ON REPRESENTING CLAIMS FOR COHERENT RISK MEASURES

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Abstract. We consider the problem of representing claims for coherent risk measures. For this purpose we introduce the concept of (weak and strong) time-consistency with respect to a portfolio of assets, generalizing the one defined in Delbaen [7].

In a similar way we extend the notion of m-stability, by introducing weak and strong versions. We then prove that the two concepts of m- stability and time-consistency are still equivalent, thus giving necessary and sufficient conditions for a coherent risk measure to be represented by a market with proportional transaction costs. We go on to deduce that, under a separability assumption, any coherent risk measure is strongly time-consistent with respect to a suitably chosen countable portfolio, and show the converse: that any market with proportional transaction costs is equivalent to a market priced by a coherent risk measure, essentially establishing the equivalence of the two concepts.

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1. Introduction

The biggest practical success of Mathematical Finance to date is in explaining how to hedge against contingent claims (and thus how to price them uniquely) in the context of a complete and frictionless market.

Two relatively recent developments in Mathematical Finance are the introduction of the concept of coherent risk measure and work on trading with (proportional) transaction costs. Both of these developments seek to deal with deviations from the idealised situation decribed above.

Coherent risk measures were first introduced by Artzner, Delbaen, Eber and Heath [1], in order to give a broad axiomatic definition for monetary measures of risk.

In their fundamental theorem, Artzner *et al.* showed that such a coherent risk measure can be represented as the supremum of expectation over a set of test probabilities.

Key words: reserving; hedging; representation; coherent risk measure, transaction costs; time-consistency; m-stability.

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Thus the setup includes superhedging under the class of all EMMs (in an incomplete, frictionless market).

Recent work on trading with transaction costs by Kabanov, Stricker, Rasonyi, Jouini, Kallal, Delbaen, Valkeila and Schachermayer, amongst others ([13], [14], [12], [8], [18]), lead to a necessary and sufficient condition for the closure of the set of claims attainable for zero endowment to be arbitrage-free (Theorem 1.2 of [11]) and a characterisation of the 'dual' cone of pricing measures (consistent price processes) ([18]).

In this paper, we consider a coherent risk measure as a pricing mechanism: in other words we assume that an economic agent is making a market in (or at least reserving for) risk according to a coherent risk measure, ρ say.

So, we consider the risk value of a financial claim as the basic price for the associated contract. Unfortunately such a pricing mechanism is not closely linked to the notion of hedging, and so the price evolution from trading time to maturity time is not well-defined. For example, taking the obvious definition for ρ_t—the price of risk at time t—it is not necessarily true that $\rho = \rho \circ \rho_t$ (see Delbaen [7]). Indeed, Delbaen has given a necessary and sufficient condition for ρ to be time-consistent in this way: the m-stability property ([7]), and this condition is easily violated. Notice that, in the absence of m-stability, reserving is not possible (without 'new business strain'), since the time 0 price of (reserving for) the time-t reserve for a claim X may (and sometimes will) be greater than the time 0 reserve for X.

Our preliminary results in this paper are as follows:

(1) we introduce a generalisation of the concept of numéraire suitable for the context of coherent risk measures (equation (4.1)) and give a characterisation of such numéraires (Theorem 4.1);
(2) we show (in equation (4.4) and Lemma 4.4) how to define a v-denominated risk measure with the same acceptance set as ρ, where v is the final value of a positive claim or of a different currency.

Then we pursue the idea of pricing using several currencies/commodities/denominations.

If we do this, then the option of creating reserves in several currencies becomes available. Moreover, the possibility of trading between currencies or commodities in order to hedge a contingent claim also appears.

Our main results are as follows:

(3) in Theorem 7.11 we give a necessary and sufficient condition (which generalises Delbaen's m-stability property) for time-consistency with respect to a portfolio of assets (we term this weak representation);
(4) in Theorems 7.12 and 7.16, we give two necessary and sufficient conditions (the first akin to Schachermayer's description of the cone of consistent price processes) for the attainability of all acceptable claims purely by trading in a portfolio of assets;
(5) in Theorem 7.27 we show that, under a separability condition, all acceptable claims may be attained by trading in a fixed countable collection of assets;

(6) finally, we show, in Theorem 8.3, that every arbitrage-free market corresponding to trading with transaction costs in fact corresponds to the representation of a coherent risk measure using a set of commodities/numéraires .

2. Preliminaries

The paper is organized as follows: in section 3 we recall properties of a conditional coherent risk measure. In section 4, we consider a one-period market, defined by a coherent risk measure, and define $\mathcal{N}_0$, the set of all numéraires in which we can trade in this market. Given a numéraire $v \in \mathcal{N}_0$, we define the v-denominated coherent risk measure ρ^v. Remark that its value in cash (that is to say, pieces of paper which pay 1 unit of account $\mathbf{1}$ at time T i.e. Zero Coupon Bonds), given by $\rho(\rho^v(X)v)$, may be different from $\rho(X)$. We discuss, in an appendix, the equivalence classes of numéraires, where such prices are the same.

Next, we consider the general multi-period model and define $\mathcal{N}$, the set of all numéraires in which we can trade in every time period. In section 5 we introduce the two concepts of *time-consistency* and *m-stability* with respect to a portfolio of assets in $\mathcal{N}$. This new version of time-consistency generalizes the one introduced in Delbaen [7], and allows us not only to consider cash-flows but also the possibility of investing in other assets. In order to show the link between these two properties in section 7, we start with the case where the portfolio of assets V is finite and then consider the cone $\mathcal{A}(V)$ of all portfolios in assets V, attainable from non-positive endowment. Then we extend these results to the case where the portfolio of assets is countable. The result in the finite case is based on the results of section 6, where we consider a more general cone $\mathcal{B}$ of portfolios attainable from non-positive endowment. We will see that the notion of *decomposability* of the cone $\mathcal{B}$, translated to the case $\mathcal{B} = \mathcal{A}(V)$, is equivalent to the time-consistency property of the cone $\mathcal{A}$ with respect to V.

We assume that we are equipped with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, \mathbb{P})$, where $\mathcal{F}_0$ is not necessarily trivial.

Now recall the setup from Schachermayer's paper [18]: we may trade in d assets at times $0, \dots, T$. We may burn any asset and otherwise trades are given by a bid-ask process π taking values in $\mathbb{R}^{d\times d}$, with π adapted to $(\mathcal{F}_t)_{t=0}^T$. The bid-ask process gives the (time t) price for one unit of each asset in terms of each other asset, so that

$$\pi_t^{i,i} = 1, \forall i,$$

and $\pi_t^{i,j}$ is the (random) number of units of asset i which can be traded for one unit of asset j at time t. We assume (with Schachermayer) that we have "netted out" any advantageous trading opportunities, so that, for any t and any $i_0, \dots, i_n$:

$$\pi_t^{i_0,i_n} \leq \pi_t^{i_0,i_1} \dots \pi_t^{i_{n-1},i_n}.$$

The time t trading cone, K_t, consists of all those random trades (including the burning of assets) which are available at time t. Thus we can think of K_t as consisting of all those random vectors which live (almost surely) in a random closed convex cone $K_t(\omega)$, where, denoting the ith canonical basis vector of $\mathbb{R}^d$ by e_i, $K_t(\omega)$ is the finitely-generated convex (hence closed) cone with generators $\{e_j - \pi_t^{i,j}(\omega)e_i, 1 \leq i \neq j \leq d;\ \text{and}\ -e_k, 1 \leq k \leq d\}$. We shall say that η is a self-financing process if $\eta_t - \eta_{t-1} \in K_t$ for each t, with $\eta_{-1} \stackrel{def}{=} 0$.

It follows that the cone of claims attainable from zero endowment is $K_0 + \ldots + K_T$ and we denote this by $\mathcal{B}(\pi)$. Note that $-K_t$ is the time-t solvency cone of claims, i.e. all those claims which may be traded to 0 at time t. Note also that, following Kabanov et al. [15], Schachermayer uses "hat" notation (which we have dropped) to stress that we are trading physical assets and uses $-K$ where we use K.

We shall show in section 8 that, by adding an extra period, we may represent $\mathcal{B}(\pi) \cap \mathcal{L}^\infty$ by a coherent risk measure and a new (final) set of prices for the vector of assets. More precisely, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathcal{F} \subset \tilde{\mathcal{F}}$ and with $\tilde{\mathbb{P}}$ coinciding with $\mathbb{P}$ on $\mathcal{F}$, a vector of strictly positive random variables $V = (v^1, \ldots, v^d) \in \mathcal{L}^1(\tilde{\mathbb{P}}; \mathbb{R}^d)$ and a set of probability measures $\mathcal{Q}$, defined on $\tilde{\Omega}$, such that:

$$\mathcal{B}(\pi) \cap \mathcal{L}^\infty = \left\{ X \in \mathcal{L}^\infty(\mathbb{P}; \mathbb{R}^d) :\ \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(X.V) \leq 0 \right\}.$$

3. Conditional coherent risk measures.

In the paper we will be dealing with pricing monetary risks in the future and, in general, in the presence of partial information. Accordingly, we recall in this section the definition and the main result on the characterization of a **conditional** coherent risk measure. This concept was introduced by Wang [19] and has been further elaborated upon within different formal approaches by Artzner et al. [2], Riedel [16], Weber [20], Engwerda et al. [5], Scandolo [17], Detlefsen and Scandolo [4].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with $\mathcal{F}_0 \subset \mathcal{F}$ a sub-σ-algebra. Throughout this section we consider the mapping $\rho_0 : \mathcal{L}^\infty(\mathcal{F}) \to \mathcal{L}^\infty(\mathcal{F}_0)$.

Definition 3.1. *(See Detlefsen and Scandolo [4]) We say that the mapping ρ_0 is a relevant, conditional coherent risk measure with the Fatou property if it satisfies the following axioms:*

1. *Monotonicity: For every $X, Y \in \mathcal{L}^\infty(\mathcal{F})$,*
$$X \leq Y \text{ a.s} \Rightarrow \rho_0(X) \leq \rho_0(Y) \text{ a.s.}$$
2. *Subadditivity: For every $X, Y \in \mathcal{L}^\infty(\mathcal{F})$,*
$$\rho_0(X + Y) \leq \rho_0(X) + \rho_0(Y) \text{ a.s.}$$
3. *$\mathcal{F}_0$-Translation invariance: For every $X \in \mathcal{L}^\infty(\mathcal{F})$ and $y \in \mathcal{L}^\infty(\mathcal{F}_0)$,*
$$\rho_0(X + y) = \rho_0(X) + y \text{ a.s.}$$
4. *$\mathcal{F}_0$-Positive homogeneity: For every $X \in \mathcal{L}^\infty(\mathcal{F})$ and $a \in \mathcal{L}^\infty_+(\mathcal{F}_0)$, we have*
$$\rho_0(a\,X) = a\,\rho_0(X) \text{ a.s.}$$
5. *The Fatou property: a.s $\rho_0(X) \leq \liminf \rho_0(X_n)$, for any sequence $(X_n)_{n \geq 1}$ uniformly bounded by 1 and converging to X in probability.*
6. *Relevance: for each set $F \in \mathcal{F}$ with $\mathbb{P}[F|\,\mathcal{F}_0] > 0$ a.s, $\rho_0(1_F) > 0$ a.s.*

We point out that, in accordance with our aim of interpreting ρ_0 as a **pricing** mechanism, we have introduced a change of sign in Definition 3.1, so $X \mapsto \rho_0(-X)$ is a conditional coherent risk measure in the sense of [4], for example.

Proposition 3.2. *(See Detlefsen and Scandolo* [4]*) Let the mapping ρ_0 be a relevant conditional coherent risk measure satisfying the Fatou property. Then*

(1) *The acceptance set*

$$\mathcal{A}_0 \stackrel{def}{=} \{X \in \mathcal{L}^\infty(\mathcal{F})\ ;\ \rho_0(X) \leq 0\ a.s\}$$

is a weak-closed convex cone, arbitrage-free, stable under multiplication by bounded positive $\mathcal{F}_0$-measurable random variables and contains $\mathcal{L}^\infty_-(\mathcal{F})$.*

(2) *There exists a convex set of probability measures $\mathcal{Q}$, all of them being absolutely continuous with respect to $\mathbb{P}$ and containing at least one equivalent probability measure, such that for every $X \in \mathcal{L}^\infty(\mathcal{F})$:*

$$\rho_0(X) = \textit{ess-sup}\,\{\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_0)\ ;\ \mathbb{Q} \in \mathcal{Q}\}\,. \tag{3.1}$$

Definition 3.3. *Given a conditional coherent risk measure ρ_0, we define $\mathcal{Q}^{\rho_0}$ as follows:*

$$\mathcal{Q}^{\rho_0} = \left\{\mathbb{Q} \ll \mathbb{P}\ ;\ \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{A}_0^*\right\}, \tag{3.2}$$

where $\mathcal{A}_0^*$ is the polar cone of $\mathcal{A}_0$. Conversely, given $\mathcal{Q}$ a collection (not necessarily closed, or convex) of probability measures absolutely continuous with respect to $\mathbb{P}$, we define

$$\rho_0^{\mathcal{Q}}(X) = \textit{ess-sup}\,\{\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_0)\ ;\ \mathbb{Q} \in \mathcal{Q}\}\,. \tag{3.3}$$

The set $\mathcal{Q}^{\rho_0}$ is the largest set $\mathcal{Q}$ for which $\rho_0 = \rho_0^{\mathcal{Q}}$.

4. Characterization of numéraires

First, we do the following:

(1) we fix a relevant, coherent risk measure with the Fatou property, $\rho : \mathcal{L}^\infty \to \mathbb{R}$ with acceptance set $\mathcal{A}$ (recall that $\mathcal{A} = \{X :\ \rho(X) \leq 0\}$ and that $\rho(X) = \inf\{c :\ X - c\mathbf{1} \in \mathcal{A}\}$) and test probabilities $\mathcal{Q}$, a maturity time T and a unit of account $\mathbf{1}$ (a currency e.g pounds sterling). The unit of account $\mathbf{1}$ is interpreted as a contract that pays one pound at time T, i.e. a zero coupon bond with redemption value of one pound.
(2) we suppose that trading is frictionless at time T and then for any claim or asset $\hat{X}$, we denote by X its value in terms of the unit of account $\mathbf{1}$ at time T.

It is necessary first to characterize assets which give rise to the same acceptance sets as ρ. Note that, since the proofs in this section are almost all straightforward, we give most of them in an appendix—any missing proofs will be found in Appendix A.

4.1. **The one-period model.** Recall that $\mathcal{A}$ is an arbitrage-free, closed, convex cone in $\mathcal{L}^\infty$ which contains $\mathcal{L}^\infty_-$.

Since $\mathcal{F}_0$ is not necessarily trivial, time zero may be understood to be some time in the future and we *interpret* $\mathcal{A}_0$ as the set of claims acceptable at time zero, using the definition in Proposition 3.2, so that $\rho_0(X) = \text{ess-sup}\,\{\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_0)\,;\,\mathbb{Q}\in\mathcal{Q}\}$, and $\mathcal{A}_0 = \{X\in\mathcal{L}^\infty : \mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_0)\le 0 \text{ for all } \mathbb{Q}\in\mathcal{Q}\} = \{X : \rho_0(X)\le 0 \text{ a.s.}\}$.

In this one-period market governed by the pricing mechanism ρ_0, to say that $\hat{v}$ is a *numéraire* at time zero, means that for any claim $\hat{X}$, there exists an $\mathcal{F}_0$-measurable number, λ, of contracts, each paying v at maturity time T, such that the final position $X-\lambda\, v$ is admissible. We think of $X-\lambda\, v$ as being obtained as the net payoff from a futures contract which agrees to exchange λ units of $\hat{v}$ for the claim $\hat{X}$ at the maturity date T.

Thus, we *define* $\mathcal{N}_0$, the set of all numéraires, at time zero by:

$$\mathcal{N}_0 = \{v\in\mathcal{L}^\infty_+ :\ \mathcal{L}^\infty = \mathcal{A}_0 + \mathcal{L}^\infty(\mathcal{F}_0)v\}, \tag{4.1}$$

and, since $\mathcal{L}^\infty_-\subset\mathcal{A}_0$, it follows that

$$\mathcal{N}_0 = \{v\in\mathcal{L}^\infty_+ :\ \mathcal{L}^\infty = \mathcal{A}_0 + \mathcal{L}^\infty_+(\mathcal{F}_0)v\}. \tag{4.2}$$

Now we characterise the numéraires.

Theorem 4.1. *Suppose that $v\in\mathcal{L}^\infty_+$, then $v\in\mathcal{N}_0$ if and only if*

$$\lambda_0(v) \stackrel{def}{=} \textit{ess-inf}_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0) = -\rho_0(-v) > 0 \; a.s,$$

and

$$\frac{1}{\lambda_0(v)} \in \mathcal{L}^\infty.$$

Given a numéraire, we may, of course use it as a new unit of account. First we define a measure that prices claims (expressed in units of account $\mathbf{1}$), in terms of contracts which pay the new numéraire.

Definition 4.2. *Let $v\in\mathcal{L}^\infty_+$. The mapping $\tau:\mathcal{L}^\infty(\mathcal{F})\to\mathcal{L}^\infty(\mathcal{F}_0)$ is said to be a v-*denominated, conditional, relevant, coherent risk measure with the Fatou property, with respect to $\mathcal{F}_0$ *if it satisfies properties 1, 2, 4, 5 and 6 of Definition 3.1 and $\mathcal{F}_0$-translation invariance with respect to v, i.e for every $X\in\mathcal{L}^\infty(\mathcal{F})$ and $y\in\mathcal{L}^\infty(\mathcal{F}_0)$, we have*

$$\tau(X+yv) = \tau(X) + y \; a.s. \tag{4.3}$$

It is easily shown that, for each $v\in\mathcal{N}_0$ and each $X\in\mathcal{L}^\infty$, the set

$$\Theta(X,v) \stackrel{def}{=} \{m\in\mathcal{L}^\infty(\mathcal{F}_0) :\ X-mv\in\mathcal{A}_0\},$$

is closed in $\mathcal{L}^\infty$ and is a lattice with respect to the minimum relation, i.e for all $m,m'\in\Theta(X,v)$ we have $\min(m,m')\in\Theta(X,v)$. We may thus define the mapping $\rho^v_0:\mathcal{L}^\infty\to\mathcal{L}^\infty(\mathcal{F}_0)$ by

$$\rho^v_0(X) = \text{ess-inf}\{m\in\mathcal{L}^\infty(\mathcal{F}_0) :\ X-mv\in\mathcal{A}_0\}, \tag{4.4}$$

and observe that $X-\rho^v_0(X)v\in\mathcal{A}_0$ a.s.

Example 4.3. *We fix our unit of account* $\mathbf{1} \equiv 1$ *pound sterling and then another currency, let us say the US dollar, will be denoted by* $\hat{\delta}$ *where* δ *is the dollar/pound exchange rate* at maturity. *So, if we assume that* $\delta \in \mathcal{N}_0$*, and* $\hat{X}$ *is a claim, with a value in pounds at maturity* X*, then* $\rho_0^{\delta}(X)$ *is the number of contracts in dollars (the amount in dollars promised at time 0 and payable at maturity or the dollar-denominated futures price) we seek as payment to make the risk* X *acceptable.*

We now proceed to show that a numéraire has all the requisite properties.

Lemma 4.4. *Let* $v \in \mathcal{N}_0$*. The mapping* ρ_0^v *defined in (4.4), is a* v*-denominated conditional, relevant, coherent risk measure which satisfies the Fatou property. Moreover*

(i) $\mathcal{A}_0 = \{X \in \mathcal{L}^\infty : \ \rho_0^v(X) \leq 0 \ a.s.\}$.
(ii) *For all* $X \in \mathcal{L}^\infty$,

$$\rho_0^v(X) = \textit{ess- sup}\left\{\frac{\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0)}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)} : \mathbb{Q} \in \mathcal{Q}\right\}.$$

Remark 4.5. *Let* τ *be a* v*-denominated conditional coherent risk measure (with respect to* $\mathcal{F}_0$*), then* $\tau = \zeta^v$ *where the conditional coherent risk measure* ζ *is defined by:*

$$\zeta(X) = \textit{ess-inf}\{m \in \mathcal{L}^\infty(\mathcal{F}_0) : \ \tau(X-m) \leq 0 \ a.s\}.$$

This follows from (i) of Lemma 4.4.

Remark 4.6. *Let* $v \in \mathcal{N}_0$ *and define the set of probabilities,*

$$\mathcal{Q}^v \stackrel{def}{=} \left\{\mathbb{R} : \frac{d\mathbb{R}}{d\mathbb{Q}} = \frac{v}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)} \textit{ for some } \mathbb{Q} \in \mathcal{Q}\right\}.$$

So for all $X \in \mathcal{L}^\infty$ *we have*

$$\rho_0^v(X) = \textit{ess-sup}_{\mathbb{Q}\in\mathcal{Q}^v}\mathbb{E}_{\mathbb{Q}}\left(\frac{X}{v}\middle|\mathcal{F}_0\right) \stackrel{def}{=} \rho_0^{(v)}\left(\frac{X}{v}\right),$$

where the mapping $\rho_0^{(v)}$ *is a conditional coherent risk measure with test probabilities* $\mathcal{Q}^v$*. Notice that we can arrive at a price for the claim* X *(in terms of contracts in* v*) in two different ways: firstly, by applying the* v*-denominated coherent risk measure* ρ_0^v*, and secondly, by converting the value* X *to its value (at maturity) –* X/v*—when expressed in terms of the new unit of account* $\mathbf{1}' \equiv \hat{v}$*, and then applying* $\rho_0^{(v)}$ *to the value* X/v*.*

Remark 4.7. *The value of a given risk as provided by our coherent risk measure* ρ_0*, does not incorporate the risk coming from a change over time in the value of the unit of account itself; in other words, the randomness of the discount rate is not taken into account. Now, by working with the discounted counterpart* $\tilde{\rho}_0$ *of* ρ_0*, the value of money over time does come into play. To express* $\tilde{\rho}_0$ *in terms of* ρ_0 *and the interest rate* r*, we denote by* $\hat{\mathbf{1}}_0$ *the contract that delivers* $1+r$ *at maturity time for one unit invested at time zero and then* $\mathbf{1}_0 = 1+r$ *(expressed in units of account* $\mathbf{1}$ *at maturity time). For a contract that pays* X *at maturity time, we have on the one hand, the price* $\tilde{\rho}_0(X/(1+r))$ *is the cash value payable at time zero and on the other hand, under the assumption that*

$1+r \in \mathcal{N}_0$, $\rho_0^{1+r}(X)$ is the number of contracts in $\hat{\mathbf{1}}_0$ (each paying $1+r$ at maturity time) so we see that:

$$\tilde{\rho}_0(X/(1+r)) = \rho_0^{1+r}(X).$$

4.2. **The multi-period model.** Now, for $t = 0, 1, \ldots, T$, we define the set of claims attainable for 0 at time t:

$$\mathcal{A}_t = \{X \in \mathcal{L}^\infty \,;\, \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) \le 0 \text{ for all } \mathbb{Q} \in \mathcal{Q}\},$$

and $\mathcal{N}_t$, the set of all numéraires at time t, is defined as the set of $v \in \mathcal{L}^\infty_+$ such that $\mathcal{L}^\infty = \mathcal{A}_t + \mathcal{L}^\infty(\mathcal{F}_t)\, v$. We define $\mathcal{N} = \bigcap_{t=0}^{T} \mathcal{N}_t$ and henceforth refer to it as the set of numéraires and any element of it as a numéraire. Note that $\mathcal{N}_T = \{X \in \mathcal{L}^\infty_+ : \text{ ess-inf } X > 0\}$.

