IMPLIED DISTRIBUTIONS IN MULTIPLE CHANGE POINT PROBLEMS

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ABSTRACT. A method for efficiently calculating marginal, conditional and joint distributions for change points defined by general finite state Hidden Markov Models is proposed. The distributions are not subject to any approximation or sampling error once parameters of the model have been estimated. It is shown that, in contrast to sampling methods, very little computation is needed. The method provides probabilities associated with change points within an interval, as well as at specific points.

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1. INTRODUCTION

This paper investigates some exact change point distributions when fitting general finite state Hidden Markov models (HMMs), including Markov switching models. Change point problems are important in various applications, including economics (Hamilton 1989; Chib 1998; Sims and Zha 2006) and genetics (Durbin, Eddy, Krogh, and Mitchison 1998; Eddy 2004; Fearnhead and Liu 2007). In many instances change point problems are framed as HMMs (Chib 1998; Frühwirth-Schnatter 2006; Fearnhead and Liu 2007), however, to date, the characterisation of the change point distributions implied by these models has been performed using sampling methods (Albert and Chib 1993; Cappé, Moulines, and Rydén 2005), exact posterior sampling (Fearnhead 2006; Fearnhead and Liu 2007), or the distributions are ignored and deterministic algorithms such as the Viterbi algorithm (Viterbi 1967) or posterior (local) decoding (Juang and Rabiner 1991) are used to determine the implied change points. The distributions in this paper are exact in that, conditioned on the model and parameters, they completely characterise the probability distribution function without requiring any asymptotic or other approximation or being subject to any sampling error. It will be shown that these distributions can be calculated in many cases using a small number of calculations compared to those needed to yield an approximate distribution through sampling.

The method can be used to determine probabilities of whether a change in regime has occurred at any particular time point. This will be evaluated through a concept called the change point probability (CPP), which is a function of the marginal probabilities of particular time ordered change points occurring at certain times. The marginal probabilities are determined after finding the joint and conditional distributions of multiple change points. Using the marginal distributions allows a probabilistic quantification of the relationship between changes in the behaviour described by the model and real-life events occurring at specific times.

A model might be deemed to capture the influence of an event causing a change in the data when the probability distribution of a change point around the time point of the event is peaked.

In contrast, a model for which the probability of a change in regime is more uniform indicates that the regime specified by the model is not particularly affected at that or any other particular point. To illustrate this point, US Gross National Product (GNP) regime switches will be examined in relation to the CPP of recession starts and ends as determined by the National Bureau of Economic Research (NBER). The NBER can be seen as providing external estimates of change points based on overall economic data. Comparing change points determined by the NBER and those determined by maximisation of local posterior state probabilities (Hamilton 1989) leads to the surprising result that regime switches determined from the latter method may not be a useful metric. In contrast, the CPP gives the exact probability of a change occurring at any time point or interval, given the model.

HMMs are widely used in statistics and engineering; see MacDonald and Zucchini (1997) and Cappé, Moulines, and Rydén (2005) for good overviews on the current state of the art both in theory and applications of them. To locate change points (or equivalently to perform data segmentation) HMMs are generally trained on data and then applied to test data (Rabiner 1989; Durbin et al. 1998). The methodology of this paper is appealing in that it allows complete quantification of uncertainty in test data analysis. This methodology is generic in that it depends only on the Markovian nature of regime switches and the ability to generate posterior probabilities, and not upon the structure of any particular HMM model.

The structure of the paper is as follows. Section 2 derives the methods to find change point distributions from data via HMMs and the use of waiting time distributions. Also in that section, the joint and marginal distributions of a set of change points are derived and the concept of CPP is defined. Section 3 contains applications of the methodology to the GNP data given by Hamilton (1989). Section 4 contains a few concluding remarks.

In the appendices, a basic smoothing algorithm for Markov switching models will be given, completing an algorithm given by Kim (1994) so that the smoother contains all needed terms.

2. Waiting time distributions and change points

The methods that will be presented here can be applied to general finite state Hidden Markov Models (including Markov switching models) with the general form:

$$\begin{aligned} y_t &\sim f(S_{t-r:t}, y_{1:t-1}), \\ P[S_t|S_{-r+1:t-1}] &= P[S_t|S_{t-1}], \quad t = 1, \ldots, n. \end{aligned} \tag{2.1}$$

The data y_t from time 1 to time n is distributed conditional on previous data and r previous switching states $S_{t-r}, \ldots, S_{t-1}$ in addition to the current state S_t (as well as model parameters, the values of which are implicitly assumed to be fixed). Here, $y_{t_1:t_2} = y_{t_1}, \ldots, y_{t_2}$ with $S_{t_1:t_2}$ defined analogously. This general form is equivalent to assumption **Y2** in Frühwirth-Schnatter (2006, p. 317). For simplicity, the switching states $\{S_t\}$ are assumed to be a first-order Markov chain with finite state space $\mathcal{S}$, but extension to higher-order Markov structures is straightforward. A given initial distribution π for $S_{-r+1:0}$ is also assumed. With suitable modification, the above model may also include exogenous variables. No assumption on the distribution of the noise in the system is made other than that the posterior probabilities of the states (probabilities conditional on the data) must exist.

Definition A run of length k in state s is defined to be the consecutive occurrence of k states that are all equal to s, i.e. $S_{t-k+1} = s, \ldots, S_t = s$ (c.f. Feller (1968), Balakrishnan and Koutras (2002)).

Now for $m \geq 1$, let $W_s(k, m)$ denote the waiting time of the mth run of length at least k in state s, and let $W(k, m)$ denote the waiting time for the mth run of length at least k of any state $s \in S$. Note that $W(k, m)$ is invariant under any state re-labelling, whereas $W_s(k, m)$ is not.

Consider, for example, the case of growing ($s = 1$) and falling GNP ($s = 0$). $W_0(k, 1)$ is then the first time that a period of falling $(\overbrace{0 \ldots 0}^{=k})$ GNP has occurred in $\{S_t\}$ whereas $W(k, 1)$

is the first time either a period of falling GNP or growing $(\overbrace{1 \dots 1}^{=k})$ GNP occurs. By changing the value of k, shorter or longer length periods can be investigated.

A change point at time t is typically defined to be any time at which $S_{t-1} \neq S_t$, the beginning of a run of length at least one. A special case is given in Chib (1998), where the states are required to change in ascending order. However, a more general definition is allowed here, where a change point is defined to have occurred when a change persists at least k time periods, $k \geq 1$. A classic example of when the generalised definition is needed is the common definition of a recession, where two quarters of decline are required ($k = 2$) before a recession is deemed to be in progress. Let $\tau_i^{(k)}$, $i = 1, \dots, m$ be the time of the ith change point under this generalised definition. Then

$$P[\tau_i^{(k)} = t] = P[W(k, i) = t + k - 1]. \tag{2.2}$$

Equation (2.2) follows because the ith run of length at least k occurs at time $t + k - 1$ if and only if the switch into that regime has occurred $k - 1$ time points earlier. When $k = 1$, it is assumed in this work that a change point of some sort has occurred at $t = 1$, i.e. $P[W(1, 1) = 1) = 1] = P[\tau_1^{(1)} = 1] = 1$, and hence the ith change point using the common definition will be equivalent to $(i + 1)$st change point ($\tau_{i+1}^{(1)}$) as defined here. Since a regime or run of length at least k can continue, $P[\tau_{i+1}^{(k)} > n] > P[\tau_i^{(k)} > n] > 0$.

