ON THE ERGODIC THEORY OF CELLULAR AUTOMATA AND TWO-DIMENSIONAL MARKOV SHIFTS GENERATED BY THEM

by

MARK ALEXANDROVICH SHERESHEVSKY

Thesis submitted for the degree of Doctor of Philosophy at the
University of Warwick, Mathematics Institute

Dedicated to
my wife Natasha,
my son Alexey
and my parents:
Valentina Borisovna
and
Alexandr Veniaminovich

SUMMARY

In this thesis we study measurable and topological dynamics of certain classes of cellular automata and multi-dimensional subshifts.

In Chapter 1 we consider one-dimensional cellular automata, i. e. the maps $T: P^{\mathbb{Z}} \to P^{\mathbb{Z}}$ (P is a finite set with more than one element) which are given by $(Tx)_i = F(x_{i+l}, ..., x_{i+r})$, $x = (x_i)_{i \in \mathbb{Z}} \in P^{\mathbb{Z}}$ for some integers $l \leq r$ and a mapping $F: P^{r-l+1} \to P$. We prove that if F is right- (left-) permutative (in Hedlund's terminology) and $0 \leq l < r$ (resp. $l < r \leq 0$), then the natural extension of the dynamical system $(P^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ is a Bernoulli automorphism (μ stands for the $(1/p, ..., 1/p)$-Bernoulli measure on the full shift $P^{\mathbb{Z}}$). If $r < 0$ or $l > 0$ and T is surjective, then the natural extension of the system $(P^{\mathbb{Z}}, \mathcal{B}, \mu, T)$ is a K-automorphism. We also prove that the shift $\mathbb{Z}^2$-action on a two-dimensional subshift of finite type canonically associated with the cellular automaton T is mixing, if F is both right and left permutative. Some more results about ergodic properties of surjective cellular automata are obtained

Let X be a closed translationally invariant subset of the d-dimensional full shift $P^{\mathbb{Z}^d}$, where P is a finite set, and suppose that the $\mathbb{Z}^d$-action on X by translations has positive topological entropy. Let G be a finitely generated group of polynomial growth. In Chapter 2 we prove that if growth(G) < d, then any G-action on X by homeomorphisms commuting with translations is not expansive. On the other hand, if growth(G) = d, then any G-action on X by homeomorphisms commuting with translations has positive topological entropy. Analogous results hold for semigroups.

For a finite abelian group G define the two-dimensional Markov shift $X_G = \{ x \in G^{\mathbb{Z}^2} : x_{(i,j)} + x_{(i+1, j)} + x_{(i, j+1)} = 0 \text{ for all } (i, j) \in \mathbb{Z}^2 \}$. Let μ_G be the Haar measure on the subgroup $X_G \subset G^{\mathbb{Z}^2}$. The group $\mathbb{Z}^2$ acts on the measure space (X_G, μ_G) by shifts. In Chapter 3 we prove that if G_1 and G_2 are p-groups and $E(G_1) \neq E(G_2)$, where E(G) is the least common multiple of the orders of the elements of G, then the shift actions on (X_{G_1}, μ_{G_1}) and (X_{G_2}, μ_{G_2}) are not measure-theoretically isomorphic. We also prove that the shift actions on X_{G_1} and X_{G_2} are topologically conjugate if and only if G_1 and G_2 are isomorphic.

In Chapter 4 we consider the closed shift-invariant subgroups $X_{\langle f \rangle} = \langle f \rangle^{\perp} \subset (\mathbb{Z}_p)^{\mathbb{Z}^2}$ defined by the principal ideals $\langle f \rangle \subset \mathbb{Z}_p[u^{\pm 1}, v^{\pm 1}] \cong ((\mathbb{Z}_p)^{\mathbb{Z}^2})^{\wedge}$ with $f(u, v) = c_f(0, 0) + c_f(1, 0)u + c_f(0, 1)v$, $c_f(i, j) \in \mathbb{Z}_p \setminus \{0\}$, on which $\mathbb{Z}^2$ acts by shifts. We give the complete topological classification of these subshifts with respect to measurable isomorphism.

CONTENTS

ACKNOWLEDGEMENTS

First of all I wish to express my deep gratitude to my supervisor Professor Klaus Schmidt for his advice, encouragement and lots of stimulating and fruitful discussions. It was he, who made me feel the beauty of the multi-dimensional symbolic dynamics. I am very thankful to Doctor Anthony Manning for moral support and permanent interest to my work, I am also thankful to him for his critically reading the manuscripts of Chapters 1 - 3. I would like to thank Professor William Parry for interesting conversations, and to Doctor Yurii Suhov for having arranged my seminar in Cambridge. I cordially thank my friends Igor, Eugene and Wim for everything they did for me. I acknowledge the financial support from the University of Warwick. Finally, I thank Professor David Elworthy and Professor Anatoly Vershik without whom I would never be able to come here.

DECLARATION

Chapters 1 - 3 of this thesis is my own work. Chapter 4 is a joint work with Klaus Schmidt.

The material of Chapters 1 - 4 constitutes the papers [S1], [S2], [S3], [SchS], respectively. All these papers have been accepted for publication.

INTRODUCTION

The subject of this thesis is cellular automata and some dynamical systems closely connected with them. In recent development of the ergodic theory there has been a growing interest in exploring dynamics of automorphisms and endomorphisms (i. e. invertible and non-invertible continuous shift-commuting maps) of subshifts and especially of the full shift (see [BK], [C], [CP], [G], [H], [Hu], [Li], [N], [ShR], [Wo]). Endomorphisms (automorphisms) of the full shift are called cellular automata. Cellular automata owe a great deal of their popularity to the numerous applications they have in physics, chemistry, computer science etc. (see e.g. [Wo]).

To the best of my knowledge, the mathematical study of one-dimensional cellular automata dates back 1969, when the cornerstone paper by G. A. Hedlund [H] appeared. Later his results were extended by E. M. Coven and M. E. Paul [CP] to endomorphisms of irreducible subshifts of finite type. They proved, in particular, that an endomorphism of an irreducible subshift of finite type is surjective if and only if it preserves the measure of maximal entropy on the subshift. Hence, a cellular automaton is surjective if and only if it preserves the equidistributed Bernoulli measure on the full shift, and we can consider a surjective cellular automaton as an endomorphism of a Borel measure space. The ergodic properties of general surjective cellular automaton seem currently intractable. Therefore, it is of interest to have at least certain classes of such maps whose dynamics can be understood. These have been provided in Chapter 1 of the thesis. The right/left permutative cellular automata are proved to be k-mixing for any $k \geq 1$. Under some additional assumption we prove them to have Bernoullian natural extension. For a surjective cellular automaton T such that $(Tx)_i$ depends only upon the coordinates

x_j with $j > i$ (or only upon the coordinates x_j with $j < i$) we prove that its natural extension has K-property. The results of Chapter 1 answer some questions raised in [ShR].

The study of cellular automata is closely connected with another interesting field of modern ergodic theory - multi-dimensional subshifts (in particular, of finite type). 'The study of multidimensional Markov shifts, while still at the early stage is bound to grow in importance...'(Jack Feldman). A survey of this fascinating topic with no indication yet of a satisfactory general theory is given by K. Schmidt in [Sch1].

In Chapter 2 we deal with automorphisms and endomorphisms of general subshifts of positive entropy of the d-dimensional full shift. We consider the continuous actions of a finitely generated (semi-) group of polynomial growth by such automorphisms (endomorphisms) and address the question of how the degree of polynomial growth of G can affect the dynamical properties of the action. It turns out that such an action cannot be expansive, if the degree of polynomial growth of G is smaller than d, and must have positive entropy if the degree is precisely d. This implies, in particular, that, firstly, there does not exist an expansive automorphism (endomorphism) of a d-dimensional subshift of positive entropy if $d > 1$, and, secondly, if a one-dimensional subshift admits an expansive automorphism (endomorphism) with zero entropy, then the subshift itself has zero entropy. These statements are somewhat complementary to the result due to A. Fathi saying that if a compact topological space admits an expansive homeomorphism with zero entropy, then its topological dimension is zero.

An important class of two-dimensional ssft (of zero entropy) is those that can be obtained by taking the inverse limit of a surjective cellular automaton. For instance, given a finite group G with the identity e, we define the ssft

$X_G = \{ x = (x_{(i,j)})_{(i,j)\in\mathbb{Z}^2} \in G^{\mathbb{Z}^2} : x_{(i,j)}\, x_{(i+1,j)}\, x_{(i,j+1)} = e \text{ for all } (i,j) \in \mathbb{Z}^2 \}$. The Haar measure on G gives rise to a well defined shift-invariant probability measure μ_G on X_G. When G is abelian, X_G is a subgroup of the full shift $G^{\mathbb{Z}^2}$ and μ_G is identical to the Haar measure on X_G. The following question attributed to H. Furstenberg is formulated in [Sch1, p. 61]: do there exist two algebraically non-isomorphic finite groups G_1 and G_2 with $|G_1| = |G_2|$ such that the shift $\mathbb{Z}^2$-actions on (X_{G_1}, μ_{G_1}) and (X_{G_2}, μ_{G_2}) are measurably isomorphic ? We address this question in Chapter 3 and prove that the answer is *no*, if G_1 and G_2 are abelian p-groups with $E(G_1) \neq E(G_2)$, where $E(G)$ is the least common multiple of orders of elements of G. This result is complementary to that of T. Ward [War]. In Chapter 3 we also give the complete topological classification of subshifts X_G with G abelian by showing that the shift $\mathbb{Z}^2$-actions on (X_{G_1}, μ_{G_1}) and (X_{G_2}, μ_{G_2}) are topologically isomorphic if and only if $G_1 \cong G_2$.

In Chapter 4 we consider another class of two-dimensional ssft with zero entropy. Let $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. The character group of the compact 0-dimensional group $(\mathbb{Z}_p)^{\mathbb{Z}^2}$ can be canonically identified with the ring $\mathbb{Z}_p[u^{\pm1}, v^{\pm1}]$ of Laurent polynomials in two commuting variables with coefficients from the finite field $\mathbb{Z}_p$. For any ideal $I \subset \mathbb{Z}_p[u^{\pm1}, v^{\pm1}]$ we denote by X_I the annihilator $I^{\perp}$ of I. It is easily seen that X_I is a ssft and, simultaneously, a subgroup in the full shift $(\mathbb{Z}_p)^{\mathbb{Z}^2}$. The first example of this kind (with $p = 2$, $I = \langle 1 + u + v \rangle$) was considered by F. Ledrappier in [L]. Recently this class of ssft have been studied by B. Kitchens and K. Schmidt in [KS1], [KS2] (see also [LSW]) who call them *Markov subgroups*. Some conjugacy invariants for Markov subgroups $X_{\langle f\rangle}$, $f \in \mathbb{Z}_p[u^{\pm1}, v^{\pm1}]$, where $\langle f\rangle$ is the principal ideal generated by f, have been found in [KS1], [KS2]. It was shown, in particular, that the convex hull of the support $S(f)$ of a polynomial $f \in \mathbb{Z}_p[u^{\pm1}, v^{\pm1}]$ is a measurable (and, hence, topological) invariant for the Markov subgroup $X_{\langle f\rangle}$. But the invariants of B. Kitchens and K. Schmidt do not distinguish between the Markov subshifts arising from the polynomials with identical (up to a translation) supports. In Chapter 4 we give a complete classification (with respect to measurable isomorphism) of Markov subgroups $X_{\langle f\rangle}$ with $S(f) = \{(0,0), (1,0), (0,1)\}$. Namely, we prove that $X_{\langle f_1\rangle}$ and $X_{\langle f_2\rangle}$ with $S(f_1) = S(f_2) = \{(0,0), (1,0), (0,1)\}$ are measurably isomorphic, if and only if the ideals $\langle f_1\rangle$ and $\langle f_2\rangle$ are identical.

CHAPTER 1. ERGODIC PROPERTIES OF CERTAIN SURJECTIVE CELLULAR AUTOMATA

1.1 INTRODUCTION

Let $P = \{0, 1, \dots, p-1\}$ with some integer $p > 1$, and let $X = P^{\mathbb{Z}}$ (i. e. X is the space of all doubly-infinite sequences $x = (x_i)_{i \in \mathbb{Z}}$, $x_i \in P$). Equip X with the product topology. Then X is a totally disconnected compact space. A continuous map $T: X \to X$ commuting with the shift τ defined by $(\tau x)_i = x_{i+1}$, $i \in \mathbb{Z}$, is called a (one-dimensional) *cellular automaton*. Here, for the sake of brevity, the cellular automata will often be referred to as CA-maps. Dynamical systems of this type are of great importance for many applications and have been undergoing extensive numerical exploration during the last decade (see e. g. [Wo]). Mathematical study of cellular automata was initiated by Hedlund and coworkers [H] and then continued in various directions by Coven and Paul [C], [CP], Gilman [G], Hurley [Hu], Lind [Li], Shirvany and Rogers [ShR] and others.

From the viewpoint of ergodic theory among the cellular automata the surjective ones seem to be of special interest for they always preserve the equidistributed Bernoulli measure μ on X (this fact was first observed in [CP]). Remark that μ is exactly the normalized Haar measure on X considered as the compact abelian group $(\mathbb{Z}_p)^{\mathbb{Z}}$. Here we prove strong ergodic properties (such as Bernoulli and Kolmogorov properties) of the natural extension of the measure theoretic endomorphism $(X, \mathcal{B}, \mu, T)$ for some classes of surjective cellular automata T ($\mathcal{B}$ stands for the Borel σ-algebra on X). We generalize a result of Ledrappier [L] about mixing of a 2-dimensional subshift of finite type defined by a cellular automaton. Some questions raised in [ShR] are answered in this chapter.

1.2 PRELIMINARIES

It is well known that every CA-map can be represented as a 'sliding-block code'.

Proposition 1.2.1 ([H, Theorem 3.4]). A map $T: X \to X$ is a cellular automaton, if and only if there exist $l, r \in \mathbb{Z}$ with $l \leq r$ and a mapping $F: P^{r-l+1} \to P$ such that

$$(Tx)_j = F(x_{j+l}, \dots, x_{j+r}) \tag{1.1}$$

for every $x \in P^{\mathbb{Z}}$ and every $j \in \mathbb{Z}$.

The mapping F is usually called *the (generating) rule* of the CA-map. To emphasize the generating rule F and the numbers l and r we shall use the notation $T_{F[l,r]}$ for the CA-map defined by (1.1). Notice that $T_{F[l+i,r+i]} = \tau^i \circ T_{F[l,r]} = T_{F[l,r]} \circ \tau^i$, $i \in \mathbb{Z}$.

For any $s \geq 0$ one can 'extend' a mapping $F: P^m \to P$ to the mapping $F_s: P^{m+s} \to P^{1+s}$ defined by $F_s(x_1, \dots, x_{s+m}) = (y_1, \dots, y_{1+s})$, where $y_j = F(x_j, \dots, x_{j+m-1})$, $1 \leq j \leq 1+s$. The surjectivity property of CA-maps admits a nice characterization in terms of the sets $F_s^{-1}(y_1, \dots, y_{1+s})$.

Proposition 1.2.2 ([H, Theorem 5.4].) The CA-map $T_{F[l,r]}: X \to X$ is surjective if and only if $\operatorname{card}(F_s^{-1}(y_1, \dots, y_{1+s})) = p^{m-1}$ for all $s \geq 0$ and all $(y_1, \dots, y_{1+s}) \in P^{s+1}$.

From this one gets the fact we have already mentioned.

Proposition 1.2.3 (cf. [CP, Theorem 2.1]). The CA-map $T_{F[l,r]}$ is surjective if and only if it preserves the measure μ, i.e. $\mu(T_{F[l,r]}^{-1} A) = \mu(A)$ for every $A \in \mathcal{B}$.

Given a mapping $F: P^m \to P$ and a block $(\hat{x}_1, \dots, \hat{x}_{m-1}) \in P^{m-1}$ define a map $F^+_{(\hat{x}_1, \dots, \hat{x}_{m-1})}: P \to P$ by setting

$$F^+_{(\hat{x}_1, \dots, \hat{x}_{m-1})}(x_m) = F(\hat{x}_1, \dots, \hat{x}_{m-1}, x_m).$$

Likewise, for any $(\hat{x}_2, \dots, \hat{x}_m) \in P^{m-1}$ we put

$$F^{-}_{(\hat{x}_2,\dots,\hat{x}_m)}(x_1) = F(x_1,\hat{x}_2,\dots,\hat{x}_m)\ .$$

We say the rule is *right (resp., left) permutative,* if the mapping $F^{+}_{(\hat{x}_1,\dots,\hat{x}_{m-1})}$ (resp., $F^{-}_{(\hat{x}_2,\dots,\hat{x}_m)}(x_1)$) is a permutation of P for every $(\hat{x}_1,\dots,\hat{x}_{m-1}) \in P^{m-1}$ (resp., $(\hat{x}_2,\dots,\hat{x}_m) \in P^{m-1}$). This terminology is, essentially, due to Hedlund [H].

Proposition 1.2.4 ([H, Theorem 6.6]). If the rule $F: P^{r-l+1} \to P$ is right or left permutative, then the CA-map $T_{F[l,r]}: X \to X$ is surjective.

Hence, any cellular automaton generated by a rule which is right or left permutative preserves μ.

A rule $F: P^m \to P$ can be iterated in the following way. Define inductively the k-th iteration $F^k: P^{k(m-1)+1} \to P$ of the rule F

$$F^k(x_1,\dots,x_m,\dots,x_{m+(m-1)},\dots,x_{k(m-1)+1}) =$$
$$= F^{k-1}(F(x_1,\dots,x_m), F(x_2,\dots,x_{m+1}),\dots, F(x_{(k-1)(m-1)+1},\dots,x_{k(m-1)+1}))\ . \quad (1.2)$$

The permutativity property is preserved by iterating the rule.

Lemma 1.2.5. If the rule $F: P^m \to P$ is right (left) permutative, then so is its k-th iteration $F^k: P^{k(m-1)+1} \to P$ for each integer $k \geq 1$.

Proof. The lemma is immediately proved by induction on k. It suffices to observe that

$$\left(F^{k+1}\right)^{+}_{(\hat{x}_1,\hat{x}_2\dots,\hat{x}_{(k+1)(m-1)})}(x_{(k+1)(m-1)+1}) =$$
$$= \left(F^{k}\right)^{+}_{F(\hat{x}_1,\dots,\hat{x}_m)\dots F(\hat{x}_{k(m-1)},\dots,\hat{x}_{(k+1)(m-1)})}\left(F(\hat{x}_{k(m-1)+1},\dots,\hat{x}_{(k+1)(m-1)},x_{(k+1)(m-1)+1})\right)$$

□

The k-th iteration of F turns out to generate the k-th iteration of the CA-map generated by F.

Lemma 1.2.6. The k-th iteration $T^k_{F[l,r]}$ of the CA-map $T_{F[l,r]}$ generated by the rule F coincides with the CA-map $T_{F^k[kl,kr]}$.

Proof. Immediately follows from (1.1) and (1.2). □

1.3 RESULTS

In [SA] we considered the cellular automata with bipermutative (i. e. both right and left permutative) rules and proved them to be topologically equivalent to one-sided Bernoulli shifts .

Theorem 1.3.1 ([SA]). If $l < 0 < r$ and F is bipermutative, then $(X, T_{F[l,r]})$ is topologically conjugate to the one-sided full shift $(\Sigma^{+}_{(p^{|l|+r})}, \sigma^{+})$ and the equidistributed Bernoulli measure μ is the (unique) measure of maximal entropy for $(X, T_{F[l,r]})$.

This result has been generalized by M. Nasu [N].

It is not hard to give a counter-example showing that the statement of Theorem 1.3.1 is no longer true if we only demand F to be right *or* left permutative. For instance, let $p = 2$ and $F(x_{-1}, x_0, x_1) = x_{-1} + x_0 x_1 \pmod 2$. Then it is easy to check that the CA-map $T_{F[l,r]}$ is not expansive (see [DGS] or Chapter 2 for the definition) for any l, r ($r - l = 2$) and, therefore, cannot be topologically conjugate to a one-sided Bernoulli shift. Here we prove that the measure-theoretic conjugacy to a Bernoulli shift still holds, if we put the additional condition $0 \le l < r$ (resp. $l < r \le 0$).

Since a surjective cellular automaton $T_{F[l,r]}$ is, in general, non-invertible, we will consider the natural extension $(\hat{X}, \mathcal{B}, \hat{\mu}, \hat{T}_{F[l,r]})$ of the endomorphism $(X, \mathcal{B}, \mu, T_{F[l,r]})$. *The natural extension* $(\hat{M}, \hat{\mathcal{A}}, \hat{\nu}, \hat{T})$ of the endomorphism $(M, \mathcal{A}, \nu, T)$ is defined as follows (see, e. g. [CFS]). Let

$$\hat{M} = \{ \hat{x} = (x^{(0)}, x^{(1)}, x^{(2)}, \dots) : \ x^{(m)} \in X, \ T x^{(m+1)} = x^{(m)} \text{ for all } m \in \mathbb{Z}_+ \},$$

$\hat{\mathcal{A}}$ be the σ-algebra generated by the sets of the form $C^{(m)} = \{ \hat{x} \in \hat{M} : x^{(m)} \in C \}$, where $C \in \mathcal{A}$, $m \ge 0$, and define the measure $\hat{\nu}$ by setting $\hat{\nu}(C^{(m)}) = \nu(C)$. Now the automorphism $\hat{T}$ of the measure space $(\hat{M}, \hat{\mathcal{A}}, \hat{\nu})$ is defined by

$$\hat{T}(x^{(0)}, x^{(1)}, x^{(2)}, \dots) = (Tx^{(0)}, Tx^{(1)}, Tx^{(2)}, \dots) = (Tx^{(0)}, x^{(0)}, x^{(1)}, \dots).$$

Recall that the automorphism $(M, \mathcal{A}, \nu, T)$ is said to be *Bernoulli*, if it is isomorphic to a Bernoulli shift (see [O]).

Theorem 1.3.2. Suppose that at least one of the following conditions is satisfied :

(RP) $0 \le l < r$ and the rule $F: P^{r-l+1} \to P$ is right permutative ;

(LP) $l < r \le 0$ and the rule $F: P^{r-l+1} \to P$ is left permutative ;

Then the natural extension $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T}_{F[l,r]})$ of the dynamical system $(X, \mathcal{B}, \mu, T_{F[l,r]})$ is a Bernoulli automorphism.

Recall that the dynamical system (M, $\mathcal{A}$, ν, T) is k-mixing ($k \ge 1$), if for $A_0, A_1, \dots, A_k \in \mathcal{A}$ we have

$$\lim_{n_1, n_2, \dots, n_k \to \infty} \nu (A_0 \cap T^{-n_1} A_1 \cap \dots \cap T^{-(n_1 + \dots + n_k)} A_k) =$$

$$= \nu(A_0)\, \nu(A_1) \dots \nu(A_k). \qquad (1.3)$$

If we omit the conditions $l \ge 0$ ($r \le 0$) of the Theorem 1.3.2 we still can prove the k-mixing for all $k \ge 1$:

Theorem 1.3.3. Suppose that at least one of the following conditions is satisfied :

(RP') $0 < r$ and the rule $F: P^{r-l+1} \to P$ is right permutative ;

(LP') $l < 0$ and the rule $F: P^{r-l+1} \to P$ is left permutative ;

Then the dynamical system $(X, \mathcal{B}, \mu, T_{F[l,r]})$ is k-mixing for all $k \ge 1$.

The automorphism ($\hat{M}$, $\hat{\mathcal{A}}$, $\hat{\nu}$, $\hat{T}$) is said to be a *K-automorphism* (see [CFS] for details), if there exists a sub-σ-algebra $\mathcal{K}$ of $\hat{\mathcal{A}}$ such that the following conditions are satisfied

$$\text{(K)} \qquad \mathcal{K} \subset \hat{T}\mathcal{K}, \qquad \bigvee_{n=0}^{\infty} \hat{T}^{n}\mathcal{K} = \hat{\mathcal{A}}, \qquad \bigcap_{n=0}^{\infty} \hat{T}^{-n}\mathcal{K} = \mathcal{N}(\hat{M}),$$

where $\mathcal{N}(Y)$ denotes the trivial σ-algebra on Y (the inclusion and the equalities are meant to hold modulo null-sets).

Theorem 1.3.4. If $0 < l \le r$ or $l \le r < 0$ and the CA-map $T_{F[l,r]}$ is surjective, then the natural extension $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T}_{F[l,r]})$ of the dynamical system $(X, \mathcal{B}, \mu, T_{F[l,r]})$ is a K-automorphism.

In view of the classical result of Rohlin [R], Theorem 1.3.4 yields the following

Corollary 1.3.5. Under the conditions of Theorem 2.4 the dynamical system $(X, \mathcal{B}, \mu, T_{F[l,r]})$ *is k-mixing for all* $k \geq 1$.

Under the additional assumption $p = 2$, Shirvani and Rogers proved in [ShR] the 1-mixing property for the classes of CA-maps considered in the theorems above. Our Theorem 1.3.3 and Corollary 1.3.5 answer in the affirmative the question posed in [ShR] whether these systems are k-mixing for all $k \geq 1$.

It was also conjectured in [ShR] that every surjective CA-map, except those of the form $(T x)_i = \pi(x_i)$, where $\pi : P \to P$ is a permutation, is ergodic with respect to μ. We remark that this is not the case. The counter example (see Proposition 1.3.6 below) was actually considered in [C], but in a slightly different context and without indication of its non-ergodicity.

Proposition 1.3.6. Let $p = 2$. *The surjective CA-map* $T_{F[0,2]}$ *generated by the rule* $F(x_0, x_1, x_2) = x_0 + x_1(x_2 + 1) \pmod 2$ *is non-ergodic with respect to the measure* μ.