Definition 4.8. *For all $v \in \mathcal{N}$ and $t = 0, 1, \ldots, T$, we define the mapping $\rho_t^v : \mathcal{L}^\infty(\mathcal{F}_T) \to \mathcal{L}^\infty(\mathcal{F}_t)$ by*

$$\rho_t^v(X) = \textit{ess-inf}\{m \in \mathcal{L}^\infty(\mathcal{F}_t) : \; X - mv \in \mathcal{A}_t\}\,.$$

Notice that $\rho_T^v(X) = \frac{X}{v}$.

Corollary 4.9. *For all $v \in \mathcal{N}$ and $t = 0, 1, \ldots, T$, the mapping ρ_t^v as defined in Definition 4.8, is a v-denominated, conditional, relevant, coherent risk measure with the Fatou property (with respect to $\mathcal{F}_t$), given by*

$$\rho_t^v(X) = \textit{ess-sup}\left\{\frac{\mathbb{E}_{\mathbb{Q}}(X|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{Q}}(v|\,\mathcal{F}_t)};\, \mathbb{Q} \in \mathcal{Q}\right\},$$

and

$$\mathcal{A}_t = \{X \in \mathcal{L}^\infty : \; \rho_t^v(X) \le 0 \;a.s\}.$$

Proof. Immediate consequence of Lemma 4.4. □

Lemma 4.10. *For all $X \in \mathcal{L}^\infty$, $0 \le t \le t+s \le T$ and $v, w^1, ..., w^n \in \mathcal{N}$ we have:*

$$\rho_t^v(X) \le \rho_t^v\left(\sum_{i=1}^{n} \rho_{t+s}^{w^i}(X_i) w^i\right),$$

whenever $X_1, \ldots, X_n \in \mathcal{L}^\infty$ and $X = X_1 + ... + X_n$.

Proof. Similar to the proof of assertion 2 in Lemma A.2. □

Remark 4.11. *In particular, if $v \in \mathcal{N}$, then*

$$\rho_t^v(X) \le \rho_t^v\left(\rho_{t+1}^v(X)v\right).$$

5. Time-consistency properties

As we discussed in the introduction, the essential element of pricing or hedging in a financial market is to build a financing strategy that starts with the price of a claim and ends with a value equal to the claim itself at maturity. Speaking loosely, if this strategy is built by trading in a specific set of assets $V = (v^1, \ldots, v^d)$, we shall say that the claim is *represented* by the vector V.

Delbaen [7] gave the following

Definition 5.1. *A coherent risk measure ρ is said to be* time-consistent *if for all $0 \leq t \leq t+s \leq T$ we have $\rho_t \circ \rho_{t+s} = \rho_t$.*

In [10], Jacka and Berkaoui proved that within a "coherent risk measure market", the property of time-consistency of ρ is equivalent to saying that any bounded claim is represented by the unit of account $\mathbf{1}$. More precisely, for any claim $X \in \mathcal{L}^\infty$, there is a sequence $(X_t)_{t=0,1,\ldots,T-1}$, with $X_t \in \mathcal{A}_t \cap \mathcal{L}^\infty(\mathcal{F}_{t+1})$ for each t, such that $X = \rho(X) + X_0 + \ldots + X_{T-1}$. We can think of X_t as being the net payoff at time $t+1$ from a contract entered into at time t or, in the context of an insurance company making reserves or a financial institution marking to market, X_t is the difference between reserves for the claim X at times t and $t+1$.

In this section we generalize this concept and define strong and weak time-consistency with respect to a portfolio of numéraires $U \subset \mathcal{N}$.

Definition 5.2. Weak time-consistency *Let $U \subset \mathcal{N}$. We say that $\mathcal{A}$ is weakly U-time-consistent if for each $v \in U$, $t \in \{0, 1, \ldots, T-1\}$ and $X \in \mathcal{L}^\infty$, there exist sequences $X_n \in \mathcal{L}^\infty$, $Y^n_{t+1} \in \mathcal{L}^\infty(\mathcal{F}_{t+1}; \mathbb{R}^n)$ and $V^n = (v^1, \ldots, v^n)$ (where $\{v^1, \ldots, v^n\} \subset U$) such that*

(i) *the sequence X_n converges weakly* to X in $\mathcal{L}^\infty$,*
(ii) $X_n - Y^n_{t+1}.V^n \in \mathcal{A}_{t+1}$,

and

(iii) $\rho^v_t(X) = \liminf_{n\to+\infty} \rho^v_t\left(Y^n_{t+1}.V^n\right)$.

In particular we say that $\mathcal{A}$ is weakly v-time-consistent when $U = \{v\}$ and weakly time-consistent when $U = \{1\}$.

Notice that if properties (i)–(iii) in Definition 5.2 hold for some $v \in U$ and for all t then they hold for each $v \in U$.

We shall show that this definition generalises Delbaen's in Theorem 5.10.

Example 5.3. *The coherent risk measure ρ, associated to a singleton test probability set $\mathcal{Q} = \{\mathbb{P}\}$, is weakly time-consistent.*

Example 5.4. *Consider a binary branching tree with two branches. So $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F}_0$ is trivial, $\mathcal{F}_1 = \sigma(\{1, 2\}, \{3, 4\})$ and $\mathcal{F}_2 = 2^\Omega$. Equating each probability measure $\mathbb{Q}$ on Ω with the vector of probability masses $(\mathbb{Q}(\{1\}), \mathbb{Q}(\{2\}), \mathbb{Q}(\{3\}), \mathbb{Q}(\{4\}))$, take*

$\mathcal{Q} = co(\mathbb{Q}_1, \mathbb{Q}_2)$, where $\mathbb{Q}_1 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4})$ and $\mathbb{Q}_2 = (\frac{1}{2}, \frac{1}{8}, \frac{3}{16}, \frac{3}{16})$ (here co denotes the convex hull). Let ρ be the associated coherent risk measure.

Denoting $X \in L^\infty$ by the corresponding (lower case) vector (x_1, x_2, x_3, x_4) (so that $X(i) = x_i$) we see that

$$\rho(X) = \max\left(\frac{1}{3}x_1 + \frac{1}{6}x_2 + \frac{1}{4}x_3 + \frac{1}{4}x_4, \frac{1}{2}x_1 + \frac{1}{8}x_2 + \frac{3}{16}x_3 + \frac{3}{16}x_4\right),$$

and

$$\rho_1(X)(\omega) = \begin{cases} \max(\frac{2}{3}x_1 + \frac{1}{3}x_2, \frac{4}{5}x_1 + \frac{1}{5}x_2) : & \omega \in \{1, 2\} \\ \frac{1}{2}x_3 + \frac{1}{2}x_4 : & \omega \in \{3, 4\} \end{cases}$$

Take $X = (3, 4, 0, 0)$ to see that $\rho \circ \rho_1 \neq \rho$:

$$\rho_1(X) = \begin{cases} \max(\frac{10}{3}, \frac{16}{5}) = \frac{10}{3} : & \omega \in \{1, 2\} \\ 0 : & \omega \in \{3, 4\}, \end{cases}$$

and $\rho(\rho_1(X)) = \max(\frac{5}{3}, \frac{25}{12}) = \frac{25}{12}$, whereas $\rho(X) = \frac{5}{3}$.

Now, setting $v = 1 + 1_{\{1\}}$ and $\tilde{x} = \frac{1}{2}(x_3 + x_4)$, it is easy to check that

$$X = W_X + Z_X + \Delta_X,$$

where

$$W_X = 2x_2 - x_1,\ \Delta_X = \frac{1}{2}(x_3 - x_4)(1_{\{3\}} - 1_{\{4\}}),\ Z_X = ((x_1 - x_2)1_{\{1,2\}} + (x_1 + \tilde{x} - 2x_2)1_{\{3,4\}})v.$$

We claim that ρ is weakly $(1, v)$-time-consistent.

Proof. To check this, first take $V^n = V = (1, v)$ and $Y_2^n = Y_2 = (X, 0)$ for each n, then $X - Y_2.V = 0 \in \mathcal{A}_2$. Now take $Y_1^n = Y_1 = (W_X, \frac{Z_X}{v})$ for each n, then $X - Y_1.V = \Delta_X$ and $\rho_1(\Delta_X) = 0$, so $\Delta_X \in \mathcal{A}_1$. Finally, $\rho_1(X) = \rho_1(Y_2.V)$ (obviously) while it is easy to see that $\rho(Y_1.V) = \rho((x_1, x_2, \tilde{x}, \tilde{x})) = \rho(X)$ so the result follows. □

Definition 5.5. Strong time-consistency *Let $U \subset \mathcal{N}$. We say that $\mathcal{A}$ is strongly U-time-consistent (or strongly time-consistent with respect to U) if for all $v \in U$, $t = 0, 1, \ldots, T-1$ and $X \in \mathcal{L}^\infty$, there exist sequences $X_n \in \mathcal{L}^\infty$, $Y_t^n \in \mathcal{L}^\infty(\mathcal{F}_t; \mathbb{R}^n)$ and $V^n = (v^1, \ldots, v^n) \subset U$ such that*

(i) *the sequence X_n converges weakly* to X in $\mathcal{L}^\infty$,*
(ii) *$X_n - Y_t^n.V^n \in \mathcal{A}_{t+1}$,*

and

(iii) *$\rho_t^v(X) = \liminf_{n\to+\infty} \rho_t^v(Y_t^n.V^n)$.*

We say that $\mathcal{A}$ is strongly v-time-consistent when $U = \{v\}$ and strongly time-consistent when $U = \{1\}$.

Remark 5.6. *Strong time consistency says that we may trade at each time t in assets in U to approximate X for an initial endowment which approximates the 'price' of X.*

Example 5.7. *Define $\Omega = \{0, 1\}$, $T = 1$, $\mathbb{P}$ uniform and $v(0) = 1$, $v(1) = 2$. Then the coherent risk measure ρ, associated to the singleton test probability set $\mathcal{Q} = \{\mathbb{P}\}$, is strongly $(1, v)$-time-consistent.*

Proof. For any $X \in \mathcal{L}^\infty$, there exists $\alpha, \beta \in \mathbb{R}$ such that $X = \alpha + \beta\, v$. Then $\rho_0(X) = \rho_0(\alpha + \beta\, v)$ and $X - (\alpha + \beta\, v) = 0 \in \mathcal{A}_1$. □

Example 5.8. *Suppose that $\mathbb{P}$ is the unique EMM for a (vector) price process $S \in \mathcal{L}^\infty$, then the coherent risk measure ρ, associated to the singleton test probability set $\mathcal{Q} = \{\mathbb{P}\}$, is strongly (S_T)-time-consistent.*

Proof. Notice that $\mathcal{A}_t = \{Z \in L^\infty : Z_t \stackrel{def}{=} \mathbb{E}_\mathbb{P}[Z|\mathcal{F}_t] \le 0\}$, and $\rho_t^v(Z) = \frac{\mathbb{E}_\mathbb{P}[Z|\mathcal{F}_t]}{\mathbb{E}_\mathbb{P}[v|\mathcal{F}_t]}$. It follows from martingale representation that for each $X \in L^\infty$ there is an adapted, self-financing, process $(\theta_t)_{t=0,\dots T}$ such that

$$X_t \stackrel{def}{=} \mathbb{E}_\mathbb{P}[X|\mathcal{F}_t] = \theta_0.S_0 + \sum_{s=0}^{t-1} \theta_s.(S_{s+1} - S_s).$$

Moreover, since θ is self-financing, it follows that

$$X_t = \theta_0.S_0 + \sum_{s=0}^{t-1} \theta_s.(S_{s+1} - S_s) + \sum_{s=0}^{t-1} (\theta_{s+1} - \theta_s).S_{s+1} = \theta_t.S_t.$$

Consequently, if we set $Y_t^n = \theta_t$ and $V^n = S_T$, then

$$\mathbb{E}_\mathbb{P}[X - Y_t.V|\mathcal{F}_t] = \mathbb{E}_\mathbb{P}[X - \theta_t.S_T|\mathcal{F}_{t+1}] = X_{t+1} - \theta_t.S_{t+1} = (\theta_{t+1} - \theta_t).S_{t+1} = 0 \in \mathcal{A}_{t+1},$$

since θ is self-financing. Moreover,

$$\rho_t^v(X) = \frac{\mathbb{E}_\mathbb{P}[X|\mathcal{F}_t]}{\mathbb{E}_\mathbb{P}[v|\mathcal{F}_t]} = \frac{\theta_t.S_t}{v_t} = \frac{\mathbb{E}_\mathbb{P}[\theta_t.S_T|\mathcal{F}_t]}{v_t} = \frac{\mathbb{E}_\mathbb{P}[Y_t.S_T|\mathcal{F}_t]}{v_t} = \rho_t^v(Y_t.V),$$

establishing that $\mathcal{Q}$ is strongly (S_T)-time-consistent. □

Remark that Definitions 5.2 and 5.5 can be unified in a single definition.

Definition 5.9. *Let $U \subset \mathcal{N}$ and $\eta = 0$ or 1. We say that $\mathcal{A}$ is (η, U)-time-consistent if for all $v \in U$, $t = 0, 1, \dots, T - \eta$ and $X \in \mathcal{L}^\infty$, there exists $X_n \in \mathcal{L}^\infty$, $Y_{t+\eta}^n \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}; \mathbb{R}^n)$ and $V^n = (v^1, \dots, v^n) \subset U$ such that X_n converges weakly* to X in $\mathcal{L}^\infty$,*

$$X_n - Y_{t+\eta}^n.V^n \in \mathcal{A}_{t+1},$$

and

$$\rho_t^v(X) = \liminf_{n \to +\infty} \rho_t^v\left(Y_{t+\eta}^n.V^n\right).$$

In particular we say that $\mathcal{A}$ is (η, v)-time-consistent when $U = \{v\}$ and η-time-consistent when $U = \{1\}$. The weak and strong versions of time-consistency are obtained respectively by setting $\eta = 1$ and $\eta = 0$.

From the previous definitions, it's clear that strong time-consistency implies weak time-consistency. Weak time-consistency can be restated in the following way.

Theorem 5.10. *Let $U \subset \mathcal{N}$. The following statements are equivalent:*

(i) *the cone $\mathcal{A}$ is weakly U-time-consistent;*
(ii) *for all $v \in U$, $t = 0, 1, \dots, T-1$ and $X \in \mathcal{L}^\infty$, there exist sequences $(X_{n,1}, \dots, X_{n,n})$, with each $X_{n,i} \in \mathcal{L}^\infty$, and $V^n = (v^1, \dots, v^n)$, with each $v^i \in U$, such that the sequence $X_n \overset{def}{=} X_{n,1} + \dots + X_{n,n}$ converges weakly* to X in $\mathcal{L}^\infty$ and*

$$\rho_t^v(X) = \liminf_{n \to +\infty} \rho_t^v \left(\sum_{i=1}^n \rho_{t+1}^{v^i}(X_{n,i}) v^i \right).$$

Proof. The implication (ii) $\Rightarrow$ (i) follows when we take

$$Y_{t+1}^n = \left(\rho_{t+1}^{v^1}(X_{n,1}), \dots, \rho_{t+1}^{v^n}(X_{n,n}) \right).$$

For the implication (i)$\Rightarrow$ (ii), first consider $X \in \mathcal{L}^\infty$ with $\rho_t^v(X) = 0$. By assumption, there exist sequences $X_n \in \mathcal{L}^\infty$, $Y_{t+1}^n \in \mathcal{L}^\infty(\mathcal{F}_{t+1}; \mathbb{R}^n)$ and $V^n = (v^1, \dots, v^n) \subset U$ such that the sequence X_n converges weakly* to X in $\mathcal{L}^\infty$, and, defining

$$Z^n \overset{def}{=} X_n - Y_{t+1}^n . V^n,$$

we have

$$Z^n \in \mathcal{A}_{t+1}$$

and

$$0 = \rho_t^v(X) = \liminf \rho_t^v \left(Y_{t+1}^n . V^n \right). \tag{5.1}$$

By the Fatou property we have:

$$\rho_t^v(X) \le \liminf \rho_t^v(X_n) = \liminf \rho_t^v \left(Y_{t+1}^n . V^n + Z^n \right).$$

Expressing $Y_{t+1}^n . V^n + Z^n$ as $(Y_{t+1}^n . V^n + \rho_{t+1}^v(Z^n)v) + (Z^n - \rho_{t+1}^v(Z^n)v)$ we obtain, by subadditivity,

$$0 = \rho_t^v(X) \le \liminf \left\{ \rho_t^v \left(Y_{t+1}^n . V^n + \rho_{t+1}^v(Z^n)v \right) + \rho_t^v \left(Z^n - \rho_{t+1}^v(Z^n)v \right) \right\}.$$

Now, since $Z^n \in \mathcal{A}_{t+1}$, it follows that $\rho_{t+1}^v(Z^n) \le 0$. Then, by Remark 4.11

$$\rho_t^v \left(Z^n - \rho_{t+1}^v(Z^n)v \right) \le \rho_t^v \left(v \rho_{t+1}^v \left(Z^n - \rho_{t+1}^v(Z^n)v \right) \right) = 0.$$

It follows that

$$0 = \rho_t^v(X) \le \liminf \rho_t^v \left(Y_{t+1}^n . V^n + \rho_{t+1}^v(Z^n)v \right) \le \liminf \rho_t^v \left(Y_{t+1}^n . V^n \right) = 0,$$

the second inequality following from subadditivity and the fact that $Z^n \in \mathcal{A}_{t+1}$, while the last equality is directly from equation 5.1.

Now, defining $X_{n,i} = Y_{t+1}^{n,i} v^i$ for $i = 1, \dots, n$ and $X_{n,0} = Z^n$ with $v^0 = v$, we see that

$$0 = \rho_t^v(X) = \liminf \rho_t^v \left(\sum_{i=0}^n \rho_{t+1}^{v^i}(X_{n,i}) . v^i \right),$$

with

$$\sum_{i=0}^{n} X_{n,i} = Y_{t+1}^{n}.V^{n} + Z^{n} = X_{n},$$

so we have established the implication in the case where $\rho_t^v(X) = 0$.

Now for a general $X \in \mathcal{L}^\infty$, define $\tilde{X} = X - \rho_t^v(X)v$. From the above we have

$$\rho_t^v(\tilde{X}) = \liminf \rho_t^v\left(\sum_{i=0}^{n} \rho_{t+1}^{v^i}(\tilde{X}^{n,i}).v^i\right),$$

with the sequence $\tilde{X}^n \stackrel{def}{=} \sum_{i=0}^{n} \tilde{X}^{n,i}$ converging weakly* to $\tilde{X}$ in $\mathcal{L}^\infty$. We deduce, using $\mathcal{F}_0$-translation invariance with respect to v (equation (4.3)), that

$$\rho_t^v(X) = \liminf \rho_t^v\left(\sum_{i=0}^{n} \rho_{t+1}^{v^i}(\tilde{X}^{n,i}).v^i + \rho_{t+1}^v(\rho_t^v(X)v)v\right),$$

with the sequence $\sum_{i=0}^{n} \tilde{X}_{n,i} + \rho_t^v(X)v$ converging weakly* to X in $\mathcal{L}^\infty$. □

Example 5.11. *Recall Example 5.4. It is straightforward to check that* $v\rho_1^v(Z_X) = Z_X$ *while* $\rho_1(W_X + \Delta_X) = W_X$ *and* $\rho(W_X + Z_X) = \rho(X)$ *so that (as we saw before)* ρ *is weakly* $(1, v)$*-time-consistent.*

Remark 5.12. *We shall prove later in Theorem 7.8 that the weak time-consistency introduced in Definition 5.2 is equivalent to that introduced by Delbaen in* [7].

Example 5.13. *Consider a binary branching tree with two branches. So* $\omega = \{1, 2, 3, 4\}$, $\mathcal{F}_0$ *is trivial,* $\mathcal{F}_1 = \sigma(\{1, 2\}, \{3, 4\})$ *and* $\mathcal{F}_2 = 2^\Omega$*, and take* $\mathbb{P}$ *uniform. Equating each probability measure* $\mathbb{Q}$ *on* Ω *with the corresponding vector of probability masses, define the set*

$$\mathcal{Q} = co\left(\left\{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right\},\left\{\frac{1}{4},\frac{1}{4},\frac{3}{8},\frac{1}{8}\right\},\left\{\frac{3}{8},\frac{1}{8},\frac{1}{4},\frac{1}{4}\right\},\left\{\frac{3}{8},\frac{1}{8},\frac{3}{8},\frac{1}{8}\right\}\right),$$

Then the associated coherent risk measure ρ*, is weakly time-consistent.*

Proof. For $\mathcal{F}_1$-measurable Y, we have

$$\rho(Y) = \frac{1}{2}(Y(1) + Y(3)) = \frac{1}{2}(Y(2) + Y(4)),$$

while for all X

$$\rho_1(X)(\omega) = \begin{cases} \max\left(\frac{3X(1)+X(2)}{4}, \frac{X(1)+X(2)}{2}\right) : & \text{for } \omega \in \{1,2\} \\ \max\left(\frac{3X(3)+X(4)}{4}, \frac{X(3)+X(4)}{2}\right) : & \text{for } \omega \in \{3,4\}. \end{cases}$$

It easily follows that $\rho = \rho \circ \rho_1$. □

In [7], it was shown that weak time-consistency (with respect to 1) is equivalent to m-stability of the corresponding test probabilities. In order to generalise this result to our context, we define m-stability with respect to a portfolio of assets in a similar way.