Other distributions can be calculated from the waiting time distribution. For example, the distribution of the maximal length of a particular regime $R_s(t)$ up to time t is given by

$$\begin{aligned} P[R_s(t) = k] &= P[R_s(t) \geq k] - P[R_s(t) \geq k + 1] \\ &= P[W_s(k, 1) \leq t] - P[W_s(k + 1, 1) \leq t]. \end{aligned} \tag{2.3}$$

Analogously, the probability that the maximal length of any regime is k, $P[R(t) = k]$, can be defined in terms of $P[W(k, 1) \leq t]$.

2.1. **Methods to Calculate Waiting Time Distributions for the First Change Point.** Finite Markov chain imbedding (Fu and Koutras 1994; Aston and Martin 2007) will be used to compute distributions associated with the regime periods. The method involves imbedding the $\{S_t\}$ sequence into a new Markov chain $\{Z_t\}$ with a larger state space. Though $\{S_t\}$ forms a homogeneous Markov chain, conditioning on the data induces rth order inhomogeneous dependence, i.e. the posterior transition probabilities $P[S_t|S_{t-r:t-1}, y_{1:n}]$ are transition probabilities of an inhomogeneous rth order Markov process (Cappé, Moulines, and Rydén (2005) and Appendix A).

The state space of $\{Z_t\}$ (which is denoted by $\mathcal{Z}_s$ or $\mathcal{Z}$ depending on whether runs of a particular state s or a run of any state is of interest) will consist of vector states of the form $((s_{t-r+1}, \ldots, s_t), j)$. The component $(s_{t-r+1}, \ldots, s_t) \in \mathcal{S}^r$, necessary due to the rth-order dependence of states conditional on the data $y_{1:n}$, gives the values of the last r states at time t, $t = 0, \ldots, n$. The component j, $j = 0, 1, \ldots, k$, gives the length of the current run of a particular state s ($j = \max_{1 \leq \phi \leq k} : S_t = s, S_{t-1} = s, \ldots, S_{t-\phi+1} = s$ if $S_t = s$, $j = 0$ otherwise), or of the current value of S_t if general runs are of interest. If $k > r$,

$$\mathcal{Z}_s = \bigcup_{j=0}^{r-1} \left\{ \bigcup_{s_{t-r+1:t}: s_l = s, l=t-j+1,\ldots,t;\ s_l \in \mathcal{S}, l=t-r+1,\ldots,t-j} (s_{t-r+1:t}, j) \right\} \cup \left(\bigcup_{j=r}^{k} ((s, \ldots, s), j) \right), \tag{2.4}$$

and if $k \leq r$,

$$\mathcal{Z}_s = \bigcup_{j=0}^{k} \left\{ \bigcup_{s_{t-r+1:t}: s_l = s, l=t-j+1,\ldots,t;\ s_l \in \mathcal{S}, l=t-r+1,\ldots,t-j} (s_{t-r+1:t}, j) \right\}, \tag{2.5}$$

where any strings $s_{a:b}$ with $a > b$ or any s_b with $b < t - r + 1$ are ignored. (Notice that in (2.4) and (2.5), some states are needed only for the initialisation stage when $t < r$).

When $j = k$, a run of length k or longer has occurred. The set A of states with $j = k$ are absorbing, i.e., once entered, the sequence remains in that state with probability one. The state space $\mathcal{Z}$ for calculating $P[W(k,1) \leq t]$ is then

$$\mathcal{Z} = \bigcup_{s \in \mathcal{S}} \mathcal{Z}_s. \tag{2.6}$$

Let z^* represent the size of either $\mathcal{Z}_s$ or $\mathcal{Z}$ as appropriate. As the components of states of $\{Z_t\}$ are functions of states $\{S_t\}$, the $|S|$ non-zero row entries in the $z^* \times z^*$ transition probability matrix M_t for transitions from transient states of $\{Z_t\}$ are completely determined by the posterior transition probabilities $P[S_t|S_{t-r:t-1}, y_{1:n}]$. Specifically,

$$P[Z_t = ((s_{t-r+1}, \ldots, s_t), j)|Z_{t-1} = ((s_{t-r}, \ldots, s_{t-1}), l), y_{1:n}]$$
$$= P[S_t = s_t|S_{t-1} = s_{t-1}, \ldots, S_{t-r} = s_{t-r}, y_{1:n}] \tag{2.7}$$

for appropriate values of j which are determined in the following manner: For transient states of $\mathcal{Z}_s$, $j = l+1$ when $s_t = s$, and $j = 0$ if $s_t \neq s$. For transient states of $\mathcal{Z}$, $j = l+1$ when $s_t = s_{t-1}$, and $j = 0$ if $s_t \neq s_{t-1}$. In Appendices A and B, a completion of Kim's algorithm is given for the purpose of calculating the posterior transition probabilities of (2.7).

The initial probability distribution for Z_0 is contained in the $1 \times z^*$ row vector ψ_0, which has non-zero probabilities

$$\psi_0((s_{-r+1}, \ldots, s_0), 0) = P[Z_0 = ((s_{-r+1}, \ldots, s_0), 0)] = \pi(s_{-r+1}, \ldots, s_0). \tag{2.8}$$

From the well-known Chapman-Kolmogorov equations for Markov chains (Feller 1968), it follows that the $1 \times z^*$ probability vector ψ_t of Z_t lying in its various states at time $t \geq 1$ is given by

$$\psi_t = \psi_0 \prod_{\tau=1}^{t} M_\tau. \tag{2.9}$$

$P[W_s(k,1) \leq t]$ can then be calculated as

$$P[W_s(k,1) \leq t] = P[Z_t = A] = \psi_t U(A), \tag{2.10}$$

with the analogous result holding for $P[W(k,1) \leq t]$, where $U(\Omega)$ is a $z^* \times 1$ column vector with ones in the locations of the set of states Ω and zeros elsewhere. Equation (2.10) holds since the Markov chain $\{Z_t\}$ is in an absorbing state if and only if a run of length at least k has occurred. Combining (2.9) and (2.10), for $t \geq k$,

$$P[W_s(k,1) = t] = P[W_s(k,1) \leq t] - P[W_s(k,1) \leq t-1] = \psi_0 \left(\prod_{l=1}^{t-1} M_l \right) (M_t - I)U(A), \tag{2.11}$$

where I is a $z^* \times z^*$ identity matrix.

2.2. **Methods to Calculate Waiting Time Distributions for Multiple Change Points.** In this subsection, a method is given to calculate joint probabilities associated with change points through augmenting the state spaces $\mathcal{Z}_s$ and $\mathcal{Z}$. Manipulations of the joint probabilities will lead to an algorithm for computing marginal change point distributions. The algorithm obviates the need to repeat states for each of the $i = 1, \ldots, m$ change point occurrences.

2.2.1. *Setup of Markov chain for Distributions Associated with Multiple Change Points.* A set of states C, called continuation states, is added to $\mathcal{Z}_s$ and $\mathcal{Z}$, and the respective sizes z^* are incremented by the number of continuation states. The role of the continuation states is that once the ith run of length at least k has occurred, a new Markov chain $\{Z_t^{(i+1)}\}$ is started to determine probabilities associated with the next occurrence of a run of the desired length. The continuation states serve to initialise the new chain $\{Z_t^{(i+1)}\}$, and indicate that run i is still in progress and needs to end before the $(i+1)$st run can begin.

