskii, (1976) Fennema and Tatre, (1985), Presmeg, (1986a), are examples of researchers who have considered mental representation in non-geometrical contexts. This study continues in the latter tradition and considers the association between different kinds of mental representations and the construction of elementary arithmetical concepts.

The early development of arithmetic has a physical counterpart originating in the real world so it has visual elements. We only have to think of counting to see this. However, within arithmetic, actions on physical objects can lead to the development of procedures through which processes are named and symbolised and conceptualised.

Vergnaud (1988) has suggested that different kinds of mathematical behaviours and different levels of such behaviours are tied to understanding of mathematical concepts and that such understanding is only implicit through a child's behaviour. Identified as a mental representation that possesses criteria for membership and properties which specify its relationship with other concepts, the notion of concept supports the compression of knowledge into meaningful 'chunks' which can be defined, interpreted, related and reasoned with. Not only may they form a basis for communication but through conceptual combination they may act as building blocks for higher order concept.

In a mathematical context, Vergnaud (1988) identified a concept as a triplet set:

(i) the set of situations that make the concept meaningful in a variety of aspects,

(ii) the set of operation invariants (properties, relationships, objects, theorems in action) that are progressively grasped by students in a hierarchical fashion and

(iii) the set of linguistic and non-linguistic symbols that represent those invariants and are used to point at them, to communicate and discuss about them, and therefore to represent situations and procedures. (Vergnaud, 1988, p. 45)

However, just because a concept exists and a child uses it does not mean that a child is fully aware of the relationship between the ideas and the way they behave. This theme was elaborated upon by Skemp (1971):

> The criterion for having a concept is not that of being able to say its name, but that of behaving in a way indicative of classifying new data according to the similarities which go to form this concept. (Skemp, 1986, p. 26).

In elaborating on the distinction between the idea and its name Skemp suggested that naming, which is identified as the sound, sign or symbol associated with the concept, can play a useful, indeed sometimes essential part, in the formation of new concepts and its communication. Hearing the same name in connection with different experiences helps us to collect ideas in our minds and "helps us abstract their intrinsic similarities" (Skemp, 1986, p. 23). Seeing concepts in two forms — those derived from our sensory and motor experiences of the outside world and those abstracted from other concepts — Skemp sees those that are "more abstract" being "more removed from experience of the outside world" (Skemp, 1986, p. 24). Within mathematics, concepts were far more abstract than those of everyday life and that the direction of learning was, for the most part, in the direction of still greater abstraction.

The interrelationship of concepts, Skemp calls a "schema" (Skemp, 1986, p. 37). Functioning as the integrator of existing knowledge, a schema is a tool for future learning and an enabler for "relational" understanding. As such, it may be considered as a major instrument in adaptability. However, whilst a schema may be the most effective organiser of existing knowledge, its very strength may be the source of its potential downfall; a strong tendency may emerge towards the self-perpetuation of existing schema. It may then be necessary to change the structure of the schema. This may be difficult and if it fails, the new experience can no longer be successfully interpreted. Adaptive behaviour may break down.

> The central importance of the schema as a tool of learning means that inappropriate early schemas will make the assimilation of later ideas much more difficult, perhaps impossible.
>
> (Skemp, 1986, p. 48)

2.3 CONCEPT DEVELOPMENT AND ELEMENTARY ARITHMETIC

2.3.1 A Piagetian Perspective

The notion of numerical concepts being formed from actions with physical objects forms the background for the conceived cognitive development of simple arithmetic (see, for example, Piaget, 1965; Steffe, *et al*, 1983; Kamii, 1985; Gray & Tall, 1994). However, although we know that there is a cognitive shift between carrying out actions on objects and the formation of numerical concepts, how such a process takes place remains the subject of theory and open to debate. We know the process takes place but we do not know how it takes place.

Piaget believed that learning as well as performing mathematics was a matter of active thinking and operating in the environment. It was not a matter of passively noting or even memorising what was presented — activity *with* objects was seen to be indispensable for the comprehension of arithmetical relations. This activity was strongly linked to 'physical experience' which "consists of acting *on* objects to discover the properties of the objects themselves" (Piaget, 1973, p. 80). Here we have further reference to the two strands of mathematical growth: the verbal/sequential giving rise to arithmetic and later symbolic developments such as algebra, and the perceptual/manipulative associated with early spatial ideas, emanating from the same source: "elementary mathematics begins with perceptions of and actions on objects in the external world" (Tall, 1995. p. 61).

Piaget believed that the ability to think mathematically was based on basic processes, the 'logico-mathematical' experiences, which consisted of gathering information "not from the physical properties of particular objects, but from the actual actions (or more precisely their co-ordinations) carried out by the child on the objects" (Piaget, 1973, p. 80). He saw the co-ordination of such actions as the roots of mathematical structure and his concern was the way such co-ordinations became mental operations. He spoke of "actions that are destined to become interiorized as operations" (Beth & Piaget, 1966, p. 251). Such 'operations', which could be carried out in thought as well as physically, were identified as "an action that can be internalised" (Piaget, 1971, p. 21) to become "thematised objects of thought or assimilation" (Piaget, 1985, p. 49).

2.3.2 Reconstructing knowledge

The fundamental Piagetian perspective is that new knowledge is, in part, constructed by the learner through the use of "active methods" and these "require that every new truth to be learned be rediscovered or at least reconstructed by the student" (Piaget, 1976, p. 15). Such a view promoted the notion that Piaget was "the most systematic theorist of constructivism" (Vergnaud, 1987, p. 43).

Constructivism may be seen to be "an abstract philosophical stance about knowledge and its relation to the world and to people's attempts, through their experiences, to try and rationalise the world" (Jaworski, 1988, p. 242) whilst the constructivist approach to education

> "is predominantly interested in the students conceptual structures and operations and focuses on behavioural manifestations only insofar as they serve the teacher or experimenter to infer the student's understanding". (von Glasersfeld, 1988, p. 7).

One implication of Piaget's work and the constructivist perspective is that the knowledge and beliefs that learners bring to a given learning situation can influence the meanings that they construct in that situation. Insofar as such a philosophy provides a basis for the current enquiry, it is allowed to form a platform for discussion. However, such a view must go hand-in-hand with the additional belief that the perspective of such meaning is affected by cognitive styles which influence what it is the learner selects from the knowledge base, how it is stored in the long-term memory and how it is used in a new situation. It is from activities associated with this knowledge base that in the context of elementary arithmetic the child is expected to create mental representations which are a platform for further development.

These mental representations may be seen to be the product of a suitable form of abstraction of which Paiget identified three, each of which contributed to qualitatively different levels of thought. Whilst 'empirical abstraction' "derived its knowledge from the properties of objects" (Beth & Piaget, 1966, pp. 188–189), 'pseudo-empirical abstraction', an intermediary between 'empirical abstraction' and a third kind, 'reflective abstraction', "teased out properties that the action of the subjects have introduced into objects" (Piaget, 1985, pp. 18–19). 'Empirical abstraction' has special connotations for this study since it is concerned with the association between specificity and generality in relation to the common properties of objects.

Piaget believed that the key to processes through which actions were projected to thought or mental representation was deemed to be 'reflective abstraction' (Piaget, 1973, p. 81) which was the precursor to the "the development of cognitive structures" (Piaget, 185, p. 143). The idea was meant to reflect both the process through which the action was projected to thought or mental representation, and the sense of the reorganisation associated with mental activity which reconstructs at a higher level everything drawn from the co-ordinations of actions. Seen essentially as the self referential use of existing structures to construct new ones by observing one's thoughts and abstracting from them, reflective abstraction involves the construction of relationships between and amongst objects. However, such relationships may not have an existence in external reality; the relationship only exists in the minds of those that can create it between objects. Kamii (1985) suggests that 'constructive abstraction' might be a more appropriate term than 'reflective abstraction' since this would indicate that this form of abstraction is a construction of the mind rather than something that exists in objects.

3.2.3 Actions and Mentally Created Objects

Substantial interest in the cognitive development of mathematics has focused on the relationship between actions and the entities formed from abstraction. For some, grammatical metaphors sharpened the subtle changes that form the basis for numerical constructs. Dienes (1960) described how a predicate (or action) becomes the subject of a further predicate which may in turn become the subject of another and so on. The qualitative benefit from making predicates the servant rather than the master of thought were clear:

> People who are good at taming predicates and reducing them to a state of subjection are good mathematicians. (Dienes, 1960, p. 21)

Davis (1984), using a similar metaphor, signalled the qualitative changes arising from the transformation of actions into objects of thought:

> the procedure, formerly only a thing to be done — a verb — has now become an object of scrutiny and analysis; it is now, in this sense, a noun. (Davis, 1984, p. 30)

Piaget's notions of the way in which dynamic actions may become conceptual entities through the process of interiorisation is now associated with contemporary terms such as "encapsulation" (Dubinsky, 1991) and "reification" (Sfard, 1991). However, what is interesting about such notions in the context of elementary mathematics is that although they indicate that there has been cognitive shift which has "encapsulated" an action into "an object of thought", there are no clear indicators which provide insight into the nature of the process. Once again it is a case of recognising that it is done yet possessing very little insight of how it is done.

Interestingly, Dubinsky associates the notion of reflective abstraction with encapsulation, in a way which "is trying to tell us what needs to happen" (Dubinsky, 1992, p. 103). Indeed, his goal in elaborating a general theory is:

> ... to isolate small portions of this complex structure [the interrelationship of schemas] and give explicit descriptions of possible relations between schemas [a more or less coherent collection of objects and processes]. When this is done for a particular concept, we call it a genetic decomposition of the concept. We should also point out that although we only give, for each concept, a single genetic decomposition, we are not claiming that this is the genetic decomposition, valid for all students. Rather it represents one reasonable way in which a student may construct a concept. (Dubinsky, 1992, p. 102)

Even though most pedagogy may be associated with a form of genetic decomposition which may seem a reasonable way for a 'typical student' to construct a concept we are generally unaware of the kind of mental representation associated with pedagogic practice.

> When one teaches a part of a larger mathematical structure, even in a mathematically "honest" way, is the partial structure not qualitatively different from the way it is later to be understood as a component of the whole mathematical edifice. In other words, can we be sure that such an approach is not creating double work for the child?
>
> (Resnick & Ford, 1981, p124)

In this study one portion of the complex structure of mathematics is isolated, that of elementary number development, and a reasonable way in which the learner may construct the concept is considered (Steffe, Richards, Cobb & von Glasersfeld, 1983; Gray & Tall, 1994). By considering children's mental representations it tries to make sense of the influence that *their* partial structures may have on successful understanding of more complex structures. Though within a limited range, that of elementary number combinations and approaches to two and three digit computation, the outcome may provide insights into qualitatively different kinds of mental representation that they bring to the process of encapsulation.

2.3.4 Creating a Mental Object

A recurring theme in the discussion so far is that although we know that there is a cognitive shift which 'encapsulates' mathematical actions as mathematical concepts, a sense of the behaviour associated with the phase of transition, in contrast to the two end points of the transition, is not a regular feature of the literature. Whether we refer to 'interiorisation' or 'encapsulation', we are left wondering about the processes which are manifested during these shifts.

Sfard's (1991) discussion of the notion of reification is one attempt to respond to this. She suggests that "in order to speak about mathematical *objects*, we must be able to deal with the products of some *processes* without bothering about the processes themselves" (Sfard, 1991, p. 10). Thus we begin with "a process performed on familiar objects" (Sfard & Linchevski, 1994, p. 64) which is "condensed" by being seen purely in terms of "input/output without necessarily considering its component steps" and then "reified" by converting "the already condensed process into an object-like entity". Such a process is identified in the context of *operational* and *structural* conceptions of mathematics. The first focuses on processes, the second on objects (Sfard, 1989, 1991, 1994). Sfard emphasises that the operational approach — constructing new objects through carrying out processes on known objects — must precede a structural approach to the new objects themselves. The three phase development — *interiorisation* of the process, *condensation* by squeezing the sequence of operations into a whole and *reification* a qualitative change manifested by the ontological shift from *operational* thinking (focusing on mathematical processes) to *structural* thinking (focusing on properties of, and relationships between mathematical objects) — contributes to our understanding of the cognitive shift associated with interiorisation in a way in which other theories do not.

Gray, Pinto, Pitta & Tall, (submitted) indicate that there is a potential flaw in theories which see the construction of new mental objects through actions on familiar objects

> If objects can only be constructed from cognitive actions on already established objects, where do the initial objects come from? This question is not appropriate in Piaget's theory, for both empirical abstraction and pseudo-empirical abstraction build mental entities through co-ordinating perceptions and actions on physical objects... In the case of Sfard's theory, most of the applications considered involve older individuals who will have already constructed a number of cognitive objects. Dubinsky, on the other hand, deals with the cognitive construction of a "permanent object" through "encapsulating the process of performing transformations in space which do not destroy the physical object" (Dubinsky, Elterman &

Gong, 1988, p. 45). This starts with an initial *physical* object that is not part of the child's cognitive structure and theorises about the construction of a *cognitive* object in the mind of the child. (Gray, Pinto, Pitta & Tall, submitted. (Pre-print p. 8))

2.3.5 Cognitive Constructions In Elementary Arithmetic

Cognitive structures associated with elementary arithmetic provide clear examples of the way in which physical objects play a fundamental role in the construction of cognitive objects. Associated with meaningful counting, objects and actions can form platforms for the 'encapsulation' of numerical concepts.

The development of early counting skills (for example, Steffe, von Glasersfeld, Richards & Cobb, 1982; Fuson, 1982; Gelman & Gallistel, 1986), the variety of strategies used to solve context-based and context-free numerical problems (for example, Carpenter, Hiebert & Moser, 1981; Hiebert, Carpenter & Moser, 1982; Gray, 1991), the results of instruction in the development of higher order strategies (for example, Secada, Fuson & Hall, 1983; Fuson, 1986; Steinberg, 1985), the order in which children acquire number concepts (Denvir & Brown, 1986a, 1986b) and children's efforts to discover new strategies in elementary arithmetic (Seigler & Jenkins, 1989), have all contributed towards our understanding of the ways in which children develop and understand numerical concepts.

However, it is suggested here that only the contribution of Steffe *et al* (1983) provides a detailed analysis of the way in which actions with physical objects are metamorphosised through a variety of increasingly abstract representations to form numerical concepts. Through such transformations, it is possible to act upon the results of carrying out the processes without bothering about the processes themselves. Predating contemporary theories associated with the dynamism of process/concept ambiguity the contribution of Steffe *et al*'s finely-grained analysis of counting units provides valuable insights into numerical development. At the same time, a re-analysis of their work seems to provide phases for the cognitive change that may fit Sfard's notions of 'interiorisation' and 'condensation'.

Children's early dependence on counting units is clearly associated with a counting action. The "counting types model" outlines a change in the nature of the object upon which the action is focused. Decreasing dependence on *perceptual* material permits children to eventually count *figural* representations of perceptual material; the counting process continues in the absence of the actual items. *Motor acts*, such as pointing, nodding and grasping, that accompany the counting process, can be taken as further substitute units for perceptual items. Dependence on these three forms of unit is further reduced by the realisation that the utterance of a number word, the *verbal unit*, can be taken as a substitute for countable items that could have been co-ordinated with the uttered number sequence. The concept of unit becomes wholly abstract when the child no longer needs any material to create countable items, nor is it necessary to use any counting process.

The notion of "wholly abstract" may be difficult to discern. Responses to combinations may be automatic. Only when there is evidence that learners use combinations they know, to establish those they do not know through the use of symbolic transformations or the use of derived facts, may the notion of "wholly abstract" take on meaning. Within Steffe *et al's* analysis we see that apart from the final 'stage', the notion of the change in unit does not lead to qualitative change in the process — each procedure is an analogue for the process of counting although there is increasing compressibility until the concept of number is formed. Gray & Tall (1994), using a coarser analysis, suggest that 'procedural compression' accounts for the transformation of lengthy counting procedures into 'known facts'.

2.3.6 The Role of Symbolism and the Notion of Procept

The manner in which children's understanding of concepts is mediated through their mental representations of verbal and written symbols is pivotal to the development of this study. In the Piagetian sense a symbol is a signifier that bears a figurative resemblance to the thing represented. As such it can be invented by the child and does not need to be taught (see, for example Hughes, 1986). However, "without an idea attached a symbol is empty, meaningless" (Skemp, 1986, p. 65); they are the essential means through which concepts are communicated but the "full developments of symbolic functions depends on the corresponding development of concepts to a degree when they can be used independently of any exemplar or embodiment" (Skemp, 1979, p. 161).

Symbols can be thought of as intellectual tools which serve the public function of recording what is already known in order to share and communicate it, and the personal function of organising and manipulating ideas. Language is a mechanism for supplying verbal symbols which represent concepts and can be used for the internalised representation of these concepts (Austin & Howson, 1979). Mathematical symbols are an efficient means of storing and conveying information, not least because they allow the compression of a lot of information into a small space (Pimm, 1987), facilitate and encourage chunking (Ernest, 1987), and elevate mathematical activity to a new plane (Resnick & Ford, 1981). "Complex ideas or mental ideas can be chunked and thus represented by physical notations which, in turn can be reflected up or manipulated to generate new ideas" (Harel & Kaput, 1991, p. 88). The paradox is that symbolism is both the reason for the power of mathematics and the reason for its complexity for many trying to learn it (Cockcroft, 1982).

Addressing the reasons for the paradox, Gray & Tall (1991) consider the duality of mathematical symbolism, it represents either a process to *do* or a concept to *know*. To emphasise this duality they coin the term *procept.*, a combination of process and concept represented by the same symbol.

> It is the *construction of meaning* for such symbols, the processes required to compute them, and the higher mental processes required to manipulate them, that constitute the abstraction of mathematics. Indeed the *ambiguity* of notation to describe either process or product, whichever is more convenient at the time, proves to be a valuable thinking device for the professional mathematician. (Gray & Tall, 1991, p. 73)

Gray & Tall use the notion of procept in a manner which relates to the "concept image", consisting of "all of the mental pictures and associated properties and processes related to the concept in the mind of the individual" (Tall & Vinner, 1981, p. 152). In elementary arithmetic procepts start as simple structures and grow in interiority with the cognitive growth of the child (Gray, Pitta & Tall, 1997). Once the notion of procept has been verbalised it can assist in explaining what occurs in the learning of mathematics, or rather, mathematical procepts.

Interpreting mathematical symbolism as a procedure or as a procept lead to the notion of a *proceptual divide* between the less successful and the more successful (Gray & Tall, 1991, 1994). On the one hand we see a cognitive style strongly associated with invoking the use of processes; on the other, a style more in tune with notions of procedural encapsulation. Those performing the latter have a cognitive advantage. They derive considerable mathematical flexibility from the cognitive links relating process and concept.

2.4 A SPECTRUM OF THOUGHT

2.4.1 Forms of Mathematical Knowledge

Believing that human knowledge is essentially active, Piaget (1971) made a distinction between two types of knowledge: the figurative and the operative. In the field of cognition he described the figurative functions as perception, imitation and mental imagery. Operative aspects of thought included the actions which transform objects or states and included intellectual operations. He indicated that the essential aspect of thought was its operative rather than its figurative aspect.

In mathematics there has been a long-standing distinction between types of knowledge, characterised by Ryle (1949) as "knowing that" and "knowing how". Skemp (1978) placed these in the context of notions of "relational understanding — 'knowing both what to do and why'" and "instrumental understanding — 'rules without reason'"

(Skemp, 1976, p. 20). Hiebert and Lefevre (1986), using the notions of "procedural knowledge" and "conceptual knowledge", bring together a set of contemporary studies on the knowledge debate to explore the connections between them.

Procedural knowledge has two components: the formal language or symbol system of mathematics and the algorithms or rules for completing the mathematical task. The latter, which are usually hierarchically arranged, prescribe instructions on how to complete tasks which operate on objects. The objects themselves can be distinguished as the written symbols of mathematics or objects that are non-symbolic in that they are concrete or mental representations.

Almost in contrast, conceptual knowledge makes use of the underlying relationships which exist within or between the objects themselves. It is described as knowledge that can be though of as:

> "... a connected web ... a network in which the linking relationships are as prominent as the discrete pieces of information ... a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is part of conceptual knowledge only if the holder recognises its relationship to other pieces of information."
>
> (Hiebert & Lefevre, 1986, pp. 3-4)

Carpenter (1986) emphasises the difficulties in sharpening the relationship between procedural and conceptual thinking: a persistent problem is the ability to measure conceptual knowledge directly,

> "It is often inferred through the observation of particular procedures for which it presumably is a prerequisite but there is strong evidence that major advances in solving addition and subtraction problems are characterised both by more sophisticated procedures and more elaborated conceptual knowledge and that both need to be taken into account to understand children's problem solving processes and to plan for instruction." (Carpenter, 1986, p. 121).

There are strong parallels between Hiebert and Lefevre's notions of procedural and conceptual knowledge and Sfard's (1991) notions of operational and structural knowledge. Though she suggests that it is "practically impossible to instantly pin-point all of the subtle aspects [between the two] let alone formulate exact definitions" (p. 4), it is clear that the operational conception of a mathematical notion is associated with qualities similar to procedural knowledge — "processes, algorithms, and actions" — and is characterised by its "dynamic, sequential and detailed" nature. In contrast, structural conceptions of mathematics are more "abstract, more integrated and less detailed" and thus possessing interactive potential (Sfard, 1991, p. 4).

The initial feelings that may be taken from the theorised views of mathematical knowledge is that there is some kind of dichotomy which implies that one form of thinking may be better than the other. However, when we consider the distinctions between instrumental and relational knowledge we see that the definition of the former is subsumed within the latter. Indeed, Carpenter (1992) emphasises that both procedural and conceptual knowledge are required for mathematical expertise, and that there is a spectrum associated with the strength of the inter-relationship between the two. What does become of crucial importance is the nature of the interaction between the two. The arguments seem to suggest that building meaning before practising rules is most beneficial (see, for example, Brownell & Chazal, 1935; Goldin, 1987). The reluctance of some learners to give up, for example, well-rehearsed counting procedures is not only well documented (see, for example Steinberg, 1985; Thornton, 1990) but those who are bound by conventional trains of thought find "it is hard to even switch from a harder to an easier method if the first is habitual and familiar and the second new and unfamiliar. One method of solution may be an obstacle to another" (Krutetskii, 1976, p. 338).

2.4.2 Qualitatively Different Thinking

Sfard's suggestion that "there is a deep ontological gap between operational and structural conceptions [of mathematical entities]" (Sfard, 1991, p. 4) may be applied to any of the theories associated with the extremes of mathematical behaviour. We have seen that Piaget subscribed to the view that there needs to be a qualitative change in the nature of the entity:

> mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget, 1972, p. 70)

The cognitive shift associated with this form of movement has been variously described as 'interiorisation', 'encapsulation' and 'reification'. Gray & Tall's (1994) notion of procept has not only helped to explain what is happening in the learning of mathematical procepts but is accompanied with the hypothesis that there are qualitative differences in thinking associated with the extremes of mathematical achievement.

These differences in the context of elementary arithmetic are revealed by Gray (1991). His empirical evidence indicated that some children wished to remain at a procedural level which, in terms of information processing, makes things very difficult for them, whilst others operated at a conceptual level which was more flexible. The notion of different cognitive styles leading to diverging outcomes came from the observation that the less able, who relied extensively on counting procedures, were "making things more difficult for themselves and as a consequence become less able" (Gray, 1991, p. 570). In contrast, the ability to "compress the long sequences [of procedures] appeared to be almost intuitive to the above-average child" (Gray, 1991, p. 570). In the context of the proceptual divide it was hypothesised that what might be a continuous spectrum of performance tends to become a dichotomy in which those who begin to fail are consigned to become procedural (Gray & Tall, 1994).

2.5 CHAPTER SUMMARY

After considering the notion of mathematical concepts this chapter started from a Piagetian perspective to consider the inter-relationship between actions and objects of thought. Such an experience has implications for development in arithmetic. Interactions with the real world form a basis for the development of a symbolic world which has the potential to link processes and relationships in a flexible cognitive structure. Transformations associated with this process may be manifested within process/object theories. These provide some insight into the cognitive shifts that need to take place between the use of an arithmetical procedure and the formation of concepts represented by symbols that may be manipulated as if they were mental objects. However, the theories do not provide any insights into the nature of mental representations that may support or hinder such a shift.

Theories which refer to the cognitive shift from process to object are process driven but they form an important backdrop for the theory of procepts. Procepts are dynamic and generic — "things" that are the source of great flexibility and power. The problem in the cognitive context is to identify why some children implicitly seem to recognise this fact but others do not.

Symbolic representations of numerical concepts share a common theme — the properties by which the physical objects that are the catalysts for the transformation are described and classified need to be ignored. Attention needs to focus on the actions on objects. Knowledge growth stems

> "not from the physical properties of particular objects but from the actual actions carried out by the child on the objects" (Piaget, 1973, p. 80)

It is this which has the potential to create an 'object of the mind' possessing properties associated with new classifications and new relationships. 'Encapsulation' theories — and here the word is used as a matter of convenience — have intrinsic differences but also share common ground in attempting to account for process/object links. However, notions such as 'interiorisation' or 'repeatable actions' may lead to quantifiable differences in procedure but not qualitative differences in thinking. They can remain at a generalist level, providing no insight as to why qualitative differences in thinking may arise. Proceptual differences do provide some insight but proceptual strength lies in an ability to recognise the generic nature of symbolism and utilise the flexibility associated with it. What is now of interest is the kind and use of mental representations and how they may form the basis for mathematical notions.

CHAPTER 3
CONSTRUCTING ARITHMETIC KNOWLEDGE: MEMORY, MENTAL REPRESENTATIONS AND ARITHMETIC

3.1 INTRODUCTION

> "Words and language written or oral, seem not to play any role in my thinking. The psychological constructs which are the elements of thought are certain signs or pictures, more or less clear, which can be reproduced and combined at liberty"
>
> (Einstein, personal communication to Hadamard, Hadamard, 1945, p. 82)

3.1.1 Chapter Overview

In Chapter 2, the review focused on the way in which actions on objects may become encapsulated as objects of the mind. The newly formed objects have the potential to support 'cognitive economy' — they minimise cognitive effort by representing aspects of our world in the most informative yet economical way. Such a view was put forward by Sfard (1991), for example, who indicates that the different kinds of conception (structural and operational) "... manifest themselves in the special representations of which people avail themselves while processing knowledge mentally... [and that] some kinds of inner representation fit one type of conception better than the other" (Sfard, 1991, p. 6). Later, in the discussion of the notion of procept, we saw that there is a homomorphic relationship between the notion of 'object' and that of 'concept'.

This study seeks to investigate how the quality of this notion may differ among children who display qualitatively different thinking in aspects of elementary arithmetic. An underlying theme is the study of qualitative difference associated with level of achievement in elementary arithmetic. The study considers the way in which such performance is mediated by mental representations which themselves have different qualities. This chapter therefore constitutes a more specialised review of the literature relevant to the three themes:

- First, it will review contemporary psychological standpoints on the structure and process of memory (Section 3.2).by considering notions of short-term and long-term memory and identifying the difference between episodic and semantic memory

- This is then placed in the context of mental representations (Section 3.3) which will examine the nature of mental representations and psychological classifications that are relevant to the analysis of the classificatory data within the study.

- The third theme, Section 3.4, will consider associated work on memory and mental representation within mathematics education.

A concluding section, Section 3.5, will draw together themes identified within Chapters 2 and 3 namely the way in which arithmetical concepts are formed and the possible role of mental representations in their formation.

Actions and Representations

An investigation into the notion of qualitatively different thinking associated with mental representations must invoke some sense of the interpretations and relationships that children evoke from mathematical activity. There appears to be an implicit assumption within pedagogy that as long as we explain things the learner will come to act as the expert does. In mathematics considerable effort is spent on refining a hierarchy of development. This is made explicit in, for example, National Curriculum requirements and it is also implicit in Dubinsky's notion of genetic decomposition. Combined with such a hierarchy are associated activities and actions, which, from the pedagogues' point-of-view, are intended to help the learner make sense of the material. For example, considerable emphasis is placed on the skills associated with counting but it is a future intention to put aside this counting. It is a means to and end but it is to be hoped that children will eventually understand the concept of number in a qualitatively different way.

Holt (1982) draws our attention to the learning paradox associated with the experts' interpretation of the concept and the relationship that they may see in an analogue or instructional representation of that concept:

> Bill and I were excited about the [Cuisenaire] rods because we could see strong connections between the world of rods and the world of numbers. We therefore assumed that children, looking at the rods and doing things with them, could see how the world of numbers and the world of numerical operations worked. The trouble with this theory is that Bill and I *already* knew how the world of numbers worked. We could say, "Oh, the rods behave just the same way as numbers do." But if we hadn't known how numbers behaved, would looking at the rods have enabled us to find out? (Holt, 1982, p. 139)

What seems to be important is what it is that the child critically extracts from the representation and what it is (s)he forms a mental representation of.

Breaking down the mathematics curriculum into manageable parts and then presenting models or metaphors associated with each of these parts raise an important issues in learning. They call on the child to construct internal representations of external representations in such a way that they are assumed to be the correct constructions. However, even though it is not clear how those who are successful manage to pick out those relationships which are self evident to the initiated from the multitude of alternatives (Cobb, Yackel & Wood, 1992), we are even less certain about what is being selected by those who do not make the linkage.

The relationship between the intuitive psychological knowledge structures which enable children to acquire mathematical knowledge from actions on representations and formal mathematical structures needs a sharper focus. Intuitive knowledge arises from the possession of an organised set of associations, propositions and relations and is seen to be similar to the qualitative knowledge that individuals possess about a situation (Behr, Harel, Post & Lesh, 1992). Qualitative knowledge is seen to be "... knowledge that 'belongs' to the individual, is constructed from real experience and provides for considerable flexibility in thought" (ibid, p. 321). It may well be the case that intuitive psychological structures enable some children to establish mental representations which support mathematical learning yet encourage others to create mental representations which discourage the development of mathematical flexibility. In essence, their qualitative knowledge is different.

3.2 MEMORY

3.2.1 Memory as a Constructive Process

One of the outcomes of learning is remembered knowledge (Byers & Erlwanger, 1985). The notion of "memory" is a general label for different forms of acquisition, retention and utilisation of information skills and knowledge. Although Piaget and constructivist theorists are less interested in questions which provide some notion of how knowledge is stored in the mind, they see memory as a constructive or reconstructive process. Complex information is structured to impose some meaning upon it and this implies some modification of the information that can be remembered. Their theories emphasise the role of elaboration, interpretation and reconstruction based upon prior knowledge such that a memory is composed of externally-derived sensory information which is integrated with internally-derived knowledge.

Whilst he did not accept Piaget's stage theory Ausubel (1968) accepted the notions of assimilation and accommodation could support meaningful learning. This was a process through which new knowledge was absorbed by connecting it to some existing relevant aspect of the individual's knowledge structure. However, if there were no relevant concepts already in the mind to which the new knowledge could be linked, it would have to be learned by rote and stored in an arbitrary and disconnected manner.

One advantage of the inclination to create connections between new and existing knowledge is that well-connected knowledge is better remembered (Bruner, 1960; Baddeley, 1976). Memory, if viewed as a reconstructive process, involves the same cognitive activity as understanding. Indeed, this may well be the rationale for Skemp's (1978) distinction between instrumental and relational understanding. An entire network of knowledge is less likely to deteriorate than an isolated piece of knowledge and retrieval of information is enhanced if it is connected to a larger network; there are more routes of recall. The possessor of a conceptual structure which is established through learning relational mathematics "can (in principle) produce an unlimited number of plans for getting from any starting point...to any finishing point" (Skemp, 1978, p. 25).

3.2.2 Long-term and Short-term Memory

Traditionally memory has been subdivided into short-term and long-term memory, the essence of each being embodied in James's (1890) notions of 'primary memory' and 'secondary memory'. The former is concerned with information that remains in consciousness after it has been processed, the latter with events that have left consciousness.

Though the notion of a short-term memory store was used to account for data from cognitive tasks in which the subjects' sole task was to remember various items of information, one of the major differences between it and the long-term memory store is capacity. Though there are no known limits on that for the former, the latter is limited (Miller, 1956). However, despite strong empirical support for a distinction between short-term and long-term memory stores (see, for example, Atkinson & Shiffrin, 1968), evidence that short-term memory does not operate in a single, uniform fashion independently of long-term memory was obtained by Warrington and Shalice (1972).

Baddeley and Hitch (1974) and Baddeley (1986) argued that the concept of a unitary short-term memory store should be replaced by a working memory system consisting of three components: a modality-free central executive resembling attention, an articulatory or phonological loop which temporarily holds information in a phonological form and a visuo-spatial scratch-pad (sketch pad) which is specialised for temporary spatial and/or visual coding. The central executive, used when dealing with most cognitively demanding tasks, is seen to be the most important component of working memory and though it has limited capacity, it co-ordinates the activities of the other two more specialised components. These are seen to be 'slave' systems used by the central executive for specific purposes. Hitch (1978) indicates how the

> "... executive is responsible for both the control of mental processes and short term retention, with one strategy for the latter being the use of the articulatory loop as a storage device through the process of rehearsal." (Hitch, 1978, p. 303)

Drawing extensively on psychological research associated with the working memory system Logie, Gilhooly and Wynn (1994) indicate that the articulatory loop comprises two sub-systems: an active sub-vocal rehearsal process and a passive, phonologically–based store. The former is closely linked to the speech system and whilst the contents of the passive store are subject to decay, they can be refreshed and maintained by sub-vocal rehearsal. The visuo-spatial component of working memory also appears to have two subsystems: one that retains visual material such as colour and shape and one that retains spatial information such as movement through space. Halford, Murray, Maybery, O'Hare and Grant (1994) suggest that whilst short-term memory can be identified with the articulatory loop, their evidence "casts doubt on the hypothesis that short-term memory and working memory constitute a single system" (p. 1352).

3.2.3 Memory Systems

Memory systems have in common the ability to retain and make available for use in on-going behaviour and cognitive functioning the effects of earlier behaviour and experiences (Tulving, 1990). The classification of forms of learning and memory has evolved around three hypothetical systems — procedural memory, semantic memory and episodic memory — which would appear to constitute a hierarchy in which forms emerging early in evolution represent the lower levels and forms evolving later represent the higher levels.

Cohen (1984) suggests that procedural memory represents a lower more general level of the classificatory hierarchy since it is thought to have appeared early in evolution and is shared in various forms by most living organisms. This form of memory enables organisms to retain learned connections between stimuli and response and it can only be expressed in terms of specific responses or behaviours. Unlike the other two forms of memory system, "acquisition of most procedural-memory responses or skills occurs slowly" (Tulving, 1990, p. 221).

Paivio (1986) argues that two sources of information contribute to performance in any memory task: one external and one internal. The internal source consists of the long term memory representations that are activated by the presented material in the context of which it occurs. This internal source "contains" two types of representational information, "one being information that cannot be attributed to any external episodic source and the other, information that can be attributed to such a source" (Paivio, 1986, p. 140). The notion that there are two sources of information and two types of representational information leads Paivio to accept the defining characteristics of episodic and semantic memory proposed by Tulving (1972) who suggests that the two are "parallel and partially overlapping information processing systems".

3.2.3.1 Episodic Memory

Described as a memory system which makes it possible for a person to remember concrete personal events or events dated in the subjective past, episodic memory entails a conscious experience of a unique kind. It would seem to have an autobiographical flavour about it, referring to the storage of specific events or episodes which occur in a particular place at a particular time.

3.2.3.2 Semantic Memory

In contrast to episodic memory, semantic memory may not have any personal relevance to the individual. Nor need it refer to the past or any other time in the individual's existence. Knowledge not attributable to specific learning episodes — our general knowledge of the world and of language — is deemed to be semantic. Semantic memory has been defined as:

> "... a mental thesaurus, organised knowledge that a person possesses about words and other verbal symbols, their meanings and referents, about relations amongst them, and about rules, formulas, and algorithms for the manipulation of these symbols, concepts and relations. Semantic memory does not register perceptual properties of inputs, but rather cognitive referents of input signals." (Tulving, 1972, p. 386)

Thus, semantic memory makes possible the cognitive representation of "things" and events and the utilisation of the information represented in the absence of the original stimuli. The semantic memory system allows the individual to construct mental models of both concrete and abstract aspects of the world.

3.3 MENTAL REPRESENTATIONS

3.1 Seeking a Definition

The New Shorter English Dictionary, (1993, p. 2553) typically defines representation as "an image, likeness or reproduction of a thing". Two ideas emerge from such a definition which are of consequence for this study. Firstly, a representation is something presented to the mind and it can be physical since it can be in some material or tangible form, or it can be mental. Secondly, the nature of the something or the 'thing' is of interest. This could be a likeness, a picture, a description, a figure or a symbol and it is obvious, therefore, that the 'thing' can vary in its abstractness — it could range from a real object, to a picture, to a linguistic description or a symbol.

Eysenck and Keane (1995, p. 207) suggest that the contents of the mind might be "object-like entities that are related together in various ways by conceptual relations".

An extensive part of Chapter 2, and in particular Section 2.2.3, discussed how actions on 'things' had the potential to be encapsulated into such 'objects'. The discussion focused on the way that actions with the physical 'thing' may be compressed to form the mental 'thing' which is the basis for the "concept image" of the number procept (Section 2.2.7). When the 'thing' is at its most abstract, its proceptual qualities offer flexibility. However, of course, the mental representation formed from the physical action may itself vary in abstractness from an analogue of the real object, to a picture, to a linguistic description or a symbol. Such a variation implies that the nature of the 'thing' that dominates the mental representation may not only differ by its degree of abstraction from the original one associated with the action, but it is also conjectured that it will imply that the mental representation may be put to different uses. It is hypothesised that this in turn, will have implications for the use of the working memory. Indeed, our concern reflects that of Bruner who, drawing upon the work of his colleague Miller, indicates that:

> "We are indeed limited in our span. Let me only suggest here that compacting or condensing is the means whereby we file our seven slots with gold rather than dross"
>
> (Bruner, 1968, p. 12)

We too are concerned with the difference between the 'gold' and the 'dross' and the implications of this difference for cognitive development in mathematics.

3.3.2 The Brunerian Perspective

Mental representation was one focus of study for Bruner (1968) who suggested three ways in which humans conserve past experience by "translate[ing] experience into a model of the world" (p. 10): through enactive, iconic and symbolic representation. Interestingly Bishop (1980) claims that Bruner developed his ideas in the context of algebraic mathematics.

Enactive representation is seen as a mode of representing past events through appropriate motor response and "is based, it seems, upon a learning of responses and forms of habituation" (Bruner, 1968, p. 11). Resnick and Ford (1981) believe that this mode may well be what we see in children who use finger tapping to support addition strategies; counting, for these children, may still be represented as a motor act.

Iconic representation takes us a step away from the concrete and the physical to the realm of mental imagery. Bruner suggests that iconic representation is what happens when a child pictures an operation or manipulation as a way of not only remembering the act but also of recreating it mentally when necessary. Symbolic representation, is based upon the translation of experience into language. For Bruner representations in the form of words or language are the hallmarks of symbolic representation and such representations would appear to be the point where concept formation, in the sense defined by Skemp (1971) and Vergnaud (1988), actually takes place. Skemp makes a distinction between two forms: the verbal and the visual. The former refers to the spoken and the written word whilst the latter is "clearly exemplified by diagrams of all kinds" (Skemp, 1971; p. 88). Tall (1994) suggests that Bruner's symbolic mode must in essence be a visuo-symbolic mode in which the symbols are written, drawn or seen.

Bruner considered that these representations grew in a sequence in the individual, first enactive, then iconic and finally symbolic representation. The latter has a power of its own which depends less on the first two. One wonders whether this takes account of the "personal style" of the individual. There seems to be an assumption that the potential to move through the hierarchy may be the same for all. Additionally, in the context of teaching and learning, it is possible to see inappropriate decisions made on the assumption that children have moved through a particular stage of development. This is an issue which is partially addressed in the empirical work of this study. It may well be that learners do not naturally develop within a sequence but become caught up in 'less' sophisticated representations that may be habitualised. The evidence within this study will suggest that in the context of elementary arithmetic, even when it is assumed that children are in a symbolic state they may in fact be viewing numerical symbols in an enactive or an iconic way.

3.3.3 Mental Representations and their Interpretation

Paivio (1986) indicates that historically mental representations have been interpreted by analogy with physical representations and the most obvious distinction is that some physical representations may be picture-like while others are language-like. Picture-like representations may include photographs, drawings, maps and diagrams whilst language-like representations include natural human language as well as more formal systems such as mathematics. Picture-like representations have been variously described as having analogue, iconic, continuous and referentially isomorphic properties whereas language-like representations are characterised as being non-analogue, non-iconic, digital or discrete, frequently arbitrary and propositional. The notions of picture-like, analogue, iconic and isomorphic all imply that such representations map onto the representation of objects or events in a non-arbitrary way. In the case of language-like representations, the relationship is completely arbitrary. Of course, the implication of the differences in these qualities is that representations may also vary in their level of concrete-abstractness. Thus, at one extreme, we have highly concrete, iconic, modality-specific representations whilst at the other extreme we have completely abstract, amodal representations that are only arbitrarily related to real world objects.

Though there are several representational concepts associated with representational theories, each may differ in its level of abstractness and the ease in which it may be observed, for example, the notions of prototype (see, for example, Rosch, 1975), exemplar (Smith & Medin, 1981), schema, (see, for example, Bartlett, 1932), propositional representation (see, for example Pylyshyn, 1984) and mental image. In this study the notion of mental representation is closely tied with the notion of mental image of which the visual image features strongly. This is not to say that, for example propositional representations were not considered. It is simply because, in common with other researchers, analysis of the empirical data revealed that it is too difficult to say with certainty whether or not particular instances are propositional representations or not.

3.3.4 Mental Images

Pylyshyn (1973) indicates that "Any analysis in the nature and role of mental imagery is fraught with difficulty. The concept is difficult to pin down". However,

> "The issue is not whether imagery exist but what is the actual significance of the properties and presentations found in the image or those which underlie its visible structure. Is an image really an image and in essence spatial, or is it just a wrinkle in consciousness better understood through some other underlying encoded abstract propositional form."
>
> (Ahsen, 1990, p. 53)

Mead (1934) defined imagery as:

> "... an experience that takes place within the individual, being by its nature divorced from the objects that would give it a place in the perceptual world, but it is a representational reference to such objects. The content of this imagery is varied. It may be of vision and contact or of other senses... playing the same part as that played by objects... What characterises it is it appearance in the absence of the objects to which it refers..." (Mead, 1934, p. 223)

The crucial issue to arise from this definition is the belief that mental objects should be given equal attention to that given to "real" objects. Piaget and Inhelder (1967) discussed the relationship between the two:

> Perception is the knowledge of objects resulting from direct contact with them. As against this, representation or imagination involves the evocation of objects in their absence or, when it runs parallel to perception, in their presence. It completes perceptual knowledge by reference to objects not actually perceived...Now in all probability the image, an internalised imitation, is consequently derived from motor activity, even though its final form is that of a figural pattern traced on the sensory data. (Piaget & Inhelder, 1967, p. 17)

An image may be created through critical abstraction from a situation, thus before an object is seen, by amalgamating parts previously generated from experience.

However, such an image may be completely unrelated to the object and in conflict to what is later discovered to be the object (Kosslyn, 1983).

In one sense, Mead (1934) approached the essence of the notion of image as a "picture in the mind" through his reference to an image as the result of "vision and other sense". Bugelski (1971) urged researchers to think of imagery in terms of the process of imaging rather than as "pictures in the head" but the common interpretation that it was such led to Pylyshyn's (1973) critique of the notion of mental image as a theoretical construct to describe a form of memory representation. Paivio (1979) interpreted imagery as a dynamic process that included motor as well as sensory components requiring interpretation by several objective indicators rather than subjective reports of pictures alone.

3.3.5 Mental Images: A distinct type of Representation?

3.3.5.1 Conceptualising Mental Images

Paivio (1986) indicates that imagery is the oldest and most persistent of all specific representational concepts but its history has been marked by repeated controversy (see, for example, Pylyshyn, 1973). Even though there has been a resurgence in interest in the phenomena, there is the belief that the "concept of mental image has become less clear as more progress has been made on the research front ... it is not such a simple nor a unified concept" (Cooper, 1995, p. 1575).

An underlying problem with the notion of imagery is the ordinary informal meaning of the word 'image'. The associated language and qualitative descriptions of images rely heavily on the picture metaphor. Pylyshyn (1978) suggested that "Even when visual imagery is clearly implicated the underlying representation is best characterised as abstract and conceptual" (p. 18). He argued that the problems associated with conceptualising mental representations are not simply related to form — whether or not the mental representation can be either verbal or imaginal — but whether or not the concept of imagery can be used as a construct in psychological theories of cognition without further reduction. In his view, it cannot:

> As long as we recognise that people can go from mental pictures to mental words or vice versa, we are forced to conclude that there must be a representation (which is more abstract and not available to conscious experience) which encompasses both. (Pylyshyn, 1973, p. 5)

3.3.5.2 Propositional Representations

To overcome the problem Pylyshyn proposed that imagery possesses propositonal or descriptive characteristics that are abstract and neutral in respect to stimulus modality. Considered to be explicit, discrete and abstract entities, propositional representations "are language like representations that capture the ideational content of the mind" (Eysenck, 1995, p. 206). Two issues are worth commenting upon here. Firstly the notion of 'description' "may carry too much undesirable excess meaning ... [the language like representations] are never accessed in a fixed serial order" (Pylyshyn, 1973, p. 12) . The notion of propositional is used to provide a sense of what may be formally adequate to describe certain types of cognitive activity. Secondly, there is no implication that the "language like" properties of propositional representations may be equated with "tokens of actual sentences in some natural language" (Pylyshyn, 1973, p. 12).

An interesting point that emerges from a propositional view of mental representations is the close association which may be made with the review in Chapter 2. For example, Pylyshyn (1984) refers to the way in which 'cognitive representations' — a term he uses instead of images — become 'reified' as "concrete entities" (see also Section 2.2.4). Propositional theorists tend to assert that their descriptive elements, for example, mathematical symbols, are the underlying entities (Paivio, 1986). Without reference to the notion of propositions, Dörfler (1993) presents a critical discussion of associated views within mathematics education. His personal perspective indicates that not only did his

"... subjective introspection never permit [him] to find a trace like a mental object for, say, the number 5"... [but he] felt no need for mental or abstract objects." (p. 152)

Tall's (1995) consideration of the status of mental objects in the mathematical context indicates that:

"... although we may not have anything in our mind which is like a physical object we have symbols which we can manipulate *as if* they were mental objects." (Tall, 1995, p. 65)

In the context of the notion of procept this point was further elaborated in Gray, Pitta and Tall (1997):

In this sense there is no claim that there is a "thing" called "a mental object" in the mind. Instead a symbol is used which can be *spoken*, *heard*, *written* and *seen*, which is capable of evoking appropriate processes to carry out necessary manipulations in the mind of the individual and which can be communicated and shared with others.

(Gray, Pitta & Tall, 1997, p. 116-117)

Propositional representation is possibly one of the most versatile representational constructs because it can be used to describe any kind of representational information. However, although one of the most centrally used theoretical constructs, it suffers from two main deficiencies. Firstly, the sole use of the notion to understand knowledge and mental representation would involve too much detail. Secondly, properties of propositional representation may only be tested indirectly — there is a distinct lack of evidence on properties from direct tests.

3.3.5.3 Dual Coding Theory

Essentially propositional theories are theoretical constructs devoted to internally consistent models of representational processes. Empirical evidence has been used to determine the minimal basic differences between these and imagistic representations (see, for example, Paivio, 1979, 1986, 1990). Paivio assumes that mental representations have their developmental origin in perceptual, motor and affective experience and that they retain these experientially — derived characteristics. His Dual Coding Theory (1986) suggests that there are two separate but interdependent symbolic systems for the representation and processing of information. A verbal system deals with linguistic information and stores it in an appropriate verbal form whilst a separate non-verbal system carries out image-based processing and information. Both systems are then further divided into sub-systems that process either verbal or non-verbal information in the different modalities. In providing a sense of his theoretical position Paivio (1986) writes:

> "I will often refer to the non-verbal (symbolic) subsystem as the imagery system because its critical functions include the analysis of scenes and the generation of mental images (both functions encompassing other sensory modalities in addition to the visual). The language specialist system will be referred to as the verbal". (Paivio, 1986, pp. 53-54)

The notion of two separate subsystems implies that the two systems were assumed to be structurally and functionally distinct, differing in the nature of their representational units[0] and the way in which these units were organised. Being functionally distinct implied that they could be independently active or function in parallel so that activity within either could activate activity in the other. The structural representations of the theory refer to relatively stable long-term memory information corresponding to perceptually identifiable objects and activities. Ahsen's (1990) critical comments focus upon the theory's parallelism and not upon the hypothesised independence of action.

3.3.5.4 A Computational Model of Imagery

In contrast to the view that there are two separate but interdependent symbolic systems Kosslyn (1980, 1994) presents a compromise position which although it indicates that mental images are assumed to function like pictures, they are assumed to be partly based on propositional representations. Kosslyn's view is that imagery is worth examining as a separate construct since it has its own privileged properties.

[0] The units referred to by Pavio – imagens and logogens – are "intended to distinguish the underlying (hypothetical) cognitive representations from their expressions as consciously experienced images and inner speech, or overt behaviours such as drawing and speech" (Pavio, 1986, p. 59). There is no intention that they imply a fixed entity of some fixed size and character. He suggests that they may be interpreted in the sense of "chunk".

Based on empirical evidence, Kosslyn's theory claims that images are represented in a special, spatial medium which has four properties: Firstly, it functions as a space and therefore preserves the spatial relationships of the objects it presents. Secondly, it does not necessarily present images in uniform resolution — the highest resolution is at the centre, the medium has a grain. Thirdly, the larger the grain the lower the detail, and fourthly, once the image is generated on the medium it begins to fade. The consequence of these properties is that images can represent a whole object or parts of the object. More specifically some image files characterise a 'skeletal image' which depicts the basic shape of the object but lacks many of the object's details. The embellishment of this 'skeletal image' is possible through connection with information in the propositional files. It is through such a relationship that Kosslyn's theory conjectures a process that generates images from the two data structures within the long term memory.

3.3.6 Classifying Mental Representations

The notion that only one kind of image exists has received extensive comment in the literature. Indeed, not only may we see distinctions between common and bizarre images (McDaniel & Einstein, 1991), memory and imagination images (Griffiths, 1927) and non-active and enactive imagery (Richardson, 1980; Goosens, 1994) but further distinctions may be made between general and specific images and personal and impersonal images (De Beni, 1988).

Drake's (1996) somewhat seminal article, in that it attempts to break away from the science-based or quantitative paradigm for studying imagery, draws attention to three levels of imagery. At Level 1, which was common to all within her study, subjects reported very concrete, visual images in which they were either observers or participants. Such images were regarded as a tool or programme to achieve a particular goal and were primarily visual and from the subjects' known physical world. Level 2, images were usually concrete and remained highly pictorial but they may include an image that acts as a symbol and not a compete picture. Such images were primarily visual but could come from other modalities. Images classified as Level 3 could be very abstract and formed from all modalities.

De Beni and Pazzaglia (1995) indicate that distinctions between 'general' and 'specific' images could be refined to suggest the existence of three different categories of images: general, which represent a concept without any reference to a particular example, specific, representing a single well-defined example of the concept without reference to a specific episode and autobiographic which involve the subject without reference to a specific episode. Kossyln (1994) considers these to be a special case of specific images: the only difference being that they are enlarged through the addition of the self schema. De Beni and Pazzaglia identified a fourth category of image: episodic–autobiographic images. These represented the occurrence of a single episode in the subjects life connected to the concept and are generated in episodic memory.

These classifications were established within a frame identified as non-contextualised and contextualised images. The former would appear to contribute to the encoding of distinctive, item-specific information whilst the latter would appear to contribute to item-specific and relational encoding.

Cornoldi, De Beni and Pra Baldi (1988) indicate that mental images spontaneously evoked from a single verbal cue were general, specific and autobiographical[1] in decreasing proportions. De Beni and Pazzaglia (1995) suggest that given a short time span (10 seconds), the generation process associated with specific, autobiographic and

[1]De Beni *et al.* refer to autobiographical images as self-referencing images. They do not dispute the fruitfulness of their use in supporting memory performance. They do question the meaning that may be given to the 'autobiographic image' category. They suggest that there is a distinction between images referring to a single episode in the subjects life (episodic–autobiographic) and those that actually involve the subject without a precise episodic reference (autobiographic images). They report on the limited use of autobiographic images, given their refined classification.
The current study (Pitta, 1998) also noted their limited occurrence and through a re-analysis of each response so classified, it was decided that the category would not be helpful in the overall discussion. Similarly, given that many of the verbal and visual cues presented in the study are arithmetical, De Beni's notion that the contextual image contributes towards relational encoding would be better served if there was a distinction made between episodic images, which refer to some scene or sequence of scenes, and generic image which more explicitly makes relational connections. The sense in which episodic images is used is based on the fact that they demonstrate a scene or a sequence of scenes (episodes). There is no implication that episodic mental representation is derived in episodic memory.

contextual images "starts from the formation of a general skeleton-image (the integration in short term memory of information derived from long-term memory)" (p. 1361). With a longer time span (40 seconds) the "image can either be enriched by more detail or, in the case of contextualised images, by the insertion of the imagined noun within a network of relationships with other objects" (De Beni & Pra Baldi, 1988, p. 1364). On the other hand "Episodic-autobiographic images seem to have a completely different case, not only because of the used memory component (episodic autobiographic) but also because of their generation process. The latter is not an enrichment of the general image, but a different process form the beginning: given the verbal cue, a search takes place among the autobiographic memories related to the cue, leading to the choice of the one considered the most representative" (pp. 1361-1362)

De Beni and Pazzaglia's (1995) work is of particular interest here. It came to light after completion of the pilot study and confirmed the findings of that study, that the traditional task of evoking mental representations from non-related nouns elicits the generation of different kinds of image. As a consequence, analysis of the main study took note of the different classifications used by De Beni.

De Beni and Pazzaglia (1988) note that "distinctions between different kinds of images... could be a way of interpreting studies on the selective involvement of the right and left hemispheres on imagery" (p.1360). Neurological and neuropsychological research data shows a greater involvement of the right hemispheres in autobiographical and general image generation whilst the left appears to have more involvement in specific image generation.

3.3.7 Memory and Mental Representations

The interesting issue that arises from the previous discussion is how mental representations are associated with memory. Baddeley (1966a, 1966b) suggests different forms of representation are associated with different memory stores. Words may be stored in the verbal long term memory based on their meaning and in the short term memory based on the acoustic and phonological properties. However, not many models clearly explain the relationship. The issue has proven to be both complex and controversial (Craig & Lockhart, 1972).

Hitch, Brandimonte and Walker (1995) present a review of the literature associated with the relationship which indicates the nature of a controversy. They indicate that visual images are generated as a direct result from perception as a record of recent visual experience or in the visual long term memory. Baddeley's (1986) model, suggests that visual images are constructed, manipulated and maintained by the visuo-spatial sketch pad. In this respect it seems to share characteristics with the spatial medium identified by Kosslyn (1983). However, the controversy regarding storage indicates that on the one hand both short term and long term memory stores the same type of representations. On the other, it is suggested that information regarding surface characteristics is stored in visual short term memory stores whilst abstract and surface descriptions are in visual long term stores. It is the latter view which is subscribed to by Hitch *et al* reaffirming the opinion of Kosslyn (1980).

3.4 MEMORY, MENTAL REPRESENTATIONS AND ARITHMETIC

3.4.1 Some Initial Considerations

In this section an attempt is made to draw together evidence from mathematics education and psychology to provide a platform for the empirical work that follows. Psychological research has identified the importance of imagery in cognitive development — children use it more in their thinking than adults (Kosslyn, 1980). Its role in a child's thought processes cause it to have far-reaching consequences on his/her concepts and reasoning (Bruner, Oliver & Greenfield, 1966; Piaget & Inhelder, 1971) and therefore images place major constraints on cognitive processes.

In this study the concept of mental imagery is closely related to the concept of mental representations. The relationship between different forms of mental representation, symbolic, verbal and analogical may be seen through the presentation and solution of arithmetical facts (Dehaene & Cohen, 1994). Gonzalez and Kolers (1982) also argued that the surface characteristics of different number systems may also affect these mental operations. Children's internal representation of numbers are often highly imaginative and unconventional and built up over time (Thomas, Mulligan & Goldin, 1995) but the possession of an image of a mathematical idea implies that the individual does not need actions or the specific instances of image making (Pirie & Kieran, 1994). Thus, though they may be eidetic, in the sense that they are visual images fully formed from something presented (Mason, 1992), mental interaction with the eidetic detail in the mind can continue in the absence of the stimulus (Ahsen, 1996) though classification of this phenomena is a problem (Gregg, 1990).

3.4.2 Mental Representations and Mathematics

Within the field of mathematics education it has been argued that spatial abilities, imagery and/or visualisation play an important role in mathematical thinking (Lean & Clements, 1981; Wheatly, 1990). Clements and Battista describe images as "internally perceived, holistic representations of objects that are isomorphic to their referents "(p. 444) and they are mentally changed by continuous transformations corresponding to physical transformations. However, much of the research associated with imagery has taken place in the field of visualisation, a term used to describe

> "the process of producing or using geometrical or graphical representations of mathematics concepts, principles or problems whether hand drawn or computer generated."
>
> (Zimmermann & Cunningham, 1991, p. 2).

Zazkis, Dubinsky & Dautermann (1996) describe visualisation as

> "an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses." (p. 441)

This definition is one which applies to the act of mentally constructing or the act of externally representing but in association with much in mathematics education it strongly implies a visual element and could almost disassociate itself from other, equally important forms of mental representation. Presmeg (1986) talks about 'visualisers' and 'non visualisers' whilst Bishop (1980) claims that visualisation plays such a large part in mathematical thinking because it is a subject which is concerned with objectivising and representing abstractions from reality.

It is perhaps because of this concern with visualisation that a considerable effort in mathematics education has been devoted to visual and analytic approaches to mathematics (see for example, Clements, 1984; Eisenberg & Dreyfus, 1986; Presmeg, 1986; Mundy, 1987). Collectively these studies indicate that visualisers are seriously under-represented amongst high achievers, that gender seems to be one of the factors that influences an individuals predisposition towards visualisation but that gender differences can be reduced through practice. However, Lean and Clements (1981) found that students who process mathematical information visually are not as successful as those who process information by verbal-logical means, whilst Hershkowitz (1989) claimed that if an individual over relies on an a single image of an object they may find it difficult to deal with a problem solving situation.

On a pedagogic front visualisation is seen as one of several forms of mathematical representation and it is most successful when linked with other forms of mathematical thinking (Hughes-Hallet, 1991), supports an oscillation between analytical and visual representation of problems (Zimmerman, 1991) and can help in the performance of algebraic computations (Goldenberg, 1988).

Bishop's comments not withstanding, the role of imagery and other mental representations have not, until recently, figured extensively in models of mathematical understanding. The inter-relationship between visualisation and analysis has been a focus of attention by Zaskis *et al* (1996) and Fennema (1979) claimed that all mathematical tasks require spatial thinking. Tartre(1990) argued that spatial orientation skill seemed to be involved in the understanding of non-geometrical problems and also in the linking of these to previously acquired knowledge. Though images in particular are referred to by Sfard (1991) little insight into their use is provided. Perhaps the most comprehensive account of their formation in a mathematical context comes from Pirie and Kieran (1994) who claim that following a period of 'primitive knowing' comes a period of 'image making' and then 'image having':

> "Image having is the level at which the learners actually have some images for concept and thus they no longer need to rely on the actions that occasioned the understanding and can carry and use the ideas they have constructed. This does not imply, however, that their images are complete, appropriate or even sufficient for the work in hand. Many learners develop strong early attachments to particular dominant images and this can seriously hamper later growth of understanding" (Pirie & Kieran, 1994, p. 247)

However, there remains no clear consensus about the overall role of mental representations particularly in a non–geometrical context. Perhaps one of the main reasons is that within mathematics education terms associated with mental representation have been used with different meaning:

> "There is no general agreement about the terminology to be used in this field: It may happen that an author uses, for instance, the term "visualisation" and another uses "spatial thinking", but we find that they are sharing the same meaning for different terms. On the other hand, a single term, like "visual image", may have different meanings if we take it from different authors. Such an apparent mess is merely a reflection of the diversity of areas where visualisation is considered relevant and the variety of specialists who are interested in it."
> (Guttierez, 1996, p. 4)

3.4.3 Mental Representations and Mathematical thinking

Kruteskii, (1976), referred to two different modes of thought: the verbal-logical and visual-pictorial. He argued that the balance between these two modes of thought determined how an individual operates with mathematical ideas and each can describe different types of learners. He identified three types: the 'analytic' those who prefer the verbal-logical modes of thought, 'geometric', those who prefer the visual-pictorial mode, and 'harmonic', those who use both. His results indicate that 1489 of the students could be classified as 'harmonic' whereas he identified only ten 'analytic', and thirteen 'geometric' thinkers. Krutetskii argued that the ability to visualise abstract mathematical relationships was not a necessary component in the structures of mathematical abilities. Interestingly, however, he did conclude that mathematical ability can take shape at a very early age

> "... and for the most part, in the form of computational abilities — abilities to operate with numbers" (p. 222).

This is not to say that computational abilities are true mathematical abilities but Krutetskii believed that on the basis of computational abilities real mathematical abilities may be formed.

3.4.4 Arithmetic and Memory

Bull and Johnston (1997) provide a sound overview of the research considering the links between arithmetic and the different memory components of long term memory, short term memory, and working memory.