Proof. Since F is left permutative, the surjectivity of the CA-map follows from Proposition 1.2.4. From [C] it follows that $T_{F[0,2]}$ is not topologically transitive. Hence, it is not ergodic with respect to μ, because $\operatorname{supp} \mu = X$. $\square$

Every cellular automaton $T_{F[l,r]}$ gives rise in a natural way to a certain two-dimensional subshift of finite type (see [Sch1] and Chapter 4 for the definition). Namely, define

$$\Omega_{F[l,r]} = \Big\{ \omega = (\omega_{(i,j)})_{(i,j) \in \mathbb{Z}^2} : \\ \omega_{(i,j)} \in P,\ \omega_{(i,j+1)} = F(x_{(i+l,j)}, \dots, x_{(i+r,j)}),\ \text{for all } (i,j) \in \mathbb{Z}^2 \Big\}.$$

The set $\Omega_{F[1,r]}$ is a closed subset of $P^{\mathbb{Z}^2}$ invariant under the two shifts $\tau_H, \tau_V : P^{\mathbb{Z}^2} \to P^{\mathbb{Z}^2}$ defined by $(\tau_H x)_{(i,j)} = x_{(i+1,j)}$, $(\tau_V x)_{(i,j)} = x_{(i,j+1)}$, $(i,j) \in \mathbb{Z}^2$. Obviously, the space $\Omega_{F[1,r]}$ may be identified with the inverse limit $\hat{X} = \varprojlim\left(X, T_{F[1,r]}\right)$ which is just the space of the natural extension considered above. This identification enables us to turn the subshift $\Omega_{F[1,r]}$ into the measure space $(\Omega_{F[1,r]}, \hat{\mathcal{B}}, \hat{\mu})$. The shifts τ_H and τ_V generate a measure preserving $\mathbb{Z}^2$-action $\mathcal{T}$ on $(\Omega_{F[1,r]}, \hat{\mathcal{B}}, \hat{\mu})$ defined by $(m,n) \mapsto \mathcal{T}^{(m,n)} = \tau_H^m \circ \tau_V^n$. Recall, that the $\mathbb{Z}^2$-action τ is mixing precisely if $\lim_{\|(m,n)\| \to \infty} \hat{\mu}(A_0 \cap \mathcal{T}^{(m,n)} A_1) = \hat{\mu}(A_0)\hat{\mu}(A_1)$ for $A_0, A_1 \in \hat{\mathcal{B}}$. For a general discussion of mixing group actions see e. g. [Sch2].

Theorem 1.3.7. *If the rule F is bipermutative and $r > 1$, then the $\mathbb{Z}^2$-action $\mathcal{T}$ on $(\Omega_{F[1,r]}, \hat{\mathcal{B}}, \hat{\mu})$ is mixing and the automorphism $\tau_H^m \circ \tau_V^n$ is Bernoulli for all $(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}$.*

Note that Theorem 1.3.7 generalizes Ledrappier's result [L], which established the mixing property for the $\mathbb{Z}^2$-action $\mathcal{T}$ on $(\Omega_{F[0,1]}, \hat{\mathcal{B}}, \hat{\mu})$ with p = 2 and $F(x_0, x_1) = x_0 + x_1 \pmod 2$.

1.4 PROOFS OF THE THEOREMS

First of all we recall some definitions and facts from the theory of Bernoulli automorphisms (see [O] for details).

The partitions $\xi = \{C_i\}$ and $\eta = \{D_j\}$ of the measure space $(M, \mathcal{A}, \nu)$ are said to be *ε-independent* $(\varepsilon \geq 0)$, if $\sum_{i,j} |\nu(C_i \cap D_j) - \nu(C_i)\nu(D_j)| \leq \varepsilon$. The partitions are *independent,* if they are 0-independent. Let T be an automorphism of the space $(M, \mathcal{A}, \nu)$. A partition $\xi = \{C_i\}$ is said to be *Bernoulli* for T, if all its shifts $T^n \xi = \{T^n C_i\}$ are pairwise independent. A partition $\xi = \{C_i\}$ is *weakly Bernoulli* for T, if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that the partitions $\bigvee_{k=-n}^{0} T^{k}\xi$ and $\bigvee_{k=N}^{N+n} T^{k}\xi$ are ε-independent for all $n \geq 0$. The automorphism T is Bernoulli, if and only if it has a generator ξ which is (weakly) Bernoulli for T.

The following theorem of D. Ornstein enables us to establish the Bernoullicity without finding a (weakly) Bernoulli generator.

Theorem 1.4.1 (see [O]). Let $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n \leq \ldots$ *be an increasing sequence of finite partitions of the measure space* $(M, \mathcal{A}, \nu)$ *with* $\bigvee_{n=0}^{\infty} \xi_n = \varepsilon$ *(ε stands for the partition of M into singletons) and let* ξ_n *be weakly Bernoulli for the automorphism T of this space. Then T is Bernoulli.*

Now we are going to construct such a sequence of finite weakly Bernoulli partitions for the natural extension of the system $(X, \mathcal{B}, \mu, T_{F[l,r]})$, where the CA-map $T_{F[l,r]}$ satisfies the conditions of Theorem 1.3.2.

Let the automorphism $(\hat{M}, \hat{\mathcal{A}}, \hat{\nu}, \hat{T})$ be the natural extension of the endomorphism $(M, \mathcal{A}, \nu, T)$. Given a partition ξ of the space $(M, \mathcal{A}, \nu)$ we define the partition $\xi^{(m)}$ of the space $(\hat{M}, \hat{\mathcal{A}}, \hat{\nu})$ to be the one consisting of the sets $C^{(m)}$ (see § 1.3) with $C \in \xi$.

Given integers $i_- \leq i_+$ and a word $(x_{i_-}, \ldots, x_{i_+}) \in P^{i_+ - i_- + 1}$ we define the cylinder set $C_{i_-}[x_{i_-}, \ldots, x_{i_+}] = \{ y \in X : y_j = x_j \text{ for all } i_- \leq j \leq i_+ \}$. Let $\xi(i_-, i_+)$ denote the partition of X into the sets of the form $C_{i_-}[x_{i_-}, \ldots, x_{i_+}]$, (cylinder sets) where $(x_{i_-}, \ldots, x_{i_+})$ runs over the set $P^{i_+ - i_- + 1}$.

Lemma 1.4.2. If a rule $G : P^{R-L+1} \to P$ *is right (resp. left) permutative, then the partitions* $\xi(i_1, j_1)$ *and* $T_{G[L,R]}^{-1}\xi(i_2, j_2)$ *are independent, whenever* $R > j_1 - i_2$ *(resp.* $L < i_1 - j_2$ *).*

Proof. We shall carry out the proof for a right permutative rule. The other case is treated exactly in the same manner. Let $i = \min\{ i_1 , i_2 + L \}$. It follows from the right permutativity of F that for any words $(x_i , \dots , x_{i_2+R-1}) \in P^{i_2+R-i}$ and $(v_{i_2} , \dots , v_{j_2}) \in P^{j_2 - i_2 + 1}$ we have

$$C_i [x_i , \dots , x_{i_2+R-1}] \cap T^{-1}_{G[L,R]}(C_{i_2} [v_{i_2} , \dots , v_{j_2}]) = \\ = C_i [x_i , \dots , x_{i_2+R-1}, x_{i_2+R} , \dots , x_{j_2+R}] ,$$

where x_{t+R}, $i_2 \le t \le j_2$ are defined inductively by

$$x_{t+R} = \left(G^{+}_{(x_{t+L} ,\dots, x_{t+R-1})} \right)^{-1} (v_t) .$$

Hence,

$$\mu \left(C_i [x_i , \dots , x_{i_2+R-1}] \cap T^{-1}_{G[L,R]} (C_{i_2} [v_{i_2} , \dots , v_{j_2}]) \right) = (1/p)^{j_2 + R - i + 1} .$$

Since $i_2 + R - 1 \ge j_1$, the set $C_{i_1}[u_{i_1} , \dots , u_{j_1}]$ for each $(u_{i_1} , \dots , u_{j_1}) \in P^{j_1 - i_1 + 1}$ is a disjoint union of $p^{i_2 + R - 1 - j_1 + i_1 - i}$ sets of the form $C_i [x_i , \dots , x_{i_2+R-1}]$. This implies

$$\mu \left(C_{i_1} [u_{i_1} , \dots , u_{j_1}] \cap T^{-1}_{G[L,R]}(C_{i_2} [v_{i_2} , \dots , v_{j_2}]) \right) = (1/p)^{(j_1-i_1+1) + (j_2-i_2+1)} = \\ = \mu (C_{i_1} [u_{i_1} , \dots , u_{j_1}]) \, \mu (C_{i_2} [v_{i_2} , \dots , v_{j_2}]) .$$

Thus, the partitions $\xi(i_1 , j_1)$ and $T^{-1}_{G[L,R]} \xi(i_2 , j_2)$ are independent.

Lemma 1.4.3. If the rule $F: P^{r-l+1} \to P$ *satisfies the conditions (RP) (resp. (LP)) of Theorem 1.3.4, then the partitions* $\xi(i_1 , j_1)$ *and* $T^{-m}_{F[l,r]} \xi(i_2 , j_2)$ *are independent, whenever* $m r > j_1 - i_2$ *(resp.* $m l < i_1 - j_2$*).*

Proof. This follows from Lemma 1.4.2. To see this it suffices to observe that by Lemmas 1.2.5 and 1.2.6 $T^{m}_{F[l,r]} = T_{F^m [ml, mr]}$ with the right (resp. left) permutative rule F^m . □

Lemma 1.4.4. Suppose that either of the conditions (RP), (LP) holds.

Then for every $i \geq 0$ *the partition* $\xi(-i,\, i)_i = \bigvee_{s=0}^{i} \xi(-i,\, i)^{(s)}$ *of the measure space* $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ *is weakly Bernoulli for the automorphism* $T_{F[l,r]}$.

Proof. Suppose that the condition (RP) holds and let T stand for $T_{F[l,r]}$. We observe that $T^{-1}\xi(i,\, j) \leq \xi(i+l,\, j+r)$ for all $i \leq j$. As $l \geq 0$, we have $T^{-1}\xi(i,j) \leq \xi(i,j+r)$ and, hence, $\bigvee_{k=0}^{n} T^{-k}\xi(i,j) \leq \xi(i,j+nr)$.

For any $n \geq 0$ we have

$$\bigvee_{k=-n}^{0} T^{k}\xi_i(-i,i) = \bigvee_{k=-n}^{0} T^{k}\Big(\bigvee_{s=0}^{i} \xi(-i,i)^{(s)}\Big) =$$

$$\bigvee_{k=0}^{n} T^{-k}\Big(\bigvee_{s=0}^{i} T^{-s}\xi(-i,i)^{(i)}\Big) = \bigvee_{j=0}^{n+i} T^{-j}\xi(-i,i)^{(i)} \geq$$

$$\leq \xi(-i,i+r(n+i))^{(i)}, \qquad (1.4)$$

$$\bigvee_{k=N}^{N+n} T^{k}\xi(-i,i)_i = \bigvee_{k=N}^{N+n} T^{k}\Big(\bigvee_{s=0}^{i} \xi(-i,i)^{(i)}\Big) =$$

$$= \bigvee_{j=N}^{N+n+i} \xi(-i,i)^{(j)} \leq \xi(-i,i+r(n+i))^{(N+n+i)} . \qquad (1.5)$$

From Lemma 1.4.3 it follows that the partitions $\xi(i_1,j_1)^{(m)}$ and $\xi(i_2,j_2)^{(m')}$ of the measure space $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$ are independent, whenever $(m-m')r > j_1 - i_2$. Therefore, if $Nr > i(2+r)$, then $\xi(-i,i+r(n+i))^{(i)}$ and $\xi(-i,i+r(n+i))^{(N+n+i)}$ are independent for all $n \geq 0$. It is known that if the partitions η_1 and η_2 are independent and $\zeta_k \leq \eta_k$, $k = 1,2$, then ζ_1 and ζ_2 are independent. Now from (1.4) and (1.5) we conclude that for all $i \geq 0$ there exists $N > 0$ such that for every $n \geq 0$ the partitions $\bigvee_{k=-n}^{0} T^{k}\xi(-i,i)_i$ and $\bigvee_{k=N}^{N+n} T^{k}\xi(-i,i)_i$ are independent. Thus, for every $i \geq 0$ the partition

$\xi(-i,i)_i$ is weakly Bernoulli for the automorphism $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T}_{F[l,r]})$.

We have proved the lemma in the case (RP). In the case when (LP) holds the proof is exactly the same, but 'left-sided'. □

Proof of Theorem 1.3.2. It is easily seen that $\xi(-i,i)_i \le \xi(-i+1, i+1)_{i+1}$ for all $i \ge 0$ and $\bigvee_{i=0}^{\infty} \xi(-i,i)_i = \varepsilon$. Thus, by virtue of Lemma 1.4.4, we are under the conditions of Theorem 1.4.1 which yields the Bernoullicity of the automorphism $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu}, \hat{T}_{F[l,r]})$. □

Proof of Theorem 1.3.3. We consider the case (RP'). Throughout the proof T stands for $T_{F[l,r]}$. To prove that T is k-mixing it is sufficient to verify (1.3) for cylinder sets $A_0, A_1, \dots, A_k$ (see [Bi, Theorem 1.2]). In fact, we show that for any $A_0 = C_{i_0}[x^0_{i_0}, \dots, x^0_{j_0}]$, $A_1 = C_{i_1}[x^1_{i_1}, \dots, x^1_{j_1}], \dots$ $\dots, A_k = C_{i_k}[x^k_{i_k}, \dots, x^k_{j_k}]$, there exists $m > 0$ such that the equality

$$\mu(A_0 \cap T^{-n_1} A_1 \cap \dots \cap T^{-(n_1 + \dots + n_k)} A_k) = \mu(A_0)\mu(A_1)\dots\mu(A_k)$$

holds for all $n_1, n_2, \dots, n_k > m$.

Let m be a positive number greater than $\max\{(j_{s-1} - i_s)/r : 1 \le s \le k\}$. Take arbitrary integers $n_1, n_2, \dots, n_k > m$ and denote $N_s = \sum_{j=1}^{s} n_j$ for $1 \le s \le k$ and $N_0 = 0$. Then we have $i_s + N_s r > j_{s-1} + N_{s-1} r$, $1 \le s \le k$. We also put $i_- = \min\{ i_s + l N_s : 0 \le s \le k \}$ and $j_+ = j_k + r N_k$. Using Lemmas 1.2.5' and 1.2.6 one checks easily that the set $A_0 \cap T^{-n_1} A_1 \cap \dots \cap T^{-(n_1 + \dots + n_k)} A_k$ is the union of cylinder sets $C_{i_-}[y_{i_-}, \dots, y_{j_+}]$ taken over all blocks $(y_{i_-}, \dots, y_{j_+}) \in P^{j_+ - i_- + 1}$ satisfying the following conditions:

(C) $\quad y_t = x^0_t$ for $i_0 \le t \le j_0$;

$$y_t = \left((F^{N_s})^+_{y_{t-N_s(r-1)} \cdots y_{t-1}} \right)^{-1} (x^s_{t-N_s l})$$

for $i_s + N_s r \le t \le j_s + N_s r$, $1 \le s \le k$.

Since the segments $[\, i_s + N_s r,\ j_s + N_s r\,]$, $s = 0, 1, \dots, k$ do not overlap each other, the number of blocks $(y_{i_-}, \dots, y_{j_+})$ satisfying the conditions (C) equals $p^{\,j_+ - i_- + 1 - \Sigma_{s=0}^{k}(j_s - i_s + 1)}$ (because y_t may be chosen arbitrarily if and only if $t \in [\, i_-, j_+\,] \setminus \bigcup_{s=0}^{k} [\, i_s + N_s r,\ j_s + N_s r\,]$). This gives us the desired equality:

$$\mu(A_0 \cap T^{-N_1} A_1 \cap \dots \cap T^{-N_k} A_k) = p^{-\Sigma_{s=0}^{k}(j_s - i_s + 1)} = \mu(A_0)\,\mu(A_1) \dots \mu(A_k).$$

The case (LP′) is proved similarly. □

Proof of Theorem 1.3.4. Throughout the proof we write T for $T_{F[l,r]}$. Suppose that $l > 0$. Let $\xi(i,\infty) = \bigvee_{j=i}^{\infty} \xi(i,j)$ and denote by $\mathcal{C}(i,\infty)^{(m)}$ the σ-sub-algebra of $\hat{\mathcal{B}}$ defined by the measurable partition $\xi(i,\infty)^{(m)}$ of the space $(\hat{X}, \hat{\mathcal{B}}, \hat{\mu})$. Set $\mathcal{K} = \bigvee_{k=0}^{\infty} \hat{T}^{-k} \mathcal{C}(0,\infty)^{(0)}$. We now verify that $\mathcal{K}$ satisfies the conditions (K) for a K-automorphism (see § 1.3). The inclusion $\mathcal{K} \subset \hat{T}\mathcal{K}$ is obvious. Directly from the definition of the CA-map $T = T_{F[l,r]}$ we observe that $\mathcal{C}(0,\infty)^{(m+k)} \supset \hat{T}^{-k} \mathcal{C}(-kl,\infty)^{(m+k)} = \mathcal{C}(-kl,\infty)^{(m)}$ for all $m, k \geq 0$. It follows that

$$\bigvee_{n=0}^{\infty} \hat{T}^{n} \mathcal{K} = \bigvee_{n=-\infty}^{\infty} \hat{T}^{n} \mathcal{C}(0,\infty)^{(0)} \supset \bigvee_{k=0}^{\infty} \bigvee_{m=0}^{\infty} \mathcal{C}(-kl,\infty)^{(m)} = \hat{\mathcal{B}}.$$

(this actually means that any $\omega \in \Omega_{F[l,r]}$ is completely determined by its values in a right half-plane). By the well known Kolmogorov's zero-one law (see [CFS]) for the Bernoulli shift $(X, \mathcal{B}, \mu, \tau)$ we have $\bigcap_{n=0}^{\infty} \mathcal{C}(n,\infty) = \mathcal{N}(X)$. Hence, $\bigcap_{n=0}^{\infty} \mathcal{C}(n,\infty)^{(0)} = \mathcal{N}(\hat{X})$. This yields

$$\bigcap_{n=0}^{\infty} \hat{T}^{-n} \mathcal{K} = \bigcap_{n=0}^{\infty} \bigvee_{k=n}^{\infty} \hat{T}^{-k} \mathcal{C}(0,\infty)^{(0)} \subset \bigcap_{n=0}^{\infty} \mathcal{C}(nl,\infty)^{(0)} = \mathcal{N}(\hat{X})$$

which completes the proof (the case $r < 0$ is treated quite similarly). □

Proof of Theorem 1.3.7. Clearly, it is sufficient to show that if F is bipermutative and $r > 1$, then the action of the semigroup $\mathbb{Z} \times \mathbb{Z}_+$ on the measure space $(X, \mathcal{B}, \mu)$ defined by $(m, n) \mapsto \tau^m \circ T^n_{F[l, r]}$, $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ is mixing, i. e.

$$\lim_{\max\{|m|, n \to \infty\}} \mu(A_0 \cap T^{-n}_{F[l, r]} \tau^m A_1) = \mu(A_0)\, \mu(A_1) \qquad (1.6)$$

for all A_0, $A_1 \in \mathcal{B}$. In the proof of Lemma 1.4.2 we showed that for any two cylinder sets $A_0 = C_{i_0}[u_{i_0}, \dots, u_{j_0}]$ and $A_1 = C_{i_1}[v_{i_1}, \dots, v_{j_1}]$ the equality $\mu(A_0 \cap T^{-1}_{G[L, R]} A_1) = \mu(A_0)\, \mu(A_1)$ holds whenever G is right permutative and $R > j_0 - i_1$ or G is left permutative and $L < i_0 - j_1$. For any $m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ we have $T^n_{F[l, r]} \circ \tau^m = T_{F^n[nl + m, nr + m]}$. Since either $l < 0$ or $r > 0$, for every sequence $(m_k, n_k) \in \mathbb{Z} \times \mathbb{Z}_+$, $k = 1, 2, \dots$ with $\max\{|m_k|, n_k\} \to \infty$ there is $K > 0$ such that either $n_k r + m_k > j_0 - i_1$ or $n_k l + m_k < i_0 - j_1$ holds for all $k > K$. Taking into account that F is both right and left permutative we obtain (1.6) for the case when A_0 and A_1 are cylinder sets. But this is enough, because the σ-algebra $\mathcal{B}$ is generated by such sets.

The Bernoulli property of $\tau_H^m \circ \tau_V^n$ for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ follows from Theorems 1.3.1 and 1.3.2, and from the fact that the automorphism T^{-1} is Bernoulli, whenever T is (see [O]). $\square$

CHAPTER 2.
EXPANSIVENESS, ENTROPY AND POLYNOMIAL GROWTH FOR GROUPS ACTING ON SUBSHIFTS BY AUTOMORPHISMS

2.1 INTRODUCTION

The study of automorphisms and endomorphisms (i. e. continuous shift commuting maps, invertible or non-invertible) of the full shift and its subshifts was begun by Hedlund and coworkers [H] and Coven and Paul [CP]. In this chapter we consider an action of a finitely generated group G of polynomial growth by automorphisms of a subshift of the d-dimensional full shift. If the subshift has positive topological entropy, we find that the degree of polynomial growth of the G may strongly affect certain dynamical properties of the action. More precisely, if growth(G) < d, then the action cannot be expansive. On the other hand, if the growth(G) is exactly d, then an expansive action must have positive topological entropy. The results still hold for finitely generated semigroups of polynomial growth acting by endomorphisms on a subshift with positive entropy. In particular, this implies that, firstly, there are no expansive endo- (or auto-) morphisms of such subshifts with $d > 1$ and, secondly, that if $d = 1$, then any expansive endo- (or auto-) morphism has positive entropy.

The obstruction to expansiveness we have obtained here supplements, to some extent, the following fact established by A. Fathi.

Proposition 2.1.1 *([F, Corollary 5.6]).* If a compact topological space admits an expansive homeomorphism with zero entropy, then its topological dimension is zero.

Related topics are also studied in [BL].

2.2. PRELIMINARIES

Subshifts. Let P be a finite set with cardinality Card(P) > 1. For any $d \geq 1$ we put

$$\Omega_d(P) = \left\{ x = (x_{\mathbf{n}})_{\mathbf{n} \in \mathbb{Z}^d} : x_{\mathbf{n}} \in P \text{ for all } \mathbf{n} \in \mathbb{Z}^d \right\}.$$

In other words, $\Omega_d(P)$ is just the set $P^{\mathbb{Z}^d}$ of all maps $x : \mathbb{Z}^d \to P$. Provide $\Omega_d(P)$ with the product topology. Then $\Omega_d(P)$ is a compact Hausdorff zero-dimensional space. Define, for every $\mathbf{m} \in \mathbb{Z}^d$, a homeomorphism $\tau^{\mathbf{m}} : \Omega_d(P) \to \Omega_d(P)$ by setting $(\tau^{\mathbf{m}} x)_{\mathbf{n}} = x_{\mathbf{n}+\mathbf{m}}$ for all $x \in \Omega_d(P)$ and $\mathbf{n} \in \mathbb{Z}^d$. The correspondence $\mathbf{m} \mapsto \tau^{\mathbf{m}}$ defines a continuous $\mathbb{Z}^d$-action τ on $\Omega_d(P)$. The pair (X, τ_X), where X is a closed τ-invariant subset of $\Omega_d(P)$ and $\tau_X = \tau | X$ is the restriction of the $\mathbb{Z}^d$-action τ to X, is called a *subshift.* A continuous map $T : X \to X$ commuting with τ (i. e. $T \circ \tau^{\mathbf{m}} = \tau^{\mathbf{m}} \circ T$ for all $\mathbf{m} \in \mathbb{Z}^d$) is called an *automorphism of the subshift* if T is invertible. If T is non-invertible it is called an *endomorphism of the subshift.*

Expansiveness. Let X be a compact topological space and let G be a group acting on X by homeomorphisms. We denote the action by T and the homeomorphism corresponding to $g \in G$ by T^g. The action T is said to be *expansive,* if there is a closed neighbourhood $V \subset X \times X$ of the diagonal $\Delta_X = \{ (x, y) \in X \times X : x = y \}$ such that for $\hat{T} = T \times T : X \times X \to X \times X$ we have $\bigcap_{g \in G} \hat{T}^g V = \Delta_X$. If the space X is equipped with a metric ρ, then expansiveness means that there is an *expansive constant* $c > 0$ such that

$$x, y \in X, \; x \neq y \text{ implies } \rho(T^g x, T^g y) > c \text{ for some } g \in G .$$

In a similar way one defines (positive) expansiveness for semigroups acting on a compact space by (non-invertible) continuous maps.

Entropy (see [M-O], [Fe] for details). It is known that a countable group G is *amenable* if and only if it contains a sequence of finite subsets $A_1 \subset A_2 \subset \ldots \subset A_n \subset \ldots \subset G$ with the properties:

(i) $\bigcup_{n \geq 1} A_n = G$; (ii) $\lim_{n \to \infty} |gA_n \Delta A_n| / |A_n| = 0$ for every $g \in G$.

Such a sequence is called *Fölner sequence.* Let T be a continuous action of an amenable group G on a compact topological space X. For a finite open cover ξ of X and a finite set $A \subset G$ let $\xi^A = \bigvee_{g \in A} (T^g)^{-1} \xi$. For a finite open cover η of X let $\mathcal{N}(\eta)$ stand for the minimal cardinality of subcovers of η. Choose a Fölner sequence $\{A_n\}_{n \geq 1}$ in G and a finite open cover ξ of X and define.

$$h(T, \xi) \doteq \limsup_{n \to \infty} |A_n|^{-1} \log \mathcal{N}(\xi^{A_n}) .$$

This value does not depend on the particular Fölner sequence $\{A_n\}_{n \geq 1}$. Then the topological entropy of the action T is defined as

$$h(T) = \sup\{ h(T, \xi) : \xi \text{ is a finite open cover of } X\}.$$

The *topological entropy* for continuous actions of amenable semigroups is defined analogously.

Groups of polynomial growth (see [B], [Gr] for details). Consider a group G generated by a finite subset $F \subset G$. Define the norm $\| g \|_F$ of an element $g \in G$ with respect to F to be the least integer $\ell \geq 0$ such that g can be expressed as a product $f_1 f_2 \dots f_\ell$ with each $f_i \in F \cup F^{-1}$. We denote by $B_F(m)$, $m \geq 1$ the 'closed' ball of radius m, i. e. $B_F(m) = \{ g \in G : \| g \|_F \leq m \}$ and set $\beta_F(m) = \mathrm{Card}(B_F(m))$. Following [B], we say that G has *polynomial growth of degree k* if there exist constants $A, C > 0$ such that $A m^k \leq \beta_F(m) \leq C m^k$ for all $m \geq 1$. It is easily seen that this notion does not depend on the choice of F. In this case we write growth(G) = k. One can show that if G has polynomial growth, then the balls $B_F(m)$, m = 1, 2, ... form a Fölner sequence in G, and, therefore, G is amenable.

For instance, it was proved in [B] (see also [Gr]) that if G is a finitely generated nilpotent group with lower central series $G = G_1 \supset G_2 \supset \dots \supset G_{p+1} = \{e\}$, then growth(G) = $\sum_{q \geq 1} q\, r_q$, where $r_q = \mathrm{rank}(G_q / G_{q+1})$. Clearly, growth($\mathbb{Z}^d$) = d.

In the same manner the degree of polynomial growth can be defined for finitely generated semigroups.

2.3. RESULTS

Let G be a finitely generated (semi-) group with growth(G) = k and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$.

Theorem 2.3.1. If $k = d$, then for any expansive action T of the (semi) group G by automorphisms (endomorphisms) on the subshift X we have $h(T) > 0$.