The continuation states $((s_{t-r+1}, \ldots, s_t), -1) \in C$ correspond to absorbing states $((s_{t-r+1}, \ldots, s_t), k) \in A$, with -1 indicating that a run continues and must end for the next run to begin. The (less than full rank) $z^* \times z^*$ matrix Υ defined by

$$\Upsilon(z_1, z_2) = \begin{cases} 1 & \text{if } z_1 \in A \text{ and } z_2 \in C \text{ is the corresponding state} \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

maps probabilities of being in the states of A into probabilities for being in the corresponding states of C.

The transition probability matrices M_t are revised to account for the continuation states. Continuation states may only be entered from other continuation states. The generic non-zero transition probabilities beginning in a continuation state $((s_{t-r+1}, \ldots, s_t), -1) \in C$, and conditional on the data are of the form

$$P[Z_t = ((s_{t-r+1}, \ldots, s_t), j) | Z_{t-1} = ((s_{t-r}, \ldots, s_{t-1}), -1), y_{1:n}]$$
$$= P[S_t = s_t | S_{t-r} = s_{t-r}, \ldots, S_{t-1} = s_{t-1}, y_{1:n}], \tag{2.13}$$

where the appropriate values of j for (2.13) are determined by:

(1) If $s_t = s_{t-1}$, $j = -1$ for both $\mathcal{Z}_s$ and $\mathcal{Z}$;

(2) If $s_t \neq s_{t-1}$, then $j = 0$ for $\mathcal{Z}_s$, and $j = 1$ for $\mathcal{Z}$.

The transition probabilities for the rest of the states in either $\mathcal{Z}_s$ or $\mathcal{Z}$ are unchanged.

2.2.2. *Computation of Joint, Conditional and Marginal Distributions.* The joint distribution of the first m change points can be factorised as

$$P[\tau_m^{(k)} = t_m, \ldots, \tau_1^{(k)} = t_1] = P[\tau_1^{(k)} = t_1] \prod_{i=2}^{m} P[\tau_i^{(k)} = t_i | \tau_{i-l}^{(k)} = t_{i-1}, \ldots, \tau_1^{(k)} = t_1]. \tag{2.14}$$

By (2.11),

$$P[\tau_1^{(k)} = t_1] = P[W(k,1) = t_1 + k - 1] = \psi_0 \left(\prod_{l=1}^{t_1+k-2} M_l \right) (M_{t_1+k-1} - I) U(A). \tag{2.15}$$

To calculate $P[\tau_i^{(k)} = t_i | \tau_{i-1}^{(k)} = t_{i-1}, \ldots, \tau_1^{(k)} = t_1]$, define

$$\xi_{t_1+k-1}^{(1)} = \psi_0 \left(\prod_{l=1}^{t_1+k-2} M_l \right) (M_{t_1+k-1} - I). \tag{2.16}$$

and for $i = 2, \ldots, m$

$$\xi_{t_i+k-1}^{(i)} = \left(\frac{\xi_{t_{i-1}+k-1}^{(i-1)} \Upsilon}{\xi_{t_{i-1}+k-1}^{(i-1)} U(A)} \right) \left(\prod_{l=t_{i-1}+k}^{t_i+k-2} M_l \right) (M_{t_i+k-1} - I). \tag{2.17}$$

By (2.15) and (2.16), $P[\tau_1^{(k)} = t_1] = \xi^{(1)}_{t_1+k-1}U(A)$. The vectors $\left(\frac{\xi^{(i-1)}_{t_{i-1}+k-1}\Upsilon}{\xi^{(i-1)}_{t_{i-1}+k-1}U(A)}\right)$ serve as the initial distribution for the excursion of the Markov chain $\{Z_t^{(i)}\}$ beginning in a continuation state at time $t_{i-1}+k-1$, analogous to ψ_0 for the first chain at time zero, and

$$P[\tau_i^{(k)} = t_i | \tau_{i-1}^{(k)} = t_{i-1}, \ldots, \tau_1^{(k)} = t_1] = \xi^{(i)}_{t_i+k-1}U(A). \tag{2.18}$$

Thus, for $i = 2, \ldots, m$, the joint probability

$$\begin{aligned}
&P[\tau_i^{(k)} = t_i, \ldots, \tau_1^{(k)} = t_1] \\
&= P[W(k,i) = t_i + k - 1, \ldots, W(k,1) = t_1 + k - 1] \\
&= \prod_{q=1}^{i} \xi^{(q)}_{t_q+k-1}U(A) \\
&= \psi_0 \prod_{q=1}^{i-1}\left[\left(\prod_{l=t_{q-1}+k}^{t_q+k-2} M_l\right)(M_{t_q+k-1} - I)\Upsilon\right]\left(\prod_{l=t_{i-1}+k}^{t_i+k-2} M_l\right)(M_{t_i+k-1} - I)U(A),
\end{aligned} \tag{2.19}$$

where $t_0 \equiv 1 - k$ for convenience and $\left(\prod_{l=a}^{b} M_l\right) =$ if $b \leq 0$.

Marginal distributions for change point $\tau_i^{(k)}$, or equivalently the marginal waiting time distribution for the first ith run occurrence, $i = 2, \ldots, m$, can then be written as

$$\begin{aligned}
P[\tau_i^{(k)} = t_i] &= P[W(k,i) = t_i + k - 1] \\
&= \sum_{1\leq t_1 < t_i} \cdots \sum_{t_{i-2} < t_{i-1} < t_i} P[W(k,i) = t_i + k - 1, \ldots, W(k,1) = t_1 + k - 1] \\
&= \sum_{1\leq t_1 < t_i} \cdots \sum_{t_{i-2} < t_{i-1} < t_i} \prod_{j=1}^{i} \xi^{(j)}_{t_j+k-1}U(A) \\
&= \left\{\sum_{t_{i-2}<t_{i-1}<t_i} \cdots \left\{\sum_{t_1<t_2<t_i}\left\{\sum_{1\leq t_1<t_i} \xi^{(1)}_{t_1+k-1}U(A)\right\}\xi^{(2)}_{t_2+k-1}U(A)\right\}\cdots\right\}\xi^{(i)}_{t_i+k-1}U(A)
\end{aligned} \tag{2.20}$$

with the marginal distribution $P[\tau_1^{(k)} = t_1]$ given by (2.15).

Equation (2.20) suggests the use of some form of sum-product algorithm (MacKay 2003) for its calculation. Let $\psi_t^{(i)}$ be row vectors carrying probabilities for the Markov chain $Z_t^{(i)}$, i.e. the joint probability that the $(i-1)$st run has occurred by time t, and that the chain is in state z at time t (so that $\psi_t^{(1)} = \psi_t$). The marginal distributions are then

$$P[\tau_i^{(k)} = t_i] = P[W(k,i) = t_i + k - 1] = (\psi_{t_i+k-1}^{(i)} - \psi_{t_i+k-2}^{(i)})U(A), \tag{2.21}$$

the probability of being absorbed at time t, and a similar formula holds for $W_s(k,i)$. Two operations need to be carried out to update $\psi_{t-1}^{(i)}$ to $\psi_t^{(i)}$: (1) Due to the Markovian nature of the system, we must multiply by the transition probability matrix M_t, and (2) Absorption probabilities for the $(i-1)$st run occurrence must be incremented since they serve as initial probabilities when waiting for the occurrence of the ith run. These operations may be carried out simultaneously for $i = 1, \dots, m$ by replacing (2.9) with matrix computations.