How solutions to basic number combinations are stored in memory is a contentious issue. Much of the research on fact retrieval has involved a chronometric approach in which reaction times for different number combinations have been used to infer the nature of the processing involved and thus, by implication, provide some notion of what is remembered (e.g. Groen & Parkman, 1972; Ashcraft, 1982; Siegler & Shrager; 1984). Children with poor arithmetical skills show a lack of automaticity in retrieving numbers and number combinations from long term memory evidenced through slow item identification and through the use of slow inefficient counting strategies rather than direct memory retrieval. This slowness may be caused by a lack of familiarity with the material. In Ashcraft's view even a simple and basic phenomenon such as incidental learning would predict eventual memory for most, if not all, of the simple addition facts. However, the slowness may represent a more serious problem in the ability to automate facts in long-term memory (Geary, 1993)

Bull and Johnston argue that these deficits often mediate the observed deficits found in short-term memory span. Geary (1991) has suggested that a component of developmental difficulties in mathematics is a working memory deficit. Pitta and Gray (1997) suggest that contrary to this low achievers show an extra-ordinary use of working memory. They hypothesise that the problem is associated with its use and not so much its capacity, a theme that is expanded upon within this study. Bull and Johnson are also sceptical of an explanation of difficulties associated with short-term memory deficit.

Perhaps the most interesting aspect of cognitive psychologists views of the relationship between memory and the acquisition of basic arithmetic facts is that they do not seem to appreciate that difficulties may be associated with qualitatively different thinking (Gray & Tall, 1994) or with many of the other issues considered by mathematics educators.

3.4.4.1 Arithmetic and Working Memory

Towse & Hitch, (1995) suggest that working memory does two things; it stores and process information It is evident that a complex task such as mental arithmetic requires that information is temporarily stored whilst new information is being processed. Baddeley's (1986) notion that working memory has three main components, the central executive, the articulatory loop and the visuo-spatial sketch pad (Section 3.2.2) is to be seen as an integrated system whereby in the context of working memory the central executive ofloads some of its short term storage functions to the articulatory loop and the visuo-sketchpad. In the context of elementary arithmetic it is hypothesised that the role of the central executive is to monitor and retrieve information associated with the process, for example addition. The articulatory loop and the sketch-pad store specific numbers involved in the calculation. Since the articulatory loop is involved in the storage of verbal information, which is subject to decay, it may be refreshed by sub-vocal rehearsal.

Recently Logie, Gilhooly and Wynn (1994) have discussed the functions of working memory in more detail and they claim that:

> "The subvocal rehearsal companion of working memory provides a means of maintaining accuracy in mental arithmetic [whereas] the extent to which people spontaneously rely on visuo-spatial temporary storage or visual imagery is still very much open to debate. Indeed it may be that imagery offers a number of strategies available and that only some individuals would choose to use imagery in laboratory studies of mental arithmetic." (p. 395 and 397)

3.4.5 Arithmetic and Mental Representations

3.4.5.1 Mental Representations and Children's Development

Hughes (1986), has described young children's spontaneous representations of quantity associated with addition and subtraction. Using four main categories to classify their responses, idiosyncratic, pictographic, iconic and symbolic, he indicated how the younger children produced mostly iconic and idiosyncratic representations whereas the older children favoured pictographic and symbolic.

Thomas, Mulligan and Goldin (1996) have investigated children's internal representations of the base ten numeration system and how these change through a period of teaching. Concentrating only on the children's visual images, which were unrelated to any mathematical task or numerical relationship and which were inferred from drawings with supportive verbal evidence, they concluded that children's internal systems of representation of numbers go through three stages:

> "(i) an incentive/semiotic stage, in which characters and configurations in a new system are first given meaning in relation to previously constructed representations,
>
> (ii) a structural development stage, where the previously existing system functions as a kind of template on which the new system is modelled,

(iii) an autonomous stage, where the new system of representation functions independently of its precursor and can assume new meanings in new contexts."

(Thomas, Mulligan & Goldin, p. 308)

However, not only may there be stages of development but there are also indications that imagery is related to quality of understanding (Brown & Presmeg, 1993). Relational thinkers appear to use more "abstract imagery such as dynamic and pattern imagery", whereas those who are less relational depend more on "concrete and memory images".

3.4.5.2 Arithmetic, Mental Representations and Process in the Mind

Dehaene and Cohen (1994) argue that the relationship between different forms of representation, symbolic, verbal and analogical may be seen through the presentation and solution of arithmetic facts.

The form which mental representations of numerical ideas take in the mind had been considered by Seron, Pesenti, Noel, Deloche & Cornet (1992). Their subjects reported seeing simple digits or numbers, numbers transformed into patterns as found on a die, numbers with colour and numbers as on a number line. They suggest that quantity directly represented by "patterns of dots, or other things such as the alignment of apples or a bar of chocolate (p. 168) may be deemed to be analogical. They have also attempted to discuss their role in number processing and the relationship of these epiphenomenal visuo-spatial representations and contemporary theoretical models on number representation and calculation discussed by neurologists (Dehaene & Cohen, 1991). Though it would seem that the occurrence of analogical forms of imagery may be rare it is not exceptional. Such forms emphasise the nature of surface characteristics associated with a transformation of the object that is the numeral into an object which is analogical. No longer is the subject dealing with an encapsulated entity. It is possible that surface characteristics associated with either the analogue or with the number system (see Gonzalez & Kolers, 1882) may have a significant effect on mental calculation.

It is a hypothesis of this study that if an individual's mental representation is of a form which does not embrace an encapsulated object but encourages procedural characteristics displayed through the use of mental analogue in visual or verbal form or in fact the real item it will have a significant effect on mental calculation.

3.5 REVIEW SUMMARY

It was the intention of Chapters 2 and 3 to bring forward theories and empirical evidence from mathematics education and psychology to establish a unified platform considered essential for the development of this study.

Chapter 2 of this review considered the nature of arithmetical concepts and theories associated with the construction of numerical concepts. Starting from a Piagetian perspective it considered theories which hypothesise that mathematical actions are 'interiorised' as mental objects which are named and manipulated in such a way that they themselves become objects which are acted upon or act as partial operators which quantify new actions. The problem is that the various descriptors which are used to denote the cognitive shift between process and concept do not, of themselves, give insight into the way this shift takes place. We remain very much at the level of knowing it happens but not how it happens.

The symbolism that represents these objects was seen to be ambiguous in nature. It carries proceptual ambiguity which in any given situation may be used to extract either the procedural or the conceptual qualities inherent within it. Because of this characteristic it can evoke qualitatively different thinking which may lead to a proceptual divide.

It is a conjecture of this study that different kinds of mental representations play an important part in providing the flexibility to utilise this ambiguity. Different kinds of visual image have been shown to exist. Since this study takes the view that visual imagery is only one aspect of the field of mental representations the issue is whether or not mental representations are of different kinds. Different kinds of mental representation seem to be associated with different memory components.

In the field of elementary arithmetic it seems likely that children may use different memory systems to arrive at solutions. 'Known facts' may be automatically retrieved from long term memory whereas alternative strategies may involve different components of memory. Different memory systems may also result in the projection of qualitatively different mental representations. Whilst it is not possible to examine directly these sorts of links it is possible to obtain a sense of the relationship between different kinds of mental representation and the predisposition towards qualitatively different modes of thinking in elementary arithmetic. If there is a link, it is suggested that the occurrence of a proceptual divide in elementary arithmetic is a consequence of a disposition towards the kinds of mental representation which do not filter out surface characteristics of the object and/or the action. This may mitigate against the formation of mental representations supportive of the process of encapsulation.

*

CHAPTER 4
ESTABLISHING A METHODOLOGY

4.1 INTRODUCTION

> "I make no apology for including data collected under natural or semi-structured conditions from barmen and divers, mothers in labour and first-aiders, and indeed anyone who has to make use of his or her memory in the rich and complex conditions of the real world."
>
> (Baddeley, 1997, p. vi)

In Chapter 2 we saw how the development of arithmetical concepts may be accounted for the cognitive shift associated with the interiorisation of actions. 'Encapsulation' theories — and here the word 'encapsulation' is used as a matter of convenience to indicate that there is a qualitative change in thinking — may indicate that something happens but it does not indicate what this something is. Chapter 3 broadened the perspective of the discussion to consider memory and mental representation and their associated links with arithmetic. The fundamental thesis of this study is that particular kinds of mental representation may be associated with qualitatively different thinking in elementary arithmetic. The purpose of this chapter is to provide the methodological basis through which this thesis is examined.

In investigating what it is that children do in their heads, the purpose of this study is not to build upon studies highlighting neurological connections although the importance of these is recognised. The intention is to consider whether or not different kinds of mental representation may be identified within children of primary school age, to identify whether these different kinds of representation may be associated with the children's level of arithmetical achievement, and to provide some explanation for the way different kinds of mental representation may contribute towards qualitatively different thinking in elementary arithmetic. In carrying out the task, there is an attempt to touch something that cannot be touched, to see something that cannot be seen and to hear something that is articulated in the mind. This necessarily calls on several influences that have directed the methodology of the study:

(i) how we account for differences in children's arithmetical behaviour may be seen to have direct links with actions interiorised as concepts (Piaget, 1965), different forms of mathematical understanding (Skemp, 1977) and qualitatively different forms of mathematical thinking (Gray & Tall, 1994). The added dimension presented by this study is to take forward the latter by considering mental representation associated with extreme levels of arithmetical achievement and to consider the contribution that different kinds of mental representation may make to thinking in elementary arithmetic,

(ii) the way in which children use their knowledge inevitably falls within the information processing paradigm (see, for example, Davis, 1983). However, an investigation which seeks to discover how this knowledge is constructed and the way in which mental representations may contribute to this construction draws upon a constructivist methodology,

(iii) insight into different kinds of mental representations draws upon psychological influences (see, in particular, De Beni & Pazzaglia, 1995),

(iv) the common qualities inherent in the kinds of mental representation identified are offered as one explanation for qualitative differences in thinking. The phenomenographic orientation provides an approach which supports the classification of data obtained through semi-structured interviews,

(v) a fundamental issue is the way in which children deal with elementary arithmetic, and therefore the methodology draws on approaches used by Carpenter and Moser (1981), Seigler and Jenkins (1989) and Gray (1991).

The current chapter presents the theoretical framework which guided the empirical work. In addition it provides a general account of the method relevant to the pilot study, with some indications of the way this was modified for the main study.

Considerations relating particularly to the Main Study are given within Chapter 6.

The Chapter is structured to provide an indication of the focus of the research (4.2) and the integrated way in which the methodology is developed (4.3). The way in which data was collected (4.4), including the nature of the three item banks, the visual (4.4.3), the verbal (4.4.5) and the numeric (4.4.7) precedes initial discussion of the classification of the material (4.6.1). Specific details relating to classification of the children's responses to the visual and verbal elements of the study are not included in this chapter. For the benefit of the reader it is felt to be more appropriate that these are discussed where they are more relevant (The Pilot Study, 5.5.1 and 5.6.1; the Main Study, 6.2.5.2). The limitations of the research method are considered in Section 4.7 whilst a Chapter Summary is presented in 4.8.

Three studies formed the basis for the reported results. Each used a sample of children whose achievement was at the extremes of the arithmetical spectrum. Following an analysis of the quality of the children's achievement, the classifications of Gray and Tall (1994) were used to identify tendencies towards procedural or proceptual thinking. The preliminary study, which was initially designed to consider possible links between proceptual thinking in whole number arithmetic and children's cognitive development in rational number, revealed a wealth of unexpected information associated with children's mental representations. The pilot study, which took note of the issues revealed by the preliminary study, established a format for data collection and analysis which was further developed in the main study. The latter took more detailed note of the relevant psychological research in the area of mental representations.

4.2 A FOCUS FOR THE RESEARCH

The thesis of this study is that arithmetical achievement is influenced by the kind of the mental representations that children form. The work of Skemp (1976) suggests that there is a spectrum of understanding related to mathematical knowledge. As we have seen, this is a theme that has contemporary interpretations which emphasise the separate and/or integrated roles that procedural and conceptual understanding may play in cognitive development. Gray and Tall (1994) suggest that interpretations of mathematical symbolism lead to qualitative differences in thinking manifested in a spectrum of performance. This study seeks to link in this work by examining the way in which different kinds of mental representation may be associated with this spectrum. It is hypothesised that these differences will be related to qualitative differences in the interpretation of mathematical symbols in a proceptual sense.

The study takes the view therefore that:

- children use qualitatively different kinds of mental representation to support their thinking in elementary arithmetic and
- these differences will be manifested in children who display qualitatively different thinking in the context of elementary arithmetic.

The identification of a qualitative relationship between distinctive features associated with cognitive structures and different levels of mathematical achievement is an aim of the study. At issue is the influence that different kinds of mental representation have on the quality of mathematical thinking in an arithmetical context. It will be suggested that children who have difficulty with elementary arithmetic do so because they are building mental representations of a qualitatively different kind from those of their more able peers. It is conjectured that some mental representations, together with the use to which they are put, facilitate the process of encapsulation whilst others may inhibit it.

Thus the study seeks to gain responses to the following questions:

(i) What kinds of mental representations are projected by children at different levels of mathematical achievement?

(ii) What kinds of mental presentation may be associated with children's arithmetical computation?

(iii) What is the association between these different kinds of mental representations and children's qualitatively different approaches to elementary arithmetic?

The study may be seen to have two distinct components: a general arithmetic component which may be clearly linked with previous work on qualitatively different thinking in arithmetic, and a psychological component which draws heavily on earlier work in psychology but extends it through the addition of items which are more clearly arithmetical in nature.

4.3 RESEARCH FRAMEWORK

4.3.1 The Disguised Nature of Mental Representations

Mental representations may appear to be well wrapped possessions, covered in many fine layers and sometimes even hidden in discrete packages. We may believe it is possible to shake the package to find out what is inside but by doing this we run the risk of breaking it. Because of their disguised nature, the interpretation of mental representations is subject to limitations. Nevertheless, this study is an attempt to uncover the layers in order to establish the relationship between children's qualitatively different approaches to simple arithmetic and the quality of mental representations. In doing so our primary source of data is children's verbal and written descriptions.

Uncovering a well wrapped possession or shaking it may be good enough in the above analogy to describe how one may attempt to find what the possession is. However, a methodology that permits us to come closer to reality not only needs to take into consideration the disguised nature of mental representations but also what it is that we might expect to learn from children's responses.

4.3.2 Metacognition

Since one aspect of this study, the arithmetical component, involves children thinking about their own thinking, the notion of metacognition is implicit. It is exemplified in responses to the invitation to *"Tell me how you did that."* However, the study does not take the direct path of eliciting knowledge and beliefs of their own cognitive processes from subjects since these qualities are not often apparent, particularly in young children — metacognitive knowledge can be extremely limited and frequently erroneous (Morris, 1984).

Almost all of our cognitive processes take place without us being aware of what they involve or how they work. This is particularly true of young children since they appear to be notoriously poor at monitoring in domains other than arithmetic and fail to self-regulate their behaviour (Beal, 1987, 1990; Markham, 1979). More recent commentary indicates that particularly in mathematical strategy use, children possess and use metacognition to their advantage (Carr, Alexander & Folds–Bennet, 1994), they possess metacognitive knowledge about mathematical strategies (Garofalo & Lester, 1985) and metacognition helps older children to represent and solve mathematical problems (Schoenfeld, 1987). When evaluation and regulation skills are taught, children show improved metacognitive monitoring of problem-solving (Charles & Lester, 1984), although the ability to take account of on-line feedback seems to develop with age (Schoenfeld, 1992). Metacognitive knowledge related to mathematics is therefore useful when children are undertaking complex problems and using higher-order cognitive strategies. Garafalo and Lester (1985) have identified two primary aspects of metacognition: knowledge of cognition and regulation of cognition. The former includes knowledge of the way in which a strategy is used and the latter our reflection on when it is best to use it and how. In this study, an invitation to respond to the question "What was happening in your head?" will inevitably involve the latter. It is from the response to this question that the former will be inferred.

4.3.3 Information Processing

The structure of knowledge in the mind and the mechanisms by which that knowledge is manipulated, transformed and generated is the focus of attention of information processing methodologies. The central idea is that humans are information processors who construct symbolic representations of the world in their minds (Newall & Simon, 1972). This view sees thinking about and acting in the world as mental operations on such representations, followed by taking actions externally that correspond to the results of the mind's internal working. The two different modes of information processing distinguished by MacInnes and Price (1987) — discursive information processing (symbolic, linguistic or verbal) and imagery processing (which includes perceptual or sensory representations in working memory that are used in much the same way as perceptions of external stimuli) — are clearly important features for this study. Taking a standpoint based on symbolic interpretation presents a perspective by which knowledge is situational and personal. Learning takes place by construction as a consequence of experience; knowledge structures can be hypothesised for particular learners and a learner's performance on named tasks may be used to verify the content and organisation of knowledge.

Distinctions made in this study consider differences associated with 'bottom-up' and 'top-down processing', the former considering the extent to which low-level analysis of the physical features of the sensory input may build towards more abstract interpretations, the latter concerned with the way in which more detailed processing may give rise to aspects which have lower-level qualities. In this sense, the study considers qualitative differences in processing behaviour which may be determined by the stimulus data or the concept that is generated from that data.

4.3.4 Importing from Psychology

In mathematics education we may see methodologies associated with mental representation based on instrument response and analysis followed by interview protocols of selected students (see, for example, Presmeg, 1986), questionnaire (Presmeg & Bergsten, 1995) and task-based problem solving interviews (Thomas, Mulligan & Goldin, 1996). Psychological evidence is frequently obtained through controlled laboratory-like experiments. Since this study is attempting to add to our knowledge of the way children think about mathematics by reflecting upon the way that they think about other aspects of their environment, it seems most appropriate that we turn to psychological paradigms. In doing so we have three objectives in mind:

(i) a framework for the items which will be the source of our data,

(ii) a classification of item responses and

(iii) a deeper insight into the construction of numerical concepts.

The method used within this study provides an opportunity for children to project their mental representations associated with a wide range of objects and to consider processing associated with a series of numerical problems.

To respond to the first objective we draw upon psychological methodologies, particularly item bank structure and delivery. In doing this we are aware that there may be differences in style between what may essentially be regarded as cognitive psychology and our involvement in educational research. Differences in methodology and theoretical perspectives may be marked. For example, the very essence of this study as an essentially qualitative study may remain unconvincing to cognitive psychologists whose emphasis may be on scientific study (see, for example, Entwistle & Marton, 1990). In addition, cognitive psychologists tend to focus on common cognitive processes with little or no regard for individual differences. This study seeks to discover differences at the extremes of arithmetical achievement in an attempt to give some insight in to why the extremes may exist. We seek to explain it and later to have some influence on it.

There is a growing body of opinion in psychology which suggests that the study of mental representations needs a radical shift from the science-based quantitative model (Giorgi, 1987; Allender, 1991; Drake, 1996). Drake, for example, indicates that imagery in the education context and imagery in the clinical setting are not successfully linked and learning associated with the laboratory experiments used within psychology does not correspond with the way individuals learn in real life.

It is not the role of this study to directly make such links but it draws on aspects of research methodologies that have been used in the clinical setting and transfers them to the classroom. It makes one considerable departure from the work of cognitive psychologists – it does not attempt to build a general model established from *common* cognitive processes but instead attempts to consider *differences* in behaviour exhibited by children at the extremes of the mathematical achievement spectrum. Given that the study is intended to explore forms of mental representation and their use and to tie these to mathematical achievement, an orientation towards a phenomenographical perspective of qualitative research appeared to be the best fit for the issues under consideration.

4.3.5 A Phenomenographical Orientation

A central feature of the study is the classification and analysis of responses that children at different ends of the spectrum of arithmetical achievement give to a series of external stimuli. Through this analysis, common qualities inherent in the mental representations are offered as *one* explanation for qualitative differences in thinking that may exist. The phenomenographical orientation provides an approach which supports such classification.

Phenomenography may be defined as "the describing of things which have been brought to light and made clear" (Neuman, 1987, p. 65). One of the problems of this study is that 'bringing to light and making clear' mental representations depends extensively on the classification and the analysis which are formed from the subjective views of the researcher. Phenomenographic approaches provide a mechanism for dealing with the extensive amount of qualitative data that are generated. However, rather than being a full-fledged methodology, the approach is more of a methodological orientation. It attempts to define the ways in which things appear to people and to discern and characterise the qualitatively different ways in which the various phenomena are experienced and conceptualised.

An important aspect of this orientation is the level of description to which it refers. Conceptions — thought of as a unit of description — are accepted and described as data given. However, relationships between different conceptions and the search for deeper structures between the system of conceptions lead to explanations that may be given in more generic terms. The responses that emerge to the items of the study can be conceived of as expressions of the hidden structures of meaning. When analysed within a framework which takes into account the context in which item responses occur, different forms of response may be seen as an expression of one and the same structure.

Thus, the general method used is qualitative although, where it is deemed appropriate differences may be considered using inferential tests for classificatory data. The data elaborated are the statements of the subjects who were interviewed. A semi–clinical interviewing approach is used since one of the intentions is to understand how interviewees conceive the issues under examination.

4.3.6 Semi–Clinical Interviewing

Opper (1977) describes how the dialogue and conversation associated with the delivery of a problem or task provides the subject with every opportunity to display behaviour from which mental mechanisms may be inferred. Thus, variants of a 'thinking aloud' approach (Newall & Simon, 1972) guides the efforts behind this study — subjects were to examine and report on their own mental processes and experience. However, think-aloud verbalisations only provide information about the sequence of reported thoughts mediating a cognitive activity. Such an approach may be reasonable when considering problem-solving activity but for the purposes of this study, more detailed information of the structure of thoughts and the processes which generate them are required. For this reason subjects were required to go beyond thinking aloud and to provide descriptions of their thought processes.

Consequently, to gain a sense of children's mental representations and the strategies used to solve the arithmetical problems, a form of 'structured' and 'open interviewing' techniques (Cohen & Manion, 1985, p. 309) were used. The interviews had a structured component in that in the 'visual/verbal' component, the children were asked to provide responses to the presented items, and in the 'arithmetical' component, answers to sets of predetermined questions. However, where appropriate, subjects were asked to provide additional information or reflect on what had been done to resolve each question such that the integration of the structured and open interview components may be best described by using the term 'revised clinical interview'.

It is recognised that providing an explanation for thinking may disrupt and alter thought processes and retrospectively given explanations may often reflect speculation rather than direct knowledge of underlying properties and processes. Consequently the approach used, draws very closely on the notion of clinical interviewing which permits variation in questions in accordance with the subject's response. This has been a common methodology for probing mental representations and understanding in mathematics (for example, Brown and Presmeg, 1993; Presmeg and Solano, 1995; Hall, 1995; Irwin, 1995; Ginsburg, Kossan, Schwartz and Swanson, 1983). Interviewers have an opportunity to interact and ask for either clarification or explanations during the interview.

The clinical interview technique was originally used by Piaget to achieve three aims which are pivotal to the study of cognitive development: a description of intellectual activity, the specification of the nature and organisation of cognitive processes and an evaluation of a child's level of cognitive competence. In this study such a technique is primarily used to gain insight into the way in which children mentally represent auditory, visual and numerically symbolic stimuli. Designed as an unstructured and open-ended method, the approach is intended to give the child his or her "natural inclination" but the structured aspect does not detract from the overall intention of the interviewer which is to be "led and take account of the whole of the mental context" (Piaget, 1929, p. 8). Verbal, written and diagrammatic items provide the main prompts for discussion.

Hunting and Doig (1995) indicate that clinical interviewing is a tool which can be powerfully applied in practice because its methodology is "closely attuned to a fundamental activity of teaching and learning — interactive communication" (p. 119). Indeed, clinical interviews have provided powerful case study data associated with children's failure to learn mathematics (for example, Allerdice and Ginsburg, 1983) used in longitudinal constructivist teaching experiments (Cobb & Steffe, 1983; Hunting, Davis & Pearne, 1995) and provided the basis for the theory which underpins this study (Gray & Tall, 1994). In the field of mental representations such interviews have frequently been supported by observation and analysis of children's drawings and the way in which they are used (for example, Hughes, 1986; Fennema & Tarte, 1985). Additionally, in this study evidence about individuals was collected from a variety of other sources, including student records and through consultation with teachers.

4.3.7 Reliability and Validity

It is recognised that data obtained from semi-clinical interviews are generated in a particular way, serve a special purpose, are looked at from a particular perspective and reported from a specific point-of-view. Therefore it may be argued that it is impossible to have one hundred percent valid and reliable results. In any given study the researcher has to decide what degree of unreliability and invalidity s(he) will regard as acceptable and also be aware of the limitations that his or her work has. It is an over riding intention of this study to be 'true' to the children and ourselves: to see, hear and feel what is happening in the children's minds.

Of course, a key issue in a study of this sort is whether or not another researcher, working independently, would find the same categories and conceptions that are found here. The words of Johanssen, Marton and Svensson (1985) may be taken as response to this desire:

> We consider the category of description a discovery, why should we require two researchers to make the same discovery independently? On the other hand, once the discovery has been made we should certainly be able to communicate it, and other researchers should be able to use the intellectual tools that are supposed to be the outcome of this kind of research and be able to replicate and confirm our discoveries. Consequently, what we want to ascertain is that once categories of description are made explicit, other researchers should be able to identify them when they are applicable in varying contexts. In accordance with this, indicators of reliability should not concern the extent to which categories are discovered independently, but the extent to which they are identified once they have been specified. (p. 251)

Thus there may not be any reliability of discovery, only a reliability of identification. In actual fact it is not the categories of description that are discovered but conceptions. In this study it was an analysis of all verbal responses that led to categories associated with mental representations. That conceptions were reliable was established firstly through the range of items used to elicit these — the visual, verbal and arithmetical components — and secondly through the somewhat lengthy periods of time between first and final interviews (in some cases this extended to six months). Such a time lapse served to demonstrate the remarkable consistency in children's responses. Only after repeated analysis of the verbal comments were the most powerful descriptive concepts and categories for discussion of the results identified. The reliability associated with classifying individual responses was verified by the separate classification of my supervisor.

Of course not all categories were 'discovered'. For example, those which identified children's approaches in simple arithmetic stemmed from the work of Gray (1991) and Gray and Tall (1994), whilst those which were used to identify objects of thought, for example, counting units, stemmed from the work of Steffe, von Glasersfeld, Richards, and Cobb, (1983). Psychology was the platform from which some other classifications were identified (see, for example, De Beni & Pazzaglia, 1995). Therefore we may see the study offering support to existing classifications but as also qualifying these to provide an overview of the qualitative differences in children's mental representations. Where it is appropriate, children's verbal comments, which were transcribed word-for-word are provided as examples of classifications and in this way the reader herself or himself may be seen as a verifier.

The issue of validity in a study of this sort is not any easier to resolve than that of reliability. The only way in which the interpretations themselves may be seen as valid or not is in whether or not they 'make sense'. If some of the notions cannot be understood as parts of a structured whole, the final model is not intelligible. If data collected by another researcher can be interpreted in terms of the model created here, this makes both of the compared studies intelligible.

However, there exists another form of validation: validation of the responses given by the subjects. How do we know that the response given is a manifestation of the conception held by the subject and not simply a 'random' answer or an explanation given to communicate in that moment? In order to clarify these questions, we can refer to Piaget's (1979) way of identifying "genuine" convictions:

- The consistency with which one child provides responses to the stimuli and the consistency among answers given by many children.
- The appearance of some form of evolution in responses.

There are three additional comments that may be made on this front:

(i) The nature of the responses given by individual children would appear to be remarkably consistent with those given by children in other cultures (Kucerova, 1997)

(ii) Responses given by the children carried similarities to those considered by Seron *et al* (1992), Hitch *et al* (1995), Logie *et al* (1994) De Beni *et al* (1995)

(iii) Some of the 'more' idiosyncratic representations appeared to be more common than originally thought and when attempted by the interviewers seemed to have considerable plausibility.

In this study our concern has not focused on providing an explanation or a cause for any one response. Rather, it has been to identify what a particular response means. Our objective is to respond to all responses and to collectively draw these into the phenomena which may provide some account of children's qualitatively different mathematical behaviour.

4.4 DATA COLLECTION

4.4.1 Refining the Approach

Based on the findings of a preliminary study and a pilot study the data collection instruments were refined, in order to provide a better focus upon the nature of children's mental representations in verbal/visual context-free and arithmetical-context items. Through each study, qualitative data were obtained which formed the basis for classificatory analysis.

4.4.1.1 Preliminary Study

The preliminary study consisted of a qualitative component which sought to investigate the relationship between children's mental representations in whole number arithmetic and their conceptual structures in rational number. A small sample of children in the age range of eight to eleven years and who could be located at the extremes of the achievement spectrum in elementary arithmetic formed the sample for the semi-clinical interviews. Qualitative data in the form of descriptive responses to verbally and visually presented material provided a sense of the different forms of mental representation. These descriptive comments indicated that children at the extremes of arithmetical achievement reported mental representations which, on the one hand, were concrete and detailed, whilst on the other, were more abstract and associated with symbolism.

4.4.1.2 Pilot Study

Drawing on the experience of the preliminary study, the pilot study considered the quality and use of the mental representations by children at the extremes of arithmetical achievement. The research issue was directed towards the different kinds and uses of mental representations associated with words, symbols and icons and with the children's solutions to elementary arithmetic combinations. These were considered through responses to four item banks: a verbal context-free component, a visual context-free component, a verbal arithmetical-context component and a visual arithmetical-context component. Repeated analysis of the children's responses to the verbal and visual phases provided a mechanism for classifying the most powerful descriptive concepts. The results indicated that those at the lower end of the spectrum tended to possess mental representations which referred to specific detail and frequently were active, whilst those at the higher end tended to project deeper meaning which being of a generic character project relational qualities.

4.4.1.3 Main Study

The main study grew out of the pilot study. Again, it considered children at the extremes of arithmetical achievement. It was directed towards the use of a more psychological approach in presentation of expanded visual and verbal item banks and in the classification of the responses. Its method its discussed in detail in Chapter 6.

4.4.2 Evoking Mental Representations

A major difficulty associated with work on mental representations is that in attempting to understand and communicate them to others we rely on words (see Mead, 1938; Stas & Lohr, 1979; Ahsen, 1987). However, psychological and mathematics education theories associated with mental representations have in common the notion that a mental representation corresponds to a description and/or drawing (see, for example Kosslyn, 1980; Pylyshyn, 1973; Pirie & Kieran, 1994; Thomas, Mulligan, & Goldin, 1995). The general methodological and conceptual obstacles are however, challenging problems in their own right :

> "... the study of imagery has always been plagued by the difficulty of finding a relatively direct and unambiguous observation criterion for imagery activity". (Paivio, 1986, p. 3)

It is in a similar context that the scope of this study is extended to cover the investigation of mental representation. Our investigation suffers from all of the problems associated with an investigation of imagery. Thus, two features govern the framework for the development of items that formed the basis for the analysis. The first is firmly embedded in psychological approaches which evoke imagery through a verbal and visual stimulus (see, for example, Stillman & Kemp, 1996). The second is that the relationship between mental representations and achievement should focus on two theoretical issues:

- the existence of different kinds of mental representation may be characterised by different generation processes, the functional properties of these representations and the likelihood that different forms can be grouped into a number of categories, and

- the relationship between an emphasis on one or more of these categories and the level of numerical achievement.

4.4.3 Developing the Verbal Item Bank

A modified version of the defining feature approach (see, for example, Roth & Bruce, 1995) was used to gain a sense of what it is children feel is important to communicate when faced with conceptual labels in verbal form.

In the preliminary study, the items chosen for the verbal phase of the investigation were established by considering those common in the mathematics curriculum associated with learning fractions. It was later extended to include two words chosen almost at random from within a child's normal experience. In all, the item bank consisted of five nouns, 'half', 'four ninths', 'fraction', 'sweet' and 'ball'. The purpose of the initial group of 'numerical' nouns was to gain insight into children's understanding in the sense established by Gray and Tall, (1995): "What does the word ... mean to you?". It was the result of children's claims about what they 'saw' in their heads that prompted the inclusion of the two concrete nouns.

The observations from the preliminary study led to the adoption of two principles which guided the development of the verbal phase of the pilot study:

(i) concrete words, in the sense that they denote things that can be perceived by one of the sense modalities, evoke images more readily than other words (Paivio, Yuille & Rogers, 1968; Paivio, 1971).

(ii) conceptual labels which had more abstract meaning are strongly associated with mathematical ideas. Formed from the encapsulation of numerical processes, such labels require the cognitive reconstruction of associated actions to form the basis for new conceptual entities. These have mental qualities associated with concreteness but they cannot be perceived directly by the senses. Little psychological evidence is available to establish how such labels may be determined within a framework of concreteness and associated imagery. Indeed a list of 925 words constructed with such qualities (Paivio, Yuille, & Rogers, 1969) contained no such item. Here, then, we were somewhat 'in the dark'. The final range of words selected were somewhat arbitrary but based upon discussion with teachers of their perception of children's experience with them. These words covered such notions as "five", "half", "number" and "fraction".

The full list used for the pilot study now consisted of the words 'ball', 'car', and 'triangle', (concrete nouns associated with a tendency to evoke imagery), 'five', 'half' and 'four ninths', 'number ' and 'fraction' (nouns which have proceptual qualities).

In the main study additional words were included and, in particular, the numerical range was considerably extended to reflect the full age/achievement range of the children. In addition, the evidence from the pilot study tentatively indicated a relationship between such words as 'five' and 'number', 'ball' and 'football'. Such parings were increased and made more explicit to gain a deeper sense of this relationship. The complete list now included:

- nouns which evoked imagery: 'ball', 'dots', 'football', whilst 'dog', 'table' were paired with 'animal' and 'furniture' and
- numerical nouns: 'five', 'seven', 'thirty three', 'ninety nine', 'half', 'three quarters', 'three eighths', 'nought point seven five', 'number' and 'fraction'.

4.4.4 Presenting the Verbal Item Bank

In the pilot study the presentation of each word was accompanied by two invitations:

(i) *What comes to mind when you hear the word ... ?*

(ii) *If ET (a creature from another world which children would be familiar with) came to ask you what the word meant what would you say to him?*

The form of the questions was designed to:

- gain a sense of the initial mental representations when triggered by a verbal stimulus,
- find the best way to describe or define the word under consideration and
- indicate what it was that they considered important enough about the word to transmit to someone who did not understand it.

The notion of '*comes to mind*' was used so that children would not be directed to talk about something 'seen' in the mind. The study was not only concerned with visual imagery but with all the kind of mental representations (in any modality) that different children formed of the words. Directing them towards visual imagery may have firstly confused the issue of 'free expression' by restricting access to something that may not automatically happen and secondly also caused some children difficulty in terms of understanding.

The notion of "what comes to mind" does have its weakness however. It does not clearly indicate what is the initial mental representation and how children progressed from there, since children are talking for different lengths of time. Moreover, it does not allow the analysis to consider the deeper differences between 'bottom-up' or 'top-down' developments. The indication from the pilot study was that there were differences between the two extremes which were reflected by 'horizontal' responses which were qualitatively similar and 'vertical' responses which were qualitatively different. It was therefore decided to sub-divide this first invitation into two parts: a 'first response' and a 'free talk' component. Since the first component considered only initial mental representation this extended the opportunity for "free talk" and provided an opportunity for enrichment of the initial response with greater detail or additional information through a network of other relationships (see, for example, De Beni & Pazzaglia, 1995). In the pilot study, children were given about 30 seconds to respond to the one question. In the main study this component was adapted to fit psychological norms (see, De Beni & Pazzaglia, 1995; Paivio, Yuille and Madigan, 1968) and children were given exactly 30 seconds to talk freely about the stimulus. The basic question structure reflects a desire to provide children with an extended opportunity to talk about the item in question. Three questions now formed the basis for responses:

(i) *What is the first thing that comes to mind when you hear the word...?*

(ii) *Talk for 30 seconds about what comes into your mind about the word.*

(iii) *If ET came to ask you what the word meant ,what would you say to him?*

Several other issues arose during this phase of the study which required clarification during the main study. Therefore the series of main questions was supplemented by questions which sought to investigate: "*what was in your head as you were saying this?*". This question was intended to establish the form of the mental representation since informal trials had indicated that children responded by saying, "*I saw it*" (visual image), "*I saw the words* (visual image), "*I spoke it* (language-like), or, indeed, "*Nothing!*". It also gave the possibility for a more detailed description of the mental representation and the way it was functioning.

4.4.5 The Visual Item Bank

In the field of psychology it is common to consider the processing of information presented in words and pictures (Lupker, 1985). Pictures stimulate recall much more easily than words (Eysenck & Keane, 1995) and there are indications that they produce higher vividness than auditory stimulus (Tracy, Roesner & Kovac, 1988). Although these notions provided a rationale for the presentation of the visual items, in this study they were presented to establish:

(i) the kind of mental representation triggered by visual presentation (since they were not asked to describe) and

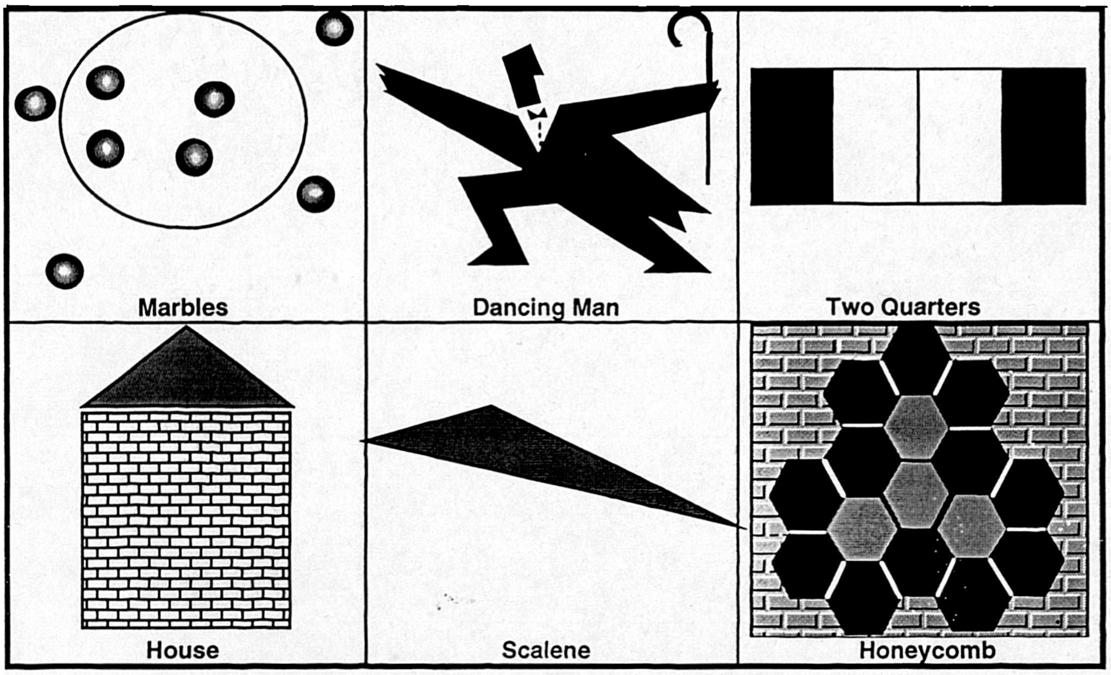
(ii) the underlying properties of the mental representation which were extracted from the surface features or from the meaning of the visual stimulus.

In addition, there were three other considerations:

(i) to identify whether or not children at extremes of numerical achievement create mental representations which are focused on different aspects of the items and

(iii) to identify qualitative links between children's responses to verbal and visual stimuli.

In addition to the numerical symbols '5', '$\frac{3}{4}$', '1995', and '3÷4' a range of 'pictures', only one of which was used in the pre-pilot (window), were separately trialed and included in the range of items. These are shown in Figure 4. (The names were not included in the presentation. They were in fact some suggestions from children given during the pre-pilot trial).



4.1: Visual component: The iconic items

Three of these visual components were dropped from the main study: '1995', 'house', and 'scalene'. This was done for four reasons. Firstly, the results suggested that the inclusion of real pictures and an expansion of the numerical component would provide a firmer basis for drawing conclusions. Secondly the icon items did not possess a degree of ambiguity which was apparent in the others. Thirdly the study did not intend to draw any comparisons between mental representations of geometrical and non-geometrical concepts: thus house and scalene could be dropped. A fourth and probably the most important reason, was to include items which were paired with verbal items so that direct comparisons of mental representations could be made.

The indications from the pilot study were that 'low achievers' tended to focus on the detail of the objects and build stories around them whilst 'high achievers' focused on the objects as representations of other ideas. It was therefore considered appropriate to include actual items and pictures in the main study. The visual phase of the main study now comprised 16 items in three forms:

Symbolic items: 5, 99, 3÷4, $\frac{1}{2}$, $\frac{3}{4}$, 0.75,

Iconic items: 'dots', 'marbles', 'dancing man', 'honeycomb' and 'two quarters',

Pictures associated with words in the verbal phase: 'football', 'dinning room', 'furniture', 'table' and ball'.

Each of the symbolic and iconic items of the visual component and one of the pictures (football) were presented on card of size approximately 7cm by 7cm. The other pictures were given on 20 x 15 cm card. The names, derived from common interpretations by children in the pre-pilot, were not included on the cards.

4.4.4.6 Presenting the Visual Item Bank

In the pilot study the format of presentation invited children to "*Look at this and tell me what comes to your mind*". For reasons similar to those discussed within the verbal component (Section 4.4.4.), once again, this invitation was subdivided into two parts for the main study:

(i) *Look at this and tell me what is the first thing that comes to mind.*

(ii) *When I tell you, close your eyes and put this in your mind. Talk to me for 30 seconds. Do it now.*

The evidence from the pilot study was that some children claimed to have seen a visual image associated with the item in question. An additional question was therefore added in the main study: "*As you were talking to me what was happening in your head?*". This question helped to identify what it was a child directed attention towards and clarified whether or not a visual image was reported and if possible give some more details about the mental representation and how it was functioning.

Although the notion of a 'picture' was used with the children, the visual items could be identified in three forms: iconic, symbolic and pictorial.

4.4.7 The numerical components

Arithmetical processing, and different levels of achievement as a result of this processing, comprise a key issue in this study. Here, interest lies in three notions:

(i) the strategies children of different achievement levels use to solve a range of addition and subtraction combinations. Through determining these, it was possible to identify procedural and proceptual differences and thus confirm levels of achievement on which children were originally selected. In the main study, it was possible to identify the consequences of these qualitative differences when they occurred in elementary arithmetic,

(ii) the mental representations children reported when using these strategies (modality, content, components) and

(iii) the relationship between the quality of these representations and the mental representations reported for the verbal and visual items.

Two item banks formed the basis for this aspect of the study: a verbally presented component and a visual component.

4.4.7.1 Verbally Presented Items

Following Gray (1991) and Gray and Tall (1994), children were presented with a range of context-free addition and subtraction combinations. In the pilot study these were given at two levels of difficulty: number combinations to 10 and number combinations to 20. In the main study, these were extended to include addition and subtraction combinations to 100 and word problems.

The numerical items were drawn from various years of the Standard Attainment Tasks of Key Stages One and Two published by the Schools Curriculum and Assessment Authority (SCAA, 1992, 1993, 1994).

4.4.7.2 Visually Presented Items

These were not included within the pilot test. Evidence gained from the verbal presentation of number combinations used there indicated that children at the lower end of the achievement spectrum had difficulty remembering the combinations, particularly if they translated symbolism into processes to do. The visual presentation of the numerical component provided a basis for comparison of children's representations stimulated by the verbal and visual modes of presentation. Therefore, the purpose of giving combinations in visual form was partly to relieve the pressure on short term memory, partly to act as a validator in considering children's overall strategy to obtain solutions, partly to obtain insight into what children described as happening in their heads and partly to have analogous phases in the arithmetical phase to those of the free context phase (visual, verbal) so that comparisons could be made.

4.5 SAMPLING

4.5.1 Considerations

We saw earlier that it is a common feature in psychology to seek commonalty in cognitive processes and design models of behaviour associated with them. Drawn from the premise is that there are qualitative differences in children's information processing which in the context of arithmetic gives some the basis for flexibility in thought and others an over-reliance on procedural approaches, the fundamental thesis of this study suggests that these differences may be linked to different kinds of mental representation. It is this hypothesis which guided the selection of the samples, that they should display quantitative differences in mathematical achievement. Thus children were drawn from extremes of the mathematical spectrum.

One additional requirement directed the nature of the sample. Gray & Tall (1994) hypothesise that the occurrence of a proceptual divide causes a divergence in children's approaches to elementary arithmetic, the consequences of which may be seen when higher levels of thought are required. Those who are procedural remain procedural with inevitable consequences of failure. Those who are proceptual have the flexibility which could bring success. Are mental representations projected by young children different from those projected by older children and how are tendencies towards particular kinds of representation associated with the notions of the proceptual divide? To begin to answer this question the sample was selected to reflect the year groups within the study schools.

Because of the nature of the interactions between interviewer and interviewee, it seemed necessary for the children to be fairly articulate. In the event, because of the types of school from which samples were drawn we did not need to give further consideration to this aspect.

4.5.2 The Schools

Children who formed the samples for the study were drawn from two schools within the English Midlands. Both schools, fairly representative of schools at the edge of small towns, were selected because teachers in these schools had participated in INSET courses at the University of Warwick and displayed a keen interest in mathematical development. Equi-distant on an east/west axis from the University of Warwick, the catchment areas of both were very similar, both receiving children from the towns and the immediate rural surroundings. The distinctive difference between them was one of nature and size. School A, used for the pre-pilot and the pilot study, was deemed to be a 'middle school'. The children attending the school were aged 8.0 to 11.11, providing samples from Year 4 (median age 8.6) to Year 7 (median age 11.6). Entry into each year group comprised about 120 children. School B, used for the main study, was a primary school and it had a yearly average entry of 32 children. The sample of children used for the study was drawn from Year 3 (age 7.0–8.11) to Year 6 (age 10.0–11.11).

4.5.3 The Children

In both cases children selected for the sample were chosen on the basis of their level of mathematical achievement. These were established through level of achievement in the Standard Assessment Tasks of England and Wales (SCAA, 1994, 1996), supplemented by other standardised tests scores, for example, Mathematical Concepts and Skills components of the Richmond Attainment Tests (Hieronymus & Lidquist, 1974), together with results from teachers' regular assessment programmes. The pilot study contained three children from each end of the achievement spectrum for four year groups. For the sake of convenience, throughout the study we refer to these children as 'high achievers' and 'low achievers'. These terms are used for simplicity to identify children's levels of achievement against criteria based norms. They do not reflect any opinions on the underlying ability of the children, nor do they imply that any longer-term prognosis is being made concerning their eventual levels of achievement in mathematics.

The main study started with four children in each year group, again equally sub-divided into 'high' and 'low achievers'. Interviews for the main study were carried out over a nine month period. This inevitably meant some movement of children which limited the numbers in some year groups, particularly amongst the 'low' achievers of Year 4. In addition, it became evident throughout the interviews that because of the relatively small size of the population (N approximately 30), the extremes of achievement were being masked. Eventually, the verbally administered arithmetical component of this study was used to identify the two highest and two lowest achievers in each year group. It is this selection that forms the basis for reporting in the main study.

4.5.4 Interview procedures

Apart from the pre-pilot study, which comprised an average of two interviews with each child, children who formed the sample for the pilot study and the main study were interviewed at different times for each series of items that comprised the individual components of the studies. For the pilot study, this meant at least three interviews with each child of approximately half an hour duration whilst the main study involved six interviews each of a similar average time. The pilot interviews were carried out over the period March to September 1995 whilst the main study interviews were held during March to November 1996.

Each interview was prompted with a reminder to the children that our interest lay in the way that they thought about things. Each item which promoted discussion was delivered separately and, where appropriate, clinical interviewing was used to expand on children's responses. All interviews were either tape-recorded or video-recorded and transcribed to provide the evidence for further analysis. Field notes taken at each interview were used to support these transcriptions.

4.6 CLASSIFYING RESPONSES

4.6.1 Some Pragmatic Considerations

Classifying the wealth of data available may be considered at three levels:

(i) the identification of strategies used to solve arithmetical combinations,

(ii) classification of quality of response to the verbal and visual items and

(iii) classification across all of the items based on notions of "what was happening in the child's head".

From a pragmatic point-of-view, and to ease the strain on the reader, it is felt to be more appropriate if the classifications associated with the verbal and visual phases of this study are best considered in the sections dealing with the Pilot Study (Chapter 5) and the verbal and visual components in the sections on the Main study (Chapters 6 and 7). Similarly, it is also felt to be more appropriate that classification of responses to the question "What was happening in your head as you were talking to me"? would also be best considered in the relevant Chapters 6, 7 and 8.

There are two reasons for this. Firstly the classifications associated with the verbal and visual phases were subject to change, partly because of the nature of the invitations that prompted discussion but mostly because those associated with the main study are more closely related to the psychological literature, particularly that of De Beni and Pazzaglia (1995). This is not to say that the change negates the original classifications, far from it. The original classifications arise from more simplistic, almost one-sentence responses where it was relatively easy to identify classifications that have an independent quality. Those in the main study are more complex and fit better the discussions that take place. Both add to the overall conclusions which indicate that the mental representations of 'high achievers' may have a general foundation but also a generic and/or proceptual quality. Those of the 'low achievers', seen to be more descriptive and episodic, once again may originate in generality yet focus upon specificity.

4.6.2 Classifying Number Combinations to 20

The introduction to each bank of items involving the presentation of the numerical combinations reminded the children that the object of the interview was for the interviewer to gain some sense of the approach the child used to solve particular items and to gain some indication of what was happening in the head.

The classification of strategies for number combinations to 10 and to 20 followed those given by Gray (1991) in arithmetical context-free situations. To obtain the solution to any one problem based on the addition and subtraction facts to 20 a child may use any one of four basic strategies. The child may:

- know how to count: **count-all,**
- conceptualise the value of at least one set and use the appropriate counting procedure: **count-on** (or count-up / count-back),
- use any other known number facts: **derived fact** and

- know directly: **known fact.**

4.6.3 Classifying Number Combinations in excess of 20

To classify number combinations in excess of 20, our approach drew upon the reported evidence of Oliver, Murray and Human (1990) and Gray (1994). Both addition and subtraction combinations were similarly treated as either sequence counting or transformation approaches.

4.6.3.1 Sequence Counting

Though a distinction is made between sequencing forward or backwards in ones, the manner in which this approach is used is at its most sophisticated when children use accumulation or iterative strategies. Here, a gradual approach is made to the final answer by a series of increasingly better approximations. Most frequently, this was exemplified in sequencing in tens and then in ones in the following manner:

24 + 43: <u>40</u>, 50, 60,...64...65, 66, 67

The subtractive equivalent would be sequence decrementing in tens and/or ones.

4.6.3.2 Transformation Approaches

This classification was used whenever combinations were rearranged before any attempt at the computation. For example:

39 + 26: 30 + 20 =50 9 + 6 = 15 50 + 15 = 65

or 30 + 20 =50 9 + 6 = 15 50 + 10 = 60 60 + 5 = 65

The subtractive equivalent would involve an equivalent strategy using either a transformation, for example:

64 – 26: 60 – 22 =38 38 + 4 = 42

or an approach which involves 'equal addition', for example:

29 – 6: 10 – 7 = 3 3 + 20 = 23

4.6.3.3 The Algorithmic approach (Known fact right to left)

This classification was used whenever the child, mentally or with pen and paper, used an approach which could be identified as a decomposition or equal addition approach (adding 10). Where it was used, a left to right strategy was identified as were the more common misunderstandings, for example smallest from largest irrespective of which number was the subtrahend and minuend.

4.7 RESEARCH LIMITATIONS

To gain some sense of the issues that form the focus of this study, we make the assumption that report, description and external representations in the forms of words, diagrams, drawings and actions provide an indication of the nature of the mental representations possessed by young children. The way in which the child communicates, whether vocal, visual or motor, reveals something of how s(he) has internally represented similar information. However, no precise claims can be made about the nature of mental representations.

We use 'free talk' introspection and clinical interviewing to gain the data for the study. These approaches may lead a child to make a verbal statement or provide a written representation which complements his/her mental representation. However, not only may children have different interpretations of what has been asked of them but their responses may require interpretation by the interviewer. Such interpretation may be problematic.

There is no presumption that children who work with counters or fingers represent all quantities internally as mental images of counters, fingers or other manipulatives. What is presumed is that children who report such images represent the quantities and actions differently for themselves than do children who interact only with the symbols. Although Kaput (1992) provides a notion of the distinctions that are central to this study — the world of mental operations and the world of physical operations — he also provides the note of caution in which the framework of the analysis must be viewed. Mental representations are always hypothetical whereas physical representations are usually observable.

The analysis and categorisation of children's reported behaviour has been selected and organised by the researcher. Thus, theoretical constructs placed on the analysis are not entirely free of the observer's expectations and theories, however objective these may have appeared to be at the time of observation. The study attempts to establish new theoretical perspectives to explain behaviour. The data established from classifications may also have been subject to expectations and theories. However, it is believed that qualitative differences in thinking cannot be automatically inferred from discontinuity in quantitative measurement if such measurement comes from a complex hierarchy of classification. It is partly for this reason that the study attempts to view the differences in children's behaviour in a qualitative way.

Any attempts at description will be constrained by the aspects of data collection and the levels of reporting which are subsequently presented. Thus, a possible source of error may concern the nature of the interaction: it could cause children to provide different interpretations and employ different approaches during the interviews to those that may have been used if the interviewer was not present. Any request to the child that s(he) reflect on what has been done or provide some insight into the things that happened in their mind may alter the nature of the response. Any effort to explain or provide a description which relates to reality can be very different from what has actually taken place in the mind. It is for this reason that revised clinical interviewing is used. It attempts to minimise misunderstanding, particularly in the arithmetical component. In most cases where children used unsolicited verbal utterances and in some cases where they appeared to use physical manipulation, additional verbal interaction served to clarify the interviewer's interpretation of the approach used.

Using metacognitive knowledge as a mechanism for understanding thinking is fraught with difficulty but, on balance, it has provided some considerable insights into learning and thinking. The mere act of explaining may influence what it is that is explained; there may be subconscious actions that are not reported. The effects of such influences are minimised through the range of items that form the stimuli for discussion. It is this range which provides the background for our discussion of commonalty in approach.

Perhaps the most over-riding limitation of the study is that it serves to clarify conceptions of qualitatively different thinking amongst children at extremes of achievement in only *two* schools. It does not say anything about the greater proportion of the children, that great mass who may be deemed to be reaching expected levels of achievement. However, it is strongly hypothesised that looking at the extremes may well signal some of the conflicts that are occurring in the remainder of the spectrum.

The study is reporting what is happening now. It does not attempt to consider the external factors that may influence the way that children think. Neither does it attempt to imply anything about the quality of the pedagogy that children have been subject to although, again, it is believed that the results will have implications for pedagogy. We may take the narrowest view of the importance of the results, that they only reflect particular cultural, environmental, pedagogic and even genetic influences, or an amalgam of all of these. This may be so but the fact remains that children are seeing things differently and they are doing things differently. This has consequences for their behaviour in arithmetic and it may well have consequences for a longer-term prognosis of achievement and its quality. It may also have consequences for the way in which we consider the development of a curriculum which, it is assumed, is out there for all.

4.8 CHAPTER SUMMARY

In this chapter we have considered the rationale and background for the method used in the study. It has been a wide-ranging chapter as there have been many influences on the design and implementation of the study. Drawing together influences that come from mathematics education and psychology has not been an easy task. To then place these within a perspective that covers information processing, constructivism and phenomenographic approaches has added to the complexity. Weaknesses in methodological design are seldom apparent when we embark on a project — they become clearer as the project reaches fruition. Nevertheless, it is worth repeating that the method used was a serious attempt to remain true to the children, to ourselves and to the nature of the enquiry. It has been a task that has encountered difficulty and caused frustration. Nevertheless, it is with confidence that the empirical results from the study are now presented.

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CHAPTER 5
PILOT STUDIES

5.1 INTRODUCTION

> "I was seeing the number and then it sort of went and faded away. I then tried to think of things connected with that." (Hannah, Year 6)

The chapter is developed through two themes. Firstly it reports the outcomes of the preliminary study (Section 5.2) which triggered the nature of the overall investigation. It was not the original intention to investigate the topic that led to the current study. However the preliminary study raised issues that seemed worthy of further consideration. The second theme in the chapter is that of the pilot study (Section 5.3). This takes forward the issues raised in the preliminary study by considering children's mental representations within elementary arithmetic and the way in which these representations were used (Section 5.4) The general pattern that emerges from the use of these representations is then associated with the kinds of mental representation projected from verbally and visually presented items (Sections 5.5 and 5.6)

The final conclusions (Section 5.7) suggested that children identified as 'low achievers' in elementary arithmetic appear to be unable to detach themselves from a search for substance and physical identity. 'High achievers', on the other hand, seem more attuned to talking about items at an impersonal level, temporarily ignoring substance and description to concentrate upon abstract and relational qualities. It is conjectured that these fundamental differences contribute towards divergence in elementary arithmetic.

5.2 PRELIMINARY STUDY

5.2.1 Introductory Comments

The preliminary study was designed to consider the relationship between children's understanding of whole number arithmetic and their understanding of the field of rational numbers. It featured semi-clinical interviewing methods to gain an understanding of children's mental representations associated with the names and written symbols linked to areas of arithmetic and rational number that they had varying degrees of experience in. The children's levels of mathematical achievement reflected a spectrum of performance, such that within each of four year groups the children were drawn from the near extremes. Semi-clinical interviewing gave the opportunity to extend the nature of the interaction that led to two observations:

(i) children who displayed qualitatively different forms of mathematical thinking called upon different kinds of mental representation to support or direct their thinking and

(ii)) the kinds of representations projected by the children may possibly transcend the arithmetical and non-arithmetical boundary.

5.2.2 Method

It is not the intention here to report on the preliminary study in great depth. It is sufficient to consider it in a brief overview which shows how the development of item banks, the selection of the sample and the general results informed the preparation of the pilot study.

Visual and verbal elements of the study were initially seen to serve the purpose of allowing children to indicate what they thought of when presented with each stimulus. These were intentionally limited in scope and reflected an initial interest in mental representations and rational number. Consequently verbal items chosen in consultation with teachers included the notions of 'fraction', 'half' and 'four ninths'. These were later supplemented with the nouns 'ball' and 'sweet'. The picture  (later to be known as window) was given as a visual stimulus.

The presentation of each verbal item was accompanied by the question "*What comes to mind when you hear the word...?*". The visual stimulus was accompanied by the question "*What comes to mind when you see this?*". These questions provided children with an opportunity to project their mental representation of an item through 'free talk' in a semi-clinical interviewing situation. On a separate occasion, the children were verbally presented with a selection of elementary number combinations representative of those listed in Section 5.4.1. It is not the intention here to report in detail on these since that would not serve any real purpose. However, it is important to note that the approaches children used to solve these combinations was consistent with those reported in Gray (1991) and Gray and Tall (1994).

Children from each of four year groups representing the ages 8+, 9+, 10+ and 11+ years was selected by considering achievement in formal tests. The school used for the preliminary study was also the school to be used for the pilot study. A large middle school in a small town in the English Midlands it had a mixed catchment area and drew its children from the town suburbs, the town itself and the immediate rural surroundings. One sample was drawn from the upper quartile of the achievement range of each year group of the school. These are identified as 'high achievers'. 'Low achievers' were drawn from the lower quartile of each year group. Teachers were expressly requested not to identify the 'best' or the 'worst' children within a year group. Finally eight children, one from each extreme of the four year groups representing children aged 8+ to 11+, were interviewed.

Although the children were interviewed four times, and an extensive range of items were used on these occasions, only part of two of those interviews, a visual and a verbal component, are considered here. In reporting the findings we draw on part of two half-hour interviews with each child. Each interview was audio-taped and was supported with field notes. Initially, one interviewer talked to a child while a second monitored the interview, collected field notes and later sought clarification on some of the issues. The study was carried out during March and May 1995.

5.2.3 Results

The most striking result to emerge from the study was that not only may children possess different mental representations but children at the extremes of arithmetical achievement appeared to have qualitatively different representations. For example, when responding to the verbal stimulus 'half', 'low achievers' focused on active aspects such as "*just cut something*"(11+)[1], the something variously described as "*a sweet*", "*a cake*", "*an apple*" or "*a bar of chocolate*". 'High achievers' attempted to identify symbolic connections such as "*one over two*", "*four out of eight*", together with standard geometrical representations. Three of these four children considered equivalences in terms of fraction and decimal.

Similarly, when responding to the item 'ball', 'low achievers' referred to a specific ball which they then described through reference to colour and/or visible marks. "*It has Aston Villa written on it*" (9+) and "*It has white hexagons and black pentagons*"(10+). Alternatively, and this was a feature of the 10+ and 11+ children, they provided several examples of different balls such as "... *football, basketball, netball*"(11+). In contrast, the 'high achievers' provided more abstract comments: "*It's a sphere or a round thing you play with.*"(11+). From this context, these children then seemed able to put this abstract notion into different environments, such as games, and in two instances, the planets.

Responses to the visual item showed similar trends. The 'low achievers' tried to give the representation colour and detail or place it in a context. Thus the rectangle was variously described as "*lift doors*" (child 8+), "*a window*"(child 9+), "*a football pitch*"(child, 10+) and "*an elevator*"(child, 11+). Each specific part of the 'picture' became a detail to talk about: "*these* [the dark parts] *are the curtains that cover the window*"(8+) or "*they are flaps that open to see out*"(9+). The representation was described as if it were a picture out of focus. It was as if the children were concentrating

1 Within this phase of the study the age of the child is abbreviated to N+. This will reflect the fact that children who would reach their eighth birthday during the school year was 8+ and in Year 4, 9+ children from Year 5, 10+ children from Year 5 and 11+ children from Year 7.

on the 'picture' in an effort to fit it to the environment that they knew. Presented as an outline the children appeared to want to overload it with detail in an attempt to make it look real.

'High achievers' tended to see the parts of the picture in relation to each other. All of them chose to talk about the word 'half' or 'fraction'. If they did put detail on, it was linked to a mathematical notion: "*these are the curtains that cover half of it*"(child, 8+). Although they did place real life perspectives on the picture they also talked about more abstract issues: "*It is a half because it is two and four*" (child, 11+), "*two fourths, these are equivalent to one half. The one half was the first thing seen. The fourth can be denominator and the two the numerator*"(10+).