Theorem 2.3.2. If $k < d$, then any action T of the (semi) group G by automorphisms (endomorphisms) on the subshift X is not expansive.

In the case $G = \mathbb{Z}^k$ (or $\mathbb{Z}_+^k$) the theorems above describe dynamical properties of the joint action of k commuting automorphisms (endomorphisms) of the subshift $X \subset \Omega_d(P)$ according as $k = d$ or $k < d$. In particular for $G = \mathbb{Z}$ we have

Corollary 2.3.3. Let $d = 1$ and $X \subset \Omega_1(P)$ be a subshift with $h(\tau_X) > 0$. Let T be an expansive endomorphism or automorphism of X. Then $h(T) > 0$.

Corollary 2.3.4. Let $d > 1$ and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$. Then there are neither expansive endomorphisms nor expansive automorphisms of X.

We remark that if we drop the condition that the entropy of the subshift X be positive, then the last statement is no longer true. To give a counterexample consider the 2-dimensional subshift of finite type

$$X = \{x \in \Omega_d(\mathbb{Z}/2\mathbb{Z}) : x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} = 0 \ (\mathrm{mod}\ 2),\ \forall (i,j) \in \mathbb{Z}^2\}.$$

This subshift was first studied by Ledrappier [L] (see also [Sch1]) and was shown to have zero entropy ($h(\tau_X) = 0$). On the other hand one readily verifies that the automorphism $T = \tau^{(1,1)}|_X$ and the endomorphism S, defined by $(Sx)_{(i,j)} = x_{(i-1,j-1)} + x_{(i,j)} + x_{(i+1,j+1)}$ are expansive.

Also, it should be mentioned that our results can be reformulated as follows. Let Y be an arbitrary compact zero-dimensional topological space and

$m \mapsto \sigma^m$ be a continuous expansive $\mathbb{Z}^k$-action on Y. Then one can construct a finite topological generator (cf. [DGS], [W]) $\mathcal{U} = \{U_i\}$ with $U_i \cap U_j = \emptyset$ for $i \neq j$, by means of which we define (in the standard way) the topological conjugacy between (Y, τ) and a d-dimensional subshift (X, τ_X). Suppose $h(\sigma) > 0$ and let G again be a finitely generated group with growth(G) = k. From Theorems 2.3.1 and 2.3.2, respectively, we have.

Proposition 2.3.5. *Suppose $k = d$. Let T be an expansive continuous action of the (semi-) group G on Y commuting with σ. Then $h(T) > 0$.*

Proposition 2.3.6. *Suppose $k < d$. Then any continuous action of the (semi) group G on Y commuting with σ is not expansive.*

2.4. PROOFS OF THEOREMS

The proofs will be carried out for the case where G is a group, the 'non-invertible' case where G is a semigroup is treated similarly.

First we prove the following technical result.

Lemma 2.4.1 (cf. [DGS]; p. 109). Let T be an expansive continuous action of a group G on a compact metric space (X, ρ) and let $c > 0$ be an expansive constant for T. Let $A_1 \subset A_2 \subset ... \subset A_n \subset ... \subset G$ be a sequence of subsets of G such that $\bigcup_{n \geq 1} A_n = G$. Then for any $\delta > 0$ there exists an integer $M = M(\delta) \geq 1$ such that $\rho(x, y) \geq \delta$ implies existence of some $g \in A_M$ satisfying $\rho(T^g x, T^g y) > c$.

Proof. Suppose, on the contrary that for some $\delta > 0$ there exist sequences $x_n, y_n \in X$, $n = 1, 2, ...$ satisfying $\rho(x_n, y_n) \geq \delta$ but with $\rho(T^g x_n, T^g y_n) \leq c$ for all $g \in A_n$. By compactness the sequence $(x_n, y_n) \in X^2$ contains a convergent subsequence. To avoid double indices we assume that the sequence (x_n, y_n) itself converges to a pair $(x, y) \in X^2$. Take any $g \in G$. Then $g \in A_n$, and hence $\rho(T^g x_n, T^g y_n) \leq c$, for all sufficiently large n. Since the action is continuous, we have $\rho(T^g x, T^g y) \leq c$. The last inequality holds for every $g \in G$ which, in view of expansiveness, implies $x = y$. But this is impossible, because we must have $\rho(x, y) \geq \delta$. The contradiction proves the lemma. $\square$

Now we provide the topological space $\Omega_d(P)$ with a metric. For $n = (n_1, \dots, n_d)$ we put $|n| = \max_{1 \leq i \leq d} |n_i|$. Let α be a fixed real number with $0 < \alpha < 1$ and put $L(x, y) = \min\{|n| : x_n \neq y_n\}$ for $x \neq y$. We define the metric ρ_α on $\Omega_d(P)$ by putting

$$\rho_\alpha(x, y) = \begin{cases} \alpha^{L(x,y)}, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Every subshift $X \subset \Omega_d(P)$ is given the induced metric.

Then we introduce a family of partitions of X which will be of substantial use in the proofs of the main results.

Given integers $a_i \leq b_i$, $1 \leq i \leq d$ define the rectangle

$$R(a, b) = \{ n \in \mathbb{Z}^d : a_i \leq n_i \leq b_i \text{ for all } 1 \leq i \leq d \},$$

$a = (a_1, \dots, a_d)$, $b = (b_1, \dots, b_d)$. We introduce the partition $\xi(a, b)$ of the space $\Omega_d(P)$ into compact open sets of the form

$$C(a, b; \hat{u}) = \{ x \in \Omega_d(P) : x_n = u_n \text{ for all } n \in R(a, b) \},$$

where $\hat{u} = (u_n)_{n \in R(a,b)} \in P^{R(a,b)}$. The partition $\xi(a, b)$ induces a partition of the subshift $X \subset \Omega_d(P)$ consisting of the non-empty intersections of X with elements of $\xi(a, b)$. We denote this partition of X by $\xi_X(a, b)$ and its elements by $C_X(a, b; \hat{u})$. Note that the partition may also be considered as an open cover.

Let $\{ R_k = R(a^{(k)}, b^{(k)}) \}_{k=1,2,\dots}$ be an increasing sequence of finite rectangles with $R_k \nearrow \mathbb{Z}^d$. Then, the topological entropy of the $\mathbb{Z}^d$-action τ_X on the subshift X is given by the formula (see [Sch1])

$$h(\tau_X) = \lim_{k \to \infty} (\mathrm{Card}(R_k))^{-1} \log \mathrm{Card}(\xi_X(a^{(k)}, b^{(k)})), \tag{2.1}$$

where the limit always exists and does not depend on the sequence $\{R_k\}$.

In what follows we write $\zeta \leq \eta$ for the partitions ζ and η, if η is finer than ζ and $\bigvee_{j \in J} \eta_j$ for the refinement of the partitions $\{\eta_j\}_{j \in J}$.

Let G be a finitely generated group with a fixed finite system of generators F. The proofs of Theorems 2.3.1 and 2.3.2 are based on the following

Lemma 2.4.2. Let $X \subset \Omega_d(P)$ *be a subshift and* T *be an expansive action of the group G by automorphisms on* X *with the expansive constant* α^{r+1} *with some* $r \in \mathbb{Z}_+$. *Then there exists an integer* $M \geq 1$ *such that*

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m}, \mathbf{m}) \geq \xi_X(-\mathbf{m} - \mathbf{1}, \mathbf{m} + \mathbf{1}) \qquad (2.2)$$

for all $m \geq r$ *(where* $\mathbf{m} = (m, \dots, m)$, $\mathbf{1} = (1, \dots, 1)$ *).*

Proof. Let us apply Lemma 2.4.1 to the action T on the subshift X with the metric ρ_α taking $\delta = c = \alpha^{r+1}$ and $A_n = B_F(n)$, $n = 1, 2, \dots$. Put $M = M(\delta)$. Choose arbitrarily two points $x, y \in X$ lying in different elements of $\xi_X(-\mathbf{r} - \mathbf{1}, \mathbf{r} + \mathbf{1})$. Then $\rho(x, y) \geq \alpha^{r+1}$ and, by Lemma 1, we have $\rho(T^g x, T^g y) > \alpha^{r+1}$ for some $g \in B_F(M)$. This means that $T^g x$ and $T^g y$ lie in different elements of the partition $\xi_X(-\mathbf{r}, \mathbf{r})$. Hence, x and y are in different elements of $\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{r}, \mathbf{r})$, whenever they are in different elements of $\xi_X(-\mathbf{r} - \mathbf{1}, \mathbf{r} + \mathbf{1})$. This implies

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{r}, \mathbf{r}) \geq \xi_X(-\mathbf{r} - \mathbf{1}, \mathbf{r} + \mathbf{1}) .$$

Now we prove (2.2) by induction on m. Suppose that (2.2) holds for some $m \geq r$. Then, since $\tau X = X$, and $T \circ \tau = \tau \circ T$, we have

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m} + \mathbf{j}, \mathbf{m} + \mathbf{j}) \geq \xi_X(-\mathbf{m} + \mathbf{j} - \mathbf{1}, \mathbf{m} + \mathbf{j} + \mathbf{1})$$

for every $\mathbf{j} \in \mathbb{Z}^d$. From this it follows that

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m} - \mathbf{1}, \mathbf{m} + \mathbf{1}) \geq \bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-\mathbf{m} + \mathbf{j}, \mathbf{m} + \mathbf{j}) \geq$$

$$\geq \xi_X(-\mathbf{m} + \mathbf{j} - \mathbf{1}, \mathbf{m} + \mathbf{j} + \mathbf{1}) . \qquad (2.3)$$

for every $j \in \mathbb{Z}^d$ with $|j| = 1$. Obviously, $R(-m-2, m+2) = = \bigcup_{|j|=1} R(-m+j-1, m+j+1)$ and, therefore, $\xi_X(-m-2, m+2) = = \bigvee_{|j|=1} \xi_X(-m+j-1, m+j+1)$. From this and (2.3) we have

$$\bigvee_{g \in B_F(M)} (T^g)^{-1} \xi_X(-m-1, m+1) \geq \xi_X(-m-2, m+2).$$

So, we have derived (2.2) for $m+1$ which completes the proof. □

Lemma 2.4.2 can be strengthened as follows.

Lemma 2.4.3. Under the conditions of Lemma 2.4.2 we have

$$\bigvee_{g \in B_F(sM)} (T^g)^{-1} \xi_X(-m, m) \geq \xi_X(-m-s, m+s) \tag{2.4}$$

for all $m \geq r$ and all $s \geq 1$ (where $s = (s, \dots, s)$).

Proof. First we observe that for any $a, b \in \mathbb{N}$ we have
$B_F(a) B_F(b) = \{ g_1 g_2 \in G : g_1 \in B_F(a),\ g_2 \in B_F(b) \} = B_F(a+b)$,
in particular, $B_F((s+1)M) = B_F(sM) B_F(M)$. Using Lemma 2.4.2 we write

$$\bigvee_{g \in B_F((s+1)M)} (T^g)^{-1} \xi_X(-m, m) =$$

$$= \bigvee_{g \in B_F(sM)} (T^g)^{-1} \Big(\bigvee_{h \in B_F(M)} (T^h)^{-1} \xi_X(-m, m) \Big) \geq$$

$$\geq \bigvee_{g \in B_F(sM)} \xi_X(-m-1, m+1),$$

and (2.4) follows by induction on s. □

Now the proofs of the main theorems are almost immediate.

Proof of Theorem 2.3.1. Let G be a group generated by a finite set $F \subset G$ with growth$(G) = d$ and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$. Let T be an expansive action of G by automorphisms on X. We have $A m^d \le \beta_F(m) \le C m^d$, $m \ge 1$ with some constants $A, C > 0$, where $\beta_F(m) = $ $= \mathrm{Card}(B_F(m))$. Using Lemma 2.4.3 and (2.1) we find that for any $m \ge r$

$$h(T, \xi_X(-m, m)) \ge$$

$$\ge \limsup_{s \to \infty} \beta_F(sN)^{-1} \log \mathcal{N}\Big(\bigvee_{g \in B_F(sN)} (T^g)^{-1} \xi_X(-m, m) \Big) \ge$$

$$\ge (2/M)^d \lim_{s \to \infty} \big((sM)^d / \beta_F(sM) \big) (2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)) \ge$$

$$\ge (2/M)^d C^{-1} h(\tau_X).$$

Hence, $h(T) \ge (2/M)^d C^{-1} h(\tau_X)$ and $h(\tau_X) > 0$ implies $h(T) > 0$. □

Proof of Theorem 2.3.2. Let G be a group generated by a finite set $F \subset G$ with growth$(G) = k < d$ and $X \subset \Omega_d(P)$ be a subshift with $h(\tau_X) > 0$. Suppose T is an expansive action of G by automorphisms on X. Using Lemma 2.4.3 again we obtain that for any $m \ge r$

$$h(T, \xi_X(-m, m)) \ge$$

$$\ge \limsup_{s \to \infty} \beta_F(sN)^{-1} \log N\Big(\bigvee_{g \in B_F(sN)} (T^g)^{-1} \xi_X(-m, m) \Big) \ge$$

$$\ge (2/M)^d \lim_{s \to \infty} \big((sM)^d / \beta_F(sM) \big) (2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)).$$

Thus, we have $h(T, \xi_X(-m, m)) = \infty$, since growth$(G) < d$ implies $(sM)^d / \beta_F(sM) \to \infty$ and $(2(m+s)+1)^{-d} \log \mathcal{N}(\xi_X(-m-s, m+s)) \to h(\tau_X) > 0$. But this is impossible in view of the obvious inequality $h(T, \eta) \le \log \mathrm{Card}(\eta)$ which holds for any finite open cover η. This contradiction completes the proof. □

CHAPTER 3.
ON THE CLASSIFICATION OF SOME TWO-DIMENSIONAL MARKOV SHIFTS WITH GROUP STRUCTURE

3.1. INTRODUCTION AND THE MAIN RESULTS

In Chapters 5 - 7 of [Sch1] K. Schmidt gave probably the first survey of higher-dimensional Markov shifts. It contains a lot of intriguing questions exhibiting the striking lack of general theory in this part of modern ergodic theory. One of the questions raised in [Sch1] is discussed in this chapter.

First of all, we recall the general definition of a Markov shift in arbitrary dimension. Let F be a finite set. A closed shift-invariant set $X \subset F^{\mathbb{Z}^d}$ ($d \geq 1$) is called a d-dimensional Markov shift (or subshift of finite type), if there exists a finite subset $D \subset \mathbb{Z}^d$ and a set $P \subset F^D$ such that X consists precisely of those points $x \in F^{\mathbb{Z}^d}$ for which the coordinate projection of x onto every translate of D results in an element of P.

Let G be a finite group with the identity e and define the two-dimensional Markov shift

$$X_G = \{ x = (x_{(i,j)})_{(i,j) \in \mathbb{Z}^2} \in G^{\mathbb{Z}^2} : x_{(i,j)} x_{(i+1,j)} x_{(i,j+1)} = e \text{ for all } (i,j) \in \mathbb{Z}^2 \}.$$

Although X_G is not a subgroup of $G^{\mathbb{Z}^2}$ unless G is abelian, the Haar measure λ_G on G always gives rise to a well defined shift-invariant probability measure μ_G on X_G. Indeed, an element $x \in X_G$ is completely determined by values of the coordinates $x_{(i,0)}, i \in \mathbb{Z}$ and $x_{(0,-j)}, j \in \mathbb{Z}_+$. So, the Haar measure $(\lambda_G)^{\mathbb{Z}} \otimes (\lambda_G)^{\mathbb{Z}_+}$ on the compact group $G^{\mathbb{Z}} \oplus G^{\mathbb{Z}_+}$ determines (by means of the natural homeomorphism between $G^{\mathbb{Z}} \oplus G^{\mathbb{Z}_+}$ and X_G) a probability measure μ_G on X_G. If G is abelian, then X_G is a (compact) subgroup of $G^{\mathbb{Z}^2}$ and the Haar measure on

X_G is precisely μ_G. The horizontal and the vertical shifts $\sigma_G, \tau_G : X_G \to X_G$ defined by $(\sigma_G x)_{(i,j)} = x_{(i+1,j)}$ and $(\tau_G x)_{(i,j)} = x_{(i,j+1)}$ are commuting automorphisms of the measure space (X_G, B_G, μ_G) where $\mathcal{B}_G$ denotes the Borel σ-algebra on X_G. The measure preserving $\mathbb{Z}^2$-action $\mathcal{T}_G$ on $(X_G, \mathcal{B}_G, \mu_G)$ generated by σ_G and τ_G is called the shift action on X_G. Ergodic properties of the action were first studied by F. Ledrappier [L] for the case $G = \mathbb{Z}_2$. He showed the shift action to have zero entropy and to be mixing, but not 2-mixing. In fact, zero entropy and mixing holds for any group G (cf. [Sch1], [Sch2]) and 2-mixing fails whenever G is abelian (see [War]). It is not hard to show that $h_{\mu_G}(\sigma_G) = h_{\mu_G}(\tau_G) = \log|G|$. So, we have

Proposition 3.1.1 *(cf. [War]). If the shift actions $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ and $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$ are isomorphic, then $|G_1| = |G_2|$.*

After this discussion we formulate the following question which was originally raised by H. Furstenberg (see [Sch1, p. 61]).

Question 3.1.2. *Let G_1 and G_2 be finite groups with $|G_1| = |G_2|$. Are the $\mathbb{Z}^2$-actions $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ and $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$ isomorphic ?*

We remark, that the invariants called relative entropies that have been introduced by B. Kitchens and K. Schmidt [KS2] do not distinguish between such actions.

The question has also been studied by T. Ward in [War] where he proves that if G_1 has the property that $\{ghg^{-1}h^{-1} : g, h \in G_1\} = G_1$, i. e. G_1 is "strongly non-abelian", and G_2 is any abelian group, then $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ and $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$ are non-isomorphic. However, the arguments of [War] are not applicable to the case when G_1 and G_2 are both abelian.

Here we deal only with abelian groups. We prove a theorem which enables us to measurably distinguish between $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ and

$(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$, say, for $G_1 = (\mathbb{Z}_p)^n$ and $G_2 = \mathbb{Z}_{p^n}$ for any prime p. As far as the question of topological conjugacy is concerned, we have been able to describe the situation completely: algebraically (non-) isomorphic abelian groups give rise to topologically (non-) isomorphic Markov shifts. We conjecture that the measure-theoretic picture is exactly the same.

In order to formulate the result precisely, let us recall some definitions from basic group theory (see e. g. [Schn]). The group G is called a p-group, if $|G| = p^s$, $s \in \mathbb{N}$ for a prime p. For any finite abelian group G the least common multiple of orders of its elements is called the exponent of G. We denote the exponent of G by E(G). Clearly, $g^{E(G)} = e$ for every $g \in G$. Furthermore, E(G) always divides $|G|$. We prove that the number E(G) is an invariant for the shift action $(X_G, \mathcal{B}_G, \mu_G, \mathcal{T}_G)$ as long as G is a p-group.

Theorem 3.1.3. *Let G_1 and G_2 are abelian p-groups. If the $\mathbb{Z}^2$-action $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ is a factor of the $\mathbb{Z}^2$-action $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$, then $E(G_1) \leq E(G_2)$. In particular, if they are isomorphic, then $E(G_1) = E(G_2)$.*

The theorem may be illustrated by the following

Corollary 3.1.4. *For an integer $m > 1$ let $G_1 = (\mathbb{Z}_p)^m$, $G_2 = (\mathbb{Z}_p)^{m-2} \oplus \mathbb{Z}_{p^2}, \ldots, G_{m-1} = \mathbb{Z}_p \oplus \mathbb{Z}_{p^{m-1}}, G_m = \mathbb{Z}_{p^m}$. Then the $\mathbb{Z}^2$-actions $(X_{G_i}, \mathcal{B}_{G_i}, \mu_{G_i}, \mathcal{T}_{G_i})$ and $(X_{G_j}, \mathcal{B}_{G_j}, \mu_{G_j}, \mathcal{T}_{G_j})$ $(1 \leq i, j \leq m)$ are not isomorphic unless $i = j$.*

The theorem above may be considered as a step towards proving the following.

Conjecture 3.1.5. *If G_1 and G_2 are any finite abelian groups, then the $\mathbb{Z}^2$-actions $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ and $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$ are isomorphic if and only if $G_1 \cong G_2$.*

We 'justify' this conjecture by proving its topological counterpart.

Theorem 3.1.6. If G_1 and G_2 are any finite abelian groups, then the topological $\mathbb{Z}^2$-actions $(X_{G_1}, \mathcal{T}_{G_1})$ and $(X_{G_2}, \mathcal{T}_{G_2})$ are topologically conjugate if and only if $G_1 \cong G_2$.

We mention an obvious "generalization" of our main results. It is easily seen from the proofs (see §§ 3.3-3.5) that they are valid for any "3-dot" Markov shift with "zero sum" law. More precisely, let

$X_{G,D} = \{ \mathbf{x} = (x_{(i,j)})_{(i,j)\in \mathbb{Z}^2} \in G^{\mathbb{Z}^2} : \prod_{(m,n) \in D} x_{(i+m,\, j+n)} = e \text{ for all } (i,j) \in \mathbb{Z}^2 \}$,

where $D = \{ (m^{(0)}, n^{(0)}), (m^{(1)}, n^{(1)}), (m^{(2)}, n^{(2)}) \} \subset \mathbb{Z}^2$ is any non-degenerate triangle, i. e. the vectors $(m^{(1)}-m^{(0)}, n^{(1)}-n^{(0)})$ and $(m^{(2)}-m^{(0)}, n^{(2)}-n^{(0)})$ are linearly independent over $\mathbb{Z}$. (The case we consider above and below is the one with $D = \{ (0,0), (1,0), (0,1) \}$.) Theorems 3.1.3 and 3.1.6 still hold for shift $\mathbb{Z}^2$-actions on such Markov shifts.

Concluding this Section we make a remark concerning notation. As from now on we will be concerned only with abelian groups, we will use the additive notation for the group operations (in particular, 0 will stand for the identity element).

3.2. SOME FACTS ABOUT BINOMIAL COEFFICIENTS

It can be easily proved by induction (see [War] for the proof) that

$$x_{(0,m)} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} x_{(k,0)} \cdot \qquad (3.1)$$

for any $\mathbf{x} \in X_G$, $m \in \mathbb{N}$. Since the long-range dependence of the coordinates in X_G involves the binomial coefficients, we investigate their divisibility properties, looking for possible cancellations in (3.1).

Lemma 3.2.1. Let p be a prime. Then for all integers $s \geq 1$ and $n \geq s$ we have $p^s \mid \binom{p^n}{k}$ *for each* $k \in \{0, 1, \dots, p^n\} \setminus \{j p^{n-s+1} : 0 \leq j \leq p^{s-1}\}$ ($a \mid b$ means a divides b).

Proof. See Appendix.

This lemma tells us, in particular, that the binomial coefficient $\binom{p^n}{p^{n-s}}$ is divisible by p^s. One can show that, in fact, p^s is the maximal power of p that divides the coefficient.

Lemma 3.2.2. Let p be a prime. Then for all integers s, n with $0 \leq s \leq n$ *the binomial coefficient* $\binom{p^n}{p^{n-s}}$ *is divisible by* p^s *but not divisible by* p^{s+1}

Proof. See Appendix.

3.3. CYLINDER SETS

Let G be a finite abelian group and X_G be the two-dimensional Markov shift defined in §3.1. With $E \subset \mathbb{Z}^2$ a finite set, let $\pi_E : X_G \to G^E$ be the projection map onto E. Then $\pi_E(X_G)$ is a subgroup of G^E. We denote $\pi_E(X_G)$ by X_G^E and for every $\alpha \in X_G^E$ define the cylinder set $C_G^E(\alpha) = \{x \in X_G : \pi_E(x) = \alpha\}$. Clearly, $\mu_G(C_G^E(\alpha)) = |X_G^E|^{-1}$. Define the triangles $E(m,n,s) = \{(i,j) \in \mathbb{Z}^2 : i \geq m,\ j \geq n,\ (i-m)+(j-n) < s\}$, $(m,n) \in \mathbb{Z}^2$, $s \in \mathbb{N}$, and set $T(\ell) = E(-\ell, -\ell, 4\ell)$, $\ell \in \mathbb{N}$. We will basically deal with the cylinder sets of the form $C_G^E(\alpha)$. It is clear that for any finite set $E \subset \mathbb{Z}^2$ there is $\ell \in \mathbb{N}$ such that $E \subset T(\ell)$. Therefore, the σ-algebra B_G is the completion of the σ-algebra generated by the collection of cylinder sets $\{C_G^{T(\ell)}(\alpha) : \alpha \in X_G^{T(\ell)}, \ell \in \mathbb{N}\}$.

The set $X_G^{T(\ell)}$ is just the set of all allowed configurations over $T(\ell)$, for every such a configuration can be extended to an element of X_G. So,

$$X_G^{T(\ell)} = \{\alpha \in G^{T(\ell)} : \alpha_{(i,j)} + \alpha_{(i+1,j)} + \alpha_{(i,j+1)} = 0 \text{ for all } (i,j) \in T(\ell-1)\},$$

if $\ell > 1$, and $X_G^{T(1)} = G$. Obviously, an element of $X_G^{T(\ell)}$ is completely determined by its values on the bottom of the triangle $T(\ell)$, i.e. by $\alpha_{(-\ell,-\ell)}, \alpha_{(-\ell+1,-\ell)}, \ldots$ $\ldots, \alpha_{(3\ell,-\ell)}$. Moreover, for every $(a_{-\ell}, \ldots, a_{3\ell}) \in G^{4\ell+1}$ there exists a unique $\alpha \in X_G^{T(\ell)}$ with $\alpha_{(i,0)} = a_i$, $-\ell \le i \le 3\ell$. From this we obtain $|X_G^{T(\ell)}| = |G|^{4\ell+1}$ and, hence, $\mu_G(C_G^{T(\ell)}(\alpha)) = |G|^{-4\ell-1}$ for all $\alpha \in X_G^{T(\ell)}$. For a set $E \subset \mathbb{Z}^2$ define its translate $E+(s,t)$ by $(s,t) \in \mathbb{Z}^2$ setting $E+(s,t) = \{(i+s, j+t) : (i,j) \in E\}$. One easily finds that

$$|X_G^{T(L) \cup (T(\ell)+(k,0))}| = |X_G^{T(L)}||X_G^{T(\ell)+(k,0)}| = |X_G^{T(L)}||X_G^{T(\ell)}|,$$

whenever $k > 4L$. Therefore, for all $k > 4L$, $\alpha \in X_G^{T(L)}$, $\beta \in X_G^{T(\ell)}$ we have

$$\mu_G(C_G^{T(L)}(\alpha) \cap C_G^{T(\ell)+(k,0)}(\beta)) = \mu_G(C_G^{T(L)}(\alpha))\, \mu_G(C_G^{T(\ell)}(\beta)).$$

We now summarize this discussion in the following

Lemma 3.3.1. Let A be a union of cylinder sets of the form $C_G^{T(L)}(\alpha)$, $\alpha \in X_G^{T(L)}$ and B be a union of cylinder sets of the form $C_G^{T(\ell)}(\beta)$, $\beta \in X_G^{T(\ell)}$. Then the sets A and $\sigma^{-k}(B)$ are independent with respect to the measure μ_G, whenever $k > 4L$.