Let Ψ_t, $t = 0, \dots, n$ be $m \times z^*$ matrices with ith row $\psi_t^{(i)}$. The initial matrix Ψ_0 then has as its first row ψ_0, with the remaining rows being composed of zeroes since the probability is zero that a run has occurred at time $t = 0$. The algorithm for $t = 1, \dots, n$ is

$$\Psi_t = \Psi_{t-1}M_t, \tag{2.22}$$

$$\psi_t^{(i)} \leftarrow \psi_t^{(i)} + \psi_{t-1}^{(i-1)}(M_t - I)\Upsilon, \quad i = 2, \dots m, \tag{2.23}$$

where (2.22) is related to computing the matrix product $\left(\prod_{j=t_{i-1}+k}^{t_i+k-2} M_j\right)$ of (2.19) while (2.23) is related to computing $(M_{t_{i-1}+k-1} - I)\Upsilon$, $i = 2, \dots m$.

Even though the algorithm is non-linear, it can be determined in almost linear time, as the non-linear update step (2.23) is just a simple alteration to entries in the matrix Ψ_t, requiring only linear time computations.

Using the calculations given above, the distribution of the number of regime changes $P[N_s(k) = i]$ into a particular state is given by

$$P[N_s(k) = i] = P[W_s(k,i) \le n] - P[W_s(k,i+1) \le n] \quad i = 0, \dots, \zeta + \lfloor \frac{n-(k+1)\zeta}{k} \rfloor, \tag{2.24}$$

where $\lfloor x \rfloor$ indicates the integer part of x and $\zeta = \lfloor \frac{n}{k+1} \rfloor$. In practice, the value at which $P[W_s(k,i) \le n]$ becomes negligible will be $i \ll \zeta + \lfloor \frac{n-(k+1)\zeta}{k} \rfloor$. Analogous results hold for probabilities $P[N(k) = i]$ associated with the number of change points in the data, by considering $P[W(k,i) \leq n]$ for $i = 0, \ldots, \lfloor n/k \rfloor$.

By using the setup above it is also possible to determine distributions associated with the time when regimes are left. Let $W_s^e(k,i)$ be the time that the ith run in state s ends, with $W^e(k,i)$ being defined in an analogous fashion for the ith run over all states. Regime i ends when $Z_t^{(i+1)}$ leaves the continuation states. Thus for $t = 1, \ldots, n-1$,

$$P[W_s^e(k,i) = t] = \psi_t^{(i+1)} (I - M_{t+1}) U(C), \quad i = 1, \ldots, m-1. \tag{2.25}$$

again with an analogous result for $P[W^e(k,i) = t]$.

2.3. **Change Point Probability.** Changes in regime are often deemed qualitatively to coincide with external events such as the start or end of a recession, or a political or historical event such as the Oil Crisis or September 11. Change point probabilities (CPPs) quantify the chance that a switch occurs at a particular time point or within a particular interval. Since only one regime switch can occur at any particular point, a CPP at time t may be computed by summing the probability of the ith change point occurring at that time over i:

$$\text{CPP}_s(t,k) = \sum_i P[W_s(k,i) = t+k-1] \tag{2.26}$$

and

$$\text{CPP}_s^e(t,k) = \sum_i P[W_s^e(k,i) = t], \tag{2.27}$$

with analogous definitions for probabilities $\text{CPP}(t,k)$ and $\text{CPP}^e(t,k)$ of change points associated with any state of $\mathcal{S}$.

When $k = 1$ and $t < n$, $\text{CPP}^e(t,1) = \text{CPP}(t+1,1)$, since the end of one regime guarantees the start of another. However when $k > 1$, this is not necessarily the case, as it can take more than one time period before a new regime of length at least k appears.

Probabilities $\text{CPP}(t_1 : t_2, k)$ that at least one change point occurs in $t_1, t_1+1, \ldots, t_2$ may be computed using the framework of the algorithm (2.22-2.23). If $\psi_t^{(i)}(z)$ is the component of $\psi_t^{(i)}$ corresponding to state z, then, since $\sum_{z\in\mathcal{Z}\setminus A}\left[\psi_t^{(1)}(z)\right]+\sum_{z\in A}\left[\psi_t^{(1)}(z)\right]=1$ and $\sum_{z\in\mathcal{Z}\setminus A}\left[\psi_t^{(i)}(z)\right]=\sum_{z\in A}\left[\psi_t^{(i-1)}(z)\right]$,

$$\sum_{i=1}^{m}\sum_{z\in\mathcal{Z}\setminus A}\psi_t^{(i)}(z)+\sum_{z\in A}\psi_t^{(m)}(z)=1. \tag{2.28}$$

Let $\tilde{\psi}_t(z)=\sum_i\psi_t^{(i)}(z)$, for $z\in\mathcal{Z}\setminus A$, then

$$\text{CPP}(t_1:t_2,k)=\tilde{\psi}_{t_1-1}\left(\prod_{l=t_1}^{t_2}M_l\right)U(A) \tag{2.29}$$

in an analogous fashion to (2.11), and similarly for $\text{CPP}_s(t_1:t_2,k)$. By a similar argument and taking care with the continuation states C, $\text{CPP}_s^e(t_1:t_2,k)$ and $\text{CPP}^e(t_1:t_2,k)$ can also be found. A small note is that $\tilde{\psi}_{t_1-1}$ is not strictly a (initial) distribution due to the term $\sum_{z\in A}\psi_t^{(m)}(z)$. However, (2.29) still gives the exact probability.

2.4. **Computational Considerations.** Given the prevalence of Bayesian techniques in the analysis of Markov switching models (see Frühwirth-Schnatter (2006) for the latest on these techniques), it is of interest to compare the computational cost of calculating the waiting time and change point distributions through the exact scheme above versus drawing samples from the posterior distribution of the states. Of course, in terms of error for fixed parameters, the two approaches cannot be compared as the exact distribution is not subject to any sampling error.

For both drawing conditional samples of the underlying states and the exact scheme, a pass through a Markov chain is necessary. Every state in either approach has at most $|\mathcal{S}|$ possible transition destinations, so all that needs to be compared is the size of the state spaces associated with the two techniques.

For drawing conditional samples (for example through Markov Chain Monte Carlo), in general, a state space of size $|\mathcal{S}|^r$ is needed as the order of dependence of the posterior Markov chain for the model given in (2.1) is r. For the exact scheme with $k>r$, the state space $\mathcal{Z}_s$ (the number of states given in (2.4) plus one for the continuation state) is of size

$$\sum_{l=0}^{r} |\mathcal{S}|^l + (k-r+1) = \frac{1-|\mathcal{S}|^{r+1}}{1-|\mathcal{S}|} + (k-r+1) < |\mathcal{S}|^{r+1} + (k-r+1)$$

while for $\mathcal{Z}$, the size needed is at most

$$|\mathcal{S}|^r + \frac{1-|\mathcal{S}|^{r+1}}{1-|\mathcal{S}|} + |\mathcal{S}|(k-r+1) < |\mathcal{S}|^r(|\mathcal{S}|+1) + |\mathcal{S}|(k-r+1).$$

Thus when $k > r$, if $k \ll |\mathcal{S}|^r$, at most $|\mathcal{S}|m$ (and often less) equivalent sample computations are needed to calculate the marginal waiting time distributions. Of course as k increases, the number of states will increase, but this is only at a linear rate proportional to k.

For $k \leq r$, the size of state space required for $\mathcal{Z}_s$ and for $\mathcal{Z}$ respectively are

$$(\sum_{i=r-k}^{r} |\mathcal{S}|^i) + |\mathcal{S}|^{r-k},$$

and

$$|\mathcal{S}|^r + (\sum_{i=r-k+1}^{r} |\mathcal{S}|^i) + |\mathcal{S}|^{r-k+1}.$$

Comparing the computational cost for a standard change point analysis with $k = 1$ and m change points, the number of computations required to calculate the exact marginal distributions is the same as for drawing $3m$ simulation samples, which is small in the usual case when m is small. Note that with the sampling approach, the precise amount of sampling needed can be difficult to quantify given convergence and approximation issues.