Definition 5.14. Weak m-stability *Let $U \subset \mathcal{N}$ and $P \subset \mathcal{L}^1_+$. We say that P is weakly U-m-stable if for all $t = 0, 1, \ldots, T$, whenever $Z^1, \ldots, Z^k \in P$ are such that there exists $Z \in P$, a partition $F_t^1, \ldots, F_t^k \in \mathcal{F}_t$, $\alpha^1, \ldots, \alpha^k \in \mathcal{L}^0(\mathcal{F}_t)$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and $Y \overset{def}{=} \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i$ satisfies $\mathbb{E}(Zv|\mathcal{F}_t) = \mathbb{E}(Yv|\mathcal{F}_t)$, for all $v \in U$, then we have $Y \in P$. In particular we say that P is weakly v-m-stable when $U = \{v\}$ and weakly m-stable when $U = \{1\}$.*

Example 5.15. *Let $\mathbb{M}(S)$ denote the set of all EMMs of a strictly positive bounded $\mathbb{R}^d$-valued process $(S_t)_{t=0,1,\ldots,T}$ with $S^1_\cdot \equiv 1$. Then $\mathbb{M}(S)$ is weakly m-stable, on identifying probability measures with their densities with respect to $\mathbb{P}$.*

Proof. Let $\mathbb{Q}, \mathbb{Q}^1, \ldots, \mathbb{Q}^k$ be in $\mathbb{M}(S)$. Let their respective densities be $Z, Z^1, \ldots, Z^k$, with each $Z^i > 0$ and fix the partition $F_t^1, \ldots, F_t^k \in \mathcal{F}_t$. Define the probability measure $\mathbb{R}$ having the following density:

$$Y = \sum_{i=1}^k 1_{F_t^i} \frac{Z_t}{Z_t^i} Z^i.$$

Then for $s \geq t$ we have $\mathbb{E}_{\mathbb{R}}(S_{s+1}|\mathcal{F}_s) = \sum_{i=1}^k 1_{F_t^i} \mathbb{E}_{\mathbb{Q}^i}(S_{s+1}|\mathcal{F}_s) = S_s$ and for $s < t$ we have

$$\mathbb{E}_{\mathbb{R}}(S_{s+1}|\mathcal{F}_s) = \mathbb{E}(Y_t S_{s+1}|\mathcal{F}_s)/Y_s = \mathbb{E}(Z_t S_{s+1}|\mathcal{F}_s)/Z_s = \mathbb{E}_{\mathbb{Q}}(S_{s+1}|\mathcal{F}_s) = S_s.$$

Thus $\mathbb{R} \in \mathbb{M}(S)$. □

We may define strong m-stability in a similar fashion.

Definition 5.16. Strong m-stability *Let $U \subset \mathcal{N}$ and $P \subset \mathcal{L}^1_+$. We say that P is strongly U-m-stable if for all $t = 0, 1, \ldots, T-1$, whenever $Z^1, \ldots, Z^k \in P$ are such that there exists $Z \in P$, a partition $F_t^1, \ldots, F_t^k \in \mathcal{F}_t$, $\alpha^1, \ldots, \alpha^k \in \mathcal{L}^0_+(\mathcal{F}_{t+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and $Y \overset{def}{=} \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i$ satisfies $\mathbb{E}(Zv|\mathcal{F}_t) = \mathbb{E}(Yv|\mathcal{F}_t)$, for all $v \in U$, then we have $Y \in P$. In particular we say that P is strongly v-m-stable when $U = \{v\}$ and strongly m-stable when $U = \{1\}$.*

Example 5.17. *Denoting by $\mathbb{M}(S)$ the set of all EMMs of a strictly positive bounded $\mathbb{R}^d$-valued process $(S_t)_{t=0,1,\ldots,T}$ with $S^1_\cdot \equiv 1$, the set $\mathbb{M}(S)$ is strongly S_T-m-stable, on identifying probability measures with their densities with respect to $\mathbb{P}$.*

Proof. Fix $t \in \{0, 1, \ldots, T-1\}$ and let $\mathbb{Q}, \mathbb{Q}^1, \ldots, \mathbb{Q}^k$ belong to $\mathbb{M}(S)$. Let their respective densities be $Z, Z^1, \ldots, Z^k$, with each $Z^i > 0$, and let $\alpha^i \in \mathcal{L}^0_+(\mathcal{F}_{t+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$. Fix the partition $F_t^1, \ldots, F_t^k \in \mathcal{F}_t$ such that $Y \overset{def}{=} \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i$ satisfies $\mathbb{E}(ZS_T|\mathcal{F}_t) = \mathbb{E}(YS_T|\mathcal{F}_t)$. Define the probability measure $\mathbb{R}$ by its density Y. We want to prove that $\mathbb{R} \in \mathbb{M}(S)$. Now, for $s \geq t+1$ we have:

$$\mathbb{E}_{\mathbb{R}}(S_T|\mathcal{F}_s) = \sum_{i=1}^k 1_{F_t^i} \frac{\mathbb{E}(Z^i S_T|\mathcal{F}_s)}{Z_s^i} = \sum_{i=1}^k 1_{F_t^i} \mathbb{E}_{\mathbb{Q}^i}(S_T|\mathcal{F}_s) = S_s,$$

and for $s \le t$ we have first

$$\mathbb{E}_{\mathbb{R}}(S_T|\mathcal{F}_t) = \frac{1}{Y_t}\mathbb{E}(YS_T|\mathcal{F}_t) = \frac{1}{Z_t}\mathbb{E}(ZS_T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(S_T|\mathcal{F}_t) = S_t.$$

Then

$$\mathbb{E}_{\mathbb{R}}(S_T|\mathcal{F}_s) = \mathbb{E}_{\mathbb{R}}(\mathbb{E}_{\mathbb{R}}(S_T|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}_{\mathbb{R}}(S_t|\mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}(S_t|\mathcal{F}_s) = S_s.$$

□

Example 5.18. *We consider a probability space $\Omega = \mathbb{Z}\backslash\{0\}$ with $\mathbb{P}$ defined by $\mathbb{P}(\omega) = 2^{-(1+|\omega|)}$, $\mathcal{F} = \mathcal{F}_2 = 2^{\Omega}$, $\mathcal{F}_1 = \sigma(\{\omega, -\omega\};\ \omega \in \mathbb{Z}_+)$ and $\mathcal{F}_0$ trivial. Then every set of probability measures $\mathcal{Q}$ is strongly U-m-stable, where $U = \{v^{\omega};\ \omega \in \mathbb{Z}\}$ with $v^{\omega} = 1+1_{\{\omega\}}$.*

This result follows from the fact that the only way we can have $(Zv)_t = (Yv)_t$ for all $v \in U$ (with Y non-negative and $Z \in \mathcal{Q}$) is if $Y = Z$.

The last two definitions can also be unified in a single definition.

Definition 5.19. *Let $U \subset \mathcal{N}$ and $P \subset \mathcal{L}^1_+$. We say that P is (η, U)-m-stable with $\eta = 0, 1$ if for each $t = 0, 1, \ldots, T-1+\eta$, whenever $Z^1, \ldots, Z^k \in P$ are such that there exists $Z \in P$, a partition $F^1_t, \ldots, F^k_t \in \mathcal{F}_t$, $\alpha^1, \ldots, \alpha^k \in \mathcal{L}^0_+(\mathcal{F}_{t-\eta+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and $Y \stackrel{def}{=} \sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i$ satisfies $\mathbb{E}(Zv|\mathcal{F}_t) = \mathbb{E}(Yv|\mathcal{F}_t)$ for all $v \in U$, then we have $Y \in P$. In particular we say that P is (η, v)-m-stable when $U = \{v\}$ and η-m-stable when $U = \{1\}$.*

The following theorem gives some simpler conditions for m-stability

Theorem 5.20. *Let $U \subset \mathcal{N}$ and $P \subset \mathcal{L}^1_+$. The following are equivalent:*

(i) *P is (η, U)-m-stable;*

(ii) *for each $t \in \{0, 1, \ldots, T-1+\eta\}$, whenever $Y, W \in P$ are such that there exists $Z \in P$, a set $F \in \mathcal{F}_t$, $\alpha, \beta \in \mathcal{L}^0_+(\mathcal{F}_{t-\eta+1})$ with $\alpha Y, \beta W \in \mathcal{L}^1$ and*

$$X \stackrel{def}{=} 1_F\alpha Y + 1_{F^c}\beta W \tag{5.2}$$

satisfies $\mathbb{E}(Xv|\mathcal{F}_t) = \mathbb{E}(Zv|\mathcal{F}_t)$ for all $v \in U$, then we have $X \in P$.

(iii) *for each $t \in \{0, 1, \ldots, T-1+\eta\}$, whenever $Y, W \in P$ are such that there exists $Z \in P$, a set $F \in \mathcal{F}_t$, and for each $v \in U$ there is an $R^v_{t+1-\eta} \in L^1_+(\mathcal{F}_{t-\eta+1})$ with $R^v_{t+1-\eta}Y, R^v_{t+1-\eta}W \in \mathcal{L}^1$, $\mathbb{E}[R^v_{t+1-\eta}|\mathcal{F}_t] = 1$ and such that*

$$X \stackrel{def}{=} (Zv)_t R^v_{t+1-\eta}\left(1_F\frac{Y}{(Yv)_{t+1-\eta}} + 1_{F^c}\frac{W}{(Wv)_{t+1-\eta}}\right) \tag{5.3}$$

is the same for each $v \in U$, then we have $X \in P$.

(iv) *For each stopping time $\tau \le T-1+\eta$, whenever there exist Z and W in P and $R^v_{\tau+1-\eta} \in L^1_+(\mathcal{F}_{\tau+1-\eta})$ such that $\mathbb{E}[R^v_{\tau+1-\eta}|\mathcal{F}_\tau] = 1$ and*

$$X = W\frac{(Zv)_\tau R^v_{\tau+1-\eta}}{(Wv)_{\tau+1-\eta}} \tag{5.4}$$

is the same for each $v \in U$, then we have $X \in P$.

Proof. ((i)⇔ (ii)) The forward implication is trivial. Now suppose that property (ii) holds. Fix $t = 0, 1, \dots, T$ and take $Z, Z^1, \dots, Z^k$ in P such that there exists a partition $F_t^1, \dots, F_t^k \in \mathcal{F}_t$, $\alpha^1, \dots, \alpha^k \in \mathcal{L}^0(\mathcal{F}_{t-\eta+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and $Y \overset{def}{=} \sum_{i=1}^k 1_{F_t^i}\alpha^i Z^i$ satisfies $\mathbb{E}(Zv|\,\mathcal{F}_t) = \mathbb{E}(Yv|\mathcal{F}_t)$ for all $v \in U$. We want to prove that $Y \in P$. First using the property in (ii) we have

$$Y^1 \overset{def}{=} 1_{F_t^1}\alpha^1 Z^1 + 1_{(F_t^1)^c} Z \in P,$$

and by induction

$$Y^{i+1} \overset{def}{=} 1_{F_t^{i+1}}\alpha^{i+1} Z^{i+1} + 1_{(F_t^{i+1})^c} Y^i \in P.$$

We prove easily that

$$Y^i = \sum_{j=1}^{i} 1_{F_t^j}\alpha^j Z^j + 1_{G_t^i} Z,$$

with $G_t^i = \cap_{j=1}^i (F_t^j)^c = \cup_{j=i+1}^k F_t^j$ for $i \le k-1$ and $G_t^k = \emptyset$. Then $Y = Y^k \in P$, establishing (i).

((ii)⇔ (iii)) Assume that (iii) holds. Observe that (5.2) implies that

$$1_F(Xv)_{t+1-\eta} = 1_F\alpha(Yv)_{t+1-\eta} \text{ and } 1_{F^c}(Xv)_{t+1-\eta} = 1_{F^c}\beta(Wv)_{t+1-\eta},$$

for each $v \in U$. It follows that

$$1_F\alpha = 1_F\frac{(Xv)_{t+1-\eta}}{(Yv)_{t+1-\eta}} \text{ and } 1_{F^c}\beta = 1_{F^c}\frac{(Xv)_{t+1-\eta}}{(Wv)_{t+1-\eta}}.$$

Setting

$$R^v_{t+1-\eta} = \frac{(Xv)_{t+1-\eta}}{(Xv)_t} = \frac{(Xv)_{t+1-\eta}}{(Zv)_t},$$

we see that (5.3) holds establishing (ii).

Conversely, assuming (ii), if (5.3) is satisfied, then

$$1_F\frac{X}{Y} = 1_F\frac{(Zv)_t R^v_{t+1-\eta}}{(Yv)_{t+1-\eta}},$$

and

$$1_{F^c}\frac{X}{Y} = 1_{F^c}\frac{(Zv)_t R^v_{t+1-\eta}}{(Wv)_{t+1-\eta}},$$

for each choice of $v \in U$. Setting these common values to α and β respectively, we see that (5.2) holds and $(Xv)_t = (Zv)_t$ for each v, so that $X \in P$, establishing (iii).

((iii)⇔ (iv)) Suppose that (iv) holds, then, setting

$$\tau = t1_{F^c} + (T-1+\eta)1_F$$

in (5.4) we see that

$$\tilde{X} = 1_{F^c}(Zv)_t R^v_{t+1-\eta}\frac{W}{(Wv)_{t+1-\eta}} + 1_F(Zv)_{T-1+\eta}R^v_T, \tag{5.5}$$

and $\tilde{X} \in P$. Now take $\tilde{Z} = \tilde{X}$, $\tilde{W} = Y$ and $\tilde{\tau} = t1_F + (T-1+\eta)1_{F^c}$ in (5.4). We obtain

$$X = (Zv)_t R^v_{t+1-\eta}(1_F \frac{Y}{(Yv)_{t+1-\eta}} + 1_{F^c}\frac{W}{(Wv)_{t+1-\eta}})$$

and $X \in P$, establishing (iii).

Conversely, suppose that (iii) holds. We prove (iv) by backwards induction on a lower bound for τ. The property is immediate for $\tau = T-1+\eta$. Now suppose that (iv) holds whenever $\tau \geq k+1$ a.s., and that the stopping time $\tilde{\tau}$ satisfies $\tilde{\tau} \geq k$ a.s. Define $F^c = (\tilde{\tau} = k)$ (so that $F = (\tilde{\tau} \geq k+1)$) and set

$$\tau^* = \tilde{\tau}1_F + (T-1+\eta)1_{F^c}.$$

Suppose that $W, Z \in P$ and $R_{\tilde{\tau}+1-\eta} \in \mathcal{F}_{\tilde{\tau}+1-\eta}$ satisfy the hypotheses of (iv) then

$$W\frac{(Zv)_{\tilde{\tau}} R^v_{\tilde{\tau}+1-\eta}}{(Wv)_{\tau+1-\eta}}1_F,$$

is independent of v so Y defined by

$$Y = W\frac{(Zv)_\tau R^v_{\tau+1-\eta}}{(Wv)_{\tau+1-\eta}}1_F + W1_{F^c}$$

is also independent of v. Now

$$Y = W\frac{(Zv)_{\tau^*} R^v_{\tau^*+1-\eta}}{(Wv)_{\tau^*+1-\eta}},$$

where $R^v_{\tau^*+1-\eta} = R^v_{\tau+1-\eta}1_F + \frac{(Zv)}{(Zv)_{T-1+\eta}}1_{F^c}$. It is easy to check that $\mathbb{E}[R^v_{\tau^*+1-\eta}|\mathcal{F}_{\tau^*}] = 1$ so, by the inductive hypothesis, $Y \in P$

Now substitute Y, W, Z and F in (5.4), with $t = k$ and $R^v_{k+1-\eta} = \frac{(Zv)_{k+1-\eta}}{(Zv)_k}1_F + R^v_{\tilde{\tau}+1-\eta}1_{F^c}$ to see that $X = W\frac{(Zv)_{\tilde{\tau}} R^v_{\tilde{\tau}+1-\eta}}{(Wv)_{\tilde{\tau}+1-\eta}}$ and (by (iii)) $X \in P$, which establishes the inductive step. □

Example 5.21. *Recall the risk measure in Example 5.4. Any element of $\mathcal{Q}$ may be written as $\mathbb{Q}_\lambda = (\frac{1}{3} + \frac{1}{6}\lambda, \frac{1}{6} - \frac{1}{24}\lambda, \frac{1}{4} - \frac{1}{16}\lambda, \frac{1}{4} - \frac{1}{16}\lambda)$.*

We shall show that $\mathcal{Q}$ is weakly $(1,v)$-m-stable by using criterion (iv) of Theorem 5.20.

We take the uniform measure on $\Omega = \{1,2,3,4\}$ as our reference measure $\mathbb{P}$. Then (equating p.m.s and their densities) Z and W may be written as $\frac{d\mathbb{Q}_\lambda}{d\mathbb{P}}$ and $\frac{d\mathbb{Q}_\mu}{d\mathbb{P}}$ respectively.

Now $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}} = (\frac{4}{3} + \frac{2}{3}\theta, \frac{2}{3} - \frac{1}{6}\theta, 1 - \frac{1}{4}\theta, 1 - \frac{1}{4}\theta)$, while $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}|_{\mathcal{F}_1} = (1 + \frac{1}{4}\theta, 1 + \frac{1}{4}\theta, 1 - \frac{1}{4}\theta, 1 - \frac{1}{4}\theta)$. Since $v = 1 + 1_{\{1\}}$, it follows that $\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}v = (\frac{8}{3} + \frac{4}{3}\theta, \frac{2}{3} - \frac{1}{6}\theta, 1 - \frac{1}{4}\theta, 1 - \frac{1}{4}\theta)$; $(\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}v)_1 = (\frac{5}{3} + \frac{7}{12}\theta, \frac{5}{3} + \frac{7}{12}\theta, 1 - \frac{1}{4}\theta, 1 - \frac{1}{4}\theta)$ and $(\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}v)_0 = (\frac{4}{3} + \frac{1}{6}\theta, \frac{4}{3} + \frac{1}{6}\theta, \frac{4}{3} + \frac{1}{6}\theta, \frac{4}{3} + \frac{1}{6}\theta)$.

Now suppose that τ is a stopping time and consider $\frac{Z_\tau}{W_\tau}$ and $\frac{(Zv)_\tau}{(Wv)_\tau}$. A quick check shows that

$$\frac{Z_\tau}{W_\tau} = (\frac{4+2\lambda}{4+2\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu})1_{(\tau=2)}$$
$$+(\frac{4+\lambda}{4+\mu}, \frac{4+\lambda}{4+\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu})1_{(\tau=1)}$$
$$+(1,1,1,1)1_{(\tau=0)},$$

while

$$\frac{(Zv)_\tau}{(Wv)_\tau} = (\frac{4+2\lambda}{4+2\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu})1_{(\tau=2)}$$
$$+(\frac{5+\frac{7}{4}\lambda}{5+\frac{7}{4}\mu}, \frac{5+\frac{7}{4}\lambda}{5+\frac{7}{4}\mu}, \frac{4-\lambda}{4-\mu}, \frac{4-\lambda}{4-\mu})1_{(\tau=1)}$$
$$+(\frac{4+\frac{1}{2}\lambda}{4+\frac{1}{2}\mu}, \frac{4+\frac{1}{2}\lambda}{4+\frac{1}{2}\mu}, \frac{4+\frac{1}{2}\lambda}{4+\frac{1}{2}\mu}, \frac{4+\frac{1}{2}\lambda}{4+\frac{1}{2}\mu})1_{(\tau=0)}.$$

Now equating $\frac{Z_\tau}{W_\tau}$ *and* $\frac{(Zv)_\tau}{(Wv)_\tau}$*, we see that if* $\mathbb{P}(\tau=0)>0$ *we must have* $4+\frac{1}{2}\lambda = 4+\frac{1}{2}\mu \Rightarrow \lambda=\mu$*, whilst if* $\mathbb{P}((\tau=1)\cap(\omega\in\{1,2\}))>0$ *we must have* $\frac{5+\frac{7}{4}\lambda}{5+\frac{7}{4}\mu} = \frac{4+\lambda}{4+\mu} \Rightarrow \lambda=\mu$*. Thus, either* $Z=W$ *or* $\tau\geq 1$ *and* $(\tau=1)\subseteq\{3,4\}$*. Now, assuming that* $Z\neq W$*, since* $(\tau=1)\in\mathcal{F}_1$ *we see that either* $\tau=2$ *or* $\tau = 1_{\{3,4\}} + 2.1_{\{1,2\}}$*. In either case,* $W_\tau = W$ *and* $Z_\tau = Z$ *so that* X*, defined by* $X = W\frac{Z_\tau}{W_\tau} = W\frac{(Zv)_\tau}{(Wv)_\tau}$ *is equal to* Z *and hence* $\mathcal{Q}$ *is weakly* $(1,v)$*-m-stable.*

Remark 5.22. *Weak m-stability (with respect to* $\{1\}$*) of a set* $P\subset\mathcal{L}^1_+$*, introduced here, coincides with m-stability as defined in Delbaen* [6]*: for all* $Z,W\in P$ *with* $W>0$ *a.s and all stopping times* τ*, we have*

$$X \overset{def}{=} Z_\tau\frac{W}{W_\tau}\in P.$$

The weak-m-stability property was first established for the collection of EMMs for a vector valued price-process in Jacka [9]*.*

Remark 5.23. *It is easy to see from condition (iv) in Theorem 5.20 that* P *is* (η,U)*-m-stable* $\Leftrightarrow$ Pv^{-1} *is* (η,Uv)*-m-stable, for any* $v\in\mathcal{N}$*.*

Lemma 5.24. $\mathcal{A}$ *is strongly time-consistent with respect to* $\mathcal{N}$*.*

Proof. Let $X\in\mathcal{L}^\infty$ and $t=0,1,\dots,T$. Then for $\lambda = 1+\|X\|_{\mathcal{L}^\infty}$ we have $v\overset{def}{=}X+\lambda\in\mathcal{N}$ and consequently

$$\rho_t(X) = \rho_t(Y.V),$$

with $Y=(-\lambda,1)$ and $V=(1,v)$. $\square$

Remark 5.25. *In Lemma 5.24, we proved that each claim X can be "hedged" by a portfolio of two assets. Later (in Theorem 7.27) we shall prove that hedging can be performed uniformly via a countable portfolio of assets .*

6. Results on multidimensional closed convex cones

cone $\mathcal{A}$ with respect to a finite portfolio We now introduce the concept of representation of a cone with respect to a collection of assets V. As we will see later in the next section, this new concept coincides with the concept of decomposition that we will analyze in this section.

We consider $\mathcal{B}$, a weak*-closed convex cone in $\mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d)$ which is arbitrage-free.

The canonical example is where $\mathcal{B}$ is the collection of admissible portfolios of the assets in V:

$$\mathcal{A}(V) = \{X \in \mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d) : X.V \in \mathcal{A}\},$$

is the set of all portfolios in (assets in the collection) V that are admissible.

In the interests of presentation, we relegate most of the proofs of results in this section to Appendix B.

First we introduce some definitions.