3.4 OBSTRUCTIONS TO MEASURABLE CONJUGACY (PROOF OF THEOREM 3.1.3)

Let G_1 and G_2 be finite abelian p-groups with $E(G_i) = p^{s_i}$, $i = 1, 2$. Suppose $s_1 > s_2$ which means that in the group the identities $p^{s_i} g = 0$, $g \in G_i$ $(i = 1, 2)$ hold, but for each r not divisible by p^{s_1} there exists $g(r) \in G_1$ such that $r\,g(r) \neq 0$. In view of the cancellations guaranteed by Lemma 3.2.1, for all $x \in X_{G_i}$, $i = 1, 2$ we can rewrite (3.1) (with $m = p^n$, $n \geq s_i$) as follows

$$x_{(0,p^n)} = (-1)^{p^n} \sum_{j=0}^{p^{s_i-1}} \binom{p^n}{j\,p^{n-s_i+1}} x_{(j\,p^{n-s_i+1},0)} \,. \tag{3.2}$$

Notice that for $x \in X_{G_1}$ the sum above consists of $p^{s_1-1}+1$ summands including the $p^{s_2-1}+1$ summands of the sum for $x \in X_{G_2}$. Furthermore, those members of the sum for $x \in X_{G_1}$ not included in the sum for $x \in X_{G_2}$ are not all identically zero. Indeed, by virtue of Lemma 3.2.2, $\binom{p^n}{p^{n-s_2}}$ is not divisible by p^{s_1} and hence the summand $\binom{p^n}{p^{n-s_2}} x_{(p^{n-s_2},0)}$ is not zero when $x_{(p^{n-s_2},0)} = g^* = = g\left(\binom{p^n}{p^{n-s_2}}\right) \in G_1$.

From (3.2) for all $\beta \in X_{G_i}^{T(\ell)}$ and $p^{n-s_i+1} > 4\ell$ we obtain

$$\tau_{G_i}^{-p^n} C_{G_i}^{T(\ell)}(\beta) = \bigcup_{\bar{\alpha}} \bigcap_{j=0}^{p^{s_i-1}} \sigma_{G_i}^{-jp^{n-s_i+1}} C_{G_i}^{T(\ell)}(\alpha_j) \,, \tag{3.3}$$

where the union is taken over all $\bar{\alpha} = \{\alpha_j : 0 \leq j \leq p^{s_i-1}\} \in (X_{G_i}^{T(\ell)})^{p^{s_i-1}}$ such that $\beta = (-1)^{p^n} \sum_{j=0}^{p^{s_i-1}} \binom{p^n}{j\,p^{n-s_i+1}} \alpha_j$.

Now we are in a position to establish two facts featuring the difference between the measurable dynamics of X_{G_1} and X_{G_2} which prevents them being isomorphic.

Lemma 3.4.1. There exist sets A_j $(0 \leq j \leq p^{s_1-1})$, $B \in \mathcal{B}_{G_1}$ such that

$$\limsup_{n\to\infty} \mu_{G_1}\left(\tau_{G_1}^{-p^n} B \cap \left(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_1}^{-k\,p^{n-s_2+1}} A_{k\,p^{s_1-s_2}} \right)\right) > 0, \tag{3.4}$$

but

$$\limsup_{n\to\infty} \mu_{G_1}\Big(\tau_{G_1}^{-p^n} B \cap \Big(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_1}^{-j p^{n-s_1+1}} A_j\Big)\Big) = 0 . \qquad (3.5)$$

Proof. We put $a_j = 0 \in G_1$ for all $0 \le j \le p^{s_1-1}$ except for $j = p^{s_1-s_2-1}$, and $a_{p^{s_1-s_2-1}} = g^* \in G_1$. Then we set $A_j = C_{G_1}^{T(1)}(a_j)$ $(= \{x \in X_{G_1} : x_{(0,0)} = a_j\})$ and $B = C_{G_1}^{T(1)}(0)$. We have

$$\sum_{k=0}^{p^{s_2-1}} \binom{p^n}{k p^{n-s_2+1}} a_{k p^{s_1-s_2}} = 0 , \qquad (3.6)$$

$$\sum_{j=0}^{p^{s_1-1}} \binom{p^n}{j p^{n-s_1+1}} a_j = \binom{p^n}{p^{n-s_2}} g^* \neq 0 . \qquad (3.7)$$

In view of (3.3), the formulas (3.6) and (3.7) give (3.4) and (3.5), respectively. □

Lemma 3.4.2. Let the sets A_j $(0 \le j \le p^{s_1-1})$, $B \in \mathcal{B}_{G_2}$ satisfy

$$\limsup_{n\to\infty} \mu_{G_2}\Big(\tau_{G_2}^{-p^n} B \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-k p^{n-s_2+1}} A_{k p^{s_1-s_2}}\Big)\Big) > 0 . \qquad (3.8)$$

Then they satisfy

$$\limsup_{n\to\infty} \mu_{G_2}\Big(\tau_{G_2}^{-p^n} B \cap \Big(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_2}^{-j p^{n-s_1+1}} A_j\Big)\Big) > 0 . \qquad (3.9)$$

Proof. Fix arbitrary sets A_j $(0 \le j \le p^{s_1-1})$, $B \in \mathcal{B}_{G_2}$ satisfying (3.8) and denote $\mu_{G_2}(A_j) = \xi_j$ (clearly, $\xi_j > 0$, $0 \le j \le p^{s_1-1}$) and $\eta_n = \mu_{G_2}\Big(\tau_{G_2}^{-p^n} B \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-k p^{n-s_2+1}} A_{k p^{s_1-s_2}}\Big)\Big) > 0$. Then, since $\mathcal{B}_{G_2}$ is the completion of σ-algebra generated by the cylinder sets, there exists $\ell \in \mathbb{N}$ and sets $A_{j,\varepsilon}$ $(0 \le j \le p^{s_1-1})$, $B_\varepsilon \in \mathcal{B}_{G_2}$ which are unions of cylinder sets of the form $C_{G_2}^{T(\ell)}(\alpha)$ and satisfy $\mu(A_j \Delta A_{j,\varepsilon}) < \varepsilon$ $(0 \le j \le p^{s_1-1})$ and $\mu(B \Delta B_\varepsilon) < \varepsilon$. It follows that

$$\mu_{G_2}\Big(\Big(\tau_{G_2}^{-p^n} B \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-kp^{n-s_2+1}} A_{kp^{s_1-s_2}}\Big)\Big) \Delta$$

$$\Delta \Big(\tau_{G_2}^{-p^n} B_\varepsilon \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-kp^{n-s_2+1}} A_{kp^{s_1-s_2},\varepsilon}\Big)\Big)\Big) < (p^{s_2-1}+2)\varepsilon, \quad (3.10)$$

$$\mu_{G_2}\Big(\Big(\tau_{G_2}^{-p^n} B \cap \Big(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_2}^{-jp^{n-s_1+1}} A_j\Big)\Big) \Delta$$

$$\Delta \Big(\tau_{G_2}^{-p^n} B_\varepsilon \cap \Big(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_2}^{-jp^{n-s_1+1}} A_{j,\varepsilon}\Big)\Big)\Big) < (p^{s_1-1}+2)\varepsilon. \quad (3.11)$$

for all $n \in \mathbb{N}$. We have $A_{j,\varepsilon} = \bigcup_{\alpha \in Q(j)} C_{G_2}^{T(\ell)+(s,t)}(\alpha)$, $0 \le j \le p^{s_1-1}$, $B_\varepsilon = \bigcup_{\beta \in R} C_{G_2}^{T(\ell)+(s,t)}(\beta)$ where $Q(j)$ and R are subsets of $X_{G_2}^{T(\ell)}$. From (3.3) for all n with $p^{n-s_2+1} > 4\ell$ we obtain

$$\tau_{G_2}^{-p^n} B_\varepsilon \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-kp^{n-s_2+1}} A_{kp^{s_1-s_2},\varepsilon}\Big) =$$

$$= \bigcup \bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-kp^{n-s_2+1}} C_{G_2}^{T(\ell)+(s,t)}(\alpha_{kp^{s_1-s_2}}), \quad (3.12)$$

where the union is taken over all $(\alpha_0, \alpha_{p^{s_1-s_2}}, \dots, \alpha_{p^{s_1-1}}) \in Q(0) \times Q(p^{s_1-s_2}) \times \dots$ $\dots \times Q(p^{s_1-1})$ such that $(-1)^{p^n} \sum_{k=0}^{p^{s_2-1}} \binom{p^n}{kp^{n-s_2+1}} \alpha_{kp^{s_1-s_2}} \in R$. We denote $\eta_{\varepsilon,n} = \mu_{G_2}\Big(\tau_{G_2}^{-p^n} B_\varepsilon \cap \Big(\bigcap_{k=0}^{p^{s_2-1}} \sigma_{G_2}^{-kp^{n-s_2+1}} A_{kp^{s_1-s_2},\varepsilon}\Big)\Big)$. From (3.10) we find that $\eta_{\varepsilon,n} > \eta_n - (p^{s_2-1}+2)\varepsilon$. Using Lemma 3.3.1, one easily obtains

$$\mu_{G_2}\left(\tau_{G_2}^{-p^n} B_\varepsilon \cap \left(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_2}^{-jp^{n-s_1+1}} A_{j,\varepsilon}\right)\right) = \eta_{\varepsilon,n} \prod_{\substack{j=0,\\ p^{s_1-s_2} \nmid j}}^{p^{s_1-1}} \mu_{G_2}(A_{j,\varepsilon}) >$$

$$> (\eta_n - (p^{s_2-1}+2)\varepsilon) \prod_{\substack{j=0,\\ p^{s_1-s_2} \nmid j}}^{p^{s_1-1}} (\xi_j - \varepsilon),$$

($a \nmid b$ means a does not divide b) for all n with $p^{n-s_1+1} > 4\ell$. Now (3.11) yields

$$\mu_{G_2}\left(\tau_{G_2}^{-p^n} B \cap \left(\bigcap_{j=0}^{p^{s_1-1}} \sigma_{G_2}^{-jp^{n-s_1+1}} A_j\right)\right) >$$

$$> (\eta_n - (p^{s_2-1}+2)\varepsilon) \prod_{\substack{j=0,\\ p^{s_1-s_2} \nmid j}}^{p^{s_1-1}} (\xi_j - \varepsilon) - (p^{s_1-1}+2)\varepsilon \ ,$$

But, since $\limsup_{n\to\infty} \eta_n > 0$ and $\xi_j > 0$, lim sup of the right-hand side of this inequality can be made positive by choosing ε small enough. Hence, lim sup of the left-hand side is positive. This completes the proof. □

The Lemmas 3.4.1 and 3.4.2 show that the system $(X_{G_1}, \mathcal{B}_{G_1}, \mu_{G_1}, \mathcal{T}_{G_1})$ cannot be a factor of the system $(X_{G_2}, \mathcal{B}_{G_2}, \mu_{G_2}, \mathcal{T}_{G_2})$, if $E(G_1) > E(G_2)$. This proves Theorem 3.1.3. □

3.5 OBSTRUCTIONS TO TOPOLOGICAL CONJUGACY (PROOF OF THEOREM 3.1.6).

Let G be a finite abelian group and $h \in G$ be an element of order 3. Then $X_G \ni \mathbf{h}$ with $h_{(i,j)} = h$ for all $(i,j) \in \mathbb{Z}^2$ (h is a fixed point of the shift action on X_G) and we define a continuous shift-commuting map $L_h : X_G \to X_G$ by $(L_h x)_{(i,j)} = x_{(i,j)} + h$, $(i,j) \in \mathbb{Z}^2$.

Lemma 3.5.1. Let G_1 and G_2 be finite abelian groups. Then a continuous factor map $\varphi : (X_{G_1}, \mathcal{T}_{G_1}) \to (X_{G_2}, \mathcal{T}_{G_2})$ is a group homomorphism or $L_h \circ \varphi$ with some $h \in G_2$ is a group homomorphism.

Proof. This proof develops some ideas of [KS1]. For any abelian G let 0_G be the zero element of the group X_G (i.e. the configuration of all zeros). Take a continuous factor map $\varphi : (X_{G_1}, \mathcal{T}_{G_1}) \to (X_{G_2}, \mathcal{T}_{G_2})$. We assume that $\varphi(0_{G_1}) = 0_{G_2}$ and prove that φ is a group homomorphism. If $\varphi(0_{G_1}) = \mathbf{h}$ with $\mathbf{h}_{(i,j)} = h$ for all $(i,j) \in \mathbb{Z}^2$ (the fixed point 0_{G_1} must be sent to a fixed point), we just pass to the continuous factor map $L_{-h} \circ \varphi$ which already satisfies $L_{-h} \circ \varphi(0_{G_1}) = 0_{G_2}$.

First we shall prove the statement under the assumption that G_2 is a p-group. We shall make use of the well known fact that φ is a block map, i.e. there exists $\ell \in \mathbb{N}$, and a mapping $F : X_{G_1}^{T(\ell)} \to G_2$ such that $(\varphi(\mathbf{x}))_{(i,j)} = F(\pi_{T(\ell)}(\tau^j \sigma^i \mathbf{x}))$.

Let $o, \alpha, \beta \in X_{G_1}^{T(\ell)}$, where o is the zero element of the group $X_{G_1}^{T(\ell)}$. Let T stand for $T(\ell)$ and denote $A = C^T_{G_1}(\alpha)$, $B = C^T_{G_1}(\beta)$, $O = C^T_{G_1}(o)$. Next, let $E(G_2) = p^s$ and define $A_n = A \cap \left(\bigcap_{k=1}^{p^{s-1}} \sigma_{G_2}^{-kp^{n-s+1}} O \right)$ and $B_n = B \cap \left(\bigcap_{k=1}^{p^{s-1}-1} \sigma_{G_2}^{-kp^{n-s+1}} O \right) \cap \tau_{G_2}^{-p^n} O$. It follows from Lemma 3.3.1 that the sets A_n and B_n are non-empty, whenever $p^{n-s+1} > 4\ell$. Now we fix $\mathbf{x} \in A_n$ and $\mathbf{y} \in B_n$ $(p^{n-s+1} > 4\ell)$. Observe that $\varphi(0_{G_1}) = 0_{G_2}$ implies $F(o) = 0$. Hence, $\varphi(\mathbf{x})_{(kp^{n-s+1}, 0)} = 0$ for $1 \le k \le p^{s-1}$ and $\varphi(\mathbf{y})_{(kp^{n-s+1}, 0)} = 0$ for $1 \le k \le p^{s-1} - 1$, $\varphi(\mathbf{y})_{(0, p^n)} = 0$. Applying (3.2) to the points $\varphi(\mathbf{x}), \varphi(\mathbf{y}) \in X_{G_2}$ we obtain

$$F(\alpha) + \varphi(\mathbf{x})_{(0, p^n)} = 0, \quad F(\beta) + \varphi(\mathbf{y})_{(p^n, 0)} = 0 \tag{3.13}$$

On the other hand, considering the point $x+y \in X_{G_1}$ we find

$$F(\alpha+\beta)+\varphi(x)_{(0,p^n)}+\varphi(y)_{(p^n,0)} = 0 \tag{3.14}$$

Comparison of (3.13) and (3.14) gives $F(\alpha+\beta)=F(\alpha)+F(\beta)$. Thus, F is a homomorphism of $X_{G_1}^{T(\ell)}$ onto G_2 and, therefore, φ is a homomorphism of X_{G_1} onto X_{G_2}. The lemma is proved in the case where G_2 is a p-group.

Now consider the general case. Any finite abelian group G_2 can be presented as a direct product (sum)

$$G_2 = \bigoplus_{i=1}^{r} G_2(p_i)$$

where $G_2(p_i)$ is a p_i-group, $i = 1, \dots, r$. It is easily seen that the group X_{G_2} is the direct product of its $\mathcal{T}_{G_2}$-invariant subgroups corresponding to the factors $G_2(p_i)$ of G_2, i. e.

$$X_{G_2} = \bigoplus_{i=1}^{r} X_{G_2(p_i)} .$$

Any continuous factor $\varphi: (X_{G_1}, \mathcal{T}_{G_1}) \to (X_{G_2}, \mathcal{T}_{G_2})$ gives rise to r continuous factors $\varphi_i = pr_i \circ \varphi : (X_{G_1}, \mathcal{T}_{G_1}) \to (X_{G_2(p_i)}, \mathcal{T}_{G_2(p_i)})$, where $pr_i: X_{G_2} \to X_{G_2(p_i)}$ is the natural projection. According to what is proved above, for each i the map φ_i is a group homomorphism. Hence, so is φ. □

Lemma 3.5.2. With G a finite abelian group, the group $Fix(\tau_G \circ \sigma_G) = \{x \in X_G : \tau_G \circ \sigma_G(x) = x\}$ *is isomorphic to* $G^2 = G \oplus G$.

Proof. Let $x \in Fix(\tau_G \circ \sigma_G)$ and denote $x_{(0,0)} = g$ and $x_{(1,0)} = h$. Then we must have $x_{(0,1)} = k = -g-h$ and, by $\tau_G \circ \sigma_G$-invariance, $x_{(i,i)} = g$ and $x_{(i+1,i)} = h$. Then we must have $x_{(i,i+1)} = k$ for all $i \in \mathbb{Z}$. Following the rule $x_{(i,j)} + x_{(i+1,j)} + x_{(i,j+1)} = 0$ and the $\tau_G \circ \sigma_G$-invariance, we are led to the unique configuration x which has $x_{(i,j)}$ being equal to g, h or k according as $i-j$ is congruent to 0, 1 or 2 mod 3. So, the projection $\pi_{\{(0,0),(1,0)\}}$: $Fix(\tau_G \circ \sigma_G) \to G^2$ gives the desired isomorphism □

Proof of Theorem 3.1.6.

Suppose that $\varphi : (X_{G_1}, \mathcal{T}_{G_1}) \to (X_{G_2}, \mathcal{T}_{G_2})$ is a topological conjugacy. Then, by Lemma 3.5.1, $L_h \circ \varphi$ for some $h \in G_2$ is a group isomorphism. Clearly, it induces a group isomorphism between $\mathrm{Fix}(\tau_{G_1} \circ \sigma_{G_1})$ and $\mathrm{Fix}(\tau_{G_2} \circ \sigma_{G_2})$. By virtue of Lemma 3.5.2, this implies $(G_1)^2 \cong (G_2)^2$ and, therefore, $G_1 \cong G_2$ This completes the proof of the theorem. □

APPENDIX TO CHAPTER 3 : PROOFS OF THE NUMBER-THEORETIC LEMMAS

We begin by establishing the equality

$$\binom{q^{n+1}}{k} = \sum_{\substack{0 \le k_1, \dots, k_q \le q^n \\ k_1 + \dots + k_q = k}} \prod_{i=1}^{q} \binom{q^n}{k_i} \qquad (3.A.1)$$

for all integers $q > 0$, $n \ge 0$, $0 \le k \le q^n$. Indeed, we have

$$(a+b)^{q^{n+1}} = \sum_{k=0}^{q^{n+1}} \binom{q^{n+1}}{k} a^k b^{q^{n+1}-k} , \qquad (3.A.2)$$

but, on the other hand,

$$(a+b)^{q^{n+1}} = \left((a+b)^{q^n}\right)^q = \left(\sum_{k=0}^{q^n} \binom{q^n}{k} a^k b^{q^n-k} \right)^q =$$

$$= \sum_{k_1=0}^{q^n} \cdots \sum_{k_q=0}^{q^n} \binom{q^n}{k_1} \cdots \binom{q^n}{k_q} a^{k_1+\dots+k_q} b^{q^{n+1}-(k_1+\dots+k_q)} . \qquad (3.A.3)$$

Comparison of (3.A.2) and (3.A.3) yields (3.A.1).

We use (3.A.1) to prove Lemma 3.2.1.

Proof of Lemma 3.2.1.

Let $\mathfrak{D}_s$ stand for the claim of the lemma for a fixed s. First we prove $\mathfrak{D}_1$ by induction on n. It is well known (see [HW], Theorem 75]) that any prime p divides the binomial coefficient $\binom{p}{k}$ for any $k = 1, \dots, p-1$. Suppose $\mathfrak{D}_1$ to be true for some n. Let $k \in \{0, 1, \dots, p^{n+1}\} \setminus \{j\, p^n : 0 \le j \le p\}$. Then for every $(k_1, \dots, k_p) \in \{0, 1, \dots, p^n\}^p$ such that $k_1 + \dots + k_p = k$ there exists ℓ, $1 \le \ell \le p$, with $k_\ell \in \{1, \dots, p^n - 1\}$ and, hence, $p \mid \prod_{i=1}^{p} \binom{p^n}{k_i}$. From (3.A.1) we conclude that $p \mid \binom{p^{n+1}}{k}$ for such k. Now suppose $k = j\, p^n$ for some $0 < j < p$. We put $K = \{(k_1, \dots, k_p) \in \{0, 1, \dots, p^n\}^p : \prod_{i=1}^{p} k_i = k\}$ and $K_0 = \{(k_1, \dots, k_p) \in K : k_i \in \{0, p^n\}$ for all $i = 1, \dots, p\}$. Clearly, $|K_0| = \binom{p}{j}$ and $\prod_{i=1}^{p} \binom{p^n}{k_i} = 1$ for each $(k_1, \dots, k_p) \in K_0$. From the induction hypothesis we have $p \mid \prod_{i=1}^{p} \binom{p^n}{k_i}$ for each $(k_1, \dots, k_p) \in K \setminus K_0$. Thus, writing

$$\binom{p^{n+1}}{k} = \binom{p}{j} + \sum_{(k_1, \dots, k_p) \in K \setminus K_0} \prod_{i=1}^{p} \binom{p^n}{k_i} ,$$

where the both summands on the right-hand side are divisible by p, completes the proof of the claim $\mathfrak{D}_1$.

Now in order to prove the lemma by induction on s we need to derive $\mathfrak{D}_{s+1}$ from $\mathfrak{D}_s$. So, let $p^s \mid \binom{p^n}{k}$ for all $n \ge s$ and all $k \in \{0, 1, \dots, p^n\} \setminus \{j\, p^{n-s+1} : 0 \le j \le p^{s-1}\}$. Take any $m \ge s+1$ and any $k \in \{0, 1, \dots, p^m\} \setminus \{j\, p^{m-s} : 0 \le j \le p^s\}$ and put $K = \{(k_1, \dots, k_p) \in \{0, 1, \dots, p^{m-1}\}^p : \prod_{i=1}^{p} k_i = k\}$. Then for every $(k_1, \dots, k_p) \in K$ there exists ℓ $(1 \le \ell \le p)$ such that k_ℓ is not divisible by $p^{m-s} = p^{(m-1)-s+1}$. Since $m - 1 \ge s$ our assumption yields $p^s \mid \binom{p^{m-1}}{k_\ell}$. Next, we partition the set K into two sets:

$K_1 = \{(k_1, \dots, k_p) \in K : |\{\ell : 1 \le \ell \le p,\ p^{m-s}$ does not divide $k_\ell\}| = 1\}$,
$K_2 = \{(k_1, \dots, k_p) \in K : |\{\ell : 1 \le \ell \le p,\ p^{m-s}$ does not divide $k_\ell\}| \ge 2\}$.

Obviously, $K_1 \cap K_2 = \emptyset$, $K_1 \cup K_2 = K$. From the observation above we see that $p^s \mid \prod_{i=1}^{p} \binom{p^{m-1}}{k_i}$ for all $(k_1, \dots, k_p) \in K_1$ and $p^{2s} \mid \prod_{i=1}^{p} \binom{p^{m-1}}{k_i}$ for all $(k_1, \dots, k_p) \in K_2$ (note that $2s \geq s+1$). Furthermore, $|K_1|$ is divisible by p, since K_1 is the disjoint union of the sets $K_1(\ell) = \{ (k_1, \dots, k_p) \in K_1 :\ p^{m-s}$ does not divide $k_\ell \}$, $\ell = 1, \dots, p$, which have equal cardinalities. Thus, we have

$$\binom{p^m}{k} = \sum_{(k_1,\dots,k_p)\in K_1} \prod_{i=1}^{p} \binom{p^{m-1}}{k_i} + \sum_{(k_1,\dots,k_p)\in K_2} \prod_{i=1}^{p} \binom{p^{m-1}}{k_i},$$

where the sums over K_1 and K_2 are both divisible by p^{s+1}. This completes the proof of the lemma. $\square$

Proof of Lemma 3. 2. 2.