All these calculations presume that the state space model is of a general structure. Models such as the change point model of Chib (1998) would require significantly less computation for the exact distributional method given the structure in the model.

3. GNP Analysis

The Markov switching model, a particular form of HMM, which relaxes the assumption of independence between the observed data, is popular in economics. One of the first uses of Markov switching models was to investigate the business cycle. Hamilton (1989) analysed logged and differenced quarterly GNP data for the time period 1951:II to 1984:IV. He showed that a two-state mean switching AR(4) is a good model for business cycles. The model is

$$
\begin{aligned}
y_t &= \alpha_{S_t} + z_t, \\
z_t &= \phi_1 z_{t-1} + \cdots + \phi_4 z_{t-4} + \varepsilon_t, \\
\varepsilon_t &\overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2),
\end{aligned}
\tag{3.1}
$$

where y_t represents the logged and differenced GNP data, S_t is the underlying state that affects the mean α_{S_t}, z_t is an AR(4) process with parameters $\phi_1, \ldots, \phi_4$ and ε_t is a Gaussian white noise process with zero mean and variance σ^2. In this paper, the underlying states are initialised in the steady state distribution for the chain, and a reduced model is used for $y_1, \ldots, y_4$, so that any ϕ_i associated with $y_{-3}, \ldots, y_0$ is set to zero. This initialisation differs slightly from that in Hamilton (1989), where the first r data points were designated as $y_{-r+1:0}$ and thus the full model was applied to a slightly smaller range of data. The effect of which initialisation is used becomes negligible after the first few time points.

The likelihood function of the model is

$$
\begin{aligned}
&f(y_t | S_{t-4:t}, y_{1:t-1}) \\
&\quad = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{\left[(y_t - \alpha_{S_t}) - \phi_1(y_{t-1} - \alpha_{S_{t-1}}) - \cdots - \phi_r(y_{t-4} - \alpha_{S_{t-4}})\right]^2}{2\sigma^2} \right\}.
\end{aligned}
\tag{3.2}
$$

The state space of $\{S_t\}$ is $\{0, 1\}$, where 0 corresponds to a regime of falling GNP and 1 to a regime of growing GNP. The analysis is conditional on the parameter estimates given in Hamilton (1989), who also conditioned posterior state probabilities on those values.

In Hamilton's analysis, the state at time t is the value that maximises the posterior probability $P[S_t = s_t | y_{1:n}]$, and (implicit) change points into a recession are time points where the state switches from one to zero. In general, depending on the properties of the transition matrix for S_t, this method for determining states, known as local or posterior decoding (Juang and Rabiner 1991; Durbin et al. 1998), does not necessarily give a posterior state sequence that can occur with positive probability. Determining change points through local decoding neither takes into account the common definition of a recession, which requires at least two quarters of negative growth, nor does it properly account for the exponential number of possible state sequences and their associated probabilities.

In the present work, to calculate distributions of changes into recessions, the imbedding state space $\mathcal{Z}_0$ is constructed as outlined in Section 2. The value of $k = 2$ used here corresponds to the common definition of requiring two quarters of falling GNP for a recession to be confirmed. The posterior transition probabilities $P[S_t = s_t | S_{t-1} = s_{t-1}, \ldots, S_{t-r} = s_{t-r}, y_{1:n}]$ are used in the transition probability matrix of the process Z_t. The resulting distributions are plotted in Figures 1-3.

[Figure 1 about here.]

Different statistics were examined to determine whether their distributions seem plausible in light of economic theory. For example, it would be expected that the number of contraction periods in the data, $N_0(2)$, should have a unimodal distribution that is quite peaked, as a high variability in $N_0(2)$ would indicate the model does not explain the data very well. This is indeed the case (Figure 1). The probability of seven contractions is almost 0.5 (seven was also the number of change points in Hamilton's analysis), with a probability of about 0.35 of eight, but little probability of any other number. This corresponds well with the fact that the NBER also found seven periods of recession in the data.

[Figure 2 about here.]

The distributions of the length of the longest continuously falling (Figure 2a) or continuously growing (Figure 2b) period of GNP were computed, using state space $\mathcal{Z}_1$ for the imbedding Markov chain associated with growing GNP. The length distributions will not necessarily be unimodal as the data may well support quite different length periods being present in relation to the location and number of regime switches that are present. This is borne out in the computed distribution. The longest continuous fall in GNP is moderately stable at around 7-8 quarters

(Figure 2). However, this is not the case when looking at the longest continuous rise in GNP, where the case can be made that it lasted anywhere from 20-40 quarters, although 33-35 quarters are most likely. The mean of 30.0 quarters in this instance is not particularly informative, given the multimodal nature of the distribution. This large variability and the multimodal nature of the distribution suggests that trying to use the model to determine the longest period of expansion might prove to be more difficult than one might hope.

[Table 1 about here.]

[Figure 3 about here.]

Whereas Figures 1-2 give insight into particular distributions, Figure 3 gives more information about the specific structure of the changes in regime for GNP. The first thing to note is that the third period of falling GNP may occur with reasonable probability in two quite different places. This indicates that the second period could be two distinct periods rather than only one. The bimodal nature then carries on from this point, indicating that regimes of falling GNP could begin and end at two different places. A sustained period of growth is then indicated from the early 1960's until just before 1970, with little variability in this assessment, which concurs with the NBER economic thinking about the time period. For the remainder of the plots in Figure 3, it can be seen that the distribution peaks are still fairly distinct, meaning that the model is characterising regimes as being either falling or growing without too much overlap. This assessment concurs with economic thinking on the nature of the business cycle, in that high frequency oscillations are not likely, suggesting that the model is still capturing properties of the data even towards the end of the data period. Probabilities for regime periods are only plotted for cases with at least 0.05 total probability. There is only a 0.018 chance of there being ten or more periods of falling GNP in the data so only the first nine change point distributions are plotted.

[Figure 4 about here.]

In Hamilton's original paper, the Hidden Markov AR(4) model was shown to coincide qualitatively well with all the NBER determinations of recessions but no quantitative analysis is possible using their methodology. Table 1 gives the CPPs according to the model of peaks and troughs of the business cycle occurring at those determined by the NBER. As can be seen, the model produces a quantitatively good fit for most of the NBER determinations but some NBER peaks are not particularly probable under the model, especially the second and sixth peaks. These two recessions were of particular interest in the Hamilton (1989) analysis as they were associated with the Suez Crisis and the Iranian Revolution, respectively. The present analysis shows that the NBER recession dates closest to these two events likely do not reflect the immediate effect of the two events. It is also interesting to note that while there are two quarters difference between the locations of the NBER peak (1957.III) and the peak in the posterior state probabilities (1957.I) for recession two, and three quarters difference (NBER: 1980.I, Posterior Decoding: 1979.II) for recession six, the probability of recession six starting at the NBER point is higher than the probability of recession two starting at the NBER point. This shows that when trying to determine whether events are explained by the model, exact CPP are better than using distance from the time of a peak in the posterior state probabilities for the event of interest. It is also interesting to note that the NBER troughs for the business cycle are more stable under the model than the peaks. This can be seen in the graph in Figure 3. In addition to the point estimates, interval probabilities of a change point being at most one quarter different from the NBER dates are also given in Table 1, yielding a confidence interval for the NBER dates in addition to the point probabilities.