Definition 6.1. Weak decomposition *We say that the cone $\mathcal{B}$ is weakly decomposable if*

$$\mathcal{B} = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{B})},$$

where $K_t(\mathcal{B}) = \mathcal{B}_t \cap \mathcal{L}^\infty(\mathcal{F}_{t+1};\mathbb{R}^d)$ and

$$\mathcal{B}_t = \{X \in \mathcal{L}^\infty;\ \alpha X \in \mathcal{B} \text{ for all } \alpha \in \mathcal{L}^\infty_+(\mathcal{F}_t)\}.$$

We say that $(K_t(\mathcal{B}))_{t=0,\dots,T-1}$ is the weak decomposition of the cone $\mathcal{B}$. Note that we use $\oplus$ simply to denote a sum of subsets of a vector space.

Remark 6.2. *Thus $\mathcal{B}$ is weakly decomposable if it can be obtained by one-period bets in the various assets/currencies. See also Definition 7.1 for the motivation for this concept.*

Example 6.3. *The acceptance set of a weakly time-consistent coherent risk measure is weakly decomposable (see Jacka and Berkaoui* [10]*).*

Definition 6.4. Strong decomposition *We say that the cone $\mathcal{B}$ is strongly decomposable if*

$$\mathcal{B} = \overline{\oplus_{t=0}^{T} \mathcal{C}_t(\mathcal{B})},$$

where $\mathcal{C}_t(\mathcal{B}) = \mathcal{B}_t \cap \mathcal{L}^\infty(\mathcal{F}_t;\mathbb{R}^d)$. We say that $(\mathcal{C}_t(\mathcal{B}))_{t=0,\dots,T}$ is the strong decomposition of the cone $\mathcal{B}$.

Remark 6.5. *Thus $\mathcal{B}$ is strongly decomposable if it can be obtained by instantaneous exchanges of assets. See also Definition 7.2 for the motivation for this concept.*

Example 6.6. *The cone $\mathcal{B}(\pi) \cap \mathcal{L}^\infty$, defined in section 2, is strongly decomposable.*

For the sake of simplification later in the proofs we introduce a unified definition of weak and strong decomposition.

Definition 6.7. *We say that the cone $\mathcal{B}$ is η-decomposable with $\eta \in \{0,1\}$ if*

$$\mathcal{B} = \overline{\oplus_{t=0}^{T-\eta} K_t^\eta(\mathcal{B})},$$

where $K_t^\eta(\mathcal{B}) = \mathcal{B}_t \cap \mathcal{L}^\infty(\mathcal{F}_{t+\eta};\mathbb{R}^d)$. We say that $(K_t^\eta(\mathcal{B}))_{t=0,\dots,T-\eta}$ is the η-decomposition of the cone $\mathcal{B}$. Remark that the weak and strong decomposition are respectively associated to $\eta = 1$ and $\eta = 0$.

Now we define η-stability in this multidimensional context:

Definition 6.8. *Let D denote a subset in $\mathcal{L}^1_+(\mathcal{F};\mathbb{R}^d)$. We say that D is η-stable with $\eta \in \{0,1\}$ if for all $t = 0,1,\dots,T$ whenever $Z^1,\dots,Z^k \in D$ are such that there exists $Z \in D$, a partition $F_t^1,\dots,F_t^k \in \mathcal{F}_t$ and $\alpha^1,\dots,\alpha^k \in \mathcal{L}^0(\mathcal{F}_{t-\eta+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and $Y \stackrel{def}{=} \sum_{i=1}^k 1_{F_t^i}\alpha^i Z^i$ satisfies $\mathbb{E}(Z|\mathcal{F}_t) = \mathbb{E}(Y|\mathcal{F}_t)$, then we have $Y \in D$.*

Definition 6.9. *For all $t = 0,1,\dots,T$, we define:*

$$D_{(t)} = \overline{conv}\left\{\alpha Z : Z \in D,\, \alpha \in \mathcal{L}^\infty_+(\mathcal{F}_t)\right\}.$$

Theorem 6.10. *Let $D \subset \mathcal{L}^1_+(\mathcal{F};\mathbb{R}^d)$. The following are equivalent:*

(i) *D is η-stable;*

(ii) *for each $t \in \{0,1,\dots,T-1+\eta\}$, whenever $Y,W \in D$ are such that there exists $Z \in D$, a set $F \in \mathcal{F}_t$, $\alpha,\beta \in \mathcal{L}^0_+(\mathcal{F}_{t-\eta+1})$ with $\alpha Y, \beta W \in \mathcal{L}^1$ and*

$$X \stackrel{def}{=} 1_F\alpha Y + 1_{F^c}\beta W \tag{6.1}$$

satisfies $\mathbb{E}(X|\mathcal{F}_t) = \mathbb{E}(Z|\mathcal{F}_t)$ then we have $X \in D$.

(iii) *for each $t \in \{0,1,\dots,T-1+\eta\}$, whenever $Y,W \in D$ are such that there exists $Z \in D$, a set $F \in \mathcal{F}_t$, and for each $i \in \{1,\dots,d\}$ there is an $R^i_{t+1-\eta} \in L^1_+(\mathcal{F}_{t-\eta+1})$ with $R^i_{t+1-\eta}Y^i, R^i_{t+1-\eta}W^i \in \mathcal{L}^1$, $\mathbb{E}[R^i_{t+1-\eta}|\mathcal{F}_t] = 1$ and such that*

$$R^i_{t+1-\eta}Z^i_t\left(1_F\frac{1}{Y^i_{t+1-\eta}} + 1_{F^c}\frac{1}{W^i_{t+1-\eta}}\right)$$

is the same for each i, then X, given by

$$X^i = Z^i_t R^i_{t+1-\eta}(1_F\frac{Y^i}{(Y^i)_{t+1-\eta}} + 1_{F^c}\frac{W^i}{(W^i)_{t+1-\eta}}) \tag{6.2}$$

is in D.

(iv) *For each stopping time $\tau \le T-1+\eta$, whenever there exist Z and W in D and $R^i_{\tau+1-\eta} \in L^1_+(\mathcal{F}_{\tau+1-\eta})$, such that $\mathbb{E}[R^i_{\tau+1-\eta}|\mathcal{F}_\tau] = 1$ and*

$$\frac{Z^i_\tau R^i_{\tau+1-\eta}}{W^i_{\tau+1-\eta}}$$

is the same for each i, then X, defined by

$$X^i = W^i\frac{Z^i_\tau R^i_{\tau+1-\eta}}{W^i_{\tau+1-\eta}}, \tag{6.3}$$

is in D.

The proof is essentially the same as that of Theorem 5.20.

Now we give some results on η-stability. To facilitate this we introduce the following equivalence relation on $\mathcal{L}^1(\mathcal{F};\mathbb{R}^d)$ as follows: $Z \equiv_{t,\eta} Z'$ if there exists $\alpha_t \in \mathcal{L}^0_+(\mathcal{F}_t)$ with $\alpha_t\, Z' \in \mathcal{L}^1$ such that $Z_{t+\eta} = \alpha_t\, Z'_{t+\eta}$.

Lemma 6.11. *Let $D \subset \mathcal{L}^1_+$ be an η-stable cone for some $\eta \in \{0,1\}$, then, defining*

$$M^\eta_t(D) = \{Z \in \mathcal{L}^1(\mathcal{F};\mathbb{R}^d) :\ Z \equiv_{t,\eta} Z' \textit{ for some } Z' \in D\}.$$

we have

$$D = \cap_{t=0}^{T-\eta} M^\eta_t(D).$$

Proof. See Appendix B □

Lemma 6.12. *Let D be a subset in $\mathcal{L}^1$ and define: $[D] = \cap_{t=0}^{T-\eta} R^\eta_t$ where*

$$R^\eta_t = \{Z \in \mathcal{L}^1 :\ Z_{t+\eta} = Z'_{t+\eta} \textit{ for some } Z' \in D_{(t)}\} = \overline{conv}(M^\eta_t(D),$$

where $D_{(t)}$ is defined in Definition 6.9.

Then

1. *$[D]$ is the smallest η-stable closed convex cone in $\mathcal{L}^1$, containing D.*
2. *$D = [D]$ if and only if D is an η-stable closed convex cone in $\mathcal{L}^1$.*

Proof. See Appendix B □

Next we characterize the η-decomposability of the cone $\mathcal{B}$. In what follows we shall denote the polar cone of $\mathcal{B}$ by $\mathcal{B}^*$ (note that, by assumption, $\mathcal{B}$ is weak* closed and so $\mathcal{B}^* \subset \mathcal{L}^1(\mathcal{F};\mathbb{R}^d)$).

Theorem 6.13. *$\mathcal{B}$ is η-decomposable $\Leftrightarrow$ $\mathcal{B}^*$ is η-stable.*

Proof. See Appendix B □

Remark 6.14. *From Lemma 6.11 we see that if a subset $D \subset \mathcal{L}^1$ is η-stable, then its polar cone D^* in $\mathcal{L}^\infty$ is η-decomposable.*

We establish some further results about η-decomposability in Appendix C.

Now, we give a useful characterisation of $\mathcal{C}_t(\mathcal{B})$:

Lemma 6.15. *Let $X \in \mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d)$. Then the following assertions are equivalent:*

1. *$X \in \mathcal{C}_t(\mathcal{B})$,*
2. *$X \in \mathcal{L}^\infty(\mathcal{F}_t;\mathbb{R}^d)$ and $Z_t.X \le 0$ a.s. for all $Z \in \mathcal{B}^*$.*
3. *$\mathbb{E}[W.X|\,\mathcal{F}_t] \le 0$ for all $W \in \mathcal{L}^1$ such that $W_t = Z_t$ for some $Z \in \mathcal{B}^*$.*

Proof. See Appendix B □

Definition 6.16. *For a fixed $t \in \{0, \ldots, T\}$, we say that a closed convex cone $\mathcal{H}$ in $\mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d)$ is an $\mathcal{F}_t$-cone (or a t-cone) if $\alpha\mathcal{H} \subset \mathcal{H}$ for each $\alpha \in \mathcal{L}^\infty_+(\mathcal{F}_t)$.*

Remark 6.17. *The property of being a t-cone is like an $\mathcal{F}_t$-measurable version of the convex cone property.*

It follows from Theorem 4.5 and Corollary 4.7 of [11] *that $\mathcal{H}$ is a t-cone if and only if there is a random closed convex cone, C in $\mathbb{R}^d$ such that*

$$\mathcal{H} = \{X \in \mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d) : \ X \in C \ a.s.\}.$$

Theorem 6.18. *$\mathcal{B}$ is strongly decomposable if and only if there exists a collection $(\mathcal{H}_t)_{t=0}^T$, with each $\mathcal{H}_t$ being a t-cone in $\mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d)$ such that:*

$$\mathcal{B}^* = \bigcap_{t=0}^{T} \{Z \in \mathcal{L}^1(\mathcal{F}; \mathbb{R}^d);\ Z_t \in \mathcal{H}_t\}. \tag{6.4}$$

Proof. Suppose that $\mathcal{B}$ is strongly decomposable and define

$$\mathcal{H}_t = \overline{\{\alpha Z_t : \ \alpha \in \mathcal{L}^\infty_+(\mathcal{F}_t),\ Z \in \mathcal{B}^*\}}.$$

Thanks to Lemma 6.15, we have (6.4). Conversely, remark that

$$\{Z \in \mathcal{L}^1;\ Z_t \in \mathcal{H}_t\}^* \subset \mathcal{C}_t(\mathcal{B}).$$

Indeed define $N_t = \{Z \in \mathcal{L}^1;\ Z_t \in \mathcal{H}_t\}$, and let $X \in \mathcal{L}^\infty$ such that $\mathbb{E}(Z.X) \le 0$ for all $Z \in N_t$. Since $Z - Z_t \in N_t$ for all $Z \in \mathcal{L}^1$ we deduce that $X \in \mathcal{L}^\infty(\mathcal{F}_t)$. For all $Z \in \mathcal{B}^*$ and $\alpha \in \mathcal{L}^\infty_+(\mathcal{F}_t)$, we have $\mathbb{E}(Z.\alpha X) = \mathbb{E}(\alpha Z.X) \le 0$ since $\alpha Z \in N_t$. The result follows. □

7. Representation

In a frictionless market with d assets $S^1, \ldots, S^d$ and under the no-arbitrage property of the price process $S_t = (S^1_t, \ldots, S^d_t)$, any bounded claim X is represented by these assets: i.e. there exists an $\mathbb{R}^d$-valued strategy β_t and a scalar x such that:

$$X = x + \sum_{t=0}^{T} \beta_t.S_T,$$

with $\beta_t.S_t \le 0$ a.s for all $t \in \{0, \ldots, T\}$.

In the presence of transaction costs, we define the cone

$$\mathcal{A} = (\mathcal{B}(\pi) \cap \mathcal{L}^\infty).V = \{X.V : \ X \in \mathcal{B}(\pi) \cap \mathcal{L}^\infty\},$$

where $\mathcal{B}(\pi)$ and V are as defined in section 2. Then any bounded claim X is represented by the contracts $v^1, \ldots, v^d$. In other words, there exists an $\mathbb{R}^d$-valued strategy β_t and a scalar x such that:

$$X = x + \sum_{t=0}^{T} \beta_t.V,$$

with $\beta_t.Z_t \le 0$ a.s for all $t \in \{0, \ldots, T\}$ and for all Z in the polar of $\mathcal{B}(\pi) \cap \mathcal{L}^\infty$, where $Z_t \overset{def}{=} \mathbb{E}(Z|\mathcal{F}_t)$.

In the presence of a conditional coherent risk measure ρ associated with an acceptance set $\mathcal{A}$, trading can take place between numéraires or portfolios of numéraires. In this section we introduce the concept of representation of the cone $\mathcal{A}$ with respect to a set of contracts with lifetime equal to zero or 1: 'weak representation' and 'strong representation'.

7.1. **The finite case.** We assume, for now, that the fixed portfolio $V \subset \mathcal{N}$ of d assets contains the unit of account $\mathbf{1}$, and, indeed, that $v_1 = \mathbf{1}$. Recall also that each element of V is bounded and bounded away from 0 (a.s.).

Recall that

$$\mathcal{A}(V) = \{X \in \mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d) : X.V \in \mathcal{A}\},$$

is the set of all portfolios in (assets in) V that are admissible, and

$$\mathcal{A}_t(V) = \{X : \alpha X \in \mathcal{A}(V) \text{ for all } \alpha \in L^\infty_+(\mathcal{F}_t)\}.$$

Definition 7.1. *We say that the cone $\mathcal{A}$ is weakly represented by the $\mathbb{R}^d$-valued vector of assets V if the cone $\mathcal{A}(V)$ is weakly decomposable, i.e*

$$\mathcal{A}(V) = \overline{\oplus_{t=0}^{T-1} K_t(\mathcal{A}, V)},$$

where $K_t(\mathcal{A}, V) \overset{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_{t+1};\mathbb{R}^d)$.

Thus weak representation means that every element of $\mathcal{A}$ is attainable by a collection of one-period bets in units of V at times $0, \ldots, T-1$ and trades at times $0, \ldots, T$.

Definition 7.2. *For $\eta \in \{0, 1\}$, we say the cone $\mathcal{A}$ is strongly represented by the $\mathbb{R}^d$-valued vector of assets V if the cone $\mathcal{A}(V)$ is strongly decomposable, i.e*

$$\mathcal{A}(V) = \overline{\oplus_{t=0}^{T} \mathcal{C}_t(\mathcal{A}, V)},$$

where $\mathcal{C}_t(\mathcal{A}, V) \overset{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_t;\mathbb{R}^d)$.

Thus strong representation means that every element of $\mathcal{A}$ is attainable by a collection of trades in units of V at times $0, \ldots, T$.

We again unify the two concepts in the following:

Definition 7.3. *We say that the cone $\mathcal{A}$ is η-represented by the $\mathbb{R}^d$-valued vector of assets V (with $\eta \in \{0, 1\}$), if the cone $\mathcal{A}(V)$ is η-decomposable, i.e*

$$\mathcal{A}(V) = \overline{\oplus_{t=0}^{T-\eta} K^\eta_t(\mathcal{A}, V)},$$

where $K^\eta_t(\mathcal{A}, V) \overset{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_{t+\eta};\mathbb{R}^d)$.

Theorem 7.4. *Under our assumptions on V, if D is a convex cone in $L^\infty(\mathcal{F})$ then, defining $D(V) = \{X \in \mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d) : X.V \in D\}$, we have*

$$D(V)^* = D^*V \overset{def}{=} \{ZV : Z \in D^*\}.$$

In particular, the polar of the cone of portfolios $\mathcal{A}(V)$ is given by:

$$\mathcal{A}(V)^* = \mathcal{A}^*V \overset{def}{=} \{ZV : Z \in \mathcal{A}^*\}. \tag{7.1}$$

Proof. First, take $Z \in D^*$; then, for any $X \in D(V)$, $\mathbb{E}ZV.X \leq 0$ since $X.V \in D$. It follows that $ZV \in D(V)^*$, and so we conclude that

$$D(V)^* \supset VD^*.$$

To prove the reverse inclusion, denote the ith canonical basis vector in $\mathbb{R}^d$ by e_i. Now note first that, since $V.(\alpha(v_ie_j - v_je_i)) = 0$, $\alpha(v_ie_j - v_je_i) \in D(V)$ for any $\alpha \in L^\infty$. It follows that if $Z \in D(V)^*$ then $Z.(v_ie_j - v_je_i) = 0$ and so any $Z \in D(V)^*$ must be of the form WV for some $W \in \mathcal{L}^1(\mathcal{F}_T)$. Now given $C \in D$, take X such that $X.V = C$ (which implies that $X \in D(V)$), then $0 \geq \mathbb{E}WV.X = \mathbb{E}WC$ and, since C is arbitrary, it follows that $W \in D^*$. Hence $D(V)^* \subset VD^*$. $\square$

To complete the link between the results in the previous section and this one we state the following:

Lemma 7.5. *Suppose that $\mathcal{B}$ is a weakly* closed convex cone in $\mathcal{L}^\infty(\mathcal{F};\mathbb{R}^d)$.*

Define the cone $\mathcal{B}.V$ by

$$\mathcal{B}.V = \{X.V : X \in \mathcal{B}\}.$$

Then

$$\mathcal{B} \subset (\mathcal{B}.V)(V)$$

and

$$\mathcal{B} = (\mathcal{B}.V)(V) \text{ if and only if } \alpha(v_j\, e_i - v_i\, e_j) \in \mathcal{B} \text{ for all } \alpha \in \mathcal{L}^\infty_+(\mathcal{F}).$$

In this case the set $\mathcal{B}.V$ is weakly closed.*

Proof. The first inclusion is immediate from the definition of $(\mathcal{B}.V)(V)$.

Define $w_{ij} \stackrel{def}{=} v_j\, e_i - v_i\, e_j$ and suppose that $\mathcal{B} = (\mathcal{B}.V)(V)$, then for all $\alpha \in \mathcal{L}^\infty_+(\mathcal{F})$ we have $\alpha w_{ij}.V = 0 = 0.V$. So $\alpha w_{ij} \in (\mathcal{B}.V)(V) = \mathcal{B}$.

Conversely, suppose that $\alpha w_{ij} \in \mathcal{B}$ for all $\alpha \in \mathcal{L}^\infty_+(\mathcal{F})$ and all pairs (i,j). It follows by the same argument as in the proof of Lemma 7.4 that $\mathcal{B}^* = VC$, for some closed convex cone $C \subset L^1(\mathcal{F})$. Now suppose that $Z \in \mathcal{B}^*$, so that $Z = VW$ for some $W \in C$. It follows that $\mathbb{E}Z.X = \mathbb{E}WV.X \leq 0$ for all $X \in B$ and thus, since X is arbitrary, that $W \in (\mathcal{B}.V)^*$. Hence $\mathcal{B}^* \subset V(\mathcal{B}.V)^*$.

Now we've already observed that

$$\mathcal{B} \subset (\mathcal{B}.V)(V)$$

so

$$\mathcal{B}^* \supset (\mathcal{B}.V)(V)^*.$$

But by Lemma 7.4, $(\mathcal{B}.V)(V)^* = V(\mathcal{B}.V)^*$ and so

$$\mathcal{B}^* = V(\mathcal{B}.V)^*.$$

Taking polar cones once more we see that, since $\mathcal{B}$ is weakly* closed,

$$\mathcal{B}^{**} = \mathcal{B} = (V(\mathcal{B}.V)^*)^* = (\mathcal{B}.V)(V)^{**} = \overline{(\mathcal{B}.V)(V)}.$$

Finally, since $\mathcal{B} \subset (\mathcal{B}.V)(V)$, we conclude that

$$\mathcal{B} = \overline{(\mathcal{B}.V)(V)} = (\mathcal{B}.V)(V).$$

To see that $\mathcal{B}.V$ is closed, let $x^n \in \mathcal{B}.V$ be a sequence which converges to x, then $\frac{x^n}{v_1} e_1 \in (\mathcal{B}.V)(V) = \mathcal{B}$ converges to $\frac{x}{v_1} e_1 \in \mathcal{B}$ (since $\mathcal{B}$ is closed). Hence $x = \frac{x}{v_1} e_1.V \in \mathcal{B}.V$. □

Next we give the equivalence between representation of the cone $\mathcal{A}$ by the finite portfolio V and V-m-stability of its polar cone.

Theorem 7.6. *$\mathcal{A}$ is strongly (resp. weakly) represented by V if and only if $\mathcal{A}^*$ is strongly (resp. weakly) V-m-stable.*

Proof. This is an immediate consequence of Theorem 6.13 and (7.1) of Theorem 7.4. □

Remark 7.7. *Example 5.4 explicitly gives the weak representation of an element of $\mathcal{A}$ for the given risk measure.*

Now we show the relationship between representation of the cone $\mathcal{A}$ by a finite portfolio V and V-time consistency.

Theorem 7.8. *Let $V \subset \mathcal{N}$, then $\mathcal{A}$ is strongly (resp. weakly) V-time-consistent if and only if it's strongly (resp. weakly) represented by V.*

Proof. Suppose that $\mathcal{A}$ is (η, V)-time-consistent. Fix $t \in \{0, \ldots, T-\eta\}$ and $X \in \mathcal{A}_t$, so there exists two sequences $X_n \in \mathcal{L}^\infty$ and $Y^n_{t+\eta} \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}; \mathbb{R}^d)$ such that

$$X_n - Y^n_{t+\eta}.V \in \mathcal{A}_{t+1}$$

and the sequence X_n converges weakly* to X in $\mathcal{L}^\infty$ with

$$\rho_t(X) = \liminf \rho_t\left(Y^n_{t+\eta}.V\right).$$

Therefore, for all $\varepsilon > 0$, there exists some $N \geq 1$ such that for all $n \geq N$

$$\rho_t(X) + \varepsilon \geq \rho_t\left(Y^n_{t+\eta}.V\right).$$

Now we can write $X_n - \varepsilon = (X_n - Y^n_{t+\eta}.V) + (Y^n_{t+\eta}.V - \varepsilon)$ with $X_n - Y^n_{t+\eta}.V \in \mathcal{A}_{t+1}$ and $Y^n_{t+\eta}.V - \varepsilon \in K^\eta_t(\mathcal{A}, V).V$. So $X_n \in K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}$. By taking the limit we see that $X \in \overline{K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}}$ and then $\mathcal{A}_t = \overline{K^\eta_t(\mathcal{A}, V).V + \mathcal{A}_{t+1}}$. By induction on $t \in \{0, \ldots, T\}$ we get $\mathcal{A} = \overline{\oplus_{t=0}^{T-\eta} K^\eta_t(\mathcal{A}, V).V}$.