For an integer *a* and a prime p we write $\mathrm{ord}_p(a)$ for the maximal power of p that divides a, i. e. $\mathrm{ord}_p(a) = \max\{ m \in \mathbb{N} :\ p^m \mid a \}$. To prove the lemma we have to show that $\mathrm{ord}_p(\binom{p^n}{p^{n-s}}) = s$. First we compute $\mathrm{ord}_p(p^n!)$. We have

$$\mathrm{ord}_p(p^n!) = \sum_{a=1}^{p^n} \mathrm{ord}_p(a) = \sum_{k=1}^{n} k\,|\{ a :\ 1 \leq a \leq p^n :\ \mathrm{ord}_p(a) = k \}| =$$

$$= n\,|\{ a :\ 1 \leq a \leq p^n,\ p^n \mid a \}| + \sum_{k=1}^{n-1} k \Big(|\{ a : 1 \leq a \leq p^n, p^k \mid a \}| - |\{ a : 1 \leq a \leq p^n, p^{k+1} \mid a \}| \Big) =$$

$$= n + \sum_{k=1}^{n-1} k\,(p^{n-k} - p^{n-k-1}).$$

Thus, for all $n \in \mathbb{N}$

$$\mathrm{ord}_p(p^n!) = n + (p-1) \sum_{k=1}^{n-1} k\, p^{n-k-1}. \qquad (3.A.4)$$

$\mathrm{ord}_p((p^n - p^{n-s})!)$ can be computed in the same manner taking into account that

$$|\{a : 1 \le a \le p^n - p^{n-s}, p^k \mid a\}| = \begin{cases} p^{n-k} - p^{n-k-s}, & \text{if } 1 \le k \le n-s-1 \\ p^{n-k} - 1, & \text{if } n-s \le k \le n-1 \\ 0, & \text{if } k \ge n \end{cases}.$$

We obtain

$$\mathrm{ord}_p((p^n - p^{n-s})!) = (p-1)\sum_{k=n-s}^{n-1} k\,p^{n-k-1} + (p-1)\sum_{k=1}^{n-s-1} k\,(p^{n-k-1} - p^{n-k-s-1}) =$$

$$= (p-1)\sum_{k=1}^{n-1} k\,p^{n-k-1} - (p-1)\sum_{k=1}^{n-s-1} k\,p^{n-s-k-1}.$$

From this and (3.A.4) it follows that

$$\mathrm{ord}_p((p^n - p^{n-s})!) = \mathrm{ord}_p(p^n!) - \mathrm{ord}_p(p^{n-s}!) - s.$$

Finally, we have

$$\mathrm{ord}_p\left(\binom{p^n}{p^{n-s}}\right) = \mathrm{ord}_p(p^n!) - \mathrm{ord}_p(p^{n-s}!) - \mathrm{ord}_p((p^n - p^{n-s})!) = s.$$

This proves the lemma. □

CHAPTER 4.
THE COMPLETE CLASSIFICATION OF " THREE DOT " MARKOV SUBGROUPS OF $(\mathbb{Z}_p)^{\mathbb{Z}^2}$

4.1. INTRODUCTION AND THE MAIN RESULT

Let $p \in \mathbb{N}$ be a prime number and denote $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. Consider the zero-dimensional compact group $\Omega_p = (\mathbb{Z}_p)^{\mathbb{Z}^2}$ and its character group $(\Omega_p)^\wedge = \oplus_{\mathbb{Z}^2} \mathbb{Z}_p$ which can be identified with the ring $\mathfrak{R}_p = \mathbb{Z}_p[u^{\pm 1}, v^{\pm 1}]$ of Laurent polynomials in the two commuting variables u, v with coefficients from the finite field $\mathbb{Z}_p$. We identify the polynomial $f(u, v) = \sum_{(i,j) \in S(f)} c_f(i, j)\, u^i v^j$, where $S(f) \subset \mathbb{Z}^2$ is a finite set called the support of f, with the character $\chi_f \in (\Omega_p)^\wedge$ defined by $\chi_f(x) = \sum_{(i,j) \in S(f)} c_f(i, j)\, x_{(i,j)}$, $x \in \Omega_p$. It is easy to see that for any ideal $I \subset \mathfrak{R}_p \cong (\Omega_p)^\wedge$ the annihilator $I^\perp = \{ x \in \Omega_p : \chi_f(x) = 0 \text{ for all } f \in I \}$ is a closed shift-invariant subgroup of Ω_p. In particular, for the principal ideal $I = \langle f \rangle$ we have

$$\langle f \rangle^\perp = \{ x \in \Omega_p : \textstyle\sum_{(i,j) \in S(f)} c_f(i, j)\, x_{(i+m, j+n)} = 0 \text{ for all } (m, n) \in \mathbb{Z}^2 \}.$$

We will use the notation $X_{\langle f \rangle}$ for the two-dimensional subshift of finite type $\langle f \rangle^\perp$.

Let $\mathcal{B}_{\langle f \rangle}$ be the Borel σ-algebra of $X_{\langle f \rangle}$ and $\mu_{\langle f \rangle}$ be normalised Haar measure on $\mathcal{B}_{\langle f \rangle}$. The horizontal and the vertical shifts $\sigma, \tau : \Omega_p \to \Omega_p$ defined by $(\sigma x)_{(i,j)} = x_{(i+1, j)}$ and $(\tau x)_{(i,j)} = x_{(i, j+1)}$ are group automorphisms preserving the subgroup $X_{\langle f \rangle}$ and, of course, $\mu_{\langle f \rangle}$. The dynamics of the measurable $\mathbb{Z}^2$-action $\mathcal{T}_{\langle f \rangle}$ generated by the restrictions of σ and τ on the measure space $(X_{\langle f \rangle}, \mathcal{B}_{\langle f \rangle}, \mu_{\langle f \rangle})$ have been recently studied by K. Schmidt and B. Kitchens in [Sch], [KS1], [KS2] (see also [W] and [Sh] for related results). One of the most interesting and hard questions here is whether or not two dynamical systems $(X_{\langle f \rangle}, \mathcal{B}_{\langle f \rangle}, \mu_{\langle f \rangle}, \mathcal{T}_{\langle f \rangle})$ and $(X_{\langle g \rangle}, \mathcal{B}_{\langle g \rangle}, \mu_{\langle g \rangle}, \mathcal{T}_{\langle g \rangle})$ with $f \neq g$ are measurably (topologically) isomorphic. It was proved in [KS1] (see also [KS2]) that the convex hull of the support S(f)

(up to a translation, of course) of the polynomial f is an invariant of measurable (and, hence, topological) conjugacy. K. Schmidt and B. Kitchens conjectured that $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ are measurably isomorphic if and only if $\langle f\rangle = \langle g\rangle$, but their invariants do not distinguish between the actions arising from polynomials f and g with the same support. Here we prove the above conjecture for the class of systems arising from polynomials with the support { (0,0), (1,0), (0,1) }.

We denote $\mathfrak{L}_p = \{ f \in \mathfrak{R}_p : S(f) = \{(0,0), (1,0), (0,1)\}\}$, i. e. $\mathfrak{L}_p$ is the subset of $\mathfrak{R}_p$ consisting of all polynomials f of the form $f(u,v) = c_f(0,0) + c_f(1,0)\,u + c_f(0,1)\,v$ with $c_f(i,j) \in \mathbb{Z}_p^{\times} = \mathbb{Z}_p \setminus \{0\}$. The main result of this Chapter is the following

Theorem 4.1.1. *The dynamical systems $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$, $f, g \in \mathfrak{L}_p$ are measurably isomorphic if and only if $\langle f\rangle = \langle g\rangle$, i. e. if and only if $f = a g$ for some $a \in \mathbb{Z}_p^{\times}$.*

Our strategem of proving this theorem is to gradually reduce the problem of measurable isomorphim to the problem of topological and, then, algebraic isomorphism.

4.2. MEASURABLE ISOMORPHISM IMPLIES TOPOLOGICAL CONGUGACY

Theorem 4. 2. 1. *Let $\varphi : X_{\langle f\rangle} \to X_{\langle g\rangle}$ be a measurable isomorphism of the dynamical systems $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$, $f, g \in \mathfrak{L}_p$, then there exists a shift commuting homeomorphism $\varphi' : X_{\langle f\rangle} \to X_{\langle g\rangle}$ such that $\varphi(x) = \varphi'(x)$ for $\mu_{\langle f\rangle}$-almost all $x \in X_{\langle f\rangle}$.*

Corollary 4. 2. 2. *If the dynamical systems $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$, $f, g \in \mathfrak{L}_p$ are measurably isomorphic, then they are topologically conjugate.*

Now we introduce some more notations and make some observations which will be used throughout the Chapter. With $E \subset \mathbb{Z}^2$ a finite set, let

$\pi_E : X_{\langle f\rangle} \to (\mathbb{Z}_p)^E$ be the natural projection onto the coordinates contained in E. Then $\pi_E(X_{\langle f\rangle})$ is a subgroup of $(\mathbb{Z}_p)^E$. We denote $\pi_E(X_{\langle f\rangle})$ by $X^E_{\langle f\rangle}$ and for every $\alpha \in X^E_{\langle f\rangle}$ define the cylinder set $C^E_{\langle f\rangle}(\alpha) = \{x \in X_G : \pi_E(x) = \alpha\}$. Clearly, $\mu_{\langle f\rangle}(C^E_{\langle f\rangle}(\alpha)) = |X^E_{\langle f\rangle}|^{-1}$. Define the triangles

$$E(m,n,s) = \{(i,j) \in \mathbb{Z}^2 : \ i \geq m,\ j \geq n,\ (i-m)+(j-n) < s\},$$

$(m,n) \in \mathbb{Z}^2$, $s \in \mathbb{N}$, and set $T(\ell) = E(-\ell, -\ell, 4\ell)$, $\ell \in \mathbb{N}$. We will basically deal with the cylinder sets of the form $C^{T(\ell)}_{\langle f\rangle}(\alpha)$. It is clear that for any finite set $E \subset \mathbb{Z}^2$ there is $\ell \in \mathbb{N}$ such that $E \subset T(\ell)$. Thus we have:

Observation 4.2.3. *The σ-algebra $\mathcal{B}_{\langle f\rangle}$ is the completion of the σ-algebra generated by the collection of cylinder sets $\{C^{T(\ell)}_{\langle f\rangle}(\alpha) : \alpha \in X^{T(\ell)}_{\langle f\rangle}, \ell \in \mathbb{N}\}$.*

The following statement can be easily proved by the same argument as Lemma 3.3.1 in Chapter 3 of the Thesis

Lemma 4.2.4. *Let A and B be unions of cylinder sets of the form $C^{T(\ell)}_{\langle f\rangle}(\alpha)$, $\alpha \in X^{T(\ell)}_{\langle f\rangle}$ with the same $\ell \in \mathbb{N}$. Then the sets A and $\sigma^{-m}(B)$ are independent with respect to the measure $\mu_{\langle f\rangle}$, whenever $m > 4\ell$.*

Next, we observe that for the polynomial $f(u,v) = c_f(0,0) + c_f(1,0)\,u + c_f(0,1)\,v \in \mathfrak{L}_p \subset \mathfrak{R}_p$ we have $(f(u,v))^{p^n} = c_f(0,0) + c_f(1,0)\,u^{p^n} + c_f(0,1)\,v^{p^n}$, but $f^{p^n} \in \langle f\rangle$ and, hence,

$$c_f(0,0)\,x_{(i,j)} + c_f(1,0)\,x_{(i+p^n,j)} + c_f(0,1)\,x_{(i,j+p^n)} = 0$$

for all $x \in X_{\langle f\rangle}$, $(i,j) \in \mathbb{Z}^2$, $n \in \mathbb{N}$. From this one easily obtains the following

Lemma 4.2.5. *Let $\alpha, \beta, \gamma \in X^{T(\ell)}_{\langle f\rangle}$ and $p^n > 4\ell$. Then the intersection $C^{T(\ell)}_{\langle f\rangle}(\alpha) \cap \sigma^{-p^n} C^{T(\ell)}_{\langle f\rangle}(\beta) \cap \tau^{-p^n} C^{T(\ell)}_{\langle f\rangle}(\gamma)$ is equal to $C^{T(\ell)}_{\langle f\rangle}(\alpha) \cap \sigma^{-p^n} C^{T(\ell)}_{\langle f\rangle}(\beta)$, if $c_f(0,0)\,\alpha + c_f(1,0)\,\beta + c_f(0,1)\,\gamma = 0$ in the group $(\mathbb{Z}_p)^{T(\ell)}$, and is empty, if $c_f(0,0)\,\alpha + c_f(1,0)\,\beta + c_f(0,1)\,\gamma \neq 0$.*

The proof of Theorem 4.2.1 is based on the following rather surprising fact

Lemma 4.2.6. *Let $\eta : X_{\langle f\rangle} \to X_{\langle f\rangle}$ be a shift-commuting automorphism of the measure space $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle})$, $f \in \mathfrak{L}_p$ and suppose that η has finite order r (i. e. $\mu_{\langle f\rangle}(E \,\Delta\, \eta^r(E)) = 0$ for each $E \in \mathcal{B}_{\langle f\rangle}$). Then for any set $A \in \mathcal{B}_{\langle f\rangle}$ with $\mu_{\langle f\rangle}(A) > 0$ satisfying*

$$\lim_{n\to\infty} \mu_{\langle f\rangle}(A \cap \sigma^{-p^n} A \cap \tau^{-p^n}\eta(A)) = \mu_{\langle f\rangle}(A)^2, \tag{4.1}$$

there exists a subgroup $H \subset X_{\langle f\rangle}$, *such that* $\mu_{\langle f\rangle}(Q \Delta A) = 0$ *for some coset* $Q \in X_{\langle f\rangle}/H$.

Proof. Without changing $\langle f\rangle$ we may assume that the polynomial f has $c_f(0,1) = -1$. To simplify notations we put $c_f(0,0) = a$, $c_f(1,0) = b$ so that $f(u,v) = a + bu - v$ and $X_{\langle f\rangle} = \{x \in \Omega_p : x_{(i,j+1)} = a\,x_{(i,j)} + b\,x_{(i+1,j)}$ for all $(i,j) \in \mathbb{Z}^2\}$. Also, since f is fixed throughout the proof, we will drop the subscript $\langle f\rangle$ in $X_{\langle f\rangle}$, $\mathcal{B}_{\langle f\rangle}$, $\mu_{\langle f\rangle}$, $C^{T(\ell)}_{\langle f\rangle}(\alpha)$ etc..

Fix an arbitrary $\varepsilon > 0$. Then, by virtue of Observation 4.2.3, for the set $A \in \mathcal{B}$ there is $\ell = \ell(\varepsilon) \geq 0$ and sets A_ε, $A_{\eta,\varepsilon}$ of the form

$$A_\varepsilon = \bigcup_{\alpha\in I} C^{T(\ell)}(\alpha), \quad A_{\eta,\varepsilon} = \bigcup_{\gamma\in J} C^{T(\ell)}(\gamma), \qquad I, J \subset X^{T(\ell)}$$

such that $\mu(A \Delta A_\varepsilon) \leq \varepsilon$, $\mu(\eta(A) \Delta A_{\eta,\varepsilon}) \leq \varepsilon$. Observe that the following equality holds

$$\int_{A_\varepsilon} \mu\{y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon}\}\, d\mu(x) =$$
$$= \lim_{n\to\infty} \mu(A_\varepsilon \cap \sigma^{-p^n}(A_\varepsilon) \cap \tau^{-p^n}(A_{\eta,\varepsilon})). \tag{4.2}$$

Indeed, using Lemmas 4.2.4, 4.2.5 for all $n \in \mathbb{N}$ with $p^n > 4\ell$ we have

$$\mu(A_\varepsilon \cap \sigma^{-p^n}(A_\varepsilon) \cap \tau^{-p^n}(A_{\eta,\varepsilon})) = \mu\Big(\bigcup_{\substack{\alpha,\beta\in I,\\ a\alpha+b\beta\in J}} (C^{T(\ell)}(\alpha) \cap \sigma^{-p^n} C^{T(\ell)}(\beta))\Big) =$$
$$= \bigcup_{\substack{\alpha,\beta\in I,\\ a\alpha+b\beta\in J}} \mu(C^{T(\ell)}(\alpha))\,\mu(C^{T(\ell)}(\beta)) = \sum_{\alpha\in I} \mu(C^{T(\ell)}(\alpha)) \bigcup_{\substack{\beta\in I,\\ a\alpha+b\beta\in J}} \mu(C^{T(\ell)}(\beta)),$$

which is equal to the integral in the left-hand side of (4.2), since given an $x \in C^{T(\ell)}(\alpha) \subset A_\varepsilon$ a point $y \in X$ satisfies $ax + by \in A_{\eta,\varepsilon}$ iff $y \in C^{T(\ell)}(\beta)$ where β satisfies $a\alpha + b\beta \in J$. This proves (4.2). Because the inequality

$$\mu((A \cap \sigma^{-p^n}(A) \cap \tau^{-p^n}(\eta(A))) \Delta (A_\varepsilon \cap \sigma^{-p^n}(A_\varepsilon) \cap \tau^{-p^n}(A_{\eta,\varepsilon}))) \leq 3\varepsilon$$

obviously holds for all $n \in \mathbb{N}$, the equality (4.2) yields

$$\mu(A)^2 = \lim_{n\to\infty} \mu(A \cap \sigma^{-p^n}(A) \cap \tau^{-p^n}(\eta(A)) \leq$$
$$\leq \lim_{n\to\infty} \mu(A_\varepsilon \cap \sigma^{-p^n}(A_\varepsilon) \cap \tau^{-p^n}(A_{\eta,\varepsilon})) + 3\varepsilon =$$

$$= \int_{A_\varepsilon} \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \} \, d\mu(x) + 3\varepsilon . \qquad (4.3)$$

For $E \subset X$, $z \in X$, $h \in \mathbb{Z}_p$ we denote $E + z = \{ x + z : x \in E \}$, $hE = \{ hx : x \in E \}$. Whenever E is measurable and $h \neq 0$, we have $\mu(E + z) = \mu(hE) = \mu(E)$ (since $x \to hx$ is a group automorphism of X, it preserves the Haar measure μ). Clearly, $\{ y \in E : ax + by \in E' \} = E \cap b^{-1}(E' - ax)$ for any $E, E' \subset X$. The inequalities $\mu(A \Delta A_\varepsilon) \leq \varepsilon$, $\mu(\eta(A) \Delta A_{\eta,\varepsilon}) \leq \varepsilon$ imply $\mu((A \cap b^{-1}(\eta(A) - ax)) \Delta (A_\varepsilon \cap b^{-1}(A_{\eta,\varepsilon} - ax))) \leq 2\varepsilon$ and, hence,

$$| \mu\{ y \in A : ax + by \in \eta(A) \} - \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \} | \leq 2\varepsilon$$

for all $x \in X$. This enables us to make the following estimate

$$\begin{aligned}
&\left| \int_A \mu\{ y \in A : ax + by \in \eta(A) \} \, d\mu(x) - \int_{A_\varepsilon} \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \} \, d\mu(x) \right| \leq \\
&\leq \int_{A \setminus A_\varepsilon} \mu\{ y \in A : ax + by \in \eta(A) \} \, d\mu(x) + \\
&+ \int_{A_\varepsilon \setminus A} \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \} \, d\mu(x) + \\
&+ \left| \int_{A_\varepsilon \cap A} (\mu\{ y \in A : ax + by \in \eta(A) \} - \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \}) \, d\mu(x) \right| \leq \\
&\leq \varepsilon + \varepsilon + \int_{A_\varepsilon \cap A} | \mu\{ y \in A : ax+by \in \eta(A) \} - \mu\{ y \in A_\varepsilon : ax+by \in A_{\eta,\varepsilon} \} | \, d\mu(x) \leq \\
&\leq 4\varepsilon .
\end{aligned}$$

Along with (4.3) this estimate gives us

$$\begin{aligned}
\mu(A)^2 &\geq \int_A \mu\{ y \in A : ax + by \in \eta(A) \} \, d\mu(x) \geq \\
&\geq \int_{A_\varepsilon} \mu\{ y \in A_\varepsilon : ax + by \in A_{\eta,\varepsilon} \} \, d\mu(x) - 4\varepsilon \geq \\
&\geq \mu(A)^2 - 3\varepsilon - 4\varepsilon = \mu(A)^2 - 7\varepsilon .
\end{aligned}$$

Since $\varepsilon > 0$ can be chosen arbitrarily, we obtain

$$\int_A \mu\{ y \in A : ax + by \in \eta(A) \} \, d\mu(x) = \int_A \mu(A \cap b^{-1}(\eta(A) - ax)) \, d\mu(x) = \mu(A)^2 ,$$

which means that for μ-almost all $x \in A$ we have $\mu(A \cap b^{-1}(\eta(A) - ax)) = \mu(A)$ and, hence, $\mu(A \Delta b^{-1}(\eta(A) - ax)) = 0$, or, in terms of the indicator functions, $\mathbb{1}_A(z) = \mathbb{1}_{b^{-1}(\eta(A) - ax)}(z) = \mathbb{1}_{\eta(A)}(ax + bz)$ for μ-almost all z (we write $\mathbb{1}_E$ for the indicator function of the set E). Finally, we express this fact as follows

$$\mathbb{1}_A(z) = \mu(A)^{-1} \int \mathbb{1}_{\eta(A)}(ax + bz) \, \mathbb{1}_A(x) \, d\mu(x) . \qquad (4.4)$$

Now we are going to use Fourier analysis on the compact abelian group

$\rho_E(\cdot) = \mu(E)^{-1}\mathbb{1}_E(\cdot)$ be the normalized indicator function of $E \subset X$. For the Fourier transform $\hat{\rho}_A : \hat{X} \to \mathbb{C}$ of the function $\rho_A \in \mathcal{L}^1(X, \mu)$ we have (taking into account (4.4))

$$\begin{aligned}
\hat{\rho}_A(\chi) &= \int \overline{\langle z, \chi \rangle}\, \rho_A(z)\, d\mu(z) = \mu(A)^{-1} \int \overline{\langle z, \chi \rangle}\, \mathbb{1}_A(z)\, d\mu(z) = \\
&= \mu(A)^{-2} \int \overline{\langle z, \chi \rangle} \Big(\int \mathbb{1}_{\eta(A)}(ax + bz)\, \mathbb{1}_A(x)\, d\mu(x) \Big)\, d\mu(z) = \\
&= \mu(A)^{-2} \iint \overline{\langle b^{-1}(ax+bz), \chi \rangle}\, \overline{\langle -b^{-1}ax, \chi \rangle}\, \mathbb{1}_{\eta(A)}(ax + bz)\, \mathbb{1}_A(x)\, d\mu(z)\, d\mu(x) = \\
&= \int \overline{\langle b^{-1}y, \chi \rangle}\, \rho_{\eta(A)}(y)\, d\mu(y) \int \overline{\langle -b^{-1}ax, \chi \rangle}\, \rho_A(x)\, d\mu(x) = \\
&= \hat{\rho}_{\eta(A)}(\chi \circ b^{-1})\, \hat{\rho}_A(\chi \circ (-b^{-1}a)),
\end{aligned}$$

for any character $\chi \in \hat{X}$, where $\chi \circ h$ $(h \in \mathbb{Z}_p)$ is the character given by $\langle x, \chi \circ h \rangle = \langle hx, \chi \rangle$ (we use the symmetric notation $\langle x, \chi \rangle$ for the value of the character χ on the element $x \in X$ which is justified by the Pontryagin duality). Putting $d = b^{-1}$, $e = -b^{-1}a \in \mathbb{Z}_p$, we obtain

$$\hat{\rho}_A(\chi) = \hat{\rho}_{\eta(A)}(\chi \circ d)\, \hat{\rho}_A(\chi \circ e), \tag{4.5}$$

for all $\chi \in \hat{X}$. The equation (4.5) holds for every set $A \in \mathcal{B}$ satisfying (4.1). But since η is shift-commuting and measure-preserving, the set $\eta^k(A)$ satisfies (4.1) whenever A does. Therefore, (4.5) remains true, if we replace A by $\eta^k(A)$ for any $k \in \mathbb{Z}$. Thus,

$$\hat{\rho}_{\eta^k(A)}(\chi) = \hat{\rho}_{\eta^{k+1}(A)}(\chi \circ d)\, \hat{\rho}_{\eta^k(A)}(\chi \circ e), \quad \chi \in \hat{X},\ k \in \mathbb{Z} \tag{4.6}$$

Clearly, $|\hat{\rho}_A(\chi)| \le 1$, $\chi \in \hat{X}$. Using (4.6), we will prove that in fact $|\hat{\rho}_A(\chi)|$ can take only two values: 0 and 1. Indeed, we have $|\hat{\rho}_A(\chi)| = |\hat{\rho}_{\eta(A)}(\chi \circ d)|\, |\hat{\rho}_A(\chi \circ e)|$. Suppose for some $m \in \mathbb{N}$ we have $|\hat{\rho}_A(\chi)| \le |\hat{\rho}_{\eta^m(A)}(\chi \circ d^m)|\, |\hat{\rho}_A(\chi \circ e^m)|$. Then, in view of (4.6), this implies $|\hat{\rho}_A(\chi)| \le |\hat{\rho}_{\eta^{m+1}(A)}(\chi \circ d^{m+1})|\, |\hat{\rho}_A(\chi \circ e^{m+1})|$. By induction, this proves the following inequality

$$|\hat{\rho}_A(\chi)| \le |\hat{\rho}_{\eta^k(A)}(\chi \circ d^k)|\, |\hat{\rho}_A(\chi \circ e^k)|, \quad k \in \mathbb{Z}. \tag{4.7}$$

For $k = r(p-1)$, where r is the order of η, (4.7) yields $|\hat{\rho}_A(\chi)| \le |\hat{\rho}_A(\chi)|^2$ which implies $|\hat{\rho}_A(\chi)| \in \{0, 1\}$.

It is a consequence of a classical result in Harmonic Analysis (see

[HR], p. 257) that the subset $\Gamma = \{\chi \in \hat{X} : |\hat{\rho}_A(\chi)| = 1\}$ is a subgroup and the restriction of $\hat{\rho}_A : \hat{X} \to \mathbb{C}$ to Γ is a character on Γ. Let $H = \Gamma^{\perp} \subset X$ be the annihilator of Γ. By virtue of the Pontryagin duality, there exists a coset $Y \in X/H$ such that $\hat{\rho}_A(\chi) = \langle \mathbf{x}_*, \chi \rangle$ for all $\chi \in \Gamma$ with an arbitrarily chosen $\mathbf{x}_* \in Y$, and from what we have proved above it follows that $\hat{\rho}_A(\chi) = 0$ for all $\chi \in \hat{X} \setminus \Gamma$. But it is easy to see now that $\hat{\rho}_A$ is identical to the Fourier transform of the function $\rho_{H-\mathbf{x}_*}(\mathbf{x}) = \mu(H)^{-1} \mathbb{1}_{H-\mathbf{x}_*}(\mathbf{x})$. By the Inversion Theorem, the functions ρ_A and $\rho_{H-\mathbf{x}_*}$ are equal as elements of $\mathcal{L}^1(X, \mu)$. Hence, A differs from the coset $H-\mathbf{x}_*$ by a set of measure zero. □

Lemma 4.2.6. Any subgroup H of $X_{\langle f\rangle}$ with $\mu_{\langle f\rangle}(H) > 0$ (and, hence, any coset $Q \in X_{\langle f\rangle}/H$) is a closed and open subset of $X_{\langle f\rangle}$.

Proof. One can easily show that the map $\Delta_H : X_{\langle f\rangle} \to \mathbb{R}$ defined by $\Delta_H(\mathbf{x}) = \mu_{\langle f\rangle}(H \,\Delta\, (H+\mathbf{x}))$ is continuous. Since $\mu_{\langle f\rangle}(H) > 0$, the closed (by continuity) set $\{\mathbf{x} \in X_{\langle f\rangle} : \Delta_H(\mathbf{x}) = 0\}$ is precisely the subgroup H. H is also open, because its complement is a finite union of cosets which are closed, since H is. □

Thus, we have shown that any set of positive measure, satisfying (4.1) differs from a closed-open set by a set of measure 0. Now we proceed to proving Theorem 4.2.1.