Figure 4 shows that the time locations for change points are grouped, although it should be remembered that this plot, unlike those of Figure 3, is not a distribution over time due to the multiple regimes in the data, but rather a graph of the CPP at each time point. The probabilities are moderately peaked, indicating that there are only a few times where change points are likely to occur. It is of interest to note however from both this graph and those in Figure 3 that the fourth change point into a recessionary state as determined by Hamilton's analysis (posterior decoding) occurs between two peaks (at 1969.III). Thus this is actually an unlikely time (0.18

from Figure 4) for the change to have occurred, with it much more likely to have occurred either just before or just after this date (the NBER peak is just after at 1969.IV). This illustrates that change points determined through posterior decoding are not necessarily the most likely ones over all possible state sequences.

4. Concluding Remarks

In this paper, methods for calculating change point distributions in general finite state HMMs (or Markov switching models) have been presented. A derived link between waiting time distributions and change point analysis has been exploited to compute probabilities associated with change points. The methodology provides a means of improved inference, as change points determined by maximising the conditional probability at each time point of states given the data can be misleading. As a by-product of deriving the theoretical basis for the approach, smoothing algorithms in the literature have been investigated and a correction to the algorithm given in Kim (1994) has been presented in Appendix B.

Functions of run distributions have been examined. It would be straightforward to extend the ideas to patterns that are more complex than runs, and to models with multiple regimes. The definition of change points has been generalised to force a sustained change before a change point is counted, however the normal definition of change point (without the first point being deemed a change point) as a switch from the current regime can be handled by a slight modification of the methodology, namely by adding a continuation state for the zeroth occurrence.

If the distribution of the change points is of fundamental interest, and a suitable prior distribution $\pi(\theta)$ is known for the parameters θ, then a Bayesian approach can be considered:

$$P[\tau_i = t|y_{1:n}] \propto \int_\theta P[\tau_i = t|y_{1:n}, \theta]\pi(\theta)d\theta = \int_\theta P[W(1,i) = t|y_{1:n}, \theta]\pi(\theta)d\theta. \tag{4.1}$$

Numerical integration over the parameter space has been used in change point analysis in the past (Fearnhead and Liu 2007). Equation (4.1) does not include the unknown states explicitly; they only occur through terms that may easily be computed exactly. Thus state sampling is not needed, especially if a suitable importance sampling distribution can be found to compute the integral. Most sampling schemes such as MCMC need the likelihood to include the unknown states, a much higher dimensional problem than when only the change point times are considered.

While the distributions given in this paper can be approximated by sampling the unknown states, there are several disadvantages to this approach. Repeated sampling over the set of unknown states is computationally expensive and introduces sampling error into the computations, whereas in the present work $P[W(1,i) = t|y_{1:n}, \theta], i = 1, \ldots, m$ are calculated exactly, at a computational complexity equivalent to at most drawing $3m$ samples in the standard change point approach. In addition, the problem of lack of invariance of change point results under state re-labelling is avoided using the presented method.

The CPP point distribution could alternatively be found, without using the presented techniques, by finding the posterior distribution of $S_{t-\max(k,r)}, \ldots, S_{t-1}, S_t$ using smoothing techniques similar to the smoother given in the appendix, but the computation time would then be exponential in k. Also, the methodology presented in this paper is necessary if change point distributions for each individual occurrence are required, and in that case CPP would require no extra computation other than summing probabilities over time points.

All the analysis and computational steps of this paper have been conditioned on the parameter values in the model. The effect of parameter estimation on the distributions calculated has not been explicitly considered; plug-in MLEs have been used, as is often done with other techniques such as local decoding or the Viterbi algorithm. However, in a small stability study using asymptotic MLE distributions and Monte Carlo integration for the GNP data, the CPPs were found to be robust to small fluctuations in the parameter estimates (data not shown). In addition, in applications where the parameters are determined using training data and the model is then applied to test data, the distributions of the change point locations in the test data are exact.

It should also be noted that many Markov switching models, such as state space switching models (Kim 1994), are analysed through the use of approximations to obtain the smoothed state probabilities. This is done to allow computation in situations where complexity becomes very large when taking advantage of algorithms such as the Kalman filter and also when r is not a fixed value but is dependent on the data length (Frühwirth-Schnatter 2006), for example in moving average Markov switching models. There have been several suggestions for ways to implement approximations (Kim 1994; Billio and Monfort 1998) to the likelihood function in such cases. These techniques can also be used to approximate the smoothed transition probabilities needed in this paper.

In conclusion, methods for further investigation of change points implied by Hidden Markov models (including Markov switching models) have been presented. The methods are ways of detecting and evaluating change points that are implied by a model. In addition, formulas are given to facilitate evaluation of joint and marginal change point distributions. The widespread use of Markov switching models and change point models in statistics and econometrics, along with hidden state segmentation models in computer science, will provide many possible applications of this work.

Appendix A. Smoothed Transition Probabilities

In order to calculate the smoothed transition probabilities conditional on the data, the following lemma is needed. It is already well known that the posterior transition probabilities form a Markov process (Cappé, Moulines, and Rydén 2005), so only the order of dependence needs to be determined.

Lemma A.1. *For a first order Markov switching model with rth order lag dependence in the data, conditional on $y_{1:n}$, $\{S_t\}$ forms an rth order Markov process:*

$$P[S_t|S_{-r+1:t-1}, y_{1:n}] = P[S_t|S_{t-r:t-1}, y_{1:n}] \tag{A.1}$$

The proof of this lemma is straightforward from the definitions of the Markov switching model.

The transition probabilities in Lemma A.1 are given by:

$$P[S_t|S_{t-r:t-1}, y_{1:n}] = \frac{P[S_{t-r:t}|y_{1:n}]}{\sum_{s_0 \in \mathcal{S}} P[S_{t-r:t-1}, S_t = s_0|y_{1:n}]}, \tag{A.2}$$

where $P[S_{t-r:t}|y_{1:n}]$ is the quantity calculated in Lemma B.2.

Appendix B. Posterior Transition Probabilities for Markov Switching Models

In this section, the smoother algorithm and proof of the Markovian nature of the posterior transition probabilities for Markov switching models are outlined. Posterior transition probabilities are needed to derive the exact distributions. The same general form of Markov switching models given in (2.1) will be assumed throughout. The filtering algorithm in Hamilton (1989) is used and output $P[S_{t-r:t}|y_{1:t}]$ and $P[S_{t-r:t}|y_{1:t-1}]$ for $t = 1$ to n is stored for use in the smoothing algorithm.

B.1. Smoothing Algorithm.

The following lemma will be needed for the proof of the smoothing algorithm for Markov switching models.