5.2.4 Issues Arising from the Preliminary Study

Several issues emerged from the analysis of the responses cited in this preliminary study. When responding to the verbal items:

- 'low achievers' tended to discuss very specific examples of the item and provide extensive details of surface features,
- 'high achievers' tended to talk in more abstract terms about them and
- where it was appropriate, for example, when considering 'fraction' and 'half', the 'high achievers' were likely to refer to symbolism whereas 'low achievers' did not.

When responding to the visual items:

- 'low achievers' gave descriptions using any context, but, these were divorced from formal abstract vocabulary or notions such as number or shape. Unlike their more able peers, they were distracted by colour and/or the need to invent a story and
- the more able children immediately said "half" and then gave possible embellishments such as colour and/or shape.

Above the age of 9+ the clear distinction between 'low' and 'high achievers' was shown by the ability of the latter to draw symbolism into the discussion and relate this to actions on objects. It is conjectured that this provided 'high achievers' with a clear advantage over 'low achievers'. Being able to see the fraction as a symbol did not limit them to a representation which they then had to free themselves from in order to discuss a new problem situation. In contrast, the mental representations of the 'low achievers' tended to be of an iconic or pictorial form. It is conjectured that this placed them at some disadvantage. New or different representations in a new problem situation may not fit the mental representation they already possessed.

5.2.5 Informing the Pilot Study

Though the item bank in the preliminary study was very limited, it suggested that qualitative differences in children's mental representations may exist and these differences may have common features. In addition, the format of the interviews suggested that any emphasis on arithmetical strategies should also include an attempt to establish what it was that children were doing in their heads as they were using their strategy. The pre-pilot had also provided an opportunity to establish an appropriate length of time for each interview – about 30 minutes. More importantly it had indicated that the key interview questions needed to be broadened in scope to include a first reaction to the stimulus and an opportunity to provide an explanation of it.

5.3 PILOT STUDY

5.3.1 Purpose of the Pilot Study

As a result of the preliminary study, it was hypothesised that there was a relationship between numerical achievement and the quality of the mental representation that formed the basis for the mental processing of elementary number combinations. It was further hypothesised that children at the extremes of a spectrum of numerical achievement possess qualitatively different kinds of mental representation and used these representations in a different way. Thus, two research issues governed the preparation, delivery and analysis of the pilot study:

(i) What kind of mental representation did children project in elementary arithmetic?

(ii) Were there differences in the nature of the representations that may be associated with children of different levels of numerical achievement?

Four other questions, associated with these issues also directed the focus of attention within the pilot study:

- what kind of the mental representation is dominant when children deal with elementary arithmetic?
- how is this mental representation used?
- what kind of mental representation is associated with numerical items in a context free situation?
- is their any relationship between the kinds of mental representations associated with numerical and non numerical items presented in a verbal and a visual fashion?

Thus objectives for the pilot study were:

- to investigate kinds of the mental representations possessed by children across a range of ages at extremes of arithmetical achievement,
- to classify these representations and to identify links between them and numerical achievement.

5.3.2 Study Design

Much of the study design has already been discussed in Chapter 4 (Section 4.4). Suffice to say here that four interview phases — verbal, visual and two arithmetical — were used to gather qualitative data for classification and analysis. Each interview phase lasted approximately half an hour. At the start of each interview, children were told how they may help in the study and invited to talk clearly about any thoughts they had associated with the things they would see, hear or do. All interviews were audio-taped and some were video-taped. The transcribed comments were supported by field notes. Several of the initial interviews in each phase were conducted by two interviewers, one carrying out the interview, the other obtaining field notes. At the end of these 'joint' interviews, the interviewing procedure was discussed and, if necessary, modified so that within the semi-clinical interview paradigm, there was uniformity in framing the general pattern of questioning to obtain consistent data.

5.3.3 The Sample

The sample was selected from school A as that described in Chapter 4, (Section 4.5.2.) Three children from the achievement extremes of four year groups — giving a total of 24 — 12 'high achievers' and 12 'low achievers', were selected for the sample. Children were selected from the extremes of numerical achievement in each year group thus giving 'high' and 'low achievers' from Y4 to Y7 (Age 9+ to 12+). Level of achievement was identified largely through performance on the two mathematical components of the Richmond Tests in Basic Skills (France, Hieronymus and Lidquist, 1974). The average Richmond Test scores for all of the selected 'high achievers' was 130, the maximum. Since several children in each year group received this score, other information such as informal test results and teacher assessment was used to make the final selection. The 'low achievers' were all selected from within two special classes run for '*children who had difficulty in mathematics*'. Both of these classes covered two year groups, one being Year 4 and Year 5 the other Year 6 and Year 7. The scores of the 'low achievers' ranged from 75 to 85 on the Richmond Tests. Since Richmond Tests had a standardised mean for each age group of 100, with a standard deviation of ±15, it would be reasonable to claim that the high achievers were two standard deviations or more above the standardised mean, whilst the low achievers were between one and two standard deviations below the mean.

All children received parental permission to take part in the investigation. Because of factors beyond the control of the investigator, interviews with only four children within Year 7 were completed, fortunately two from each achievement group. This made a complete sample of 22 children: 11 'high achievers' and 11 'low achievers', three from each achievement extreme of each year group, apart from those in Year 7.

5.4 RESULTS OF THE ARITHMETICAL PHASE

The item banks were presented in the order verbal, visual, number combinations to twenty and finally number combinations in excess of 20. However, the analysis of the results is presented in such a way that the general mathematical thinking of the children in the context of the numerical items is considered first. Through this route we confirm our selection of the children, their qualitative differences in thinking and establish a platform for further discussion.

5.4.1 The Number Combinations

The number combinations to 20 were drawn from Gray (1991) and were chosen because they were shown to be capable of discriminating between procedural and proceptual thinkers (Gray & Tall, 1994). They reflected items common to the Standard Assessment Tasks (SATs, of 1992, 1993 and 1994 and could thus be used to identify children's levels of achievement in elementary arithmetic though this was based on the subjective standards applied within the SATs. Thus, at Level 2, the average child of Year 2 (age 7.11 years) would be expected to know the number combinations to 10 whilst at Level 3 the average child (age 8.11 years) would be expected to know the number combinations to 20. Earlier research showed this was not the case and neither was it true of these children even of those who were in Year 7. Children used alternative strategies, count-all, count-on and derived fact.

The individual items which made the component were:

Level 2: Addition combinations to 10.

2+1   3+5   8+2   4+4   0+2   6+3   5+4   5+0   7+2.

Level 2: Subtraction combinations to 10.

3–2 5–4 6–3 9–5 3–3 6–0 8–2 9–8 7–5.

Level 3: Addition combinations to 20.

12+1 13+5 18+2 14+4 10+2 3+16 15+4 9+8 4+7 8+6.

Level 3: Subtraction combinations to 20.

13–2 15–4 16–3 15–9 16–10 12–8 18–9 20–8 17–13 19–17.

Children were presented with the items orally and asked to mentally complete the combination as quickly as possible using what they thought was the best approach. There was no time restriction. The current interest lay in the strategy used and what it was children thought was going on in their heads as they worked towards a solution.

In addition to the above, the children were also presented with a range of two-digit addition and subtraction combinations drawn from the 1992 and 1993 SATs:

18+5; 24+43; 16+41; 39+26; 80–30; 29–6; 26–19

Solution strategies were classified as 'known', 'transformation', 'accumulation' and mental 'analogue' of the written algorithm. The interview procedure was similar to that carried out for the more elementary combinations.

5.4.2 Solution Strategies

These are examined in two phases, those that were apparent with number combinations to 20 and those associated with the two digit combinations.

5.4.2.1 Number Combinations to 20

Figure 5.1 illustrates the strategies that the various groups of children used to solve the elementary number combinations. The results are remarkable for the similarity they bear to Gray's (1991) even though, compared to that study, in each group there appears to be improvement in levels of sophistication. It is hypothesised that differences between the earlier work and the current results could be attributed to the considerable emphasis on learning of number combinations in UK schools over the past few years. Each of the percentages is calculated by considered the ratio of frequency of each strategy as a ratio of the product of the number of combinations presented to each child with each category (N=9) and the number of children who responded in each year group. (Raw scores may be seen in Appendix 1.1 and calculated percentages in Appendix 1.2)

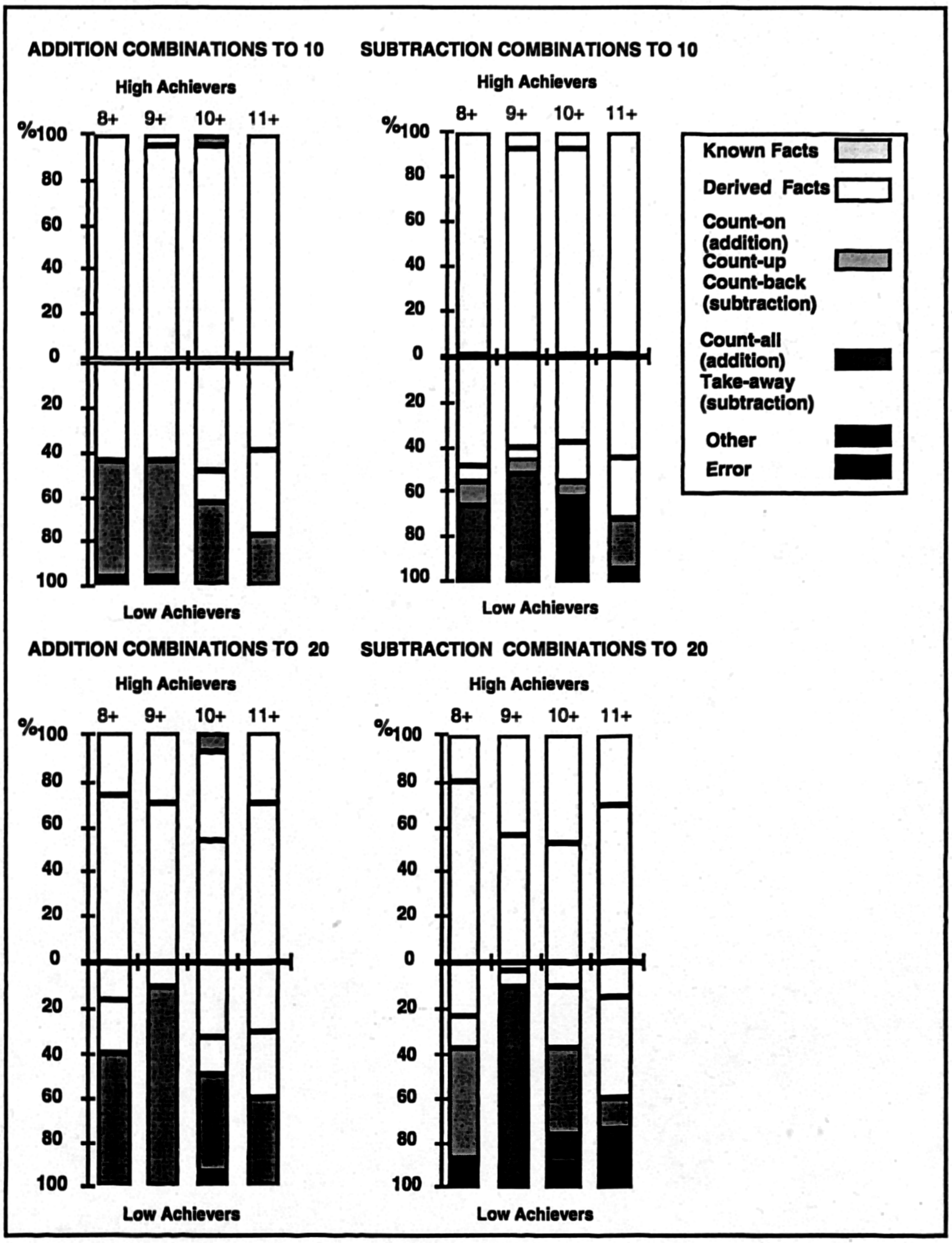

Figure 5.1: Children's solution strategies to elementary number combinations. Comparisons by level of achievement.

Nevertheless, similar pictures emerge:

- 'Below average' achievers rely extensively on procedural approaches. Of particular interest is the use of 'take away' by the 9+ children,

- Counting procedures are almost non-existent amongst the 'high achievers',
- 'High achievers' extensively use derived facts for number combinations between 10 and 20, and
- Even when they have a fairly extensive repertoire of known facts, 'low achievers', apart from the oldest children, tend to make very limited use of derived facts.

This range of solution strategies confirms earlier evidence that there is a qualitative difference in the way that children handle elementary number combinations. Although the differences are less marked than those previously reported, the results have to be placed within the current climate. However, even though it is expected that the 'average' child should know these combinations by the age of 8+, amongst the 'low achievers', only the 11+ children used retrieval methods (either 'automatic' knowing or the use of known knowledge) for over 50% of the combinations to 20. At this level, their overall strategy use does not match that of the youngest 'high achievers' who solved 100% of these combinations through retrieval approaches.

5.4.3.2 Two digit addition and subtraction

Figure 5.2 indicates the approaches the children used to solve the two-digit addition and subtraction combinations. Percentages are calculated in the same way as those in Figure 5.1 where N=10 for addition and subtraction combinations. Again Appendices 1.1. and 1.2. refer.

Clearly when the children attempted these combinations, there were not only quantitative differences in the level of achievement but also qualitative ones in approach. The 8+, 9+ and 10+ 'low achievers' had considerable difficulty obtaining correct solutions, not so much because the children could not develop a strategy but because of the combined effort associated with remembering the actual number pairs and using the strategy. Where they were successful, 'low achievers' obtained solutions through accumulation strategies based on counting in ones or sequence counting in 10s, for example: 39 +26 became "**39**, 40, 45, 50, 55, 60, **65**". Fingers were extensively used to support these approaches but there was very little evidence of transformation strategies.

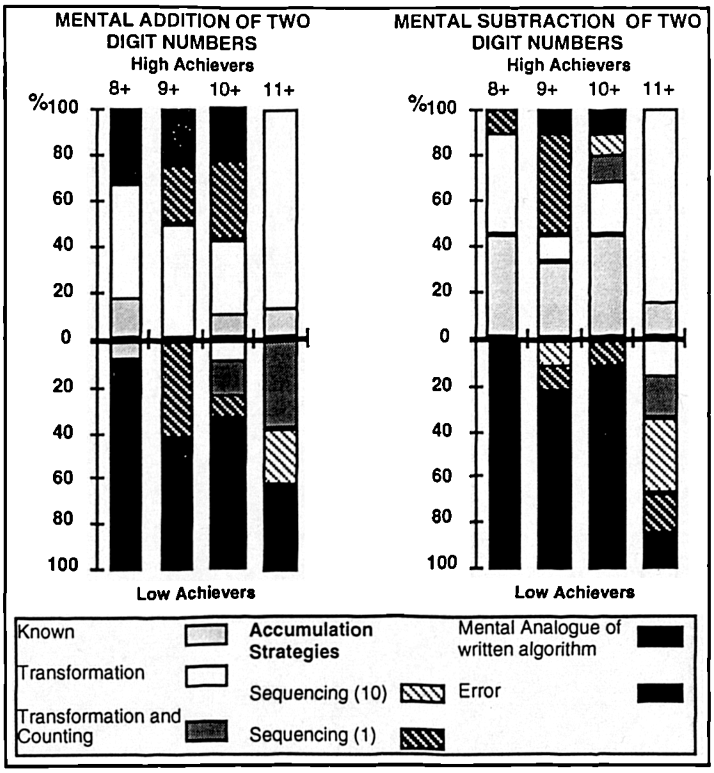

Figure 5.2: Children's solution strategies to two digit addition and subtraction combinations. Comparisons by level of achievement.

'High achievers' used transformation strategies, all except one completing combinations this way. There was some evidence that younger 'high achievers' and older 'low achievers' used a mental analogue of a paper and pencil approach, particularly for addition.

5.4.4.3 Summary

Overall the differences in the strategies the children used to solve the number combinations were seen to be such that:

(i) as combinations became more difficult there was a steady increase in the use of counting procedures amongst the 'low achievers' but these strategies, though relatively successful for small combinations, did not appear to provide a sound platform for a competent level of achievement with the two-digit combinations

(ii) although 'low achievers' can make use of known facts to obtain solutions to some that are not known, there was very little evidence that this approach was utilised with two-digit combinations by making use of transformation strategies.

(iii) 'high achievers' made extensive use of the facts they know to build the facts that they did not know. Strategies developed to do this were effectively modified for two-digit combinations.

The evidence suggests that there were qualitative differences in the children's approach to elementary number combinations. This is manifest in the extensive differences that occurred in the way children approached the mental addition and subtraction of two-digit combinations.

5.4.3 Representations in Elementary Arithmetic

It was of particular interest to build on the children's different strategy use by considering the form of representation used to solve each combination. This was resolved in two ways, firstly by observing children's actions and secondly by considering their response to the question *"What was happening in your head?"*. Figure 5.3 provides a broad indication of the nature of the representations used by the children to solve the number combinations to 20. It indicates the percentage distribution of children's use of representations when solving number combinations to 10 and 20. The total responses for addition and subtraction have been combined. (Appendix 1.3)

Unclassified responses were those where it was difficult to obtain a clear notion of the form of the representations. Unfortunately, clear indications that a child had said something in his or her mind were not obtained at this stage but the omission was corrected in the main study. The children's approaches to the numerical items indicated that not only did the children use a variety of strategies but also they used a range of representations to support these strategies. Although perceptual items dominated the processing activity of the 'low achievers', mental representations in visual form were in evidence across the age and achievement range.

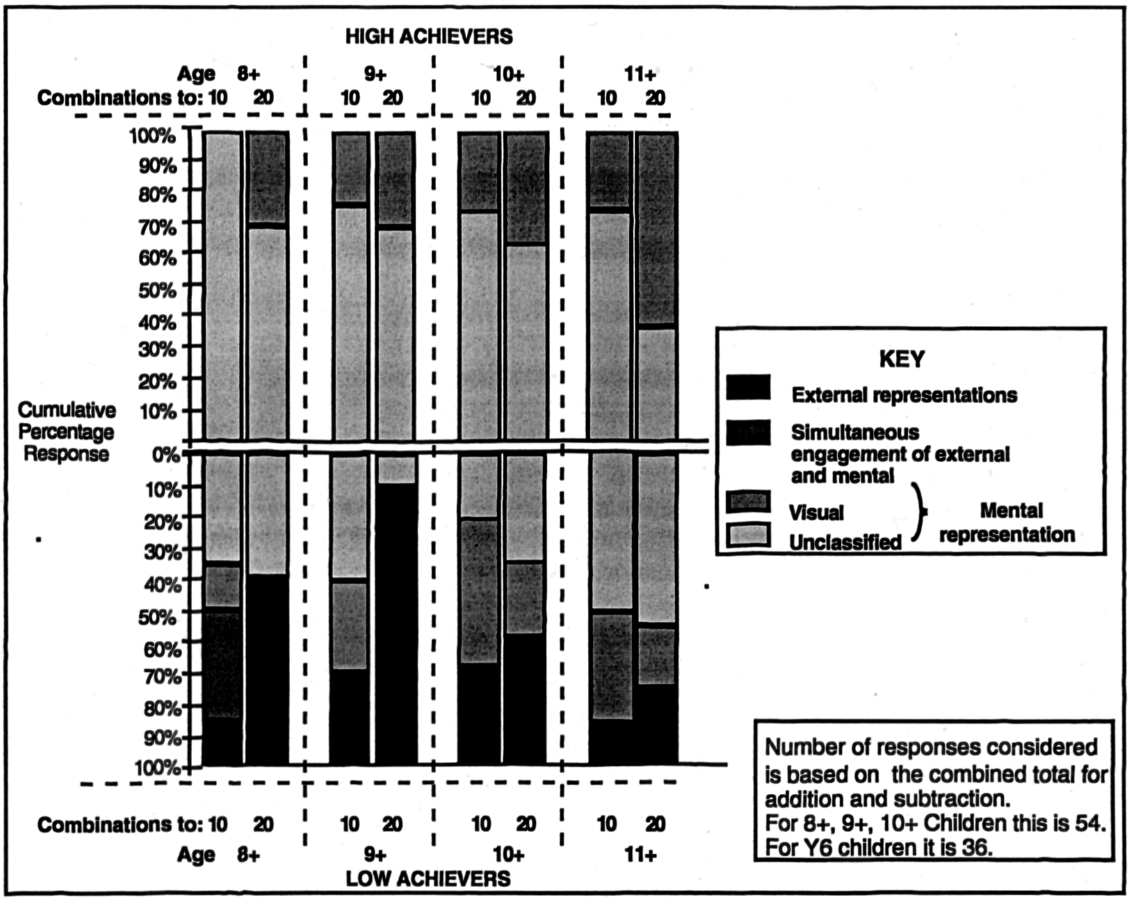

Figure 5.3: Children's use of representations when solving elementary number combination

It was relatively common for the 'low achievers' to report visual imagery for the addition and subtraction combinations to 10, sometimes this involving simultaneous engagement with perceptual items, but the combinations to 20 only evoked projections of visual images from amongst the 10+ and 11+ children. No 'high achievers' used external representations to solve any combination but symbols that were "*seen*" were reported across the spectrum of the age range.

Essential differences between the two groups indicated that:

- 'low achievers' made greater use of perceptual items as the combinations became more difficult. Such an increase was associated with a decline in the reported use of visual imagery and

- 'high achievers' made more use of 'symbolic imagery' as combinations became more difficult.

However, as we shall see, the mental representations reported by the two groups were qualitatively different. The evidence could, of course, be pointing towards two different things. Taking the achievement groups as distinct groups, may add to the belief that there are qualitatively different kinds of mental representation associated with qualitatively different forms of thinking. Taking the children as a whole could provide evidence that imagery is a mediator between the use of perceptual objects and automatic symbolic manipulation (see, also Steffe *et al*, 1983). What we may be seeing are snap shots of different stages in the abstract development of the number concept. Although this view is not discounted, the evidence does seem to suggest that the children are projecting qualitatively different kinds of representation, one kind that supports the notion of imagery as mediator, the other that the imagery is used for a different purpose.

There were also clear differences in the way children with different levels of achievement talked about automatics responses. 'Low achievers' tended to use the phrase "*thought it*", high achievers the words "*knew it*". Generally, in such instances, there was a tendency for 'low achievers' to indicate that they had no visual representation whereas there was evidence that high achievers "*saw*" symbols.

Visual images reported by 'high achievers' was almost always symbolic in form but those reported by 'low achievers' were dominated by the analogical form — the images appeared to either have the attributes of actual objects or icons or were used in a way that reflected the use of actual objects. Frequently highly detailed, these images demonstrated the need to concretise mathematical symbols. The mental representations that were formed became part of active mental actions which seemed indispensable to the solution procedures.

5.4.3.1 Analogical Images

Mental representations described by 'low achievers' were associated with visual images which were essentially analogues of objects. They appeared to be of three different forms: those based on discrete objects such as counters and marbles, those based upon 'linked' discrete items such as fingers and those based on analogues of the number track or number line.

Figure 5.4, a diagrammatic representation of visual images described by 'low achievers', illustrates the way in which visual images of counters or marbles appeared to play a role in numerical processing. The sequence is not presented as a hierarchy but as a demonstration of the way in which some of the 'low achievers' concretised numerical symbols.

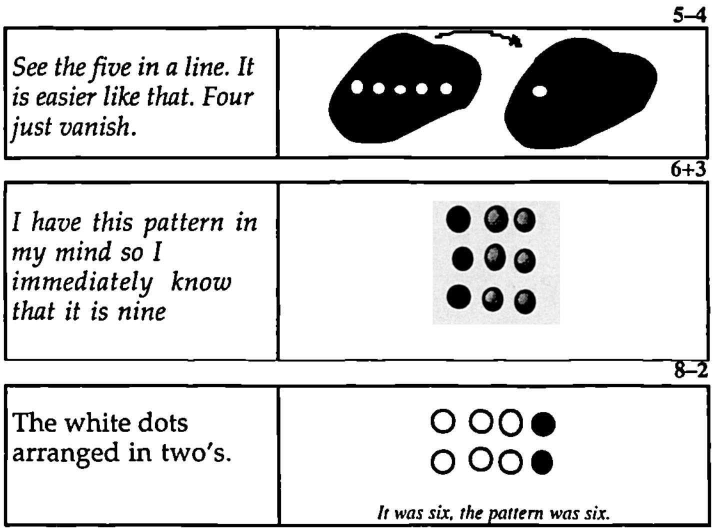

Figure 5.4: Analogical images based upon discrete objects

In the first example, 5–4, a 10-year-old boy described how five "*white counters*" in a row "*flashed up*", only to be almost instantly replaced by the single counter. The 'objects' seen in the other two instances, by a 9-year-old, were successively described as " *black marbles*" and "*white dots but two turned black*". It is conjectured that these kinds of mental representation could support the later development of known facts and derived facts. The examples given could support the view that the children knew the responses but were not confident enough to rely on the knowledge. Several of the 'low achievers' indicated that "*...even if you know something it could be wrong so it is better to work it out*". Most frequently that meant to carry out an action and count.

Though the examples given above were not associated with counting, most of the visual images described by 'low achievers' were. In some instances marbles or counters were used in a dynamic way to reflect a count–on procedure. Figures 5.5 and 5.6 indicate diagrammatic copies of representations described and drawn by a 9-year-old and an 11-year-old. These are associated with the solutions to 9–5 and 7+4 .

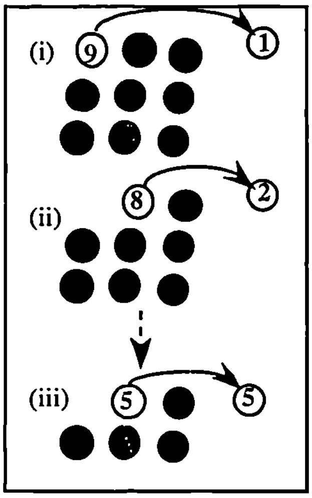

Figure 5.5:
Active analogical image associated with 9–5.

The 9-year-old described how he saw nine "*marbles*", "*eight black ones and one with a nine written on it*". They were arranged as in Figure 5.5 (i). The counter with 9 on it moved and the 9 changed to a 1. Now there were seven black counters and one with 8 written on it. This moved and the 8 became a 2 (Figure 5.5(ii)). This carried on until the image was of four black counters and one with a 5 written on it. The 5 moved but this time the number did not change (Figure 5.5(iii)). The remaining pattern was known because "*three and one make four*".

Figure 5.6 indicates how an 11-year-old used a similar approach to find the solution to 7+4. First the "black" 7 appeared with "*four white balls*". One of the balls had an 8 written above it and the 8 moved to take the place of the 7 which disappeared, There were now three white balls the one nearest the 8 having a 9 written over it. This now moved to take the place of the eight, and so on until there were no more balls to move. The common feature of both approaches was the process of decrementing the imagined objects until the solution was obtained.

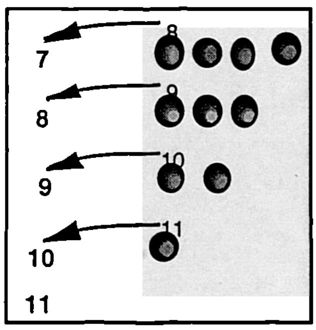

Figure 5.6:
Active analogical image associated with 7+4

It was common for children to describe how they obtained a solution by using *"fingers in my head*". One 11-year-old indicated that the detail of these was dependent on the difficulty of the combination that had to be worked out. For example, when attempting to find the sum of 3–2, the child *"just saw three lines and took away two*". The lines were described *"like fingers but not joined together*". When solving 9–5, the 'lines' were *'joined together to become fingers*'. However, now instead of fingers, the child saw a hand. However, the hand had four long fingers and one small stump (the thumb) sticking out of the side. Unfortunately this stump was often omitted in the counting procedure so that, inevitably, solutions to subtraction combinations to 10 were initially always one less than the actual answer. Each time she used this approach the child checked the answer with her actual fingers – now it was always right. This child made no attempt to use mental representations with combinations between 10 and 20. She always used her fingers. She explained that

> "If somebody does not allow me to use my fingers, I would try to get a number of some things into my mind and do the sum. The things I would try to think of are books, bags, letters etc." (Emmenda, age 11)

Children describing these forms of mental representations appeared to be carrying out mental actions which were imitations of a procedure they could use with real objects. Descriptions by children using real items and descriptions by children who claimed to see mental representations of these items were remarkably similar but there were difficulties associated with the use of the latter – a double counting procedure. This involved incrementing one set of objects and at the same time decrementing another set, both being mental analogues of physical objects. To overcome these difficulties, some children not only described active visual images but also used external representations such as fingers. Descriptions associating mental representations with physical action is an example of what is termed *simultaneous engagement.* The evidence was that the physical action was not a focus of attention in the perceptual sense: The operation was purely tactile and used to support concentration which centred on seeing the objects *"in my head*".

Images of discrete objects like counters provided some children with a degree of flexibility not associated with more static imagery like an analogical number line or fingers in the head. We can leave the explanation to a 9-year-old:

> "[with] the dots...it's...it's easier because you don't have to keep on thinking, 'No it's that one I need to move, no it's that one or that one' because it doesn't really matter which one you move". (Amanda, age 9)

5.4.3.2 'Low achievers' and Images with Symbols

In some instances symbols became the units for mental counting but it is conjectured that these took an iconic form. They replaced the mental analogues associated with marbles and counters but they were used in the same way. An eight year old described how she found the solution to 3 + 5 (Figure 5.7).

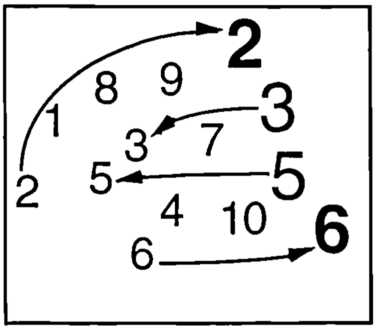

Figure 5.7:
Active image using symbols to find the sum of 3 + 5.

The child described how all of the numbers were going around in her head in a circle: "*the number I want moves out and I count them. Then they go back and new numbers go out.*" In this case, it was first the '3' and the '5'. These became "*blacker*" than the other numbers. The 3 moved back and was replaced by 2 and the 5 moved back to be replaced by 6. The '2' then became a '1' and the '6' a '7' and then the '1' became '0' and the '7' an '8'. "*That is the answer*".

An 11-year-old gave Figure 5.8 as a diagrammatic representation of what he saw in his head when subtracting 20–8. He described the two 'number tracks' as "*two calculators going around in opposite ways in my head*". The figures '8' and '20', were the first to appear, and then these were "*crossed out*" to be replaced by '7' and '19'. This process continued until '0' and '12' were reached.

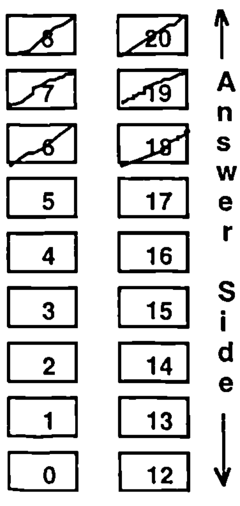

Figure 5.8:
A mental calculator

An interesting feature of the children's use of analogical images was the difference in their quality of detail. The more a child's use of visual representation oscillated with the use of perceptual items, the greater was the level of detail associated with that imagery. However, analogical images of older children appeared to possess greater sophistication. They appeared to reflect the child's growing awareness of number order as seen in the number track or a number line but the representation was used for counting. External manifestations of this could be seen in sub-vocal utterances associated with motor acts such as nodding or eye movement. In other instances, some children indicated that they "*... just counted it in my head*".

Movement was often associated with the images of 'low achievers' and the notion of "*spinning*" seemed to be a common feature of their descriptions. Even when adding 2+1, a 9-year-old reported seeing all of the operation symbols "*spinning around on one side and a big black 3 on the other*". In some instances images of symbols were associated with approximation. When adding 6+3, another 9-year-old reported seeing "*a jumble of numbers with 8 and 9 standing out because they are near the answer*". This was a similar response to that given by a 12-year-old who, when doing the same combination, reported an image that consisted of 3, 6, 9, 12, 15 and 18. "*All the numbers were in the three times table*". Whilst the "*three and the six stayed there because they were part of the nine, the twelve, fifteen and the eighteen just fall away*".

5.4.3.3 Images with Symbols

Of the 11 children identified as 'high achievers', only one (age 9+) did not describe any visual images whilst another (age 10+) indicated that she may see the solutions before she said them but she was not sure about this. The other children described visual images most frequently associated with number combinations at Level 3, combinations between 10 and 20, than at Level 2, the elementary combinations to 10. In all cases these representations were associated with numerical symbols and they were more strongly associated with solutions obtained through the use of derived facts.

'*Seeing*' symbols were largely reported in two contexts:

- immediately following the verbal delivery of the combination by the interviewer, nine of the children reported seeing the expression or a transformation of the expression "*in my mind*" and
- a visual image of the solution "*I saw the answer before I told you*" (eight children).

Seeing the two, input and solution, at the same time was common for those children who indicated that they knew the solution to combinations at Level 3. Transformations were frequently reported when derived facts were used:

- Presented with 9+8, a child in the 8+ group stated "*In my mind I saw 8+10*". When presented with 15–9, the same child indicated how she "*... thought 15–10 and saw 5, thought 10–9 and saw 1, and then I saw 6*".
- When presented with 18–9, a child in the 9+ group reported seeing 9+9.
- Another 9+ child when solving 3+16 indicated how he saw first 3+6, then 9. He then reported: "*the one was out of mental view so I needed to remember to bring it back*". He finally reported seeing 19.
- When dealing with 17–13 an 11+ child indicated that she saw 7–3 and 4. She knew that the 10s cancelled each other. Indeed, her additional comments provided some insight into the way in which a large proportion of the children found solutions to the two-digit mental addition and subtraction combinations:

  > "The ones I know automatically, if I see them, I see them horizontally. If I am not 100% sure or have to work it out I will have to see them vertically". (Betty, age 11+)

Frequently the approach used when symbolic representations were seen vertically was associated with the representation which was common when performing the written algorithm.

Only one 'high achiever' (9+) reported visual imagery that rapidly mapped numerals onto an identified string of digits. The magnitude of the solution was put into relation with the magnitude of one of the givens or with a transformation of one of the givens. For example, when obtaining the solution to 9 + 8, this child reported seeing a partial number track. The greater proportion of the track was very hazy and his focus of attention was directed to two numbers that stood out from the rest. These were 10, formed from 8+2, and 17 (10+7). The transformation was completed before the number track was 'seen'. An important difference between this child's visual image of the number track and the similar images of the 'low achievers' was that the number track was not used to obtain the solution. It acted as a representation of *analogical magnitude* (Dehaene & Cohen, 1995) which may appear for "*more difficult*" combinations but it was not used to support a counting procedure.

A 10-year-old girl demonstrated interesting insights into proprioceptive use of fingers. The only 'high achiever' who used any counting when obtaining solutions to the number combinations at level 3 (13 + 5 and 4 + 7), she explained how she counted in her head:

> "I sort of feel my fingers moving. I don't move them or touch them. I sort of send a message on my fingers and I feel it when I count". (Vicki, age 10 +)

5.4.3.4 Mental Representations and Two Digit Combinations

All of the two-digit combinations were presented verbally and the children were requested to attempt them mentally.

Of the eleven children who were 'low achievers', three did not do any of the two-digit mental computations, three were stopped after making a strenuous effort on the first one (18–5) and three had less than half correct. As one 9-year-old child said when trying 16+41, "*I used my fingers but I didn't know when to stop*". Only the two 11-year-old children obtained correct solutions to all of the combinations.

It was the general pattern for the 'low achievers' to attempt to complete these combinations using some form of counting and usually fingers were used to support this. However, one of the 11-year-old children did report one incidence of mental representations associated with the use of symbolism. Solving 18+5 the child stated how he saw the sum arranged in vertical fashion transformed into 15+8, which disappeared to give the answer 16 underneath. He then saw 16+7 which in turn disappeared to give the answer 17. This very slow process continued until he reached 22 + 1 which disappeared to give 23. He then saw 23+0 which also disappeared to give the final answer 23.

Amongst the 'high achievers', 97% of the responses to the two-digit component were correct. Of the eleven children, eight gave some indication of the mental representations used to obtain solutions. Two of the 8-year-old children reported using a mental representation isomorphic to a written pencil and paper approach. The remaining children talked of '*seeing the symbols being worked on*', "*I saw the two numbers [to be added together] and then they went away and I remembered them*" and "*I saw the answer*". However, the common feature which emerged from the interviews with these children was the sense that they oscillated between *seeing* and *thinking*. Six of them ranging across the spectrum of age talked of:

(i) seeing the combination they worked on and/or seeing the results of that work and

(ii) thinking about answers to combinations.

For example, a 9-year-old saw the combination 39+26 and saw the answer 65. He "*thought about 3+2 and 9+6 . I knew the answers to those*"

5.4.3.5 Mental Representations and Mental arithmetic: A Summary

The actions in simple arithmetic are meant to be the platforms from which children can give meaning to symbols. However, to do this there needs to be a qualitative shift in thinking — the action needs to be encapsulated as a concept. Different kinds of mathematical behaviour and different levels of such behaviours are tied to understanding of mathematical concepts and such understanding is only implicit through a child's behaviour.

The different mental representations projected by the children in this sample highlights three things:

- Children who were having some difficulty appeared to focus on mental representation of real objects and actions with them. The objects in the mind were 'real' things that have 'body' and the actions were analogues of those that children may carry out with real things. The more that the child's use of imagery oscillated with the use of perceptual items, the greater the level of detail that was reported. Even when they use symbolism such children used the symbols as countable items. The children seemed unable to detach themselves from the search for substance — *the mental analogues projected by 'low achievers' and the actions associated with these analogues appear to be essential for solving the problem.*

- Those children who were 'more successful' used the knowledge they knew to build on knowledge they did not know. In doing so, their mental representations were based upon the use of symbols. The detail associated with particular procedures was filtered out to give a focus on the more abstract qualities of the symbols. A common word used to describe the use of symbolic images by the 'high achievers' was the word '*flashing*'. Visual mental representations seemed to come and go very quickly. Such a notion leads to the hypothesis that the symbolic images used by these children may be seen as *thought generators. Visual images that 'high achievers' associate with symbols appear to flash as memory reminders momentarily coming to the fore as new transformations or precursors of verbal comment.*

- Numerical combinations solved with the use of mental representations by the 'low achievers' seemed to be limited to a narrow range. This may be partly related to the difficulty of holding information in their minds. It is conjectured that their mental representations are essential to the act of counting which is primarily the main act of obtaining a solution. It is further conjectured that 'high achievers' used mental representations, particularly that in visual form, to hold information and to signal what had to be done with the information.

Table 5.1 draws together the main characteristics of the two groups of children when they attempted the arithmetical combinations.

Low Achievers	High Achievers
•Concrete •Superfluous information •"Horizontal" thinking —qualitatively similar, directed towards personalised procedures associated with variations of the pereceptual/figural items •Imitation	• Abstract and symbolic • Information rejected • "Vertical" thinking. Attention directed towards known facts and/or transformations • Thought generator

Table 5.1: Children's representation in mental arithmetic

We see that the mental representations generated by the two groups differed considerably. Amongst 'low achievers', we see the 'things' that are the focus of attention being real and physical, for example, fingers or mental representations which are analogues of those real things. At times, these things might simply be replaced by labels, the number symbols, but then these labels are manipulated as if they were real things. There is little qualitative change in the nature of the thing, counter, marble, finger or label – all are used for counting. Though an individual's procedure may, in isolation, be called idiosyncratic, there was sufficient commonalty in these idiosyncratic behaviours to signal other concerns: it indicates a tendency to concretise and little attempt to filter out information. This, of course, causes problems. What may be successful at one level is not nearly so successful at the next stage of difficulty. The consequence is that many who seriously attempt mental methods need to fall back on perceptual items as things gets harder. Eventually they do not appear to have a satisfactory strategy for solving the 'most difficult' problems.

In contrast, 'high achievers' appeared to focus on those abstractions that enabled them to make choices — they displayed the ability to reject information or at least to select that information that was relevant to the situation. Such differences appear to have long-term and more fortunate consequences for the children's numerical achievement.

5.5 THE VERBAL PHASE

The range of items that formed this component of the interview series has been discussed elsewhere (Section 4.4.3). It is sufficient to remind the reader that eight nouns, 'ball', 'car', 'triangle' (concrete nouns), 'five', 'half', 'four ninths' (abstract 'numerical' nouns), 'number' and 'fraction' (collective nouns), were discussed with the children using a semi-clinical interviewing procedure. Discussion was stimulated through two core questions:

(i) "*What comes to mind when you hear the word...?*"

(ii) "*If ET came and asked you what a ... was, what would you say to him?*'

The purpose of these questions was:

- to give children an opportunity to report and describe their mental representations
- to give children an opportunity to explain, describe or define the word under consideration and
- to give children an opportunity to indicate what it was that they considered important enough about the word to transmit to someone who did not understand it.

5.5.1 Classifying the Responses

Repeated analysis of children's responses to the free talk components and the descriptive component of the verbally presented items indicated that the most powerful descriptive concepts and categories for children's responses could be categorised under five major headings:

1. The item was **not known** or not recognised by the respondent. Common responses to the word 'four ninths' — particularly from 'low achievers'— included "*I haven't heard of it*" or "*I don't know*" came under this category.

2. The object of discussion was either **associated** with another object which then became the focus of attention, or the object was placed within a *context* which provided an opportunity to embellish comments referring to the original. For example, the word "ball" evoked the response from an 8-year-old 'low achiever': "*Picture of people playing football, playing with the football, throwing it*". This response was seen as placing the notion of a "ball" within a context. Similarly, the word 'fraction' could be placed within the context of mathematics or associated with "*some kind of sum*".

3. The object of discussion generated a well-defined **example** or examples. The word 'car' evoked many singular examples which were often personalised: "*a white Astra–our car*", " *A blue car, one wheel cap missing–mum's car*". 'Fraction' was exemplified by "*half*" and by several examples, often qualitatively similar: "*half a door, half a box, half a picture*". In some instances, it was generalised and associated with a symbol: "*Could be anything split into two and written one over two*".

4. The item in question evoked **descriptive** comments which were associated with either its external qualities, embellished through the use of imagination, or provided insight into the deeper meaning of the object.

   External visual qualities were frequently used by 'low achievers' to describe the items: 'car' was described as "*four wheels, steering wheel, seat belts*", 'fraction' was identified as "*two numbers with a line in the middle*" or "*a number on another number*" whilst even the word 'number' was described through its visual characteristics: "*some are straight (1, 4), some are bent and lines (5), some are circles* (8, 6)".

The tendency to provide insight into deeper meaning could be seen in responses to the verbally presented items which tended to be substitutes for actual definitions: 'car' was described "*as a vehicle for transporting people*", 'five' was identified as "*... the number between four and six telling an amount of something*", four-ninths was identified as "*four parts of the whole when the whole is nine parts*" whilst fraction was described as "*the number of equal parts taken from an equally divided whole*". Frequently, such comments gave an indication of implied action.

5. Only relevant to the numerical items, a category identified as **proceptual** was founded on equivalence and/or notions of procedural encapsulation. For example, 'half' was identified as "*The whole divided by two, 50% of 100, one over two, nought point five*".

These five core categories were themselves sub-divided so that quantifiable and/or qualitative differences within particular categories were identified. A full list of the categories used in the analysis of the verbal items is seen in Table 5.2.

Core Category	Final Classification	Description
1 **Not Known**	**1.0 Not Known**	Response indicate no meaning or sense of recognition.
2 **Association and Contextualised**	**2.0 Association and Contextualised**	Response does not pin-point meaning. It indicates conjecture and/or provides an associative theme or context.
3 **Example**	**3.1 Single Example**	Response provides a single example which is not symbolic.
	3.2 Multi-Examples	Response provides several examples but no symbolic ones.
	3.3 Symbolic Example	Response provides a symbolic example.
4 **Descriptive**	**4.1 Visual**	Response indicates surface characteristics or qualities.
	4.2 Imaginative Extensions	The item forms the basis for a response embellished with imaginative aspects and/or concrete extensions.
	4.3 Insight into abstract qualities	Response provides a description of non-visual characteristics. It provides insight into meaning and relationship. May tend towards a definition.
5 **Proceptual interpretation**	**5.0 Proceptual interpretation**	Emphasis on equivalence and interpretation.

5.2 Classification of responses to verbally presented items

5.5.2 The "Free Talk" Phase

Though the verbal items provided an opportunity for a range of responses, in each group, responses are in fact remarkable both for their consistency and for the different foci of attention placed on them. Between groups there are qualitative differences.

The 'free talk' phase was introduced with the request that children indicate *"What comes to mind when you hear the word...?"*.

Figure 5.8 is a diagrammatic representation of the percentage of children's classified responses to this request. (Refer Appendix 1.4). The analysis is presented in such a way that it is based on an interpretation where over 10% of responses fell into a particular classification. By 'trimming' the results in this way the representation displays trends which highlight differences between children in different levels of numerical achievement.

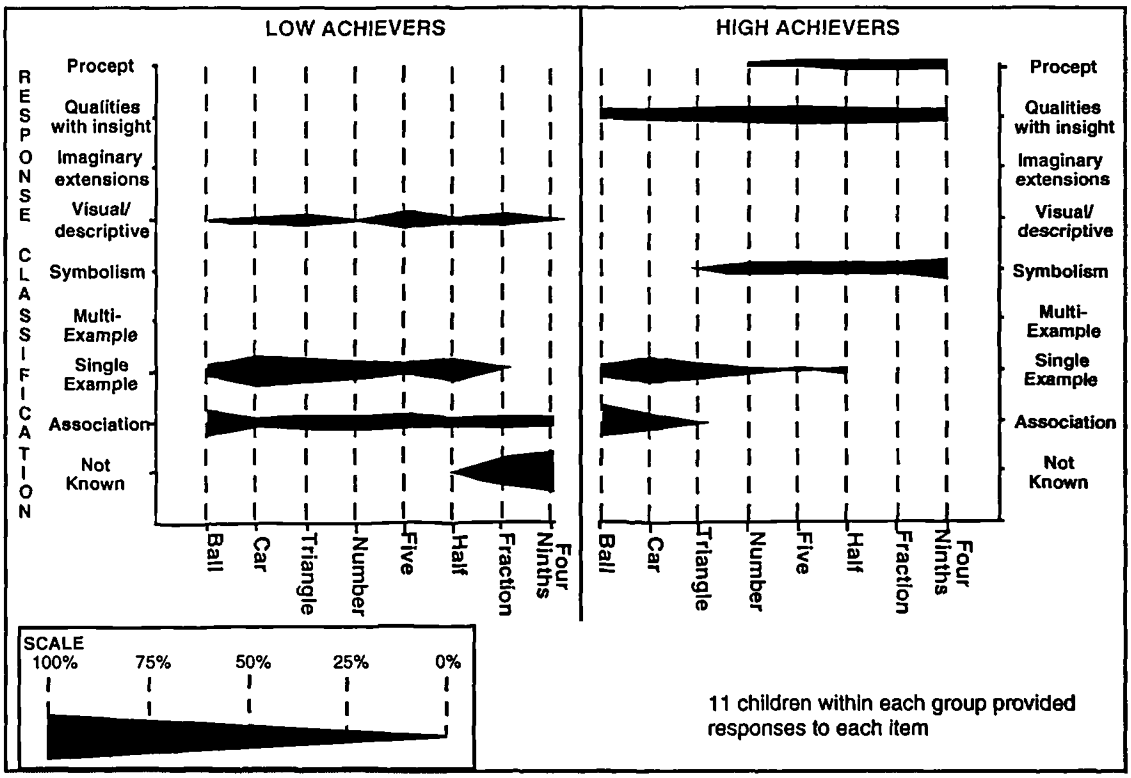

Figure 5.8: General trends identified from the analysis of children's 'free recall' responses to the verbally presented items

5.5.2.1 General trends of responses from 'low achievers'

Amongst 'low achievers', responses classified as 'association' or 'single example' tended to dominate with, for example, 67% of the responses to 'ball' and 58% of the responses to 'half' being classified as the former, and 58% of the responses to 'car' being classified as the latter. Qualitative differences in responses associated with age were limited and may only be considered in very general terms. For example, placing the item 'car' in a context was a trend noticed amongst the 8-year-old children – "*My mum driving the car*" and "*Car driving along the road with my mum, dad, brother and me*". Most other responses to this item spanned the age groups to identify a particular car. Responses to the notion of 'fraction' were age-related, with two out of three of the 8-year-old and the 9-year-old groups not providing a response. Two out of three 11-year-old children provide a visual description of the representation for a fraction: "... *numbers with a line between*", "... *a number on another number*".

A feature of the embellishments that 'low achievers' gave to the items was the need to 'concretise'. This was a feature of responses classified as 'association/context'(2.0) and 'single example' (3.1). "*People playing football*", "*bouncing a ball*", "*my car*" and "*speeding down the motorway*" were examples of comments which not only concretised but also reflected aspects of an episode (a scene or a sequence of scenes) associated with the concrete nouns.

5.5.2.2 'Low Achievers' Responding to the Numerical Items

Similar features seemed to dominate 'low achievers'' responses to the 'numerical' items. However, there also seemed to be a greater need to provide drawings which would clarify their responses. It was almost as if they did not feel they could articulate without expanding the dimension under consideration. However, these written representations not only gave the children an opportunity to expand meaning but also provided us with a sense of their understanding.

Figure 5.9 illustrates the way that some of the children illustrated their responses to the notion of 'five'.

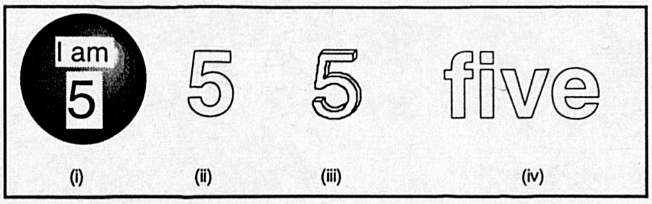

Figure 5.9: 'Low achievers' representations of 'five'

It was often related to a birthday but more frequently children liked to represent it as a "*bubble number*" (5.9(ii)) or even as a three dimensional block (5.9(iii)). A child who was very concerned about her spelling ability wrote out the word slowly as in 5.9(iv).

These items were not classified as symbolic mental representations since it became quite apparent through the interview process that these children's mental representations of five drew on iconic features which possessed picture-like qualities. It was the surface characteristics of the symbol which were the focus of discussion. These aspects were also evident when some of the children talked about 'number'. Many talked about "*seeing*" many numbers, and for some, these were dynamic and associated with movement (Figure 5.10)

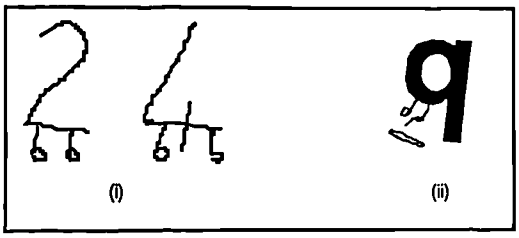

Figure 5.10: Representations associated with the word "number"

The 11-year-old child who drew 5.10(i) talked about "*numbers running around in my head*". Figure 5.10(ii) constructed by a 10-year-old child illustrated his statement that "*nine is a number and I see nine doing maths*".

Only one of these children (10+) provided any reference to symbolism when they talked about "half". The concept was always concretised with reference to circles, shapes, oranges, cakes, chocolates etc., but it was also associated with the notion of a "*little bit*". In some instances, it could be seen that children focused on the detail of actions, such as "cutting", "*breaking*" and much less frequently "*folding*".

Figure 5.11(i) illustrates an 8-year-old child's representation of 'half'. This child did

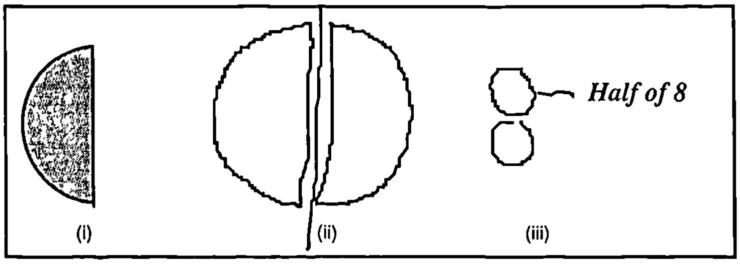

Figure 5.11: Representations associated with the word "half"

not associate the "*shape*" with a whole that produced it and neither could she see that her shape could be a whole. It was drawn without any referent.

In the case of 5.11(ii), the 11-year-old boy who drew it described the half as "*flat one side, round the other with a straight line in the middle*". Figure 5.11(iii) is an interesting representation from a 10-year-old girl who was asked to show a half of eight. It is not incorrect if she sees the symbol as an icon or picture, but she also proceeded in giving half of 2 in the same manner.

5.5.2.3 'High Achievers' Responding to the Items

In Figure 5.8, we can see that the 'high achievers' display similar qualities to the 'low achievers' when they talked about the concrete nouns 'car' and 'ball'. Again, responses focused on a particular ball and actions with a ball, but there was evidence, not age-related, of children getting to the heart of the notion: "... *a round spherical object that you play with*" (Classification 4.3). Similarly, the notion of 'car' was frequently associated with the family and a particular example of a car was high on the agenda but there was evidence of attempts to provide a definition (4.3).

In their responses to the 'numerical items', 'high achievers' aged eight and nine tended to focus on symbolism (3.3) whilst the older children tended to focus on qualities with insight (4.3). In contrast to the responses of the 'low achievers', those of 'high achievers' were seldom classified in the categories 'association/context' (2.0) and neither did they consider 'single examples' of items (3.1).

Most children relied on the use of language to give some notion of their thinking. Given the freedom to provided written representation of the words, few chose to use it. Where it was used, particularly with the concept of 'fraction', some interesting developments were identified (Figure 5.12).

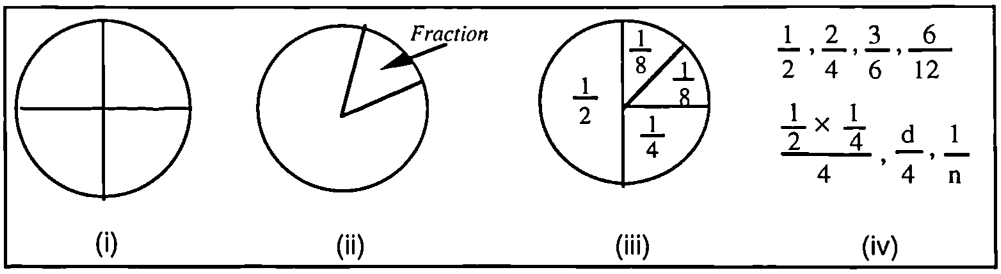

Figure 5.12: 'High achievers'' representations associated with the word 'fraction'.

An 8-year-old described a fraction as "*a circle with two lines in*" (5.12 (i)). Another eight-year-old represented the notion of a "*little bit*" (5.12 (ii)). However, by 9-year-plus it was more common for children to use symbolism and identify the relationships between fractions. The child who drew Figure 5.12 (iii) indicated how he saw "*all of these things in the shape*" whilst the 11-year-old responsible for Figure 5.12(iv) indicated how he first "... *thought of half, equivalence, division, denominator... thought of the general idea with symbols, didn't think of shapes. The symbol on the top, the numerator, represents the number of parts taken from an equi-divided whole, the denominator*". Such a qualitative change over time was not apparent in any of the comparisons with the 'low achievers'' responses.

Because of their emphasis on symbolism, 'high achievers' were far more able to face the notion of 'four-ninths', described by one 10-year-old as "*an alien fraction because it is not easy to represent with a shape*". With the exception of two 8-year-olds who were unable to provide any insight into the notion, the other 'high achiever' mentioned symbolism when referring to this idea. The only thing that any of the low achievers could do was to attempt to create a picture of four-ninths — something they found extremely difficult to do.

5.5.2.4 Some Initial Conclusions

One initial inference drawn from the analysis of the free recall element of the verbally — presented items was that the 'high achievers' seemed to shift the focus of attention of their mental representations so that there appeared to be qualitative differences in their response to concrete and abstract notions. They could talk in abstract terms about abstract ideas. 'Low achievers', on the other hand, provided essentially qualitatively similar responses. Frequently, through association or context they appeared to change the object of the discussion to elaborate on the new idea. Most of their discussion involved a need to concretise items. This was to be their theme when trying to provide explanations of the objects for the alien 'ET'.

5.5.3 The "Explanation" Phase

Prompted by the invitation to explain the object to 'ET', the purpose of this second element of verbal presentation was to provide children with an opportunity to demonstrate the best way that they could find to explain it. What was it that they considered important about the object?

Figure 5.13 provides a diagrammatic presentation of the classification of children's responses to this aspect of the interview. (Again refer to Appendix 1.4).

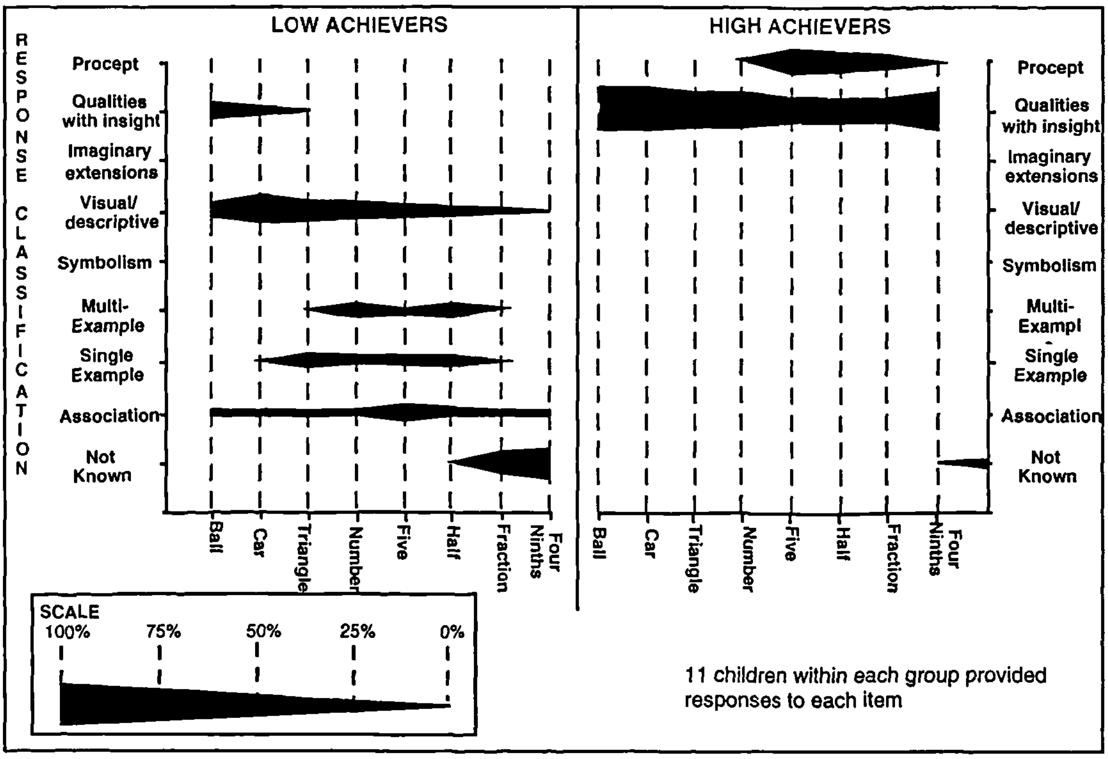

Figure 5.13: General trends identified from the analysis of children's explanations associated with the verbally presented items.

The striking difference between Figure 5.8 and Figure 5.13 is the shift in focus of the 'high achievers'. Only two categories dominated their explanations to ET: 'qualities with insight' (4.3) and proceptual considerations (5.0). These characteristics can be seen in marked contrast to the range of responses given by the 'low achievers' where, the extensive 'not known' category not withstanding, 'association/context' (2.0), the use of examples (3.1 and 3.2) and 'visual characteristics' (4.1) tended to dominate.

5.5.3.1 Qualitative Changes in Response

What is of particular interest is the change in the quality of the children's responses. These changes may be considered on a global rather than a specific level.

It is important to note that the questions served two different purposes. The first was designed to obtain a sense of the mental representation that children associated with a particular item, the second was designed to establish what children felt was more important in order to explain an item.

Qualitative changes in the nature of the responses were identified in 51% of those responses given by 'low achievers' and 77% of those given by 'high achievers'. Amongst the 'low achievers' this change was essentially towards the provision of visual, concrete and active characteristic of items. Amongst the 'high achievers', it was essentially only those responses not initially identified as 'insight' which did not change. All of the responses to the concrete nouns originally characterised as 'association' were now classified within category 4.3, demonstrating insight into abstract qualities.

5.5.4 Drawing Some Strands Together

The verbally presented items provided a basis for the two groups of children to give a mental representation and what they felt was worth talking about in order to explain it to the uninitiated. In the context of the sample, differences between the two groups indicate that there are qualitative differences in the form of mental representations that each possesses.

Low Achievers	High Achievers
• Concretised • Detail, description and association • Horizontal thinking directed towards surface features	• Concentrate on abstract qualities • Information ignored • Vertical thinking directed towards core features or definitions

Table 5.3: Children's representations associated with verbally presented items.

'Low achievers' focused on surface features or the association of the presented object with other objects through which they could create an alternative to talk about. Focusing on these examples or the descriptions associated with them while they talked in 'free talk' or to ET, seemed to limit their ability to make the generalisations necessary to provide adequate information to an uninitiated person. Perhaps, more importantly it also seems to be a reflection of their inability to make insightful and purposeful transformations. It is conjectured that *the tendency of the 'low achievers' 'to provide an association or give a context does not appear to form a platform which is powerful enough to provide insightful links.*

The overall qualitative differences seen between the 'free talk' and 'ET' responses of the 'high achievers' suggests that they have a better ability to make transformations on the items under discussion. The use of 'association' amongst the 'high achievers' appeared to be of a different quality. It could form *a platform powerful enough for insightful transformations.* Similarly, the tendency of the 'high achievers' to turn their initial examples into statements possessing insightful qualities suggests that these examples may be prototypical in nature.