Conversely, fix $t \in \{0, \ldots, T-1\}$ and $X \in \mathcal{A}_t$ with $\rho_t(X) = 0$, then there exists a sequence

$$X_n \in \oplus_{s=t}^{T-\eta} K^\eta_s(\mathcal{A}, V).V,$$

which converges weakly* to X in $\mathcal{L}^\infty$. So there exists $Y^n \in K^\eta_t(\mathcal{A}, V)$ such that $Z^n = X_n - Y^n.V \in \mathcal{A}_{t+1}$. We conclude that

$$0 = \rho_t(X) \leq \liminf \rho_t(X_n) = \liminf \rho_t\left(Y^n.V + Z^n\right) \leq \liminf \rho_t\left(Y^n.V\right) \leq 0,$$

so $\rho_t(X) = \liminf \rho_t\left(Y^n.V\right)$. Now for all $X \in \mathcal{L}^\infty$ we have $X - \rho_t(X) \in \mathcal{A}_t$ and from previously

$$\rho_t(X - \rho_t(X)) = \liminf \rho_t\left(Y^n.V\right),$$

therefore

$$\rho_t(X) = \liminf \rho_t\left(Y^n.V + \rho_t(X)\right) = \liminf \rho_t\left((Y^n + \rho_t(X)e_1).V\right).$$

□

Remark 7.9. *The following assertions are obviously equivalent:*

(1) *$\mathcal{A}$ is η-represented by V and the cone $K^\eta(\mathcal{A}, V).V$ is closed.*
(2) *For all $t \in \{0, \ldots, T-\eta\}$, $v \in V$ and $X \in \mathcal{L}^\infty$, there exists some $Y \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}; \mathbb{R}^d)$ such that $X - Y.V \in \mathcal{A}_{t+1}$ and*

$$\rho_t^v(X) = \rho_t^v(Y.V).$$

In particular, if either of the statements (1) or (2) holds then, for the case of weak representation, we have that for all $t \in \{0, \ldots, T-1\}$, $v \in V$ and $X \in \mathcal{L}^\infty$, there exists some $X_1, \ldots, X_d \in \mathcal{L}^\infty(\mathcal{F})$ such that $X = \sum_{i=1}^d X_i$ and

$$\rho_t^v(X) = \rho_t^v\left(\sum_{i=1}^d \rho_{t+1}^{v_i}(X_i)v_i\right).$$

Remark 7.10. *Since the cone $K(\mathcal{A}, \{1\}) = K(\mathcal{A})$ is closed, the time-consistency property introduced by Delbaen is equivalent to the weak 1-time consistency property.*

Given a probability measure $\mathbb{Q} << \mathbb{P}$, denote the Radon-Nikodym derivative (or density) of $\mathbb{Q}$ with respect to $\mathbb{P}$ by $\Lambda^{\mathbb{Q}}$ and denote the density of the restriction of $\mathbb{Q}$ to $\mathcal{F}_t$ by $\Lambda_t^{\mathbb{Q}}$ (so that $\Lambda_t^{\mathbb{Q}} = \mathbb{E}_{\mathbb{P}}[\Lambda^{\mathbb{Q}}|\mathcal{F}_t]$).

We now state the equivalence for weak representation:

Theorem 7.11. *Let V be a finite subset of $\mathcal{N}$, then $\mathcal{A}$ is weakly represented by V if and only if* **whenever** *$\mathbb{Q}, \mathbb{Q}' \in \mathcal{Q}$, with $\mathbb{Q}' \sim \mathbb{P}$, and τ is a stopping time such that*

$$\mathbb{E}_{\mathbb{Q}}[V|\mathcal{F}_\tau] = \mathbb{E}_{\mathbb{Q}'}[V|\mathcal{F}_\tau],$$

then the p.m. $\hat{\mathbb{Q}}$, given by

$$\Lambda^{\hat{\mathbb{Q}}} = \frac{\Lambda^{\mathbb{Q}'}}{\Lambda_\tau^{\mathbb{Q}'}}\Lambda_\tau^{\mathbb{Q}}$$

is an element of $\mathcal{Q}$.

Proof. This is an easy corollary of Theorems 7.6 and 5.20. □

Now we give two key equivalences for strong representation.

Theorem 7.12. *$\mathcal{A}$ is strongly represented by V if and only if there exists a collection $(\mathcal{H}_t)_{t=0}^T$, with each $\mathcal{H}_t$ being a t-cone in $\mathcal{L}^1(\mathcal{F}_t; \mathbb{R}^d)$ such that:*

$$\mathcal{A}^* = \bigcap_{t=0}^T \{Z \in \mathcal{L}^1(\mathcal{F});\ \mathbb{E}(ZV|\mathcal{F}_t) \in \mathcal{H}_t\}.$$

Proof. This is an immediate consequence of Theorem 6.18. □

Remark 7.13. *Thanks to Corollary 4.7 of* [11], *we can interpret a t-cone in* $\mathcal{L}^1(\mathcal{F}_t;\mathbb{R}^d)$ *as the collection of all elements of* $\mathcal{L}^1(\mathcal{F}_t;\mathbb{R}^d)$ *which lie almost surely in a random, closed, convex cone in* $\mathbb{R}^d$.

Example 7.14. *If* $\mathcal{Q}=\{\mathbb{P}\}$ *where* $\mathbb{P}$ *is the unique EMM for the vector price process* $(S_t)_{0\le t\le T}$ *(so that* $\mathcal{A}$ *is strongly represented by* S_T*), then, with* $\mathcal{H}_t=\{\alpha S_t:\ \alpha\in L^\infty_+(\mathcal{F}_t)\}$, $\mathcal{A}^*=\bigcap_{t=0}^T\{Z\in\mathcal{L}^1(\mathcal{F});\,\mathbb{E}(ZS|\mathcal{F}_t)\in\mathcal{H}_t\}$. *We leave the proof of this statement to the reader.*

Example 7.15. *We consider a binary branching tree again. Our sample space is* $\Omega=\{1,2,3,4\}$ *with* $\mathbb{P}$ *uniform,* $\mathcal{F}=\mathcal{F}_2=2^\Omega$, $\mathcal{F}_1=\sigma(\{1,2\},\{3,4\})$ *and* $\mathcal{F}_0$ *trivial. Equating each probability measure* $\mathbb{Q}$ *on* Ω *with the corresponding vector of probability masses, take*

$$\mathcal{Q}=co\left(\left\{\frac{1}{2},\frac{1}{2},0,0\right\},\left\{0,0,\frac{1}{2},\frac{1}{2}\right\},\left\{\frac{1}{3},\frac{1}{6},\frac{1}{6},\frac{1}{3}\right\},\left\{\frac{1}{6},\frac{1}{3},\frac{1}{3},\frac{1}{6}\right\}\right),$$

$v=1+1_{\{1,3\}}$ *and* $V=(1,v)$. *Then* $\mathcal{Q}$ *is strongly* V*-m-stable.*

Proof. Denoting the convex cone generated by a set S by $cone(S)$, define $D_0=cone\left(\{1,\frac{1}{2}\}\right)$,

$$D_1(.)=cone\left(\left\{1,\frac{1}{3}\right\},\left\{1,\frac{2}{3}\right\}\right),$$

$$D_2(1)=D_2(3)=cone\left(\{1,1\}\right),$$

and

$$D_2(2)=D_2(4)=cone\left(\{1,0\}\right).$$

It is not hard to show that

$$\mathcal{A}^*=\bigcap_{t=0}^{2}\left\{Z\in\mathcal{L}^1:\ \mathbb{E}(ZV|\mathcal{F}_t)\in D_t\text{ a.s}\right\}.$$

The result follows from Theorem 7.12. □

Theorem 7.16. $\mathcal{A}$ *is strongly represented by* V *if and only if whenever* $\mathbb{Q},\mathbb{Q}'\in\mathcal{Q}$, *with* $\mathbb{Q}'\sim\mathbb{P}$, τ *is a stopping time with* $\tau\le T-1$ *a.s. and* $\hat{\mathbb{Q}}$ *satisfies*

$$\frac{\Lambda^{\hat{\mathbb{Q}}}}{\Lambda^{\hat{\mathbb{Q}}}_{\tau+1}}=\frac{\Lambda^{\mathbb{Q}'}}{\Lambda^{\mathbb{Q}'}_{\tau+1}},\tag{7.2}$$

and

$$\Lambda^{\hat{\mathbb{Q}}}_\tau=\Lambda^{\mathbb{Q}}_\tau,\tag{7.3}$$

then

$$\mathbb{E}_{\hat{\mathbb{Q}}}[V|\mathcal{F}_\tau]=\mathbb{E}_{\mathbb{Q}}[V|\mathcal{F}_\tau]$$

implies that $\hat{\mathbb{Q}}$ *is an element of* $\mathcal{Q}$.

Proof. This is an easy corollary of Theorems 7.6 and 5.20 on noticing that equations (7.2) and (7.3) are equivalent to saying that

$$\Lambda^{\hat{\mathbb{Q}}} = R_{\tau+1}\frac{\Lambda^{\mathbb{Q}'}}{\Lambda^{\mathbb{Q}'}_{\tau+1}}\Lambda^{\mathbb{Q}}_{\tau}$$

for some $R_{\tau+1} \in \mathcal{L}^1(\mathcal{F}_{\tau+1})$ with $\mathbb{E}[R_{\tau+1}|\mathcal{F}_\tau] = 1$.

□

Example 7.17. *We consider a binary branching tree on two time steps, but with one node pruned. Thus, our sample space is $\Omega = \{1, 2, 3\}$ with $\mathbb{P}$ uniform, $\mathcal{F} = \mathcal{F}_2 = 2^\Omega$, $\mathcal{F}_1 = \sigma(\{1, 2\}, \{3\})$ and $\mathcal{F}_0$ trivial. Equating each probability measure $\mathbb{Q}$ on Ω with the corresponding vector of probability masses, take*

$$\mathcal{Q} = co\left((\frac{1}{2},\frac{1}{3},\frac{1}{6}),(\frac{1}{3},\frac{2}{9},\frac{4}{9})\right),$$

$v = 1 + 1_{\{1\}}$ *and* $V = (1, v)$*. Then* $\mathcal{Q}$ *is strongly* V*-m-stable.*

Proof. Take a stopping time $\tau \le 1$. Since $\mathcal{F}_0$ is trivial it is clear that either $\tau = 0$ a.s. or $\tau = 1$ a.s.

A generic element of $\mathcal{Q}$ may be written as $\mathbb{Q}_\lambda = (p_\lambda, q_\lambda, r_\lambda) = (\frac{1}{2} - \frac{\lambda}{6}, \frac{1}{3} - \frac{\lambda}{9}, \frac{1}{6} + \frac{5\lambda}{18})$. Denoting a generic p.m. on $(\Omega, \mathcal{F})$ by $\mathbb{Q}$ by (p, q, r), and taking $\mathbb{P}$ to be the uniform measure on Ω we see that

$$\Lambda^{\mathbb{Q}} = (3p, 3q, 3r),\ \Lambda^{\mathbb{Q}}_1 = (\frac{3p+3q}{2}, \frac{3p+3q}{2}, 3r) \text{ and } \frac{\Lambda^{\mathbb{Q}}}{\Lambda^{\mathbb{Q}}_1} = (\frac{2p}{p+q}, \frac{2q}{p+q}, 1).$$

Notice that (for any value of λ)

$$\frac{\Lambda^{\mathbb{Q}_\lambda}}{\Lambda^{\mathbb{Q}_\lambda}_1} = (\frac{6}{5}, \frac{4}{5}, 1).$$

First suppose that the equations (7.2) and (7.3) are satisfied with $\mathbb{Q}' = \mathbb{Q}_\mu$ and $\mathbb{Q} = \mathbb{Q}_\lambda$ and $\tau = 1$. Then $\Lambda^{\hat{\mathbb{Q}}}_1 = \Lambda^{\mathbb{Q}_\lambda}_1$ and so $\hat{p} + \hat{q} = p_\lambda + q_\lambda$. Moreover, $1 + \frac{\hat{p}}{\hat{p}+\hat{q}} = \mathbb{E}_{\hat{\mathbb{Q}}}[v|\mathcal{F}_1] = \mathbb{E}_{\mathbb{Q}_\lambda}[v|\mathcal{F}_1] = 1 + \frac{p_\lambda}{p_\lambda+q_\lambda}$. It follows that $\hat{\mathbb{Q}} = \mathbb{Q}_\lambda$ and so $\hat{\mathbb{Q}} \in \mathcal{Q}$.

Now suppose that $\tau = 0$ and equations (7.2) and (7.3) are satisfied with $\mathbb{Q}' = \mathbb{Q}_\mu$ and $\mathbb{Q} = \mathbb{Q}_\lambda$. Equating $\frac{\Lambda^{\hat{\mathbb{Q}}}}{\Lambda^{\hat{\mathbb{Q}}}_1}$ and $\frac{\Lambda^{\mathbb{Q}_\mu}}{\Lambda^{\mathbb{Q}_\mu}_1}$, we see that $\frac{\hat{q}}{\hat{p}} = \frac{2}{3}$. Then, equating $\mathbb{E}_{\hat{\mathbb{Q}}}[v]$ and $\mathbb{E}_{\mathbb{Q}_\lambda}[v]$ we see that $1 + \hat{p} = 1 + p_\lambda$. It follows that $\hat{\mathbb{Q}} = \mathbb{Q}_\lambda$ and so $\hat{\mathbb{Q}} \in \mathcal{Q}$ once more.

□

7.2. **The countable case.** Recall (Definition 7.3), that if V is a vector of assets in $\mathcal{N}$ then $K^\eta_t(\mathcal{A}, V) \stackrel{def}{=} \mathcal{A}_t(V) \cap \mathcal{L}^\infty(\mathcal{F}_{t+\eta}; \mathbb{R}^d)$ and $\mathcal{A}$ is η-represented by V iff $\mathcal{A}(V) = \overline{\oplus^{T-\eta}_{t=0} K^\eta_t(\mathcal{A}, V)}$. It is clear that this is true if and only if $\mathcal{A} = [\mathcal{A}, V]^\eta \stackrel{def}{=} V.\overline{\oplus^{T-\eta}_{t=0} K^\eta_t(\mathcal{A}, V)}$

We extend this as follows:

Definition 7.18. *Let $U \subset \mathcal{N}$. We say that $\mathcal{A}$ is η-represented by U if for all $X \in \mathcal{A}$, there exists a sequence $X_n \in \mathcal{A}$ which converges weakly* to X in $\mathcal{L}^\infty$ such that for all $\varepsilon > 0$, there exists $n \geq 1$ and a finite set $V^\varepsilon \subset U$ such that $X_n - \varepsilon \in [\mathcal{A}, V^\varepsilon]^\eta$.*

Next we prove the equivalence between the U-time-consistency of the cone $\mathcal{A}$ and U-stability of its polar cone $\mathcal{A}^*$ when U is countable. In order to do this we introduce the following relations:

Definition 7.19. *For $Z, Z' \in \mathcal{L}^1$, we say that $Z \equiv_{t,\eta,U} Z'$ for $\eta = 0, 1$ if there exists $\alpha \in \mathcal{L}^0_+(\mathcal{F}_t)$ with $\alpha Z' \in \mathcal{L}^1$ such that $\mathbb{E}(Zu|\, \mathcal{F}_{t+\eta}) = \alpha\, \mathbb{E}(Z'u|\, \mathcal{F}_{t+\eta})$ for all $u \in U$.*

Proposition 7.20. *Let $P \subset \mathcal{L}^1_+(\mathcal{F})$ be an (η, U)-m-stable cone, then $P = \cap_{t=0}^{T-\eta} M_t^\eta(P, U)$ where*

$$M_t^\eta(P, U) = \{Z :\ Z \equiv_{t,\eta,U} Z' \text{ for some } Z' \in P\}.$$

Proof. The proof is essentially the same as that of Lemma 6.11. □

Proposition 7.21. *Let $P \subset \mathcal{L}^1_+(\mathcal{F})$ and U be a set of assets. Define*

$$[P, U]^{(\eta)} \stackrel{def}{=} \cap_{t=0}^{T-\eta} R_t^\eta(P, U),$$

where

$$R_t^\eta(P, U) = \left\{Z :\ Z \equiv_{t,\eta,U} Z' \text{ for some } Z' \in P_{(t)}\right\}.$$

Then

1. *$[P, U]^{(\eta)}$ is the smallest (η, U)-m-stable closed convex cone in $\mathcal{L}^1$, containing P.*
2. *P is an (η, U)-m-stable closed convex cone if and only if $P = [P, U]^{(\eta)}$.*

Proof. The proof follows that of Lemma 6.12 very closely. □

From now on we fix a countable set of assets $U \subset \mathcal{N}$ and a sequence of finite sets of assets U^n, increasing to U with $U^0 = \{1\}$.

Theorem 7.22. *Let P denote a* **closed** *convex cone in $\mathcal{L}^1_+$, then the following are equivalent*

- (i) *P is strongly (resp. weakly) U-stable;*
- (ii) *there exists a decreasing sequence $(P^n)_{n\geq 0}$ such that P^n is strongly (resp. weakly) U^n-stable for each n and $P = \bigcap_{n\geq 1} P^n$.*

Proof. Remark that the implication (ii) $\Rightarrow$ (i) is straightforward. Now suppose that P is (η, U)-stable for $\eta \in \{0, 1\}$. Define $P^n = [P, U^n]^{(\eta)}$. We may check easily that the sequence P^n is decreasing and $P \subset \bigcap_{n\geq 1} P^n$. We shall show that $\bigcap_{n\geq 1} P^n \subset P$. Let $Z \in \bigcap_n P^n$ with $\mathbb{E}(Z) > 0$ (if not $Z = 0$ and then $Z \in P$), then for all $n \geq 1$ and for all $t \in \{0, \ldots, T - \eta\}$, there exists $Z^{n,t} \in P_{(t)}$ such that:

$$\mathbb{E}(Zu|\, \mathcal{F}_{t+\eta}) = \mathbb{E}(Z^{n,t}u|\, \mathcal{F}_{t+\eta})$$

for all $u \in U^n$. For all t, there exists a sequence of positive real numbers a_t^n such that the sequence $f^{n,t} = \sum_{k\geq n} a_t^k Z^{k,t}$ converges in $\mathcal{L}^1$ to some $Z^t \in P_{(t)}$. So for all $t \in \{0, \ldots, T-1\}$, we have:

$$\sum_{k\geq n} a_t^k \mathbb{E}(Zu|\,\mathcal{F}_{t+\eta}) = \mathbb{E}(f^{n,t}u|\,\mathcal{F}_{t+\eta}),$$

for all $u \in U^n$. By taking the limit we see that the sequence $\sum_{k\geq n} a_t^k$ converges to some $a_t = \mathbb{E}(Z^t)/\mathbb{E}(Z)$ and then

$$\mathbb{E}(Zu|\,\mathcal{F}_{t+\eta}) = \frac{1}{a_t}\mathbb{E}(Z^t u|\,\mathcal{F}_{t+\eta}),$$

for all $u \in U$. Thus $Z \in [P, U]^{(\eta)} = P$. □

Proposition 7.23. *$\mathcal{A}$ is strongly (resp. weakly) U-time-consistent if and only if there exists an increasing sequence $(\mathcal{A}^n)_{n\geq 0}$ of acceptance sets, with $\mathcal{A}_n$ strongly (resp. weakly) U^n-time-consistent for each n, such that $\mathcal{A} = \overline{\bigcup_{n\geq 1}\mathcal{A}^n}$.*

Proof. Suppose that $\mathcal{A} = \overline{\bigcup_{n\geq 1}\mathcal{A}^n}$ and let ρ_t and ρ_t^n be respectively the coherent risk measures associated to the sets $\mathcal{A}_t$ and $\mathcal{A}_t^n$. Since for all $n \geq 1$, we have $\mathcal{A}^n \subset [\mathcal{A}, U^n]^\eta$, we can suppose from now on that $\mathcal{A}^n = [\mathcal{A}, U^n]^\eta$. First we prove that for all $X \in \mathcal{L}^\infty$, the sequence $\rho_t^n(X)$ converges a.s to $\rho_t(X)$. Remark that for all n, we have $X - \rho_t^n(X) \in \mathcal{A}^n \subset \mathcal{A}^{n+1}$ and then $\rho_t^{n+1}(X - \rho_t^n(X)) = \rho_t^{n+1}(X) - \rho_t^n(X) \leq 0$ and the sequence $\rho_t^n(X)$ is decreasing. Moreover for all n,

$$\rho_t^n(X) \geq \rho_t(X),$$

Define

$$\rho_t^\infty(X) = \liminf \rho_t^n(X) \geq \rho_t(X).$$

Since $\mathcal{A} = \overline{\bigcup_{n\geq 1}\mathcal{A}^n}$ we deduce that $\mathcal{A}_t = \overline{\bigcup_{n\geq 1}\mathcal{A}_t^n}$ and then for $Y = X - \rho_t(X) \in \mathcal{A}_t$, there exists a sequence $Y^n \in \mathcal{A}_t^{k_n}$ with $k_n \geq n$ such that Y^n converges weakly* to Y in $\mathcal{L}^\infty$. Therefore

$$\rho_t^\infty(Y) \leq \liminf \rho_t^\infty(Y^n) \leq \liminf \rho_t^{k_n}(Y^n) \leq 0.$$