Proof of Theorem 4.2.1. Let $\varphi : X_{\langle f\rangle} \to X_{\langle g\rangle}$ be a measurable isomorphism of the dynamical systems $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$, $f, g \in \Omega_p$. Any cylinder set $C_f = C^{T(\ell)}_{\langle f\rangle}(\alpha) \subset X_{\langle f\rangle}$ obviously satisfies

$$\lim_{n\to\infty} \mu_{\langle f\rangle}(C_f \cap \sigma^{-p^n}(C_f) \cap \tau^{-p^n}\xi(C_f)) = \mu_{\langle f\rangle}(C_f)^2 ,$$

where $\xi : X_{\langle f\rangle} \to X_{\langle f\rangle}$ is given by $\xi(\mathbf{x}) = -c_f(0,1)^{-1}(c_f(0,0) + c_f(1,0))\,\mathbf{x}$. Clearly, if $c_f(0,0) + c_f(1,0) \neq 0$, then ξ is a shift-commuting automorphism of finite order ($\xi^{p-1}(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$). Hence, so is the automorphism $\eta = \varphi \circ \xi \circ \varphi^{-1} : X_{\langle g\rangle} \to X_{\langle g\rangle}$. For any cylinder set $C = C^{T(\ell)}_{\langle f\rangle}(\alpha)$ the image $\varphi(C_f) \in \mathcal{B}_{\langle g\rangle}$ has positive measure and satisfies

$$\lim_{n\to\infty} \mu_{\langle g\rangle}(\varphi(C_f)\cap\sigma^{-p^n}\varphi(C_f))\cap\tau^{-p^n}\eta(\varphi(C_f))) = \mu_{\langle g\rangle}(\varphi(C_f))^2,$$

Now Lemmas 4.2.5 and 4.2.6 tell us that $\varphi(C_f)$ coincides with a closed and open set up to a set of measure 0. The "mirror image" of the above argument shows that for any cylinder set $C_g \subset X_{\langle f\rangle}$ the pre-image $\varphi^{-1}(C_g) \in \mathcal{B}_{\langle f\rangle}$ coincides with a closed and open open set up to a set of measure 0.

Thus, φ establishes a well defined one-to-one correspondence between the set algebras $\mathcal{CO}_{\langle f\rangle}$ and $\mathcal{CO}_{\langle g\rangle}$, where $\mathcal{CO}_{\langle f'\rangle} \subset \mathcal{B}_{\langle f'\rangle}$ is the algebra of clopen (closed and open) sets of $X_{\langle f'\rangle}$, i. e. the algebra generated by cylinder sets. This correspondence defines an isomorphism F_{co} between the algebras of simple continuous functions $\mathfrak{C}_{co}(X_{\langle f\rangle})$ and $\mathfrak{C}_{co}(X_{\langle g\rangle})$ (the algebra $\mathfrak{C}_{co}(X_{\langle f'\rangle})$ is generated by the indicator functions of clopen subsets of $X_{\langle f'\rangle}$). Since $\mathfrak{C}_{co}(X_{\langle f'\rangle})$ is dense in the Banach algebra $\mathfrak{C}(X_{\langle f'\rangle})$ of continuous functions on $X_{\langle f'\rangle}$, the isomorphism $F_{co}: \mathfrak{C}_{co}(X_{\langle f\rangle}) \to \mathfrak{C}_{co}(X_{\langle g\rangle})$ has a unique extension to an isomorphism of the Banach algebras $F: \mathfrak{C}(X_{\langle f\rangle}) \to \mathfrak{C}(X_{\langle g\rangle})$. In its turn, F induces a homeorphism $\Phi: \mathfrak{M}(\mathfrak{C}(X_{\langle f\rangle})) \to \mathfrak{M}(\mathfrak{C}(X_{\langle g\rangle}))$ between the maximal ideal spaces of the Banach algebras. But, according to the Gelfand theory (see [Ru], p.271), $\mathfrak{M}(\mathfrak{C}(X_{\langle f'\rangle}))$ is (homeomorphic to) $X_{\langle f'\rangle}$ Thus, we have obtained a homeomorphism $\Phi: X_{\langle f\rangle} \to X_{\langle g\rangle}$ which defines the same correspondence between the set algebras $\mathcal{CO}_{\langle f\rangle}$ and $\mathcal{CO}_{\langle g\rangle}$ as φ. It is clear that if we throw away the set $N = \bigcup_{C\in \mathcal{CO}_{\langle f\rangle}} (C \,\Delta\, \varphi^{-1}(\Phi(C)))$, then $\varphi(x) = \Phi(x)$, on the complement $X_{\langle f\rangle}\backslash N$. But N is a countable union of sets of measure 0 and, therefore, $\mu_{\langle f\rangle}(N) = 0$ completing the proof.

In the argument above we used the fact that $c_f(0,0)+c_f(1,0) \neq 0$. But it is easy to se that, unless $p = 2$ (the case $p = 2$ is of no interest for us, since $\mathfrak{L}_2$ consists of only one polynomial), at least one of the following inequalities holds: $c_f(0,0)+c_f(1,0) \neq 0$, $c_f(0,0)+c_f(0,1) \neq 0$, $c_f(1,0)+c_f(1,0) \neq 0$. If, say, $c_f(1,0)+c_f(1,0) \neq 0$ is the case, then we just have to use a modified version of Lemma 4.2.6, with

$$\lim_{n\to\infty} \mu_{\langle f\rangle}(\eta(A) \cap \sigma^{-p^n}A \cap \tau^{-p^n}A) = \mu_{\langle f\rangle}(A)^2,$$

instead of (4.1) (the proof of the lemma remains essentially the same). $\square$

4.3. TOPOLOGICAL CONGUGACY IMPLIES ALGEBRAIC ISOMORPHISM

In the previous section we proved that any measurable isomorphism of the dynamical systems $(X_{\langle f\rangle}, \mathcal{B}_{\langle f\rangle}, \mu_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle})$ and $(X_{\langle g\rangle}, \mathcal{B}_{\langle g\rangle}, \mu_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$, $f, g \in \mathfrak{L}_p$ must coincide almost everywhere with a homeomorphism. This enables us now to be concerned only with the question of topological conjugacy. In this section we consider the shift action $\mathcal{T}_{\langle f\rangle}$ on $X_{\langle f\rangle}$ as a topological dynamical system. Theorem 4.3.1 below provides the further reduction. Namely, it turns out that a topological conjugacy between the Markov subgroups must respect their algebraic structure.

Remark that the abelian group $\Omega_p = (\mathbb{Z}_p)^{\mathbb{Z}^2}$ can in a natural way be considered as an (infinite dimensional) vector space over the finite field $\mathbb{Z}_p$. and any Markov subgroup $X_{\langle f\rangle}$ becomes a subspace of Ω_p. The following theorem is a generalization of Observation 4.1 in [KS1].

Theorem 4.3.1. *Let $f, g \in \mathfrak{L}_p$ and let $\varphi : (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ be a continuous factor map. Then there exists $h \in \mathbb{Z}_p$ such that $(\varphi(x))_{(i,j)} = (\psi(x))_{(i,j)} + h$, $x \in X_{\langle f\rangle}$, $(i,j) \in \mathbb{Z}^2$, where $\psi : (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ is a homomorphism of vector spaces .*

Proof. For any $f' \in \mathfrak{R}_p$ let $0 \in X_{\langle f\rangle}$ be the configuration of all zeros (the zero element of $X_{\langle f'\rangle}$). Obviously 0 is a fixed point of the action $\mathcal{T}_{\langle f'\rangle}$ and it is the only one, unless $f'(1,1) = 0$; if $f'(1,1) = 0$, then for each $h \in \mathbb{Z}_p$ the action $\mathcal{T}_{\langle f'\rangle}$ has the fixed point $\mathbf{b} \in X_{\langle f\rangle}$ all coordinates of which equal h. The factor map $\varphi : (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ sends the fixed point $\mathbf{0} \in X_{\langle f\rangle}$ to a fixed point $\mathbf{h} \in X_{\langle g\rangle}$, hence $\varphi(\mathbf{x}) = \psi(\mathbf{x}) + \mathbf{h}$, $\mathbf{x} \in X_{\langle f\rangle}$, where $\psi : (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ is a factor map with the additional property $\psi(0) = \mathbf{0}$.

Let $f, g \in \mathfrak{L}_p$ and let $\psi : (X_f, \mathcal{T}_f) \to (X_g, \mathcal{T}_g)$ be a continuous factor map such that $\psi(0) = 0$. Clearly, there exists $\ell \in \mathbb{N}$ and a map $\Psi : X_{(f)}^{T(\ell)} \to \mathbb{Z}_p$ such that $(\psi(\mathbf{x}))_{(i,j)} = \Psi(\pi_{T(\ell)}(\tau^i \sigma^j \mathbf{x}))$, $\mathbf{x} \in X_{\langle f\rangle}$, $(i,j) \in \mathbb{Z}^2$. Take $\theta, \alpha, \beta \in X_{(f)}^{T(\ell)}$, where θ is the zero element of the group $X_{(f)}^{T(\ell)}$ and α, β are just arbitrary, and denote: $O = C_{(f)}^{T(\ell)}(\theta)$, $A = C_{(f)}^{T(\ell)}(\alpha)$, $B = C_{(f)}^{T(\ell)}(\beta)$. According to Lemma 4.2.4 the intersections $A \cap \sigma^{-p^n} O$ and $B \cap \tau^{-p^n} O$ are non-empty for n large enough, and we fix $x \in A \cap \sigma^{-p^n} O$ and $y \in B \cap \tau^{-p^n} O$. We also consider the point $x + y \in X_{\langle f \rangle}$. We have

$$c_f(0,0)\,(\psi(x))_{(0,0)} + c_f(1,0)\,(\psi(x))_{(p^n,0)} + c_f(0,1)\,(\psi(x))_{(0,p^n)} = 0,$$
$$c_f(0,0)\,(\psi(y))_{(0,0)} + c_f(1,0)\,(\psi(y))_{(p^n,0)} + c_f(0,1)\,(\psi(y))_{(0,p^n)} = 0,$$
$$c_f(0,0)\,(\psi(x+y))_{(0,0)} + c_f(1,0)\,(\psi(x+y))_{(p^n,0)} + c_f(0,1)\,(\psi(x+y))_{(0,p^n)} = 0.$$

On the other hand, $(\psi(x))_{(0,0)} = \Psi(\alpha)$, $(\psi(y))_{(0,0)} = \Psi(\beta)$, $(\psi(x+y))_{(0,0)} = \Psi(\alpha+\beta)$, $(\psi(x))_{(p^n,0)} = \Psi(\theta) = 0$, $(\psi(y))_{(0,p^n)} = \Psi(\theta) = 0$, $(\psi(x+y))_{(p^n,0)} = (\psi(y))_{(p^n,0)}$, $(\psi(x+y))_{(0,p^n)} = (\psi(x))_{(0,p^n)}$, where the equality $\Psi(\theta) = 0$ follows from $\psi(0) = 0$. Thus, we obtain $c_f(0,0)\Psi(\alpha+\beta) =$ $= c_f(0,0)\Psi(\alpha) + c_f(0,0)\Psi(\beta)$ and $\Psi(\alpha+\beta) = \Psi(\alpha) + \Psi(\beta)$, where α and β are arbitrary. This shows that $\psi(x+y) = \psi(x) + \psi(y)$, $x, y \in X_{\langle f \rangle}$, which also implies $\psi(d\,x) = d\,\psi(x)$, $x \in X_{\langle f \rangle}$, $d \in \mathbb{Z}_p$. $\square$

Corollary 4.3.2. Let $f, g \in \Omega_p$ and let $\varphi : (X_{\langle f \rangle}, \mathcal{T}_{\langle f \rangle}) \to (X_{\langle g \rangle}, \mathcal{T}_{\langle g \rangle})$ be a topological conjugacy. Then there exists $h \in \mathbb{Z}_p$ such that $(\varphi(x))_{(i,j)} =$ $= (\psi(x))_{(i,j)} + h$, $x \in X_{\langle f \rangle}$, $(i,j) \in \mathbb{Z}^2$, where $\psi : (X_{\langle f \rangle}, \mathcal{T}_{\langle f \rangle}) \to (X_{\langle g \rangle}, \mathcal{T}_{\langle g \rangle})$ is an isomorphism of vector spaces .

4.4. ALGEBRAIC CLASSIFICATION

Now we give the complete classification of the "three dot" Markov subgroups with respect to a shift commuting vector space isomorphism. In view of the results of the previous sections, this algebraic classification immediately gives a measurable classifications of such dynamical systems.

***Theorem** 4.4.1. Let $f, g \in \Omega_p$ and let $\varphi : (X_{\langle f \rangle}, \mathcal{T}_{\langle f \rangle}) \to (X_{\langle g \rangle}, \mathcal{T}_{\langle g \rangle})$ be an isomorphism of vector spaces commuting with shifts. Then $\langle f \rangle = \langle g \rangle$, i. e. $f = d\,g$ for some $d \in \mathbb{Z}_p^{\times}$.*

Proof. Observe that for any $f \in \Omega_p$ the set $\mathrm{Fix}(X_{\langle f \rangle}, \tau \circ \sigma) =$ $= \{ x \in X_{\langle f \rangle} : \tau \circ \sigma(x) = x \}$ is a subspace of the vector space $X_{\langle f \rangle}$ isomorphic to $(\mathbb{Z}_p)^2$, the isomorphism being given by the projection π_S, where $S = \{ (0,0),$

$(1,0)\}$. The restriction of σ onto $\mathrm{Fix}(X_{\langle f\rangle}, \tau\circ\sigma) \cong (\mathbb{Z}_p)^2$ is the linear operator given by the matrix

$$M_f = \begin{pmatrix} -c_f(0,0)\, c_f(1,0)^{-1} & -c_f(0,1)\, c_f(1,0)^{-1} \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{Z}_p).$$

Now let $f, g \in \Omega_p$ and suppose that there exists a topological conjugacy $\varphi : (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$. Then from Lemma 4.3.2 it follows that $\varphi(x) = \psi(x) + b$, $x \in X_{\langle f\rangle}$, where $\psi: (X_{\langle f\rangle}, \mathcal{T}_{\langle f\rangle}) \to (X_{\langle g\rangle}, \mathcal{T}_{\langle g\rangle})$ is an isomorphism of vector spaces. Clearly, ψ induces an isomorphism between $\mathrm{Fix}(X_{\langle f\rangle}, \tau\circ\sigma)$ and $\mathrm{Fix}(X_{\langle g\rangle}, \tau\circ\sigma)$ which can be considered as an invertible linear operator on $(\mathbb{Z}_p)^2$ given by a matrix $R \in GL(2, \mathbb{Z}_p)$. As ψ is shift-commuting, R must satisfy the equation $R\, M_f = M_g\, R$. This implies $\mathrm{tr}(M_{f_1}) = \mathrm{tr}(M_{f_2})$ and $\det(M_f) = \det(M_g)$. Thus, we have

$$c_f(0,0)\, c_f(1,0)^{-1} = c_g(0,0)\, c_g(1,0)^{-1}, \quad c_f(0,1)\, c_f(1,0)^{-1} = c_g(0,1)\, c_g(1,0)^{-1},$$

hence, $c_f(0,0)\, c_g(0,0)^{-1} = c_f(1,0)\, c_g(1,0)^{-1} = c_f(0,1)\, c_g(0,1)^{-1}$, i. e. $g = a f$ for some $a \in \mathbb{Z}_p^{\times}$ which completes the proof. $\square$

Now our Main Result (Theorem 4.1.1) follows immediately from Corollary 4.2.2, Corollary 4.3.2 and heorem 4.4.1.

REFERENCES

[B] H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. London Math. Soc. **25** (1972), 603- 614.

[Bi] P. Billingsley, *Ergodic Theory and Information*, John Wiley & Sons, New York, 1965

[BK] M. Boyle and W. Krieger, *Periodic points and automorphisms of the shift*, Trans. Amer. Math. Soc. **302** (1987), 125-149

[BL] M. Boyle and D. Lind, *Expansive subdynamics of $\mathbf{Z}^n$-actions*, In preparation

[CFS] I. P. Cornfeld, S.V. Fomin, and Ya. G. Sinai, *Ergodic Theory*, Springer-Verlag, New York, 1981

[C] E. M. Coven, *Topological entropy of block maps*, Proc. Amer. Math. Soc. **78** (1980), 590-594

[CP] E. M. Coven and M. Paul, *Endomorphisms of irreducible subshifts of finite type*, Math. Syst. Theory **8** (1974), 167-175.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic Theory on Compact Spaces*, Lecture Notes in Math. **527**, Springer-Verlag, New York, 1976.

[F] A. Fathi, *Expansiveness, hyperbolicity and Hausdorff dimension*, Commun. Math. Phys. **126** (1989), 242-261.

[Fe] J. Feldman, *Ergodic theory of continuous group actions*, Astérisque 49 (1977), 61 - 74.

[G] R. H. Gilman, Classes of linear automata, Ergod. Theory and Dynam. Syst. **7** (1987), 105-118

[Gr] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. IHES **53** (1981), 53-73.

[HW] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, 1975

[H] G. A. Hedlund, *Endomorphisms and automorphisms of the shift dynamical systems*, Math. Syst. Theory **3** (1969), 320-375.

[HR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, Vol. II*, Springer-Verlag, Berlin, 1970

[Hu] M. Hurley, *Attractors in cellular automata*, Ergod. Theory and Dynam. Syst. **10** (1990), 131-140

[KS1] B. Kitchens and K. Schmidt, *Markov subgroups of $(\mathbf{Z}_2)^{\mathbf{Z}^2}$*, Contemp. Math., vol. 135 (1992), 265-283

[KS2] B. Kitchens and K. Schmidt, *Mixing sets and relative entropies for higher-dimensional Markov shifts.* University of Warwick, Preprint, 1991

[L] F. Ledrappier, *Un champ markovien peut être d'entropie nulle et mélangeant*, C. R. Acad. Sci. Paris, Ser. A, **287** (1978), 561-562

[Li] D. Lind, *Applications of ergodic theory and sofic systems to cellular automata*, Physica 10D (1984), 36-44

[LSW] D. Lind, K. Schmidt and T. Ward, *Mahler measure and entropy for commuting auiomorphisms of compact groups*, Invent. Math. **101** (1990), 593-629

[M-O] J. Moulin-Ollagnier, *Ergodic Theory and Statistical Mechanics*, Lecture Notes in Math. **1115**, Springer-Verlag, New York, 1985.

[N] M. Nasu, *Textile systems for endomorphisms and automorphisms of the shift*, Mie University, Preprint, 1992

[O] D. S. Ornstein, *Ergodic Theory, Randomness and Dynamical Systems*, Yale University Press, New Haven, Connecticut, 1974

[R] V. A. Rohlin, *Exact endomorphisms of a Lebesgue space*, Amer. Math. Soc. Translations (2) **39** (1964), 1-36

[Ru] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973

[Schn] E. Schenkman, *Group Theory*, Van Nostrand, 1965

[Sch1] K. Schmidt, *Algebraic ideas in ergodic theory*, CBMS Regional Conf. Ser. in Math., No. 76, Amer. Math Soc., Providence RI, 1990

[Sch2] K. Schmidt, *Asymptotic properties of unitary representations and mixing*, Proc. London Math. Soc. 48 (1984), 445-460

[Sch3] K. Schmidt, *Automorphisms of compact abelian groups and affine varieties*, Proc. London Math. Soc., **61** (1990), 480 - 496

[SchS] K. Schmidt and M. A. Shereshevsky, *New obstructions to measurable and topological conjugacy of some Markov subgroups of* $(\mathbf{Z}_p)^{\mathbf{Z}^2}$, Israel J. Math. (to appear)

[S1] M. A. Shereshevsky, *Ergodic properties of certain surjective cellular automata*, Monatsh. Math, (to appear)

[S2] M. A. Shereshevsky, *Expansiveness, entropy and polynomial growth for groups acting on subshifts by automorphisms*, Indag. Math., (to appear)

[S3] M. A. Shereshevsky, *On the classification of some two-dimensional Markov shifts with group structure*, Ergod. Th. & Dyn. Syst., (to appear)

[SA] M. A. Shereshevsky and V. S. Afraimovich, *Bipermutative cellular automata are topologically conjugate to the one-sided Bernoulli shift*, Random & Comput. Dynamics, 1 (1992) 91-98

[ShR] M. Shirvani and T. D. Rogers, *On ergodic one-dimensional cellular automata*, Commun. Math. Phys. 136 (1991) 599-605

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer-Verlag, New York, 1982.

[War] T. Ward, *An algebraic obstruction to isomorphism of Markov shifts with group alphabet*, Bull. London Math. Soc. (to appear)

[Wo] S. Wolfram, *Universality and complexity in cellular automata*, Physica 10D (1984), 1-35

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Chapter 4

Three Dimensional Bénard Convection

4.1 Introduction

In this chapter we apply some of the results of chapters 2 and 3 to the Bénard convection problem in a 3-dimensional box. In section 1.1 we gave a brief description of bifurcations occuring in a 2-dimensional vertical cross section of the domain. By exploring the third direction we find a much richer structure of patterns, specialy when the horizontal cross section is a square.

In section 4.2 we state the Boussinesq approximation of the equations for time independent convection in a box with a mixture of Neumann and Dirichlet boundary conditions. Then we define a scaling that makes the quantities nondimensional.

In section 4.3 we set the symmetry context where these equations should be viewed. It will be shown that the problem has more symmetries than the group that leaves the domain invariant. These symmetries are found by defining a periodic extension by reflection across the boundaries in such a way that the regularity of the solutions is preserved. The extended problem satisfies periodic boundary conditions on a larger domain and the symmetries are a 3-torus extension of the group that leaves the domain invariant. Then, the solutions that satisfy the original boundary conditions are constrained by invariance under a group of reflections.

Sections 4.4 and 4.5 deal with bifurcation problems. The construction described above will be used to give a general form of the reduced bifurcation equations. In appendices A and B, normal forms and universal unfoldings are given together with some bifurcation diagrams.

Sections 4.6, 4.7 and 4.8 are concerned with the well known method of Liapunov-Schmidt reduction that gives an exact Taylor expansion of the bifurcation equations. From the previous sections we have the information about which terms must be calculated. By taking the result to appendix A or B we get the bifurcation diagrams.

4.2 The Boussinesq Equations

As in Golubitsky *et al.* [11], the Boussinesq approximation of the equations for the Bénard convection in the box $[0,\pi\ell_1]\times[0,\pi\ell_2]\times[0,\pi\ell_3]$ may be written as

$$
\begin{aligned}
&\frac{1}{\sigma}\left(\frac{\partial v}{\partial t}+(v.\nabla)v\right) = -\nabla p+\Theta g+\Delta v\\
&\operatorname{div} v = 0\\
&\frac{\partial\Theta}{\partial t}+(v.\nabla)\Theta = Rv_3+\Delta\Theta
\end{aligned}
$$

where $v=(v_1,v_2,v_3)$ is the velocity vector, Θ describes the deviation of temperature, p is the pressure and the parameters R and σ are, respectively, the Rayleigh number and the Prandtl number. In order to nondimensionalize the quantities, we scale the domain variables as

$$\xi_1\mapsto\frac{1}{\ell_1}\xi_1 \qquad \xi_2\mapsto\frac{1}{\ell_2}\xi_2 \qquad \xi_3\mapsto\frac{1}{\ell_3}\xi_3,$$

the deviation of temperature, the pressure and the Rayleigh and Prandtl numbers the are scaled as

$$\Theta\mapsto\ell_3^3\Theta \qquad p\mapsto\ell_3 p \qquad R\mapsto\ell_3^2R \qquad \sigma\mapsto\frac{1}{\ell_3}\sigma.$$

By denoting $u=(v,\Theta,p)$ and applying the scaling above to the time independent Boussinesq equations we get

$$
\begin{aligned}
\Phi_r[1](u,R) &\equiv \Delta_r v_1-\frac{1}{r_1}\frac{\partial p}{\partial\xi_1}-\frac{1}{\sigma}v.\nabla_r v_1=0\\
\Phi_r[2](u,R) &\equiv \Delta_r v_2-\frac{1}{r_2}\frac{\partial p}{\partial\xi_2}-\frac{1}{\sigma}v.\nabla_r v_2=0\\
\Phi_r[3](u,R) &\equiv \Delta_r v_3-\frac{\partial p}{\partial\xi_3}+\Theta-\frac{1}{\sigma}v.\nabla_r v_3=0\\
\Phi_r[4](u,R) &\equiv \Delta_r\Theta+Rv_3-v.\nabla_r\Theta=0\\
\Phi_r[5](u,R) &\equiv \nabla_r.v=0
\end{aligned}
\tag{4.1}
$$

where the parameter

$$r = (r_1,r_2)=\left(\frac{\ell_1}{\ell_3},\frac{\ell_2}{\ell_3}\right)$$

has been transferred from the dimensions of the domain to the equations, and Δ_r, ∇_r are, respectively, scaled laplacian and gradient as

$$
\begin{aligned}
\Delta_r &= \frac{1}{r_1^2}\frac{\partial^2}{\partial\xi_1^2}+\frac{1}{r_2^2}\frac{\partial^2}{\partial\xi_2^2}+\frac{\partial^2}{\partial\xi_3^2}\\
\nabla_r &= \left(\frac{1}{r_1}\frac{\partial}{\partial\xi_1},\frac{1}{r_2}\frac{\partial}{\partial\xi_2},\frac{\partial}{\partial\xi_3}\right).
\end{aligned}
$$

We denote the scaled domain $\Omega = [0,\pi]^3$ and the boundary conditions are a mixture of Neumann and Dirichlet as

$$\begin{array}{lll} v_1 = \dfrac{\partial v_2}{\partial \xi_1} = \dfrac{\partial v_3}{\partial \xi_1} = \Theta_{\xi_1} = p_{\xi_1} = 0 & \text{for} & \xi_1 = 0,\pi \\ \dfrac{\partial v_1}{\partial \xi_2} = v_2 = \dfrac{\partial v_3}{\partial \xi_2} = \Theta_{\xi_2} = p_{\xi_2} = 0 & \text{for} & \xi_2 = 0,\pi \\ \dfrac{\partial v_1}{\partial \xi_3} = \dfrac{\partial v_2}{\partial \xi_3} = v_3 = \Theta = p_{\xi_3} = 0 & \text{for} & \xi_3 = 0,\pi. \end{array} \tag{4.2}$$

4.3 The Appropriate Symmetry Context

Let $u \in (C^\infty(\Omega))^5$ be an arbitrary solution of $\Phi_r = 0$. We proceed by summarizing the extension method described in chapters 2 and 3 and showing that the procedure applies to this particular problem. We define an extension $\hat{u}$ by reflecting each component of u across the boundaries of the domain Ω and combine it with sign change in the case of a Dirichlet boundary condition. These reflections generate the group

$$K \;=\; \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 = \langle \kappa_1, \kappa_2, \kappa_3 \rangle$$

acting on $\hat{u}$ as

$$\begin{array}{ll} \kappa_1 : & (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (-v_1,v_2,v_3,\Theta,p)(-\xi_1,\xi_2,\xi_3) \\ \kappa_2 : & (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (v_1,-v_2,v_3,\Theta,p)(\xi_1,-\xi_2,\xi_3) \\ \kappa_3 : & (v_1,v_2,v_3,\Theta,p)(\xi_1,\xi_2,\xi_3) \mapsto (v_1,v_2,-v_3,-\Theta,p)(\xi_1,\xi_2,-\xi_3). \end{array}$$

In lemma 9 below we show that this extended function is a solution of the same operator on the larger domain

$$\hat{\Omega} \;=\; [-\pi,\pi]^3.$$

Then, by extending $\hat{u}$ periodicaly to $\mathbb{R}^3$ we still have a solution of $\Phi_r = 0$ for which we keep the notation $\hat{u}$. Lemma 10 below says that this extension preserves the regularity of the solution. The extended solution $\hat{u}$ satisfies periodic boundary conditions on $\hat{\Omega}$.