It is suggested that these characteristics are those which support the transformations that were evident in their solution to numerical combinations. Though they may provide isolated experiences they appeared more ready, or more able, to draw together knowledge and compress all of the discrete information into a generalised image which serves a dynamic purpose of acting as a 'skeleton' for other ideas. Such a 'skeleton' can form the basis for general definitions when required. The issue for the next series of items was whether or not an icon or symbol can stimulate the projection of similar kinds of mental representation.

5.6 THE VISUAL PHASE

5.6.1 Classifying the Responses

The items which formed the visual component of the Pilot Study were considered in section 4.4.4. They were numerical symbols: '5', '$\frac{3}{4}$', '1995', and '3÷4', and iconic representations: 'marbles', 'dancing man', 'window, 'house, 'scalene' and 'honeycomb'.

The basis for establishing classification of the responses to the visual items was similar to that given for the verbal items as seen in Section 5.5.1 and Table 5.3.

Whilst, qualitatively, responses to the visual items possessed similar characteristics to those given for the verbal items, there were some exceptions:

- No responses were identified in the broad heading of 'Example'. Since they were presented with a visual representation of the item in question, the children did not add additional examples.

- It was common for the 'low achievers' to create imaginative stories around the visual items. For example, 'marbles' was described as "*a game of marbles, some in a circle and others trying to hit it*" or "*balls, trying to join like a magnet but gravity stops them*". The 'dancing man' was frequently embellished though the description with obvious actions: "*Jack Frost. But he would be transparent. It is a crook running away from a robbery*" or "*somebody on a stage doing an act. He is smartly dressed*". In contrast, 'high achievers' tended to add insights with deeper meaning – 'marbles' "*... could be a fraction — four-eigths*". The 'dancing man' "*... was a symbol, because it is not a person...*", it "*... was a* *silhouette, advertising a show.*"

- Proceptual interpretation was strongly associated with the symbolic items by 'high achievers'. For example, the visual presentation of '5' could trigger "*5 digits, units, an amount, 2+3, 5+2–2, 100–95*" or '1995' reflected *"a year, 4, digits, 1,9,9,5, 19 centuries, 199 decades, 19995 ones*". The '3÷4' evoked the response *"Well, its a sum, may need to do it by long division... its 0.75, 3/4, 75%...".*

5.6.2 Considering the Results

5.6.2.1 General Trends of the Responses

Figure 5.14 illustrates the general trends observed as a result of the classification of children's responses to the symbolic and iconic items. The table is constructed in the same way as Figures 5.8 and 5.13. Appendix 1.5 provides core data.

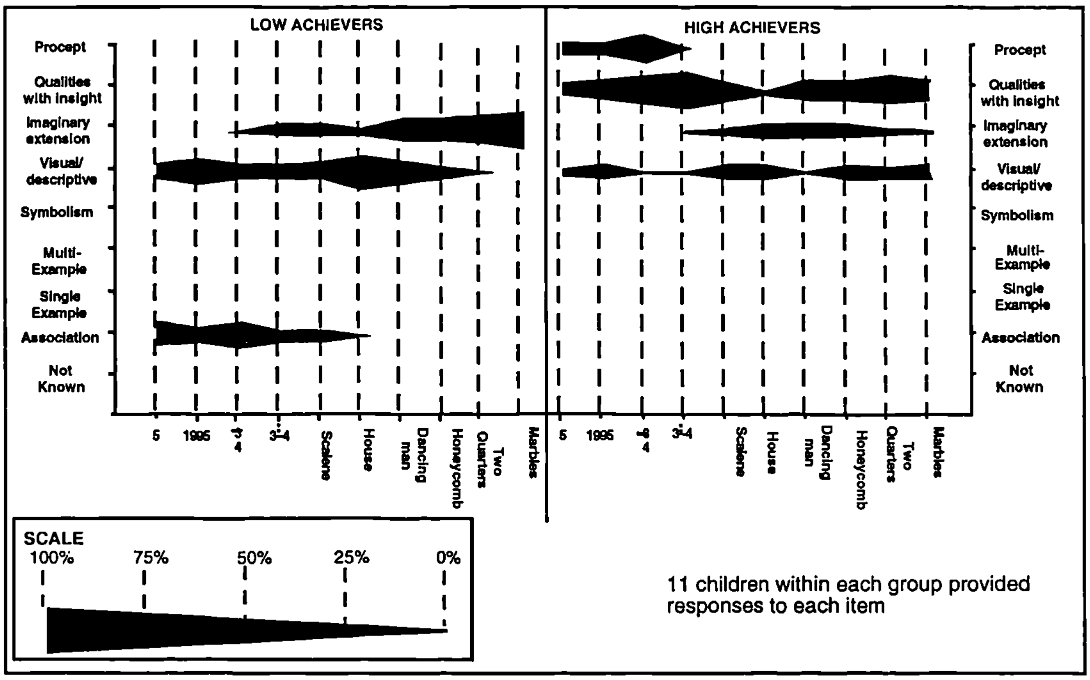

Figure 5.14: General trends identified from the analysis of children's responses to the visually presented items

In comparison to the verbally presented items there is less evidence of responses that may be deemed 'association' from 'low achievers'. When the children responded to the symbolic items, they relied heavily on 'visual description'. Iconic items tended to evoke 'imaginary extensions' and 'visual description'. In comparison, 'high achievers'' responses to the same items tended to focus on drawing out 'qualities with insight'. Depending on item difficulty, these responses either grew out of the initial observation of the visual qualities or were expanded to include visual qualities. Though responses were classified according to the overall quality of the discussion, amongst the 'high achievers' there were indications that items with underlying content tended to evoke changes in the quality of responses from, for example, "visual description" to "insight". Those which had more obvious mathematical overtones tended to evoke responses that moved in the other direction.

5.6.2.2 Comparing 'Low' and 'High Achievers'

In only two instances (out of a possible 88) did 'low achievers' provide any discussion that may be classified as 'insight'. In all other cases, they discussed the icons and the symbols within a framework which included either "visual/concrete" qualities or "imaginary extension". In general the numerical items were associated with other more concrete objects or were emphasised by their descriptive qualities.

The responses of the 'low achievers' may be seen to have two main qualities:

(i) Icons were seen as 'pictures out of focus'. They appeared to need concretising by being given colour, detail and a realistic content. This was largely achieved through the use of 'imaginary' extensions.

(ii) Symbols needed concretising. This was achieved by:

(a) associating them with concrete items, for example, *"Three can't be divided by four because there would be a remainder. You can do it with apples though"* and

(b) allowing them to become the concrete items and describe the lines and curves which were their external features.

What is of interest amongst the 'high achievers' are the indications of qualitative changes in response. Although these are not apparent within Figure 5.16, these changes appeared to display vertical movement embracing one of two directions:

(i) Top down – the direction is from more abstract qualities towards more concrete ones, for example the icon 'windows' was considered as a *"shape in four quarters, half shaded....picture of a window"*, or *"two out of four, half, cupboard, windows where they don't use shutters"*.

(ii) Bottom up – the direction is from the more concrete qualities towards the abstract ones. The more complex "honeycomb" presented evidence of such an interpretation. Initially descriptive, each quality seemed to provide cues for the next level of processing – *"hexagons, symmetrical, four light, four dark, one quarter and three-quarters"*.

It is worth noting that, at times, the distinction between classifications identified as 'qualities with insight' (4.3) and 'procept' (5.0) was frequently very fine. In the final analysis, the emphasis given to symbolic equivalences provided the distinction. For example, with the item 'three-quarters', discussion which gave the following, *"three quarters, 0.75, 75%, three over four, equivalent, four squares, three shaded"* was identified as 'procept' (5.1) whilst that given as *"a fraction, something divided by four, 3 of those parts, three quarters"* was classified as 'quality with insight' (4.3).

5.6.3. Summary of the Phase

Table 5.4 presents a summary of the main differences between the 'low' and the 'high achievers' when they responded to the visually presented items. It has been sub-divided to show the differences and/or similarities in responses when the items are identified as symbols or icons.

	Low Achievers	High Achievers
S Y M B O L S	• Order to carry out an action • Concretised by either: (a) associating with a concrete item or (b) identifying as an icon. • "Horizontal thinking"—symbol associated with personalised procedure	• Recognised as both the holder of an idea and an action • Detached from concrete qualities, associated with abstraction • "Vertical thinking" demonstrated by proceptual flexibility
I C O N S	•Interpreted as a "picture" out of focus, an incomplete concrete reality requiring focus •Given colour, detail and realism (with imagination) •Display "horizontal" thinking–'imaginary extensions' similar in quality	• Concentrate on abstract qualities • Ignore detail–concentrate on interpretation • "Vertical " thinking – free movement between abstract and descriptive aspects

Table 5.4: Children's representations associated with visually presented items

As with the numerical and verbal items, we can see the tendencies of the 'high achievers' to think in abstract ways and those of the 'low achievers' to think in concrete ways. There is little evidence of a qualitative change in the nature of the 'low achievers' discussion. They concentrated on detail or the need to add detail if it was felt to be absent. However, within this phase of the investigation, there was a sense that the items initially encouraged a similar style of thinking amongst the 'high achievers'. In one sense, the differences were not so marked. Both groups referred to detail and both supplied imaginary extensions. However, it was where the children went from there that created a difference. 'Low achievers' tended to remain at the same level of thinking. 'High achievers' moved vertically. Details of surface characteristics could be a platform for discussing more abstract qualities. Abstract qualities could be a platform for presenting surface characteristics.

5.7 CHAPTER SUMMARY

Three themes are covered the presentation in this chapter: the strategies and mental representations that children at extremes of numerical achievement used to mentally solve numerical combinations, the mental representations that these same children associated with verbally presented numerical and non-numerical nouns and the mental representation that the children associated with the visually presented icons and symbols.

The different approaches used by the children illustrated that not only was there a qualitative difference in the children's approaches to the mental arithmetic but there was also a qualitative difference in children's mental representations:

- 'Low achievers'' mental representations in elementary arithmetic were analogues of physical representations. They suggest that the children needed to concretise mathematical symbols and the representations so formed were indispensable to the solution procedures. The more that a child's use of a mental representation oscillated with the use of perceptual items, the greater the level of detail reported on mental representations.

- Mental representations reported by 'high achievers' when dealing with elementary arithmetic were symbolic in form. They were reported as "flashes" that came and went very quickly.

- 'Low achievers'' mental representations in the verbal and visual phases of the investigation indicated that the surface characteristics, links with other objects and/or events, the need to concretise, were characteristics which guided their thinking. The children tended to change the object of discussion and then elaborate on the new ideas.

- When responding to the visual and the verbal components, 'high achievers' demonstrated that they could disregard detail and focus on those abstractions that enabled them to make choices. This is not to say that the detail was then no longer available to them – it was, but in a more generalisable format. Detail could be incorporated, focused on or used if needed.

5.8 CHAPTER CONCLUSION

It is conjectured that the different mental representations identified in the two groups of children have their roots in a qualitative abstraction which is governed by a perception of personal and impersonal involvement. The mental representations of the 'high achievers' are established at a more impersonal level than those of the 'low achievers'. The more that children were able to talk about symbols at such a level the more they concentrated on their abstract qualities. This, in turn, has implications for the way in which they approach numerical combinations.

In mathematics a considerable amount of information is compressed into a simple representation, the symbol. To recognise this is to have the source of considerable power but the platform from which cognitive development associated with this power grows has three qualities:

(i) an ability to filter out superfluous information,

(ii) an ability to compress relevant information and represent it within a numerical symbol and

(ii) the ability to use the ambiguity of symbolism *and* draw on the filtered out information when it is appropriate.

These features support the numerical transformations evident in the way 'high achievers' handled their arithmetic. The evidence seems to suggest that the children appeared more ready or more able to draw together knowledge and compress all of the discrete information into mental representation which serves the dynamic purpose of acting as a 'skeleton' for other ideas. Such a skeleton can form the basis for general definitions when required.

'Low achievers' tend to concretise mathematical symbols. Their mental representations are either of concrete objects or "episodes" (scene or sequence of scenes) which are strongly associated with procedural aspects of the numerical processes. The children carry out procedures in their mind as if they were carrying out procedures with perceptual items in front of them.

In contrast the symbolic mental representations of 'high achievers' appeared to act as 'thought generators'. They appear to flash as memory reminders, momentarily coming to the fore so that new actions or transformations may take place. Such a distinction leads us to conjecture that the mental representations of the 'low achievers' are necessary for carrying out current activity whilst those of the 'high achievers' are compressions of completed activity or pointers to future action.

The evidence obtained from the series of items in the current study indicates that children who are 'low achievers' in mathematics appeared unable to detach themselves from the search for concrete substance and meaning – no information is rejected, no surface feature filtered out. When creating mental representations, they seem to focus on **visual** characteristics and **parts** of objects. The similarities between those associated with non-mathematical and mathematical items are striking. The children did not treat their mental representation as a **skeleton** on which they may pin core ideas.

The mental representation projected from the three forms of stimulus suggest that qualitative differences between the two groups of children were least apparent in the iconic items and most apparent in the verbal, symbolic and arithmetical (mental arithmetic) ones. Whilst the 'low achievers' remained at the same qualitative level throughout, the 'high achievers' tended to utilise the qualities that reflected the more 'skeletal' nature of the external representation. The more able children appeared to be looking for general descriptions, relationships and equivalences which were true, no matter which stimulus was used. 'Low achievers' seemed to place their attention on the particular. Thus, whilst the visual stimuli and the concrete nouns could stimulate amongst 'high achievers' the similar qualities to those projected by 'low achievers', more abstract notions signalled quite different behaviour.

The purpose of the following chapters will be to consider these behaviours with a different group of children. Partly, this is because the preliminary study and the pilot study involved one school in a considerable commitment and we did not wish to outdo our welcome. It is also partly because there was a wish to determine how common the differences between children were. The pilot study served its purpose. It showed that there were differences in children's mental representations and it provided a mechanism for examining and classifying these. However, the breadth of items used and the format of the questions required reconsideration. Issues had arisen which needed greater clarification. It is to these initial considerations in the context of the main study that we now turn.

*

CHAPTER 6

STRUCTURING THE MAIN STUDY

"I had to concentrate a bit harder because it was a bit hard " (Child, Year 4)

6.1. INTRODUCTION

The pilot study suggested that the mental representations of two groups of children, 'low achievers' and 'high achievers', had qualitative differences. 'Low achievers' appeared reluctant to reject information. They frequently used scenes from their known physical world to embellish their descriptions by building stories around the presented items. In contrast, 'high achievers' filtered out superficial detail to concentrate on more abstract qualities. Although they could relate to real world concepts, they were also able to form a hierarchy of ideas which allowed them to refer to objects in the abstract. The similarities and differences between the two groups could lead us to conclude that the mental representations of the low achievers are more specific and closely associated with surface characteristics whilst those of the high achievers have deeper meaning attached to them.

The thesis of this study is that different levels of achievement in elementary arithmetic may be associated with qualitatively different kinds of mental representation. The understanding developed from the pilot study leads to the formulation of supplementary hypotheses which suggest that:

- 'low achievers' draw on mental representations that are associated with specific examples, surface detail and realistic scene(s), (episodes) and
- 'high achievers' draw on mental representations which have an underlying generic nature.

Distinctions of this sort may suggest that the qualitative differences which underlie the kind of children's mental representation are of one form or the other. Such a simplistic view may suggest a dichotomy. However, as such it would not take into account the limiting factors associated with specific detail and "episodes" on the one hand and the power of generative mental representations on the other. Neither would it account for the instances where at any one time an individual may demonstrate that they possess a broad spectrum of qualitatively different mental representations. What it does suggest is that when faced with different stimuli, the underlying qualities will tend to dominate. Thus it is the purpose of this and the two subsequent chapters to consider the form of the mental representations that the children possess and the larger scale difference that may be discerned between children at both extremes of an achievement spectrum associated with elementary arithmetic.

Together with Chapters 7 and 8, this chapter will form a unified whole that reports the outcomes of the main study. Its purpose is to consider how this study was organised, how data were collected and how this data, which was in the form of verbal statements from the children, were classified. Thus this chapter re-examines some of the aspects initially considered within Chapter 4. Section 6.2.1 considers how the main study was developed. It indicates how the interview phases and the items that were part of each phase were influenced by experiences within the pilot study and it presents a consolidated version of these items (Section 6.2.2). It considers in depth the nature of the sample and its final selection (Section 6.2.3) and how the interview process operated and the way in which data were collected (Section 6.2.4). A substantial feature of the chapter is the way in which this data were organised. This is considered in Section 6.2.5. A reappraisal of the classifications used to distinguish between different kinds of mental representation draws on the experience of the pilot study and related work in the field of psychology. The classifications are clarified in section 6.2.5.2. This has the benefit of interdisciplinary strength by drawing on psychological influences, which in turn, link to the field of neuropsychology. The chapter concludes with a brief summary (Section 6.3).

6.2 PREPARING FOR THE MAIN STUDY

6.2.1 A Platform for Development

> Imagery is usually accompanied by the experience of "seeing with the mind's eye", "hearing with the mind's ear" and so on. This experience is a hallmark that specific types of activities are taking place in the brain. (Kosslyn, 1990, p. 1770)

This study broadens the scope of Kosslyn's definition so that the word imagery is interpreted as 'mental representation'. Its purpose was not to become caught up in the format of these mental representations by clarifying whether or not they were propositional or visual imagery. A guiding principle is that, irrespective of the format of the mental representation, there are other aspects of it which are not widely discussed. These may prove important to cognition particularly in our understanding of the divergence in numerical thinking. These aspects may be associated with the kind of mental representation. What is the content which guides its classification and how may it be subsequently developed and what component forms the basis for it? Consequently this aspect of the study retains the general format of the pilot study and possesses the same focus, namely to:

(i) increase our understanding of the kinds of children's mental representation,

(ii) identify whether or not children at either extreme of numerical achievement focus upon different kinds of mental representations and

(iii) identify whether different kinds of mental representations are related to achievement in early number arithmetic.

More particularly, it seeks insight into the:

- kinds of the mental representations created by verbal, visual stimuli which include pictures, icons, words and mathematical symbols,

- similarities and/or differences between children's mental representations of numerical and non-numerical words, icons and symbols,
- what it is that children think is happening in their head when subjected to the different items and
- what children see or articulate in their mind when they are dealing with mental arithmetic and
- the relationship between different kinds of mental representation and the strategies children use in elementary arithmetic.

The pilot study presented a somewhat limited impression of the mental representations of the two groups of children. Certainly, when Figure 5.3 is considered we gain the impression that a considerable amount of information about representations other than visual images is not forthcoming. It is this shortcoming that the main study attempts to overcome. Though different kind of visual images triggered from children's interpretation of numerical procepts is worthy of deeper study in its own right , the fundamental issue within this component of the study was to gain a deeper sense of the qualitative differences in the kinds of mental representation, and link these to arithmetical behaviour; to link wider aspects of 'seeing in the mind's eye' and 'hearing with their mind's ear' to the qualitative differences that have been identified in children's numerical achievement.

There is an important reason for this. Unlike concrete objects that may clearly offer a visual image, it may be more natural for mathematical objects to offer verbal representations[1] for example, counting, addition, multiplication or, indeed, other numerical relationships. Of course, this does not mean that numerical objects cannot depict visual images. On the contrary, we have already seen that they do so in the

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[1] As has been indicated in the literature review, the identification of propositional representations is difficult. This means that distinctions between propositional representation and verbal images is problematic. It is an issue that this study will not concern itself with. It will investigate the occurrence of verbal mental representations without exemplifying whether or not they are propositional representations or verbal images.

pilot study. Numerical objects offer the potential for a greater variety of qualitatively different visual images than may be offered by concrete objects. Numbers, for example, may be offered as objects themselves in a number of different formats: iconically as in a tally, in Arabic and roman symbolism, as analogues on a number line, or proceptual equivalences. Moreover, other mental objects may replace the mathematical symbols for example visual images of fingers or dots or even the real object such as fingers, blocks and other manipulatives.

The pilot study did not clearly indicate a clear sense of the way in which a child's initial representation provided a platform for development. Although apparent at one stage, the visual, that 'high achievers' were employing both 'bottom-up' and/or 'top-down' processing approaches, this information was 'volunteered' at a level that did not allow more detailed analysis. The development of a two part questioning process for the verbal phase of the main study — a 'first response' and a 'free talk' response — was believed to be one that would give an opportunity for enrichment of the initial response with greater detail or additional information through a network of other relationships. Allowing children to talk for a longer time would also allow us to investigate all the other pieces of information that are contained in the mental representation. Drake (1996) summarises issues associated with this notion by saying:

> the generation of an image promotes the development of a trace in the brain that integrates the separate components. Thus, accessing a part of the information encoded in the memory prompts the retrieval of all the other pieces of information contained in the image.
>
> (Drake, 1996, p. 7)

6.2.2 A Framework for Interviewing

A consequence of these observations was that the research questions in the three-phase interview procedure in the pilot study was expanded. These additions provided a range of items which stimulated investigation into:

(i) the content of the mental representation,

(ii) the main components of the representation – what object is the focus of discussion, what action and how may these be associated,

(iii) the modality of the representation,

(iv) the quality of representation that is initially generated and its subsequent development and

(v) the association between different kinds of representation and arithmetical achievement.

Insight into these issues was gained through the development of four interview phases:

(i) mental representations, associated with verbally presented items — the verbal, free of context phase,

(ii) mental representations associated with visually presented items — the visual, free of context phase,

(iii) mental representations associated with mental computation of verbally and visually presented items — the mental arithmetic phase and

(iv) mental representations associated with pencil and paper methods — the computational phase.

6.2.2.1 The Verbal Phase — Phase 1

The verbal phase now comprised 17 items (see also section 4.4.3): concrete nouns — dog, table, dots, football, animal, furniture, ball and the numerical items — five, seven, thirty-three, ninety-nine, half, three-quarters, three-eighths, nought point seven five, number and fraction.

These were presented around the core questions:

(i) *What is the first thing that comes to mind when you hear the word...?*

(ii) *Talk for 30 seconds about what comes in your mind when you hear the word...*

(iii) *If ET came to ask you what the word ... meant what would you say to him?*

A supplementary question, asked after the first two core questions

"As you were talking (or giving your answer to the previous question) ,what was happening in your head?"

had the purpose of widening the scope of evidence obtained within the pilot study. This had indicated that in the numerical component 'low achievers' had described visual scenes in their head (moving marbles, fingers, and static number tracks) whilst 'high achievers' had described visual 'flashes', somewhat in the form of skeletal flashes, which appeared to allow the children to generate and discuss ideas. The issue was whether or not similar sorts of mental representation were projected in the verbal component.

6.2.2.2 The Visual Phase — Phase 2

The full list of items within this phase (see also section 4.4.4) was:

- symbols: 5*, 99*, 3÷4, $\frac{1}{2}$,* $\frac{3}{4}$*. 0.75*
- iconic/pictorial items: marbles, dancing man, honeycomb, two-quarters, dots*, football*, dining room, furniture*, a table* and a ball*. (Appendix 2.1)

Those marked * were replicated in the verbal phase. This replication was an attempt to consider whether visual items triggered the same qualitative responses as the verbal items.

As with the verbal phase two core questions, followed by a supplementary question, guided the nature of the qualitative data that were obtained. The core questions:

"What is the first thing that comes to mind when you see this (the item)?"

"Look at this, when I tell you close your eyes and put this in your mind. Talk to me for 30 seconds. Do it now.".

were intended to make a clear distinction between the first mental representation and the way in which this mental representation was used for other ideas. The first question was asked immediately prior to displaying the item for a period of two seconds. This time span was regarded to be long enough to prevent perceptual difficulties and too short to make direct connections with the long term memory (Frick, 1990). The second question was given during seven second display of the item (the period it took to deliver the question to the point of "Do it now"). During the additional time it was anticipated that the children would give other pieces of information that may be contained in the mental representation. Such information would more easily enable us to identify qualitative differences in the kinds of representation projected by children from each of the two groups. In such a way we may be able to discern the sequence of development of a mental representation.

A supplementary question:

> *"As you were talking (about what came into your mind) to me what was happening in your head?"*

was grounded in a similar standpoint to that noted for the supplementary questions in the verbal phase. The question was not asked after the initial response because an assumption was made that the very first mental representation that takes place in the head is the one created by the visual perception of the visual stimuli.

6.2.2.3 The Mental (Verbal) Arithmetic Phase — Phase 3

Not as extensive as that used in the pilot study, which indicated that patterns of behaviour could be identified with a smaller range of combinations, this phase drew on a selection of combinations that were used in the 1995 Level 4 SAT (SCAA, 1995). To extend the more able children two and three-digit combinations were included. The full range of items was:

combinations to 20: 3 + 2, 9 – 7, 7 + 6, 4 + 7, 17 – 13, 12 – 8,

two digit combinations: 14 + 8, 29 – 6, 64 – 26, 27 + 62, 73 – 32, 45 + 57 and three digit combinations 188 + 267, 396 – 157.

Combinations were presented individually. Children requested solve them 'mentally' and two questions guided the discussion on each:

"Give the answer to..."

"What was happening in your head as you were solving this?"

Through their responses to the latter, the children's strategies were clarified and/or confirmed, the quality of response in the context of mental representation was identified and a clearer indication of the nature of the object that dominated and supported arithmetical thinking was established.

6.2.2.4 The Visual (with paper and pencil) Arithmetic Phase – Phase 4

Two components formed this phase of the study: an elementary component related to involving number combinations to 20 and an 'algorithmic' component in which addition and subtraction problems with at least one two-digit number were presented using standard formats. In addition to the phases given in the pilot study, this phase of the interview procedure was presented in an effort to establish parallel components between the verbal and visual elements of presentation. The items contained within the elementary phase were:

addition: 6 +3, 3 + 5, 3 + 4, 5 + 2, 4 + 5, 2 + 3 + 2, 3 + 4 + 3 and

subtraction: 9 – 2, 8 –2, 15 – 8, 13 – 5, 9 – 8.

Items in the 'algorithmic' component were:

30 +57	47 +15	274 +159	5 + 135	11+696
21 –15	82 –24	293 –185	438 – 21	687 – 47

As with all other phases this component requested that children found solutions to the combinations and then the interviewer asked:

"What was happening in your head as you were solving this?

The reasons behind the question have already been established: strategies could be clarified, the quality of response in the context of mental representation could be identified and a clearer indication of the nature of the object that dominated arithmetical thinking could be established.

6.2.3 The Sample

The broad nature of the sample was discussed in Section 4.5. Here, greater clarity of the context of the numerical achievement of the children is provided.

The school selected for the main study was more of an 'opportunity sample' based on pragmatic issues, ease of access, support for the investigation, an environment to carry out the interviews, easily accessible children and full parental support. A report based on an Ofsted inspection carried out during February and March 1996 indicated that, generally, teaching was "satisfactory to good" and that in mathematics it was "mostly satisfactory". The school's mechanisms for the assessment and reporting of children's progress were generally "less than satisfactory".

The nature of the work carried out in the school required access two days per week over a six month period. To carry out such a project, the objectives and nature of the work had to be clearly understood by the head teacher and staff involved. There needed to be a mutual commitment to the nature of the project and the lengthy period of sustained work within school. In the event, the project was fully supported by teachers and governors, and as a consequence, special facilities were always provided for the interviewing process. This made a stark contrast to that available in the over-crowded pilot school where the interview process was frequently carried out in a cloakroom at the end of a draughty corridor.

Selection of the children who formed the final sample reported in this study was based on three criteria:

(i) reported levels of numerical achievement based on criterion-referenced tests, such as the Standard Assessment tasks available at Year 2 and Year 4, and other formal tests,

(ii) teacher assessment which supported the above and

(iii) levels of achievement identified from the numerical components which formed phases five and six of the interview process.

Based on criteria (i) and (ii), the school was asked to identify 32 children reflecting the extremes of achievement — 'high achievers' and 'low achievers' — in each of the four year groups Year 3 to Year 6. The 'formal' spread of achievement for children at Year 3 was over the Level 1 and Level 2 range of the Key Stage 1 (KS1) Standard Assessment Tasks (SATs) (SCAA, 1995). Cumulative marks were used to select the eight children within this group. The selected children were within the range of $\pm 0.9\sigma$ (mean=19). Children in Year 6 were selected on the basis of their predicted scores in an imminent Key Stage 2 SAT. Allocation of marks and levels as a result of these tests (SCAA, 1996) indicated that only one child had been incorrectly selected. This child, identified as a 'low achiever', in fact achieved a Level 4 with an overall mark of 57 (mean=60). This child was eventually dropped from the analysis (presented here). In the event it proved to be less easy to obtain relevant objective evidence for the children in Year 4 and Year 5. Thus to establish the criteria against which all of the children could be measured by the same scale, even though 32 children had shared in the interview processes, it was decided that for purposes of efficiency and reliability, these 32 would be reduced to 16 based on the highest and lowest achievers in the numerical phases of the current study (i.e. phases 3 and 4). The criteria for selection was based on the percentage of correct responses. Such was the difference between children that only amongst the Year 4 'high achievers' was the difference between selected and non-selected children so marginal that additional discussion with the teacher led to the final decision.

The sample reported in this study consisted of two children whose numerical achievement reflected that of the extreme in each of the four year groups.

- In Year 3 (mean age 7.8), 'high achievers' had all achieved Level 2 in the 1995 KS1 SATs tests. The children had the highest cumulative scores and in addition obtained 77% in the Phase 3 and 4 numerical components. The 'low achievers' were drawn from those who had obtained Level 1 in the 1995 SATs and who had also scored lowest on the current test (less than 27%).

- It was harder to objectively judge children from Year 4 (mean age 8.8), although they had reflected extremes in the KS1 SATs more up-to-date objective profile of achievement was not available. It transpired that the schools current assessment profile was weaker than originally suggested. In the event four children at both extremes were selected on the basis of teacher assessment and the final two in each group selected on scores within the Phase 3 and 4 items: the two 'high achievers' scored above 97% and the two 'low achievers' below 47%.

- Similar problems emerged with Year 5 children (mean age 9.7) so the solution remained the same. 'High achievers' scored above 87% and 'low achievers' below 47%.

- The Year 6 children (mean age 10.8) were selected on the basis of their predicted SAT scores and the Phase 3 and 4 items. In the event, the selected two 'high achievers' not only the received the highest totals within the year in the 1996 SAT, but were awarded Level 5, scoring 78% (class mean 59.7, standard deviation 12.8) and achieved 100% in the investigation test. The 'low achievers', each awarded Level 3, were at the low extreme of this band, receiving total scores of 36 and 38 respectively. Both achieved below 60% in the numerical test.

6.2.4 The interviewing process

The model used in the Pilot Study was developed in this study. The four phases of items were each presented on different occasions over a period of seven months — from March to November 1996. Each interview was video-recorded and supported by field notes. The key questions that promoted initial discussion were common to all children but they were supported by supplementary questions in order to gain deeper insight when appropriate. Two interviewers carried out the interview process, sometimes together but more often independently. Joint interviews were common at the start of each phase of questions so that the agreed procedure could be jointly discussed, evaluated and, if necessary, modified. During the 'individual' interviewing times, the interviewers ensured that each child was interviewed over for at least one phase by a different interviewer. It was felt that this would contribute to the strength of discussion associated with classifying responses. Interview times were approximately half an hour for each phase. In the event that the interview process was slower than usual, a phase was split into two interview periods.

6.2.5 Classifying Responses

6.2.5.1 Rethinking

Classification of the data had three different aspects: classification of the kind of mental representations associated with the core questions in the verbal and visual phases, classification of the what it was that the child said was happening in his or her head together with factual interpretation of the type of representation (visual/verbal) and classification of the strategies used to solve the number combinations.

When the pilot study was carried out, there were no pre-determined classifications in mind for the visual and verbal phases. Through repeated analysis of the data, the classifications which formed the focus of the discussion were conceptualised and analysed. However, it transpired that these classifications had striking similarities to those of De Beni and Pazzaglia (1995). Though Drake (1996) later referred to three classifications (see section 3.3.6), these did not appear rigorous enough for this study. De Beni and Pazzaglia's work had an added bonus — their classifications related work in the neuropsychological field (see Section 3.3.7)

Even though De Beni and Pazzaglia's work had strong implications for the development of this study there were important differences. Firstly, the items used by De Beni and Pazzaglia consisted of 40 high imagery value nouns all having a "medium to high frequency in Italian". The current work used concrete and abstract nouns, the former drawn from Paivio, Yuille and Rogers, (1969), the latter from consultation with teachers regarding children's experiences in arithmetic. Secondly, the current study did not require that subjects "construct good and vivid images". The assumption here was that everyone constructs visual images. The requirement of this study for subjects to indicate "what was happening in their head" does not make this assumption. Additionally, the nature of the object was of interest. This was an important aspect of the study since indications in the pilot study show that when talking about abstract numerical nouns, children could refer to the arithmetical symbol, the general concept or to physical/figural objects. In short, given the extended nature of this enquiry, classifications which bridged the different emphases placed on data collection were required.

After careful consideration of the response data across the spectrum of items, a modified version of De Beni and Pazzaglia's (1995) classifications seemed to supply an appropriate core to link with classifications of a more particular nature, for example, children's numerical strategies, the nature of the object that guided thinking and the form of the representation.

6.2.5.2 A Classificatory Core

The pilot study used five core classifications: 'not known', 'association and contextual', 'example', 'description' and 'proceptual qualities'. Several of these, together with their more finely-graded subdivisions were closely associated with the classifications De Beni and Pazzaglia (1995) used to distinguish kinds of mental imagery. For example, their notion of 'specific' is closely related to the pilot study notion of 'example' whilst their notion of 'contextualised' fits the pilot study notion of 'association'. Of course, if it is possible to have a 'specific' mental representation, it should also be possible to have a 'general' one. In the pilot study this classification was not clearly identified but some of the qualities associated with this aspect were identified as 'association' or 'description'.

However, although essential features were common, the evidence of some of the finer classifications in the pilot study prompted a more relevant, to this study, reconstruction of De Beni and Pazzaglia's (1995) classifications. Constructing "good and vivid images' of "high value nouns" is one thing, reporting mental representations of abstract nouns such as 'five' or 'half', or indeed projecting mental representations of symbols or icons of these nouns, poses other problems.

The 'contextualised' emphasis that was given by De Beni and Pazzaglia (1995) to some responses to items within their study indicates that such images have distinctive and relational characteristics (see also section 3.6.6). However, the finer analysis through which contextualised mental representation are classified, 'item specific' and 'relational', did not satisfy the clear distinctions observed between the responses of the subjects within the current study. Contextualised representations could describe a scene or a sequence of scenes or they could have a higher order quality. Therefore the category 'episodic' was invented whereby mental representations associated with description, a scene or sequence of scenes. Such representations were most often narrated in continuous speech but it has to be made clear that 'episodic' mental representations are not created in the episodic memory. It is not a retrieval of a specific scene from the remembered past. They are called 'episodic' because they simply refer to a scene. On the other hand, 'autobiographic-episodic' mental representations are a result of autobiographic episodic memory. They denote a specific scene that occurred in the individual's life. However, there were other responses that implied the existence of a context but the structure of the responses was more fragmentary; a collection of disconnected, seemingly arbitrary, generic statements which originated from the same general concepts which seemed to serve as an anchor to this expansion in a relational way (used in Skemp's sense). Thus the notion of 'generic' representation was included. In a similar way, to identify a deeper, more proceptually oriented mental representations, which could only be identified for mathematical objects and involved the use of, or reference to, a mathematical symbol, the classification 'proceptual' was added.

Thus, drawing upon De Beni and Pazzaglia (1995) and on the rigorous appraisal of the responses, the conceptual identities which led to the core classification used within the main study are identified below:

- **General Mental Representations**

  According to De Beni and Pazzaglia (1995):

  > "general images [were identified] when the subject's description did not specify any characteristics of the noun, for instance, given the noun 'table' the description of a general image could be 'I can see a table.'" (De Beni & Pazzaglia, 1995, p. 1363)

  The children's mental representations were classified in this category when their responses indicated that they were not talking about a specific item. For example, to the item 'table', the response "*it's a surface on metal or wooden sticks*" (Y4+. 'table')[2], and to the item 'ball' the response "*It is a 3D shape, 3D circle that bounces*" (Y5+, 'ball'), were classified as 'general'.

  In the case of numerical items, the most frequent general response was a reference to the mathematical symbol or very general comments, for example,

[2] It will be usual throughout this section of the study to identify children's comments of at least a few words length as a 'quote'. These will be attributed by referring to the year and ability of the child, for example Year 4, 'low achiever' as Y4- , and the item that generated the 'quote', for example 'table'.

'fraction' was identified as "*Part of*" (Y6+, 'fraction') or in the case of an icon, the icon being given a name, for example, "*Dots*" (Y3+, 'dots').

- **Specific Mental Representations**

A specific representation

> "represents a single well-defined example of the concept without reference to a specific episode." (De Beni & Pazzaglia, 1995, p. 1360)

The criteria in this classification were extended to allow for multiple examples which were qualitatively similar. For example, a 'low achiever's' response to the word 'animal' included "*... a cheetah is one, a rabbit is one, a dog is one a cat, a Labrador, Dalmatian, owl, eagle, buzzard, etc.*"(Y3–, 'animal')

In the numerical context, children's specific responses most often arose when children were asked about "number" or "fraction", for example, "*like one, two, three, four, five, six, and ten are numbers*" (Y4–, 'number') or the word 'fraction' exemplified by "*A half*" (Y6+, 'fraction').

This category also included 'autobiographic mental representations'. According to Kosslyn (1994) the notion of autobiographic is a special case of exemplar (specific) which has been enlarged by the addition of a self-schema, for example, "*I see my ball with my name on it*" (Y3-, 'ball') and "*My pet*" (Y5+, 'dog').

- **Episodic Mental Representations**

Classification of this form was identified when the item was associated with an 'episode' (a scene or sequence of scenes) that occurred in a specific context. The classification does not refer to elements and neither is it associated with a specific event in the child's past life. For this classification to be associated with the response the item is taken from an episode, a scene or a context. Thus, one aspect of the qualities of this classification was as an active scene narrated or described in full detail, for example:

"Boys can kick it around and sometimes it can get lost over the field." (Y3–, 'ball')

"Number five. I Think of a row of numbers and light shines on number five. A light goes along and stops over the number five." (Y5–, 'number')

A further aspect of the classification was the context to which the object is connected.

"Between 6 and 8... say when you were 7 the next you will be year eight and when you were six you will be seven." (Y5–, 'seven')

Indications of the place where these items are seen were identified in this classification.

"It looks like one side of a dice. " (Y6+, visual, 'dots'),

"A number on a bus, on a door, on a sum, on a book." (Y4-,visual, '7').

In some instances, children would not only talk about an episode but emphasised it with what were termed within the pilot study 'imaginary extensions'. The children would give colour, detail, and movement to transform the object to a real scene:

"Somebody trying to get the ball in the circle, It's a game with children and then I am thinking of some people laughing at them". (Y5–, 'marbles')

"... looks like a window... Outside of the window, quite a big window, there is a pot of flowers, like roses—red and blue. Outside the windows there is a very small garden with hedges that go around the garden like nettles. The flaps are a brownish colour lines like a blind and you can see through." (Y5+, 'window')

- **Generic Mental Representations**

This classification represents productive statements that are common to the general concept. It was not a description of a sequential event which had a clear beginning and end. It appeared most often as a collection of statements that seem to have the potential to produce new ideas. Though they had a 'general' quality, the statements diverged to produce different ideas related to the item.

A typical generic response would be:

> "keeps you fit. An exciting game. Millions of fans. Important in every nation. Children and adults play it. Different types of football and balls." (Y4+, 'football')

> "... maths and writing. Seven you could be doing some adding or times and the number seven might come up. Seven is also played in sport back or shirt has one digit, phone number." (Y5+, '7').

The question "*What is the first thing that comes to mind*" often generated one word response that could equally be classified as 'general', 'generic' or 'episodic'. In such cases where it was difficult to discern how the first response could be best classified the 30 seconds response was taken into account. For example, when children were asked about the numerical word 'five', the response may have been "*Maths*" (Y5+, 'five'). This response was classified as 'generic' because it was possible that other ideas could be produced from this first response. If however during the 30 second response there was no qualitatively change and generation of new, relational ideas, for example "*You see it in writing and maths five is ... five is also*" (Y5+, '5') then such response was re-classified as episodic. The child's 'initial response' was only an expressing of a context where this word arises, thus it was classified as 'episodic'.

- **Autobiographic-episodic Mental Representations**

In the context of this study, the definition of 'autobiographic-episodic' has been taken from by De Beni and Pazzaglia (1995). Their definition allowed for the "occurrence of a single episode in the subject's life connected to the concept"

(p. 1361). Therefore, examples such as the following were classified as 'autobiographic-episodic':

> "My friend wasn't good at fractions and she had to take extra work home."
>
> (Y4+, 'fraction')

> "We have recently done reflections and they had lots of halves in them. We had to put our mirror down the side and see the rest of it. I saw lots of those."
>
> (Y4+, 'half')

De Beni & Pazzaglia (1995) maintain a distinction between 'autobiographic-episodic' images and 'autobiographic images'. In this study as we have mentioned earlier 'autobiographic' mental representations were considered as a special case of the 'specific' kind of representation. 'Autobiographic' mental representations were seen to those which involved the subject without a precise episodic reference or objects belonging to the subject. It was perceived that mental representations such as these were special cases of 'specific' enlarged with the involvement of the self-schema. On the other hand the 'autobiographic episodic' mental representations were seen as a retrieval of a specific event from the individual's autobiographic episodic memories. These were separately classified.

- **Proceptual Mental Representations**

  This classification is additional to those identified by De Beni & Pazzaglia and it is particular to the numerical items. References to mathematical relationships, processes, concepts, manipulations of mathematical symbols and/or indications of equivalence were classified as such. For example, *"it's divisible by nine"* (Y6+, '99'), *"3 parts out of 4, fraction, 0.75, more than half."* (Y6+, '3/4')

These core classifications — general, specific, episodic, generic, autobiographic-episodic and proceptual — were used to classify all responses to the core questions and all responses to the question *"What is happening in the head?"* in the first two free-context phases of the study.

6.2.5.3 Classifying the Arithmetical Component

Classification of the children's strategies to solve the arithmetical components of phases 3 and 4 was discussed within Sections 4.6.2 and 4.6.3. The reader is reminded that these drew upon the work of Carpenter and Moser (1982) and Gray (1991) to provide classifications for the elementary arithmetic, that is, count-all, count-on, derived fact and known fact. Classification of the two-digit mental combinations drew on Oliver, Murray, and Human (1990) to provide the notions of 'transformation' and 'accumulation' whilst the written component drew on a classification of the standard algorithms but identified variants such as left-to-right calculation.

In addition, drawing on the work of Steffe, Richards and Cobb (1983), the identification of the nature of the response was deemed to be either automatic, abstract, figural or perceptual or associated with verbal counting. To support these forms of representation the modality of the representation was also considered. It was from these considerations that the nature of the object was deduced. It is felt to be more appropriate that clarification of these issues is given when they are discussed in the context of the evidence (see Sections 8.2.2.2 and 8.2.2.3).

It is important to say that although the strategies were often external and one could see how the children reached their solution, mental representations were not apparent and the children were asked to report and describe them. In some instances an immediate response could give the impression that it was automatic. However, at times children would refer to a combination of symbols, or some visual images of symbols that flashed in their mind. On other occasions there might have not been an obvious external behaviour but the children would claim verbal count, use of a figural representation or feeling the answers on the fingers without moving them. On such occasions it was the interviews interpretation, based on all of the evidence to hand, including video analysis that identified an appropriate classification.

6.3 CHAPTER SUMMARY

This chapter reviewed the way in which the main study developed from the pilot study using insights gained from related work in the field of psychology. It indicated two aspects of the investigation. The first is that in using a phenomenographical approach, there is always a search for the most appropriate ways to conceptualise the data that are to hand. The response here has been to move towards an interdisciplinary paradigm that may have more to offer than that found in a single discipline. The second, by implication, gives strength to the notions of reliability and validity discussed in Section 4.3.7. The qualities distinguished by De Beni and Pazzaglia (1995) were found quite independently in the pilot study. There is little doubt that mutual recognition could provide a forum for a continuing debate. The additional distinguishing characteristics associated with the additional classifications of 'generic' and 'proceptual' serve to highlight differences in a study which is essentially looking at arithmetical achievement. The following chapter serves to present the evidence which supports this and to add a further dimension to the notion of qualitative difference in thinking.

*

CHAPTER 7

MENTAL REPRESENTATIONS AND NUMERICAL ACHIEVEMENT

I can see a row of numbers one to a hundred. They are all in the dark and then I can see a light above each number. Say number one is here [on the left] and 100 is here [on the right]. Number one comes forward and goes back, number two comes forward and goes back, then three, then four and up to 100 (Child, Y5-)

7.1 INTRODUCTION

7.1.1 The Development of the Chapter

In this chapter we discuss the different kinds of mental representation triggered by visual and verbal stimuli, projected by two different groups of children and should be seen in conjunction with Chapter 8 which considers the children's approaches to elementary arithmetic. We show that there is a qualitatively different emphasis in the kinds of mental representation projected by children at extremes in numerical achievement. Those identified as 'low achievers' project mental representations which have descriptive emphasis and are classified as 'general', 'episodic' and 'specific' mental representations. Those who are 'high achievers', whilst able to project mental representations of these kinds, also demonstrate those with relational characteristics, classified as 'generic' and 'proceptual' kinds, which provide them with the flexible interaction to move to a more abstract level of thought.

Firstly we consider the children's mental representations associated with visual items (Section 7.2) then verbal ones (Section 7.3). Similarities and differences associated with the two phases are considered in Section 7.4. In Section 7.5, we consider the specific role of visual imagery. We suggest that restricting our efforts to the identification of different kinds of visual imagery and its use in the context of elementary arithmetic will constrain our interpretations of the qualitative differences that exist in mathematical behaviour (Section 7.5.4). A necessarily limited consideration of the influence of age in Section 7.6 suggests that there may be no change in the quality of mental representation projected by 'low achievers' whereas those of 'high achievers' become increasingly more attuned to the demands of the stimulus.

It seems natural to suppose that if children are selected on the basis of their proceptual/procedural interpretation of symbolism at an operational level, such differences would be reflected in the mental representations they project when faced with arithmetical symbols. This chapter shows that it goes further than that. Not only do children at different levels of arithmetical achievement possess qualitatively different mental representations of arithmetical symbols, they also appear to possess qualitatively different mental representations of other conceptual ideas presented, free of context, in two modalities: the verbal and the visual. The evidence suggests that the difference increases as external representations become more abstract (Section 7.4.2.3). Both groups can project mental representations that are underpinned by descriptive qualities. However, whilst 'low achievers' consistently project the features that are embedded in these kinds of mental representation, 'high achievers' mentally represent stimuli differently. They project these or generate more relational mental representations to display a spectrum of quality which, it is conjectured, has a more generic core.

After discussing the representational qualities associated with children's explanations of the verbal items (Section 7.7), the chapter summary (Section 7.8) concludes by indicating that:

- 'low achievers' consistently portray 'general', 'specific' and 'episodic' mental representations. These are characterised by a descriptive emphasis either of an object, a scene or a sequence of scenes which often resemble reality and

- 'high achievers' are influenced by the nature of the external stimuli and project mental representations with relational and descriptive qualities.

7.1.2 The Research Questions

The research questions that will guide the analysis of the visual and verbal phases of the study take the following forms:

- to what extent are the different kinds of mental representation distributed across the whole group of children and what form does this distribution take between the different achievement groups?
- does the kind of mental representation differ between numerical and non-numerical items and does achievement influence the distribution?
- what are the similarities and differences between mental representations associated with the visually and verbally presented items and to what extent is this related to numerical achievement?
- what are the kinds of children's visual images and how may these be associated with 'level of arithmetical achievement'?

7.1.3 Presentation of the Data

Essentially the form of presentation will be descriptive, based upon qualitative analysis. Where it is thought appropriate, a categorical statistical test will be applied to indicate the level of significance of any differences noted. Probabilities will be presented at two levels: $p<0.01$ (significant) and $p<0.001$ (highly significant). Where differences are obvious from the descriptive presentation such a test will not be applied. This is for two reasons. First the nature of the presentation will clearly indicate that there are differences between the two groups of children and secondly, in some instances categories which summarise the mental representations of one group are not identified to any truly comparable proportion in the other.

Since it is the intention here to provide indications of the distribution of different kinds of mental representation within phases and between achievement groups, the category 'not known' will not figure within the general discussion. Only those responses where a mental representation was classified will be considered. This is not to say that 'not known' is ignored. Indeed, several children indicate that the notion of 'not known' is accompanied by "*just black*", "*flashing orange*", "*nothing came*", or in some instances " *I can not connect it with anything*" . However, discussing it as a kind of mental representation which supports our understanding of qualitative difference would provide little additional insight. However, where it is felt appropriate, introductory comments for each individual section will consider the extent that 'not known' occurred in responses.

The study conjectures that different kinds of mental representation will emerge between in children who have different levels of achievement in arithmetic. To highlight these differences, in the sense that it is identifiable patterns that are of concern, many of the figures presented in the chapter do not take into account all of the kinds of mental representation identified in a particular group or in the cumulative discussion of items in a particular phase. Those representations which occur in less than 8% of the instances are collectively identified under the headings 'other' or 'not classified'. The use of these words should not be taken to indicate that there are kinds of mental representation that have not been identified in the analysis. It simply means that the results reported are 'trimmed' to present a clearer picture. The choice of 8% may be regarded as quite arbitrary. However, careful consideration of the summarised results indicates that percentages up to this level frequently indicate truly idiosyncratic behaviours which are not shared by other children — the occurrence is marginal. To report such marginal occurrences in the analysis would present a 'fuzzier' picture than that which is emerging. The 'trimming' allows the discussion to focus upon a relationship between kinds of mental representation that do indicate differences between the children. Should the reader wish to establish how, marginal tables which indicate full distribution of kinds of representation are presented within Appendix 2.

7.2 MENTAL REPRESENTATIONS AND THE VISUAL PHASE

7.2.1 Introduction

The distribution of responses in the pilot study indicated a certain commonality in the quality of response. Both the 'high' and the 'low achievers' provided 'surface characteristics' and 'imaginative extensions' for the icons. However, whereas there was little evidence that 'low achievers' could provide qualitatively different responses, 'high achievers' demonstrated that they could generate mental representations with relational characteristics. Descriptions of 'surface characteristics' and 'imaginative extensions' were supplemented by 'insight' and 'proceptual' observations, the former common for icons, the latter for symbols. Neither of these two features were apparent amongst 'low achievers'. Their emphasis on 'surface characteristics' and 'imaginative extensions' dominated the mental representations they derived from external icons whilst 'surface characteristics' and 'association' were derived from symbolic considerations.

Though the core of the revised list of classifications was established from the De Beni and Pazzaglia (1995) classifications and were strongly associated with the reactions to verbal stimuli, in this study the full list was finalised through repeated conceptual analysis of the responses given to both the verbal and visual stimuli. The outcome suggests that features observed in the pilot study would be replicated in the current study. A high proportion of 'general', 'specific' and 'episodic' responses would be determined amongst 'low achievers'; the first since we expect the children to provide a 'general' representation without descriptive details, the second since it includes notions of 'well defined' and 'surface characteristics' and the third because it includes notions of 'imaginary extensions'. We expect 'high achievers' to provide 'generic' (qualities with insight) and proceptual qualities in greater frequency than the 'low achievers'.

7.2.2 Looking at 'Not Known'

Sixteen children responded to sixteen items in the visual phase. For the two questions:

> *"What is the first thing that comes to mind when you see this (the item)?"*
>
> *"When I tell you, close your eyes, put this in your mind and talk to me for 30 seconds. Do it now."*

the number of classified responses differed. The first question only took account of one response. Therefore, 256 responses were possible. The second gave free expression to the children, and consequently, there may be several qualitatively different kinds of response from each child. In the event 374 classified responses were noted. The distribution of different kinds of mental representation was considered after the 'not known' category was removed. Overall, 'not known' accounted for 6% of the responses identified from the first question – 16 responses equally divided between 'high' and 'low achievers'. Symbolic stimuli caused particular problems for the younger children. For example a Y3 'low achievers' reaction to '3÷4' was

> "It's a... I don't know what the sign is... we haven't done this yet. Is it subtraction – something like that." (Y3–, '3÷4')[1]

Iconic stimuli could cause difficulty for 'high achievers':

> "I can't think or see. It's so weird I cannot connect anything with it."
>
> (Y6+, 'marbles')

Although the proportion of 'not known' declined for the second response, slightly in excess of 3%, the distribution between 'high achievers' and 'low achievers' changed. The number of 'not known' responses from 'high achievers' remained the same (and

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[1] It will be usual throughout this section of the study to identify children's comments of at least one sentence length as a 'quote'. These will be attributed by referring to the year and ability of the child, for example, Year 3, 'low achiever' as Y3–, and the item that generated the 'quote', for example '3÷4'.

the same children) whilst only 2 (of eight) responses from 'low achievers' remained 'not known'. Although they could not identify or give a name to the icon, after closing their eyes, 'low achievers' talked about it in descriptive terms:

> "It is dots, like dots all over the place, with a big circle so the dots are inside the circle".
>
> (Y4–, 'marbles')

> "It is black and grey". (Y3- , 'honeycomb')

Both of these mental representations were identified as 'specific' — they recalled surface characteristics of a well-defined example. It was interesting that 'high achievers', whose initial response was 'not known', did not attempt to do this. It was as if their difficulty in providing a 'general' mental representation in the form of a name took precedence over any other aspects that might occur to them. They also, often claimed that they could not give a response because they could not "*connect anything with it*". (Y6+, marbles).

7.2.3 Distribution of Kinds of Mental Representation

7.2.3.1 A General Model

Table 7.1 indicates the overall distribution of the kinds of mental representation associated with the first and second responses of the visual phase. The table draws together responses from 'low' and 'high achievers' and indicates the number of responses and their associated percentage distribution after 'not known' is excluded. It is a summary of data presented in Appendix 2.2. N represents the total number of classifications identified for 16 children responding to 16 items.

	First thing in the mind		30 sec Free talk	
	N=239	%	N=374	%
General	90	38	67	18
Specific	72	30	159	43
Generic	9	4	17	5
Episodic	51	21	100	27
Autob. Ep.	4	2	11	3
Proceptual	13	5	19	5

Table 7.1: Overall distribution of different kinds of mental representation associated with the first two responses of the visual phase.

The dominance of 'general', 'specific' and 'episodic' kinds of mental representation is clearly apparent whilst those of the 'generic' 'proceptual' and 'autobiographic-episodic' kind are somewhat limited. It can be seen that the proportions of the 'dominant' representations change over the two responses: 'General' responses decline considerably in the second response whilst 'specific' responses increase. There is a less noticeable change in the proportions of the 'less dominant' mental representations such as 'proceptual' and 'generic'. The difference in the proportions of the more dominant kinds of mental representation is significant (χ^2=10.06, p<0.01) and largely accounted for by the decline in the proportion of the 'general' mental representations and an increase in the proportion of the 'specific' ones.

In a sense this is not a surprise. When providing a first representation, 38% of the of the representations identified were 'general'. After being asked to close their eyes and think about the item for 30 seconds this proportion was halved although it continued to represent the fact that in over one in four instances a general response was given. For example, "*fraction*" (Y4+, $\frac{1}{2}$) "*number*" (Y5+, 'five'). 43% of the classified responses for the second response were identified as 'specific'. In two out of three instances children provided this kind of representation (a total of 245 separate responses gave 374 different kinds of representation).

> "It's got 8 black dots, 16 in the corner and a thin round circle with a green background."
> (Y4-, 'marbles')

> "Green background, 15 in the corner, black writing, a nought, a little dot, a black dot next to it and the seventy five written in black, and its bubble writing." (Y4-, '0.75')

From such a generalised picture we get a sense that it is possible to distinguish between different kinds of mental representation but that it is less easy to identify the role played by those that have truly 'relational' qualities. Here the term 'relational' is used in Skemp's sense (1971) to identify those representations that are 'generic' and 'proceptual'. The over-riding sense, particularly from the second response, is that mental representations that are well-detailed ('specific' representations) dominate. If the intention was to present a model based on this generalised picture, it would not be stretching the evidence too far to say that children's mental representations associated with this visual phase are of three kinds — 'general', 'specific' and 'episodic' — a conclusion which generally supports De Beni and Pazzaglia's (1995) comments in their synthesis of traditional distinctions drawn from a metacognitive approach to imagery. However, it is not the intention of this study to identify a common model but rather to seek a model which gives some account for differences. To do this the mental representations of the 'high' and 'low achievers' are considered separately and so too are mental representations stimulated by three different forms of visual stimuli: 'picture', 'icon' and 'numerical symbol'.

7.2.3.2 A Comparative Model

Table 7.2 is established in the same way as Table 7.1, and again drawing upon the summary in Appendix 2.2, provides a comparison of the kinds of mental representation classified for the 'high' (HA) and the 'low' (LA) achievers. In each instance 8 children responded to 16 items and N indicates the number of classified representations. The distributions are in percentages.

	First thing in the mind		30 sec Free talk	
	HA	LA	HA	LA
	N=119	N=119	N=183	N=190
General	46	30	21	15
Specific	17	42	38	46
Generic	6	1	7	2
Episodic	20	22	21	32
Autob. Ep.	1	4	3	3
Proceptual	10	1	9	1

Table 7.2: Percentage distribution of the kinds of representation identified for 'high' and 'low' achievers within the visual phase.

It can be seen that the greatest proportion of mental representation associated with the 'first' responses of the 'low achievers' are identified as 'specific', and to a lesser extent, 'general' and 'episodic'. The sum of the proportions of these three kinds of mental representation accounts for 94% of the mental representations they project. In this sense, the 'low achievers' fit the generalised model identified in Section 7.2.3.1.

Though their 'specific' mental representations are often related to a description, "*dots in a number five*" (Y5-, 'dots'), they may also suggest that there is a need to identify with reality "*robber*" (Y3–, 'dancing man'), "*It's like a dancer*" (4–, 'dancing man'). Furthermore, the 'episodic' comments may provide a detailed imaginary context for this reality:

> "Somebody eating dinner on table and someone sitting down on couch reading a book, and someone in bed sleeping." (Y4–, 'furniture').

Overall, from the first responses we see that:

- though there is a similarity in the dominant kinds of mental representation projected by the two groups of children, that is 'general', 'specific' and 'episodic', the distribution of these mental representations and their association with other kinds of mental representation, for example 'generic' and 'proceptual' kinds, contributes to a highly significant difference between the two groups of children (χ^2=28.86, p<0.001),

- 96% of the mental representations projected through the initial responses of the 'low achievers' are designated as 'specific', 'general' and 'episodic' with 'specific' mental representations contribution nearly 50% of this proportion and

- though 83% of the 'high achievers' mental representations are of a similar kind over half of these are identified as 'general'.

The second response provides some interesting observations:

- Although there is almost no change in the proportion of the sum of the 'general', 'specific' and 'episodic' kinds of mental representation identified in each group, 'high achievers' had 80%, 'low achievers' had 93%, in comparison to the first response, there are highly significant differences in the contribution that each makes to this sum amongst 'high achievers' (χ^2=17.47, p<0.001), and significant differences amongst 'low achievers' (χ^2=12.359, p<0.01).

- Given more time to talk we see that, in comparison to the first responses, the smaller proportion of identified 'general' mental representations amongst 'low achievers' is accompanied by a higher proportion of 'episodic' mental representations.

- An equivalent drop in the proportion of 'general' responses amongst 'high achievers' is associated with a sharp increase in the proportion of 'specific' mental representations, the proportion having more than doubled, whilst 'general' mental representations have more than halved.

These percentage considerations can themselves mask some clearer distinctions in the occurrence of 'general' and 'specific' mental representations with the two groups in the context of the 'first thing that comes to mind'. Almost one in two of the mental representations identified from the eight 'high achievers' for the sixteen items were identified as 'general', compared to less than one in three from the eight 'low achievers'. In almost one in two instances, 'specific' mental representations were identified from the 'low achievers' compared to one in six of instances from the 'high achievers'.

Not only was the proportion of 'general' mental representations from 'high achievers' lower for the second response but the total (39) represented a decline to one in three of the number of separate responses that projected this kind of mental representation[2]. In comparison, slightly more than one in two responses now suggested a 'specific' representation.

[2] Though their may have been multiple representations recorded for any one response to question 2 any individual representation was only identified once. Therefore each representation could be identified at least 118 time from amongst the high achievers (10 don't know) or 126 times from amongst the 'low achievers' (2 don't know). The ratio's indicated are based upon the occurrence of any one mental representation across the total number of items considered.

Of course, the additional time allowed the children to contribute as much as they feel able to and consequently, from each child, there may be a sequence of responses that embrace different kinds of representation. Although the general pattern that emerges is that these visually presented items evoke an emphasis upon 'general', 'specific' and 'episodic' kinds of mental representation, indications of the occurrence of 'proceptual' and 'generic' mental representations are evident from 'high achievers'

The former, which are exemplified by the following:

> "it's a double number, it is two nines, one less than hundred, thirty three times three, one third would be thirty three, one ninth would be eleven" (Y6+, '99')
>
> "can also be a decimal and like nought point seven five. Not a whole number, not more than one. Multiply by one and add one third of it and you get one" (Y6+, '3/4').

and the latter by:

> "fraction , half, part, number and a word" (Y4+, 'half')
>
> "fractions, maths, arithmetic... decimals" (Y5+, 0.75)

were kinds of mental representations almost non-existent amongst the 'low achievers'. The notion of 'proceptual' representation did not extend across the full range of items and for this reason it is now worth considering the kinds of mental representations associated with the different groups of visual items.