Hence $\rho_t^\infty(X) \leq \rho_t(X)$ from the translation invariance property of ρ_t^∞. We deduce that $\rho_t(X) = \rho_t^\infty(X)$. Since each $\mathcal{A}^n$ is (η, U^n)-time-consistent, for all $X \in \mathcal{L}^\infty$ and $t \in \{0, \ldots, T-1\}$, there exists $X_{n,m} \in \mathcal{L}^\infty$ and $Y^{n,m} \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^n)$ such that $X_{n,m}$ converges weakly* to X in $\mathcal{L}^\infty$ when m goes to infinity, $X_{n,m} - Y^{n,m}.U^n \in \mathcal{A}_{t+1}^n \subset \mathcal{A}_{t+1}$ and

$$\rho_t^n(X) = \liminf_m \rho_t^n\left(Y^{n,m}.U^n\right).$$

Since $\rho_t^n \geq \rho_t$, we obtain

$$\rho_t^n(X) \geq \liminf_m \rho_t\left(Y^{n,m}.U^n\right).$$

We take the limit in n and get

$$\rho_t(X) = \liminf_{n,m} \rho_t\left(Y^{n,m}.U^n\right).$$

Now assume that $\mathcal{A}$ is (η, U)-time-consistent and define $\mathcal{A}^n = [\mathcal{A}, U^n]^\eta$. Fix $t \in \{0, \ldots, T-\eta\}$ and let $X \in \mathcal{A}_t$, so there exists $X_n \in \mathcal{L}^\infty$, $Y^n \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^n)$ and an $\mathbb{R}^n$-valued portfolio $V^n \subset U$, containing the unit $\mathbf{1}$ such that X_n converges weakly* to X in $\mathcal{L}^\infty$, $X_n - Y^n.V^n \in \mathcal{A}_{t+1}$ and

$$\rho_t(X) = \liminf \rho_t\left(Y^n.V^n\right).$$

Therefore for an arbitrary $\varepsilon > 0$, there exists some $N = N_\varepsilon$ such that for all $n \geq N$

$$\varepsilon + \rho_t(X) \geq \rho_t\left(Y^n.V^n\right).$$

Remark that

$$X_n - \varepsilon = (X_n - Y^n.V^n) + (Y^n.V^n - \varepsilon),$$

with $X_n - Y^n.V^n \in \mathcal{A}_{t+1}$ and $Y^n.V^n - \varepsilon \in K_t^\eta(\mathcal{A}, V^n).V^n$. Consequently

$$X_n - \varepsilon \in K_t^\eta(\mathcal{A}, V^n).V^n + \mathcal{A}_{t+1},$$

with $V^n \subset U^n$. By backwards induction on t, we deduce that for every $X \in \mathcal{A}$, there exists a sequence X_n which converges weakly* to X and for any $\varepsilon > 0$, we have:

$$X_n - \varepsilon \in \overline{\bigcup_{k\geq 1}[\mathcal{A}, U^k]^\eta} = \overline{\bigcup_{k\geq 1}\mathcal{A}^k},$$

by taking the limit in n we obtain that $X \in \overline{\bigcup_{n\geq 1}\mathcal{A}^n}$ and then $\mathcal{A} = \overline{\bigcup_{n\geq 1}\mathcal{A}^n}$. □

Theorem 7.24. *$\mathcal{A}$ is strongly (resp. weakly) U-time-consistent if and only if $\mathcal{A}^*$ is strongly (resp. weakly) U-stable.*

Proof. This is an immediate consequence of Theorem 7.22 and Proposition 7.23. □

Theorem 7.25. *$\mathcal{A}$ is strongly (resp. weakly) U-time-consistent if and only if $\mathcal{A}$ is strongly (resp. weakly) represented by U.*

Proof. Suppose that $\mathcal{A}$ is (η, U)-time-consistent. We show in the proof of Theorem 7.23 that for all $X \in \mathcal{A}$, there exists a sequence $X_n \in \mathcal{A}$ which converges weakly* to X in $\mathcal{L}^\infty$ such that for all $\varepsilon > 0$, there exists $n \geq 1$ and $V^n \subset U$ with

$$X_n - \varepsilon \in [\mathcal{A}, V^n]^\eta.$$

Conversely let U^n be a sequence of finite sets, increasing to U. By assumption for all $X \in \mathcal{A}$, there exists a sequence $X_n \in \mathcal{A}$ which converges weakly* to X in $\mathcal{L}^\infty$ such that for all $\varepsilon > 0$, there exists $n \geq 1$ and a finite subset $V^n \subset U$ such that $X_n - \varepsilon \in [\mathcal{A}, V^n]^\eta$. Then

$$\mathcal{A} \subset \overline{\bigcup_{n\geq 1}[\mathcal{A}, U^n]^\eta},$$

and consequently $\mathcal{A}$ is (η, U)-time-consistent. □

Definition 7.26. *We say that $\mathcal{A}$ is countably (resp. finitely) strongly (resp. weakly) time-consistent if there exists a countable (resp. finite) set $U \subset \mathcal{N}$ such that $\mathcal{A}$ is strongly (resp. weakly) U-time-consistent. By analogy, we say that $\mathcal{A}^*$ is countably (resp. finitely) strongly (resp. weakly) stable if there exists a countable (resp. finite) set $U \subset \mathcal{N}$ such that $\mathcal{A}^*$ is strongly (resp. weakly) U-stable.*

Theorem 7.27. *Suppose that the vector space $\mathcal{L}^1$ is separable, then the cone $\mathcal{A}$ is countably strongly time-consistent.*

Proof. Thanks to Theorem 7.24, we need only show that $\mathcal{A}^*$ is countably strongly stable. Since we assume that the space $\mathcal{L}^1$ is separable it follows that the subset $B_+ \stackrel{def}{=} \{X \in \mathcal{L}^1_+ :\ X \le 1\}$ is separable. Denote by $H = \{u_n;\, n \ge 1\}$, a countable dense set and define

$$U = \{1\} \cup \{1 + u_n :\ n \ge 1\}.$$

We may check easily that $\mathcal{A}^*$ is strongly U-stable, indeed let $t \in \{0, \dots, T-1\}$ and $Z, Z^1, \dots, Z^k \in \mathcal{A}^*$ such that there exists some $\alpha^i \in \mathcal{L}^0_+(\mathcal{F}_{t+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and a partition $F^1_t, \dots, F^k_t$ satisfying $\mathbb{E}((Z - \sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i)u|\mathcal{F}_t) = 0$ for all $u \in U$, in particular $\mathbb{E}(Z - \sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i)u_n = 0$ for all $n \ge 1$. We deduce that for all $u \in B_+$, there is a sequence $u_n \in B_+$ which converges to u in $\mathcal{L}^1$, define $f = Z - \sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i$ and $f_N = f 1_{(|f| \le N)}$ for an integer $N \ge 1$, then

$$|\mathbb{E}f(u_n - u)| \le |\mathbb{E}f_N(u_n - u)| + |\mathbb{E}(f - f_N)(u_n - u)| \le N\mathbb{E}|u_n - u| + 2\mathbb{E}|f|1_{(|f| \ge N)}.$$

We take the limit when n goes to infinity and obtain

$$\lim_{n\to\infty} |\mathbb{E}f(u_n - u)| \le 2\mathbb{E}|f|1_{(|f| \ge N)}.$$

We take the limit again a N goes to infinity to obtain

$$\lim_{n\to\infty} \mathbb{E}f(u_n - u) = 0.$$

Then $\mathbb{E}(Z - \sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i)u = 0$ for all $u \in \mathcal{L}^\infty$ and thus $\sum_{i=1}^k 1_{F^i_t}\alpha^i Z^i = Z \in \mathcal{A}^*$. □

We conclude this subsection with the following counterexample.

Counterexample 7.28. *We take an uncountable collection of independent, identically distributed Uniform[1,2] random variables indexed by $t \in [0,1]$, with the usual product measure and σ-algebra. So, to be concrete: $\Omega = [1,2]^{[0,1]}$ and we take the coordinate process $(X_t)_{t\in[0,1]}$ with $X_t(\omega) = \omega(t)$ for $\omega \in \Omega$. We define the σ-algebra*

$$\mathcal{F}_1 = \mathcal{F} = \mathcal{B}([1,2]^{[0,1]}),$$

and the probability measure $\mathbb{P}$ is the product meaure on Ω corresponding to Lebesgue measure λ on each component interval $[1,2]$. We take $\mathcal{F}_0$ to be the trivial σ-agebra. Note that the vector space $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ is not *separable.*

Now consider the coherent risk measure associated with the singleton $\{\mathbb{P}\}$: we claim that there is no countable set $U \subset \mathbb{N}$ such that $\{\mathbb{P}\}$ is U-m-stable.

To prove this, suppose that there is such a countable set U, with $U = \{u_n,\, n \ge 1\}$ where each u_n is $\mathcal{F}$-measurable. Then for each n, there exists a sequence $(X_{s^n_j})_{j\ge1}$ such that u_n is measurable with respect to $\sigma(X_{s^n_j} :\ j \ge 1)$ and so, by diagonalisation, there is a sequence $(X_{s_j})_{j\ge1}$ such that each u_n is measurable with respect to $\sigma(X_{s_j} :\ j \ge 1)$.

Take $X = X_t$ for some t with $t \notin \{s_j :\ j \ge 1\}$. By assumption, defining $V^n = (u_1, \dots, u_n)$, there is

(i) *a sequence $X_n \in \mathcal{L}^\infty$ that converges weakly* to X;*
(ii) *a sequence $Y^n \in \mathbb{R}^n$ (since $\mathcal{F}_0$ is trivial) such that $X_n - Y^n.V^n \in \mathcal{A}_1$ for each integer n; with*

(iii) $\mathbb{E}_{\mathbb{P}}(X) = \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}(Y^n.V^n)$.

Since $\mathcal{A}_1 = \mathcal{L}^{\infty}_{-}$ *we have* $X_n \leq Y^n.V^n$ *a.s. for each integer* n *and so, for all* $Z \in \mathcal{L}^1_+$ *we have* $\mathbb{E}_{\mathbb{P}}(Z\, Y^n.V^n) \geq \mathbb{E}_{\mathbb{P}}(Z\, X_n)$. *In particular, taking* $p \in \mathbb{N}$, *and setting* $Z_p = X^{2p}/\mathbb{E}(X^{2p})$, *we see that for all* p:

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}(X) &= \liminf \mathbb{E}_{\mathbb{P}}(Y^n.V^n) = \liminf \mathbb{E}_{\mathbb{P}} Z_p(Y^n.V^n)\\ &\geq \liminf \mathbb{E}_{\mathbb{P}}(Z_p\, X_n) = \mathbb{E}_{\mathbb{P}}(X^{2p+1})/\mathbb{E}_{\mathbb{P}}(X^{2p}) \geq \|X\|_{\mathcal{L}^{2p}},\end{aligned}$$

the second equality in the first line holding since Z_p *and* $(Y^n.V^n)$ *are independent and* $\mathbb{E}Z_p = 1$ *and the last inequality in the second line is an application of H older's inequality* $\|X\|_{\mathcal{L}^q} \leq \|X\|_{\mathcal{L}^{q+1}}$ *with* $q = 2p$. *We take the limit as* $p \to \infty$ *to obtain:*

$$\mathbb{E}_{\mathbb{P}}(X) = 3/2 \geq \textit{ess-sup}X = 2,$$

since X *is uniform on* $[1,2]$ *under* $\mathbb{P}$. *This is the desired contradiction.*

7.3. **The case of a finite sample space.** Here we consider the case where Ω is finite with cardinality N. We consider random variables as vectors in $\mathbb{R}^N$. It is not immediately obvious (but is, nevertheless, true) that an acceptance set $\mathcal{A}$ is finitely strongly time consistent.

Lemma 7.29. *The cone* $K^\eta(\mathcal{A}, V)$ *is closed in* $\mathcal{L}^\infty$.

Proof. Remark that $K^\eta(\mathcal{A}, V) = K^\eta_{0,T-\eta}(\mathcal{A}, V)$ where

$$K^\eta_{t,T-\eta}(\mathcal{A}, V) = K^\eta_t(\mathcal{A}, V) + \ldots + K^\eta_{T-\eta}(\mathcal{A}, V),$$

for $t \in \{0, \ldots, T-\eta\}$. We prove the closedness of the cone $K^\eta(\mathcal{A}, V)$ by backwards induction on $t = T-\eta, \ldots, 0$. For $t = T-\eta$, the cone $K^\eta_{T-\eta}(\mathcal{A}, V)$ is closed. Now suppose that the cone $K^\eta_{s,T-\eta}(\mathcal{A}, V)$ is closed for $s = T-\eta, \ldots, t+1$. Observe that

$$K^\eta_{t,T-\eta}(\mathcal{A}, V) = K^\eta_t(\mathcal{A}, V) + K^\eta_{t+1,T-\eta}(\mathcal{A}, V),$$

and that the subset

$$\mathbb{N} \stackrel{def}{=} K^\eta_t(\mathcal{A}, V) \cap -K^\eta_{t+1,T-\eta}(\mathcal{A}, V),$$

forms a vector space. We define $\mathbb{N}^\perp$ to be its orthogonal complement, then

$$K^\eta_{t,T-\eta}(\mathcal{A}, V) = K^\eta_t(\mathcal{A}, V) \cap \mathbb{N}^\perp + K^\eta_{t+1,T-\eta}(\mathcal{A}, V).$$

Now take a sequence

$$x^n = x^n_0 + x^n_1 \in K^\eta_t(\mathcal{A}, V) \cap \mathbb{N}^\perp + K^\eta_{t+1,T-\eta}(\mathcal{A}, V),$$

that converges (weakly* or in norm) to some x. We claim that the sequence x^n_0 is bounded. If not we divide both sides of the equation by the norm of x^n_0 in $\mathcal{L}^\infty$ and obtain

$$\frac{x^n}{\|x^n_0\|} = \frac{x^n_0}{\|x^n_0\|} + \frac{x^n_1}{\|x^n_0\|} \stackrel{def}{=} y^n_0 + y^n_1.$$

The sequence y^n_0 is bounded, it converges (or at least some subsequence does) to some y_0 and then the sequence y^n_1 converges to $y_1 = -y_0$. Therefore $y_0 \in \mathbb{N} \cap \mathbb{N}^\perp$ which means that $y_0 = 0$. This contradicts the fact that $\|y_0\| = 1$. Now, since the sequence x^n_0 is bounded, w.l.o.g. it converges to some x_0 and then the sequence x_1^n converges to x_1. We conclude that the sequence x^n converges to $x = x_0 + x_1$. $\square$

Lemma 7.30. *$\mathcal{A}$ is finitely strongly time-consistent.*

Proof. We may assume without loss of generality that $\mathcal{F}$ is the power set of Ω. Then every $X \in \mathcal{L}^\infty$ can be written as $X = \sum_{\omega\in\Omega} X(\omega)\, 1_{\{\omega\}}$. Define $U = \{1, 1 + 1_{\{\omega\}};\ \omega \in \Omega\}$. We need to show that $\mathcal{A}^*$ is strongly U-stable. Fix $t \in \{0, \ldots, T-1\}$, $Z, Z^1, \ldots, Z^k \in \mathcal{A}^*$ such that there exists some $\alpha^i \in \mathcal{L}^0_+(\mathcal{F}_{t+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and a partition $F_t^1, \ldots, F_t^k$ satisfying $\mathbb{E}((Z - \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i)u|\mathcal{F}_t) = 0$ for all $u \in U$,

which means that $\mathbb{E}(Z - \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i)\, 1_{\{\omega\}} = 0$ for all $\omega \in \Omega$. Consequently $Z = \sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i$ and so $\sum_{i=1}^k 1_{F_t^i} \alpha^i Z^i \in \mathcal{A}^*$. $\square$

8. Associating a coherent risk measure to a trading cone.

As promised, we now show how to represent a trading cone as (essentially) the acceptance set of a coherent risk measure[1].

Let $\mathcal{B}$ be a closed convex cone given by $\mathcal{B} = K_0 + ... + K_T$ where, as described in section 2, each K_t is generated by positive $\mathcal{F}_t$-measurable multiples of the vectors $-e_i$, $e_j - \pi_t^{ij} e_i$ for $1 \le i, j \le d$.

Recall that null strategies are elements $(\xi_0, \ldots, \xi_T)$ of $K_0 \times \ldots \times K_T$ staisfying $\sum_0^T \xi_t = 0$, and, from [11], that we may suppose without loss of generality that the null strategies of this decomposition form a vector space. Our aim in this section is to transform trading with transaction costs to a partially frictionless setting by adding a new period on the time axis and then to show that the revised trading cone is (essentially) the acceptance set of a coherent risk measure.

We introduce some notation. For $i \in \{1, \ldots, d\}$ we define the random variables $B^i \overset{def}{=} \pi_T^{1i}$ and $S^i \overset{def}{=} 1/\pi_T^{i1}$ We define the random convex sets $H = \{1\} \times \prod_{i=2}^d [S^i, B^i]$ and, for $\varepsilon > 0$,

$$H_\varepsilon = \{1\} \times \prod_{i=2}^d [(1-\varepsilon)S^i, (1+\varepsilon)B^i].$$

Let Ψ_ε be the (finite) set of extreme points of the set H_ε, i.e the 2^{d-1} random vectors of the form $(1, X_2, ..., X_d)$ where each $X_i = (1-\varepsilon)S^i$ or $(1+\varepsilon)B^i$. Let $\tilde{\Omega} = \{0,1\}^{d-1}$ and enumerate the elements of Ψ_ε as follows:

$$\Psi_\varepsilon = \{Y(\omega, \tilde{\omega}) : \tilde{\omega} \in \tilde{\Omega}\},$$

[1]A version of the results in this section originally appeared in [11]. Since they are only distantly related to the main results in that paper, we have removed them from the version of that paper which is to be submitted for publication.

where
(8.1)
$$Y(\omega,\tilde{\omega}_1,\ldots,\tilde{\omega}_{d-1}) \stackrel{def}{=} e_1 + \sum_{j=1}^{d-1} \left\{(1-\tilde{\omega}_j)(1-\varepsilon)\, S^{j+1}(\omega) + \tilde{\omega}_j(1+\varepsilon)\, B^{j+1}(\omega)\right\}\, e_{j+1}.$$

Define $\mathcal{B}^o$ to be the collection of consistent price processes for $\mathcal{B}$. Recall from [18] that this means that

$$\mathcal{B}^o = \{Z \in \mathcal{L}^1_+(\mathcal{F}_T, \mathbb{R}^d) : Z > 0 \text{ a.s. and } Z_t \in K_t^{(*)} \text{ a.s.}\},$$

where $K_t^{(*)} \stackrel{def}{=} \{X \in \mathcal{L}^1(\mathcal{F}_t, \mathbb{R}^d) : X.Y \leq 0 \text{ a.s. for all } Y \in K_t\}$.

Proposition 8.1. *Let $Z \in \mathcal{B}^o$. Then there exist strictly positive random variables $\lambda^{Z_T}(\cdot,\tilde{\omega})$ defined for each $\tilde{\omega} \in \tilde{\Omega}$ such that $\sum_{\tilde{\omega}\in\tilde{\Omega}} \lambda^{Z_T}(\omega,\tilde{\omega}) = 1$ and*

$$\frac{Z_T}{Z_T^1} = \sum_{\tilde{\omega}\in\tilde{\Omega}} \lambda^{Z_T}(\omega,\tilde{\omega}) Y(\omega,\tilde{\omega}).$$

Proof. We know from the properties of consistent price processes that for $i,j = 1,...,d$,

$$\frac{Z_T^j}{Z_T^i} \leq \pi_T^{ij} \leq \pi_T^{i1}\,\pi_T^{1j} = \frac{B^j}{S^i}.$$

In consequence, for every $i = 2,\cdots,d$ we get

$$\bar{Z}_T^i \stackrel{def}{=} \frac{Z_T^i}{Z_T^1} \in [S^i, B^i]\,,$$

and so

$$\bar{Z}_T = (\bar{Z}_T^1,...,\bar{Z}_T^d) \in H \subset H_\varepsilon.$$

Now, for $2 \leq i \leq d$, let

$$\theta(\omega,i) \stackrel{def}{=} \frac{\bar{Z}_T^i - S^i(1-\varepsilon)}{B^i(1+\varepsilon) - S^i(1-\varepsilon)},$$

and then define

$$\lambda^{Z_T}(\omega,\tilde{\omega}) \stackrel{def}{=} \prod_1^{d-1} \theta(\omega,i+1)^{\tilde{\omega}_i}(1-\theta(\omega,i+1))^{1-\tilde{\omega}_i}$$

Since $\theta(\omega,i)$ is exactly the co-efficient θ such that

$$\bar{Z}_T^i = \theta B^i(1+\varepsilon) + (1-\theta)S^i(1-\varepsilon)$$

the result follows. □

To set up the new probability space, let $\tilde{\mathcal{F}}$ be the power set of $\tilde{\Omega}$ and let $\tilde{\mathbb{P}}$ be the uniform measure on $\tilde{\Omega}$, then define $\hat{\Omega} = \Omega \times \tilde{\Omega}$, $\hat{\mathcal{F}} = \mathcal{F} \otimes \tilde{\mathcal{F}}$ and $\hat{\mathbb{P}} = \mathbb{P} \otimes \tilde{\mathbb{P}}$.

Now we define the frictionless bid-ask prices at time $T+1$ by

$$\pi_{T+1}^{ij} \stackrel{def}{=} \frac{Y_j}{Y_i}$$

(where the random vector Y is defined in (8.1)), and so K_{T+1} is the convex cone generated by positive $\mathcal{F}_{T+1}$-measurable multiples of the vectors $-e_i$ and $e_j - \pi^{ij}_{T+1} e_i$, where $\mathcal{F}_{T+1} = \mathcal{F}_T \otimes \tilde{\mathcal{F}}$. We define the new trading cone by $\mathcal{B}_{T+1} \stackrel{def}{=} \mathcal{B} + K_{T+1}$. Here we assume the obvious embedding of $\mathcal{B}$ in $L^0(\mathcal{F}_{T+1}; \mathbb{R}^d)$.

From now on, closedness and arbitrage-free properties are with respect to the vector space $\mathcal{L}^0(\mathcal{F}_{T+1})$.

Proposition 8.2. *The cone $\mathcal{B}_{T+1}$ is closed and arbitrage-free.*

Proof. We prove first that a consistent price process for the cone $\mathcal{B}$ can be extended to (be the trace of) a consistent price process for the cone $\mathcal{B}_{T+1}$.