Lemma B.1. *For a Markov switching model of lag dependence order r,*

$$f(y_{t+1:n}|S_{-r+1:t+1}, y_{1:t}) = f(y_{t+1:n}|S_{t-r+1:t+1}, y_{1:t}), \quad t > 0 \tag{B.1}$$

Proof.

$$f(y_{t+1:n}|S_{-r+1:t+1}, y_{1:t}) = \frac{\sum_{S_{t+2:n}} f(y_{1:n}, S_{-r+1:n})}{f(y_{1:t}, S_{-r+1:t+1})} \tag{B.2}$$

$$= \frac{\sum_{S_{t+2:n}} f(y_n|y_{1:n-1}, S_{-r+1:n}) \cdots f(y_{t+1}|y_{1:t}, S_{-r+1:n}) f(y_{1:t}, S_{-r+1:n})}{f(y_{1:t}, S_{-r+1:t+1})}. \tag{B.3}$$

By the model definition

$$f(y_t|y_{1:t-1}, S_{-r+1:n}) = f(y_t|y_{1:t-1}, S_{t-r:t}), \tag{B.4}$$

thus (B.3) can be written as

$$= \frac{\sum_{S_{t+2:n}} f(y_n|y_{1:n-1}, S_{n-r:n}) \cdots f(y_{t+1}|y_{1:t}, S_{t-r+1:t+1}) f(y_{1:t}, S_{-r+1:n})}{f(y_{1:t}, S_{-r+1:t+1})}, \tag{B.5}$$

which by dividing the last term in the numerator by the denominator and using the Markov property of S_t (and the independence of S_{t+1} and $y_{1:t}$ given S_t)

$$= \sum_{S_{t+2:n}} f(y_n|y_{1:n-1}, S_{n-r:n}) \cdots f(y_{t+1}|y_{1:t}, S_{t-r+1:t+1}) P[S_{t+2:n}|S_{t-r+1:t+1}, y_{1:t}] \tag{B.6}$$

$$= \frac{\sum_{S_{t+2:n}} f(y_{1:n}, S_{t-r+1:n})}{f(y_{1:t}, S_{t-r+1:t+1})} \tag{B.7}$$

$$= \frac{f(y_{1:n}, S_{t-r+1:t+1})}{f(y_{1:t}, S_{t-r+1:t+1})} = f(y_{t+1:n}|S_{t-r+1:t+1}, y_{1:t}). \tag{B.8}$$

□

Lemma B.2 (Markov Switching Model Smoothing Algorithm)**.** *For a Markov switching model of lag dependence order* r*, the joint probability of the state variables conditional on the data from time* 1 *to* n *can be computed recursively as follows:*

$$P[S_{t-r:t+1}|y_{1:n}] = \frac{P[S_{t-r+1:t+1}|y_{1:n}] P[S_{t-r:t}|y_{1:t}] P[S_{t+1}|S_t]}{P[S_{t-r+1:t+1}|y_{1:t}]}, \tag{B.9}$$

where

$$P[S_{t-r:t}|y_{1:n}] = \sum_{S_{t+1}} P[S_{t-r:t+1}|y_{1:n}]. \tag{B.10}$$

Proof.

$$P[S_{t-r:t+1}|y_{1:n}] = \frac{f(y_{t+1:n}, S_{t-r:t+1}|y_{1:t})}{f(y_{t+1:n}|y_{1:t})} \tag{B.11}$$

$$= \frac{P[S_{t-r:t+1}|y_{1:t}] f(y_{t+1:n}|S_{t-r:t+1}, y_{1:t})}{f(y_{t+1:n}|y_{1:t})}, \tag{B.12}$$

which by the Markov property of $\{S_t\}$ and the independence of S_{t+1} and $y_{1:t}$ given S_t

$$= P[S_{t-r:t}|y_{1:t}]\frac{P[S_{t+1}|S_t]f(y_{t+1:n}|S_{t-r:t+1}, y_{1:t})}{f(y_{t+1:n}|y_{1:t})}, \tag{B.13}$$

which by lemma B.1

$$= P[S_{t-r:t}|y_{1:t}]\frac{P[S_{t+1}|S_t]f(y_{t+1:n}|S_{t-r+1:t+1}, y_{1:t})}{f(y_{t+1:n}|y_{1:t})} \tag{B.14}$$

$$= P[S_{t-r:t}|y_{1:t}]\frac{P[S_{t+1}|S_t]f(y_{t+1:n}, S_{t-r+1:t+1}|y_{1:t})}{f(y_{t+1:n}|y_{1:t})P[S_{t-r+1:t+1}|y_{1:t}]} \tag{B.15}$$

$$= P[S_{t-r:t}|y_{1:t}]\frac{P[S_{t+1}|S_t]P[S_{t-r+1:t+1}|y_{1:n}]}{P[S_{t-r+1:t+1}|y_{1:t}]}. \tag{B.16}$$

By summing over all the possible states of S_{t+1},

$$P[S_{t-r:t}|y_{1:n}] = \sum_{S_{t+1}} P[S_{t-r:t+1}|y_{1:n}]. \tag{B.17}$$

□

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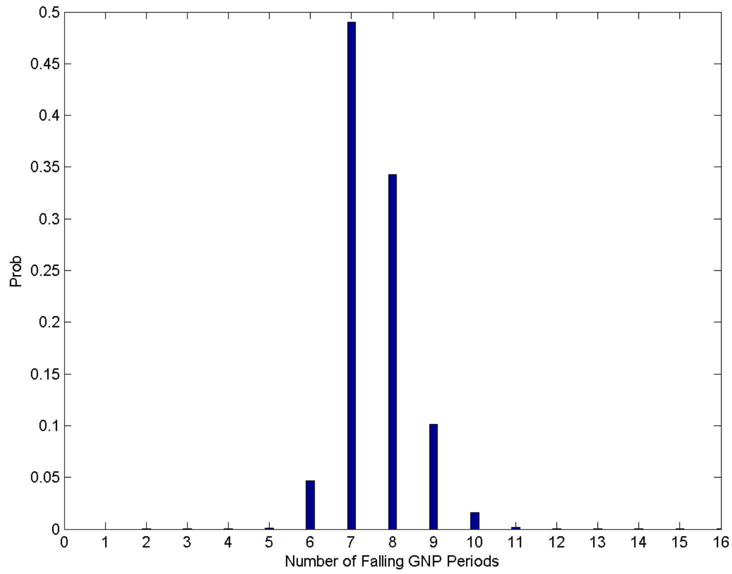
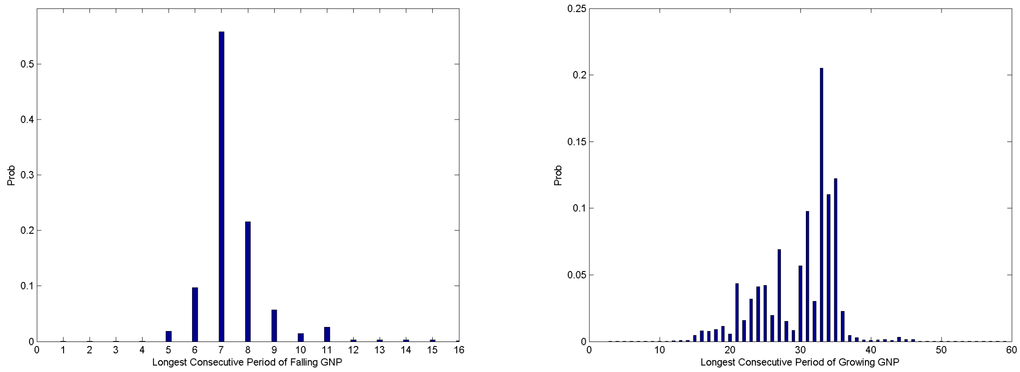


FIGURE 1. $P[N_0(2) = i], i = 1, \ldots, 16$; the distribution of the number of falling GNP periods within the time period 1951:II to 1984:IV. The graph shows that there are most likely seven falling GNP periods within the data range, but there is also a probability of about 0.35 that there are eight periods. The mean number of periods of falling GNP is 7.56.



(a) Falling GNP

(b) Growing GNP

FIGURE 2. Distributions of the longest period of falling ($R_0(n)$) and growing ($R_1(n)$) GNP within the time period 1951:II to 1984:IV. Graph (a) shows that the longest GNP falling period most probably lasts seven quarters within the data range. The mean value for the longest falling GNP period is 7.43 quarters. Graph (b) shows that the longest GNP growing period is much more variable than the longest falling GNP period. The mean value for the longest growing GNP period is 30.0 quarters.