7.2.3.3 The First Response: Distribution and Kind of Visual Stimulus

The items that formed the visual phase were themselves classified into three kinds: those that replicated the words of the verbal phase and which were mostly presented as *pictures*: 'football', 'ball', 'table', 'dining room', 'furniture' and 'dots', those that were identified as *icons*: 'dancing man' 'windows' 'honeycomb' and 'marbles', and those that were *numerical symbols:* '5', '99', '$\frac{1}{2}$', '$\frac{3}{4}$' and '0.75'.

Figure 7.1, established from Appendix 2.2, illustrates how the different kinds of mental representations derived from the first response were distributed over the groups of visual items by each ability group. The figure presents 'trimmed' results formed from the sum of the occurrences of a particular representation given as a percentage of the sum of all representations. the number of representations associated with each group is identified by N.

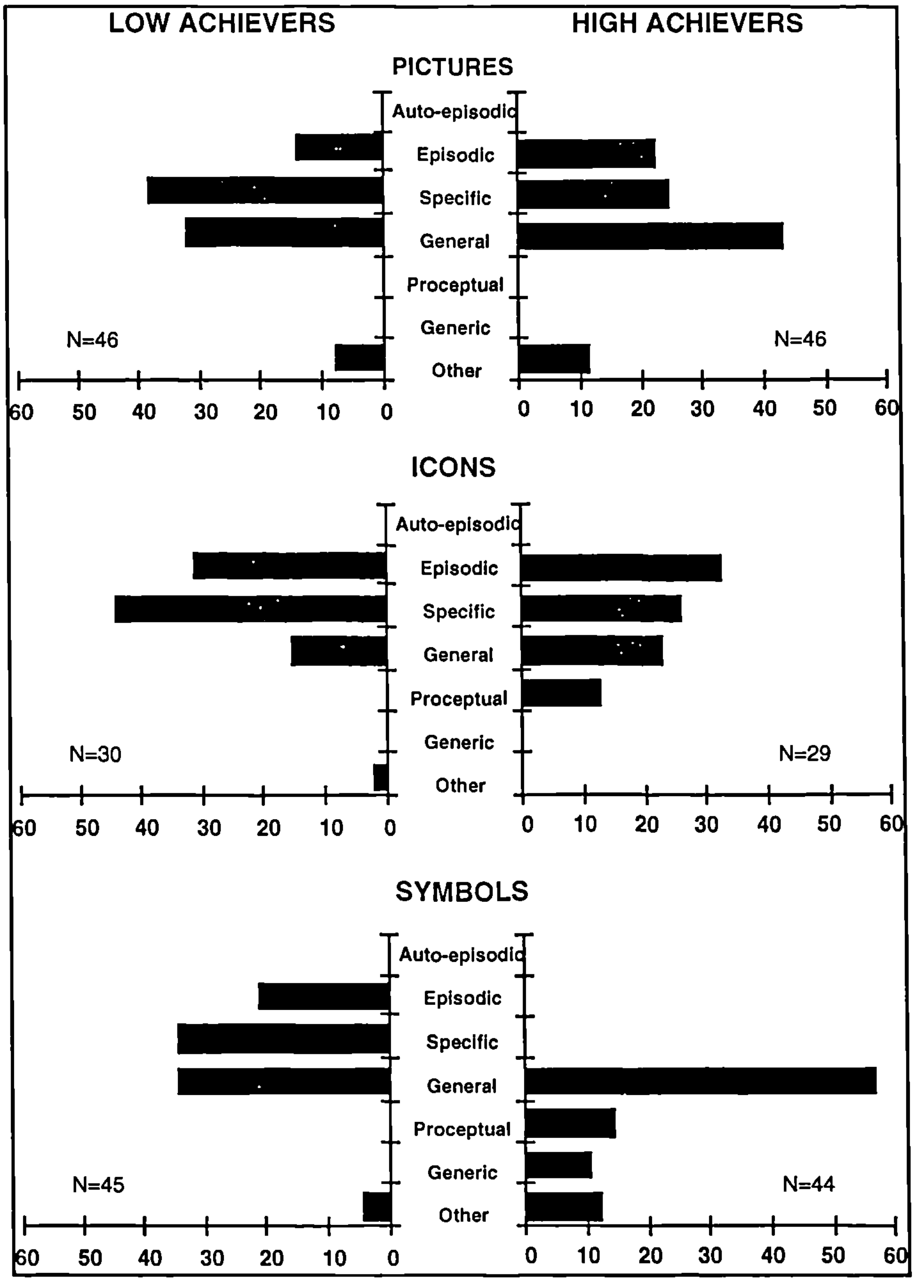

Figure 7.1: Distribution of different kinds of mental representation identified from each group's first response to groups items in the visual phase.

The features that emerge from this figure illustrate:

- the similarity in the distribution of different kinds of mental representation when the two groups considered items presented as pictures. This form of external stimuli seems to generate either 'specific', 'general' or 'episodic' mental representations,

- indications of a broader platform of mental representations amongst 'high achievers' when considering external stimuli in 'iconic' form. The mental representations of the 'low achievers' continue to be identified as 'specific' 'episodic' and 'general'. 47% of these mental representations are 'specific' whilst 50% are 'episodic' and 'general' in the ratio 2:1,

- the difference between the two groups begins to emerge with the appearance of 'proceptual' mental representations from the 'high achievers'. Only the three of the iconic items, 'windows', 'marbles' and 'honeycomb' and all the numerical symbols, had the potential to generate 'proceptual' mental representations and

- when responding to the visual presentation of symbols, the difference in the quality of the representation projected by the two groups of children is highly significantly different ($\chi^2=69.72$, $p<0.001$). 'High achievers' project mental representations of a 'general' kind (60%), together with those differentiated as 'proceptual' (16%) and 'generic' (11%). Though 'general' mental representations form a high proportion of those projected by 'low achievers' (36%), this is equal to the proportion of 'specific' representations whilst the greatest proportion of the balance are 'episodic' (23%). There is no evidence of 'proceptual' mental representations.

By considering the distribution of mental representations projected from the first responses, it can be seen that as the form of external representation changes from 'pictures' of objects through icons to numerical symbols, we note an increasing difference in the quality of the mental representations projected by the 'low' and 'high achievers'. Mental representations of the former continue to be of the 'specific', 'general' or 'episodic' kind, no matter what form of visual stimulus is given, whilst those of the latter change as the external stimuli becomes more abstract.

It is quite apparent that the overall pattern of the mental representations projected by 'high achievers' when responding to the 'symbolic' items are different to those projected by them for the 'iconic' and 'pictorial' items. Though 'general' mental representations are associated with each of the three kinds of visual representation, 'specific' and 'episodic' kinds of mental representation, though common when the children refer to the 'pictorial' and 'iconic' items, are not apparent from this analysis when the children initially form mental representations associated with the 'symbolic' items. The 'general' mental representations evoked in some instances are supported by 'generic' and 'proceptual' mental representations in others.

'Low achievers' display little qualitative change in the range of mental representations they project and quantitative change is only significant when differences between mental representations associated with pictures and icons are considered ($\chi^2=16.39$, $p<0.001$). It is largely accounted for by the increasing proportion of 'specific' mental representations and a decline in the proportions of 'general' mental representations associated with the icons

7.2.3.4 The Second Response: Distribution and Kind of Visual Stimulus

Figure 7.2 displays the distribution of mental representations identified when the children were requested to close their eyes and talk for 30 seconds about what came into their mind about the item that had been shown to them. Again this figure is constructed from the summary in Appendix 2.2. Formed in a similar way to that of Figure 7.1, this is a 'trimmed' presentation of the sum of a particular mental representation expressed as a percentage of the total number of identified representations. The number of responses considered in each group is given by N.

The figure indicates some interesting features:

- the absence of any 'general' mental representations and the high proportion of 'specific' mental representations amongst children of both groups considering the icons,

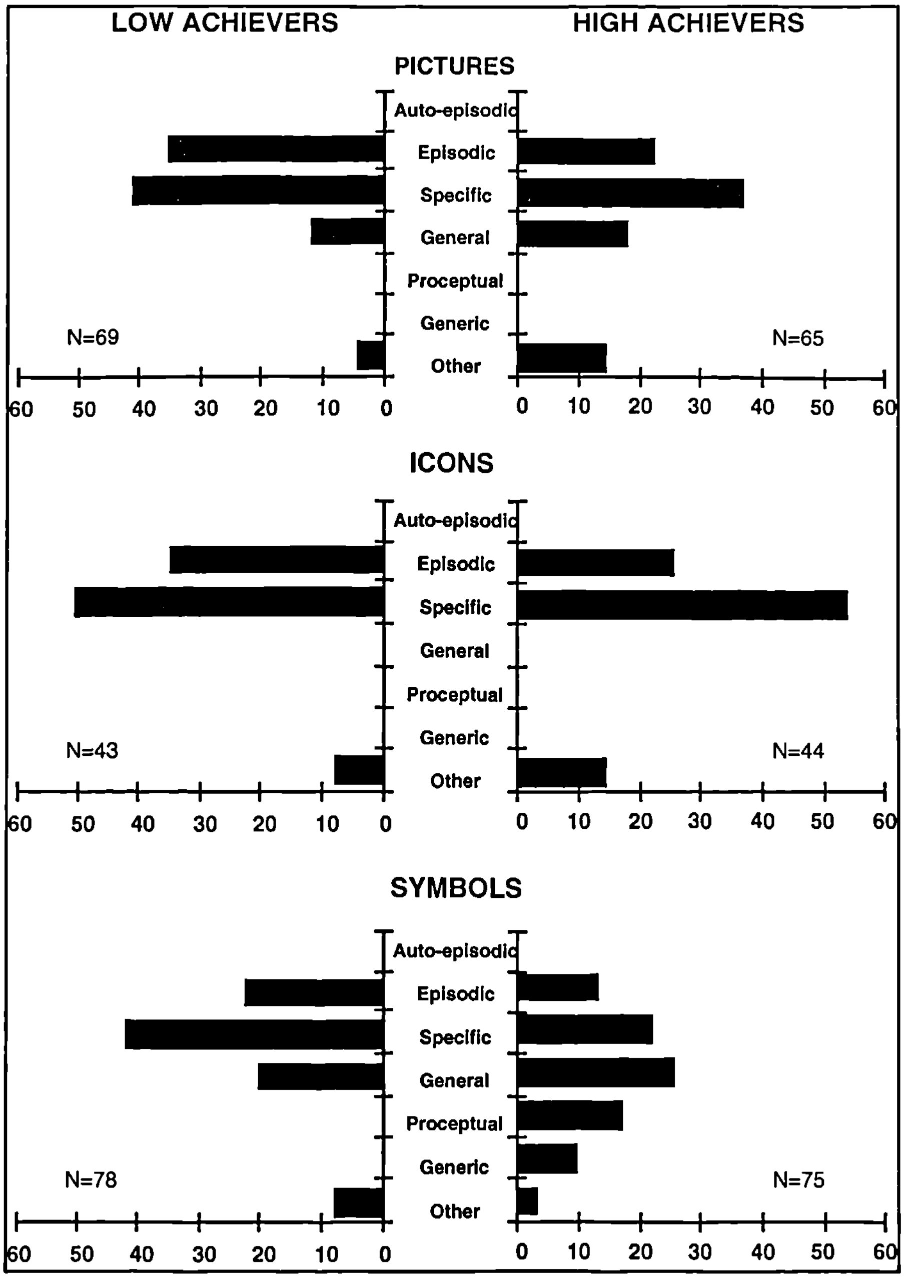

Figure 7.2: Distribution of different kinds of mental representation identified from each group's second response to items in the visual phase

- the occurrence of 'general' mental representations when children consider the pictures and

- the clear differences that emerge when presented with the symbolic items.

The mental representations projected by the 'low achievers' when they consider the symbols appear to have no qualitative difference from those that are formed from the pictorial and/or iconic items. There is no evidence of mental representations projecting 'generic' and 'proceptual' qualities from the 'low achievers'. These mental representations are apparent amongst the 'high achievers'. It also seems that for all children that the icons, though possibly more difficult to name, had more easily distinguishable surface features and characteristics which could be mentally represented easier.

Given the opportunity to say more, the children may provide different kinds of mental representations in response to any one item. 'Low achievers' continue to demonstrate that 'episodic', 'specific' and, more selectively, 'general' mental representations, form the platform for their discussion. 'High achievers' again provide evidence that they have more flexibility, this being particularly evident when their responses to the symbolic representations are considered. This 'representational flexibility' appears to have been masked when they responded to the pictorial and the iconic items, perhaps because they trigger a different form of mental search, an issue which will be considered in greater depth in Chapter 10.

Differences between the two groups of children become more apparent if the way in which the different kinds of mental representation are distributed is considered, Figure 7.3 considers the different kinds of mental representations as a proportion of the number of child responses that were given (N). (In comparison, Figure 7.2 considers each kind of mental representation as one example of the total number of different kinds of mental representation projected). This analysis, is a reappraisal of the summary within Appendix 2.2.

From Figure 7.3 we see that 'low achievers':

- respond similarly to each group of items but that 'general' mental representations are now identifiable in increasing proportions. (In Figure 7.2 they were masked through the 'trimming' procedure),

- provided 'specific' mental representations in at least three out of five instances. This kind of mental representation was dominant no matter what form of visual stimulus was given and

- provided 'episodic' mental representations in approximately one in two of the instances where they provided a response. The lowest proportion (40%) was identified during responses to the symbolic items.

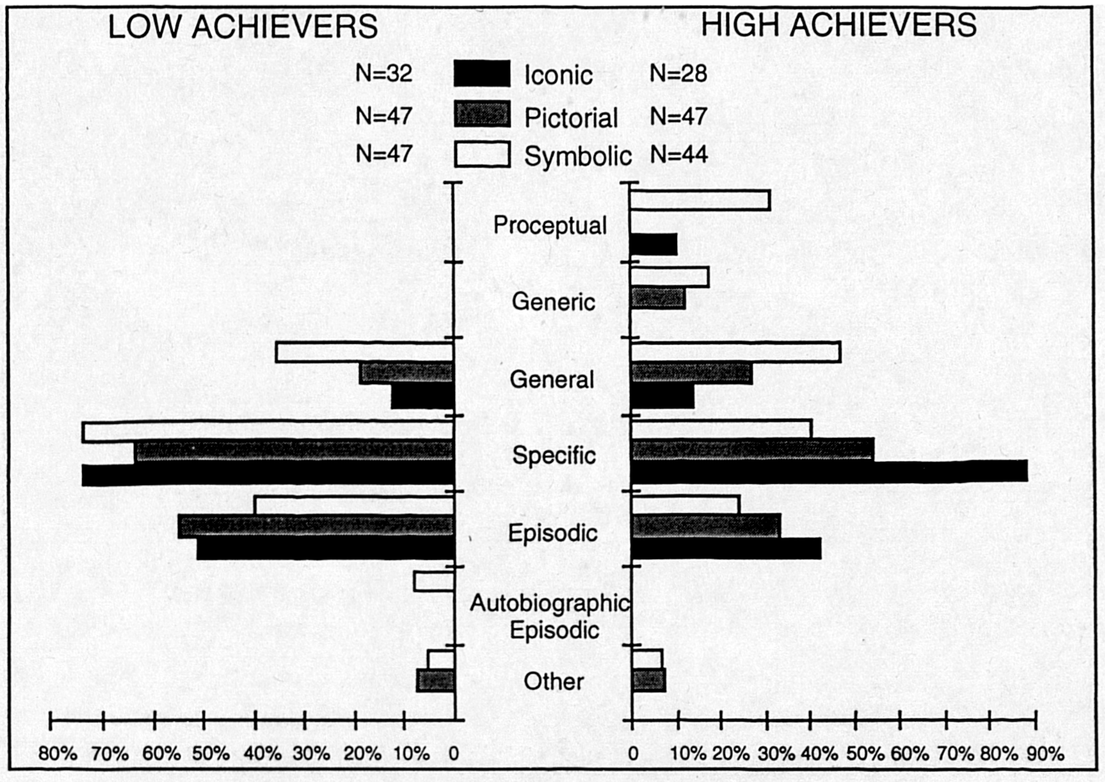

Figure 7.3: Occurrence of the different kinds of mental representation as a proportion of the number of items considered in the second response of the visual phase

There are some common features noted amongst the 'high achievers':

- They projected 'specific' mental representations in nine out of ten of the instances where they responded to an iconic item (89%). This is double the number of 'episodic' mental representations (43%) which, in turn, is almost double the sum of the 'general' and 'proceptual' mental representations (25%) identified for these items.

- as the form of the visual presentation changes from icon to picture to symbol a decline in the projection of 'specific' and 'episodic' mental representations is accompanied by an increase in 'general' mental representations. This change is also associated with an increase in 'proceptual' and/or 'generic' mental representations dependent upon the form of the visual item. Collectively, 'generic' or 'proceptual' mental representations were associated with 50% of the items in the symbolic sub-group.

Perhaps the most interesting interpretation that can be made of the analysis is the difficulty that both groups of children appear to have in detaching themselves from mental representations associated with 'specific' and 'episodic' content when mentally representing the icons. Few of the children drew upon relationships with other items to provide mental representations associated with 'generic' and/or 'proceptual' qualities.

These kinds of mental representation did not appear to form part of the 'low achievers' repertoire of mental representations when responding to the visual stimulus. In contrast, in 30% of the responses that 'high achievers' projected 'proceptual' or 'generic' mental representations. When responding to the symbolic element there is a highly significant difference between the quality of their mental representations and those projected by 'low achievers' (χ^2=40.50, p<0.001); 50% of 'high achievers' responses projected 'proceptual' or 'generic' mental representations. (No single symbol generated mental representations identified as 'generic' *and* 'proceptual'. These mental representations were mutually exclusive).

7.2.3.5 Mental Representations and the Visual Phase: A Summary

Although the evidence suggests that there are qualitative differences in the kinds of mental representation projected by 'high' and 'low achievers' in the visual phase, apart from the symbolic element, these differences are mere hints rather than clear cut, However, the qualitative difference in the visual items suggested an emerging difference in the qualitative nature of the different kinds of representation projected by the two different groups of children:

- Over the series of sub-groupings, the mental representations of the 'low achievers' maintained a consistency epitomised by the dominance of 'specific' mental representations combined with 'general' and 'episodic' ones,

- Although, overall, 'general' mental representations tended to dominate 'high achievers'' first responses, their second responses were moderated by the nature of the stimulus which in the symbolic context reflected a significantly different frame for discussion.

- Both the 'low achievers' and the 'high achievers' had difficulty attaching a 'general' mental representation to the iconic items. It is conjectured that these were not easily conceptualised since they were difficult to name. In addition, their 'specific' characteristics (shapes, colours) may have been more easily identified because of the relative simplicity of their appearance. However, naming extends the possibility of being able to capture the intrinsic qualities of the item (Skemp, 1986). Finding this difficult, the children talked about what the icon looked like.

The results of the analysis suggest that maintaining a traditional view of three different categories of mental representation conceptualised as 'general', 'specific' and 'episodic', does not support our efforts to account for qualitative differences in mathematical behaviour. The inclusion of classifications which consider truly relational representations in the form of 'generic' and 'proceptual' representations appear to go some way towards this. These are additional features that may contribute towards the qualitative differences in thinking that may account for differences in arithmetical achievement. This was a trend which received clarification during the verbal phase.

7.3 MENTAL REPRESENTATIONS IN THE VERBAL PHASE

7.3.1 Introduction

As qualitative differences in the mental representations projected by the two groups of children began to emerge through the different forms of visual representation, so too did the distribution of different kinds of mental representation associated with the verbal phase point to significant differences between the two groups of children.

These sharpen the view that 'low achievers' project mental representations of a more 'descriptive' kind whilst 'high achievers' project mental representations of a 'relational' and 'descriptive' kind. The verbal items, in the form of concrete nouns and numerical nouns, form the basis for discussion based on 'high' and 'low achievers'' first and 'free talk' responses. Once again the analysis is completed with the exclusion of 'don't know' responses.

7.3.2 Looking at 'Not Known'

9.5% of the possible number of first responses were identified as 'not known'. 'High achievers' accounted for 7% of these, 'low achievers' for 12%. Eight of the 'high achievers' 'not known' responses were associated with the numerical items, one with the non-numerical. 'Not known' responses of 'low achievers' all came from the 'numerical' category and were largely identified when the younger children tried to respond to items such as 'half', 'three-quarters' and 'nought point seven five'. Although in comparison to the visual phase the number of 'high achievers' unable to provide a response to the numerical items only increased by one (from eight to nine) the number of 'low achievers' doubled (from eight to sixteen). The longer time allowed for the 'free talk' element of the second response gave some children an opportunity to think about the items. Consequently some who had not been able to provide an initial response were now able to. Thus the proportion of 'don't know' during the second response reduced to 4% for the 'high achievers' and 8% for the 'low achievers'.

7.3.3 Distribution of Mental Representations in the Verbal Phase

This section follows the pattern of Sections 7.2.3.3 and 7.2.4.3. First it considers mental representations associated with the first response and then those associated with the second. It shows that mental representations of 'low achievers' are 'general', 'specific' and/or 'episodic' when considering numerical and non-numerical items whilst those of the 'high achievers' are more 'general' and 'generic' or 'proceptual' as the numerical items are considered.

7.3.3.1 The First Response: Distribution of Kinds of Mental Representation

The kinds of representation identified from response to the question "what comes to mind when you hear the word...?" are illustrated in Figure 7.4. which is drawn from the summarised results given in Appendix 2.2 and is constructed in a similar way to Figure 7.1.

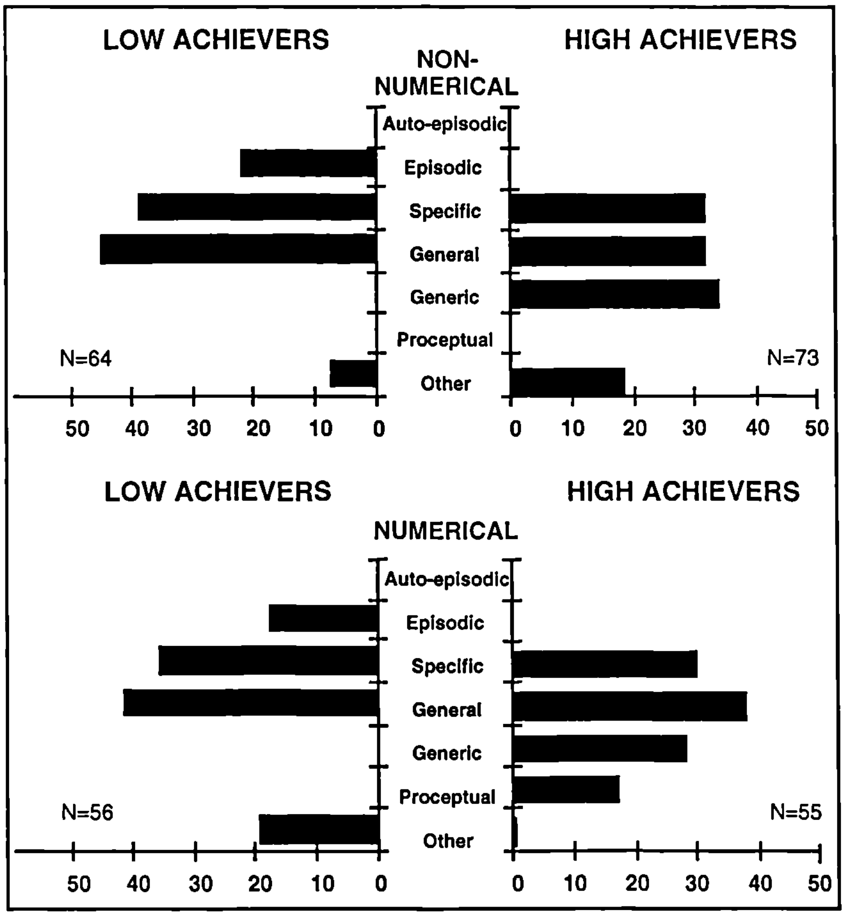

Figure 7.4: Distribution of different kinds of mental representation identified from each achievement group's first response to items in the verbal phase

It may be seen from this figure that:

- The dominance of mental representations associated with 'general', 'specific' and 'episodic' features is once again apparent in the response of the 'low achievers'
- 'high achievers' do not provide any examples of 'episodic' mental representations but 'generic' or 'proceptual' mental representations' are more in evidence than in the visual phase. These account for almost one in two of the representations associated with the numerical items.

'General', 'specific' and 'episodic' representations were typified in such responses as:

"Number 7." (Y3-, general, 'seven')

"Like 1, 2, 3, 4, 5, 6 and 10 are numbers." (Y4-, general, 'number')

"Yellow seven. Seven has got to be coloured. Bubble writing." (Y4-, specific, 'seven')

"A rottweiler." (Y5-, specific, 'dog')

"Picture of a football with people kicking it." (Y3+, episodic, 'football')

"I just think of 3/4....in my maths book." (Y5-, episodic, 'three quarters')

'Generic' responses given as a first response to the question "what is the first thing that comes to your mind..." were frequently difficult to distinguish from 'general' and 'episodic'. To classify such responses, reference was frequently made to the second response. If this indicated 'generic' or 'proceptual' qualities a decision was made to classify the first response as 'generic'. A typical one word response was a Y5+ child's response to the words 'dots' "*Braille*" which on clarification proved to be 'generic'. Proceptual representations were easier to identify: consider the response "*nought point three seven five*" to the word three eighths by a Y6+ child.

7.3.3.2 The Second Response: Distribution of Kinds of Mental Representation

Some children continued to provide one word 'general' mental representations during the 'free talk' phase of the interview. For example:

"Decimal." (Y4+, 'Nought point seven five')

"Game." (Y6-, 'football')

However, given the opportunity to provide a more extended indication of their mental representations, others, particularly 'low achievers', frequently used the opportunity to build around one idea or to provide qualitatively similar examples. The quality of 'specific' mental representations associated with the 'non-numerical' words was indicative of those associated with 'numerical' ones:

"Well... they are quite... like black. Some are black, some are different colours, some are striped. They could be big or small but colourful." (Y3-, specific, "dots")

"... a cheetah is one, a rabbit is one, a dog is one, a cat, a Labrador, a Dalmatian, owl, eagle, buzzard, etc." (Y4-, specific, 'animal')

"There is thirty three worlds or countries, thirty three bins, thirty three doors, thirty three houses and some people are thirty three."

(Y3-, specific, Thirty-three)

"'0' is a nothing number. The biggest number in nothing seventy five is the seven. I think it is a very big number and I have not heard that number before."

(Y3-, specific, 'nought point seven five')

The opportunity to be descriptive also provided an opportunity to talk about something that was previously 'not known':

"It's three... an odd number, and eight... an even number. If you add them together [by counting on fingers], it makes eleven." (Y4-, specific, 'three eighths')

The extra time also gave some the opportunity to build very vivid stories around the words. Comparable to the classification of 'imaginative extensions' within the pilot study, were now identified as 'episodic' mental representations:

"You sit at a table - tuck in your chair and you can have dinner at Christmas or a nice party and you can have all nice food and a table cloth on it." (Y3-, episodic, 'table')

"Lots of animals crowded in fields, walking about with each other playing. They crowded in a big field in a circle." (Y3-, episodic, 'animal')

Some episodic representations were very revealing:

"Adults think they are old when they are 33. But it is not *that* old." (Y4+, 'thirty three')

"Some people... even people can get to 99... even very old people." (Y5-, 'ninety nine')

Figure 7.5 indicates how kinds of representations identified through the second response were distributed as a percentage of the total number of identified representations. (Appendix 2.3)

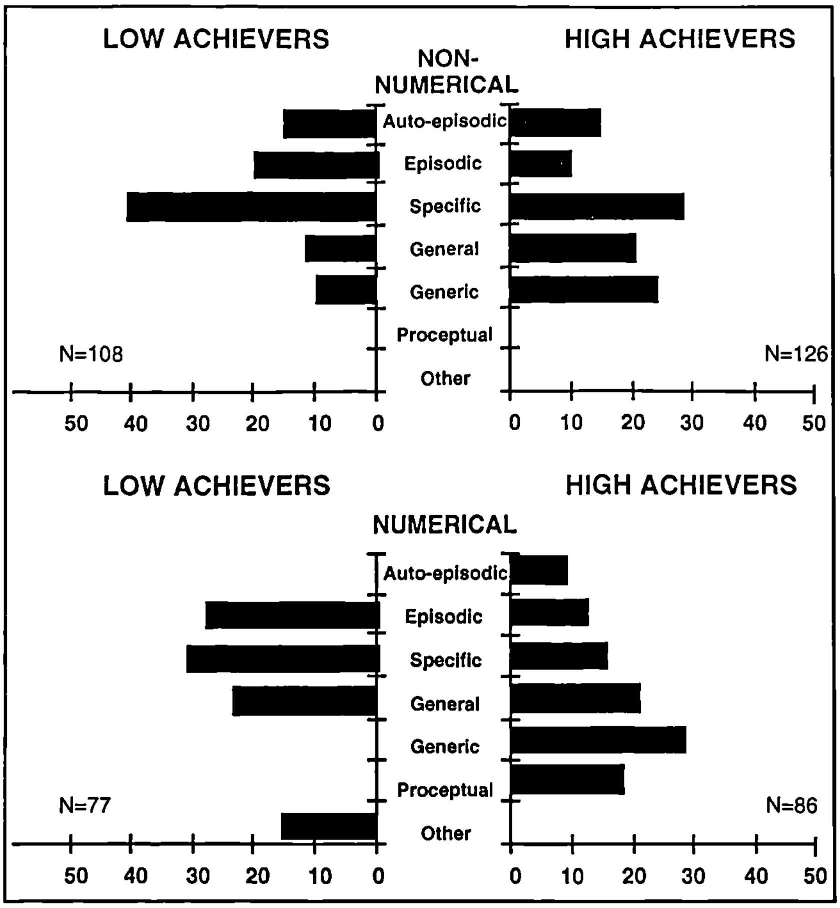

Figure 7.5: Distribution of different kinds of mental representation identified from each achievement group's second response to items in the verbal phase

Figure 7.5 indicates that:

- 'general', 'specific' and 'episodic' kinds of mental representation occur relatively extensively in both groups for the 'non-numerical' words, more so amongst the 'low achievers',
- In contrast to the visual phase, the non-numerical phase generates 'episodic autobiographic' and 'generic' mental representations amongst both groups, the latter apparent almost three time more amongst 'high achievers' than 'low achievers'.
- The occurrence of a broader range of mental representations is particularly noticeable when the 'high achievers' respond to the numerical items. However, these items elicited from 'low achievers' the familiar 'general', 'specific' and 'episodic' kinds of mental representation.

Though there appears to be a degree of homogeneity in the kinds of mental representation identified within the two groups of children for the non-numerical items, there is a significant difference between them ($\chi^2=14.5$, $p<0.01$). Although qualitatively similar they are quantifiably different. This difference is largely accounted for by differences which are reflected in the proportion of 'generic' mental representations (24% amongst 'high achievers', 10% amongst 'low'), the occurrence of fewer 'specific' mental representations from 'high achievers' than from 'low achievers' (29% compared to 42%) and fewer 'episodic' mental representations (10% compared to 21%). The interesting similarity is the occurrence of 'autobiographic-episodic' mental representations. This form of mental representation remained evident amongst the 'high achievers' numerical profile but 44% of this profile was now accounted for by 'generic' and 'proceptual' mental representations. 'Autobiographic-episodic' mental representations did not occur in any reported proportion when the visual phase was considered. These emerged when the children associated the item with a past event in their own lives:

> "When I was little I used to watch this 'Play Days' programme and it was about dots–big dice with big dots on and... I remember my brother coming home with a book showing 'little mice dotty'. In school when we use dice and full stops it just came into my mind a big dot ... a big black dot." (Y4+, 'dots')

> "I remember when we did some work on halves and quarters. It was boring because all we had to do was answer questions and draw halves and quarters which I found so boring. It was too easy because all we had to do was cover a quarter red or a half blue." (Y3+, 'three quarters')

> "Saw my birthday cake in my mind and all my friends and my mum cutting in half." (Y3-, 'Half')

'General' mental representations were frequently expressed with one word whereas 'specific' and 'episodic' mental representations were usually given as continuous pieces of prose. It was as if one idea needs to be described or narrated to make it understood. This continuous narrative differed from the way in which 'generic' and 'proceptual' representations were delivered. They were presented as a string of associated ideas:

"Keeps you fit. An exciting game. Millions of fans. Important in every nation. Children and adults play it. Different types of football and balls." (Y4+, generic, 'football')

"Dig, bury, bark, jump, fetch, ...canine." (Y6-, generic, 'dog')

"Seven is in maths and writing. Seven, you could be doing some adding or times and the number seven might come up. Seven is also played in sport... it's on the back of a shirt. It is one digit. A phone number." (Y5+, generic, 'seven')

"Shapes fraction, divide." (Y5+, generic, 'three-quarters')

Proceptual representations were unique to the numerical component:

"You can do sums like 99+1=100 or something like. You can do sums that make 99 for ex.ample 70 add 29 equals 99." (Y3+, proceptual, 'ninety-nine')

"Half way between 0-66...3x11...1/3 of 99." (Y6+, proceptual, 'thirty-three')

As seen in the discussion of the visual phase, the important issue that arises from the second response is the occurrence of each kind of mental representation as a proportion of the number of child responses. This is illustrated in Figure 7.6 which is constructed similarly to Figure 7.3 but draws upon Appendix 2.3.

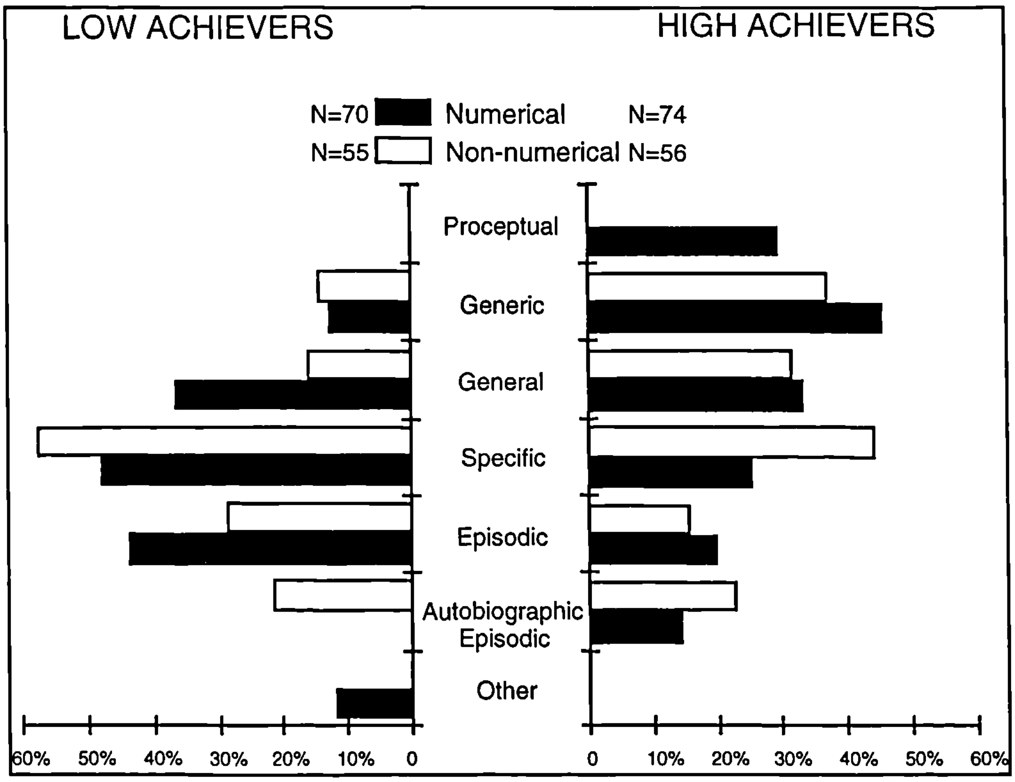

Figure 7.6: Occurrence of different kinds of mental representation as a percentage of the number of verbal items in the second response of the verbal phase.

It may be seen from Figure 7.6 that:

- there is a degree of similarity in the occurrence of 'specific' mental representations projected from both groups of children when responding to the non-numerical items: 'high achievers' in approximately one in two instances (45%), 'low achievers' in almost three in five instances (58%),

- when responding to the numerical items only, in slightly less than one in four instances did 'high achievers' project 'specific' mental representations (26%). This contrasts with one on two of the 'low achievers' (49%), and

- in over seven out of 10 instances 'high achievers' projected 'generic' or 'proceptual' mental representations when responding to the numeric items (76%). In only slightly above one in 10 instances (13%), 'low achievers' projected 'generic' mental representations but no 'proceptual' ones.

'High achievers' demonstrated that they can respond to the nature of the items in a qualitatively different way. The greater proportion of the 'low achievers'' mental representations continued to project 'general', 'specific' and 'episodic' qualities.

7.3.3.3 Mental Representations in the Verbal Phase: A summary

The distribution of different kinds of mental representation displayed by the two groups of children suggests that, once again, there are qualitative differences in the kinds of mental representation projected by the children.

- The mental representations of 'low achievers' are largely confined to 'general', 'specific' and 'episodic'. These kinds are characterised by descriptive attributes.

- 'High achievers' project a larger variety of mental representations in which 'generic' and 'proceptual' mental representations are associated with 'general' and 'specific' ones in different proportions to resonate with the concrete or abstract nature of the item under discussion.

In essence, we see that mental representations of the 'low achievers' do not show substantive change. No matter whether faced with 'numerical' or 'non-numerical' items, they mentally represent them the same way. It is conjectured that their 'expansion' of the 'general' mental representation is epitomised by 'specific' and/or 'episodic' mental representations. 'High achievers', on the other hand, appear more able to go either way. Whilst they can 'expand' 'general' mental representations in a 'specific' or 'episodic' way, they can also supply the 'generic' and 'proceptual' qualities that indicate that their mental representations are truly 'relational'.

Two questions now seem to emerge:

- to what extent are the mental representations associated with the visual phase similar to or different from those of the verbal phase? and
- to what extent may age influence the kind of mental representations?

7.4 LINKING REPRESENTATIONS IN THE VISUAL AND THE VERBAL PHASE

7.4.1 Introduction

Ten items in the visual and verbal phases of the interview process were common to both: five 'non-numeric' and five 'numeric'. These were: 5, 99, $\frac{1}{2}$, $\frac{3}{4}$ and 0.75 (numeric), 'dots', 'football', 'dining room', 'furniture', 'table' and 'ball' (visual). All of the visual items were presented as 'pictures'.

The purpose of replicating items in a different format was to consider the constancy or otherwise of the quality of the children's responses. Evidence of stability in responding to visual and verbal items is illustrated by the following, each pair of responses being given by the same child:

- "Fractions, maths, arithmetic... decimals" (Y5+, response 2, visual, generic '0.75')
  "Maths... fractions... decimals...." (Y5+, response 2, verbal, 'generic, '0.75')

- "It's one over two it could be formed in different ways, 3/6, 0.5, 4/8, 2/4, 10/20, 6/12."
(Y6+, response 2, visual, proceptual, '$\frac{1}{2}$ ')

"Divided by two... 50 over 100. Way of dividing things ... in the middle..."
(Y6+, response 2, verbal, proceptual, 'half ')

- "A football" (Y3+, response 2, visual, general, 'football')

"Round sphere shape that you can kick about."
(Y3+, response 2, verbal, general, 'football'))

However, in virtually no instances did the external stimuli in visual form trigger a qualitative difference which indicated that 'specific' or 'episodic' forms were transformed into 'generic' or 'proceptual' forms of representation in the verbal phase. Frequently, though, mental representations associated with items of the visual phase which were identified as 'general' or 'generic' mental representations, were identified as 'proceptual in the verbal phase (again paired responses are from the same child):

- "Number with 3 digits. Used in money, pounds. Used in lots of things. Not a whole number....not more than one" (Y6+, response 2, visual, generic, '0.75')

"Its a number that's very low down. It can be fraction or a decimal...its three times zero point two five, and one and a half times point five."
([A], Y6+, response 2, verbal, 'proceptual', 'nought point seven five')

- "Number... not whole... 3 quarters." (Y4+, response 2, visual, general, $\frac{3}{4}$)

"You need... half of half you need three of them to make it."
(Y4+, response 2, verbal, proceptual, three-quarters)

The visual phase did seem to give children an opportunity to provide some sense of a mental representation which was 'not known' in the verbal phase. During the visual they could focus on the detail:

"Black writing. Number 1 with a line underneath and a black two, its on a green card."
(Y4-, response 2, visual, 'specific, $\frac{1}{2}$)

"Nought is a very, very, very little number but it isn't a number just a nought. Seven and five are middle sized numbers." (Y3-, response 2, visual, 'specific', '0.75')

or misrepresenting the stimulus to talk about a completely different situation:

"If you put them together so that three is in the front and four at the back it would be thirty four. If you put them together so that the four was in the front and the three at the back it would be forty three. If you put them away it would just be three and it would just be four. "

(Y4-, response 2, visual, 'specific' $\frac{3}{4}$)

7.4.2 Comparative Distribution of Mental Representations

7.4.2.1 Representations and First Responses

Figure 7.7 presents a comparison of the distribution of the different kinds of mental representation for the first responses of the visual and verbal phases. The figure is formed by considering the percentage occurrences of each kind of representation and for the common items in the visual and the verbal phases. It is established from Appendix 2.4.

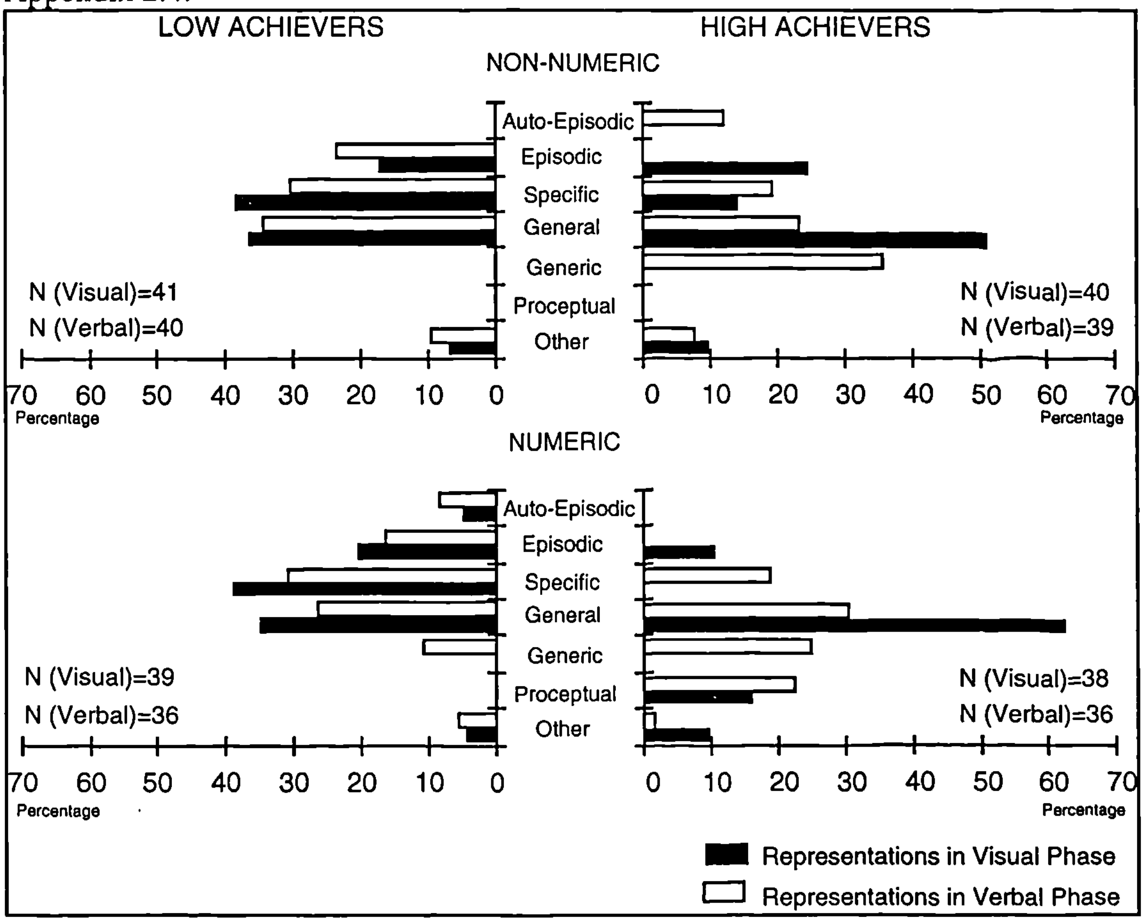

Figure 7.7: Comparison of the distribution of different kinds of representation in the verbal and visual phases: First Responses.

An examination of the mental representations that both groups provided for the first response to the non-numerical items in the visual phase indicates their similarity. Both provided indications of 'general', 'specific' and 'episodic' mental representations. 'General' mental representations formed over half of the mental representations projected by 'high achievers'(60%), those of the 'low achievers' (36%) being almost equal to the slightly more dominant 'specific' representations (38%). Comparisons made in any other way display differences within and between the two groups:

- In comparison to the 'visual' phase verbal presentation of the non-numerical items encourages 'generic' mental representations from the 'high achievers' with a 50% reduction in 'general' mental representations. There is a highly significant difference in the quality of the mental representations they project when responding to similar items in the two different phases (the visual and the verbal) (χ^2=39.88, p<0.001),

- Mental representations of the 'low achievers' which derived from the 'non-numerical' items appear to be independent of the phase which stimulates the mental representation. Both almost equally projected 'general', 'specific' and 'episodic' mental representations and

- 75% of the 'low achievers'' mental representations associated with the 'verbal-numeric' phase were identified as 'general', 'specific' or 'episodic'. In contrast, 79% of the 'high achievers'' representations were identified as 'general', 'generic' or 'proceptual'.

Thus, we see that non-numerical items, used in the visual or verbal phase, trigger a qualitatively different first mental representation in each phase. 'High achievers'' first response in the visual phase is dominantly 'general' and the first response to the verbal phase is dominantly 'generic'. On the other hand, a qualitatively similar mental representation is generated in the two phases from the 'low achievers'. This qualitative difference is widened when the numeric items are considered. The high proportion of 'general' representations associated with 'high achievers'' mental representations of the visual items (63%) is reduced and redistributed over 'proceptual' (22% compared to 17%), 'generic' (24% compared to 0%) and 'specific' (18% compared to 0%) kinds of mental representation during the verbal phase. Only marginal differences are noted in the changes displayed by 'low achievers', these being essentially slight reductions in 'general', 'episodic' and 'specific' mental representations and more significantly, the appearance of 'generic' mental representations (11%).

7.4.2.2 Representations and Second Responses

The general pattern that emerges from considering the comparative distribution of different kinds of mental representation for the first response is continued when the distribution for the 'free talk' comparisons, considered as a percentage of the total occurrence of representations, is examined (Figure 7.8, Appendix 2.4 refers).

- 'General', 'specific' and 'episodic' mental representations remain dominant amongst 'low achievers' for all groups of items: the 'visual non-numerical', 'verbal non-numerical', 'visual numerical' and 'verbal numerical'.

- Given the extra time allowed in 'free talk' a broader range of mental representations is projected by the 'high achievers'. 'General', 'specific' and 'episodic' mental representations continue to play an important part in the 'visual non-numeric' and the 'visual-numeric' phases. The proportion of 'specific' and 'episodic' mental representations is tempered by the inclusion of 'generic' mental representations in the former (24%) and 'generic' and 'proceptual' mental representations in the latter (33%). The occurrence of these mental representations has a greater influence in the 'verbal' phases such that there is a significant difference in the proportions of 'high achievers'' mental representations associated with the visual and verbal numeric phases ($\chi^2=14.07$, $p<0.01$). This is largely accounted for by the higher proportion of 'generic' mental representations (10% to 24%) and a decline in 'specific' mental representations (26% to 9%).

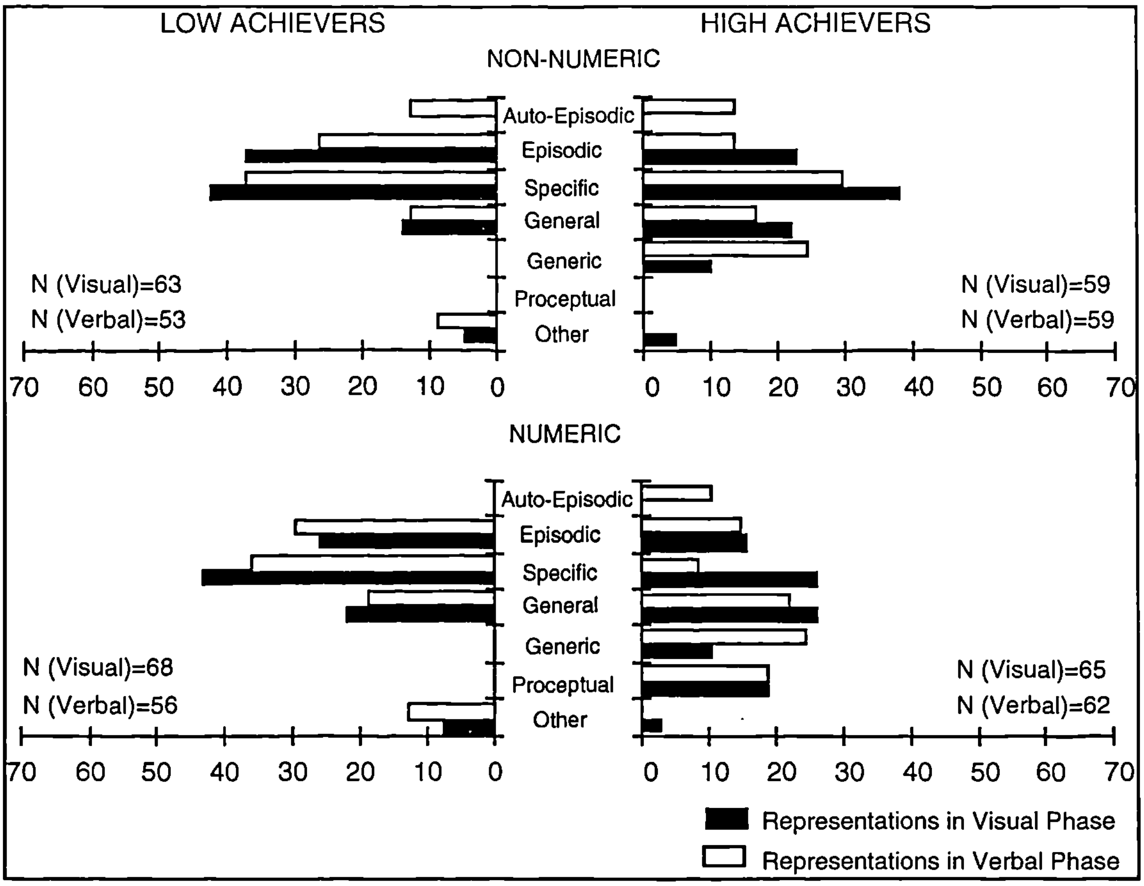

Figure 7.8: Comparison of the distribution of different kinds of representation in the verbal and visual phases: 'free talk' responses.

7.4.2.3 Summary of Comparisons

The conclusions to be drawn from this comparison are similar to those indicated in Section 7.3.3.3. 'Low achievers' seem to display the same pattern of behaviour across items. The 'high achievers' seem to respond differently to different phases. There are, of course, common features between the groups. Both project mental representations which have 'general', 'specific' and 'episodic' qualities. However, only 'high achievers' demonstrate that they can also project 'generic' *and* 'proceptual' mental representations when faced with numerical items, the cumulative total of these in the case of the verbal phase illustrating that they were projected in almost 50% of instances.

7.5 THE ROLE OF VISUAL IMAGERY

7.5.1 Introduction

The question "what was happening in your head as you were talking to me?" was associated with each of the separate components of the visual and the verbal phases. In the visual phase it was presented after the 30-second 'free talk', whilst in the verbal phases it was presented after the 'first thing that comes to mind' and the 'free talk'. It was designed to consolidate observations from the pilot study which had provided evidence of qualitative differences in children's visual imagery and the role it played in the arithmetical component. In this portion of the study there is an attempt to identify three things:

- to what extent is visual imagery reported by the children?
- to what extent does its occurrence differ in the visual and verbal phase? and
- to what extent can investigation on children's visual imagery, allow us to obtain a sense of qualitative differences among the children?

The section restricts itself to a consideration of visual imagery since the information gleaned from the children did not allow us to clearly identify 'verbal images' and 'propositional representation' or indeed offer a clear descriptions of these forms of mental representations. It was never quite clear whether children were reporting representations in a propositional form, verbal image or simply retrospectively describing their actual thinking.

7.5.2 Forms of Visual Imagery

Throughout the series of interviews, the responses of the children provided support to the observations noted in the pilot study. Visual representation could:

- be 'picture-like' or display movement as in a "*film*" or "*video*":

"I saw a picture of 99 like squiggly and misty." (Y5-, verbal, number, [1])[3]

"I saw a picture of dots keeping quite still." (Y5-, verbal, dots, [1])

"I saw the word NUMBER with numbers around it. .... with 1,2,3,4,5,.... 99, 100 and 'all the other numbers' spinning around it." (Y3-, verbal, ninety-nine, [1])

"I just think of all these.. animals. The dog stands out. I see a Golden Retriever. It is like a moving picture, with animals walking along–the dog comes forward. This is like I see 'five'." (Y5-, verbal, animal, [1])

- be strongly personalised. The children could be actors in the image or an observer in the image:

"Basketball–I like to play, sometimes baseball, sometimes rounders... I saw a baseball with a friend and me whacking the ball. In basketball I saw myself scoring, in rounders I saw the ball coming to me and me hitting it." (Y6–, visual, ball, [3])

"I see it [a picture] with me sitting down looking at all of the different kinds of furniture. I am looking all of the way around the room with all of my family sitting there." (Y4– visual, furniture, [3])

- possess colour and sharp detail:

"Saw 7 in my mind and just said the others. Seven had patterns, spots. 7 came and then went away after about 5 seconds." (Y3– visual, 7, [3])

"Numbers... a blank but one number called seven and it is bold in orange and green colour around it." (Y5+, verbal, seven, [1])

"I see the numbers 1,2,3,4,5 and all the signs add, subtract, multiply and divide. The 3 and 4 come out and so does the sign on the piece of card come out [+]. So there is the three, that sign and the four and I can see the green card behind it and the numbers are black and the sign in the middle in dark blue–not the same colour as the numbers." (Y5–visual, 3+4, [3])

- show some remarkable consistencies:

"Started seeing things. Numbers over each other–they were changing. Just changing numbers on the top like 1/2 is one over two, three over five. One was going away and another coming." ([A], Y6-, visual, $\frac{3}{4}$, [3])

3 Referencing of children's quotes indicates the year and ability group, for example Y5–, the phase of interview, 'verbal' or 'visual' and the question responded to; that is the 'first thing that came to mind' [1] or the 'free talk' of the verbal phase [2] and after the 30 second 'free talk' of the visual phase [3].

"Saw it...the numbers over each other. Didn't see it all of the time, saw all different numbers coming in, everyone I could think of like 1,2,3....that's why I said 'maths'."

([B], Y6-, visual $\frac{3}{4}$, [3])

- act as a thought generator:

"I was seeing the number and then it sort of went and faded away. I then tried to think of things connected with a half. When I saw it was one over two." (Y6+, verbal, half, [2])

"I saw that [the '5'] then it disappeared and I saw something else [road sign)], then I saw the '5', it disappeared again and I saw something else [counters]."

(Y4+, visual, '5', [3])

"A digit... a number... below 8 and above 6... use in arithmetic and counting. I saw '7'. I think about seeing seven as a 'bubble number'. It helps me to think." (Y4+ , visual, '7', [3])

- be necessary for thought:

"Saw it...the numbers over each other. Didn't see it all of the time, saw all different numbers coming in, everyone I could think of like 1,2,3....that's why I said 'maths'."

(Y6-, visual, $\frac{3}{4}$, [3])

"Number five–I think of a row of numbers and a light shines on the number five. light goes along and stops over the number five." (Y5-, verbal 'five',[2])

- and could be the source of confusion:

"I see 0.75 as a sum and then 0.75 disappears but it comes back muddled up as 57.0, then the nought. They are different colours - number changed and so did the colours. The colour is purple. First it was black and then it became 57.0 in purple." (Y5-, visual, 0.75, [3])

The children were very aware of when the visual image disappeared or when they did not see a visual image.

"It was just like a pink number of a 5 and it was staying there and then it started to fade and then it went away and then it all went black–nothing was there. All I could see was a few lines but it was mostly black." (Y5+, verbal, 'five', [1])

"I could see it and then it faded and it went orange and I just had to think about things and what to say." (Y6+, visual, 'honeycomb', [3]) "Blank – I was trying to concentrate. [When I was trying to] concentrate it was just orange and purple and all different colours. But I didn't see anything." ([A]Y5+, visual, 'marbles. [3])

"They were just thoughts. I was trying to make my thoughts into a picture but that wouldn't come- saw nothing , it was black (that is how I know it was nothing)."

([B]Y5+, visual, '99', [3])

7.5.3 The Occurrence of Visual Imagery

Of the 16 children who took part in the investigation, all indicated, that at one time or another that they "*saw it in my head*". The interpretation placed on this was that all the children were capable of creating visual images. They may have also used them to respond to a particular item. However, this reference to visual imagery was by no means uniform. Over the 33 items within the visual and the verbal phases that could be responded to, the frequency with which children indicated that they "*saw it*" varied from a minimum of three items to a maximum of 32.

Table 7.3 indicates the number of responses associated with visual imagery for each of the sub divisions used in the study. It can be seen from Table 7.3 that:

- visual imagery was reported more during the visual phase than during the verbal phase and
- 'high achievers' reported visual imagery associated with numerical components with a greater frequency than did low achievers.

	Verbal Phase 1				Verbal Phase 2				Visual Phase			
	High Achievers		Low Achievers		High Achievers		Low Achievers		High Achievers		Low Achievers	
Numerical	N=73	%	N=64	%	N=74	%	N=70	%	N=47	%	N=47	%
Raw Total/%	41	*56*	25	*39*	36	*49*	17	*24*	41	*87*	38	*81*
Non-numeric	N=55		N=56		N=56		N=55		N=31		N=30	
Raw Total/%	42	*58*	39	*61*	35	*47*	33	*47*	28	*60*	30	*64*
Iconic									N=48		N=45	
Raw Total/%									36	*77*	31	*66*

N=the number of responses considered within each sub-section after the exclusion of 'don't know'
Raw Total— number of responses within each sub-section in which visual imagery was reported.

Table 7.3 The occurrence of visual imagery in each sub-division of the study by ability group

7.5.4 Kinds of Visual Imagery

To identify different kinds of visual imagery, it was expected that visual images would possess the same qualitative characteristics as those identified for the different kinds of mental representation. Indeed, an interpretation of De Beni and Pazzaglia's (1995) classification associated with the occurrence of imagery formed the basis for classification of the mental representations. The essential additional characteristic for this component of the study was that the children indicated that they saw something in their 'mind's eye'. The following serve as selective examples:

> "Saw the picture of the pattern. Kept seeing it all of the time and kept saying things about it, like it had hexagons." (Y6+, general, visual, 'honeycomb')
>
> "Started seeing things. Numbers over each other – they were changing. Just changing numbers on the top like one half is one over two, three over five. One was going away and another coming." (Y6-, specific, visual, 3/4)
>
> "Number five–think of a row of numbers and a light shines on the number five. A light goes along and stops over the number five." (Y5-, episodic, verbal, 'five')
>
> "My brother, he is a maniac for football and I could see him running around with the balloon that he always has with his football kit, his black kit, and I could see him running around putting his hands up in the air like he always does... and I kept thinking about five [it was his fifth birthday] and it came up lots of times." (Y4+, autobiographic episodic, visual, 'five')
>
> "The three and the four come out and so does that sign on the piece of card come out. So there is the three, the sign and the four and I can see the green card behind it and the numbers are black and the sign in the middle in dark blue–not the same colour as the numbers."
> (Y5-, specific, visual, 3+4)
>
> "I saw the fraction symbol 3/4 as a picture."
> (Y5+, proceptual, verbal, 'nought point seven five')
>
> "It will be three-quarters. Saw three over four."
> (Y6+, proceptual, verbal, 'nought point seven five')

The interesting feature associated with classifying the different kinds of visual image is that in no cases, either amongst the 'high achievers' or the 'low achievers', were any visual images identified as 'generic'. In only four instances did 'high achievers' responding to the verbal phase, and in five when responding to the visual phase, provide indications of visual images identified as 'proceptual'. If the visual images identified in the pilot study were classified on the basis of the classifications used in the main study, we would see that the great majority are either 'episodic', in that they are analogues of counting episodes, or 'specific' in that they refer to 'specific' symbols. Only in those instances where children reported seeing transformations associated with derived facts may proceptual images be identified.

This leaves the classification of visual images essentially in the framework devised by De Beni and Pazzaglia (1995), 'general', 'episodic', 'specific' and 'autobiographic-episodic'. However, their framework does not take account of numerical inputs, and therefore, even though proceptual images appear to be rare, they do strongly reflect qualitatively different thinking in the numerical sense. Using the De Beni and Pazzaglia's framework, Figure 7.9 presents the occurrence of different kinds of visual imagery for each sub component of the visual and verbal phases. It is a figure of two parts:

- Firstly, it illustrates in graphical form the percentage frequency of occurrence of the different kinds of visual images identified across each of the sub-phases of the visual and verbal phases.

- Secondly, it records in tabular form the total number of responses identified within each sub-phase for each of the 'low' and the 'high achieving' groups, the sub-phases being those adjacent to the figures. It also identifies the percentage of these totals which are considered in the graphs. Visual imagery associated with proceptual considerations is not included so therefore we note changes in the percentage of visual images considered in the tables. Figures relating to the 'high achievers' are provided separately to those associated with 'low achievers'.