Let $Z_T \in \mathcal{B}^o$ and define

$$Z_{T+1} \stackrel{def}{=} 2^{d-1}\, Z^1_T\, \lambda^{Z_T}\, Y,$$

where the random variable λ^{Z_T} is given in Proposition 8.1. Then $Z_{T+1} > 0$, $Z_{T+1} \in K^*_{T+1}$ and for $X_T \in \mathcal{L}^\infty_+(\mathcal{F}_T)$ we have, by Fubini's Theorem,

$$\mathbb{E}_{\hat{\mathbb{P}}}\,(X_T.Z_{T+1}) = \sum_{\tilde{\omega}\in\tilde{\Omega}} \mathbb{E}_{\mathbb{P}}\, X_T.\left(Z^1_T\, \lambda^{Z_T}(\cdot,\tilde{\omega}) Y(\cdot,\tilde{\omega})\right) = \mathbb{E}_{\mathbb{P}}\, X_T.\left(Z^1_T \bar{Z}_T\right) = \mathbb{E}_{\mathbb{P}}\,(X_T.Z_T).$$

Consequently $Z_{T+1} \in \mathcal{L}^1$ with $Z_T = \mathbb{E}_{\hat{\mathbb{P}}}\,(Z_{T+1}|\mathcal{F}_T)$, therefore $(Z_0, \ldots, Z_T, Z_{T+1})$ is a consistent price process for the cone $\mathcal{B}_{T+1}$ and so we conclude from Theorem 4.10 of [11] that $\bar{\mathcal{B}}_{T+1}$ is arbitrage-free. We shall now show that

$$K_{T+1} \cap \mathcal{L}(\mathcal{F}_T) \subset \mathcal{B}. \tag{8.2}$$

Indeed, let $X \in K_{T+1} \cap \mathcal{L}(\mathcal{F}_T)$, so for every $n \geq 1$, we have

$$X^n \stackrel{def}{=} X\, 1_{(|X|\leq n)} \in K_{T+1} \cap \mathcal{L}^\infty(\mathcal{F}_T);$$

therefore, for any consistent price process, Z,

$$\mathbb{E}(Z_T.X^n) = \mathbb{E}(Z_{T+1}.X^n) \leq 0.$$

It follows from Theorem 4.14 of [11] that $X^n \in \mathcal{B}$ and thus, by closure, $X \in \mathcal{B}$.

Now we prove that the cone $\mathcal{B}_{T+1}$ is closed. We do this by showing that $\mathbb{N}(K_0 \times \ldots \times K_{T+1})$, the collection of null strategies of the decomposition $K_0 + \ldots + K_{T+1}$, is a vector space.

Let

$$(x_0, \ldots, x_{T+1}) \in \mathbb{N}(K_0 \times \ldots \times K_{T+1})$$

and define

$$x = x_0 + ... + x_T$$

so that $x + x_{T+1} = 0$. Then it follows (since $x \in \mathcal{L}(\mathcal{F}_T)$) that $x_{T+1} \in \mathcal{L}(\mathcal{F}_T)$ and so we conclude from (8.2) that $x_{T+1} \in \mathcal{B}$. We deduce that there exist $y_0 \in K_0, \ldots, y_T \in K_T$ such that $x_{T+1} = y_0 + \ldots + y_T$. We conclude that each $-(x_t + y_t) \in K_t$ and then, by adding x_t, respectively y_t, we conclude that both $-x_t$ and $-y_t$ are contained in K_t for $0 \leq t \leq T$.

Observe that since the time $T+1$ bid-ask prices are frictionless, it follows that every element, $u \in K_{T+1}$ can be written as $u = u_1 - u_2$, where $u_1 \in lin(K_{T+1})$, the lineality space of K_{T+1}, and $u_2 \geq 0$. If we express x_{T+1} like this, we then have that

$$0 \leq u_2 = u_1 - x_{T+1} = u_1 + x \in \mathcal{B}_{T+1},$$

so, since $\mathcal{B}_{T+1}$ is arbitrage-free, $u_2 = 0$ and therefore

$$-x_{T+1} = -u_1 \in K_{T+1}$$

(since $u_1 \in lin(K_{T+1})$). It follows that $\mathbb{N}((K_0 \times \ldots \times K_{T+1})$ is a vector space. □

Now define the subset of probabilities

$$\mathcal{Q} \stackrel{def}{=} \left\{ \mathbb{Q} : \frac{d\mathbb{Q}}{d\hat{\mathbb{P}}} = 2^{d-1} \frac{Z_T^1}{Z_0^1} \lambda^{Z_T} \; : \; Z \in \mathcal{B}^o \right\},$$

and denote by ρ the associated coherent risk measure.

Theorem 8.3. *For every $X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d)$ we have:*

$$\rho(Y.X) = \sup\{\mathbb{E}(Z_T.X) : \; Z \in \mathcal{B}^o, \, \mathbb{E}Z_T^1 = 1\}. \tag{8.3}$$

In particular

$$\begin{aligned} (8.4) \qquad & \mathcal{B} \cap \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) \\ & = \; \left\{ X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) : \mathbb{E}_\mathbb{Q}(Y.X) \leq 0 \;\; \textit{for all} \;\; \mathbb{Q} \in \mathcal{Q} \right\} \\ & = \; \left\{ X \in \mathcal{L}^\infty(\mathcal{F}_T; \mathbb{R}^d) : \rho(Y.X) \leq 0 \right\}. \end{aligned}$$

Proof. Equality (8.3) is immediate from the definition of $\mathcal{Q}$; the second equality in (8.4) follows from (8.3), while the first follows from Theorem 4.14 of [11] and the fact that, as in the proof of Proposition 8.2, $\mathbb{E}_\mathbb{Q} Y.X = \mathbb{E}_\mathbb{P} Z_T.X$ □

Remark 8.4. *If we define $\rho_t : \mathcal{L}^\infty(\mathcal{F}_{T+1}) \to \mathcal{L}^\infty(\mathcal{F}_t)$ by*

$$\rho_t(X) = \textit{ess inf}\{\lambda \in \mathcal{L}^\infty(\mathcal{F}_t) : \; \rho(c(X - \lambda)) \leq 0 \; \textit{for all} \; c \in \mathcal{L}^\infty_+(\mathcal{F}_t)\},$$

then it is easy to show that

(i) $\{X : \; cX \in \mathcal{B}_{T+1} \cap \mathcal{L}^\infty(\mathcal{F}_{T+1}; \mathbb{R}^d) \; \textit{for all} \; c \in \mathcal{L}^\infty_+(\mathcal{F}_t)\} = \{X : \; \rho_t(Y.X) \leq 0 \; a.s.\}$.

(ii) $\mathcal{C}_t^\infty \stackrel{def}{=} \mathcal{C}_t(\mathcal{B}_{T+1}) \cap \mathcal{L}^\infty(\mathcal{F}_{T+1}; \mathbb{R}^d) = \{X \in \mathcal{L}^\infty(\mathcal{F}_t; \mathbb{R}^d) : \; \rho_t(Y.X) \leq 0 \; a.s.\}$.

It follows directly from Theorem 4.16 of [11]*, that $\mathcal{C}_t^\infty$ is $\sigma(\mathcal{L}^\infty(\mathbb{P}), \mathcal{L}^1(\mathbb{P}))$-closed and hence we may apply Corollary 4.7 of* [11].

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Appendix A. Proofs and further results on numéraires

Proof of Theorem 4.1: Suppose $v \in \mathcal{N}_0$, then there exists some $\lambda \in \mathcal{L}^\infty(\mathcal{F}_0)$ such that $1-\lambda v \in \mathcal{A}_0$. Since, by the no-arbitrage property, $1_F \notin \mathcal{A}_0$ for any $F \in \mathcal{F}_0$ with $\mathbb{P}(F)>0$, we have $\lambda>0$ a.s. Now, $1-\lambda v \in \mathcal{A}_0$ means that for all $\mathbb{Q} \in \mathcal{Q}$ we have $1-\lambda\,\mathbb{E}_\mathbb{Q}(v|\mathcal{F}_0) \leq 0$, therefore a.s $\lambda_0(v) \geq 1/\lambda > 0$ and $1/\lambda_0(v) \leq \lambda$.

Now let $v \in \mathcal{L}^\infty$ be such that $\lambda \stackrel{def}{=} \lambda_0(v) > 0$ a.s and $1/\lambda \in \mathcal{L}^\infty$. Then for all $X \in \mathcal{L}^\infty$, setting $b = \|X\|_{\mathcal{L}^\infty}$, we have

$$X - \frac{b}{\lambda} v \in \mathcal{A}_0,$$

since for all $\mathbb{Q} \in \mathcal{Q}$,

$$\mathbb{E}_\mathbb{Q}\left(X - \frac{b}{\lambda} v|\mathcal{F}_0\right) \leq \mathbb{E}_\mathbb{Q}\left(X - \frac{b}{\mathbb{E}_\mathbb{Q}(v|\mathcal{F}_0)} v|\mathcal{F}_0\right) = \mathbb{E}_\mathbb{Q}(X|\mathcal{F}_0) - b \leq 0.$$

□

Proof of Lemma 4.4: The first assertion can be deduced immediately from the properties of the cone $\mathcal{A}_0$. Now we prove that

$$\mathcal{A}_0 = \{X \in \mathcal{L}^\infty : \ \rho_0^v(X) \le 0 \text{ a.s}\} \stackrel{def}{=} \mathcal{A}^v.$$

By definition of the mapping ρ_0^v we have the first inclusion $\mathcal{A}_0 \subseteq \mathcal{A}^v$. Now let $X \in \mathcal{A}^v$, then $X - \rho_0^v(X)v \in \mathcal{A}_0$ from the definition of ρ_0^v and $\rho_0^v(X)v \in \mathcal{A}_0$ from the monotonicity of ρ_0 and then $X = (X - \rho_0^v(X)v) + \rho_0^v(X)v \in \mathcal{A}_0$.

To prove (ii), define $\xi(X) \stackrel{def}{=}$ ess- sup $\left\{\dfrac{\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0)}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)} : \mathbb{Q} \in \mathcal{Q}\right\}$. Note first that $X - \rho_0^v(X)v \in \mathcal{A}_0$, so for all $\mathbb{Q} \in \mathcal{Q}$ we have

$$\mathbb{E}_{\mathbb{Q}}(X - \rho_0^v(X)v|\mathcal{F}_0) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0) - \rho_0^v(X)\,\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0) \le 0,$$

which leads us to conclude that $\xi(X) \le \rho_0^v(X)$. Now define $\tilde{X} = X - \rho_0^v(X)v$ and suppose that there exists some $\varepsilon > 0$ such that $\mathbb{P}(F^\varepsilon) > 0$ where

$$F^\varepsilon \stackrel{def}{=} \{\xi(\tilde{X}) \le -\varepsilon\} \in \mathcal{F}_0.$$

Then $(\tilde{X} + \varepsilon v)1_{F^\varepsilon} \in \mathcal{A}_0$ and consequently $\rho_0^v(\tilde{X}) \le -\varepsilon$ on F^ε. This contradicts the fact that $\rho_0^v(\tilde{X}) = 0$ a.s. We conclude then that $\xi(\tilde{X}) = \rho_0^v(\tilde{X}) = 0$. By the $\mathcal{F}_0$-translation invariance property with respect to v of ξ we conclude that $\xi(X) = \rho_0^v(X)$. □

Later we will need the following lemma.

Lemma A.1. *Let $u, v \in \mathcal{N}_0$ and $X \in \mathcal{L}^\infty$, then $(\rho_0^v(X) = 0) = (\rho_0^u(X) = 0)$ a.s.*

Proof. Fix $X \in \mathcal{L}^\infty$, define $F = (\rho_0^v(X) = 0)$ and $X^F = X1_F$. Remark that $F \in \mathcal{F}_0$ and

$$\rho_0^v(X^F) = \rho_0^v(X)1_F = 0.$$

It follows that $X^F \in \mathcal{A}_0$. Suppose that there exists some $\varepsilon > 0$ such that $\mathbb{P}(G^\varepsilon) > 0$ with $G^\varepsilon = F \cap (\rho_0^u(X) \le -\varepsilon)$. We deduce that $(X + \varepsilon u)1_{G^\varepsilon} \in \mathcal{A}_0$ and thus by subadditivity $0 = \rho_0^v(X) \le -\varepsilon\rho_0^v(u) < 0$ a.s on $G^\varepsilon \subset F$. We obtain a contradiction. Thus, for all $\varepsilon > 0$, we have $\mathbb{P}(G^\varepsilon) = 0$. Consequently, since, by part (i) of Lemma 4.4, $X^F \in \mathcal{A}_0 = \mathcal{A}^u$:

$$\mathbb{P}(F) = \mathbb{P}(F \cap (\rho_0^u(X) \ge 0)) = \mathbb{P}(F \cap (\rho_0^u(X) = 0)).$$

We deduce that $F \subset (\rho_0^u(X) = 0)$. By symmetry the result follows. □

In the following lemma, we give some properties of the mapping $v \mapsto \rho_0^v(X)$ for a fixed $X \in \mathcal{L}^\infty$.

Lemma A.2. *For all $X \in \mathcal{L}^\infty$ and $v, u, w, w^1, ..., w^n \in \mathcal{N}_0$ we have:*

(1) $\rho_0^v(u) \le \rho_0^v(w)\,\rho_0^w(u)$.
(2) $\rho_0^v(X) \le \rho_0^v\left(\sum_{k=1}^n \rho_0^{w^k}(X_k)w^k\right)$ *whenever $X = X_1 + ... + X_n$ with $X_i \in \mathcal{L}^\infty$ for each $i = 1, \ldots, n$.*
(3) *With the convention $\dfrac{0}{0} = 0$, we have*

$$\rho_0^{v+w}(X) \le \frac{\rho_0^v(X)\,\rho_0^w(X)}{\rho_0^v(X) + \rho_0^w(X)}.$$

(4) *For* $\lambda \in \mathcal{L}_+^\infty(\mathcal{F}_0)$ *such that* $\lambda v \in \mathcal{N}_0$*, we have:*

$$\rho_0^{\lambda v}(X) = \frac{1}{\lambda}\,\rho_0^v(X).$$

(5) $uv \in \mathcal{N}_0$ *iff ess-inf*$_{\mathbb{Q}\in\mathcal{Q}^v}\mathbb{E}_{\mathbb{Q}}(u|\mathcal{F}_0) > 0$ *a.s and then*

$$\rho_0^{uv}(X) = \left(\rho_0^{(v)}\right)^u (X/v).$$

Proof. To prove the assertion (1), we remark that

$$u - \rho_0^v(w)\,\rho_0^w(u)v = u - \rho_0^w(u)w + \rho_0^w(u)(w - \rho_0^v(w)v) \in \mathcal{A}_0.$$

To prove (2), we use formula (ii) in Lemma 4.4 to see that for all $\mathbb{Q} \in \mathcal{Q}$,

$$\rho_0^v\left(\sum_{k=1}^n \rho_0^{w^k}(X_k)w^k\right) \geq \frac{1}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)}\mathbb{E}_{\mathbb{Q}}\left(\sum_{k=1}^n \frac{\mathbb{E}_{\mathbb{Q}}(X_k|\mathcal{F}_0)\,w^k}{\mathbb{E}_{\mathbb{Q}}(w^k|\mathcal{F}_0)}|\mathcal{F}_0\right),$$

which means that

$$\rho_0^v\left(\sum_{k=1}^n \rho_0^{w^k}(X_k)w^k\right) \geq \frac{1}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)}\sum_{k=1}^n \mathbb{E}_{\mathbb{Q}}(X_k|\mathcal{F}_0).$$

So, since $X = X_1 + \ldots + X_n$, we have for all $\mathbb{Q} \in \mathcal{Q}$:

$$\rho_0^v\left(\sum_{k=1}^n \rho_0^{w^k}(X_k)w^k\right) \geq \frac{\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0)}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)},$$

and consequently

$$\rho_0^v\left(\sum_{k=1}^n \rho_0^{w^k}(X_k)w^k\right) \geq \rho_0^v(X).$$

To prove (3), we note that $\rho_0^v(X)$ and $\rho_0^w(X)$ have the same sign and then thanks to Lemma A.1, $\rho_0^v(X) + \rho_0^w(X) = 0$ if and only if $\rho_0(X) = 0$. Now suppose that $\rho_0(X) \neq 0$ and remark that the $\mathcal{F}_0$-measurable random variable

$$\alpha(X) \overset{def}{=} \frac{\rho_0^v(X)}{\rho_0^v(X) + \rho_0^w(X)},$$

has values a.s in the interval $[0,1]$, so

$$X - \frac{\rho_0^v(X)\rho_0^w(X)}{\rho_0^v(X) + \rho_0^w(X)}(v+w) = \alpha(X)(X - \rho_0^w(X)w) + (1-\alpha(X))(X - \rho_0^v(X)v) \in \mathcal{A}_0.$$

Assertion (4) is an immediate consequence of Lemma 4.4. For assertion (5) we have for $\mathbb{Q} \in \mathcal{Q}$:

$$\frac{\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_0)}{\mathbb{E}_{\mathbb{Q}}(uv|\mathcal{F}_0)} = \frac{\mathbb{E}_{\mathbb{R}}(X/v|\mathcal{F}_0)}{\mathbb{E}_{\mathbb{R}}(u|\mathcal{F}_0)},$$

with

$$\frac{d\mathbb{R}}{d\mathbb{Q}} = \frac{v}{\mathbb{E}_{\mathbb{Q}}(v|\mathcal{F}_0)},$$

and then $\mathbb{R} \in \mathcal{Q}^v$. We deduce that

$$\rho_0^{uv}(X) \leq \left(\rho_0^{(v)}\right)^u (X/v).$$

The reverse inequality is proved in the same way. □

Remark that at time zero, trading between two different numéraires may incur additional costs. Nevertheless the set $\mathcal{N}_0$ can be partitioned into equivalence classes so that trade is frictionless within each one of them.

Definition A.3. *Two assets $v, w \in \mathcal{N}_0$ are said to be $\mathcal{F}_0$-equivalent (or equivalent at time zero) (and we write $v \sim w$) if for all $X \in \mathcal{L}^\infty$, we have $\rho_0^v(X) = \rho_0^v(w)\,\rho_0^w(X)$.*

Some equivalent conditions for $\mathcal{F}_0$-equivalence are given below.

Lemma A.4. *Let $v, w \in \mathcal{N}_0$, then the following are equivalent:*

(1) $\rho_0^v(w)\rho_0^w(v) = 1$.
(2) $w - \rho_0^v(w)v \in lin(\mathcal{A}_0)$ *where* $lin(\mathcal{A}_0) = \mathcal{A}_0 \cap (-\mathcal{A}_0)$ *is the lineality subspace of* $\mathcal{A}_0$.
(3) $\rho_0^v(-w) = -\rho_0^v(w)$.
(4) *The two assets v and w are $\mathcal{F}_0$-equivalent.*
(5) *The two assets v and $v + w$ are $\mathcal{F}_0$-equivalent.*
(6) *For all $X \in \mathcal{L}^\infty$,*
$$\rho_0^{v+w}(X) = \frac{\rho_0^v(X)\,\rho_0^w(X)}{\rho_0^v(X) + \rho_0^w(X)},$$
with the convention $\dfrac{0}{0} = 0$.

Proof. (1) $\Rightarrow$ (2) We have $v - \rho_0^w(v)w \in \mathcal{A}_0$ by definition, so
$$-w + \rho_0^v(w)v = \rho_0^v(w)(v - \rho_0^w(v)w) \in \mathcal{A}_0.$$
Since $w - \rho_0^v(w)v \in \mathcal{A}_0$, it follows that $w - \rho_0^v(w)v \in lin(\mathcal{A}_0)$.

(2) $\Rightarrow$ (1) Conversely we have $w - \rho_0^v(w)v \in lin(\mathcal{A}_0)$ which means that $v - \frac{1}{\rho_0^v(w)}w \in \mathcal{A}_0$ and hence $\rho_0^v(w)\rho_0^w(v) \leq 1$. From assertion (1) in Lemma A.2 we deduce that $\rho_0^v(w)\rho_0^w(v) = 1$.

(2) $\Rightarrow$ (3) We have $\rho_0^v(-w + \rho_0^v(w)v) = 0$ since $-w + \rho_0^v(w)v \in lin(\mathcal{A}_0)$ and then $0 = \rho_0^v(-w + \rho_0^v(w)v) = \rho_0^v(-w) + \rho_0^v(w)$.

(3) $\Rightarrow$ (2) Conversely we have, by definition, $-w - \rho_0^v(-w)v \in \mathcal{A}_0$, so
$$-w + \rho_0^v(w)v = -w - \rho_0^v(-w)v \in \mathcal{A}_0.$$

(1) $\Leftrightarrow$ (4) to prove the forward implication, note that
$$X - \rho_0^v(w)\rho_0^w(X)v = (X - \rho_0^w(X)w) + \rho_0^w(X)(w - \rho_0^v(w)v) \in \mathcal{A}_0,$$
since $X - \rho_0^w(X)w \in \mathcal{A}_0$ and $w - \rho_0^v(w)v \in lin(\mathcal{A}_0)$. Hence $\rho_0^v(X) \leq \rho_0^v(w)\rho_0^w(X)$. Swapping the roles of v and w we obtain also that $\rho_0^w(X) \leq \rho_0^w(v)\rho_0^v(X)$ and then
$$\rho_0^v(X) \leq \rho_0^v(w)\rho_0^w(X) \leq \rho_0^v(w)\rho_0^w(v)\rho_0^v(X) = \rho_0^v(X).$$
The converse is trivial.

(3) $\Rightarrow$ (5) we have

$$\rho_0^v(-(v+w)) = \rho_0^v(-v-w) = \rho_0^v(-w) - 1 = -\rho_0^v(w) - 1 = -\rho_0^v(v+w).$$

(5) $\Rightarrow$ (3) Conversely we have $\rho_0^v(-w) = \rho_0^v(-v-w) + 1 = -\rho_0^v(v+w) + 1 = -\rho_0^v(w)$.

(1) $\Rightarrow$ (6) Assume (1) holds, then, as we have previously proved, (4) and (5) hold. Then

$$\rho_0^{v+w}(X) = \rho_0^{v+w}(v)\rho_0^v(X) = \frac{\rho_0^v(X)}{\rho_0^v(v+w)} = \frac{\rho_0^v(X)}{1+\rho_0^v(w)} = \frac{\rho_0^v(X)\rho_0^w(X)}{\rho_0^v(X)+\rho_0^w(X)}.$$

(6) $\Rightarrow$ (1) Conversely, we take $X = v+w$ and deduce assertion (1). □

Remark A.5. *Assertion* (2) *in Lemma A.4 is equivalent to*

(2′) : $w - \alpha v \in lin(\mathcal{A}_0)$ *for some* $\alpha \in \mathcal{L}_+^\infty(\mathcal{F}_0)$ *such that* $1/\alpha \in \mathcal{L}^\infty$.

Indeed for the direct implication we can take $\alpha = \rho_0^v(w)$. *For the converse, we have*

$$\rho_0^w(v) \le \frac{1}{\alpha} \le \frac{1}{\rho_0^v(w)},$$

and then $\rho_0^v(w)\rho_0^w(v) = 1$. *This implies* (2).