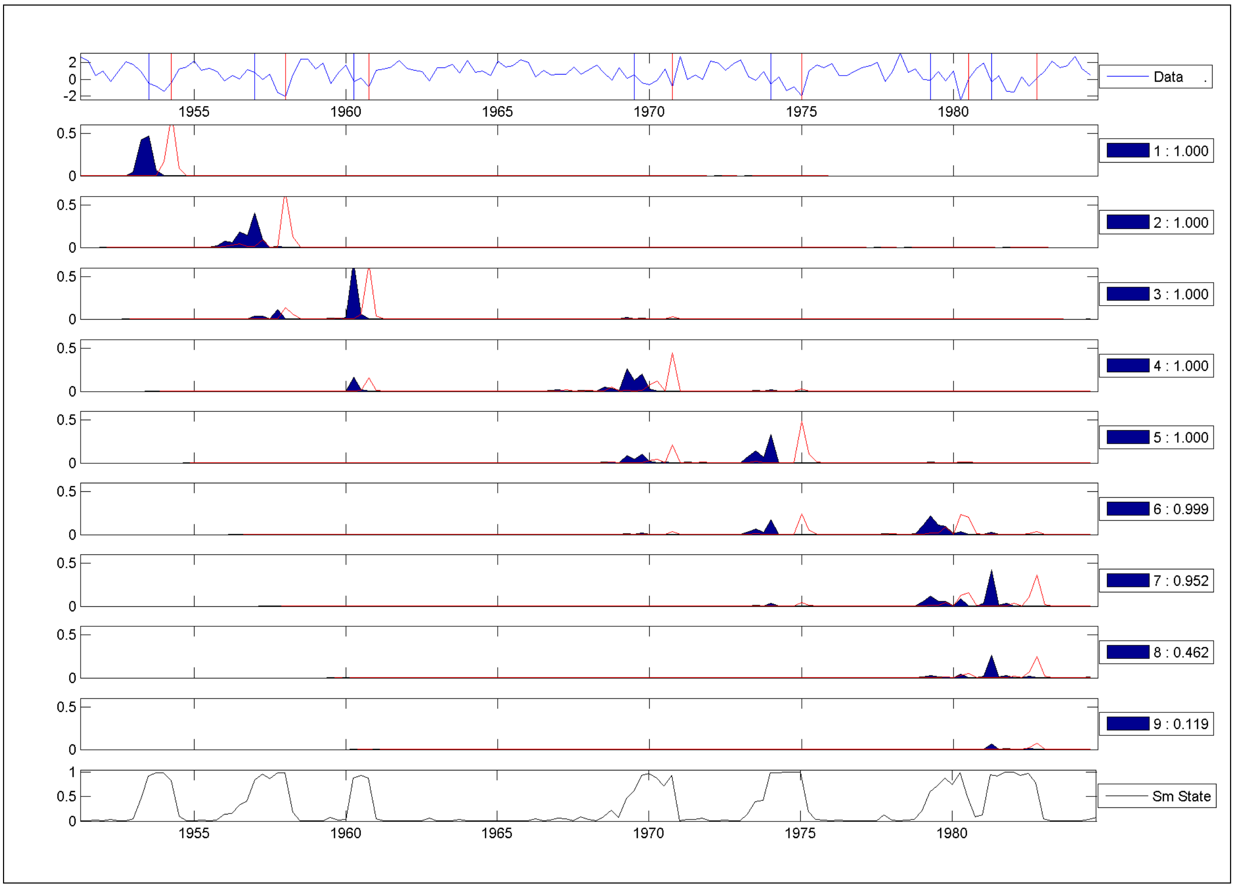


FIGURE 3. Regime Variability of GNP. This is a graphical plot to detect the variability of when different periods of falling GNP states occur in GNP data from 1951:II to 1984:IV. The numbers in the legends to the right of the graphs indicate the index of period falling GNP period under consideration, followed by the probability of at least that many periods occurring by the end of the data. The very top graph gives a plot of the logged differenced GNP data along with the start (blue line) and finish (red line) of periods where $P[S_t = 0|y_{1:n}] > 0.5$. For the subsequent graphs, the blue area indicates the distribution of the start of a period of falling GNP while the red line indicates the distribution of the end of the falling GNP regime, thus giving a measure of the variability in the length of periods of falling GNP. The ith graph gives the distribution of the ith falling GNP period having occurred at time t, for $i = 1, \ldots, m$. The final plot gives the posterior probability plot $P[S_t = 0|y_{1:n}]$, as used in posterior decoding.

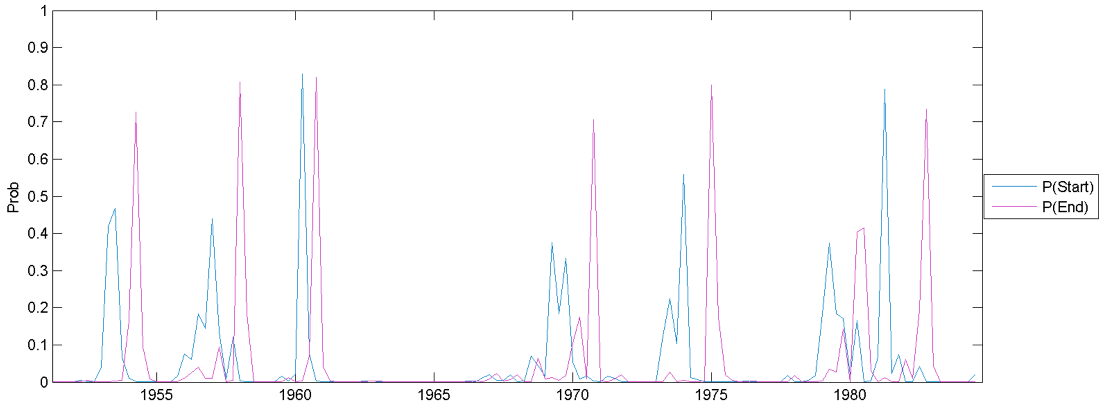


FIGURE 4. Plot of the CPP of starting (CPP_0) or ending (CPP_0^e) a falling GNP regime within the time period 1951:II to 1984:IV. The graph, while not a distribution, gives information as to the probability of a switch occurring in a particular quarter.

TABLE 1. Dating of the US business cycle peaks and troughs as determined by the NBER, along with their associated probabilities of occurring at or before each time according to the AR(4) mean switching model.

i	Peak (t_1)	$P[W_0(2,i) = t_1+1]$	$\text{CPP}_0(t_1, 2)$	CPP_0 $(t_1-1:t_1+1, 2)$	Trough (t_2)	$P[W_0^e(2,i) = t_2]$	$\text{CPP}_0^e(t_2, 2)$	CPP_0^e $(t_1-1:t_1+1, 2)$
1	1953.III	0.46	0.47	0.93	1954.II	0.72	0.73	0.99
2	1957.III	0.0036	0.0092	0.58	1958.II	0.13	0.18	0.99
3	1960.II	0.66	0.83	0.85	1961.I	0.034	0.042	0.87
4	1969.IV	0.2	0.33	0.89	1970.IV	0.44	0.71	0.72
5	1973.IV	0.065	0.1	0.45	1975.I	0.48	0.8	0.98
6	1980.I	0.0088	0.019	0.37	1980.III	0.2	0.42	0.85
7	1981.III	0.012	0.023	0.88	1982.IV	0.36	0.72	0.96