Now we define an action of the extended group $K\dot{+}\mathbf{T}^3$. This group is generated by the reflections $\kappa_1, \kappa_2, \kappa_3$ acting as above together with the translations $\theta_j \in [0,2\pi)$ for $j = 1,2,3$ acting as

$$\begin{array}{ll} \theta_1 : & \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1+\theta_1,\xi_2,\xi_3) \\ \theta_2 : & \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1,\xi_2+\theta_2,\xi_3) \\ \theta_3 : & \hat{u}(\xi_1,\xi_2,\xi_3) \mapsto \hat{u}(\xi_1,\xi_2,\xi_3+\theta_3). \end{array}$$

Given the translation invariance of the operator Φ_r, by acting on $\hat{u}$ with an element of the 3-torus $\mathbf{T}^3$ we get a solution of the periodic boundary value problem.

Up to this point we know that any smooth solution u of $\Phi_r = 0$ satisfying the boundary conditions (4.2) corresponds to a unique $K\dot{+}\mathbf{T}^3$-orbit of solutions to the same equation satisfying periodic boundary conditions on $\hat{\Omega}$. We choose from this group orbit of $\hat{u}$ all the solutions w that satisfy the boundary conditions (4.2). These are exactly the ones that are fixed by the action of K

$$\kappa w(\kappa\xi) = w(\xi)$$

for all $\kappa \in K$. These solutions belong to the subspace fixed by K, which we denote by $\mathrm{Fix}(K)$.

We proceed by stating and proving the results referred to above.

Lemma 9 *Let u be a solution of $\Phi_r = 0$. Let the group K act as above. Then κu is a solution of the same equation for all $\kappa \in K$.*

Proof Given the group structure of K it is enough to show that this result holds for the generators of K. We define an action of $\kappa_1, \kappa_2, \kappa_3$ on the operator Φ_r as

$$\begin{aligned}
\kappa_1 :\ & (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (-\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \\
\kappa_2 :\ & (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[1], -\Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \\
\kappa_3 :\ & (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[1], \Phi_r[2], -\Phi_r[3], -\Phi_r[4], \Phi_r[5]).
\end{aligned}$$

It is easy to see that Φ_r commutes with the action of K

$$\Phi_r(\kappa_j u(\kappa_j \xi), R) = \kappa_j \Phi_r(u(\xi), R) \qquad \text{for} \qquad j = 1, 2, 3. \tag{4.3}$$

By assumption, u is a solution of $\Phi_r = 0$. So

$$\Phi_r(u(\xi), R) = 0 \qquad \text{for} \qquad j = 1, 2, 3.$$

Together with (4.3) this implies that

$$\Phi_r(\kappa_j u(\kappa_j \xi), R) = 0 \qquad \text{for} \qquad j = 1, 2, 3$$

and this is what we wanted to show. □

Lemma 10 *Let $u \in (C^1(\mathbb{R}^3))^5$ be a solution of $\Phi_r = 0$. Then $u \in (C^\infty(\mathbb{R}^3))^5$.*

Proof See Field *et al.* [8]. □

Note that by this method we find symmetries that are not obvious in bounded domains. The most immediate thing to do would be to consider only the reflections that leave the equations and domain invariant. This approach would be simpler but incomplete: some translations of our extended solutions satisfy the required boundary conditions and remain hidden if we insist upon leaving the domain invariant.

Up to now we mentioned all the symmetries that do not depend on the parameter r. If we allow group actions on r there is one more symmetry: the group $\mathbf{S}_2$. This group has one generator denoted by s and acting as

$$\begin{aligned}
s :\ & (r_1, r_2) \mapsto (r_2, r_1) \\
s :\ & (v_1, v_2, v_3, \Theta, p)(\xi_1, \xi_2, \xi_3) \mapsto (v_2, v_1, v_3, \Theta, p)(\xi_2, \xi_1, \xi_3) \\
s :\ & (\Phi_r[1], \Phi_r[2], \Phi_r[3], \Phi_r[4], \Phi_r[5]) \mapsto (\Phi_r[2], \Phi_r[1], \Phi_r[3], \Phi_r[4], \Phi_r[5]).
\end{aligned}$$

By noting that the set of boundary conditions is invariant under the action of $\mathbf{S}_2$ on u we have the following

Lemma 11 *Let u be a solution of $\Phi_r = 0$ satisfying the boundary conditions (4.2). Let the group $\mathbf{S}_2$ act as above. Then su is a zero of $\Phi_{sr} = 0$ satisfying the same boundary conditions.*

Proof As we said before there is nothing to prove about the boundary conditions. So we only have to show that the s-conjugate of a solution u of $\Phi_r = 0$ is a solution of the conjugate operator Φ_{sr}. A straightforward calculation shows that the operator Φ_r commutes with the action of $\mathbf{S}_2$ as

$$\Phi_{sr}(su(s\xi), R) \quad = \quad s\Phi_r(u(\xi), R). \tag{4.4}$$

By assumption, u is a solution of $\Phi_r = 0$. So

$$\Phi_r(u(\xi), R) \quad = \quad 0.$$

Together with (4.4) this implies that

$$\Phi_{sr}(su(s\xi), R) \quad = \quad 0$$

and the result follows. □

Note that if $r_1 = r_2$ the operator Φ_r is invariant under the action of $\mathbf{S}_2$. In this case, given a solution u of $\Phi_r = 0$ there is a conjugate su satisfying the same equations. If $r_1 \neq r_2$ then su is a solution of the different equation $\Phi_{sr} = 0$.

Finaly we combine all the results obtained up to now in the following

Theorem 17 *Let Φ_r be the operator defined by (4.1). Let the groups K, $\mathbf{T}^3$ and $\mathbf{S}_2$ act as above. Then*

1. *If $r_1 \neq r_2$ ($r_1 = r_2$) every smooth K-invariant solution $\hat{u}$ of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K(\dot{+}\mathbf{S}_2)\dot{+}\mathbf{T}^3$-symmetry restricts to a smooth solution of $\Phi_r = 0$ on Ω with the boundary conditions (4.2).*

2. *Let $u \in \left(C^1(\Omega)\right)^5$ be a solution of $\Phi_r = 0$ on Ω with the boundary conditions (4.2). Then*

   - *u is smooth.*
   - *If $r_1 \neq r_2$ then u extends uniquely to a smooth K-invariant solution of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K\dot{+}\mathbf{T}^3$-symmetry. The $\mathbf{S}_2$-conjugate su is a solution of the equation defined by the $\mathbf{S}_2$-conjugate operator Φ_{sr}.*
   - *If $r_1 = r_2$ then u extends uniquely to a smooth K-invariant solution of $\Phi_r = 0$ on $\mathbb{R}^3$ with $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-symmetry.*

We observe that $u = 0$ is always a translation invariant solution of the equation $\Phi_r = 0$. We are interested in steady states bifurcating from this trivial branch when the Rayleigh number R is increased from below. We restrict our bifurcation analysis to a neighbourhood of some critical values of the unfolding parameter r. Denote $L_r = d\Phi_r$ the linearization of Φ_r about $u = 0$. In sections 4.4 and 4.5 below, and according to chapters 2 and 3, we use the symmetries in theorem 17 to get a general polynomial form for the projection of the bifurcation equations onto ker L_r. In latter sections a Liapunov-Schmidt reduction will be performed to give the exact values of the coefficients in the bifurcation equations. Finally, these equations will be put into normal form and the bifurcation diagrams read from the tables in appendix.

4.4 Bifurcations with a $K \dot{+} \mathrm{T}^3$-Symmetric Extension

Assume that r is such that when the parameter R is increased from zero, it crosses a critical value for which the linear operator L_r has a nontrivial kernel. Then there are branches of solutions bifurcating from $u = 0$. The Rayleigh number R is playing the role of bifurcation parameter and r is a 2-dimensional unfolding parameter. Recall that $r = (r_1, r_2)$ depends only on the domain before being scaled: the two components represent the aspect ratios of the horizontal dimensions of the box by the vertical dimension.

By Golubitsky *et al.* [11], bifurcations of codimension up to three are generic in problems with two unfolding parameters. Such bifurcations are expected to occur in regions of the unfolding parameter space as follows:

- Codimension one in open regions.
- Codimension two along lines.
- Codimension three at isolated points.

We begin by describing shortly the simplest codimension one bifurcations, proceeding then to the more complicated ones. The main interest of this section is the codimension three.

As in section 4.3, from the problem $\Phi_r = 0$ with the boundary conditions (4.2) we construct a larger one consisting of the same equation with periodic boundary conditions. The second problem is invariant under an action of the group $K \dot{+} \mathrm{T}^3$. We also know that in order to get explicitly all the symmetries of the first, we really need to state the extended problem and restrict the result to the subspace fixed by K. This is what we proceed to do.

4.4.1 Single Mode Bifurcations

By Golubitsky *et al.* [11], we have a codimension one bifurcation with $K \dot{+} \mathrm{T}^3$-symmetry when ker L_r is an irreducible representation of this group. Suppose that r in such that this holds for some value of the bifurcation parameter R. We may write an irreducible representation of T^3 as

$$\begin{aligned}
\theta_1: &\quad (z_1, z_2, z_3, z_4) \mapsto (e^{ik_1\theta_1} z_1, e^{ik_1\theta_1} z_2, e^{ik_1\theta_1} z_3, e^{ik_1\theta_1} z_4) \\
\theta_2: &\quad (z_1, z_2, z_3, z_4) \mapsto (e^{ik_2\theta_2} z_1, e^{ik_2\theta_2} z_2, e^{-ik_2\theta_2} z_3, e^{-ik_2\theta_2} z_4) \\
\theta_3: &\quad (z_1, z_2, z_3, z_4) \mapsto (e^{ik_3\theta_3} z_1, e^{-ik_3\theta_3} z_2, e^{ik_3\theta_3} z_3, e^{-ik_3\theta_3} z_4)
\end{aligned}$$

where k_1, k_2, k_3 are nonnegative integers and

- $z_1 = \overline{z}_4$ and $z_2 = \overline{z}_3$ if $k_1 = 0$.
- $z_1 = z_3$ and $z_2 = z_4$ if $k_2 = 0$.
- $z_1 = z_2$ and $z_3 = z_4$ if $k_3 = 0$.
- z_1, z_2, z_3, z_4 are any complex numbers otherwise.

The action of $K = \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ may be written as

$$\begin{aligned}
\kappa_1 : &\quad (z_1, z_2, z_3, z_4) \mapsto (\overline{z}_4, \overline{z}_3, \overline{z}_2, \overline{z}_1) \\
\kappa_2 : &\quad (z_1, z_2, z_3, z_4) \mapsto (z_3, z_4, z_1, z_2) \\
\kappa_3 : &\quad (z_1, z_2, z_3, z_4) \mapsto (z_2, z_1, z_4, z_3).
\end{aligned}$$

Note that bifurcating solutions have a well defined set of mode numbers $\underline{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$. These are induced by the action of $\mathbf{T}^3$ above and are associated with pattern formation that can be observed in experiments. Denote

$$\hat{W}_k = \text{span}\{z_1, z_2, z_3, z_4\}$$

and W_k the subspace fixed by K. Then we have that

$$\begin{aligned}
W_k &= \{(\text{Re}(z_1), \text{Re}(z_2), \text{Re}(z_3), \text{Re}(z_4)) | z_1 = z_2 = z_3 = z_4\} \\
&= \mathbb{R}.
\end{aligned}$$

Note that the representation $\hat{W}_k$ of the group $K\dot{+}\mathbf{T}^3$ is a particular case of that used in chapter 2 in a slightly different setting: single mode bifurcations of reaction-diffusion equations with Neumann boundary conditions on n-dimensional rectangles. By finding an isomorphism between $\hat{W}_k$ and $\ker L_r$ we will be able to use here all the results obtained in chapter 2.

Theorem 18 *Assume that the translation invariant solution $u = 0$ of $\Phi_r = 0$ undergoes a codimension one bifurcation with mode numbers $\underline{k}$ when the parameter R is increased from zero. Assume also that this occurs for some value of r such that $r_1 \neq r_2$. Then*

1. *In the extended problem, $\ker L_r$ is isomorphic to $\hat{W}_k$.*
2. *If the boundary conditions (4.2) are imposed, $\ker L_r$ is spanned by u_k with components*

$$\begin{aligned}
v_1 &= a_k[1] \sin(k_1\xi_1) \cos(k_2\xi_2) \cos(k_3\xi_3) \\
v_2 &= a_k[2] \cos(k_1\xi_1) \sin(k_2\xi_2) \cos(k_3\xi_3) \\
v_3 &= a_k[3] \cos(k_1\xi_1) \cos(k_2\xi_2) \sin(k_3\xi_3) \\
\Theta &= a_k[4] \cos(k_1\xi_1) \cos(k_2\xi_2) \sin(k_3\xi_3) \\
p &= a_k[5] \cos(k_1\xi_1) \cos(k_2\xi_2) \cos(k_3\xi_3),
\end{aligned}$$

where the $a_k[j]$ are real numbers depending on the mode numbers and the unfolding parameters r.

Proof To prove part 1 we construct an isomorphism between the two representations of the group $K\dot{+}\mathbf{T}^3$. For that we need to introduce explicitly a set of generators for $\ker L_r$. In part 2 we restrict these generators to $\mathrm{Fix}(K)$.

1. We write z_1, z_2, z_3, z_4 as functions of the domain variables ξ_1, ξ_2, ξ_3. These functions will be used to generate the eigenmodes spanning $\ker L_r$. The condition for isomorphism is that acting on the z_j as above is the same as acting on the eigenfunctions as in section 4.3. Define

$$\begin{aligned} z_1 :\ & (\xi_1, \xi_2, \xi_3) \mapsto e^{i(k_1\xi_1 + k_2\xi_2 + k_3\xi_3)} \\ z_2 :\ & (\xi_1, \xi_2, \xi_3) \mapsto e^{i(k_1\xi_1 + k_2\xi_2 - k_3\xi_3)} \\ z_3 :\ & (\xi_1, \xi_2, \xi_3) \mapsto e^{i(k_1\xi_1 - k_2\xi_2 + k_3\xi_3)} \\ z_4 :\ & (\xi_1, \xi_2, \xi_3) \mapsto e^{i(k_1\xi_1 - k_2\xi_2 - k_3\xi_3)}. \end{aligned}$$

Let $\hat{u}_k$ be an eigenmode in $\ker L_r$. Then we say that each component of $\hat{u}_k$ belongs to a space spanned, over the reals, by the z_j as

$$\begin{aligned} v_1 &\in \mathrm{span}\{-iz_1, -iz_2, -iz_3, -iz_4\} \\ v_2 &\in \mathrm{span}\{-iz_1, -iz_2, iz_3, iz_4\} \\ v_3 &\in \mathrm{span}\{-iz_1, iz_2, -iz_3, iz_4\} \\ \Theta &\in \mathrm{span}\{-iz_1, iz_2, -iz_3, iz_4\} \\ p &\in \mathrm{span}\{z_1, z_2, z_3, z_4\} \end{aligned} \tag{4.5}$$

and checking that the two actions are isomorphic is a straightforward calculation.

2. In order to restrict $\ker L_r$ to $\mathrm{Fix}(K)$ it is more convenient to write (4.5) in the equivalent form

$$\begin{aligned} v_1(\xi) &\in \mathrm{span}\left\{z_1(\xi_1 - \tfrac{\pi}{2}, \xi_2, \xi_3), z_2(\xi_1 - \tfrac{\pi}{2}, \xi_2, \xi_3), z_3(\xi_1 - \tfrac{\pi}{2}, \xi_2, \xi_3), z_4(\xi_1 - \tfrac{\pi}{2}, \xi_2, \xi_3)\right\} \\ v_2(\xi) &\in \mathrm{span}\left\{z_1(\xi_1, \xi_2 - \tfrac{\pi}{2}, \xi_3), z_2(\xi_1, \xi_2 - \tfrac{\pi}{2}, \xi_3), z_3(\xi_1, \xi_2 - \tfrac{\pi}{2}, \xi_3), z_4(\xi_1, \xi_2 - \tfrac{\pi}{2}, \xi_3)\right\} \\ v_3(\xi) &\in \mathrm{span}\left\{z_1(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_2(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_3(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_4(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2})\right\} \\ \Theta(\xi) &\in \mathrm{span}\left\{z_1(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_2(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_3(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2}), z_4(\xi_1, \xi_2, \xi_3 - \tfrac{\pi}{2})\right\} \\ p(\xi) &\in \mathrm{span}\left\{z_1(\xi_1, \xi_2, \xi_3), z_2(\xi_1, \xi_2, \xi_3), z_3(\xi_1, \xi_2, \xi_3), z_4(\xi_1, \xi_2, \xi_3)\right\}. \end{aligned}$$

The result follows by fixing these generators by the action of K in section 4.3.

□

By theorem 18, there is no loss of generality in working with the abstract representation $\hat{W}_k$. For ease of exposition the most relevant results obtained in chapter 2 will be reproduced here when appropriate. However, we should not forget the aim of this chapter: to see how the abstract settings of the previous chapters apply to a concrete physical problem. This is why at some points we go back to see what is happening in the Bénard problem.

Before proceeding to bifurcations of higher codimension we denote

$$V_k = \mathrm{span}\{u_k\},$$

which, we recall, is $\ker L_r$ when the single mode $\underline{k}$ bifurcates from $u = 0$ and the boundary conditions (4.2) are imposed.

4.4.2 Simultaneous Bifurcation of Three Distinct Modes

Now assume that $r = r_c$ is such that, by increasing the parameter R from zero, we cross a critical value R_c, for which three distinct modes $\underline{k}, \underline{l}, \underline{m} \in \mathbb{N}^3$ bifurcate simultaneously from $u = 0$. Then, with the boundary conditions (4.2), we have

$$\ker L_r \;=\; V_k \oplus V_l \oplus V_m. \tag{4.6}$$

In the associated periodic boundary value problem, $\ker L_r$ is isomorphic to $\hat{W}_k \oplus \hat{W}_l \oplus \hat{W}_m$ and $K\dot{+}\mathrm{T}^3$ acts on each irreducible block as in the previous section. We may assume, without loss of generality that

$$\mathrm{hcf}(k_j, l_j, m_j) = 1 \qquad \text{for} \qquad 1 \le j \le 3. \tag{4.7}$$

Otherwise we factor out the kernel of the group action and take it into account at the end when interpreting the results. Now (4.6) is isomorphic to

$$\mathrm{Fix}(K) \;=\; W_k \oplus W_l \oplus W_m = \mathbb{R}^3.$$

We define

$$\lambda \;=\; R - R_c$$

so that the bifurcation occurs at $\lambda = 0$. We want to know what are the symmetry constraints imposed on the bifurcation equations

$$f(x, y, z, \lambda) \;=\; 0,$$

where f maps $\mathbb{R}^3 \times \mathbb{R}$ onto $\mathbb{R}^3$.

As in chapter 2 we denote by N the subgroup $N_{K\dot{+}\mathrm{T}^3}(K)/K$, which is the normalizer of K in $K\dot{+}\mathrm{T}^3$ factored by K. The group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ and the mapping f must be equivariant under its action. As in section 2.5, the action of the normalizer N on $W_k \oplus W_l \oplus W_m$ is generated by translations of π along each of the directions chosen to generate T^3

$$\begin{aligned} \tau_1 &: \;(x, y, z) \mapsto ((-1)^{k_1} x, (-1)^{l_1} y, (-1)^{m_1} z) \\ \tau_2 &: \;(x, y, z) \mapsto ((-1)^{k_2} x, (-1)^{l_2} y, (-1)^{m_2} z) \\ \tau_3 &: \;(x, y, z) \mapsto ((-1)^{k_3} x, (-1)^{l_3} y, (-1)^{m_3} z). \end{aligned} \tag{4.8}$$

Note that by assumption (4.7) all the τ_j act nontrivially and depending on the parities of the mode numbers, one or two of them may be redundant. Therefore we have that N factored by the kernel of its action is isomorphic to

- $\mathbf{Z}_2$ if one and only one of the τ_j is redundant.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ if two and only two of the τ_j are redundant.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ if all τ_j are nonredundant.

It can be shown that if N factored by the kernel of its action is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$ then the mapping f is a generic N-equivariant and the form of the bifurcation equations is easy to obtain.

Theorem 19 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the three modes $\underline{k}, \underline{l}, \underline{m} \in \mathbb{N}^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 \neq r_2$. If all τ_j in (4.8) are nonredundant then the form of the reduced bifurcation equations is*

$$\begin{aligned} f_1(x,y,z,\lambda) = ax &= 0 \\ f_2(x,y,z,\lambda) = by &= 0 \\ f_3(x,y,z,\lambda) = cz &= 0 \end{aligned} \tag{4.9}$$

where a, b, c are functions of x^2, y^2, z^2 and λ.

Proof If all τ_j are nonredundant, the problem has $Z_2 \oplus Z_2 \oplus Z_2$-symmetry. It is easy to see that the invariant functions are all the even functions in x, y, z. Then the equivariant mappings are obtained as in theorem 11, chapter 2 and the result follows. □

In appendix A the normal form of system (4.9) and its universal unfolding are given. Tables of nondegeneracy conditions and branching equations are obtained. In section 4.6, a Liapunov-Schmidt reduction is performed. This reduction gives the values of some low order derivatives of the coefficients a, b, c. In this case derivatives up to third order are enough. Then, the tables in the appendix will be used to obtain the bifurcation diagrams.

If the action of N has a nontrivial kernel the problem is a bit more subtle. In this case the periodic extension induces extra symmetries. In chapter 2 we deal with this problem in three steps:

1. Impose explicitly the constraints on T^3-invariant monomials.
2. Restrict these constraints to Fix(K) and use an algorithm to obtain a minimal set of generators for the restricted $K\dot{+}T^3$-invariants.
3. Generate the restricted $K\dot{+}T^3$-equivariants by using the result of 2 in theorem 11.

Then the restricted $K\dot{+}T^3$-equivariants are generated by the result of step 3 modulo the ring of invariants generated by the result of step 2.

4.5 Bifurcations with a $K\dot{+}S_2\dot{+}T^3$-Symmetric Extension

The notion of codimension depends on the symmetry of the problem and, by theorem 17, the condition $r_1 = r_2$ imposes an extra S_2-symmetry. Under this condition, the unfolding is being restricted to one parameter and codimension three bifurcations are no longer generic. In this diagonal of the r-space, bifurcations are expected to occur in regions as follows:

- Codimension one along lines.

- Codimension two at isolated points.

As in section 4.4 we begin with a brief description of codimension one bifurcations. More attention will be concentrated on simultaneous bifurcations of two distinct modes by increasing the Rayleigh number R from zero. In appendix B we unfold these bifurcations with the diagonal of the r-space and then break the S_2-symmetry with the a parameter transverse to this diagonal.

As in section 4.3, by reflecting across the boundaries any solution of the problem $\Phi_r = 0$ satisfying the boundary conditions (4.2) we construct another one consisting of the same equation and periodic boundary conditions on a larger domain. The second problem is invariant under an action of the group $K \dot{+} S_2 \dot{+} T^3$. By theorem 17, the solutions satisfying the boundary conditions (4.2) are in one to one correspondence with those of the periodic boundary value problem that are fixed by the action of K.

4.5.1 Single Mode Bifurcations

Now the solution $u = 0$ undergoes a single mode bifurcation with $K \dot{+} S_2 \dot{+} T^3$-symmetry when ker L_r is an irreducible representation of this group. Assume that this holds for some r when the bifurcation parameter R crosses a critical value. As in section 4.4, bifurcating solutions have a well defined set of mode numbers $\underline{k} \in \mathbb{N}^3$. Let the group S_2 act on the mode numbers as

$$s: \ (k_1, k_2, k_3) \mapsto (k_2, k_1, k_3).$$

By noting that S_2 acts trivialy if and only if $k_1 = k_2$, we construct an irreducible representation of $K \dot{+} S_2 \dot{+} T^3$ as

- $\hat{W}_k$ if $k_1 = k_2$
- $\hat{W}_k \oplus \hat{W}_{sk}$ if $k_1 \neq k_2$

where $\hat{W}_k$ is as in section 4.4. On the other hand, when a solution with mode numbers $\underline{k}$ satisfying $\Phi_r = 0$ and the boundary conditions (4.2) bifurcates from $u = 0$ we have that

- ker $L_r = V_k$ if $k_1 = k_2$
- ker $L_r = V_k \oplus V_{sk}$ if $k_1 \neq k_2$.

As in section 4.4 it can be shown that ker L_r is isomorphic to

- $\mathrm{Fix}(K) = W_k = \mathbb{R}$ if $k_1 = k_2$
- $\mathrm{Fix}(K) = W_k \oplus W_{sk} = \mathbb{R}^2$ if $k_1 \neq k_2$

and we have the tools necessary to proceed with a more detailed study of higher codimension bifurcations.

4.5.2 Simultaneous Bifurcation of Two Distinct Modes

Assume that $r = r_c$ is such that, by increasing the parameter R, we cross a critical value R_c, for which two distinct modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ bifurcate simultaneously from $u = 0$. Then, if the boundary conditions (4.2) are imposed we have that

- $\ker L_r = V_k \oplus V_l$ if $k_1 = k_2$ and $l_1 = l_2$
- $\ker L_r = V_k \oplus V_l \oplus V_{sl}$ if $k_1 = k_2$ and $l_1 \neq l_2$
- $\ker L_r = V_k \oplus V_{sk} \oplus V_l \oplus V_{sl}$ if $k_1 \neq k_2$ and $l_1 \neq l_2$.

By analogy with the previous section we construct a direct sum of irreducible representations of the group $K\dot{+}S_2\dot{+}T^3$ which subspace fixed by K is

- $\mathrm{Fix}(K) = W_k \oplus W_l = \mathbb{R}^2$ if $k_1 = k_2$ and $l_1 = l_2$
- $\mathrm{Fix}(K) = W_k \oplus W_l \oplus W_{sl} = \mathbb{R}^3$ if $k_1 = k_2$ and $l_1 \neq l_2$
- $\mathrm{Fix}(K) = W_k \oplus W_{sk} \oplus W_l \oplus W_{sl} = \mathbb{R}^4$ if $k_1 \neq k_2$ and $l_1 \neq l_2$.