Corollary A.6. *The binary relation* $\sim$ *defined on* $\mathcal{N}_0$ *in Definition A.3, is an equivalence relation. For all* $v \in \mathcal{N}_0$, *the subset* $[v] \cup \{0\}$ *is a convex cone where* $[v] \overset{def}{=} \{w \in \mathcal{N}_0 : w \sim v\}$ *denotes the equivalence class of* v. *Moreover*

$$\overline{[v]} = \left\{x \in \overline{\mathcal{N}_0} : \exists \lambda \in \mathcal{L}_+^\infty(\mathcal{F}_0) \text{ such that } x - \lambda v \in lin(\mathcal{A}_0)\right\} \overset{def}{=} \mathcal{E}(v),$$

where the closure is taken with respect to the weak topology in* $\mathcal{L}^\infty$.

Proof. From Remark A.5 we deduce that $\sim$ is an equivalence relation and from the equivalence of properties (4) and (5) in Lemma A.4, we deduce easily that for each $v \in \mathcal{N}_0$, the subset $[v] \cup \{0\}$ is a convex cone. Now we prove that $\overline{[v]} \subset \mathcal{E}(v)$. Let w^n be a sequence in $[v]$ which converges weakly* to w in $\mathcal{L}^\infty$. The sequence

$$\alpha^n \overset{def}{=} \rho_0^v(w^n) = \frac{\mathbb{E}_\mathbb{Q}(w^n|\mathcal{F}_0)}{\mathbb{E}_\mathbb{Q}(v|\mathcal{F}_0)},$$

for a fixed $\mathbb{Q} \in \mathcal{Q}^e$, converges weakly* to

$$\alpha \overset{def}{=} \frac{\mathbb{E}_\mathbb{Q}(w|\mathcal{F}_0)}{\mathbb{E}_\mathbb{Q}(v|\mathcal{F}_0)}.$$

By working with $w^n + \varepsilon v$ and then taking the limit in ε, we suppose, without loss of generality, that $\alpha \ge \varepsilon > 0$. So the sequence $1/\alpha^n$ is bounded, thus there exists an $\mathcal{F}_0$-measurable integer-valued sequence τ_n such that the sequence α^{τ_n} converges a.s to some α and then $\alpha \in \mathcal{L}^\infty$ and $1/\alpha \in \mathcal{L}^\infty$. We know that for all $\mathbb{Q} \in \mathcal{Q}$ and $f \in \mathcal{L}^1(\mathcal{F}_0)$ we have

$$\text{(A.1)} \quad \mathbb{E}_\mathbb{Q}(f\, w^{\tau_n}) = \sum_k \mathbb{E}_\mathbb{Q}(f\, 1_{(\tau_n=k)} w^k) = \sum_k \mathbb{E}_\mathbb{Q}(f\, 1_{(\tau_n=k)} \alpha^k\, v) = \mathbb{E}_\mathbb{Q}(f\, \alpha^{\tau_n}\, v),$$

since for all k, $w^k \sim v$ and $f\, 1_{(\tau_n=k)} \in \mathcal{L}^1(\mathcal{F}_0)$. So the left hand side of (A.1) converges to $\mathbb{E}_{\mathbb{Q}}(f\, w)$ and the right hand side converges to $\mathbb{E}_{\mathbb{Q}}(f\, \alpha\, v)$. Hence $\mathbb{E}_{\mathbb{Q}}(w - \alpha\, v|\mathcal{F}_0) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ and so $w - \alpha\, v \in lin(\mathcal{A}_0)$.

Conversely, let $w \in \mathcal{E}(v)$ which means that $w = \alpha\, v + z$ with $z \in lin(\mathcal{A}_0)$. Define

$$w^n \stackrel{def}{=} w + \frac{1}{n} v = \left(\alpha + \frac{1}{n}\right) v + z \in \mathcal{N}_0,$$

then $w^n \in [v]$ from Remark A.5 and w^n converges weakly* to w in $\mathcal{L}^\infty$. □

Appendix B. proofs of results in section 6

Proof of Lemma 6.11: The inclusion $D \subset \cap_{t=0}^{T-\eta} M_t^\eta(D)$ is trivial.

Now we let $Y \in \cap_{t=0}^{T-\eta} M_t^\eta(D)$ and seek to prove that $Y \in D$. So, for all $t \in \{0, \ldots, T-\eta\}$, there exists some $Z^t \in D$, $\beta_t \in \mathcal{L}^0_+(\mathcal{F}_t)$ with $\beta_t Z^t \in \mathcal{L}^1$ such that $Z_{t+\eta} = \beta_t\, Z^t_{t+\eta}$. Define $\xi^{T-\eta} = Z^{T-\eta}$ and for $t \in \{0, \ldots, T-\eta-1\}$:

$$\xi^t = 1_{F_t} \kappa_t\, \xi^{t+1} + 1_{F_t^c}\, Z^t,$$

where $F_t = (\beta_t > 0)$ and $\kappa_t = \beta_{t+1}/\beta_t$. Remark that

$$Z_{t+1+\eta} = \beta_{t+1}\, Z^{t+1}_{t+1+\eta},$$

and

$$Z_{t+\eta} = \beta_t\, Z^t_{t+\eta}.$$

Thus

$$\mathbb{E}[\beta_{t+1} Z^{t+1} | \mathcal{F}_{t+\eta}] = \mathbb{E}[Z_{t+1+\eta}|\mathcal{F}_{t+\eta}] = Z_{t+\eta} = \beta_t\, Z^t_{t+\eta},$$

which leads us to deduce that

$$\mathbb{E}((1_{F_t} \kappa_t\, Z^{t+1} + 1_{F_t^c} Z^t)|\mathcal{F}_{t+\eta}) = Z^t_{t+\eta}.$$

Remark also that, since $D \subset \mathcal{L}^1_+(\mathcal{F}; \mathbb{R}^d)$,

$$Z = \beta_0\, \kappa_0 \times \ldots \times \kappa_{T-\eta-1}\, Z^{T-\eta}.$$

We deduce that $Z = \beta_0\, \xi^0$.

Now we prove by backwards induction on t that $Z^t_{t+\eta} = \xi^t_{t+\eta}$ and $\xi^t \in D$. For $t = T-\eta$, we have $\xi^{T-\eta} = Z^{T-\eta} \in D$ by definition. Suppose that for all $s = T-\eta, \ldots, t+1$, we have $Z^s_{s+\eta} = \xi^s_{s+\eta}$ and $\xi^s \in D$ and show that $Z^t_{t+\eta} = \xi^t_{t+\eta}$ and $\xi^t \in D$. First $\xi^t = 1_{F_t} \kappa_t\, \xi^{t+1} + 1_{F_t^c}\, Z^t$ so we get

$$\xi^t_{t+\eta} = \mathbb{E}((1_{F_t} \kappa_t\, \xi^{t+1} + 1_{F_t^c}\, Z^t)|\mathcal{F}_{t+\eta}) = \mathbb{E}((1_{F_t} \kappa_t\, Z^{t+1} + 1_{F_t^c}\, Z^t)|\mathcal{F}_{t+\eta}) = Z^t_{t+\eta}.$$

Then

$$Z^t_{t+\eta} = \mathbb{E}((1_{F_t} \kappa_t\, \xi^{t+1} + 1_{F_t^c}\, Z^t)|\mathcal{F}_{t+\eta}),$$

with $Z^t, \xi^{t+1} \in D$. By the η-stability of the subset D we deduce that $\xi^t \in D$ and consequently $Z = \beta_0\, \xi^0 \in D$ since $\xi^0 \in D$ and β_0 is a positive scalar. □

Proof of Lemma 6.12:

(1) We remark that $[D]$ is a closed convex cone in $\mathcal{L}^1$, containing D. Now we prove that $[D]$ is η-stable. Fix $t \in \{0, 1, \ldots, T-\eta\}$ and suppose that $Z^1, \ldots, Z^k \in [D]$ are such that there exists $Z \in [D]$, a partition $F_t^1, \ldots, F_t^k \in \mathcal{F}_t$ and $\alpha^1, \ldots, \alpha^k \in \mathcal{L}^0(\mathcal{F}_{t-\eta+1})$ with each $\alpha^i Z^i \in \mathcal{L}^1$ and

$$\mathbb{E}(Z|\mathcal{F}_t) = \mathbb{E}(\alpha^i Z^i|\mathcal{F}_t),$$

on F_t^i for each $i = 1, \ldots, k$. We want to prove that

$$Y \stackrel{def}{=} \sum_{i=1}^{k} 1_{F_t^i} \alpha^i Z^i \in [D].$$

Now for $s \geq t - \eta + 1$ we have

$$Y_{s+\eta} = \sum_{i=1}^{k} 1_{F_t^i} \alpha^i Z_{s+\eta}^i,$$

with $Z_{s+\eta}^i = W_{s+\eta}^i$ for some $W^i \in D_{(s)}$. Therefore

$$Y_{s+\eta} = \left(\sum_{i=1}^{k} 1_{F_t^i} \alpha^i W^i \right)_{s+\eta},$$

with $\sum_{i=1}^{k} 1_{F_t^i} \alpha^i W^i \in D_{(s)}$.

Now for $s \leq t - \eta$ we have $Y_{s+\eta} = (Y_t)_{s+\eta}$ and

$$Y_t = \sum_{i=1}^{k} 1_{F_t^i} \mathbb{E}(\alpha^i Z^i|\mathcal{F}_t) = Z_t,$$

and then $Y_{s+\eta} = Z_{s+\eta} = W_{s+\eta}$ for some $W \in D_{(s)}$. Therefore $Y \in [D]$.

(2) We prove that $D = [D] \Leftrightarrow D$ is an η-stable closed convex cone in $\mathcal{L}^1$. For the reverse implication, thanks to Lemma 6.11 we have:

$$D = D^{**} = \cap_{t=0}^{T-\eta} \overline{conv}(M_t^\eta(D)) = \cap_{t=0}^{T-\eta} \overline{R_t^\eta},$$

with $D \subset [D] \subset \cap_{t=0}^{T-\eta} \overline{R_t^\eta}$. Then $D = [D]$. The direct implication is trivial from the first assertion.

(3) To prove that $[D]$ is the smallest η-stable closed convex cone in $\mathcal{L}^1$ which contains D, simply let D' be an η-stable closed convex cone in $\mathcal{L}^1$, containing D. Then $[D] \subset [D'] = D'$.

□

Proof of Theorem 6.13: Remark that $\mathcal{B}^*$ is a closed convex cone in $\mathcal{L}^1$. We claim that if we can prove that for all $t = 0, \ldots, T-\eta$ we have $K_t^\eta(\mathcal{B}) = (M_t^\eta(\mathcal{B}^*))^*$, the result will follow thanks to Lemmas 6.11 and 6.12. To see this, note first that Lemma 6.11 tells us that $\mathcal{B}^*$ being η-stable implies that $\mathcal{B}^* = \cap M_t^\eta$. Thus if $K_t^\eta = (M_t^\eta)^*$ then $\mathcal{B} = \mathcal{B}^{**} = (\cap M_t^\eta)^* = \overline{\oplus (M_t^\eta)^*} = \overline{\oplus (K_t^\eta)}$, establishing the reverse implication. Conversely, $(M_t^\eta)^* = (\overline{conv}(M_t^\eta))^* = (R_t^\eta)^*$, so, if $\mathcal{B} = \overline{\oplus (K_t^\eta)} = \overline{\oplus (M_t^\eta)^*}$, then $\mathcal{B} = \overline{\oplus (R_t^\eta)^*}$ so $\mathcal{B}^* = \cap R_t^\eta = [\mathcal{B}^*]$ and then, by Lemma 6.12, $\mathcal{B}^*$ is η-stable.

First we prove that $M_t^\eta(\mathcal{B}^*) \subset (K_t^\eta(\mathcal{B}))^*$. Let $Z \in M_t^\eta(\mathcal{B}^*)$, then there exists some $Z' \in \mathcal{B}^*$ and $\alpha \in \mathcal{L}^0_+(\mathcal{F}_t)$ with $\alpha\, Z' \in \mathcal{L}^1$ such that $Z_{t+\eta} = \alpha\, Z'_{t+\eta}$. Take $X \in K_t^\eta(\mathcal{B})$, then

$$\mathbb{E}(Z.X) = \mathbb{E}(Z_{t+\eta}.X) = \mathbb{E}(\alpha_t\, Z'_{t+\eta}.X),$$

since $X \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^d)$. We then obtain:

$$\mathbb{E}(Z.X) = \mathbb{E}(Z'_{t+\eta}.\alpha_t\, X) = \lim_{n\to\infty} \mathbb{E}(Z'.\alpha_t\, 1_{(\alpha_t \le n)}\, X) \le 0,$$

since $\alpha_t\, 1_{(\alpha_t\le n)}\, X \in \mathcal{B}$ and $Z' \in \mathcal{B}^*$.

Now we prove that $(M_t^\eta(\mathcal{B}^*))^* \subset K_t^\eta(\mathcal{B})$. We remark that $\mathcal{B}^* \subset M_t^\eta(\mathcal{B}^*)$ and also that $\mathcal{L}^\infty_+(\mathcal{F}_t)\, M_t^\eta(\mathcal{B}^*) \subset M_t^\eta(\mathcal{B}^*)$, so $(M_t^\eta(\mathcal{B}^*))^* \subset \mathcal{B}_t$. Let $X \in (M_t^\eta(\mathcal{B}^*))^*$, we want to prove that $X \in \mathcal{L}^\infty(\mathcal{F}_{t+\eta}, \mathbb{R}^d)$. Let $Z \in \mathcal{L}^1(\mathcal{F}, \mathbb{R}^d)$, we remark that $Z - Z_{t+\eta} \in M_t^\eta(\mathcal{B}^*)$ and consequently $\mathbb{E}((Z - Z_{t+\eta}).X) \le 0$. We deduce then that $\mathbb{E}((X - X_{t+\eta}).Z) = \mathbb{E}((Z - Z_{t+\eta}).X) \le 0$ for all $Z \in \mathcal{L}^1$. Therefore $X = X_{t+\eta}$.

□

Proof of Lemma 6.15: (1) ⇔ (2). For $Z \in \mathcal{B}^*$ and $f_t^+ \in \mathcal{L}^\infty_+(\mathcal{F}_t)$ we have:

$$\mathbb{E} f_t^+(Z_t.X) = \mathbb{E} Z.(f_t^+\, X) \le 0,$$

since $f_t^+\, X \in \mathcal{B}$. Then $Z_t.X \le 0$ a.s. Conversely let $f_t^+ \in \mathcal{L}^\infty_+(\mathcal{F}_t)$ we want to prove that $f_t^+\, X \in \mathcal{B}$. Let $Z \in \mathcal{B}^*$ then

$$\mathbb{E} Z.(f_t^+\, X) = \mathbb{E} Z_t.(f_t^+\, X) = \mathbb{E} f_t^+(Z_t.X) \le 0.$$

Therefore $f_t^+\, X \in \mathcal{B}$.

(2) ⇔ (3), for all $W \in \mathcal{L}^1$ such that $W_t = Z_t$ with $Z \in \mathcal{B}^*$ and $f_t^+ \in \mathcal{L}^\infty_+(\mathcal{F}_t)$, we have

$$\mathbb{E}(f_t^+\,(W.X)) = \mathbb{E}(f_t^+\, W_t.X) = \mathbb{E}(f_t^+\, Z_t.X) \le 0.$$

Conversely we prove first that $X \in \mathcal{L}^\infty(\mathcal{F}_t)$. Remark that for every $W \in \mathcal{L}^1$ we have $\mathbb{E}(W - W_t|\, \mathcal{F}_t) = 0$ with $0 \in \mathcal{B}^*$, then $\mathbb{E}[(W - W_t).X] \le 0$. Consequently for every $W \in \mathcal{L}^1$ we get

$$\mathbb{E} W.(X - X_t) = \mathbb{E}(W - W_t).X \le 0,$$

and so $\mathbb{E} W.(X - X_t) = 0$ for every $W \in \mathcal{L}^1$. Then $X = X_t \overset{def}{=} \mathbb{E}(X|\, \mathcal{F}_t)$. Let $Z \in \mathcal{B}^*$, then

$$Z_t.X = \mathbb{E}(Z.X|\, \mathcal{F}_t) \le 0.$$

□

Appendix C. Further results on η-decomposability

Proposition C.1. *We have:*

(1) *For all* $t \in \{0, \ldots, T\}$, $(\mathcal{B}_t)^* = \overline{(\mathcal{B}^*)_{t,T}} \overset{def}{=} (\mathcal{B}^*)_{(t)}$, *where*

$$(\mathcal{B}^*)_{t,T} = \{\alpha Z;\ \alpha \in \mathcal{L}^\infty(\mathcal{F}_t),\ Z \in \mathcal{B}^*\}.$$

(2) *Define* $\mathcal{B}^\eta \stackrel{def}{=} \overline{\oplus_{t=0}^{T-\eta} K_t^\eta(\mathcal{B})}$; *then* $\mathcal{B}^\eta$ *is the largest* η*-decomposable closed convex cone in* $\mathcal{B}$.

Proof. To prove (1), we fix $t \in \{0, \dots, T\}, Z \in (\mathcal{B}^*)_{t,T}$ and $X \in \mathcal{B}_t$, then there exists some $Z' \in \mathcal{B}^*$ and $\alpha \in \mathcal{L}_+^\infty(\mathcal{F}_t)$ such that $Z = \alpha\, Z'$ and

$$\mathbb{E}Z.X = \mathbb{E}\alpha\, Z'.X = \mathbb{E}Z'.(\alpha\, X) \le 0.$$

We deduce that $(\mathcal{B}^*)_{t,T} \subset (\mathcal{B}_t)^*$.

Now let $X \in \mathcal{L}^\infty$ be such that $\mathbb{E}Z.X \le 0$ for all $Z \in (\mathcal{B}^*)_{t,T}$. Then, in particular for all $Z \in \mathcal{B}^*$ and $\alpha \in \mathcal{L}_+^\infty(\mathcal{F}_t)$ we have

$$\mathbb{E}Z.(\alpha\, X) = \mathbb{E}(\alpha\, Z).X \le 0,$$

since $\alpha\, Z \in (\mathcal{B}^*)_{t,T}$, which implies that $\alpha X \in \mathcal{B}$ for all $\alpha \in \mathcal{L}_+^\infty(\mathcal{F}_t)$ and then $X \in \mathcal{B}_t$.

To prove (2), remark that for all $t \in \{0, \dots, T-\eta\}$, we have $K_t^\eta(\mathcal{B}) \subset \mathcal{B}^\eta \subset \mathcal{B}$, then $K_t^\eta(\mathcal{B}^\eta) = K_t^\eta(\mathcal{B})$ and so $\mathcal{B}^\eta = \overline{\oplus_{t=0}^{T-\eta} K_t^\eta(\mathcal{B}^\eta)}$. Now let M be an η-decomposable closed convex cone in $\mathcal{B}$, then for all $t \in \{0, \dots, T-\eta\}$ we have $K_t^\eta(M) \subset K_t^\eta(\mathcal{B}) \subset \mathcal{B}^\eta$. We deduce that $M = \overline{\oplus_{t=0}^{T-\eta} K_t^\eta(M)} \subset \mathcal{B}^\eta$. □

Corollary C.2. *We have*

(1) *If* $\mathcal{B}$ *is* η*-decomposable, then each* $\mathcal{B}_t$ *is* η*-decomposable.*
(2) *For fixed* $t \in \{0, \dots, T\}$, *we have* $\mathcal{B}_t^\eta \stackrel{def}{=} (\mathcal{B}^\eta)_t = (\mathcal{B}_t)^\eta$.
(3) *For fixed* $t \in \{0, \dots, T\}$, $\mathcal{B}_t$ *is* η*-decomposable if and only if* $\mathcal{B}_t = \mathcal{B}_t^\eta$.

Proof. Suppose that $\mathcal{B}$ is η-decomposable, then $\mathcal{B}^*$ is η-stable. Hence for all $t \in \{0, \dots, T\}$, $(\mathcal{B}^*)_{t,T}$ is η-stable: indeed for $s \in \{0, \dots, T-\eta\}$, consider $Z, Z^1, \dots, Z^k \in (\mathcal{B}^*)_{t,T}$ and a partition $F_s^1, \dots, F_s^k \in \mathcal{F}_s$ and $\alpha^1, \dots, \alpha^k \in \mathcal{L}_+^0(\mathcal{F}_{s-\eta+1})$ with $\alpha^i Z^i \in \mathcal{L}^1$ such that $Y \stackrel{def}{=} \sum_{i=1}^k 1_{F_t^i}\alpha^i Z^i$ satisfies:

$$Z_s = \mathbb{E}(Y|\mathcal{F}_s).$$

For $s \le t+\eta-1$, we have $Y \in (\mathcal{B}^*)_{t,T}$ and for $s \ge t-\eta$, by definition of $(\mathcal{B}^*)_{t,T}$, there exists $\beta, \beta^1, \dots, \beta^k \in \mathcal{L}_+^\infty(\mathcal{F}_t)$ and $W, W^1, \dots, W^k \in \mathcal{B}^*$ such that $Z = \beta W$ and $Z^i = \beta^i W^i$. Then

$$\beta W_s = \sum_{i=1}^k 1_{F_t^i}\beta^i \mathbb{E}(\alpha^i W^i|\mathcal{F}_s),$$

which means that

$$W_s = \mathbb{E}\left(1_G \frac{\sum_{i=1}^k 1_{F_t^i}\beta^i\alpha^i W^i}{\beta} + 1_{G^c}W|\mathcal{F}_s\right),$$

where $G = (\beta > 0)$. Since $\mathcal{B}^*$ is η-stable, it follows that

$$Y' \stackrel{def}{=} 1_G Y + 1_{G^c}W \in \mathcal{B}^*,$$

and consequently $Y = \beta Y' \in \mathcal{B}_{t,T}^*$.

By Theorem 6.13, $\mathcal{B}_t$ is η-decomposable. Now to prove (2), remark that $\mathcal{B}_t^\eta$ is η-decomposable (this follows by assertion (1) since $\mathcal{B}^\eta$ is η-decomposable) and for all $s \in \{t+1, \ldots, T-\eta\}$ we have

$$K_s^\eta(\mathcal{B}_t^\eta) = K_s^\eta(\mathcal{B}^\eta) = K_s^\eta(\mathcal{B}) = K_s^\eta(\mathcal{B}_t),$$

and for $s \leq t$, we have $K_s^\eta(\mathcal{B}_t) \subset K_t^\eta(\mathcal{B}_t)$. Hence

$$\mathcal{B}_t^\eta = \overline{\oplus_{s=0}^{T-\eta} K_s^\eta(\mathcal{B}_t^\eta)} = \overline{\oplus_{s=t}^{T-\eta} K_s^\eta(\mathcal{B}_t)} = (\mathcal{B}_t)^\eta.$$

To prove (3), assume that $\mathcal{B}_t$ is η-decomposable, then $\mathcal{B}_t = (\mathcal{B}_t)^\eta = \mathcal{B}_t^\eta$. The converse is obvious. □

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