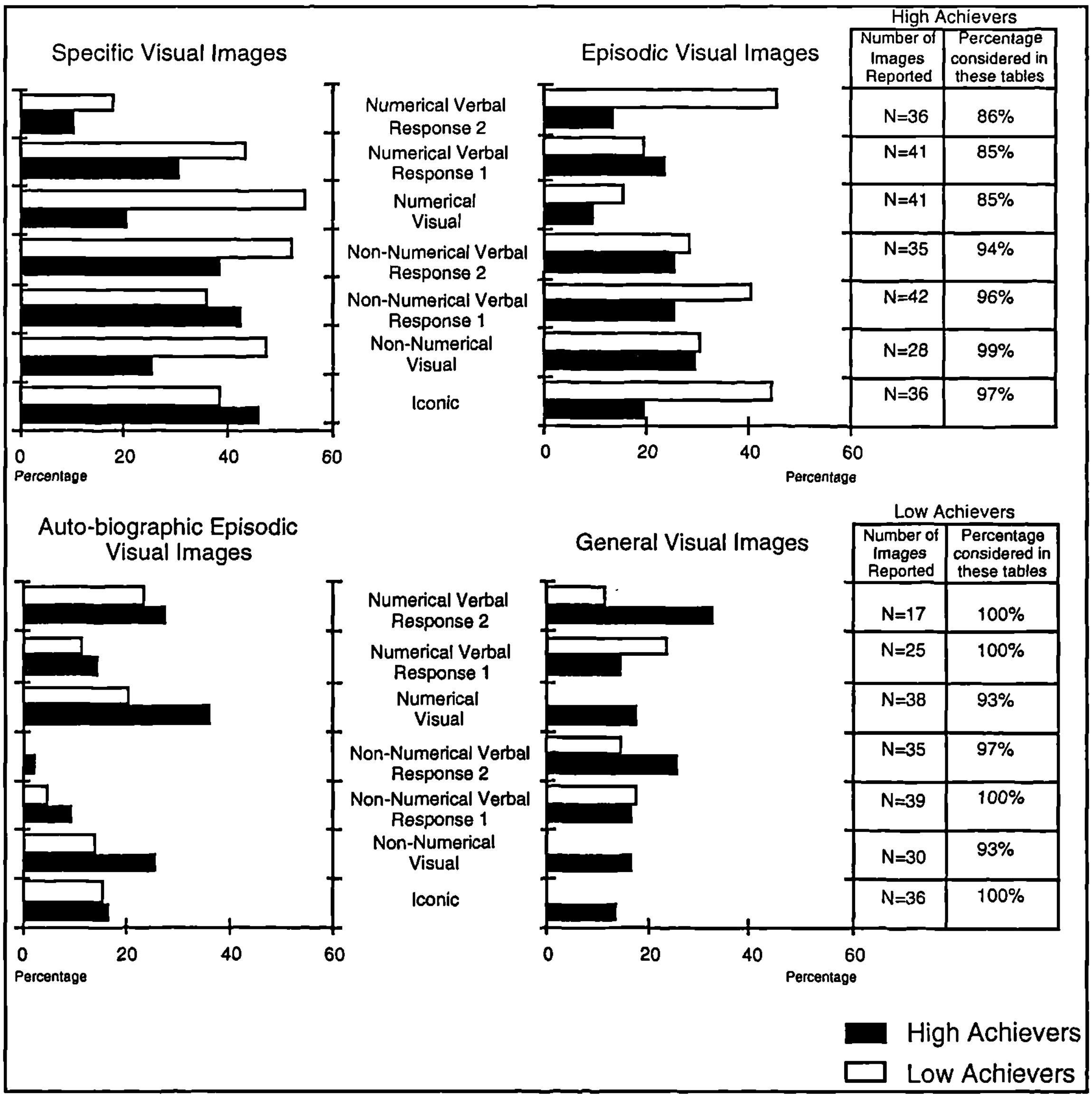

Figure 7.9: A comparison of the occurrence of different kinds of visual imagery in each sub-phase of the visual and verbal phases

The following features emerge from Figure 7.9:

- the kinds of visual imagery identified accounted for at least 93% of those identified from the 'low achievers' and at least 85% of those identified from the 'high achievers'. The balance in each case were identified as 'proceptual',

- 'specific' and 'episodic' visual images accounted for over 70% of those identified amongst the 'low achievers' in all sub-groups apart from the two numerical verbal sub-groups,

- 'episodic' and 'specific' kinds of visual image account for over 55% of the 'high achievers'' visual imagery for all sub-groups apart from the 'numerical visual' and the 'numerical verbal second response' where 'general' and 'autobiographic-episodic' visual imagery dominated, and
- overall there is a greater tendency for 'high achievers' to report 'general' and 'autobiographic-episodic' visual images than 'low achievers', 'general' visual imagery being absent from 'low achievers' in the context of visual stimuli.

7.5.5 Visual Imagery: A Summary

On balance, the investigation into visual imagery did not reveal deep differences between the two groups of children. The overall dominance of 'specific' and 'episodic' forms of imagery requires the sort of analysis that emerged in the pilot study. However the nature of this enquiry did not include the differentiation of visual imagery into analogic and symbolic forms at this stage. Neither did it reveal detail of a 'generic' and 'proceptual' kind. The conclusion that must be drawn from the section on visual imagery is that an analysis of mental representations that may include visual imagery can provide deeper insight into children's thinking than will an analysis of visual images alone.

It seems that the uses to which visual images are put are quite different in the two groups of children. The 'general', 'specific' and 'episodic' visual images identified of the 'low achievers' may be associated with mental representations–the content is the same but the mental representation may be visual or verbal. Given the discussion in the verbal and visual phases it seems that *the visual images of the 'low achievers' are used 'to be talked about' to be described and put in a context.* In contrast it is conjectured the visual imagery of the 'high achiever' is put to different use. *The visual imagery of the high achiever generates an idea and can be put aside so that they may focus on the more relational characteristics.* If this was not happening, it is conjectured that in the numerical context research concentrating on the surface characteristics of visual imagery would provide no evidence of generic or proceptual thinking whereas giving consideration to the kinds of visual imagery may. However, even this only provides limited insight in comparison to that which may be gained by considering the wider spectrum of mental representations.

7.6 MENTAL REPRESENTATIONS AND AGE DIFFERENCE

All of the discussion to date has focused on merging children who, though indicating some homogeneity in numerical achievement, are in fact drawn from four different age groups. An issue that seems to be central to the discussion on mental representations is whether or not, given enough time, all children are able to project mental representations of both descriptive and relational form. Of course one way to answer this is to consider whether or not there are changes over time. Unfortunately the current study does not provide data to answer this. Neither would a very close analysis of the 'snap shots' provided by year groups in the sample be overly helpful. Indeed, given the length of time over which the interview procedures were carried out (over six months) it may have been anticipated that in some areas, particularly those associated with the numerical components, kinds of mental representations associated with particular children may have changed. However, this was not the case: The kinds of mental representations projected by individual children showed remarkable constancy.

However, in an attempt to consider whether or not children's mental representations qualitatively change, an analysis of those projected by children within the two youngest classes, that is Y3 and Y4, was compared with those of the two eldest classes, Y5 and Y6. The achievement groupings were maintained and thus sample sizes containing four children were considered. Figures 7.10 and 7.11 indicate the similarities and differences across the two phases of the interview process.

In Figure 7.10, an age/achievement comparison of the visual phase is considered. The percentages are constructed by considering the distribution of different kinds of mental representation in each sub-phase using responses to the 30-second 'free talk' element of both the visual and the verbal phases. The selection of this single aspect of the interviewing process is undertaken since it is intended to be illustrative. Appendix 2.6 provides the wider results for all questions and phases. Again, the overall percentages are 'trimmed' to exclude those which occur in less that 8% of the instances.

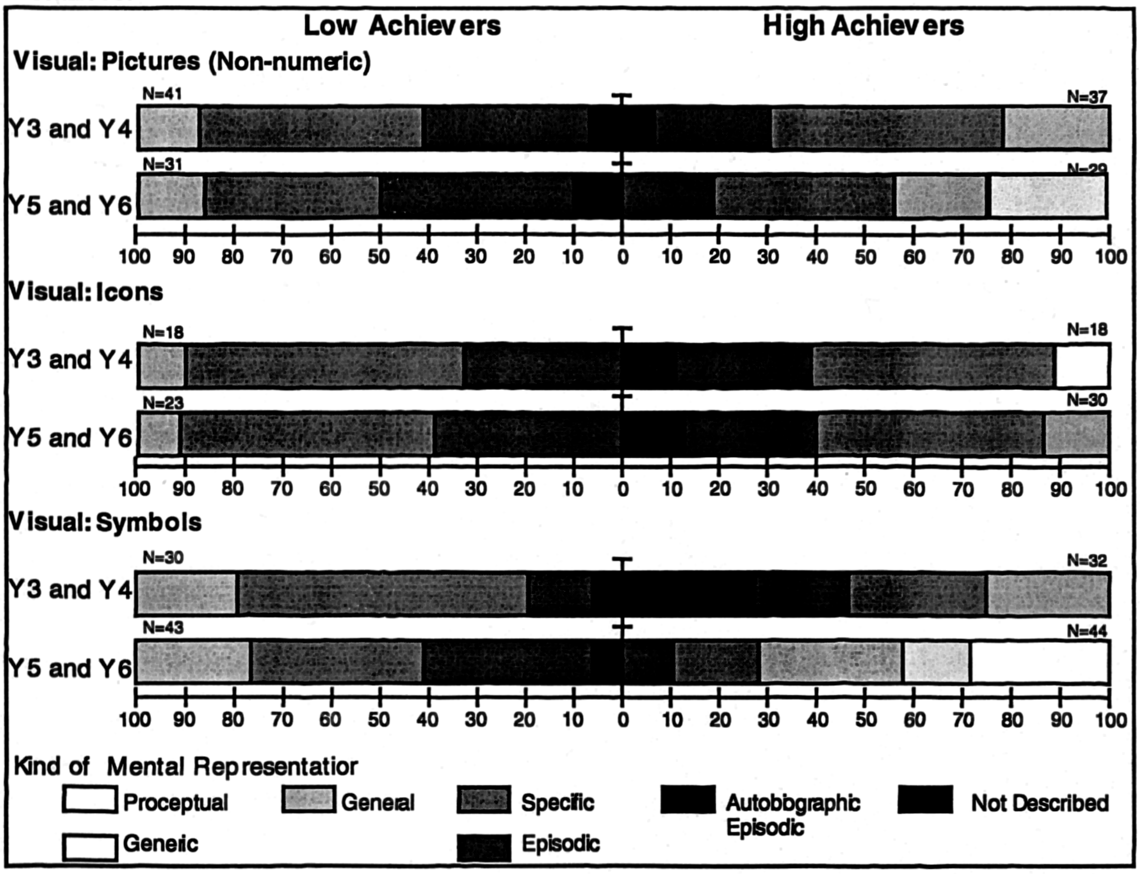

Figure 7.10. Distribution of different kinds of representation by year and achievement: The visual sub-groups.

It may be seen that:

- compared to the younger 'high achievers', those in Y5 and Y6 provide some indication of the use of a 'general' mental representation with an associated reduction in the use of 'specific' and 'episodic' mental representations when considering visual pictures. In comparison the representations of the 'low achievers' are remarkably similar,

- when responding to icons, the two different groups of 'low achievers' are again remarkably similar. Although, there is evidence that younger 'high achievers' can provide 'proceptual' and 'autobiographic-episodic' representations — 11% in both instances — the occurrence of 'specific' and 'episodic' mental representations is very similar to those of the older children (50% and 28% compared to 47% and 27%) and
- differences begin to emerge in the kinds of mental representation identified for the numeric element of the visual phase. Whilst those of the 'low achievers'' remain qualitatively similar yet proportionately different, those of the 'high achievers' reflect a qualitative change associated with the use of 'generic' (14%) and 'proceptual' (27%) mental representations by the older children and a slight increase in the proportion of 'general' representations (30% compared to 25%).

These essential differences are more marked when the representations associated with the second response of the verbal phase are considered (Figure 7.11).

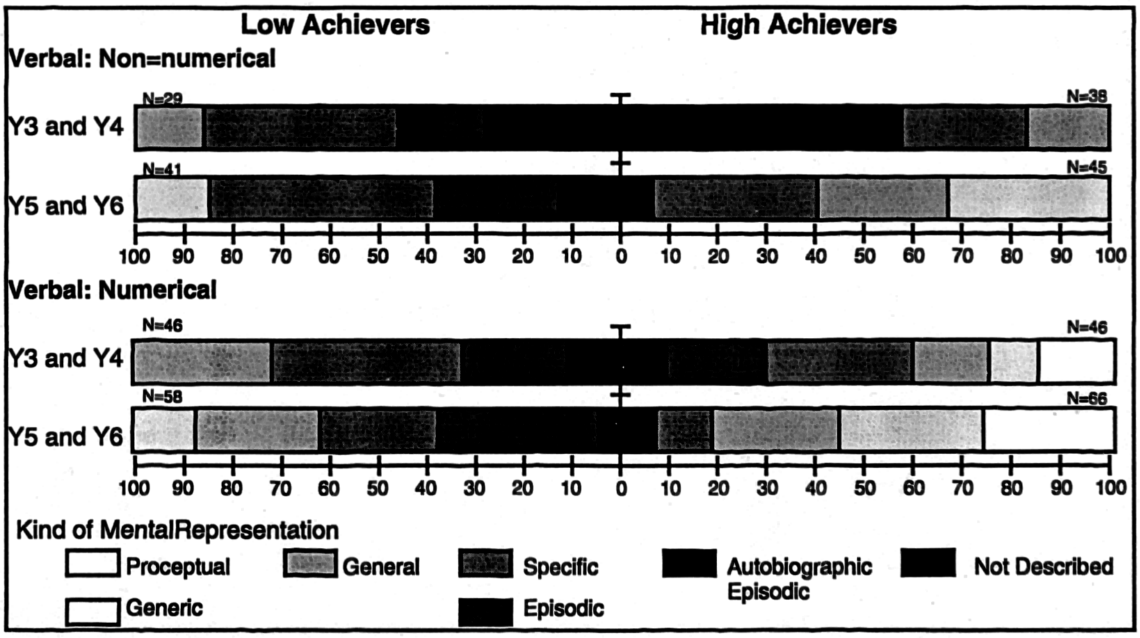

Figure 7.11. Distribution of different kinds of representation by year and achievement: The visual sub-groups.

- once again the qualitative similarity between the two groups of 'low achievers' is only impinged on by the appearance of 'generic' mental representations amongst the older children — 15% in the case of the non-numerical items and 10% in the case of the numerical ones. A further difference emerges in the tendency of younger children to extensively project 'episodic' mental representations, a tendency which is mirrored by the younger 'high achievers', and

- the differences becoming apparent between the two groups of 'high achievers' responding to the visual numeric items are increased when the children respond to the verbal numeric items. Here, we see over 80% of the older children's representations identified as 'general' (26%), 'generic'(29%) or 'proceptual' (26%). However, these qualities also emerge, though to a lesser extent (15%, 10%, 15%) amongst the younger children.

Essentially when we consider the different mental representations that may be projected by children of different age and achievement the evidence is that:

- those of the two groups of 'low achievers' remain qualitatively similar, projecting the by now familiar 'general', 'episodic' and 'specific' qualities whether prompted by visual or verbal stimuli and

- those of the 'high achievers' reflect different qualities that, on the one hand are associated with the nature of the stimulus, and on the other, are a reflection of their age. In other words, representations of the 'high achievers' illustrated more relational characteristics as they became older.

7.7 CHILDREN'S EXPLANATIONS OF WORDS

Whilst the main components of the verbal phase focused upon a first representation and 'free talk' the third component was prompted by the question:

If ET came to ask you what (the word) meant what would you say to him?

This was given to identify what it was that children considered important enough about the word to transmit to another who did not know it. The evidence from the pilot study indicated that, in their reliance upon description and examples, the 'low achievers' may have given more appropriate responses than 'high achievers' who tended to provide 'qualities with insight'. If this pattern was repeated we would expect to see 'specific' and 'episodic' comments dominant amongst 'low achievers' whilst 'generic' and 'proceptual' qualities would more likely emerge from the 'high achievers'.

Table 7.4 indicates the distribution of the individual kinds of representation identified during this phase of the interview. In six occasions 'high achievers' did not respond to the question — five to the numerical component, one to the non-numerical — and in eighteen occasions 'low achievers' did not respond — seventeen to the numerical and one to the non-numerical. The table illustrates the distribution of the remaining responses across each achievement and sub-phase where N signifies each raw total.

	High Achievers				Low Achievers			
	Numerical Items N=140		Non-Numerical Items N=83		Numerical Items N=107		Non-Numerical Items N=77	
	Total	%	Total	%	Total	%	Total	%
Autob. Ep.	0	*5*	0	*0*	4	*4*	2	*3*
Episodic	17	*12*	11	*13*	28	*26*	15	*19*
General	43	*31*	29	*35*	33	*31*	14	*18*
Specific	30	*21*	33	*40*	34	*32*	45	*58*
Generic	9	*6*	10	*12*	4	*4*	1	*1*
Proceptual	34	*24*	0	*0*	4	*4*	0	*0*

Table 7.4: Comparison of the distribution of 'descriptive' responses by achievement.

As was seen in the pilot study, the essential qualitative difference between the two groups when they respond to the numerical items is the inclusion of explanations which have 'proceptual' qualities by the 'high achievers'. Beyond that, to all intents and purposes, the responses of the 'high achievers' and the 'low achievers' are qualitatively similar, with 65% of the 'high achievers'' responses and 89% of those of the 'low achievers'', providing explanations which are 'general', 'episodic' or 'specific'.

However, as was seen earlier in the discussion of the visual and verbal phases, these proportions present a different picture if they are examined by considering them as proportions of the actual statements made (see Figure 7.12)

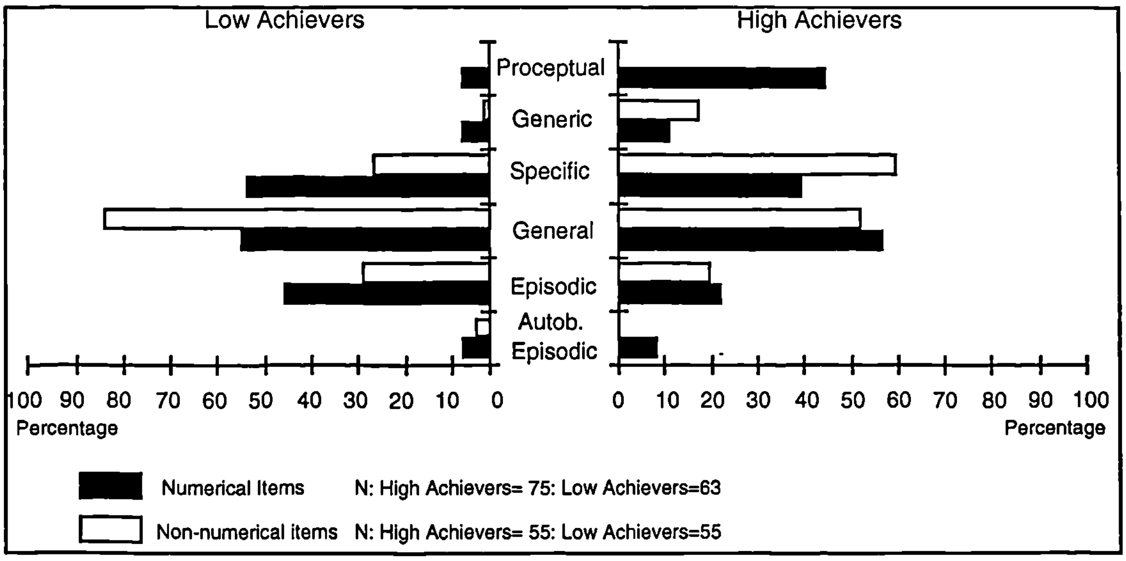

Figure 7.12: Classification and distribution of children's explanation of the verbal items.

In the pilot study it was seen that 'qualities with insight' 'and 'proceptual' qualities dominated 'high achievers'' explanations about numerical items to ET. Similarly in this study, 'generic' and 'proceptual characteristics' are considerably in evidence. Almost three out of five (58%) explanations of the numerical items are of this quality but unless ET has some relational knowledge, such explanations may not be too helpful:

> "It's 3/4 of a whole. It's sometimes is used in lots of things. We use it in supermarket...it can be fraction or decimal." (Y6+, generic, 0.75)

> "Is part of something–half of it or part of it. You can have small fraction. One six hundredth, or a large fraction, one half." (Y6+, proceptual, half)

Indeed, the 'high achievers'' description generally involved characteristics that had been evident in their mental representations of the items. Their 'general' comments required further explanation:

> "They are specks and dots and drops." (Y6+, general dots)

Numeric explanations identified as 'specific' were often process driven:

> "Have a shape, four equal parts three of them make three quarters."
>
> (Y5+, specific, three quarters)

or very descriptive:

> "Fractions is a different number on top of another number with a line going through the middle." (Y5+, specific, fraction)

The usual three characteristics again dominate the 'low achievers'' explanations. The extended use of 'general' characteristics for the non-numerical items (almost nine out of ten explanations included this characteristic) which would require further embellishment unless the item was physically shown:

> "It is an animal." (Y3-, general, dog)

> "Show him the number" (Y4-, general, ninety nine)

The explanation frequently referred to the description of one item:

> "It's round, colourful, my name on it, striped and it has got pink and orange and green and black in my name. It is big." (Y3-, specific, ball)

However, through their explanations the 'low achievers' provided a sense of their personal view of arithmetic:

> "A number is how you count." ([A], Y4-, episodic, five)

> "Six becomes before seven,... and eight becomes after seven." ([B]Y4-, episodic, seven)

Most of the responses of the children in this section are quite similar but they are again some important differences. The similarities lay on the fact that both groups of children were giving 'general', 'specific' and 'episodic' explanations for the items. However, some of the proportions of these responses appeared to be reverse to those of mental representations. 'High achievers' gave more 'specific' responses to the non-numerical items in contrast to the 'low achievers' who gave more 'general' responses. It can be argued that although the mental representations of the 'low achieves' are quite specific the 'low achievers' may not be able to identify either the most appropriate example or surface characteristics to offer as an explanation to the uninitiated. They tend to think that offering a general explanation or the name of the category that the object belongs may be a sufficient explanation. On the other hand 'high achievers place similar importance on both 'general' and 'specific' explanations. However, where the two groups of children seem to have the most striking differences is in the use of 'generic' but especially 'proceptual' responses. It appears that 'high achievers' feel it is very important to explain to the uninitiated how this object relates to other objects and present relational qualities.

7.8 CHAPTER SUMMARY

The questions which directed the study and analysis reported within this chapter sought clarification on the qualitatively different numerical thinking that is manifest in different levels of numerical achievement by:

- identifying whether or not different kinds of mental representations may be associated with these levels of achievement and whether or not visual, and verbal, numeric and non-numeric items encourage different kinds of mental representation

- seeking clarification on the nature of children's visual images associated with this achievement.

- what are the important elements that children of different levels of numerical achievement choose to talk about when providing an explanation.

A supplementary issue sought to identify whether or not age may be a factor in any differences or similarities noted.

The classifications associated with different kinds of mental representation were devised to provide a distinction between those classifications that may be more useful in identifying descriptive kinds of mental representation and those more useful in denoting relational mental representations. The notion of relational was interpreted in the way that it is defined by Skemp (1976) and was characterised by the occurrence of 'generic' and 'proceptual' mental representations.

This added dimension began to illustrate a clear distinction between those children identified as 'low achievers' and those identified as 'high achievers' to show that the more descriptive kinds of mental representation associated with the classifications identified by De Beni and Pazzaglia (1995) were common to all children:

- both 'high and 'low achievers' projected mental representations that were 'general' 'specific' and 'episodic', but
- 'low achievers' projected these for all of the stimuli presented to them
- 'high achievers' were able to draw upon the intrinsic qualities of the items to additionally project 'generic' and 'proceptual' mental representations.

However, the degree to which any qualitative differences were noted was dependent upon the nature of the stimulus and became clearly apparent as this became more abstract or language like. Thus there was a remarkable similarity between the two groups when they responded to pictorial items. The emergence of 'proceptual' mental representations from high achievers in response to the iconic items was the start of qualitative divergence in the kinds of mental representations projected by the children.

'Low achievers' responded to every group of items in a similar fashion, projecting 'general', 'specific' and 'episodic' mental representations for each different stimulus. Visual symbols prompted the same quality of mental representation as icons, numeric words the same quality as non-numeric. Any differences were essentially of a distributive nature, for example, verbal items tended to evoke more 'general' mental representations than visual items during the first response but the second response within each phase was characterised by a high proportion of 'specific' mental representations.

The qualitative differences between the mental representations projected by 'low achievers' and those projected by 'high achievers' were manifest in two ways:

- 'high achievers' used a spectrum of mental representations in a more 'integrated' way. These could be phase specific, for example, 'episodic' and 'specific' mental representations were projected with greatest frequency during the iconic phase whilst 'general' and 'generic' mental representations were evoked with greatest frequency during the non-numeric verbal phase. In general though, the more abstract the nature of the stimulus the more relational the mental representation appeared to be.

- the mental representations of the 'low achievers' were more limited in kind and were projected across the range of items without any real evidence that these children can see through the items to identify more abstract qualities.

It could be hypothesised that the 'high achievers'' different reactions to the different phases may be accounted for by the processing differences that apply between the presentation of a visual stimulus and the presentation of a verbal stimulus. All of the items in the visual phase were presented in the absence of any comment by the investigators. When presented with such a stimulus the 'high achievers' provided a 'general' mental representation, often through naming and often by giving a general comment about the item without any other characteristics, for example, five is a number. It is conjectured that this needs to be done before they carry out a mental search to retrieve 'generic' or 'proceptual qualities'. If it was too difficult to project a general mental representation we saw that 'high achievers' did not give a response. 'Low achievers' carried out their usual approach: they supplied specific mental representations associated with the surface characteristics or a specific example of the item.

The verbal stimulus provided a name and it is conjectured that 'high achievers' carried out a mental search to retrieve mental representations but having done this once they carried on doing it in an almost automatic way. 'Generic' and 'proceptual' relationships were identified with a frequency in excess of that identified for visual items leading to the conclusion that these forms of mental representation are more easily retrieved with a verbal cue or they are stored in such a way that it is more accessible to verbal stimuli.

Such a distinction has, it is conjectured, considerable implications for the way in which children view mathematical symbolism. Whilst 'low achievers' seem predisposed towards identifying this symbolism in the same way that they do everything else, the 'high achievers' draw upon its language like nature and proceptual qualities.

The way in which different kinds of visual imagery may be associated with children of different levels of numerical achievement led to the conclusion that the investigation of this form of mental representation may only tell half a story. On balance no deep differences were identified in the kinds of visual imagery projected by the two groups of children and therefore it is more profitable to consider visual images in the broader context of mental representations.

However, where differences did emerge they were within the context of the use of the imagery:

- the detail carried by the visual images of the 'low achievers' is used 'to be talked about' in an episode which can be described in a context.
- The visual imagery of the 'high achievers' was used to generate an idea and can be put aside so that they may focus on the more relational characteristics.

'High achievers' tended to give more visual imagery in the numerical items in the verbal phase whereas in the visual phase the occurrence of visual imagery amongst both groups is of very similar frequency. It may be argued that visual stimuli may cause a higher tendency to create visual images. Therefore it is concluded that the modality of a presentation (visual or verbal) may have an impact on the modality and context of the mental representation that is created. 'Low achievers' take what they see.

The different kinds of mental representation projected by the two groups did bear some relationship to the types of explanations they felt is important to give to the uninitiated. Again similarities were noted in that these explanations were heavily laden with 'general', 'specific' and 'episodic' explanations. However, it seemed that 'high achievers' not only felt it important to give a 'general' mental representation and describe the object but to also explain how the object related to other objects . As a consequence they presented relational qualities which may or may not have been helpful in the context.

The evidence within the chapter points to qualitative differences in the kinds of mental representations project by 'low achievers' and 'high achievers'. From the evidence available it is suggested that age has little influence upon the quality of mental representations projected by the 'low achievers' whereas the mental representations of the 'high achievers' being of a generic nature operate at a more relational level which may grow in complexity.

*

CHAPTER 8
ARITHMETICAL CONSIDERATIONS

"It was as if I split my mind into two to do it. I have two bits. One is remembering and the other one is talking. So the talking is working it out and then I say to the remembering part what I want to remember" (Child, Y4+)

8.1 INTRODUCTION

8.1.1 Structure of the Chapter

In Chapter 7 children's mental representations associated with two forms of external stimuli, visual and verbal, suggested that children at extreme levels of arithmetical achievement projected qualitatively different kinds of mental representation. This chapter considers children's mental representations in elementary arithmetic. It considers the qualitatively different ways that the children respond to numerical combinations in elementary arithmetic, the mental representations that they use and the modalities associated with these representations.

Building out of the work of Gray (1991) and Gray and Tall (1994), the chapter indicates how similar considerations were given to the children's numerical behaviour as a means of associating qualitatively different forms of numerical thinking to particular kinds of mental representation. The chapter builds on and expands the work of the pilot study. However, there, the focus of interest was on visual imagery but Chapter 7 showed that concentrating on this form of mental representation may limit the breadth of the discussion as to what it is that children see or articulate in their mind when dealing with arithmetic.

The hypotheses associated with this component of the study draw on previous work to confirm that children at extremes of arithmetical achievement display qualitatively different approaches to elementary arithmetic. However, the discussion builds on evidence in the pilot study to illustrate that children possess mental representations in arithmetic which vary in content, have an emphasis on different objects and are of different modalities. It continues the theme established within chapter 7 to suggest that children who have a tendency to project descriptive kinds of mental representation interpret elementary arithmetic in a procedural way whilst those who are able to project relational kinds of metal representation interpret their arithmetic in a proceptual way.

After considering the children's range of achievement (Section 8.2.1) we devote three sections to associating this achievement with the qualitatively different approaches that are observed. The first considers the approaches of the children to a range of number combinations to 20 (Section 8.2.2). It considers how children respond to a range of elementary combinations presented visually and verbally by examining the strategies they use (8.2.2.1), representations associated with these strategies (8.2.2.2) and the modalities of these representations (8.2.2.3). It concludes by indicating that qualitative differences in children's thinking derived from children's interpretations of symbolism may be associated with the quality of the mental representations that they form.

Section 8.2.2.3 extends these ideas to examine how they affect children's approaches to two-digit combinations and in section 8.2.4, three-digit combinations. The chapter concludes by suggesting that qualitatively different interpretations of numeric symbolism may be associated with a predisposition towards qualitatively different kinds of mental representation. A predisposition towards descriptive kinds of mental representations suggests that the mental representations that children form of arithmetical activity are of an 'episodic' and 'specific' kind which leads to a concentration on arithmetical procedures. A predisposition towards forming integrated mental representations projecting descriptive and relational characteristics supports the formation of those qualities of behaviour which are manifest in proceptual thinking.

8.1.2 Research Issues

In Section 4.4.7 three issues that are related to the arithmetical component of the study were identified. These lead to research questions which ask:

(i) once identified how do the qualitatively different approaches children use in elementary arithmetic relate to their approaches to more complex number combinations?

(ii) what kind of mental representation may be associated with qualitatively different approaches to elementary arithmetic? How do these change as combinations become more difficult and what is the form of the object that is the focus of attention?

The overall approach used to answer these questions has been considered in Sections 3.4. and 3.5 which established a rationale for this component and the nature of the sample. In Chapter 6, a more detailed examination of the nature of the questions and the nature of the sample was considered.

As a reminder, it can be briefly noted that two phases of questions, which replicate the visual and verbal phases of Chapter 7, were used to present a range of arithmetical combinations focusing on addition and subtraction. These combinations were used to:

(i) consider and classify the strategies that children used to resolve the addition and subtraction combinations to 20 which they could not recall,

(ii) establish and classify the way in which children used mental approaches to compute whole numbers to 100,

(iv) examine briefly the ways they may deal with mental computations beyond 100 and

(v) provide a mechanism to examine representation, modality and content in elementary arithmetic.

Additionally, the phases are used to consider what was happening in the children's heads as they were dealing with the combinations (see Section 4.6.2 4). The issues for discussion were prompted by the question: "*What was happening in your head as you were doing this?*"
The strategies that children used to obtain the solutions to the combinations were classified using count-all, take-away, count-on, count-up, count-back, derived fact, transformation, accumulation, algorithmic and known fact (see also Section 3.4.6.2 and 3.4.6.2). Classifications associated with the kind of representations were linked to whether or not the child used external or mental representations, what these involved and whether or not they 'saw' or 'said' things in his or her head (Section 8.2.2.2).

8.2 SOLVING THE NUMERICAL COMBINATIONS

8.2.1 Children's Overall Achievement on the Numerical items

Table 8.1 indicates the mean percentage of each ability groups correct responses for each of the categories of the arithmetical combinations (see Appendix 3.1 to 3.3). These are sub-divided into the visually and verbally presented items, each showing a group mean and an individual range of correct responses for the addition and subtraction combinations to 20 and for the two- and three-digit combinations. The group means are calculated by giving the total number of correct responses in each category as a percentage of the product of the number of items (N) in each group and the number of children in each achievement group (8). The range is given as the percentage of correct responses by the lowest and the highest achievers in each category.

Features which emerge from the table indicate that:

- amongst both groups of children, the mean number of responses which are correct for the addition combinations was higher than those correct for the subtraction combinations and

- as the problems increased in difficulty, the difference in the level of achievement between each group widened.

Visually Presented Items

	Combinations to 20		2 and 3 Digit Combinations	
	Addition (N=7)	Subtraction (N=5)	Addition (N=5)	Subtraction (N=5)
High Achievers	98	90	100	93
Range	92 to100	75 to100	–	60 to 100
Low Achievers	75	78	42	20
Range	14 to100	60 to100	0 to100	0 to 40

Verbally Presented Items

	Combinations to 20		2 and 3 Digit Combinations		3 Digit
	Addition (N=5)	Subtraction (N=3)	Addition (N=2)	Subtraction (N=3)	Addition and Subtraction (N=2)
High Achiever	100	92	100	92	62.5
Range	–	85 to 100	–	66 to 100	0 to 100
Low Achiever	75	71	25	21	0
Range	66 to 100	66 to 100	0 to 66	0 to 66	–

Table 8.1: Mean percentage and percentage range of each achievement groups correct responses to each category of the arithmetical combinations.

The children in the sample were finally selected on the basis of their level of achievement in these three categories of arithmetical problems so differences in achievement comes as no surprise. The immediate discussion is therefore presented as a platform for later discussion in the chapter.

Only when attempting the three-digit combinations presented verbally did the success rate of the average success rate of 'high achievers' fall below 90% (to 62.5%). The associated range, 0% to 100%, reflected the fact that the 'high achievers' from Y3 were unable to provide correct solutions to the verbally-presented addition or subtraction problems. All of the children in the group gave five correct responses to the other verbal addition combinations and the majority (six out of eight) gave correct responses to the other subtraction problems. In contrast we can note that some of the 'low achievers' had difficulty with the number combinations to 20 in each phase, these difficulties increasing to embrace the whole group when they attempted the verbally-presented three-digit combinations.

Difficulties associated with children's failure to obtain solutions to the combinations may be seen from four perspectives:

- isolated errors amongst 'high achievers' which were almost momentarily lapses in thought. Generally, during the discussion associated with each problem these children noted their error and corrected them,

- the application of fixed routine procedures to a range of problems was a feature of the strategies applied by the 'low achievers'. In some instances this began to emerge very early in the interview. For example, the child who obtained one correct solution (14%) to the visually-presented addition combinations to 20 always under-counted by one. However, the application of a difficult almost non-generalisable procedure became common in some instances when children attempted the subtraction combinations to 20. Whether or not the stimulus was visual or verbal count-back began to dominate. This caused widespread difficulty with '17–13' for all but two of the 'low achievers',

- the application of inappropriate procedures. This was seen particularly amongst 'low achievers' who frequently used an unsuccessfully adapted algorithm (e.g., smallest from largest) for the visually-presented subtraction items, and often attempted an inappropriate counting procedure for the verbally presented items and

- the difficulty associated with remembering verbally-presented items.

Each of the phases in the numerical component of the study examined the strategies that children used to solve the context-free problems, the form of representation used to support the approach and the modality of the representation. As each of the groups of number combinations is considered, the discussion will examine these other issues.

8.2.2 Number Combinations to 20

8.2.2.1 Distribution of Strategies

Table 8.2, drawn from a summary of Appendices 3.1 and 3.2 illustrates the distribution of strategies used by each group of children to solve the number combinations to 20. The identity of each strategy is based on concepts to know (the known and derived facts (KF and DF) and processes 'to do', (count-on (CO), count-up (CU), count-back (CB) and take-away (TA)). Each cell represented a mean percentage distribution of strategies for children in an achievement group. Collectively, the percentages illustrate the mean distribution of strategies for a child in that group. However, there are substantive differences associated with age, particularly amongst the 'low achievers'. The distributions are calculated by giving the total number of identified strategies as a percentage of the product of the number of items (N) in each phase and the number of children in each achievement group (8).

VISUAL

	Addition: N=7				Subtraction: N=5				
	KF	DF	CO	CA	KF	DF	CU	CB	TA
High Achievers	94	2	4	0	74	16	5	5	0
Low Achievers	29	13	53	5	28	5	46	18	3

VERBAL

	Addition: N=4				Subtraction: N=3				
	KF	DF	CO	CA	KF	DF	CU	CB	TA
High Achievers	69	31	0	0	67	21	12	0	0
Low Achievers	12	16	66	6	8	4	40	43	4

Table 8.2: The occurrence and distribution of strategies identified when 'high' and 'low achievers' solve combinations to twenty (percentages).

In Table 8.2 we see immediately the different levels of 'known facts' by the two groups of children. The qualitative difference in support strategies in the absence of 'known facts' is clear:

- a relatively high proportional use of derived facts amongst 'high achievers' and
- a relatively high proportional use of counting amongst 'low achievers'.

In only one instance when solving the verbal-addition combinations did a 'low achiever' from Y3 or Y4 use any strategy but counting. In only three instances in the whole group was any strategy but counting used for verbal-subtraction.

Thus, the essential difference between the two groups of children was the difference in the 'fall–back' strategy if combinations were not known. The 'high achievers' solved 93% of all the combinations presented using 'known' or 'derived facts'. The 'low achievers' solved 68% of the verbal-addition and subtraction combinations through counting; the younger children (Y3 and Y4) in 81% of instances, the older children (Y5 and Y6) in over 50% of instances. The older ones also knew 30% of the combinations or used derived facts for 20% of them. Such a distribution is very similar to that reported by Gray and Tall (1994).

The discussion above considers all of the strategies used in an attempt to solve the number combinations to 20 but, as can be seen in Table 8.1 the overall success rate of the 'low achievers' in any one category (visual-addition, visual-subtraction, verbal-addition and verbal-subtraction) did not rise above 78%. It is noted that in one instance, visual subtraction combinations, the success rate of the 'high achievers' fell to 90%.

The errors of the 'high achievers' were distributed across the range of strategies that they used, an error in a recalled fact (9–2=6), an inappropriately derived fact (13 – 5 leads to 14 – 4 =10) or an error in counting (17 – 13 = 14). All were later corrected (with no prompt) during the discussion but for the purposes of this study they remained as errors.

The errors of the 'low achievers' were largely accounted for by procedural difficulties associated with counting. In one quarter of all instances (i.e. for both phases of interview and for both addition and subtraction) counting procedures used by 'low achievers' terminated in an error. This could be associated with a simple miscount or it could be part of a pattern which indicates procedural inefficiency. For example, children who successfully made use of their fingers as a perceptual anchor for counting small numbers had difficulty when quantities were greater than 10. In some instances this was recognised. For example, 17 – 13 was "*too tricky to count*" (Y3-).

In other words, in several instances no suitable counting procedure was identified. In other instances difficulties associated with a double-counting procedure caused problems:

> "I just counted 1, 2, 3, 4 ... 16, 17 then I went 17, 16, 15, 14, 13 and said 5." (Y4–, 17–13)

8.2.2.2 Form of Representation Associated with the Children's Strategies

Three issues guided the enquiry into the form of mental representation that children used to solve the numerical problems:

(i) the nature of the representation and whether or not it was associated with a *visual* or *verbal* form of mental representation,

(ii) whether or not the mental representation may be associated with the kinds of mental representation considered in Chapter 7 and

(iii) whether or not a particular form and/or a particular kind of representation may be associated with the level of a child's achievement and identified strategies.

Table 8.3 (Appendix 3.1) indicates the range and proportions of representations identified. The table indicates percentages based on the ratio of the occurrence of a particular form of representation and the total number of responses in each group (N). The table distinguishes between:

- *automatic response*: These were identified wherever a child indicated that "I just knew it" and no overt or covert actions appeared to be associated with the response. The classification was always associated with a 'known fact',

- *abstract representation*: The use of the term is based on the notion of Steffe *et al* (1983) but in the current context not only did a child not require to construct countable units but the symbol was identified as the object of thought. The classification was most frequently associated with derived facts, a typical response being:

"[Said to myself] the difference between six and seven is one, so two sixes are 12 and 12 plus 1 is 13. (Y4+, verbal, 'seven add six')

- *perceptual representation*: Again, this is applied in the sense used by Steffe. Strategies are applied with the support of physical items, for example, fingers,

- *verbal counting:* Here the number word is taken as a substitute for countable items. Typical examples include:

"Five Counted in my head three, four five." (Y3-, verbal, 3+2)

"9, 8, 7, ... just said that to myself." (Y6-, verbal, 9-7)

- *figural representations.* Here the counting process continues in the absence of actual items but be associated with visual analogues of the items:

"I saw a line of numbers. It was one, two, three... 13, 14, 15. After 10, the numbers got bigger. I counted from one until I got to 8." (Y3-, visual, 3+5)

It can be seen from Table 8.3, where N is the number of responses considered that amongst the 'high achievers', the visual and the verbal phases of presentation invoked automatic and abstract interaction to an equal degree.

Representation and the Visual Phase

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
High Achievers					
Addition (N=56)	74	21	5	0	0
Subtraction (N=40)	77	13	5	5	0
Low Achievers					
Addition (N=56)	13	23	32	25	7
Subtraction (N=40)	24	3	43	20	10

Representation and the Verbal Phase

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
High Achievers					
Addition (N=32)	56	41	0	0	3
Subtraction (N=24)	71	17	8	4	0
Low Achievers					
Addition (N=32)	13	12	59	13	3
Subtraction(N=24)	8	4	63	21	4

Table 8.3: Representations in the visual and verbal numeric phases: Combinations to twenty.

Although a quantitative difference was noted between the visual and verbal-addition phases, little difference was identified, for 'high achievers' between the subtraction combinations. This suggests that when these children were not able to provide an automatic response they would use number facts that could be automatically retrieved to manipulate abstract symbols:

"Took 3 off the 5 to make 10 and then 2 off the 10 to make 8."

((A), Y3+, visual, 13-5)

"I said [to myself] 3 add 7 is 10, add 1 more is 11" ((B), Y3+, visual, 4+5)

Amongst the 'low achievers' we see that subtraction, in both phases, evoked the use of perceptual representations more than addition. Most frequently fingers, these perceptual units, became objects of thought which were sequentially tagged in the counting procedure. Tagging was mostly overt in that children looked directly at their fingers, tagging those on one hand with those on the other, or, if the number was relatively large, tagging through touching the desk or even the nose. On some occasions motor acts were used as a substitutes for tagging. These forms of counting have been identified elsewhere (Steffe *et al,* 1983; Gray & Tall, 1995). At times the counting was not associated with any obvious tagging. Children would "feel" movement in their fingers without any obvious sign that they were doing so. When this happened, it seemed that the younger the child or the more difficult the sum the more exaggerated this movement was. Therefore, while a Year 4 'low achiever' said:

"7,8,9,10,11 and was counting on my fingers." (Y4-, verbal, 4+7)

a Y6 one said:

"I was counting in my head 6, 7, 8, 9; but had it as a finger feeling." (Y6-, visual, 6+3)

The smaller combinations were often associated with verbal counting.

"I was thinking of it. Add. It is 11. I said 10, 11 to myself." (Y6-, visual, 9-2)

"5. I counted in my head 3, 4, 5." (Y3-, verbal, 3+2)

"9,8,7. I said it to myself." (Y6-, verbal, 9-7)

There were rare instances where 'low achievers' made reference to figural representations. These may be seen in the context of the analogical representations described in the pilot study (Section 5.4.3.2). In the instances associated with the number combinations to 20, these figural representations resembled the number forms of Seron *et al,* (1992) in that they were based on the number line:

> "I saw the number line in my head and saw a lot of numbers in my head on a number line. There are lots of numbers but it depends on how high the numbers are. High means how big the numbers are. This time it was three plus four. I went up to 33. If I went up to 11 I couldn't count. I didn't want to go to 10 I wanted to go to 33 but I don't know why – but it is my favourite number." (Y3-, visual, 3+4,)

This kind of figural mental representation also seems to have some relationship to Dehaene and Cohen's (1996) notion of "number analogue". However, in the context that they were given, the figural representations identified amongst the current group of 'low achievers' were epiphenomenal.

8.2.2.3 Modality of the Representations.

Figural mental representations were an example of visual representations reported by the children, but there were also verbal mental representations and perceptual representations. Distinctions between these three were essentially identified by children, indicating that they 'saw' something in their head, 'said' something to themselves or carried out some 'action' with physical objects. Over all of the items responded to by all of the children (a total of 304 responses for the verbal and visual phases) modalities identified as visual, verbal or perceptual were identified in 71% of the instances. The remainder were unclear. Two 'high achievers', responding to the visual items, provided automatic responses to every item but one, and in every case their response was not associated with a modality.

Table 8.4 indicates how the 71% of the identified modalities was distributed by achievement and by type of combination. The table is constructed by considering the verbal and the visual phases together, indicating the number of responses identifying representations (N) and presenting this as a percentage of the total number of responses from in each achievement group (88 for addition, 64 for subtraction). It is worth noting that amongst the 'low achievers', the responses are greater than 100%. This is because in nine instances for the visual phase and three for the verbal phase, children displayed at least two modalities. For example,

"I used my fingers to count but I saw the numbers in my head" (Y3–, visual, 9–8)

		Percentage of Total Responses	Percentage of identified Modality		
			Visual	Verbal	Perceptual
High Achievers	Addition N=40	45	33	60	8
	Subtraction N=24	38	46	42	13
Low Achievers	Addition N=93	106	19	46	34
	Subtraction N=60	107	22	28	50

Table 8.4: Identification of modality in addition and subtraction combinations amongst 'high' and 'low achievers': Combinations to twenty.

The first striking observation in Table 8.4 is the degree to which modalities were identified amongst 'low achievers' in comparison to 'high achievers'. It is a distinction reflecting the dominance of automatic approaches amongst 'high achievers' therefore no subsequent explanation was forthcoming. The frequent use of perceptual representations amongst 'low achievers' is a second feature worthy of note.

The distribution of modalities through each phase of the interview processes can be seen in Figure 8.1. The percentages are based on the number of modalities identified in each phase (N) where the total number of combinations within each phase are as Table 8.3. In some instances N is higher than the totals in Table 8.3 since some children used two modalities, for example, verbal and visual.

Several distinct features emerge from the table:

- the high proportion of unclear responses amongst the 'high achievers' reflected the high number of automatic responses,

- as the combinations move from the visual to the verbal phase and from addition to subtraction amongst the 'low achievers' we note a decline in the occurrence of the visual modality and increase in the occurrence of the perceptual and

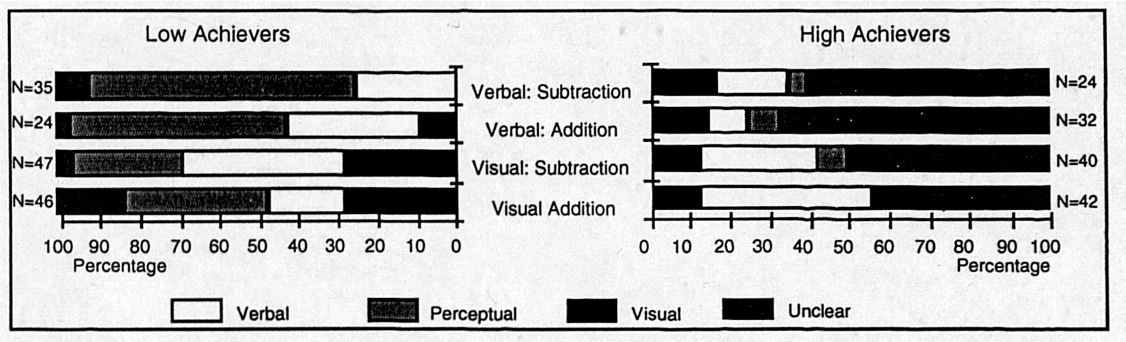

Figure 8.1: Distribution of modalities in each phase of the numerical presentations.

- in contrast, where modalities were identified we note a similar level of the use of the visual modality amongst 'high achievers' but a decline in the use of the verbal.

This tendency to display all modalities in the visual phase was very apparent amongst 'low achievers'. Five did so whilst three were verbal/perceptual. During the verbal phase, three were verbal/perceptual and three of the four youngest were essentially perceptual. Only one, a Y6 child, was totally verbal.

The emphasis on the use of perceptual items amongst the 'low achievers' reinforced the evidence that symbols need to be concretised. In these instances the objects that dominated their thinking were fingers. Where symbols dominated thinking a response was classified as 'abstract', the symbols themselves being frequently associated with the input combination or with the final solution.

"I saw the eight then the two and then I see them all together." (Y5-, visual, 8–2)

The tendency to see the input symbols and/or the final output was relatively common amongst those 'high achievers' who reported seeing symbols though other characteristics were also apparent .

"I saw picture of 9-2, black." (Y6+, visual, 9-2)

"I saw a three then I saw a nine... first three then the nine (the three as a bubble number in green and the nine as a bubble number in red inside." (Y3+, visual, 6+3)

"I saw nine and seven flash, not as a sum but just nine and seven. I then saw two much stronger." (Y6+, verbal, 9-7,)

'High achievers' also reported symbolism associated with the formation of derived facts:

"I saw 13-5 and thought I would split it up into different parts to make it easier. This stood out. Saw 13-3 then 10. Then I saw 10-2 this stood out. Then I saw 8. 13-3 and 10-2 stayed at the same time. I saw 8 on its own." (Y6+, visual, 13-5,)

The verbal modality was generally more common than the visual amongst both groups of children. In the 44% of instances where it was identified it was frequently associated with the use of a derived facts:

I said [to myself] seven and 7 and take away 1. Told you 13 (Y3+, verbal, 7+6)

Amongst the 'low achievers', the verbal modality was strongly associated with verbal counting. In this context the verbal modality would most often be in a sequential counting form where the numbers themselves would serve as countable objects. The verbal tone or double counting drove the counting act, although this sometimes lead to errors:

"Got to 13 to add up to 17... 13, 14, 15, 16, 17. 13. One 14, two 15, three 16, four 17, five... said all thin my mind." (Y6-, verbal, 17-13)

8.2.2.4 Number Combinations to Twenty: A summary

It was seen at the start of the section that 'high achievers' were more competent at the elementary arithmetic than 'low achievers'. This is not an unexpected observation because of the criteria on which the children were selected and the issue of competence was the mechanism through which other variables were considered.

'High achievers' knew, on average, between 9 and 10 of the 11 addition combinations presented to them, and six of the eight subtraction combinations. Failure to directly recall a solution evoked alternative known facts to derive those that were not known. As might be expected, it was the children in Y5 and Y6 who knew all of the combinations so the use of derived facts was more extensively noted amongst the younger children, particularly those in Y3.

In contrast, 'low achievers' could only recall between two and three of the addition combinations and on average, slightly less than two of the subtraction items. The occurrence of 'known facts' amongst the oldest 'low achievers' did not match that of the younger 'high achievers'. Although there was some evidence of the use of derived facts, particularly from children in Y5 and Y6, the usual fall-back procedure was counting.

These 'strategic' differences between the two groups of children were manifested by automatic responses or abstract representations using symbolism by the 'high achievers' and with the extensive use of perceptual counting using fingers by the 'low achievers'.

It was seen that verbal and/or visual modalities were dominant amongst 'low achievers', any relationship between the two being a product of the form of the stimuli. Amongst 'high achievers' the high verbal modality in the visual phase (43% for addition, 29% for subtraction) is possibly associated with a transformation of the visual modality into a verbal code.

Hearing the combinations triggered the 'low achievers' to do something, in this instance an overwhelming desire to count, seeing the stimulus provided slightly more flexibility or triggers an increased tendency for the mental representation to have a symbolic form. This may be linked to the fact that these children focus on the surface characteristics of objects.

It could be hypothesised that in their failure to recall combinations the representations that the 'low achievers' used, both physical and mental, were based on the general representation of a number sequence. This 'general representation' possesses 'number line' qualities. It was retrieved and then allocated the numbers that the individual needs to count. It may be seen as 'episodic' in the sense that it is a scene or a sequence of scenes — in the child's mind, the 'counting episode' needed to be performed. It may be identified as specific' in the sense that a particular number generated a particular 'line' or particular countable objects, and in most cases, these were fingers.

The 'specificity' of the number sequence is not only identified through the inclusion of the numbers that need to be counted but also through its 'figural' and/or 'perceptual' associations. Therefore, it seems that the choices the children make (consciously or subconsciously) focus on the nature of the counting procedure and representation (mental or physical) they need to support this procedure. Increasing procedural efficiency may determine both the counting strategy, for example, count-up as opposed to count-back, and the nature of the representation to be used. This may change from exaggerated 'finger movements' to just 'feeling the finger counting' and then possibly figural representations and verbal counting. This transition, if it may be identified as such, is the one so aptly identified by Steffe *et al* (1983). The essence, however, remains the same: *the strategy and the representations of 'low achievers' are based on the general representation of number sequence which becomes 'episodic' and 'specific'*. In only using counting strategies and representations of this kind, the 'low achiever' is once again illustrating a reluctance or even an inability, to search for and retrieve an appropriate number relationship that could assist in a more sophisticated approach to a combination.

In contrast 'high achievers' are using the retrieval of 'known facts', either to give an automatic response or in order to proceed to a 'derived fact'. It is conjectured that underpinning this approach is the power that emerges from their representational flexibility. They seem to be carrying out a 'search' for the most appropriate number fact that can be used. They retrieve it and either present it as the answer or manipulate it in order to reach an answer. The former is not easy to qualify, the latter is the essence of proceptual thinking. Of course, 'specific' and 'episodic' kinds of mental representation in the form of the general counting sequence or procedural manipulation are available to them if at any time they feel they need to use it.

8.2.3 Strategies and Representation: Two Digit Combinations.

8.2.3.1 The Range of Achievement

Table 8.1 illustrates the range of competencies that were noted amongst the children when they attempted to solve the two-digit addition and subtraction combinations. Whether these were presented visually or verbally, the 'high achievers' obtained a mean score of at least 92%, the greatest range of individual scores being seen in the subtraction combinations, 60% to 100%. 'Low achievers', on the other hand had a maximum mean score of 42% for the visually-presented addition combinations, this falling to 29% for the subtraction ones. The range for the former was 0% to 100% whilst for the latter it was 0% to 40%. Amongst the 'low achievers' only those in Y6 coped reasonably well with the visually presented items but even they had difficulty with the verbal-subtraction items.

8.2.3.2 Strategies used to Solve the Combinations

The fundamental difference between the two types of presentation of the two-digit combinations was that verbally-presented items required mental approaches to solve whilst the visually-presented ones could be resolved using standard written algorithms.

The fundamentally different way the two groups of children approached the verbal items accounted, to a large degree, for differences in achievement. 'High achievers' responded to all of the combinations, 2 addition and 3 subtraction. 'Low achievers' particularly the younger ones had difficulty responding to the subtraction.

- over 75% of the solutions given by 'high achievers' used transformation strategies. (N=40 responses, 16 addition, 24 subtraction) and

- over 75% of the solutions of those 'low achievers' who attempted the combinations (N=28, 13 addition, 15 subtraction) were count-based accumulation strategies. (Here the notion of 'accumulation' is used as a general term to cover addition and subtraction strategies involving incrementing or decrementing through sequencing in ones and/or 10s.) In over 75% of instances these were unsuccessful, the highest success rate being achieved by the Y5 and Y6 children.

Only 5% of a total 40 answers supplied for addition (16) and subtraction (24) were completed using accumulation strategies by the 'high achievers'. Of the balance, 76% were transformation strategies and 18% were 'known'. Whilst 'known' responses featured largely amongst the Y6 children, transformation strategies were used across the age ranges and were 100% successful:

64 – 26:	64 –20 = 44	44 - 4 = 40	40 – 2 = 38	(Y3+)
64 – 26:	64 = 50 +14	14 = 8 + 6	26 + 8 = 38	(Y4+)
64 – 26:	70 – 32 = 38			(Y6+)

With the addition combinations, there was a tendency for these strategies to be analogous to the standard written algorithms using a left-to-right approach:

45 + 57:	40 + 50 = 90	5 + 7 = 12	90 + 12 = 102	(Y3+)

There was evidence of the use of transformation strategies from two 'low achievers', one in Y5 and one in Y6. However, their efforts were not 100% successful because elements of the transformation were forgotten or misrepresented. Only when explaining the approach was this realised:

> "I took 20 away from 60, that was 40, then 4 away from 6, that is 2 and then the 2 from 4 to make 20... Oh! Its four 10s..." (Y5-, 64 – 26)

> "20 away from 60 [is] 40. Trying to repeat 40. Six away from 40 I think is 34. That is the answer" (Y6-, 64–26°

The second example given above illustrates the notion of articulatory rehearsal identified by Baddeley (1976) and discussed in Section 3.4.4.1.

Whereas in the application of their transformation approaches the 'high achievers' knew all of the relevant relationships, in half of the instances where they applied them, 'low achievers' supported the approach with some counting:

45 + 57   40 + 50 = 90   7 + 5 = 12 "(I counted this on my fingers)"   90 + 12 = 102   (Y5)

'Low achievers' used counting to support their accumulation strategies. It took two forms, counting in ones, and sequence counting in 10s and then adding the 'bits' by counting in 1s. The former was identified in 60% of instances where accumulation strategies were identified, the latter in 40%. In 84% (11) of the instances where counting in ones was used, incorrect solutions were given. Incrementing or decrementing in sequences of 10 were used in eight instances. In five of these incorrect solutions were given.

A typical successful accumulation strategy sequencing in 10s would be:

"27 + 62: 60,70,80, 87, 88, 89."   (Y5-)

The standard algorithms figured largely in children's attempt to deal with the visually presented items. However, once again there was evidence of transformation approaches by 'high achievers', often supported by a left to right approach, dealing with the 10s first:

47 +15 : 47 and 10, 57 and 5 is 62.   (Y5+)

82 −24 20 from 80 is 60, that's 62, 4 from 62 is 58.   (Y3+)

Dealing with the 10s first also featured amongst those 'low achievers'. One quarter of their responses to the addition and subtraction problems were of this pattern but all involved some form of counting either by attempting to count on the smaller number to the larger, count back from the larger or, where the approach was associated with a standard algorithm by dealing with elements of each number separately. Counting in some form or another was used by every child.

Misconceptions, in the form of smallest from larger errors, were displayed by every child. If the tendency of these children is to consider descriptive qualities of items, it may well be that this reinforces their focus of attention on the separate elements of numbers. Several children talked about the 10s as if they were ones — not 10 from 20 but one from two. It would seem that 'specific' and 'episodic' qualities associated with well rehearsed episodes may dictate the way they deal both with addition and subtraction combinations. With the former it may lead to success, with the latter it may lead to failure. 'Specific' and 'episodic' qualities may be apparent when children base their behaviour on the specific episode "largest first". It makes addition easier but it may lead to smallest from largest errors in subtraction.

The children tended to approach the verbal two digit items in the way that they had dealt with unrecalled facts to 20. The derived facts of 'high achievers' evolved into transformation strategies. 'Low achievers' either attempted to apply counting procedures which were no longer appropriate or attempted to modify these procedures to 'chunk' 10s and apply sequence counting. The general difference was one which highlighted the relational versus 'descriptive' approach – a difference between concepts 'to know' and processes 'to do'.

8.2.3.3 Forms of Representation Associated with Children's Strategies

It was inevitable that the decline in the number of 'known' responses associated with the two-digit combinations would mean a decline in the number of automatic responses. Even so, automatic responses from 'high achievers' were identified in 32% of the instances for the visual phase and 15% of the verbal phase. (There were 2 each of the visual addition and subtraction combinations and 2 verbal addition combinations and 3 verbal subtraction combinations. This gave the total number of responses in the verbal phase as N=40; 16 for addition, 24 for subtraction and within the verbal phase N= 32; 16 for addition and 16 for subtraction. All of the automatic responses were associated with subtraction, particularly 29 – 6. No automatic responses were identified amongst 'low achievers'.

Collectively the 'low achievers' did not complete as many of the two digit combinations as the 'high achievers'. The actual completed totals are seen in Table 8.5.

In both groups of children, there were distinct differences in the way that they approached the verbally and visually presented combinations:

- 'high achievers' solved the visually presented combinations largely by manipulating symbols, thus, through abstract mental representations. Of 16 responses for addition, this kind of representation was used almost three out of four times, the balance being automatic. In subtraction, abstract representations were identified three out of five times, the balance again being automatic,
- almost eight out of 10 of the verbally presented items were solved by 'high achievers' through the use of symbols, thus, an abstract representation and
- 'low achievers' did not provide any automatic responses.

Table 8.5 shows the distribution of 'low achievers' other representations used for the visual and the verbal items. It illustrates the number of responses considered in each category (N). The percentage occurrence of each representation based on this number is given and the correct number of responses as a proportion of the number of combinations solved with a particular representation is also given.

The general picture that emerges is that some of the representations used by the 'low achievers' were inappropriate attempts to generalise approaches that worked relatively successfully with smaller number. This is particularly true of the use of perceptual representations.

Representation and the Verbal Phase

Visual

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
Addition (N=15)		27	53	13	7
Correct Responses		3 of 4	3 of 8	1 of 2	None
Subtraction (N=13)		77	8	13	15
Correct Responses		None	None	1 of 2	None

Verbal

	Automatic	Abstract	Perceptual	Verbal Counting	Figural
Addition (N=14)		36	36	28	0
Correct Responses		4 of 5	None	None	None
Subtraction (N=19)		37	37	26	0
Correct Responses		2 of 7	2 of 7	None	None

Table 8.5: 'Low achievers'' representations and associated success with two-digit combinations

The proportion of abstract representations used to respond to the visual subtraction problems is no surprise. It is conjectured that children who focus on surface characteristics of numbers, so that, for example, 26 is seen as a two and a six, feel that they are doing little more than elementary number combinations to 10, which are combinations that they already know how to deal with. Of course, this leads to the 'smaller from larger' errors wherever some form of decomposition is required. For example:

"Two twos is four, is half, two. Two from eight is six." [Answer 62] (Y4-, $\begin{array}{r}82\\-24\end{array}$)

Abstract representations amongst the 'low achievers' tended to be more successful if used with an accumulation strategy based on sequencing in 10s and with verbal counting.

"47 and 10 is 57, then 58, 59, 60, 61, 62" (Y5-, visual, $\begin{array}{r}47\\+15\end{array}$

"60, 70, 80, and 9 is 89" (Y5-, verbal, 27 + 26)

The category 'perceptual representation' identifies those responses which involved the use of fingers associated with counting in ones. The representation gave a basis for some success for the visual addition, partly since the numbers were small, but no success for verbal addition unless accompanied by some sort of transformation and verbal counting. There remained procedural difficulties, however, as exemplified by the following vignette from a Y5 child. He used the fingers of both hands to add on 10 ones in two cycles until 20 was added, giving 82. Two more are then added with a final leap of 10 to 94.

"27 + 62: 63, 64, 65, ...71, 72;... 73, 74, 75,... 81,82... 83, 84, 94." (Y5-, verbal, 27 + 62)

It is difficult to ascertain the reasons for this error but perhaps some clues to the difficulty of counting emerge in the child's difficulties with a subtraction combination:

"I try to count back but I can't, even when I use my fingers." (Y5-, verbal, 64 – 26)

8.2.3.4 Modalities Associated with Two Digit Combinations

The distribution of the modalities identified in the two-digit phases is seen in Figure 8.2. The number of responses identifying modalities is seen as N. The totals are different to those expected for high achievers (N=16 in all instances apart from the verbal subtraction where N=24) and for 'low achievers' (as given in Table 8.5) because in some instances dual modalities, for example visual/verbal were identified. It is also worth noting that amongst the 'high achievers', it was frequently unclear which modality they worked in, particularly when the external stimuli was visual.

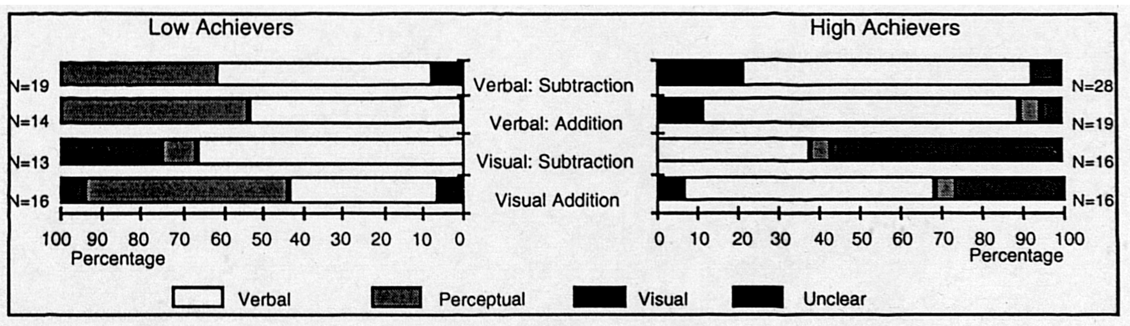

Figure 8.2: Distribution of modalities in each phase of the two digit combinations

Perceptual and verbal modalities were evident amongst the 'low achievers', but the perceptual modalities were frequently inappropriately used. There were strong suggestions that verbal modalities placed an excessive load on working memory in the verbal phase with the result that the children were frequently unable to remember the combination that they were handling.

Verbal modalities occurred with considerable frequency amongst 'high achievers' and in over one third of instances was associated with the visual modality. Children referred to 'seeing' and 'saying' and were able to make a distinction between the two:

> "I forgot the four and the six (ones) and I did 60-20=40 I knew that. I then forgot the 40. I then said four take away six that gave two to take away from 40 which is 38. I could see these (the four and six) in my mind but I wasn't using them. The things I used I couldn't see. When I used the 4 and the six and could see the 40." (Y3+, verbal, 64-26)

> "Saw 64 and 26 vertically. I did 64-20 and 44-6 but do not remember seeing these. I knew the answers to both of these." (Y6+, verbal, 64-26)

The verbal modality was used most frequently whenever 'high achievers' performed transformations on the quantities to be combined. Always preceded by the words, "I said to myself...", 100% of the solutions in the verbal phase (31 out of a possible 40) associated with transformations were obtained in the verbal modality, one-quarter of these combined with the visual modality.

8.2.3.5 Two digit combinations: A Summary

During the two-digit phase those differences apparent when children were dealing with the number combinations to 20 were manifested in strategies associated with mental representations which terminated in failure for 'low achievers' and success for 'high achievers'.

The strategies of 'low achievers' were largely dependent on perceptual representations and/or verbal counting in either vocal or sub-vocal forms. The overwhelming use of transformations by the 'high achievers' not only provided success but also reflected 'proceptual' and 'generic' characteristics which enabled them to escape from the reliance on one form of transformation. The modalities that accompanied their approaches permitted some to oscillate between seeing symbols and thinking about the solution. It adds further evidence to the conjecture that they were able to use visual symbolic images to refresh memory. 'Low achievers' seemed to use them as mental countable objects. It is a distinction that enabled 'high achievers', particularly in the verbal context, to approach the difficulties associated with three digit numbers in a positive and relational way.

8.2.4 Strategies and Representation: Three Digit Combinations.

When the two groups attempted to obtain answers to the verbal combinations, differences that had emerged in resolving the two digit combinations continued. 'Low achievers' continued to use perceptual items and verbal counting and in only one instance was a transformation approach used. The result was that the conflict between trying to remember the input, seeking an appropriate method (based substantially on the episodic approaches previously identified and specific items such as fingers) and attempting to carry out the procedure was too much:

> "You have to add that sort on your fingers but you don't have enough." (Y4-, 188 + 267)
>
> "I was trying to count up on my fingers." (Y5-, 188 + 267)
>
> "First I see 396, then I see 196 then it is all mixed up. The numbers are black and it is all white around there." (Y5-, 396 – 157)

Transformation approaches once were used by all 'high achievers' at least once but there was evidence that a mental representation associated with a standard paper and pencil method was also used:

> "I set it out as a sum in my head and did as a take away sum in the head." (Y5+, 396-157)

Every 'high achiever' in Y4, Y5 and Y6 completed every combination in using a verbal modality, four of them also reporting seeing symbolic representations in their mind at the same time. None used perceptual items.

The relationship between visual and verbal mental representations was one that clearly gave individuals the power to complete the combinations

> "I was working hard to remember the sum - but that got in the way of doing it. I saw 396 in my head and then saw 157. I saw some working out — that came and went because the numbers in the sum kept flashing on and off. When they flashed they changed bit. I tried to remember what flashed but it was hard. If I didn't see it, I thought it. Once I had done the units — which were nine — I had to remember nine 10s was really eight 10s"
>
> (Y4+, verbal, 396-157)

However, such an approach may not have been as economical as that used by children who appeared to be making transformations as they heard the problem. They did not seem to need to try and remember the separate components of the problem:

> "108, said 185+100 is 285 285+8=293." (Y6+, verbal, 293-185)

> "100 and 200 and the 300 and 70 , 370 and then add so equals 4 and 20 and 9 equals 433. I just decided to do it like that." (Y3+, verbal, 274+159)

In such instances no visual symbolic images were maintained. Everything was processed in a truly abstract fashion. However, these approaches were not efforts to replicate 'pencil and paper' methods. The oldest child not only transformed the numbers but also changed a subtraction question to an addition one which simplified the approach. However, it is hypothesised that when they tried to replicate the 'pencil and paper' methods mentally, the left-to-right approach, as illustrated above, is a more economical way to deal with the problem. The 'number word' associated with the answer is already being formed. The right-to-left approach implies that each component of the number (units, tens, hundreds etc.) is considered and named separately. Each needs to be held in memory as ones, tens or hundreds, added if necessary to the value on the left and then eventually renamed by returning back from left to right. Such a task can be quite demanding on working memory capacity.

> "What was it ? 300...? I got somewhere but lost it. 400 and was it 266? I think I got lost again. 435 - I was doing 247. I was doing one part of the sum but then I was losing the other part of the question. I decided 100+200 then the eight from 88 and the 20 from 60 that was 400. The eight from eight and seven... as I was doing it it was as if I split my mind into two to do it. I have two bits. One is remembering and the other one is talking. So the talking is working it out and then I say to the remembering part what I want to remember?" (Y4+, 188+267)

Though the above explanation is again an excellent example of Baddeley's (1986) model of working memory, such problems did not exist with the real 'pencil and paper' approaches.

8.2.4.2 Paper and Pencil Approaches to the Three-Digit Combinations

Standard algorithms were the usual approach of the 'high achievers' to solve the visually presented items although there were differences between their occurrence in subtraction (nearly four out of five instances) and in addition (one out of three). In nearly half of the instances transformation or left-to-right algorithms were used as an alternative. Amongst the 'low achievers' a mixture of left-to-right and right-to-left approaches were used and although this mixture was relatively successful in addition, it caused difficulties in subtraction, particularly where exchange was required.

Whilst every 'high achiever' used symbols and their relative values for calculation, every 'low achiever' used fingers at some point to support calculation of the separate parts. Only three errors occurred amongst the 'high achievers' whilst only one 'low achiever', a Y6 child, obtained correct solutions for the subtraction combinations. None in Y3 had any correct solutions.

Errors took the form that became apparent when the children were dealing with the two-digit combinations and could be associated with place value for addition and smaller from larger errors in subtraction. One of the 'high achievers', who once displayed this error recognised it and offered his explanation:

> "Did it the basic way - (smallest from largest). I think I know what is wrong - I haven't done those "jumping things" - going to the next "door" or "house". You have to go to the next number - you cross that out and take one off. Then you add 1 to the other number."
>
> (Y5+, 293 – 185)

8.3 CHAPTER SUMMARY

In this chapter we have considered the way in which children at different levels of achievement have approached arithmetical combinations at three levels of difficulty. It confirms the existence of qualitative differences in thinking (Gray & Tall, 1994) between 'low achievers'' over-emphasis of the routine application of counting procedures and 'high achievers'' use of the flexibility associated with numerical procepts.

The benefits of this more flexible form of thinking are clearly seen as the level of difficulty associated with the combinations increases. The indications are that a predisposition towards qualitatively different mental representations may trigger these differences. Using transformation approaches, 'high achievers' appear to oscillate between visual and verbal modalities which utilise abstract representations associated with the use of symbols in order to generate thinking and/or retrieve appropriate relationships. 'Low achievers', most frequently use perceptual representations and/or vocal or sub-vocal counting, use lengthy counting procedures as if they were generalisable processes. As the level of difficulty associated with the combinations increases these end in an inability to generate a suitable procedure or in the procedural inefficiency which causes errors.

The symbolic form of representation featured strongly amongst the 'high achievers' as was demonstrated by the occurrence of 'abstract' mental representations. Perceptual representations featured most strongly amongst the 'low achievers' and this may have been associated with vocal or sub-vocal counting.

However, an important issue for this study is not whether or not children operate their arithmetic in visual or verbal modality but the nature of the qualities associated with their mental representations. Though a proceptual divide is apparent in the qualitative differences associated with the children's arithmetic, it is conjectured that children's predisposition towards qualitatively different kinds of mental representation is important in the formation of this divide. Though modality appears to be important it does not answer fundamental questions about the proceptual divide. Of course we may see distinctions in the use of perceptual items, but distinctions between visual and verbal mental representations are less clear.

'Generic' and 'proceptual' kinds of mental representation may be more easily associated with the arithmetic that the 'high achievers' are doing. After all, the notion of 'proceptual' mental representation is characterised by those very qualities that are identified as proceptual. These are also generic in the sense that they do not need to be specific or re-enacted but rely instead upon relational qualities and the selection of appropriate general components. But how may procedural approaches to elementary arithmetic be linked to the dominant mental representations projected by the 'low achiever'? To answer this we need to look at counting from a representational perspective.

The counting number sequence may be viewed as a general mental representation that is then sequentially acted out like an episode. The procedure with which this sequential episode is re-enacted (count all, count-on, count-back) reflects the child's familiarity and competence, their personal preference (see Gray 1991) as well as the specific number combination. In the application of any procedure it is conjectured that for any arithmetical combination the general number sequence will be specified in two ways: (a) the number that counting should start and end at, (this could be one, it could be the largest quantity), and (b) the objects that will be manipulated. These objects may be verbal number names, symbols arranged in different forms, an analogue of dots or fingers or perceptual items. The transformation of numerical symbols to countable objects may be specified depending on how heavily the child leans towards perceptual representation. The more they depend on perceptual objects the more concrete their mental objects. It seems that children who are heavily relying on the counting episode have not 'encapsulated' the concept of sum as it is associated with a particular combination with the result every time they meet the combination they have to activate a procedure

Therefore it is conjectured that procedural thinkers retrieve the general number sequence and re-enact the counting episode based on the specifications formed from the problem. It may be argued this style of approach may be strongly associated with the 'episodic' and 'specific' mental representations that we saw emerging from the same children's interaction with the objects that formed the basis for discussion in Chapter 7.

This 're-enactment' of process approach contrasts starkly with the 'search and retrieve' approach used by 'high achievers'. This would appear to be a quality of thinking that is more associated with selective spontaneity and not the re-enactment of an episode that needs to be specified.

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CHAPTER 9
INDICATIVE CASE STUDIES

"I wanted to be like the clever children" (Child, Y4–)

9.1 INTRODUCTION

9.1.1 The Nature of the Case Studies

The intention of this study was to identify the different kinds of mental representation projected by children who are at the extremes of mathematical achievement. It was hypothesised that qualitative differences in mathematical behaviour may have direct links with qualitative differences in the kinds of mental representation that the two different groups of children project.

The purpose if this chapter is to consider how the observations identified within Chapter 7 and 8 may be associated with children of particular levels of achievement. The notion of 'indicative' case study is simply a mechanism for illustrating in an uninterrupted fashion the way in which 'real children' respond to both the free context and the arithmetical components of the study. 'Case study' may be a misnomer since it is the intention to provide a flavour of the way in which patterns identified within the previous two chapters may be seen in particular children and it is not to be seen as the description of a detailed interactive investigation.

The development will be tackled by first contextualising the children in terms of numerical achievement. It will then consider the kinds of mental representation portrayed by the individual and relate these to their approaches in the numerical component. In general terms the 'case studies' support the thesis of the study, namely that children's qualitative differences in thinking associated with achievement in elementary arithmetic are associated with the qualitatively different kinds of mental representation they portray. However, they will also illustrate some of the grey areas which may only be understood with reference to the broader picture. Therefore, in some instances, reference will be made to other children within the study. The chapter will conclude by considering the way in which mental representations may be linked with the proceptual divide.

9.1.2 'Typical' Children

Four children are selected to provide the basis for the discussion. One from Year 3 who is a 'low achiever' and one from Year 6 who is a 'high achiever'. These two children will demonstrate the two extremes of the sample of children considered in this study, both in respect of age and 'level of achievement'. The other two children are taken from the same year group almost at the middle of our age range, Y4, but they reflect the two extremes of achievement within one year. This selection aims to illustrate the extreme patterns of behaviour of children at the opposite extremes of age and ability as well as the extremes in the level of achievement found in children of the same age. Through such snap shots we may get a sense of where the children may be going at a later stage. All children had spent their full school careers at the same school and the children from within the same year group had been in the same class throughout and had therefore experiences of the same teachers.

9.2 CHILDREN AT THE EXTREMES

9.2.1 Natalie –Year 3 'Low Achiever'

Natalie was considered to be the worst at arithmetic in her class of 32 children. Within the arithmetical phase of this study she achieved the lowest percentage mark (33%). Her weakest component was the one with the verbally presented items where she gave 2 correct responses out of a possible 14. In the visual element of the same category she obtained 10 correct responses out of a possible 12, addition being her more successful operation (seven out of seven). She also solved correctly one two digit combination within the visual-addition category.

Overall then, we see that Natalie performed better at the visual component than the verbal, but her success in the arithmetic component was restricted to combinations below 20. She did not attempt to provide answers for combinations in excess of 20 and only in the visual phase did she attempt the addition combinations. She only recalled one fact (6 + 3, visual), and there was no attempt to derive a solution. She was totally procedural, relied extensively upon counting, including count-all, but often did not have the procedural competence which would have allowed her to obtain a correct solution.

During the verbal phase of the study Natalie's mental representations were strongly dominated by 'general', 'specific' and 'episodic' kinds which occurred with equal frequency. (Natalie did not respond to three of the numerical items, 'three quarters' 'nought point seven five' and 'fraction'. 'Three eighths' was misinterpreted as 'three and eight').'Specific' mental representations were projected for every item in the visual phase, 'general' representations were only projected in two instances. This time there were no responses recorded as 'not known'. The equal distribution of 'general' 'specific' and 'episodic' mental representations during the verbal phase took very little account of whether the item was non-numerical or numerical. Thus non-numerical items were described:

> "It can be all different colours, black, white and brown" (Dog, specific, Verbal Ph., Res 2)[1]

or associated with some event

> Lots of animals crowded in fields walking about with each other playing. Little giraffes are in field eating grass. They are crowded in a big field in a circle. I've got a cuddly toy seal it is my favourite. (Animal, episodic, Verbal Ph., Res. 2)

However, there was frequently an additional characteristic within Natalie's responses — she would often refer to how the objects are used:

> It's a circle, You can play tennis with a ball and cricket with a ball. Boys play cricket with ball you can throw it and it can get stuck on a tree." (Ball, episodic, Verbal Ph., Res. 2)

[1]The items in the parenthesis are indicating the stimulus (dog), the classification that the response got (specific), the phase in which this response was given (Verbal Ph.). If it was given as an initial response this will be classified as Res. 1, if it was given as a 30 seconds response this will be classified as Res.2, if it was given as a response to the question of 'what was in the head as you were talking' this will be classified as Visual image, and if it was a response to ET it will be classified as ET.

The visual items prompted Natalie to project 'specific' and/or 'general' mental representations in 100% of the instances where she responded. 'General' mental representations figured fairly extensively in her first response, six items out of ten, but 'specific' mental representations identified were projected during every second response.

Mental representations projected for the numerical items were qualitatively similar to those projected for non-numerical items. During the verbal phase descriptive mental representations of whole numbers gave a sense of what a symbol may look like:

> "It's like a circle in the bottom. Written they can be different." (Five, specific, Verbal Ph., Res. 1)
>
> "They are round... and shaped." (Ninety nine, specific, Verbal Ph., Res. 1)

'Episodic' mental representations contextualised the numbers:

> "7 is an easy number to work with." (Thirty three, episodic, Verbal Ph., Res. 1)

Though they drew no response in the verbal phase the items 'half', 'three quarters' and 'nought point seven five' caused Natalie to talk about fragmented parts of the symbols. Each part carried its own meaning, which was usually related to whole number arithmetic. In some instances Natalie mentally moved these symbols around to create a new symbol.

> "It's 12 and it's got 2 numbers in. One number is 1 and the other is 2."
> ($\frac{1}{2}$, specific, Visual Ph., Res. 2)
>
> "It is 34... two numbers... 3 and 4. Its like tens and units."($\frac{3}{4}$, specific, Visual Ph., Res. 2)

It is interesting to note that Natalie created new objects from those that were given. Moving the symbols around enabled her to talk about things that fitted her perception of what numbers meant.

Only on one occasion could it be claimed that Natalie brought some insight to her response to a visually presented item:

> "Its a number. Its five again but is in spots and you can have it any way you like and it would still be 5."
> (Dots, specific, Visual Ph., Res. 2)

However, given the context of her other responses it could be interpreted that Natalie is seeing five dots in the same way she views the number symbol '5'. It is conjectured that, for her, this is a more useful representation of five since it represents more graphically objects that could be used in a counting procedure. It was this approach that dominated her arithmetical phase.

During the numerical phases of the study Natalie's mental representations were once again strongly associated with the 'episodic' and 'specific' kinds of mental representation. The episodic content was always associated with counting, the specific content with either perceptual counting or the simultaneous engagement of perceptual counting and a mental analogue of a number line.

Perceptual counting was always used with verbally presented items. Her procedures worked successfully for numbers below ten, for example 3+2 and 9–7, but failed with numbers larger than ten. For example, she used the procedure for 7 + 6, first counting the seven and then trying to count-on six. When she reached ten she seemed to have no mechanism for continuing the count and simply gave the answer 11. As she explained when attempting 4 + 7:

> I just knew it was 12. I didn't have enough fingers to do it... I just knew it.

For 17 – 13 she just explained:

> "This is a tricky one–I don't know how to do it. It has completely gone."

A different picture emerged for the visually presented items. First she knew 6+3 explaining that "*My voice told me in my head*". For the other combinations she referred to the simultaneous engagement of a mental analogue of a number line. In some instances there were no overt actions identified and Natalie claimed she used the number line to solve the combinations. For example:

> "I saw a number line which started at one and went two, three, four, ... ten, eleven. I started at one and counted to five. Then I counted six, seven. The number line went away and the seven stayed with an equal sign." (Natalie, 5+2, visual[2])

A similar procedure was used for 4+5 but the number line in Natalie's head extended to fourteen. There was no claim about seeing the equal sign this time. Interestingly this simultaneous engagement, which always provided more detail than was necessary, seemed to increase her competence in the use of fingers. However, during the verbal phase it failed at the first hurdle (3+2) and was not subsequently attempted.

> "I couldn't work it out with my number line so I used fingers." (Natalie, 3+2, verbal)

Natalie's references to a number line in both the free context situations and in the arithmetical situations would seem to support Drake's (1996) claims that the mental image gives traces of what the memory holds about a concept. However, her continued reference to it in the numerical-visual phase and almost complete absence of any reference to it during the numerical-verbal phase suggests that either:

- during the visual phase her memory trace became fixed on a mental analogue and during the verbal phase on perceptual items or,
- easier items could be completed by evoking figural representations, harder ones could only be successfully completed by evoking perceptual representations. Since on balance the visual-numeric items were easier than the verbally presented ones, this would seem to fit observations within the pilot study.

9.2.2 Jeremy – Year 6 'High Achiever'

Jeremy had an outstanding record of results, 42 correct responses out of the 44 questions. Selected on the basis of his predicated SAT results in 1996 he was finally awarded Level 5 with an overall mark of 75, the second highest in the class.

[2]Reference of children's quotes indicate the name of the child, the arithmetical combination that they were asked. The words visual and verbal denote the modality in which the question was given to the children.

A classification of his strategies in the arithmetical component of these tests indicated that based upon Gray and Tall's (1994) analysis he was clearly a proceptual thinker. 41 of the 44 number combinations that he completed in this study were solved through the use of 'know facts', 'derived facts' or 'transformation' strategies. This included the 'paper and pencil' component where only two combinations were clearly identified as being solved by a standard algorithm, both addition.

An indication of his powerful and flexible thinking was given by Jeremy when asked verbally 293-185:

> "Said 185+100 is 285, 285+8=293." (Jeremy, $\begin{array}{r}293\\-185\end{array}$, visual)

It was common for him to handle two and three digit numbers in either the visual or the verbal phase by operating similar left to right transformations. Even when he completed written combinations using standard algorithmic approaches nothing was written until the final answer was obtained and then written left to right.

When looking at his claims on mental representations we find Jeremy particularly interesting. Although he gave a high proportion of automatic response to number combinations below 20, Jeremy often claimed that he saw symbolic images – the symbols flashed through his mind:

> "7 6 flashed and 13 saw strongest." (Jeremy, 7+6, verbal)

His description seems to offer some support to what Kosslyn's describes to be happening in the spatial medium in his computational model of imagery:

> "As soon as an image is generated in the medium it begins to fade and so, if the image is to be maintained in the medium, it needs to be regenerated or refreshed. A similar type of fading occurs with after-images in the visual system. When we look at bright lights and then close our eyes we see after images caused by the over-stimulation of our retinal cells. Although these after-images are not the same as visual images, they have this same quality of rapidly fading after they first appear." (Eysenck & Keane, p.222)

Other children described a similar effect:

"I was seeing the number and then it sort of went and faded away."

(Y6+, $\frac{1}{2}$, Visual Ph., Visual image)

"I see the shapes and the colours, they came close to my face, and then one in the middle started flashing and then it went out like a light bulb or something"

(Y5-, honeycomb, Visual Ph., Visual image)

"Mind kept showing the picture I looked at. When I talked about one part it went then it came back again" (Y6+, marbles, Visual Ph., Visual image)

Seeing visual images was not typical of the 'high achievers'. In most instances they did not see anything in their mind. However, when they did see symbolic images they were often of the form described by Jeremy, quick flashes of mathematical symbols. As we have seen, on the whole 'high achievers' visualised on average no more and no less than 'low achievers'. What was different was the quality of these visualisations. In Jeremy's case these symbolic images could occur at any time during the solution phase. For combinations associated with known facts the symbolic images could be the input, the output or a combination of these:

"Saw a picture of 3+5 in black . Saw 8 next 3+5 8 black. Said 8 in my head first then spoke it out." (Jeremy, visual, 3+5)

"Saw 9, flash not as sum but just 9 and 7. 2 I saw strongest." (Jeremy, verbal, 9-7,)

"Saw 45 and 57. This disappeared to see 102." (Jeremy, verbal, 45 +57)

With 'derived facts' or 'transformations' he saw the transformations:

"Saw 13-3 then 10. Then I saw 10-2 this stood out. Then I saw 8." (Jeremy, 13–5, visual)

At no time did Jeremy see or use a number analogue to actively count on. His visual images seemed to be economical, triggered thinking, a form of 'note' that refreshed his memory and assisted transformations. It could be argued that Jeremy is an example of a child using the "visual sketch pad" (likely to be what Kosslyn calls spatial medium) in the solution of his questions. His symbolic images seem to have skeletal qualities that allow him to hold information in his working memory.

There is a complete absence of procedural methods associated with counting and there is no evidence of use of external representations, figural items or verbal counting. Verbal enunciation was sometimes associated with visual images of numerical symbols that flashed (22/57). These symbolic images were either the expressions themselves or the final solution.

As combinations become more difficult we see that Jeremy's mental representations of a visual kind are being associated with mental representations of a verbal kind. Whilst the greater majority of number combinations below twenty were associated with visual mental representations no matter which modality introduced them, as Jeremy solved the mental combinations above twenty, the verbal phase, visual mental representations were associated with verbal ones:

> "I saw 64 and 26 vertically and did 64-20 and 44-6 but do not remember seeing anything. I said 60 take 20. I may have said 44 take away 6 and said 38, but I know both answers."
>
> (Jeremy, 64-26, verbal)

None of the visually presented two and three digit numbers were associated with visual mental representations but all were associated with verbal mental representations. This would suggest that the symbolic images are assisting memory. When symbols remain in front of him, as they did with the 'paper and pencil' this sort of support may not be needed.

Jeremy's obvious flexibility with numbers was not only evident in the arithmetical context questions but also in the free context questions. During Phase 1 and Phase 2 of the study most frequently gave "general", "generic" and "proceptual" responses. He projected 'general' mental representations for half of the non-numerical items in the verbal phase, whilst 'generic' and 'proceptual' mental representations were projected for 8 out of ten items in the numerical phase. In the visual phase every numerical item was associated with 'generic' or 'proceptual' mental representations, the latter being more frequent than the former.

In almost half of the instances his "generic" or "proceptual" responses would grow out of a first 'general' representation. What was even more impressive was that in some cases the first mathematical symbol that would come to his mind, when asked the question, would be of a mathematical equivalence rather than the object itself. For example:

"Half way between 0-10." (five, Verbal Ph., Res. 1)

"Nought point seven five." (Three quarters, Verbal Ph., Res. 1)

"Nought point three seven five." (Three eighths, Verbal Ph., Res. 1).

"Three quarters." (Nought point seven five, Verbal Ph., Res. 1)

"Eleven (first number I can think of that goes into 99)." (Ninety nine, Verbal Ph., Res. 1)

In four out of five of the above instances Jeremy claimed to see a visual mental representation of the object that prompted the response. These responses add strength to the conjecture that these mathematical symbols were acting as "skeletons". It was as if he was starting from these symbols and was pinning related ideas on them that resulted to a larger schema of relationships. This idea of the mathematical symbol acting as a "skeletal images" is strengthened by Jeremy's claims that he was seeing these symbols and at the same time he was thinking:

"(Saw a) picture of 99 and kept thinking about it." (99, Visual Ph., Visual Image)

"Saw 3/4 and just said things about it–like it is a fraction, it doesn't come out as a whole."
(3÷4, Visual Ph., Visual Image)

Of course Jeremy did not content himself to forming only a "skeletal image" of the object, that was often just the beginning. The complete mental representation or series of mental representations created during the 30 second responses, were formed from a collection of relationships and equivalencies; the collection was named and symbolised and then associated with other collections. This was achieved by giving more "generic" and "proceptual" responses:

"33 is another low number. It is divisible by 11 and 3. It's almost a 1/3 of 100 and times by 3 you get 1 less than 100." (33, Verbal Ph., Res. 2)

"1/2 of one and a half. Its one and a half halves ... its a low number and it is under one – it is a figure." (3/4, Verbal Ph., Res. 2)

> "3/8 is another low number. If you times it by three you get 1 and 1/8. 3/8 can be written decimal or fraction or word... 0.375, 3/8 , three eighths." (3/8, Verbal Ph., Res.2)

> "3/4 is another fraction. Can also be a decimal and like 0.75. Not a whole number, not more than one. Multiply by 1 and add one third of it and you get one." (3/4, Visual Ph., Res. 2)

It is interesting that when talking to ET the responses for the same items became more simplistic. It might have been the case that Jeremy, was trying to make things easier for ET since as we had explained to the children he "did not know anything about these objects". Consequently there is an increase in "episodic" and "specific responses" to this question :

> It is a number with 2 digits...both the same... almost the very last number with two digits....9 tens 9 unit. (99, Verbal Ph., ET)

> Its a fraction of a whole thing...if you divide by two you get half. Half times a half is a quarter. (half, Verbal Ph., ET)

His ability to create 'general', 'generic' and 'proceptual' mental representations for the arithmetical objects carries a lot of similarities with the style of responses he gives to the non-arithmetical words. Of course the only group of mental representations that does not exist for non-mathematical words are the "proceptual" ones because they apply only to the mathematical ones. However, the other trends are the same. He tends to start with a 'general' mental representation and then attaches other 'generic' ideas to that. His 'general' are minimalistic; they are 'skeletal', and carry few surface characteristics.

> "The heart beating." (animal, Verbal Ph., Res. 1)

> "Like a blob." (dots, Verbal Ph. , Res. 1)

It is very interesting that when shown the icons which had mathematical undertones Jeremy was often referring to a possible mathematical meaning by "looking through" these icons and not so much by "looking at" them:

> "A half." (window, Visual Ph., Res. 1)

> "An oval in the middle of the paper. Balls inside and balls outside. 3/4 inside and 2 or 3 outside." (marbles, Visual Ph., Res. 2)

It seems however, that although Jeremy would give these insightful responses as both first response and as a response to the 30 second question, in his mind he had a visual image of the icon and he was talking about the meaning he was attaching to the visual image:

> "I saw the shape with four different parts. Saw the two black parts on outside and started talking about it." (window, Jeremy, Visual Ph., Visual Image)
>
> "My mind kept showing the picture I looked at. When I talked about one part of it, it went and then it came back again." (marbles, Visual Ph., Visual Image)
>
> "Saw the picture of the pattern. I kept seeing it all of the time and kept saying things about it, like it had hexagons." (honeycomb, Visual Ph., Visual Image)

CHILDREN WITHIN THE SAME YEAR

After having considered two children who are at the extreme in achievement and in age, two children at the extremes in achievement but within the same year group are now considered. It is not the intention to provide the same depth of 'profile' but simply to provide a flavour of the differences in the mental representations projected by the children, to illustrate the similarities and differences that may exist between these two and the two already considered and to suggest a prognosis for future growth.

Of the two children Sonia obtained an overall percentage score of 41% in the arithmetical component of this study. She provided no correct responses to the visually and verbally presented two and three digit combinations but obtained correct solutions to and no correct responses to the verbal two digit but she obtained correct responses to a greater proportion of the combinations to twenty. Malcolm on the other hand obtained correct responses to every combination presented.

9.3.1 Sonia – Year 4 'Low Achiever'

Sonia's mental representations in both the visual and verbal phase were strongly dominated by 'general', 'episodic' and 'specific' mental representations. During the verbal phase 'specific' mental representations were projected for at least one in two of the items that were known (three of the numerical items, 'three quarters', 'nought point seven five' and 'fraction'. 'Three eighths' was misinterpreted as 'three eight's'. It is remarkable how similar this is to Natalie). Mental representations projected for the numerical component of the visual phase were almost all like Natalie's, that is 'specific', whilst once again there were no 'not known' in this phase. All of the pictures except one were associated with 'episodic' mental representations, half of them also with 'specific' mental representations. She projected the same proportion of mental representations when considering the icons but 'specific' mental representations dominated her associations with the numerical items.

Thus non-numerical items of the verbal phase were frequently described:

> "There are small ones and massive. Some are tiny and some are big."
>
> (Animal, specific, Verbal Ph., Res. 2)

or put into an episode:

> "Kick it... kick it anywhere you like and... you can play football with some players and you can kick it around. That's all. (saw people kicking the ball and loads of people saying 'Goal!' because the Blues scored a goal)." (Ball, Verbal Ph., Res 2 and visual image)

During the visual phase she would often expand on this 'episodic' mental representation to offer additional comments that would make the whole episode into an imaginary story full of detail and colour:

> "Windows are made of glass and you see a lot of things out of them, trees, grass, sometimes you see animals." (window, episodic, Visual Ph., Res. 2)

Mental representations projected for numerical items were qualitatively similar to those of the non numerical items. So Sonia would describe what a symbol looked like:

> "Is like a squiggle line and when put 2 fives together is 55."
>
> (55, specific, Visual Ph., Res. 2)

In some instances these symbols would be moved around and this could possibly be the source of confusion as well as a departure from the object of discussion:

> "It's two number like 1 and 2. I don't know why that line is there but it is just there. It could be 12 or 21." ($\frac{1}{2}$, Visual Ph., specific, Res. 2)

Earlier we saw that Natalie gave almost an identical mental representation. Both of these children referred to fragmented parts of the symbol that were moved around and the result was eventually a new symbol.

In both the visual and the verbal phases Sonia gave more episodic responses than Natalie. She would often place the number in an episode which she would then talk about:

> "Like a clock is like a clock, 0, 7, 5, I don't know what that is."
>
> (0.75, Visual Ph., episodic, Res. 1)

> "If your birthday is 7, if you are 6 and your birthday is tomorrow you will be 7... 6 comes first... then 7." (7, Verbal Ph., episodic, Res. 2)

In some instances she would place the number in the number sequence episode.

> "99 is a number. It comes before 100 and after 98... and I don't know ..."
>
> (99, Verbal Ph., episodic, Res. 2)

In the solution of her arithmetical sums Sonia relied extensively on counting procedures. Out of 29 combinations she attempted, she used some form of counting procedure for 25 of them. These included count-on, count-all, count-back and count-back-to. Counting episodes were always supported with the use of perceptual items (fingers). This procedural use started to lead to difficulties with combinations in the teens although 3+2 in the verbal phase and 5+2 in the visual phase both caused procedural difficulties. However, subtraction caused particular problems:

> "Now I've lost track. 12, 11, 10, 9, 8, 7, 6, 5, 4, 4. This is the easy way [Counting back on her fingers]." (12-8, verbal)

When she was given sums to 100 she could not deal with them and in one instance she said:

> "I don't know that, my dad might know it but not me. You have to put 73 fingers on."
>
> (73-32, verbal)

In some instances perceptual counting was replaced by verbal counting but this was always associated with small combinations presented visually. No verbal counting was associated with verbally presented items. In each case she almost instantly translated verbal symbols into perceptual items.

9.3.2 Malcolm – Year 4 'High Achiever'

Malcolm, the 'high achiever' in Year 4, obtained correct solutions to every number combination presented. However, unlike Jeremy he made considerably less use of derived facts and transformations and in one instance did use counting to help with a two digit combination. Frequently the modality Malcolm used to obtain solutions was unclear but in general the number combinations to 20 presented visually drew automatic responses. Those presented verbally were unclear but at times two digit combinations were associated with verbal mental representations. Seldom were any combinations associated with visual mental representations.

The verbal phase was characterised by 'general', 'generic' and 'proceptual' mental representations. 'Generic' mental representations projected for half of the non-numerical items. The numeric items apart from 99, evoked either 'generic' or 'proceptual' mental representations. The latter, given in half of the instances, were identified less than in Jeremy's case (four out of five instances).

In the visual phase numeric items only evoked one 'proceptual' mental representation and two 'generic' ones. 'General' forms of mental representation dominated the component. 'Specific' mental representations occurred with greatest frequency during the visual phase, all items except the numeric evoking this kind of mental representation. 'Episodic' mental representations were also strongly associated with the iconic items.

Though Malcolm seldom projected visual imagery when he did we see a good example of the way in which such imagery may be put aside to trigger other ideas. The 'generic' mental representation projected for the word 'animal' differed from the descriptive way in which the associated visual mental representation was described.

> "Some are older than others and they do different things, rabbits jump, goldfish swim, tigers run. Some animals are vegetarians, some are predators and cannibals. They all live in different climates and different habitats. The cannibals hunt to eat." (Animal, generic, Verbal Ph., Res.2)

> "I picture climates and see the animals like, tigers and see how they hunt. I was seeing the animals chasing things. This time there was more detail and I could see what they were. For the tiger I could see black stripes on the creamy orange."(Animal, episodic, Verbal Ph., Visual image)

Visual mental representations played a part in Malcolm's attempt to deal with some of the computational items. When dealing with verbal presentations of two and three digit combinations he frequently reported seeing the combinations laid out as in the standard vertical form. The combination was then dealt with as a standard algorithm. At times he reported difficulty remembering the numbers.

> "I was working hard to remember the sum — that got in the way of doing the sum in my head 396, saw 157. saw working out — it came and went because the numbers in the sum kept flashing on and off. In the flashing they changed bit. I tried to remember what flashed but it was hard. Once I had done the units — which were 9, I had to remember 9 tens was really 8 tens." (396-157, verbal)

It was a feature of Malcolm's mental arithmetic that he tried few transformation approaches.

Verbal mental representations also played a part in enabling Malcolm to project his thoughts:

> "words were coming in my mind. I could feel them and sense them there and I was selecting the ones that had something to do with dog." (dog, Verbal Ph.)

> "My inner voice helped me ... the words became clear and I said them."
> (five, Verbal Ph.)

In a similar way many of the derived facts that Malcolm gave relied on verbal mental imagery:

> "I said to myself the difference between 6 and 7 is 1 so 2 x 6 = 12, 12 +1 =13."(7+6, verbal).
>
> "I threw away the ten from the sixteen and then thought of any ten because I am thinking of 4 and 8. I thought 'five is half of ten, take one from either of the fives that makes one of them six and the other four'. So 4+6 is ten and 4+8 is 12. Now I brought back the other ten to give 22." (14+8, verbal)

9.4 Mental Representations and the Proceptual Divide.

It has been conjectured that the qualitative differences that emerge from interpretations of mathematical symbolism lead to a proceptual divide. The implications of this study suggest that this divide may not only emerge because of interpretations placed upon symbolism but it may have its roots in the qualitative differences associated with a disposition towards qualitatively different kinds of mental representation. The way in which the child 'mentally represents' will influence how s(he) comes to 'know'.

The qualitative evidence suggests that mental representations may be either 'descriptive' or 'relational'. Those which are descriptive embrace those kinds of mental representation which project 'episodic' and 'specific' characteristics. Those which are relational project 'generic' and 'proceptual' characteristics. It is conjectured that the 'general' representation links the two in a pivotal way. The essential characteristic associated with this kind of mental representation is that of 'naming', 'categorising' or giving information about the structure or meaning without specific details. However, it is not naming that is important but the relational possibilities that exist from naming (Skemp, 1971). Therefore it is possible to see the pivotal role of the 'general' representation played out in two ways. It triggers the descriptive representations associated with the name, or it triggers relational links. This difference seems to clearly exist between the 'low achievers' and the 'high achievers' and it is conjectured that the predisposition towards the one or the other is manifest in the procedural and proceptual differences that are identified in the children considered in this chapter. Those predisposed towards the descriptive focus upon an arithmetic manifest through knowing what to do. Those predisposed towards mental representations of the relational kind have 'generic' and 'proceptual' platforms which support the abstraction of the intrinsic similarities that exist in numerical concepts.

However, the notion of predisposition may be interpreted in the same way as the two extremes of a spectrum. In this context nothing could be further from the truth. It is not regarded that any one kind of mental representation is better than another but simply that one form allows us to do things that maybe the other does not. 'Episodic' and 'specific' mental representations support the communication of descriptive elements. In global terms they provide something to talk about. In arithmetical terms it is conjectured that they generate things to do. The more relational mental representations provide the basis for recognising intrinsic qualities through which we may form connections between items. In arithmetical terms it is conjectured that these support the development of transformations that build upon these qualities to provide alternatives which support success.

It would therefore seem most advantageous to be able to form mental representations which are compatible in a descriptive and a relational way. At any moment one kind may be filtered out to enable us to concentrate on the other. It is conjectured that this is exactly what is happening when 'high achievers' deal with the arithmetic. They demonstrate that throughout the range of items that formed the focus for discussion their 'specific and 'episodic' mental representations were similar to those of the 'low achievers' but as things become more abstract they could offer more. 'Low achievers' do not appear to filter, they treat everything at the same level and in an arithmetical context the availability of interrelated kinds of mental representation does not seem to be available to them.

It is suggested that the tendency of 'low achievers' to project mental representations of an 'episodic' and 'specific' kind is one of the reasons why they concentrate upon the benefit of counting. This is episodic in that it is process driven and it is specific in that it is procedurally related and takes more specific and concrete appearance. The tendency to project 'episodic' mental representations will invoke the translation of numerals with conceptual potential into identifiable processes which are activated through 'specific' mental representations associated with particular procedures and particular metal or physical objects. 'High achievers' reject the re-enactment of the 'episodic' and the 'specific' in favour of the relational but if needs be, they can reactivate it in a way that may be qualitatively similar to that of the 'low achievers'.

The differences apparent in the children within this chapter would seem to highlight the differences evoked by a predisposition towards a limited range of mental representations as compared to that which embraces the broad spectrum. It would be consoling to interpret what we see within these children as various points along a line of cognitive development. For example, Natalie's strong emphasis upon 'specific' mental representation in the visual phase slowly giving way to more relational representations as she gains in experience and knowledge. Her somewhat cautious experiments with 'specific' and 'episodic' mental representations in the verbal phase providing a platform towards the high frequency of relational representations projected by Jeremy. But this does not happen!

In identifying these children it is claimed that they are typical in the sense that they fit the model presented in the summary of Chapter 8. Several other children may have been selected to project this typicality; Anne from Year 6, whose profile displays very little difference from that of Jeremy, apart from the fact that she does not use visual mental representations. We may have picked Chara, from Year 3, Orlando, from Year 4, Ioannis from Year 5, all of whom display those characteristics which have come to be identified within 'low achievers'. But some were not so typical. Gabriel in Y6, for example, displayed all of the typical characteristics of the 'low achiever' in the numerical contexts but in the free context items he provided responses identified as 'generic' which were actually repetitious but the net effect was that his profile was raised. We may have selected Katia, a 'high achiever', but she was not typical. She projected a very high proportion of 'generic' mental representations, no 'proceptual' ones and untypically of the 'high achievers' she used a high proportion of counting although very successfully. Perhaps there is a link there! Any theoretical model which tries to account for behaviours will be suspect at its fringes when linked to reality. This is no less the case with this one but the trends are clear and these are worthy of further consideration.

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CHAPTER 10
CONCLUSIONS

"Maybe other people don't think the same, but this is what I think." (Child, Y5+)

10.1 REFLECTIONS

Two studies, drawing upon children at the extremes of numerical achievement, were designed to consider the relationship between different kinds of mental representation and numerical achievement. The first, a pilot study, indicated that the mental representations of the 'high achievers' are established at a more impersonal level than those of the 'low achievers'. 'Low achievers' appear unable to detach themselves from a search for substance and physical identity. 'High achievers'' mental representations seem more attuned to temporarily ignoring substance and description to concentrate upon abstract and relational qualities. It was conjectured that these fundamental differences contribute towards divergence in elementary arithmetic.

The second study, the main study, focused on the issues raised by the pilot study. Drawing upon the psychological study of De Beni & Pazzaglia (1995) its purpose was to:

- use a modified version of their classification to identify children's predisposition towards particular kinds of mental representation and
- to associate any differences with the children's levels of numerical achievement.

In the event this became a wide ranging study which considered the different kinds of mental representation alongside children's approaches to elementary arithmetic and their predisposition towards particular modalities and particular forms of stimuli. Though using a different but analogous form of classification to that used in the pilot study the evidence from the main study suggests that:

- children identified as 'low achievers' in numerical achievement project mental representations which have descriptive emphasis and are classified as 'general', 'episodic' and 'specific' and

- those who are 'high achievers', whilst able to project mental representations of these kinds, also demonstrate those with relational characteristics, classified as 'generic' and 'proceptual', which provide them with the flexible interaction to move to a more abstract level of thought.

It is from such evidence that the outcome of the study confirms the original hypotheses that:

- children at extreme levels of numerical achievement project different kinds of mental representations,
- children project qualitatively different kinds of mental representation to support their thinking in elementary arithmetic and
- different kinds of mental representation are associated with qualitatively different kinds of arithmetical thinking.

and that the central thesis of the study:

> *The qualitatively different thinking that children display in elementary arithmetic is linked to their predisposition towards qualitatively different kinds of mental representation*,

is upheld.

It will not be the purpose of this chapter to reconsider all of the separate summaries found at the end of relevant sections but to attempt to draw together the central issues and consider their influence on other theories associated with children's cognitive development in elementary arithmetic. The chapter will do this within two main themes. First it will review the evidence from the study (Section 10.2) by considering qualitative differences associated with the children's approaches to elementary arithmetic and their dispositions towards different kinds of mental representations. It will then seek to draw these together (Section 10.2.3) before considering the special case of visual representations. Within Sections 10.3 and 10.4 it will review some of the relevant literature in the context of the results before placing an additional perspective on the limitations of the study (Section 10.5) and considering areas where future research would be fruitful (Section 10.6).

10.2 A REVIEW OF THE EVIDENCE

10.2.1 Qualitatively Different Approaches to Arithmetic

Selection of the children within both the pilot study and the main study was on the basis of their achievement in elementary arithmetic. In this the study followed the approach used by Gray (1991) and Gray & Tall (1994). Within the current study the qualitative differences noted in those earlier studies were apparent:

- 'low achievers' rely extensively on procedural approaches that make use of counting,
- counting procedures are almost non-existent amongst the 'high achievers',
- 'high achievers' extensively use derived facts to obtain solutions to number combinations between 10 and 20 and
- even when they have a fairly extensive repertoire of 'known facts', 'low achievers', apart from the oldest children, tend to make very limited use of derived facts.

These observations had direct influence on the children's approach to mental arithmetic with two digit numbers:

- as combinations became more difficult there was a steady increase in the use of counting procedures amongst the 'low achievers'. Manifest through the use of accumulation strategies these were relatively successful for small combinations but did not provide a sound platform for a competent level of achievement with the two-digit combinations and

- 'high achievers' made extensive use of the facts they knew to build the facts that they did not know. With larger combinations these were manifest in the use of transformation strategies, an approach not usually apparent amongst 'low achievers'.

Viewed from the standpoint of the theoretical construct of procept these differences in approach to elementary arithmetic indicate that on one hand we see a wide spectrum of performance in elementary arithmetic which is based upon the interpretation of operations on numbers as a procedure associated with counting. Differences in competency in this counting lead to different levels of achievement amongst 'low achievers'. On the other hand we see a smaller range of difference amongst 'high achievers' which is based on the flexible manipulation of numerical procepts. These 'strategic' differences between the two groups of children were manifest in automatic responses or the manipulation of abstract representations using symbolism by the 'high achievers' and with the extensive use of perceptual counting using fingers by the 'low achievers'. Thus we see that qualitatively different objects form the platform for qualitatively different thinking. Amongst 'low achievers' we see the tendency to use real objects to support their numerical procedures, amongst the 'high achievers' we see the use of a symbolic 'object of thought' which behaves as if it was real.

The qualitative differences that exist in the ways that the two groups of children think about elementary arithmetic when viewed from a proceptual standpoint illustrates the effect of a proceptual divide between the use of increasingly inflexible methods in which the arithmetic becomes more and more difficult, and the more flexible approaches which can lead to long-term success.

10.2.2 Qualitatively Different Mental Representation

The evidence obtained from the pilot suggested that children who are 'low achievers' in mathematics did not detach themselves from the search for concrete substance – no information is rejected, no surface feature filtered out. Their mental representations, focus on visual characteristics, parts of objects and actions with those objects. To put it in another way the children did not appear able to focus upon those intrinsic qualities of the objects that would enable them to form relationships with other objects. Their mental representations in elementary arithmetic were analogues of physical representations and the more that a child's use of a mental representation oscillated with the use of perceptual items, the greater the level of detail reported for the mental representations. The children did not treat their mental representation as a skeleton on which they may pin core ideas, a feature that is apparent amongst the 'high achievers'.

It was to establish whether or not children had a wider disposition to these kinds of representation that the main study was devised. In doing so it established a method using methodologies drawn from closely associated areas of mathematics education and psychology in an attempt to find links between qualitatively different forms of mathematical thinking and dispositions towards qualitatively different kinds of mental representation.

The classification of different kinds of mental representation was devised to distinguish between those which were more useful in identifying descriptive kinds of mental representation and those more useful in denoting relational mental representations. It was seen that the more descriptive kinds of mental representation, the 'specific' and 'episodic' kinds, were common to all children but:

- 'low achievers' projected these for all of the stimuli presented to them , whilst
- 'high achievers' were able to draw upon the intrinsic qualities of the items to additionally project 'generic' and 'proceptual' mental representations.

The degree to which any qualitative differences were noted was dependent upon the nature of the stimulus and became clearly apparent as this became more 'abstract' or 'language like'. The remarkable similarity between the two groups when they responded to pictorial items diverged to show qualitative differences when they responded to numerical items presented in a visual or a verbal form. 'Low achievers' disposition towards the 'descriptive' kinds of mental representation meant that visual symbols prompted the same quality of mental representation as icons whilst numeric words prompted the same quality as non-numeric ones.

Qualitative differences between the mental representations projected by 'low achievers' and those projected by 'high achievers' suggested that the limited spectrum of mental representations used by the former provides some evidence that the children are either unable to, or simply choose not to, see through the items to identify more abstract qualities. 'High achievers' used a spectrum of mental representations in a more 'integrated' way. In general though, the more abstract the nature of the stimulus the more relational the mental representation appeared to be.

The results of the study also seem to be in accord to De Beni and Pazzaglia's claim that 'autobiographic episodic' responses occur with much less frequency than 'general', 'contextual' (in our case 'episodic') and 'specific' responses. They speculate that given more time visual images can be enriched by more details or by the insertion of imaginal nouns within a network of relationships with other objects. It is a speculation that the evidence of this study supports in the context of mental representations.

10.2.3 Representations and Arithmetical Achievement

Within the arithmetical phases it was shown that once again qualitatively different forms of mental representation were associated with children identified as 'high' and 'low achievers'. Classified in a way more relevant to the numerical context, the occurrence of these kinds of representations indicated that those of:

- 'high achievers' tend to be 'abstract' with a numerical symbol used as the object of thought', whilst
- those of 'low achievers' are more strongly associated with 'verbal counting',

'figural' and 'perceptual' representations.

In common with the more localised view of visual representations seen in the pilot study, within the main study it was once again noted that these representations are also used differently by the two groups. Those of the 'high achievers' appear to be referents that allow them to generate new ideas or refresh their memory. Those of the 'low achievers' appear to be indispensable and necessary for their act of counting.

To determine the link between qualitatively different forms of numerical thinking and a predisposition towards qualitatively different kinds of mental representation a relationship between the 'context free classifications' and the 'numerical classifications' must be formed. Essentially this is seen at the level of a qualitative analysis of the activity of the 'low achievers'. Their disposition towards perceptual items, figural representations and verbal counting reflects a disposition towards qualitatively similar activity albeit of different form. This activity essentially focuses upon counting. It has already been seen that this activity may be viewed as a 'general' mental representation that is then sequentially acted out like an episode. In the application of any counting procedure it has been argued that for any arithmetical combination the general number sequence will be specified in terms of a start and end and in terms of the object used to support it. Thus a predisposition towards the act of counting reflects a predisposition towards 'specific' and 'episodic' kinds of mental representation.

In contrast the 'abstract mental representations' which predominate amongst the 'high achievers' share similarities with 'generic' and 'proceptual' mental representations. These abstract representations are a manifestation of number facts, relationships and equivalencies; they are productive statements that apply to the number procept.

10.2.4 Qualitatively Differences and Visual Imagery

Had this study only reported on visual imagery the relational characteristics apparent through kinds of mental representation identified as 'generic' and 'proceptual' may have been marginalised, occurring with such infrequency that overall qualitative differences may not have emerged.

Within the pilot study the role of visual imagery within the numerical component had suggested that:

- children of different levels of achievement projected visual imagery of different form: 'Low achievers' tended to project analogical images which became more concrete as they oscillated between visual imagery and the use of perceptual items. As combinations became more difficult the occurrence of visual imagery receded whilst the use of perceptual items increased. 'High achievers' projected visual images associated with numerical symbols. Claims to have 'seen' these symbols in the mind increased with the difficulty of the numerical combination, and

- the visual images appeared to have been put to different use. Amongst 'high achievers' they appeared to be used as thought generators and/or a support of working memory, amongst 'low achievers' they appeared to be essential to thought.

However, visual imagery was reported in less 30% of the instance where 'high achievers' solved combinations mentally but more than 40% of instances where 'low achievers' did them. This left a great proportion of unclassified responses in both instances and an attempt to clarify this was made in the main study.

Whilst in the main study all of the children reported, at some time or another, 'seeing' something in their head, the frequency of occurrence for individuals was such that it may be reasonable to identify the children by the general labels 'visualisers' and 'non-visualisers'. However, such labels do not provide any indicator of the level of numerical achievement of the children. The indicative case studies reported on children at complete extremes who were both essentially high visualisers. Equally children at the extremes could be identified as non-visualisers. It is because of these that the study concludes that a focus purely upon the incidence of visually imagery does not provide any overall insight into qualitative differences between the children whereas one associated with mental representation does. Nevertheless, in the analysis of visual imagery some important features did arise:

- visual stimuli evoked more visual imagery than verbal stimuli and
- 'high achievers' tended to report it more than 'low achievers'.

Within the context free element of the main study it was quite possible for children of different levels of achievement to possess the same visual image but then attach a different meaning to it. These seems to be particularly the case with the numerical items. Visual images of a 'high achiever' may be more 'generic' or 'proceptual' whereas that of a 'low achiever' more 'specific' and 'episodic'. Therefore it may be argued that whereas the visual image of a numerical symbol may be used by a 'high achiever' to refresh memory or as a skeleton that ideas, equivalences and relationships may be attached to, the same visual image may be used in an active mental episode for a 'low achiever' or as an object whose surface characteristics may be described.

10.4 THE IMPLICATIONS

Other theories which may be closely related to the findings within this study fall into two groups, those associated with cognitive development from the standpoint of mathematics education and those associated with psychological research into the use of memory and mental representation. Within the former should be placed Piaget's notion of reflective abstraction, theories associated with notions of encapsulation, and the notion of the proceptual divide (Gray & Tall, 1994). Within the later we must consider those associated with the kind of mental representation, the occurrence of different kinds of mental representation and the implications of memory research.

It has been seen within the literature review (Chapter 2) that it is implicit in Piaget's perspective and that of the constructivists' that the knowledge and beliefs that learners bring to a learning situation can influence the meanings that they construct from that situation. The development of early numerical concepts is heavily associated with physical activity, partly from a realisation that these concepts evolve from an interaction with the environment and also possibly because they relate to the Piagetian belief that new knowledge is constructed by the learner through 'active methods'. It was also Piaget's belief that the key to the process through which the actions associated with active methods were projected to thought was 'reflective abstraction'.

This requires the ability to concentrate the mind and give careful thought to an act or idea and then to filter out irrelevancies and separate notions from their context. It involves the construction of relationships between and amongst objects and of the interrelationships of the actions on them. It would seem that such a process may work to the advantage of the 'high achievers'. The evidence would seem to suggest that their disposition towards the formation of mental representations, which integrate both descriptive and relational characteristics, seems to ensure that their construction of number concepts, which as Tall (1995) notes begins with 'pseudo-empirical abstraction'. It is conjectured that this follows a very different cognitive development from that of children whose disposition towards 'descriptive' mental representations ensures that they concentrate upon 'empirical abstraction' and/or 'pseudo empirical abstraction'.

In any context involving an action on objects, the individual has the possibility of attending to different aspects of the situation. Indeed this is a theme that Cobb, Yackel and Wood (1992) see as one of the great problems in learning mathematics particularly if learning and teaching are approached from a representational context. In their search for substance and meaning 'low achievers' would appear to be distinctly disadvantaged right at the start of their mathematical development, but it is a disadvantage that may not make itself apparent in the earlier stages of cognitive development. It may be argued a false notion of 'pseudo-empirical abstraction' may give the same result as 'reflective abstraction' if after using each form of abstraction the result of a counting action is the same.

Mental representations that are dominated by a search for 'specificity', and 'episodic activity' will lead naturally towards 'empirical' and 'pseudo-empirical abstraction'. In the case of the latter, the teasing out of properties that the "actions of the subject have introduced into objects" may lead to a form of procedural competence, such as that seen amongst some of the older 'low achievers'. However, again as has been seen, attempts to generalise these actions can be very suspect.

It is possible to associate theories which try to explain the cognitive shift from process to object with early number development. Implicit within these theories is the need for some form of 'reflective abstraction' to be associated with the processes variously described as 'interiorisation', 'encapsulation' or 'reification'. However, these theories lack any real dynamism when it comes to describing what the processes actually mean. Piaget provides some sense of what is involved in 'interiorisation' through his notion of 'reflective abstraction' whilst Sfard's (1991) notions of interiorisation and condensation provide useful intermediaries which can unwrap her notion of 'reification'. None of the theories explicitly provide a role for mental representation. Since this study is indicating that individuals may 'internalise' different things which are manifest in different kinds of mental representations it follows that 'simplistic' notions of 'encapsulation' (a word of convenience to describe all associated processes) may be an ideal that all cannot rise to. The sequential, active, 'specific' and 'episodic' mental representations of the 'low achievers' may be making encapsulation difficult if not impossible. To encapsulate a numerical process into a numerical object there must be recognition that the object can exist, can be detached from its associated action and can be used for higher order thinking. The evidence within the study suggests that 'low achievers' may not realise this. Numerical items were treated in the same way as non-numerical items. Visual symbols were associated with mental representations which were hardly different from those associated with icons and pictures. It would seem that this would have implications for the way in which they deal with their arithmetic.

The mental representations of the 'low achievers' which are characterised by description of a specific object or scene are more in tune with the sequential re-enactment of an action. The mental representations of the 'high achievers' which are more 'skeletal' in nature are more likely not to include surface detail and/or sequential enactment. It is conjectured that though 'high achievers' realise that the actions exist, could describe their purpose, and name it, they were not compressed but eliminated to allow the focus of attention to be on the more abstract qualities of their arithmetic. When 'high achievers' are encapsulating a process they may not just compress and squeeze everything down but may be actively taking some components out. It is this 'actively taking' out that guided a small teaching experiment with a child whose mental representations were analogues of real objects (See Pitta & Gray, 1997). It is an approach that may need to be considered if children are not to fail at the very first hurdle.

Qualitative differences in the interpretation of numerical symbolism have lead to the notion of the proceptual divide (Gray & Tall, 1994). The evidence within this study suggests that there may be another divide, that based on a disposition towards the formation of the qualitatively different kinds of mental representations. The qualitatively different interpretations that children place upon mathematical symbols may be consequence of this disposition. The 'low achievers' disposition towards descriptive mental representations support those qualities required for procedural thinking. Those of the 'high achievers' may be a manifestation of their ability to think proceptually, since the children were selected on that basis. In the numerical context we may not yet be as sure which comes first. However, it is conjectured that a predisposition towards the formation of qualitatively different kinds of mental representation may explain those differences which have been considered in the context of 'reflective abstraction', the process of 'encapsulation' and the ability to use the ambiguity of mathematical symbolism. The assumption that active approaches supply all children with a route towards understanding arithmetical procepts is open to question.

The evidence within the study lends support to Krutetskii's (1976) argument that distinguishing between visualisers and verbalisers may not provide the evidence to identify the more successful but it gives some ideas of how mental representations determine the type of giftedness. However, it is a conjecture arising from the study that the content, the kind and the use of the mental representation will provide a clearer indication of individuals' understanding of a topic.

10.4 THE PSYCHOLOGICAL CONNECTION

Studying the relationship of mental representations, memory and arithmetic is a combination that can provide interesting information not only in the field of cognitive development in arithmetic but also in the wider field of cognition. Investigating the cognitive development of individuals through a topic which is usually broken down into clear steps, associated with a particular kind of pedagogy, for example an introductory active phase, an intermediary iconic phase and then a strongly dominant symbolic phase may offer a clearer picture of cognitive development than that seen within other disciplines. The many faces of numbers, the ability to represent them in different formats and their concrete and abstract nature can permit exploration to satisfy different purposes.

It is with this in mind that from the results of the research study a move into the realm of speculation which bearing in mind the literature considered in Chapters 2 and 3, seems appropriate. This is done in the belief that the format (image or proposition) of a mental representation, although an important characteristic, should not stand in the way of exploring other aspects of mental representations. The kinds of mental representation (De Beni & Pazzaglia, 1995) and models of multi-component memory and the role of mental representations (Baddeley's, 1966; Hitch's *et al* , 1995) are only some examples of research which contribute to knowledge without suffering from the debate on the format of mental representations. De Beni and Pazzaglia's (1995) efforts to place an interdisciplinary perspective on their work draw upon neuropsychological evidence to support their discussion on the different ways that mental representations are formed.

They indicate that this is preceded by the formation of a general skeleton image which involve the integration in short term memory of information derived from long term memory. The one exception cited is the autobiographic-episodic mental representation which involves a search in autobiographic memories following a verbal cue and the selection of that memory considered to be most associated with the cue.

Discussions which revolve around memory and arithmetic often attempt to explain the efficiency or inefficiency of individuals based on developmental delay, short term memory span, lack of automatically, effects of familiarity with material, speed of identification, retrieval and processing (Bull & Johnston, 1997). However, what they do not seem to consider is that there are different qualities of thinking in early number arithmetic and not all individuals tackle arithmetic in the same way (Gray & Tall, 1994). Thus, whereas cognitive psychology is attempting to respond to the issue with the creation of one catholic model we argue that differences may occur due to different models of cognition, or different 'cognitive behaviours' that individuals may possess.

Geary (1991) claimed that difficulties in mathematics may be attributed to working memory deficit. A conjecture formed from the evidence of this study is that the working memory of 'low achievers' does not suffer from memory deficit but from memory use. 'Low achievers' try to manage more pieces of information than the 'high achievers' but the strain that they may be putting on their working memory may be too great to handle and they also appear to lack a sense of organisation of the memory components. Their difficulties are as much associated with the immense amount of information that they attempt to store and process within memory as well as the way they organise it. 'High achievers' on the other hand seem to select the minimum — a 'skeleton' which generates thought, refreshes memory and may be supplemented as required. This skeletal information in associated with a better organisation of the components of memory (the phonological loop, which temporarily hold information in phonological form, and the visuo-spatial sketch pad, which specialises in temporary spatial and/or visual coding) gives more power to the 'high achievers'.

Given the further development of these notions by Logie, Gilhooly and Wynn (1994) it would appear that in some instances the visuo-spatial sketch pad is spontaneously participating in the solution of arithmetical questions. This may be seen in two ways. First the responses from children who indicate that they are using dots, fingers or numbers (visual images) in the mind in order to count. Such responses indicate that these visual images were essential and actively used for their thinking. In addition, comments that visual images of numerical symbols were 'flashing' in some children's mind also constitute strong evidence in favour of visual images being functionally significant and of the visual sketch pad participating in their thinking. Functionally significant in the second instance however would imply that these images are used to generate thought or act as memory reminders.

The role of long-term and short-term memory may also provide some answers to the sorts of differences that were noted in the kinds of mental representation projected by the children. Retrieval from long term memory allows the creation of mental representations that carry both meaning and surface characteristics (Hitch *et al, 1995*). Retrieval from short term memory would emphasise information about the appearance of the objects.

The ability to create mental representations seems to be inherent to every thinking human being. It appears, however, that different stimuli may trigger different kinds of mental representations of the same object and different individuals may have different kinds of mental representations. When the children within this study were presented with a pictorial free of context stimulus and were asked to create a mental representation there was hardly any diversity in the responses between the 'high' and 'low achievers'. Hitch *et al*, argued that given a visual stimuli for a very short period of time, it is likely that the individual will create a mental representation of the recently presented item in the short term memory — it is not logical to go and retrieve something from the long term memory. This study seems to confirm this argument.

When both groups of children responded to the visual stimuli they gave a high proportion of 'general', 'specific' or 'episodic' mental representations which although differing in generality referred to the appearance of the object or of an episode that involved the object. These kinds of mental representations are coincident with those created in short term memory. Evidence that the long term memory was not involved to the same extent comes from the fact that the children did not give many mental representations which were related to meaning, equivalences or relationships of the object. However, as the stimuli of the visual phase became more 'iconic' or 'symbolic' and therefore more 'abstract' and 'language like', the diversity between the two groups increased. It may be argued that the more 'language like' the stimulus the more it encourages long term memory retrievals. It may be that the more relational mental representations are stored in such a way that they are more accessible to verbal stimuli.

When the children were asked what comes into mind when stimulated verbally the diversity was even greater. It may be hypothesised that children in the verbal phase may more directly retrieve information from the long term memory or create an object or a scene in the short term memory by incorporating information from the long term memory. The possibility of being able to generate different mental representations in these two different ways may provide a platform for this diversity. The resulting mental representation may be different because the individual may have saved or retrieved different information from the long term memory or even generated something different in the short term memory. This may be the reason that children's responses are distributed over a greater range of categories. It seems that visual stimuli may cause more children to create a mental representation in the visual short term memory whereas the verbal stimuli may more readily allow the creation of mental representations in both short term memory and long term memory.

We cannot however, exclude the possibility that children were more aware of the immediate visual mental representations rather than the immediate verbal mental representations (which may have had acoustic characteristics). It has already been noted that is easier to describe visual images rather than verbal mental representations

(Eysenck & Keane, 1995; Tracy, Roesner, Kovac, 1988). But all of these are speculation. What does emerge from this study is the fact that mental representations were more descriptive in the visual phase and as the items displayed more 'language like' and 'abstract' qualities, the mental representations of the 'high achievers' became more relational.

There seem to be two possible explanations for these differences. First the tendency towards the projection of 'descriptive' and/or 'relational' mental representations may be one that is linked to an emphasis on the use of either the short term memory or the long term memory. A tendency towards the projection of surface characteristics only may be more strongly related to short term memory. On the other hand emphasis on long term memory use may mean retrieval of appropriate mental representations from the different components of the long term memory, which could be both abstract or carry surface characteristics. Different kinds of mental representation may imply different generation processes that may emphasise long term memory retrieval or short term use. An alternative explanation suggests that both groups rely equally on long term memory but whereas 'low achievers' are only using the descriptive information of the long term memory the high achievers more readily use both the descriptive as well as the more abstract, meaning related information of the long term memory.

Another explanation associated with the different kinds of mental representations may be related to different generation processes. De Beni and Pazzaglia argue that one mental representation generation process starts from a general mental representation which is then either specified or given a context (called episode in this study) through the incorporation of surface characteristics or additional nouns (note: this is integration in the short term memory of information derived from the long term memory). This generation process seems to be the one that dominates the reactions of 'low achievers' since they are the ones given more 'specific' and 'episodic' mental representations. A second mental representation generation process is one that suggests that mental representations are the result of a search in the long term memory for the more appropriate response. 'High achievers' seem to possess and use both generation processes. They give 'general', 'specific' and 'episodic' responses and 'generic' and 'proceptual' responses. When giving 'proceptual' responses the 'high achievers' seem to be carrying out a search in the memory components and retrieve appropriate relationships and/or equivalencies related to the object. It is more difficult to speculate how 'generic' responses are generated. They could be a collection of 'general' mental representations, or the result of 'filtering out' from a variety of contexts, or a search and retrieval of the most appropriate relationships. However, both 'generic' and 'proceptual' seem to be fragmented pieces of information which have the concept as the answer. They are not given in a 'descriptive' manner but resemble more a 'bombardment of uncontrollable, relational pieces of information'. Of courses these pieces of information could be specified either with the incorporation of more details and other nouns, or they could be acted upon or operated with if needed. De Beni and Pazzaglia also relate these different kinds of mental representations to over-reliance on either the right or left hemispheres of the brain. It would seem that 'general' mental representations may be associated with the right and 'specific' and 'episodic' mental representations with the left.

If we were to characterise the a disposition towards a particular kind of the mental representation, its generation process and its use as a "cognitive behaviour", it is conjectured that children at the extremes of arithmetical achievement appear to have different 'cognitive behaviours'. This behaviour may not only be apparent in the qualitative differences that emerge when children's efforts at elementary arithmetic are considered but the evidence suggests that it is characterised by 'high achievers' being more attuned to doing a long term memory *search* whilst the 'low achievers'' emphasis on short term memory use conditions them more towards conditioned or procedural responses apparent in the arithmetical context.

10.5 RECONSIDERING THE LIMITATIONS

The theoretical argument established through the analysis of the data which formed the focus for this study must of course be seen within the context of the limitations of the study. Within Chapter 4 the specified limitations directly attributable to obtaining data which seeks to identify mental representations was considered. Two central issues were highlighted:

(i) theoretical constructs, which arise from the notion that language may reveal something of how the child has represented information internally, are not entirely free of the observer's expectations and theories, however objective these may appear to be at the time. To gain some sense of children's mental representations the assumption was made that children's projections, reports, descriptions and external representations in verbal, written and motor form can mediate the internal, mental representations. However, no precise claims can be made about the nature of mental representations and

(ii) the nature of interaction between the interviewer and the child may cause the child to use different approaches during the interview to those that may have been used if the interviewer had not been present.

The totality of the discourse which provided the data formed the basis for classification, analysis of the data and the resulting theory. However, further limitations are imposed by the sampling method and the level of reporting subsequently presented. The study itself is concerned with an analysis of behaviour and this must be placed within a context that accepts a third form of limitation: the selection of samples, their size and the instruments used.

The results reported in this study are on interviews with children who are at the extremes of numerical achievement within two school in the English Midlands. There is no attempt to project the results to all children and neither can claims be made about the greater majority who lie between these extremes. However, there is an underlying conjecture that if these behaviours are noted in these schools they may well be noted in others. In addition there is a sense that by looking at the extremes we may gain some insight into what the intermediate range of children may be doing. With such information we may be able to direct them towards more flexible ways of thinking and not leave their cognitive development to chance.

Overall the amount of information collected for this study was considerable and some of the item banks had weaknesses which only became apparent during the analysis stage, for example a more harmonic selection of the visual and verbal arithmetic items may have sharpened verbal/ visual distinctions. As they were they appeared to have no effect on identifying qualitative differences in thinking.

In one sense the selection of the sample for the main study, an opportunist one dictated by factors other than the research project, depended a little too heavily upon teacher assessment. Notions of a child's ability may not be restricted to the actual quality being considered (Secada, 1992) and they may be reflected in the child's level of achievement in the sense that the child may rise or fall to the teacher's level of expectation (Nash, 1973).

It has not been the intention of this study to indicate proportions of children that reflect different styles of thinking. Neither has it been the intention to stereotype children through their quality of thinking to establish a long term prognosis which predicts different levels of success in mathematics. However, it was the intention to indicate whether or not children project qualitatively different kinds of mental representation and then consider the relationship of these differences with achievement in elementary arithmetic. Although the results do not provide a prognosis of the longer term effects of the differences that may be seen within the same children taken over a period of time, they suggest that there are issues that are worthy of further investigation.

Although this study makes no claims about the influence that cause children to have a disposition towards particular kinds of mental representation, what external factors cause or influence them and what role the teaching plays, it is possible that the results may reflect particular cultural, environmental, pedagogic, cognitive and genetic influences.

An underlying assumption of the study has been the belief that children's comments complement their mental representations. However, the children could only comment on aspects of mental representations that they were conscious of. Mental representations, their generation, use, functioning and other mechanisms may involve a number of features that individuals may not be conscious of and therefore they could not report upon or describe. Any study which draws upon aspects of introspection is fraught with uncertainty however the emerging patterns do allow us to view the results with some confidence. What may be described in some instance as idiosyncratic behaviour is simply too common for us to ignore.

Although the overall classification was based upon repeated analysis which resulted in the determination of the best conceptual categories, it was inevitable that some responses did not easily fit these categories. In addition some responses projected characteristics appropriate to more than one class. To overcome this a second researcher independently verified all classifications and those that were problematic where allocated as a result of joint reflection upon the nature of the response.

10.6 FUTURE RESEARCH

This study considers a series of 'snap shots' from which children's disposition towards particular kinds of mental representation are considered. Very little evidence has been established which provides a sense of the initial influences which may focus this disposition, how the disposition may change over time, and how children between the extremes project different kinds of mental representation. Research within the following areas would be profitable:

- a more informed study identifying the way in which a disposition towards particular kinds of mental representations influence cognitive development in arithmetic at the earliest stages. It could be claimed that the current study has examined two forms of behaviour after some considerable experience. It has considered some reasons for success and failure. Considering how these influences affect development through a longitudinal study would provide profitable insights into the direct influences of a particular disposition, the effects of pedagogy and change over time.

- an implicit assumption in pedagogy is that concept development should be associated with a sequence of carefully planned activities linked to appropriate metaphors. A worthwhile area of study would be to consider the kinds of mental representation possessed by the pedagogue, the way in which these are translated into external representations and the resulting quality of the mental representation formed by the child and

- the whole issue of the relationship between mathematical metaphors, manipulatives, and techniques used in the mathematical classroom with mental representations and achievement is an intriguing one. Teachers are not often aware of the links. Now that guided imagery (Drake, 1996) is gaining ground as a method of teaching in other disciplines a better understanding of the use and development of visual imagery in mathematics may also give some ideas on whether and how guided imagery may be implemented in the mathematical classroom.

An issue that was not approached in this study was that associated with how the mental representations arose and what influenced them. This could be a research topic on its own. Any justification for the claim that there are similarities between the mental representations of numerical and non-numerical objects give rise to an intriguing question. "Are individual pre-disposed to mentally represent mathematical ideas in a particular way?"

At an interdisciplinary level collaboration with cognitive psychologists may offer some more extensive investigations of the involvement of long term and working memory into the kinds, generation and function of mental representations in early number arithmetic. Links could also be drawn between attention and mental representations. Neuropsychology and neurology may provide valuable insights into the involvement of different parts of brain in mental representations as well as the processing of numbers. An interesting study would seek to identify whether or not a particular "cognitive behaviours" could be accounted for by certain neurological behaviour.

Demetra Pitta

June 1998

Glossary

"I don't know what that means" (Child, Y3+)

UNDERSTANDING

Encapsulation

The cognitive process of forming a (static) conceptual entity from a (dynamic) process (Dubinsky, 1991).

Object (of thinking)

The encapsulated entity created through a mathematical process.

Procedural Knowledge

Procedural knowledge is seen as knowledge which focuses on doing through applying.

Procedural Thinking

Thinking that reflects the use of procedural knowledge.

Procept

The amalgam of three components: the *process* which produces a mathematical *object* and the *symbol* which is used to represent either the process or object (Gray & Tall, 1994).

Proceptual Divide

The divergence in thinking that stems from the flexible use of procepts as object or process on the one hand and the excessive reliance on procedures on the other (Gray & Tall, 1994).

Proceptual Thinking

Thinking which portrays the flexibility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object which may be decomposed, recomposed and manipulated at a higher level (Gray & Tall, 1994).

REPRESENTATION

Analogical mental representation

The visual images that appear to have either the attributes of actual objects, icons or symbols and are used in a way that reflects the use of these actual objects.

Autobiographic-episodic mental representation

"The occurrence of a single episode in the subject's life connected to the concept" (De Beni & Pazzaglia, 1995, p. 1361).

Autobiographic mental representation

These mental representations are of two types:

(a) involving the subject him/herself without precise reference to a single incident and

(b) involving objects belonging to the subject. (Adapted from De Beni & Pazzaglia, 1995, p. 1361)

Episodic mental representation

Classification of this kind are identified when the mental representation is associated with an episode, a scene or a sequence of scenes that occurred in a specific context. (Note: It is not a retrieval from episodic memory.)

General mental representation

"General mental representations [are identified] when the subject's description did not specify any characteristics of the item." (De Beni & Pazzaglia, 1995, p. 1363)

Generic mental representation

This classification represents productive statements that are common to the general concept. It is not a description of a sequential event which has a clear beginning and end. It appears most often as a collection of statements that seem to have the potential to produce new ideas. Though they have a 'general' quality, the statements diverge to produce different ideas related to the item.

Mental Imagery

"An experience that takes place within the individual, being by its nature divorced from the objects that would give it place in the perceptual world, but it is a representational reference to such objects. The content of this imagery is varied. It may be of vision and contact or of other senses... playing the same part as that played by objects... What characterises it is its appearance in the absence of the of the other objects to which it refers..." (G. H. Mead, 1934, p. 223)

Mental representation

A mental reference which, divorced from the objects that give it a place in the external world, is the product of imaging in any modality whether it be visual, verbal, olfactory, auditory or kinaesthetic.

Referring to mental representations provides a major benefit for this study: it is inclusive of both imagery and propositional representations but due to the nature of the study it will be very difficult, if not impossible, to identify the difference between 'verbal images' and 'propositional representations'. Therefore, referring to verbal mental representations will allow us to argue our case without getting caught up in the on going debate about the nature of 'images' and 'propositional representations'.

Perception

Perception is the knowledge of objects resulting from direct contact with them. (Piaget & Inhelder, 1967, p. 17)

Proceptual mental representation

This is an additional classification to those identified by De Beni and Pazzaglia (1995) and it is particular to the numerical items. References to mathematical processes and concepts and/or indications of equivalence and interpretation were classified as such.

Propositional representation

They are language like, in that they assert properties of objects or entities. However, propositional representations are not sentences in a natural language. (Eysenck & Keane, 1995, p.179)

Representation

"Representation is an image, likeness or reproduction of a thing".

Specific mental representation

A specific representation "represents a single well-defined example of the concept without reference to a specific episode" (De Beni & Pazzaglia, 1995, p.1360). The criteria within this classification were extended to allow for multiple examples which were qualitatively similar.

Verbal mental representation

Mental representation which occurs as a sound in the 'mind's ear'. This study will not distinguish between verbal images and propositional representations.

Visual mental representation or visual image

Mental representations which occur as a picture in the 'mind's eye'.

Visualisation

Visualisation is the comprehension and performance of imagined visual movements of objects in two and three dimensional space. The objects may resemble concrete items, pictures, icons or symbols.

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APPENDICES

APPENDIX 1.1

Pilot Study

Children's Strategies: All Number Combinations to 100

			Combinations to Ten																	
	Ach.	Age	2+1	3+5	8+2	4+4	0+2	6+3	5+4	5+0	7+2	3–2	5–4	6–3	9–5	3–3	6–0	8–2	9–8	7–5
4	A	8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
5	A	8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
6	A	8	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
10	A	9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
11	A	9	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
12	A	9	1	1	1	1	1	1	1	1	2	1	1	1	2	1	1	2	1	1
16	A	10	1	1	1	1	1	1	1	1	1	1	2	1	1	1	1	1	1	1
17	A	10	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
18	A	10	1	1	1	1	1	3	1	1	1	1	1	1	1	1	1	1	1	2
22	A	11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	A	11	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
24	A	11																		
1	C	8	1	3	3	1	1	3	3	1	3	1	1	3	1	1	1	4	3	3
2	C	8	3	3	3	3	1	3	3	1	3	4	4	4	4	1	4	4	4	4
3	C	8	3	3	1	1	0	1	1	1	1	1	1	1	2	1	1	1	1	2
7	C	9	3	3	3	1	1	3	3	1	3	4	4	2	4	1	1	1	4	4
8	C	9	3	3	3	1	1	3	3	1	4	4	4	1	4	4	4	3	1	4
9	C	9	1	1	3	1	1	3	1	1	3	1	1	1	4	1	1	3	1	4
13	C	10	1	3	1	1	1	1	1	1	3	1	3	1	3	1	1	3	3	3
14	C	10	1	3	3	1	3	2	2	2	3	IS	IS	IS	IS	IS	IS	IS	IS	IS
15	C	10	3	3	1	3	1	3	2	1	1	1	1	4	1	1	0	4	1	1
19	C	11	3	3	3	2	1	2	2	1	2	1	1	2	0	1	1	2	1	2
20	C	11	1	3	2	1	1	2	2	1	1	4	4	2	4	1	1	1	2	4
21	C	11																		

Key

1 Known

2 Transformation Strategies

3 Transformation and counting

4 Accumulation strategies: Sequencing in tens

5 Accumulation Strategy: Counting in ones

6 Analogous to written procedure

			Combinations to Twenty																				Combinations over Twenty						
	Ach.	Age	12+1	13+5	8+2	14+4	10+2	3+16	15+4	9+8	4+7	8+6	13–	15–	16–	15–	6–1	12–	18–	20–	17–13	19–17		18+5	24+43	16+41	39+26	80–30	29–6	26–19
4	A	8	1	1	1	1	1	1	1	1	2	2	1	1	1	1	1	1	1	1	1	1		1	2	2	2	2	1	2
5	A	8	1	1	1	1	1	1	1	2	1	2	1	1	1	2	1	2	2	1	1	1		1	0	2	1	1	1	2
6	A	8	1	1	1	2	1	2	2	2	1	1	1	1	2	2	1	2	1	1	1	1		2	2	6	6	2	1	5
10	A	9	1	2	1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	1	2	1		2	5	5	2	5	5	2
11	A	9	1	1	1	1	1	1	1	2	2	1	1	1	2	2	1	1	1	1	1	1		2	6	6	6	1	1	1
12	A	9	1	2	1	2	1	2	2	1	2	2	1	2	2	2	2	2	2	1	2	2		2	5	2	2	5	5	0
16	A	10	1	3	2	2	1	2	2	2	3	1	1	2	2	2	1	1	2	2	2	2		5	6	6	6	4	6	3
17	A	10	1	1	1	1	1	1	1	1	1	2	1	1	1	2	1	1	1	2	1	1		1	5	5	5	1	1	1
18	A	10	1	2	1	1	1	2	2	2	2	2	2	1	1	2	1	2	1	2	2	1		2	2	2	2	1	2	2
22	A	11	1	1	1	1	1	1	1	2	2	2	1	1	1	2	1	1	2	1	1	1		2	2	2	2	1	2	2
23	A	11	1	2	1	2	1	1	1	2	1	1	1	1	2	2	1	2	2	1	1	1		1	2	2	2	2	2	2
24	A	11																												
1	C	8	3	3	3	3	1	2	2	3	3	3	3	2	3	1	0	0	0	3	3.1	3		3						
2	C	8	3	3	3	3	3	3	3	3	3	3	3	3	3	0	3	3	3	3	3	3								
3	C	8	1	2	1	2	3	2	2	1	1	2	2	2	1	2	1	1	1	1	3.2	1								
7	C	9	3	3	3	3	3	3	1	3	3	3	2	4	4	4	4	4	4	4	4	4		5	0	5	0	0	5	0
8	C	9	3	3	3	3	3	3	3	3	3	3	4	4	4	4	4	4	4	2	4	4		5						
9	C	9	1	3	3	3	1	3	3	3	3	3	4	4	4	4	1	4	4	4	4	4		5	5	0	0	4	0	0
13	C	10	1	2	1	2	1	1	1	3	3	3	1	1	G	G	3	3	3	G	3	4		3	0	2	0	0	guess	5
14	C	10	3	3	3	IS	1	3	2	2	3	2	4	2	IS	2	2	2	2	2	4	4		3						
15	C	10	1	3	1	IS	1	3	3	3	3	1	1	3	3	2	0	3.2	2	2	3.2	3.2		5						
19	C	11	3	2	1	2	1	2	2	2	3	3	1	1	2	2	3	1	2	2	3	3		4	6	6	6	4	5	6
20	C	11	2	1	1	3	1	3	3	3	1	3	4	4	4	4	2	4	2	2	2	2		4	3	3	3	3	2	4
21	C	11																												

Pilot Study

Children's Strategies: Elementary Number Combinations to 20

RAW SCORE TOTALS

High Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	27	26	26	18
DF	0	1	0	0
CO	0	0	1	0
CA	0	0	0	0

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	27	25	25	18
DF	0	2	2	0
CO	0	0	0	0
CA	0	0	0	0

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	22	21	16	14
DF	8	9	12	6
CO	0	0	2	0
CA	0	0	0	0

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	24	17	16	14
DF	6	13	14	6
CO	0	0	0	0
CA	0	0	0	0

Low Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	12	12	13	7
DF	0	0	4	7
CO	14	14	10	4
CA	0	1	0	0
Other	0	0	0	0
Error	1	0	0	0

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	13	11	10	8
DF	2	1	5	5
CO	3	2	2	4
CA	9	13	0	0
Other	0	0	1	1
Error	0	0	9	0

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	5	3	10	6
DF	7	0	5	6
CO	18	27	13	8
CA	0	0	0	0
Other	0	0	2	0
Error	0	0	0	0

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	7	1	3	3
DF	4	2	8	9
CO	15	0	12	3
CA	0	27	4	5
Other	0	0	3	0
Error	4	0	0	0

PERCENTAGES

High Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	100	96	96	100
DF	0	4	0	0
CO				
CA				

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	100	93	93	100
DF		7	7	
CO				
CA				

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	73	70	53	70
DF	27	30	40	30
CO	0	0	7	0
CA				

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	80	57	53	70
DF	20	43	47	30
CO				
CA				

Low Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	44	44	48	39
DF	0	0	15	39
CO	52	52	37	22
CA	0	4	0	0
Other	0	0	0	0
Error	4	0	0	0

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
KF	48	41	37	44
DF	7	4	19	28
CO	11	7	7	22
CA	33	48	0	0
Other	0	0	4	6
Error	0	0	33	0

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	17	10	33	30
DF	23	0	17	30
CO	60	90	43	40
CA	0	0	0	0
Other	0	0	7	0
Error				

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
KF	23	3	10	15
DF	13	7	27	45
CO	50	0	40	15
CA	0	90	13	25
Other	0	0	10	0
Error	13	0	0	0

Pilot Study

Representations: Elementary Number Combinations to 20

RAW SCORE TOTALS

High Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	0	0	0	0
Ext. Fig	0	0	0	0
Int. Fig	0	0	2	0
Symbol	0	8	13	20

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	0	0	0
Symbo	0	5	9	15

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	5	2	0
Symbol	8	8	0	12

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	0	0	0
Symbol	9	5	5	13

Low Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	7	0	0	0
Ext	3	11	2	0
Int	5	2	9	4
Symbol	7	0	8	9

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	8	0	0	0
Ext	4	15	8	0
Int	0	0	10	4
Symbo	5	0	0	0

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	18	27	12	6
Int	0	0	3	2
Symbol	0	0	3	2

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	19	27	13	5
Int	0	0	0	4
Symbol	0	0	0	0

PERCENTAGES

High Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	0	7	0
Symbol	0	30	48	74

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	0	0	0
Symbo	0	19	33	83

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	17	7	0
Symbol	27	27	0	60

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	0	0	0	0
Int	0	0	0	0
Symbol	30	17	17	65

Low Achievers

	Addition to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	26	0	0	0
Ext	11	41	7	0
Int	19	7	33	22
Symbol	26	0	30	50

	Subtraction to Ten			
	N=3x9			N=2x9
	9+	10+	11+	12+
None	30	0	0	0
Ext	15	56	30	0
Int	0	0	37	22
Symbo	19	0	0	0

	Addition to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	60	90	40	30
Int	0	0	10	10
Symbol	0	0	10	10

	Subtraction to Twenty			
	N=3x10			N=2x10
	9+	10+	11+	12+
None	0	0	0	0
Ext	63	90	43	25
Int	0	0	0	20
Symbol	0	0	0	0

Pilot Study

Classification of Chidren's Responses to Verbal Items

BALL

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.		8	3	1					
	ET		2	1			4		5	
High Achievers	Mental Repr.		7	3					2	
	ET								12	

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	0	67	25	8	0	0	0	0	0
	ET	0	17	8	0	0	33	0	42	0
High Achievers	Mental Repr.	0	58	25	0	0	0	0	17	0
	ET	0	0	0	0	0	0	0	100	0

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.		67	25							8
	ET		17				33		42		8
High Achievers	Mental Repr.		58	25					17		0
	ET								100		0

CAR

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.		2	7			2		1	
	ET		1	1	1		7		2	
High Achievers	Mental Repr.		3	5	1		2		1	
	ET								12	

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	0	17	58	0	0	17	0	8	0
	ET	0	8	8	8	0	58	0	17	0
High Achievers	Mental Repr.	0	25	42	8	0	17	0	8	0
	ET	0	0	0	0	0	0	0	100	0

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.		17	58			17				8
	ET						58		17		24
High Achievers	Mental Repr.		25	42			17				17
	ET								100		0

TRIANGLE

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.		4	4		1	3			
	ET		2	4			5		1	
High Achievers	Mental Repr.		1	2		4	3		2	
	ET			2			1		9	

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	0	33	33	0	8	25	0	0	0
	ET	0	17	33	0	0	42	0	8	0
High Achievers	Mental Repr.	0	8	17	0	33	25	0	17	0
	ET	0	0	17	0	0	8	0	75	0

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.		33	33			25				8
	ET		17	33			42				8
High Achievers	Mental Repr.			17		33	25		17		0
	ET			17					75		8

NUMBER

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.		4	6	2					
	ET		2	2	3	1	3		1	
High Achievers	Mental Repr.		1	2	4	4				1
	ET		1		1				9	1

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	0	33	50	17	0	0	0	0	0
	ET	0	17	17	25	8	25	0	8	0
High Achievers	Mental Repr.	0	8	17	33	33	0	0	0	8
	ET	0	8	0	8	0	0	0	75	8

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.		33	50	17						0
	ET		17	17	25		25				17
High Achievers	Mental Repr.			17	33	33					17
	ET								75		25

FIVE

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	2	4			2	4			
	ET	1	5			1	3		2	
High Achievers	Mental Repr.		1			4	1		4	2
	ET						1		6	5

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	17	33	0	0	17	33	0	0	0
	ET	8	42	0	0	8	25	0	17	0
High Achievers	Mental Repr.	0	8	0	0	33	8	0	33	17
	ET	0	0	0	0	0	8	0	50	42

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.	17	33			17	33				0
	ET		42				25		17		17
High Achievers	Mental Repr.					33			33	17	17
	ET								50	42	8

FRACTION

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	6	3				3			
	ET	6	2	1			2		1	
High Achievers	Mental Repr.		1		2	4	1		2	2
	ET				3		1		6	2

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	50	25	0	0	0	25	0	0	0
	ET	50	17	8	0	0	17	0	8	0
High Achievers	Mental Repr.	0	8	0	17	33	8	0	17	17
	ET	0	0	0	25	0	8	0	50	17

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.	50	17				25				8
	ET	50	17				17				16
High Achievers	Mental Repr.				17	33			17	17	17
	ET				25				50	17	8

HALF

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.		2	7	2		1			
	ET		3	3	3		2		1	
High Achievers	Mental Repr.		1	2		3			3	3
	ET			2	2				7	1

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	0	17	58	17	0	8	0	0	0
	ET	0	25	25	25	0	17	0	8	0
High Achievers	Mental Repr.	0	8	17	0	25	0	0	25	25
	ET	0	0	17	17	0	0	0	58	8

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.		17	58	17						8
	ET		25	25	25		17				8
High Achievers	Mental Repr.			17		25			25	25	8
	ET			17	17				58		8

FOUR NINTHS

RAW DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	9	2			1				
	ET	9	3							
High Achievers	Mental Repr.	2				6			2	2
	ET	2							9	1

PERCENTAGE DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0
Low achievers	Mental Repr.	75	17	0	0	8	0	0	0	0
	ET	75	25	0	0	0	0	0	0	0
High Achievers	Mental Repr.	17	0	0	0	50	0	0	17	17
	ET	17	0	0	0	0	0	0	75	8

TRIMMED DATA

RESPONSE		1.0	2.0	3.1	3.2	3.3	4.1	4.2	4.3	5.0	Other
Low achievers	Mental Repr.	75	17								8
	ET	75	17								8
High Achievers	Mental Repr.	17				50			17		17
	ET	17							75		8

KEY

1.0	Not known
2.0	Association and contextual
3.1	Single Example
3.2	Multi-Examples
3.3	Symbolic Examples
4.1	Visual Characteristics
4.2	Imaginative extensions
4.3	Insight into abstract qualities
5.0	Proceptual

Pilot Study

Classification of Children's Responses to Visual Items

LOW ACHIEVERS

RAW DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5
5		6				3			
1995		4				5			
Three quarters		6				2	1		
3–4	1	2				1	4	1	
Marbles		1					8		
Two quarters		1				1	6	1	
Honeycomb		1				3	5		
Scalene		3				4	2		
House		1				7	1		
Dancing Man						3	5	1	

PERCENTAGE DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5
5	0	67	0	0	0	33	0	0	0
1995	0	44	0	0	0	56	0	0	0
Three quarters	0	67	0	0	0	22	11	0	0
3–4	11	22	0	0	0	11	44	11	0
Marbles	0	11	0	0	0	0	89	0	0
Two quarters	0	11	0	0	0	11	67	11	0
Honeycomb	0	11	0	0	0	33	56	0	0
Scalene	0	33	0	0	0	44	22	0	0
House	0	11	0	0	0	78	11	0	0
Dancing Man	0	0	0	0	0	33	56	11	0

TRIMMED DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5	Other
5		66				33				0
1995		44				56				0
Three quarters		67				22				11
3–4		22				22	44			11
Marbles							89			11
Two quarters							67			33
Honeycomb						33	58			11
Scalene		33				44	22			0
House						78				22
Dancing Man						33	56			11

HIGH ACHIEVERS

RAW DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5
5		1				3		4	4
1995		2				4		3	3
Three quarters						1	1	2	8
3–4								11	1
Marbles	1					4	1	6	
Two quarters						2	2	8	
Honeycomb						3	3	6	
Scalene						5	2	5	
House						7		5	
Dancing Man						1	5	6	

PERCENTAGE DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5
5	0	8	0	0	0	25	0	33	33
1995	0	17	0	0	0	33	0	25	25
Three quarters	0	0	0	0	0	8	8	17	67
3–4	0	0	0	0	0	0	0	92	8
Marbles	8	0	0	0	0	33	8	50	0
Two quarters	0	0	0	0	0	17	17	67	0
Honeycomb	0	0	0	0	0	25	25	50	0
Scalene	0	0	0	0	0	42	17	42	0
House	0	0	0	0	0	58	0	42	0
Dancing Man	0	0	0	0	0	8	42	50	0

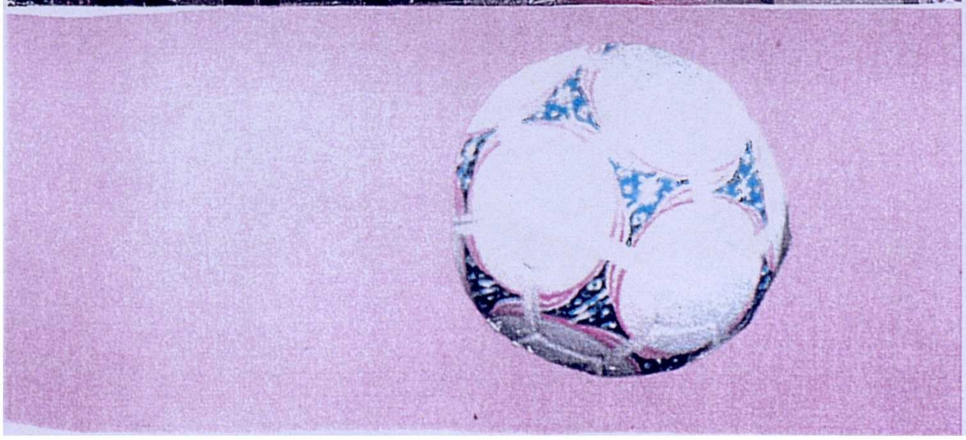
TRIMMED DATA — RESPONSE

	1	2	3	3	3	4.1	4	4	5	Other
5						25		33	33	8
1995		17				33		25	25	0
Three quarters								17	67	16
3–4								92		8
Marbles						33		50		17
Two quarters						17	17	67		0
Honeycomb						25	25	50		0
Scalene						40	17	42		1
House						58	42			0
Dancing Man							42	50		8

KEY
- 1.0 Not known
- 2.0 Association and contextual
- 3.1 Single Wxample
- 3.2 Multi-Examples
- 3.3 Symbolic Examples
- 4.1 Visual Characteristics
- 4.2 Imaginative extensions
- 4.3 Insight into abstract qualities
- 5.0 Proceptual

Main Study

Pictorial Visual Items



Main Study

Classification of Children's Responses to Visual Items

All Children, All Items In Visual Phase

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures N=94		Icons N=63		Symbols N=97		Pictures N=136		Icons N=91		Symbols N=158	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	2	2	4	6	8	8	2	1	4	4	5	3
General	36	39	12	19	43	45	22	16	8	9	38	24
Specific	30	32	22	35	18	19	56	42	48	53	48	31
Generic	3	3	1	2	5	5	7	5	0	0	10	6
Episodic	17	18	20	32	14	14	42	31	28	31	30	19
Autob. Ep.	4	4	0	0	2	2	6	4	0	0	12	8
Proceptual	2	2	4	6	7	7	1	1	3	3	15	9

High Achievers

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures N=46		Icons N=29		Symbols N=45		Pictures N=65		Icons N=44		Symbols N=75	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
General	20	43	7	24	27	60	13	20	4	9	21	28
Specific	11	24	8	28	2	4	26	40	25	57	18	24
Generic	3	7	0	0	5	11	6	9	0	0	8	11
Episodic	10	22	10	34	4	9	16	25	12	27	11	15
Autob. Ep.	1	2	0	0	0	0	3	5	0	0	3	4
Proceptual	1	2	4	14	7	16	1	2	3	7	14	19

Low Achievers

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures N=46		Icons N=30		Symbols N=44		Pictures N=69		Icons N=43		Symbols N=78	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
General	16	35	5	17	16	36	9	13	4	9	17	22
Specific	19	41	14	47	16	36	30	43	23	53	35	45
Generic	0	0	1	3	0	0	1	1	0	0	2	3
Episodic	7	15	10	33	10	23	26	38	16	37	19	24
Autob. Ep.	3	7	0	0	2	5	3	4	0	0	4	5
Proceptual	1	2	0	0	0	0	0	0	0	0	1	1

Main Study

Classification of Children's Responses to Verbal Items

High and Low achiever in Verbal Phase All Items

High Achievers

	First thing that comes to Mind				Talk freely for 30 secs.				Explain to ET			
	Numerical N=80		Non-Numerical N=56		Numerical N=118		Non-Numerical N=78		Numerical N=124		Non-Numerical N=78	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	8	*10*	1	*2*	6	*5*	0	*0*	5	*3*	1	*1*
General	24	*30*	15	*27*	25	*19*	18	*21*	43	*30*	29	*35*
Specific	19	*23*	15	*27*	19	*14*	25	*29*	30	*21*	33	*39*
Generic	18	*22*	16	*29*	34	*26*	21	*25*	9	*6*	10	*12*
Episodic	1	*1*	5	*9*	15	*11*	9	*10*	17	*12*	11	*13*
Autob. Ep.	0	*0*	4	*7*	11	*8*	13	*15*	7	*5*	0	*0*
Proceptual	11	*14*	0	*0*	22	*17*	0	*0*	34	*23*	0	*0*

Low Achievers

	First thing that comes to Mind				Talk freely for 30 secs.				Explain to ET			
	Numerical N=80		Non-Numerical N=56		Numerical N=118		Non-Numerical N=78		Numerical N=124		Non-Numerical N=78	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	16	*20*	0	*0*	10	*8*	1	*1*	17	*14*	1	*1*
General	23	*29*	22	*39*	26	*22*	9	*12*	33	*27*	14	*18*
Specific	20	*25*	19	*34*	34	*29*	32	*42*	34	*28*	45	*57*
Generic	6	*8*	3	*5*	9	*8*	8	*10*	4	*3*	1	*1*
Episodic	10	*13*	11	*20*	31	*26*	16	*21*	28	*23*	15	*19*
Autob. Ep.	3	*4*	1	*1*	4	*3*	12	*15*	4	*3*	2	*3*
Proceptual	2	*3*	0	*0*	4	*3*	0	*0*	4	*3*	0	*0*

Main Study

Comparison of the Visual and the Verbal Phases

Classification of Responses

Visual and Verbal Phase, Numerical and Non-numerical (Comparison Items) (Percentages)

High Achievers

	First thing in the mind				Talk for 30 seconds			
	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.
	Num.	Num	Non-Num	Non-Num	Num.	Num	Non-Num	Non-Num
	N=41	N=41	N=41	N=40	N=68	N=66	N=60	N=59
Not know	7	12	2	3	4	6	2	0
General	58	27	50	23	25	21	22	17
Specific	5	17	14	19	25	8	38	30
Generic	5	22	5	35	10	23	10	25
Episodic	10	2	24	8	15	14	23	14
Autob. Ep.	0	0	3	12	3	10	3	14
Proceptual	15	20	2	0	18	18	2	0

Low achievers

	First thing in the mind				Talk for 30 seconds			
	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.
	Num.	Num	Non-Num	Non-Num	Num.	Num	Non-Num	Non-Num
	N=40	N=40	N=42	N=40	N=69	N=60	N=64	N=54
Not know	3	10	2	0	1	7	2	2
General	34	24	36	35	22	18	14	13
Specific	38	28	38	31	43	34	42	37
Generic	0	10	0	8	3	5	2	9
Episodic	20	15	17	24	26	28	37	26
Autob. Ep.	5	8	5	2	4	5	3	13
Proceptual	0	5	2	0	1	3	0	0

Visual Verbal Phase Numerical and Non-numerical (Comparison Items) (Responses)

High Achievers

	First thing in the mind				Talk for 30 seconds			
	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.
	Num.	Num	Non-Num	Non-Num	Num.	Num	Non-Num	Non-Num
	N=41	N=41	N=41	N=40	N=68	N=66	N=60	N=59
Not know	3	5	1	1	3	4	1	0
General	24	11	21	9	17	14	13	10
Specific	2	7	6	8	17	5	23	18
Generic	2	9	2	14	7	15	6	15
Episodic	4	1	10	3	10	9	14	8
Autob. Ep.	0	0	1	5	2	7	2	8
Proceptual	6	8	1	0	12	12	1	0

Low achievers

	First thing in the mind				Talk for 30 seconds			
	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.	Vis. Ph.	Ver. Ph.
	Num.	Num	Non-Num	Non-Num	Num.	Num	Non-Num	Non-Num
	N=40	N=40	N=42	N=40	N=69	N=60	N=64	N=54
Not know	1	4	1	0	1	4	1	1
General	14	10	15	14	15	11	9	7
Specific	15	11	16	12	30	20	27	20
Generic	0	4	0	3	2	3	1	5
Episodic	8	6	7	10	18	17	24	14
Autob. Ep.	2	3	2	1	3	3	2	7
Proceptual	0	2	1	0	1	2	0	0

Main Study

Classification of Visual images in the Visual and Verbal Phases

High and Low achievers' Visual Images in Verbal and Visual Phases (All Items)

High Achievers

	Verbal Phase				Verbal Phase				Visual Phase					
	1st Visual Image				30sec. Visual Image				30 seconds Visual Image					
	Numerical N=41		Non-Numerical N=42		Numerical N=36		Non-Numerical N=35		Non-numerical N=28		Icons N=36		Numerical N=41	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	0	0	0	0	0	0	0	0	5	18	1	3	2	5
General	6	15	4	10	10	28	1	3	6	21	6	17	14	34
Specific	13	31	18	43	4	11	14	39	6	21	16	44	8	20
Generic	2	5	2	5	2	6	2	6	0	0	0	0	1	2
Episodic	10	24	11	26	5	14	9	26	7	26	7	19	4	10
Autob. Ep.	6	15	7	17	12	33	9	26	4	14	5	14	7	17
Proceptual	4	10	0	0	3	8	0	0	0	0	1	3	5	12

Low Achievers

	Verbal Phase				Verbal Phase				Visual Phase					
	1st Visual Image				30sec. Visual Image				30 seconds Visual Image					
	Numerical N=25		Non-Numerical N=39		Numerical N=17		Non-Numerical N=33		Non-numerical N=30		Icons N=31		Numerical N=38	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	0	0	0	0	0	0	0	0	1	3	0	0	0	0
General	3	12	2	5	4	24	0	0	4	13	5	16	8	21
Specific	11	44	14	36	3	18	18	53	14	48	12	39	21	55
Generic	0	0	0	0	0	0	1	3	0	0	0	0	0	0
Episodic	5	20	16	41	8	46	9	29	9	30	14	45	6	16
Autob. Ep.	6	24	7	18	2	12	5	15	2	6	0	0	2	5
Proceptual	0	0	0	0	0	0	0	0	0	0	0	0	1	3

Main Study
Comparative Classification of Mental Representations
Younger and Older Children: Visual Phase

Younger and older children in the Visual Phase In all Items

Younger HA

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures		Icons		Symbols		Pictures		Icons		Symbols	
	N=25		N=16		N=25		N=29		N=21		N=34	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	0	*0*	0	*0*	4	*16*	0	*0*	3	*14*	2	*6*
General	10	*40*	2	*13*	15	*60*	6	*21*	0	*0*	8	*24*
Specific	7	*28*	7	*44*	2	*8*	14	*50*	9	*43*	9	*26*
Generic	0	*0*	0	*0*	0	*0*	0	*0*	0	*0*	2	*6*
Episodic	7	*28*	6	*37*	3	*12*	7	*24*	5	*23*	6	*17*
Autob. Ep.	1	*4*	0	*0*	0	*0*	2	*5*	2	*10*	5	*15*
Proceptual	0	*0*	1	*6*	1	*4*	0	*0*	2	*10*	2	*6*

Older HA

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures		Icons		Symbols		Pictures		Icons		Symbols	
	N=29		N=15		N=29		N=38		N=31		N=46	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	1	*3*	2	*13*	0	*0*	1	*3*	1	*3*	2	*4*
General	9	*31*	5	*33*	11	*38*	7	*18*	4	*14*	13	*29*
Specific	7	*24*	1	*7*	1	*3*	14	*37*	14	*45*	8	*17*
Generic	5	*18*	0	*0*	4	*14*	9	*24*	1	*3*	6	*13*
Episodic	7	*24*	4	*27*	2	*7*	7	*18*	8	*26*	5	*11*
Autob. Ep.	0	*0*	0	*0*	0	*0*	0	*0*	1	*3*	0	*0*
Proceptual	0	*0*	3	*20*	11	*38*	0	*0*	2	*6*	12	*26*

Younger LA

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures		Icons		Symbols		Pictures		Icons		Symbols	
	N=32		N=21		N=29		N=41		N=18		N=31	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	0	*0*	2	*10*	3	*10*	0	*0*	0	*0*	1	*3*
General	8	*25*	5	*24*	6	*21*	5	*12*	2	*11*	6	*19*
Specific	16	*50*	9	*42*	16	*55*	19	*47*	10	*56*	18	*59*
Generic	0	*0*	0	*0*	0	*0*	1	*2*	0	*0*	0	*0*
Episodic	7	*22*	5	*24*	4	*14*	14	*34*	6	*33*	4	*13*
Autob. Ep.	1	*3*	0	*0*	0	*0*	2	*5*	0	*0*	2	*6*
Proceptual	0	*0*	0	*0*	0	*0*	0	*0*	0	*0*	0	*0*

Older LA

	First thing that comes to Mind						Talk freely for 30 seconds					
	Pictures		Icons		Symbols		Pictures		Icons		Symbols	
	N=22		N=16		N=25		N=31		N=23		N=43	
	Total	%	Total	%	Total	%	Total	%	Total	%	Total	%
Not Known	1	*5*	0	*0*	1	*4*	1	*3*	0	*0*	0	*0*
General	9	*40*	4	*24*	11	*44*	4	*13*	2	*9*	10	*24*
Specific	7	*32*	6	*38*	6	*24*	11	*36*	12	*52*	15	*35*
Generic	0	*0*	0	*0*	0	*0*	0	*0*	0	*0*	1	*2*
Episodic	4	*18*	6	*38*	7	*28*	12	*39*	9	*39*	15	*35*
Autob. Ep.	0	*0*	0	*0*	0	*0*	2	*6*	0	*0*	1	*2*
Proceptual	1	*5*	0	*0*	0	*0*	1	*3*	0	*0*	1	*2*

Main Study

Comparative Classification of Mental Representations

Younger and Older Children: Verbal Phase

Younger and older children in the Verbal Phase In all Items

Younger HA

	First thing in Mind				30 seconds free talk			
	Non-numerical		Numerical		Non-numerical		Numerical	
	N=28		N=41		N=38		N=46	
	Total	%	Total	%	Total	%	Total	%
Not Known	1	*4*	7	*17*	0	0	6	*13*
General	7	*25*	10	*24*	6	16	6	*13*
Specific	9	*32*	13	*31*	10	26	12	*27*
Generic	4	*14*	6	*16*	3	8	4	*9*
Episodic	4	*14*	1	*2*	7	18	8	*17*
Autob. Ep.	3	*11*	0	*0*	12	32	4	*9*
Proceptual	0	*0*	4	*10*	0	0	6	*13*

Older HA

	First thing in Mind				30 seconds free talk			
	Non-numerical		Numerical		Non-numerical		Numerical	
	N=28		N=41		N=45		N=66	
	Total	%	Total	%	Total	%	Total	%
Not Known	0	*0*	0	*0*	0	*0*	0	*0*
General	8	*29*	15	*37*	12	*27*	17	*26*
Specific	7	*24*	6	*14*	15	*33*	8	*13*
Generic	12	*43*	12	*29*	15	*33*	19	*29*
Episodic	0	*0*	1	*2*	3	*7*	5	*8*
Autob. Ep.	1	*4*	0	*0*	0	*0*	0	*0*
Proceptual	0	*0*	7	*17*	0	*0*	17	*26*

Younger LA

	First thing in Mind				30 seconds free talk			
	Non-numerical		Numerical		Non-numerical		Numerical	
	N=28		N=39		N=30		N=55	
	Total	%	Total	%	Total	%	Total	%
Not Known	0	*0*	12	*31*	1	*3*	9	*16*
General	9	*32*	9	*23*	4	*13*	13	*24*
Specific	8	*28*	10	*25*	12	*40*	18	*31*
Generic	1	*4*	3	*8*	0	*0*	1	*2*
Episodic	9	*32*	2	*5*	5	*17*	10	*18*
Autob. Ep.	1	*4*	2	*5*	8	*27*	3	*7*
Proceptual	0	*0*	1	*3*	0	*0*	1	*2*

Older LA

	First thing in Mind				30 seconds free talk			
	Non-numerical		Numerical		Non-numerical		Numerical	
	N=28		N=39		N=41		N=59	
	Total	%	Total	%	Total	%	Total	%
Not Known	0	*0*	4	*10*	0	*0*	1	*2*
General	15	*53*	10	*26*	3	*7*	13	*22*
Specific	9	*32*	9	*22*	19	*46*	16	*27*
Generic	1	*4*	4	*10*	6	*15*	7	*12*
Episodic	3	*11*	10	*26*	10	*24*	19	*32*
Autob. Ep.	0	*0*	1	*3*	3	*7*	0	*0*
Proceptual	0	*0*	1	*3*	0	*0*	3	*5*

Main Study

Strategies, Representations, Modalities and Objects:

Numerical Combinations to 20: Visual Phase

Visually presented arithmetical combinations																					
6+3 (2 sec)					Strategy					Representation					Modality				Object		
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		8	0	0	0	0	0	5	2	0	0	1	1	2	1	4	7	1	0
	LA	7		4	2	1	1	0	0	2	2	2	0	1	2	5	1	0	7	1	0
9–2 (2 secs)																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	6		6	0	0	0	1	0	6	0	1	0	0	1	2	0	6	7	0	0
	LA	5		2	0	0	3	1	1	2	0	4	0	1	3	4	1	1	7	1	0
3 + 5																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		7	0	1	0	0	0	7	0	0	0	1	1	1	1	5	7	1	0
	LA	4		2	0	3	2	0	1	0	0	3	1	3	4	4	3	1	4	3	1
8–2																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	7		7	0	0	0	1	0	7	0	1	0	0	1	3	0	5	8	0	0
	LA	8		4	0	0	3	1	0	4	0	1	1	2	4	0	2	2	5	2	1
13–5																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		2	4	1	0	0		3	2	0	0	1	2	2	1	3	6	1	0
	LA	6		1	1	0	1	5		2	0	1	1	5	2	4	5	1	4	5	1
9–8																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		8	0	0	0	0	0	8	0	0	0	0	1	0	0	7	8	0	0
	LA	7		4	1	0	1	2	0	4	1	1	1	2	2	1	2	4	6	2	1
3+4																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		7	1	0	0	0	0	7	1	0	0	0	2	1	0	5	7	0	0
	LA	6		0	1	2	4	0	1	0	1	1	1	5	1	2	5	0	2	5	1
5+2																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		7	0	0	1	0	0	7	0	0	0	1	1	0	1	6	7	0	0
	LA	6		2	0	0	6	0	0	2	0	3	1	2	2	4	2	1	5	2	1
15–8																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	TA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	7		5	2	0	1	0	0	5	2	0	0	1	2	1	1	4	7	1	0
	LA	5		0	0	2	4	2	0	0	0	1	1	7	2	0	7	0	1	7	1
4+5																					
S.	Ab.	C		KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8		8	0	0	0	0	0	8	0	0	0	0	1	0	0	7	8	0	0
	LA	6		3	3	0	2	0	0	2	3	0	1	1	4	4	1	0	3	1	1
2+3+2																					
S.	Ab.	C	NS	KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	8	2	8	0	0	0	0	0	5	3	0	0	0	1	2	0	5	8	0	0
	LA	6	3	2	1	0	4	0	0	1	3	3	0	1	1	5	1	2	7	1	0
3+4+3																					
S.	Ab.	C	NS	KF	DF/T	COf1	CO	CB	CA	Aut	Abs	VC	Fig	Per	Vis	Ver	Per	Uncl	Sym	Fin	Line
	HA	7	1	8	0	0	0	0	0	2	6	0	0	0	2	5	0	2	8	0	0
	LA	7	3	4	0	1	3	0	0	0	4	2	0	1	2	5	2	0	6	2	0

Main Study

Strategies, Representations, Modalities and Objects:

Numerical Combinations to 20: Verbal Phase

Verbally presented arithmetical combinations																				
3+2					Strategy					Representation					Modality				Object	
		C.	KF	DF	DF (LTR)	CO	CB	CA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	8	0	0	0	0	0	8	0	0	0	0	1	1	0	6	8	0	0
	LA	6	4	1	0	1	0	2	4	1	2	1	1	1	5	2	1	6	2	1
4+7																				
		C.	KF	DF	DF (LTR)	CO	B	CA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	5	3	0	0	0	0	4	4	0	0	0	1	4	0	3	8	0	0
	LA	7	0	1	0	7	0	0	0	1	6	1	0	1	2	6	0	3	6	0
7+6																				
		C.	KF	DF	DF (LTR)	CO	CB	CA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	4	4	0	0	0	0	3	5	0	0	0	1	5	0	2	8	0	0
	LA	7	0	3	0	5	0	0	0	3	5	0	0	0	3	5	0	3	5	0
14+8																				
		C.	KF	DF	DF (LTR)	CO	CB	CA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	5	3	0	0	0	0	4	4	0	0	0	1	4	0	3	8	0	0
	LA	4	0	0	0	8	0	0	0	0	6	2	0	1	2	6	0	3	6	0
9–7																				
		C.	KF	DF	DF (LTR)	CO	CB	TA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	7	0	0	1	0	0	6	0	0	1	0	1	3	0	4	8	0	0
	LA	8	2	0	0	1	5	0	2	0	3	3	0	0	3	3	2	5	3	0
12–8																				
		C.	KF	DF	DF (LTR)	CO	CB	TA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	6	1	0	1	0	0	6	1	1	0	0	0	1	1	6	7	1	0
	LA	7	0	1	0	3	3	1	0	1	6	1	0	0	2	6	0	2	6	0
17–13																				
		C.	KF	DF	DF (LTR)	CO	CB	TA	Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	6	5	2	0	1	0	0	4	3	1	0	0	2	3	1	2	7	1	0
	LA	2	0	0	0	6	1	0	0	0	6	1	0	0	1	6	0	1	6	0
27+62																				
		C.	KF	T	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	2	6	0	1	0	0	0	7	1	1	0	0	7	1	1	7	1	0
	LA	2	0	1	0	5	0	0	0	3	3	2	0	0	3	3	0	3	3	0
45+57																				
		C.	KF	T(LR*)	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	0	4	2	1	0	0	0	7	0	1	0	2	8	0	0	8	0	0
	LA	2	0	1	1	4	0	0	0	2	2	2	0	1	4	2	1	4	2	0
29–6																				
	29–6	C.	KF	T	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	4	3	0	0	1	0	4	3	0	1	0	1	5	0	2	8	0	0
	LA	3	0	2	0	0	4	0	0	2	3	2	0	1	4	3	0	5	3	0
64–26																				
		C.		T	DF (LTR)	CU/CB	B		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	8	0	8	0	0	10	0	0	8	0	0	0	4	8	0	0	8	0	0
	LA	0	0	2	0	1	1	0	0	2	3	0	0	0	2	3	0	2	3	0
73–32																				
		C.	KF	T(LR*)	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	6	1	7	0	0	0	0	2	6	0	0	0	1	8	0	0	8	0	0
	LA	2	0	0	0	2	3	0	0	3	1	3	0	0	3	1	0	3	1	0
188+267																				
		C.	KF	T(LR*)	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	5	0	7	1	0	0	0	0	7	0	0	0	3	8	0	0	8	0	0
	LA	0	0	1	0	2	0	0	0	2	2	1	0	1	3	2	0	5	2	0
396–157																				
		C.	Alg(P)	T(LR*)	DF (LTR)	CO	CB		Aut	Abs	Per	VC	Fig	Vis	Ver	Per	Und	Sym	Fin	Line
	HA	5	3	1	2	0	0	0	0	6	0	0	0	3	6	0	0	6	0	0
	LA	0	0	0	0	1	0	0	0	1	0	1	0	2	2	0	0	3	0	0

APPENDIX 3.3

Main Study

Strategies, Representations, Modalities and Objects:

Two/Three Digit Number Combinations: Visual Phase

Visually presented arithmetical combinations (written sums)																										
30+ 57 (Vertical)					STRATEGY							DIRECTION			REPRESENTATION						MODALITY			OBJECT		
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR		AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	3	2	2	0	0	0	1	0		3	2	3	3	4	1	0	0	1	4	1	2	7	1	0
LA	4	4	0	1	2	0	0	1	2	1		0	3	4	0	3	3	0	1	0	4	3	1	4	3	0
47+15(Vertically)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR		AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	1	0	4	3	0	0	0	0		1	3	4	1	7	0	0	0	0	6	0	2	8	0	0
LA	3	5	0	0	1	0	0	3	4	0		0	3	5	0	1	5	1	1	1	2	5	0	2	5	1
274+159(Vertically)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR		AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	0	0	4	0	2	0	0	0		0	0	6	0	6	0	0	0	0	6	0	2	6	0	0
LA	1	7	0	0	2	0	0	2	3	0		0	2	5	0	2	2	0	2	0	4	2	1	4	2	0
5+135 (Horizontal)																										
	C	W	KF	KF	KF	T		CO				AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	7	0	1	0	0	0	0	0		7	0	1	7	1	0	0	0	1	1	0	6	8	0	0
LA	5	3	2	0	0	2	0	2	0	0		2	0	1	2	3	1	0	1	2	3	1	1	3	1	0
11+696(Horizontal)																										
			KF	KF	KF	T	T	CO	CO	CA	CO (all)															
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR		AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	0	0	2	5	0	0	0	0	0	0	5	2	0	7	0	0	0	2	7	0	1	7	0	0
LA	4	4	0	0	1	1	0	0	1	0	2	0	1	3	0	3	0	0	3	1	6	0	0	6	0	0
21–16 (Vertically)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR	CO (all)	AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	7	1	5	0	3	0	0	0	0	0	0	5	0	3	5	3	0	0	0	0	2	0	6	8	0	0
LA	2	6	0	1	4	0	0	1	0	0	0	0	2	4	0	6	0	0	1	0	4	0	3	7	0	0
82–24 (Vertically)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR	CO (all)	AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	1	0	4	2	0	1	0	0	0	1	3	4	1	6	1	0	0	0	4	1	3	7	1	0
LA	0	8	0	0	3	0	0	1	0	0	0	0	2	3	0	4	1	0	1	0	4	1	0	4	1	0
293–189(Vertically)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR	CO (all)	AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	0	0	5	2	0	0	0	0	0	0	2	5	0	7	0	0	0	0	7	0	1	7	0	0
LA	0	8	0	2	1	0	0	2	0	0	0	0	4	1	0	4	1	0	1	0	4	1	0	5	1	0
438+21 (Horizontal)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR	CO (all)	AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	6	2	0	0	4	1	0	0	0	0	0	0	1	4	0	6	0	0	0	2	4	0	3	6	0	0
LA	3	5	0	1	3	0	0	0	0	0	0	0	2	3	0	4	0	0	0	2	3	0	2	4	0	0
687+47(Horizontal)																										
			KF	KF	KF	T	T	CO	CO	CA																
Ab.	C	W		LTR	RTL	LTR	RTL	LTR	RTL	LTR	CO (all)	AUT	LTR	RTL	Aut	Abs	Per	Fig	VC	Vis	Ver	Per	Uncl	Sym	Fin	Dots
HA	8	0	1	0	5	1	0	0	0	0	0	1	1	5	1	6	0	0	0	3	6	0	2	7	0	0
LA	3	5	0	1	2	2	0	0	0	0	0	0	3	2	0	5	0	0	0	1	4	0	1	5	0	0