Note that as explained in section 4.4.2, there is no loss of generality in assuming that

$$\begin{aligned} &\mathrm{hcf}(k_1, k_2, l_1, l_2) = 1 \\ &\mathrm{hcf}(k_3, l_3) = 1. \end{aligned}$$

As in section 4.4 it can be shown that, when the boundary conditions (4.2) are imposed, $\ker L_r$ is isomorphic to $\mathrm{Fix}(K)$ as above. From now on, without loss of generality, we work with $\mathrm{Fix}(K)$ so that several results of chapters 2 and 3 apply directly. Once more we define

$$\lambda = R - R_c$$

so that the bifurcation is located at $\lambda = 0$. We want to know what are the symmetry constraints on the bifurcation equations

$$f(x, y, \lambda) = 0,$$

where f maps $\mathrm{Fix}(K) \times \mathbb{R}$ onto $\mathrm{Fix}(K)$.

By analogy with section 4.4 we denote by N the subgroup $N_{K\dot{+}S_2\dot{+}T^3}(K)/K$ and the mapping f must be equivariant under its action. As we will see below, f is not always a generic N-equivariant. In order to write explicitly the action of N on $\mathrm{Fix}(K)$ we need to consider separately three distinct cases.

(a) $k_1 = k_2$ and $l_1 = l_2$

This case reduces to that of 3-dimensional rectangles described in section 2.4. The results are reproduced here to make this chapter complete. The group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ and its action on $W_k \oplus W_l$ is generated by translations of π along each of the directions chosen to generate $\mathbf{T}^3$

$$\begin{aligned}\tau_1: \ (x,y) &\mapsto ((-1)^{k_1}x,(-1)^{l_1}y)\\ \tau_3: \ (x,y) &\mapsto ((-1)^{k_3}x,(-1)^{l_3}y).\end{aligned}$$

The translation τ_2 is omited because it acts as τ_1. Note that all τ_j act nontrivialy and some of them may be redundant. In fact, as in theorem 5, we have that N factored by the kernel of its action is isomorphic to

- $\mathbf{Z}_2$ if all k_j have the same parity and all l_j have the same parity.
- $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ otherwise.

We proceed by giving a general form for the bifurcation equations.

Theorem 20 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 = r_2$. If $k_1 = k_2$ and $l_1 = l_2$ then the form of the reduced bifurcation equations is*

1. *If all k_j have the same parity and all l_j have the same parity*

$$\begin{aligned}f_1(x,y,\lambda) = ax + cx^{l-1}y^k &= 0\\ f_2(x,y,\lambda) = by + dx^l y^{k-1} &= 0\end{aligned}$$

where $k = max_j k_j$ and $l = max_j l_j$.

2. *Otherwise*

$$\begin{aligned}f_1(x,y,\lambda) = ax &= 0\\ f_2(x,y,\lambda) = by &= 0\end{aligned}$$

where a, b, c, d are functions of x^2, y^2 and λ.

Proof See section 2.4. □

(b) $k_1 = k_2$ and $l_1 \neq l_2$

In this case the group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_4$ and its action on $W_k \oplus W_l \oplus W_{sl}$ is generated by

$$\begin{aligned}\tau_1: \ (x,y_1,y_2) &\mapsto ((-1)^{k_1}x,(-1)^{l_1}y_1,(-1)^{l_2}y_2)\\ s: \ (x,y_1,y_2) &\mapsto (x,y_2,y_1)\\ \tau_3: \ (x,y_1,y_2) &\mapsto ((-1)^{k_3}x,(-1)^{l_3}y_1,(-1)^{l_3}y_2).\end{aligned} \tag{4.10}$$

Note that the translation τ_2 is omited because it acts as $s\tau_1 s$. As before, one of these generators may be redundant. By inspection on (4.10) we see that N factored by the kernel of its action is isomorphic to

- $Z_2 \oplus S_2$ if all k_j have the same parity and all l_j have the same parity
  or $k_1, k_3 + 1$ have the same parity and $l_1, l_2, l_3 + 1$ have the same parity.

- $Z_2 \oplus Z_2 \oplus S_2$ if all k_j have the same parity and $l_1, l_2, l_3 + 1$ have the same parity
  or $k_1, k_3 + 1$ have the same parity and all l_j have the same parity.

- D_4 if $l_1, l_2 + 1$ have the same parity and k_3 is even.

- $Z_2 \oplus D_4$ if $l_1, l_2 + 1$ have the same parity and k_3 is odd.

It can be shown that if N factored by the kernel of its action is isomorphic to $Z_2 \oplus D_4$ then the mapping f is a generic N-equivariant and the form of the bifurcation equations is easily obtained.

Theorem 21 *Assume that the solution $u = 0$ of the Boussinesq equations undergoes a simultaneous bifurcation of the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ when a shift λ of the Rayleigh number crosses zero. Assume that this happens for some value of r such that $r_1 = r_2$. If l_1, l_2 have different parities and k_3 is odd then the form of the reduced bifurcation equations is*

$$\begin{aligned} f_1(x, y_1, y_2, \lambda) &= ax = 0 \\ f_2(x, y_1, y_2, \lambda) &= (b + c\delta)y_1 = 0 \\ f_3(x, y_1, y_2, \lambda) &= (b - c\delta)y_2 = 0 \end{aligned} \tag{4.11}$$

where a, b, c are functions of x^2, $y_1^2 + y_2^2$, $(y_2^2 - y_1^2)^2$ and λ; and $\delta = y_2^2 - y_1^2$.

Proof The action of Z_2 generated by τ_2 implies that f_1 is odd in x and f_2, f_3 are even in x. Terms involving y_1, y_2 are determined by the action of D_4 generated by s, τ_3. See Golubitsky *et al.* [11], chapter XVII, section 4. □

The normal form for system (4.11) together with the universal unfolding that keeps the symmetry are given in appendix B. Tables of nondegeneracy conditions, branching equations and some bifurcation diagrams are obtained. Then the S_2 symmetry is broken with an other unfolding parameter.

If the action of N has a nontrivial kernel, a general form for the bifurcation equations cannot be given. The method for finding the generators of the restricted invariants and equivariants splits into five steps:

1. Impose explicitly the constraints on T^3-invariant monomials.

2. Restrict these constraints to Fix(K) and use an algorithm to obtain a minimal set of generators for the restricted T^3-invariants.

3. Generate the restricted T^3-equivariants by using the result of 2 in theorem 11.

4. Symmetrize the result of 2 over S_2 to obtain the restricted $K\dot{+}S_2\dot{+}T^3$-invariants.

5. Symmetrize the result of 3 over $\mathbf{S}_2$ to obtain the restricted $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-equivariants.

Then the restricted $K\dot{+}\mathbf{S}_2\dot{+}\mathbf{T}^3$-equivariants are generated by the result of step 5 modulo the ring of invariants generated by the result of step 4. Now we can justify the statment above that a general form for the bifurcation equations cannot be given: the method for obtaining them involves an algorithm. Thus, we have to work case by case if the action of the group N has a nontrivial kernel.

(c) $k_1 \neq k_2$ and $l_1 \neq l_2$

In this case the group N is isomorphic to $\mathbf{Z}_2 \oplus \mathbf{D}_4$ and its action on $W_k \oplus W_{*k} \oplus W_l \oplus W_{*l}$ is generated by

$$\begin{aligned}
\tau_1 :&\ (x_1, x_2, y_1, y_2) \mapsto ((-1)^{k_1} x_1, (-1)^{k_2} x_2, (-1)^{l_1} y_1, (-1)^{l_2} y_2) \\
s :&\ (x_1, x_2, y_1, y_2) \mapsto (x_2, x_1, y_2, y_1) \\
\tau_3 :&\ (x_1, x_2, y_1, y_2) \mapsto ((-1)^{k_3} x_1, (-1)^{k_3} x_2, (-1)^{l_3} y_1, (-1)^{l_3} y_2).
\end{aligned}$$

As before, N factored by the kernel of its action is isomorphic to some normal subgroup of $\mathbf{Z}_2 \oplus \mathbf{D}_4$ containing $\mathbf{S}_2$. Whatever parities the mode numbers have, a general form for the bifurcation equations cannot be given: we need to follow the steps 1 to 5 as in the previous case to compute the generators of the restricted invariants and equivariants.

4.6 Some Generalities About the Liapunov-Schmidt Reduction

This section gives a description of the Liapunov-Schmidt reduction as in Golubitsky and Schaeffer [28]. Then we apply it to steady-state bifurcations of the family of operators Φ_r parametrized by $r \in \mathbb{R}^2$ satisfying the boundary conditions (4.2). This procedure projects the dynamics onto the kernel of the linearized operator L_r. Denote

$$\begin{aligned}
\mathcal{X} &= (C^2(\Omega))^5 \\
\mathcal{Y} &= (C^0(\Omega))^5.
\end{aligned}$$

The operator

$$\begin{aligned}
\Phi : \mathcal{X} \times \mathbb{R} \times \mathbb{R}^2 &\to \mathcal{Y} \\
(u, R, r) &\mapsto \Phi_r(u, R)
\end{aligned}$$

in section 4.2 is a smooth C^∞ mapping and R_c, r_c are chosen such that the derivative $L = (d\Phi)_{(0,R_c,r_c)}$ is a Fredholm operator of index 0. Substituting

$$\begin{aligned}
r &\mapsto \rho + r_c \\
R &\mapsto \lambda + R_c
\end{aligned}$$

we say that a bifurcation for the parameter λ occurs at $\rho = 0$ and $\lambda = 0$ if L has a nontrivial kernel, which will be assumed latter in all applications. Because L is Fredholm, it has a finite-dimensional kernel. We choose a basis for ker L and write

$$\ker L = \operatorname{span}\{u_1, \ldots, u_n\}.$$

Also, range L has finite codimension and we may write

$$(\text{range } L)^{\perp} = \ker L^*,$$

where L^* is the adjoint of L. Now, L having index 0 means that ker L and ker L^*have the same dimension. So we choose a basis for ker L^* and write

$$\ker L^* = \text{span}\{u_1^*, \ldots, u_n^*\}.$$

We may decompose

$$\mathcal{X} = \ker L \oplus M \tag{4.12}$$
$$\mathcal{Y} = N \oplus \text{range } L. \tag{4.13}$$

Let E be the projection of $\mathcal{Y}$ onto range L and write the equations $\Phi(u, \lambda, \rho) = 0$ in the equivalent form

$$E\Phi(u, \lambda, \rho) = 0 \tag{4.14}$$
$$(I - E)\Phi(u, \lambda, \rho) = 0 \tag{4.15}$$

where I is the identity on $\mathcal{Y}$. Note that $I - E$ is the projection of $\mathcal{Y}$ onto $N = \ker L^*$. By decomposition (4.12) we may write any $u \in \mathcal{X}$ as

$$u = \sum_{j=1}^{n} x_j u_j + w$$

where $w \in M$. Denote $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $U = (u_1, \ldots, u_n)$. By the implicit function theorem, equations (4.14) may be solved as a function $w = W(x, \lambda, \rho)$. Thus there exists a function

$$W: \ \ker L \times \mathbb{R} \to M$$

such that

$$E\Phi(x.U + W(x, \lambda, \rho), \lambda, \rho) \equiv 0.$$

Substituting u we make the projection I-E by using the basis chosen for ker L and get the so called Liapunov-Schmidt reduced mapping

$$f: \ \ker L \times \mathbb{R} \times \mathbb{R}^2 \to \ker L^*$$

as

$$f_j(x, \lambda, \rho) = \langle u_j^*, \Phi(x.U + W(x, \lambda, \rho), \lambda, \rho)\rangle,$$

for $1 \leq j \leq n$. Note that the formula for f is not explicit: it depends on the implicitly defined function W. However, the derivatives of W can be expressed in terms of Φ and we can compute terms of the Taylor expansion of f. By using the symmetry constraints described in the previous sections we see which terms of the expansion we need in order to determine the bifurcation equations.

4.7 Basic Tools

The aim of this section is to extract some information from the Boussinesq equations (4.1) with boundary conditions (4.2) so that we will be ready to perform Liapunov-Schmidt reductions in the following sections. In section 4.7.1 we locate bifurcation points in the parameter space. Section 4.7.2 is concerned with symmetries of the linearized operator L_r that will simplify the calculations.

4.7.1 Location of Primary Bifurcations

Here we identify parameter values for which there is a bifurcation from the solution $u = 0$ of Φ_r satisfying the boundary conditions (4.2). So we want the values of R such that $L_r(u, R)$ has a nontrivial kernel. By theorem 18, associated to any bifurcating solution is a set of mode numbers $\underline{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$ and an eigenmode u_k with components

$$\begin{aligned}
v_1 &= a_k[1]\sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)\\
v_2 &= a_k[2]\cos(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3)\\
v_3 &= a_k[3]\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
\Theta &= a_k[4]\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
p &= a_k[5]\cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3),
\end{aligned}$$

where the $a_k[j]$ are real numbers depending on the mode numbers and the unfolding parameters r. As in section 4.4, we denote

$$V_k = \mathrm{span}\{u_k\}.$$

Let r be any point in the unfolding parameter space. By solving

$$L_r(u_k, R_k) = 0, \tag{4.16}$$

we calculate the critical Rayleigh number R_k and the coefficients a_k of the eigenmode. By denoting

$$\Lambda_k = \frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2} + k_3^2.$$

the solution of (4.16) is

$$R_k = \frac{\Lambda_k^3}{\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2}}.$$

and

$$\begin{aligned}
a_k[1] &= \frac{k_1}{r_1}(\Lambda_k^2 - R_k)\\
a_k[2] &= \frac{k_2}{r_2}(\Lambda_k^2 - R_k)\\
a_k[3] &= k_3\Lambda_k^2\\
a_k[4] &= k_3\Lambda_k R_k\\
a_k[5] &= \Lambda_k(\Lambda_k^2 - R_k).
\end{aligned}$$

As a simplification of notations, the unfolding parameters r have not kept as a subscript. However, we remark the dependence of a_k and u_k on r. From now the dependence on r should be seen implicitly on the mode numbers.

As a tool for the Liapunov-Schmidt reductions in latter section we have that when $R = R_k$, the function u_k^* with components

$$\begin{aligned}
v_1^* &= \frac{k_1}{r_1}\sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)\\
v_2^* &= \frac{k_2}{r_2}\cos(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3)\\
v_3^* &= k_3\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
\Theta^* &= (R_k - \Lambda_k^2)\cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3)\\
p^* &= k_3\cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3)
\end{aligned}$$

is an eigenmode of the adjoint operator L_r^*. We keep also the notation

$$V_k^* = \operatorname{span}\{u_k^*\}.$$

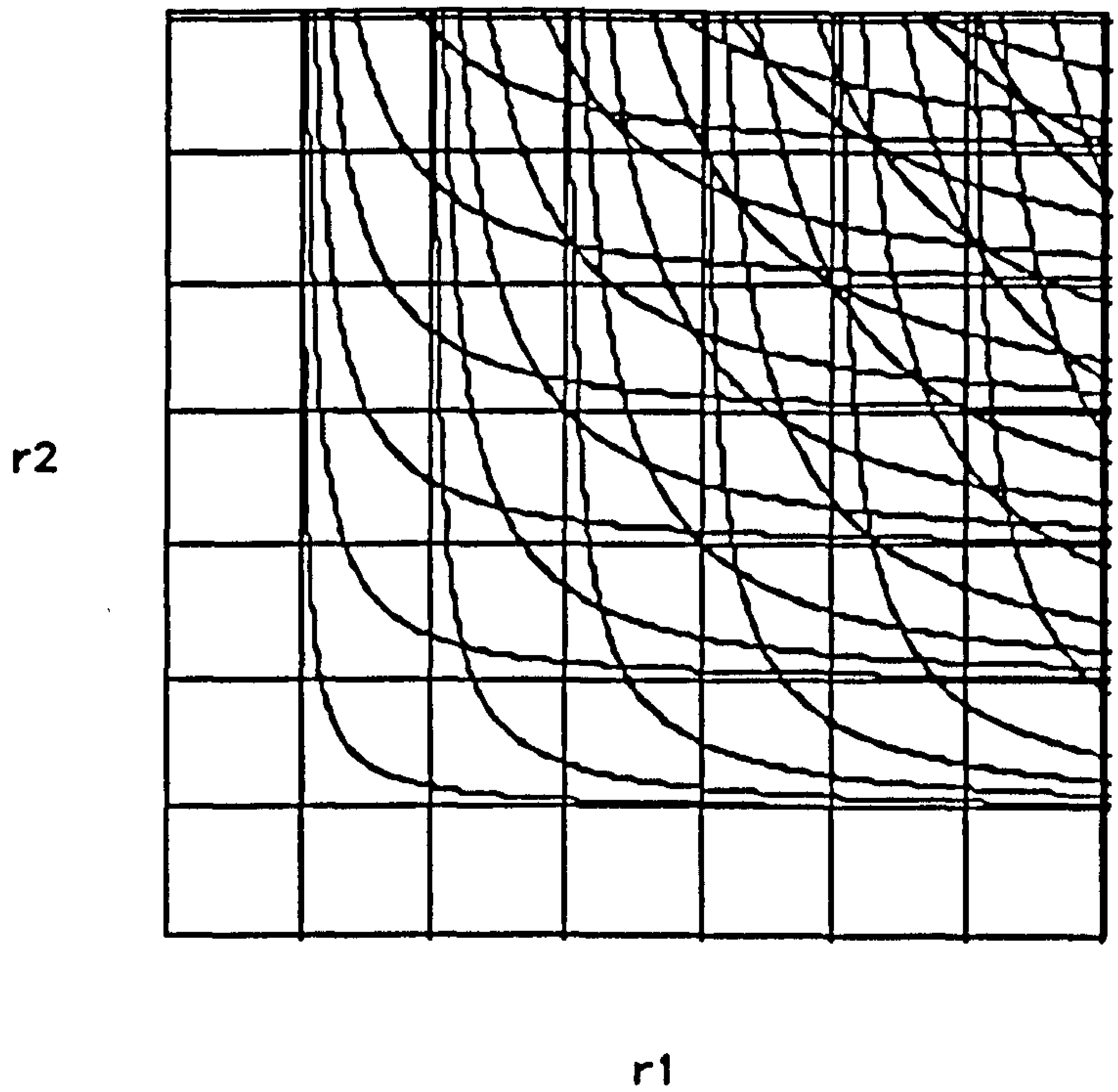
The graph of R_k is a folded surface with minimum

$$R_k^{\min} = \frac{27}{4}k_3^4 \tag{4.17}$$

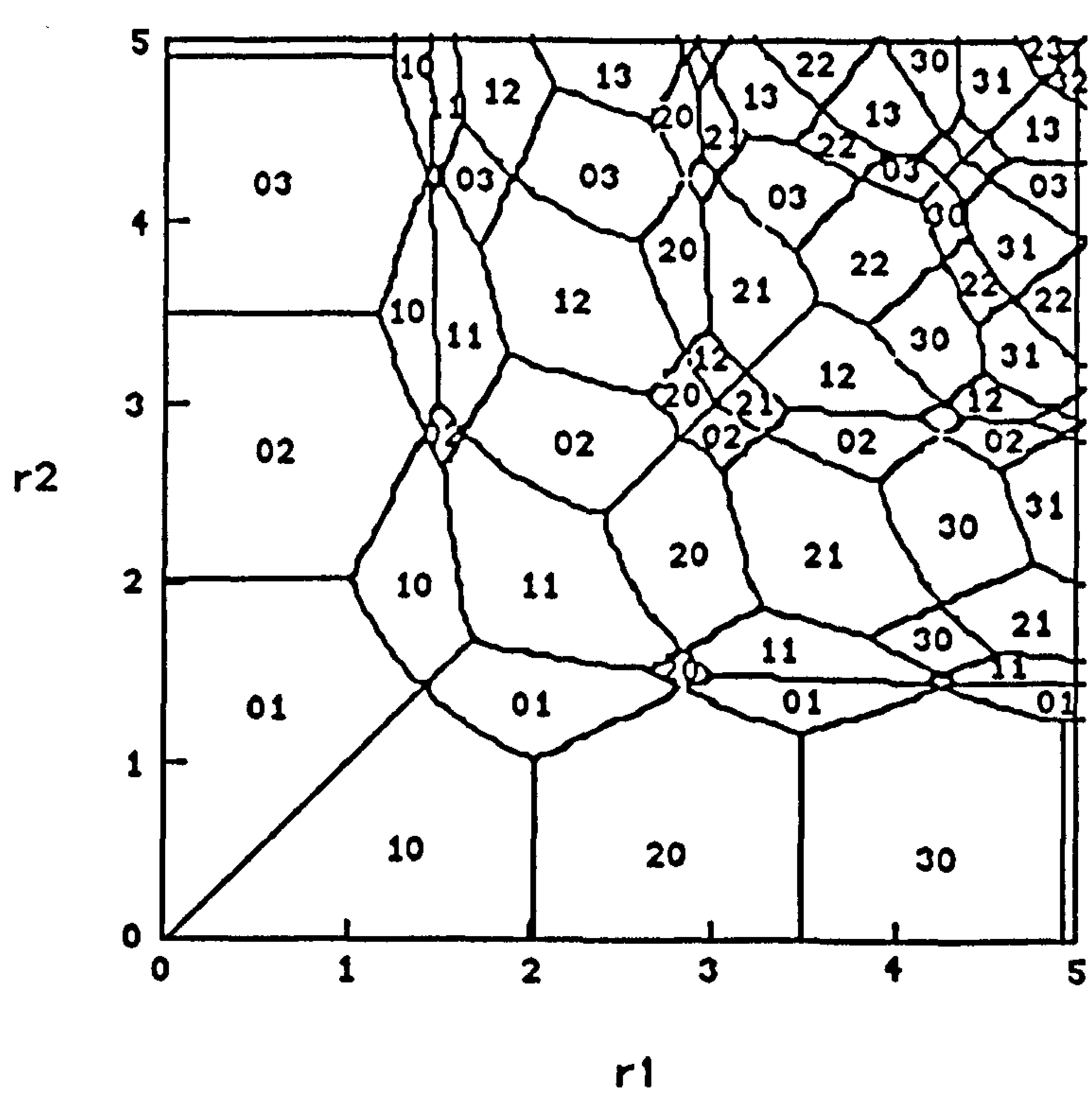
along the curve

$$\frac{k_1^2}{r_1^2} + \frac{k_2^2}{r_2^2} = \frac{1}{2}k_3^2. \tag{4.18}$$

From (4.17) we see that $R_k^{\min}$ increases very fast with k_3. From (4.18) we get that $k_1 = k_2 = 0$ when $k_3 = 0$. Thus, the natural candidates to primary bifurcations by increasing the parameter R from zero are modes such that $k_3 = 1$. For such sets of mode numbers, the curves $R_k^{\min}$ are projected in the 2-dimensional unfolding parameter space as



The unfolding parameter space is divided into regions according to the mode that bifurcates by increasing the Rayleigh number from zero. The result, obtained numerically, is as



4.7.2 S_2-Equivariance of the Linearized Operator

Let the group S_2 act on the unfolding parameters r and $\mathcal{X}$ as in previous section. By making the only generator s act on the set of mode numbers $\underline{k}$ as

$$s: \quad (k_1, k_2, k_3) \mapsto (k_2, k_1, k_3)$$

we consider implicitly an action of s on r. We observe that the solution u_k of $L_r(u) = 0$ calculated in section 4.7.1 is S_2-equivariant

$$u_{sk}(s\xi) \quad = \quad su_k(\xi).$$

This suggests another result concerning a symmetry of the linear operator L_r. Let the group S_2 act on the operator L_r as

$$s: \quad (L_r[1], L_r[2], L_r[3], L_r[4], L_r[5]) \mapsto (L_r[2], L_r[1], L_r[3], L_r[4], L_r[5]).$$

This group action is the basis for the proof of the following

Lemma 12 *Let $w \in (\ker L_r)^\perp$ be a solution of $L_r(u) = b$ where $b \in \mathcal{Y}$ is S_2-equivariant. Then w is S_2-equivariant.*

Proof The equation $L_r(u) = b$ has a unique solution in $(\ker L_r)^\perp$. Let $w(\xi)$ be this solution

$$L_r(w(\xi)) \quad = \quad b(\xi). \tag{4.19}$$

We claim that $sw(s\xi)$ is a solution of the same equation and by uniqueness it must be equal to $w(\xi)$. This gives the equivariance that we want

$$w(s\xi) \quad = \quad sw(\xi).$$

To prove the claim we observe that

$$L_{sr}(sw(s\xi)) \quad = \quad b(s\xi). \tag{4.20}$$

Now a direct calculation shows that the operator L_r commutes with the action of S_2 as

$$L_{sr}(sw(s\xi)) \quad = \quad L_r(w(\xi))$$

and by equivariance of $b(\xi)$ we have that

$$b(s\xi) \quad = \quad sb(\xi)$$

where s acts on b by permuting the first two components. Therefore, equation (4.20) is the same as (4.19) and $sw(s\xi)$ is a solution. $\square$

4.8 Interaction of Three Modes in a Box with Rectangular Cross Section

Assume that $r = r_c$ is such that three solutions with mode numbers $\underline{k}, \underline{l}, \underline{m} \in \mathbb{N}^3$ bifurcate simultaneously when the Rayleigh number is increased across a critical value $R = R_c$. As in section 4.7 we have that ker L and ker L^* are 3-dimensional spaces as

$$\begin{aligned} \ker L &= V_k \oplus V_l \oplus V_m \\ \ker L^* &= V_k^* \oplus V_l^* \oplus V_m^*. \end{aligned}$$

Let

$$\begin{aligned} \underline{k} &= (0,1,1) \\ \underline{l} &= (1,1,1) \\ \underline{m} &= (2,0,1). \end{aligned}$$

A calculation shows that these modes bifurcate simultaneously at two points in the unfolding parameter space. These points are approximatly as

	Critical point 1	Critical point 2
r_{1c}	2.634	2.992
r_{2c}	1.521	1.496

and the critical values of the bifurcation parameter are approximatly

	Critical point 1	Critical point 2
R_c	6.797	6.778

The Liapunov-Schmidt calculations will be performed for the two points in parallel since they both satisfy the requirements of the procedure described in the sequel. As in section 4.4.2, the bifurcation equations obtained by a Liapunov-Schmidt reduction have $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry. Thus, up to 3rd order they are of the form

$$\begin{aligned} f_1 &= (a_{N_1}N_1 + a_{N_2}N_2 + a_{N_3}N_3 + a_\lambda\lambda + a_{\rho_1}\rho_1 + a_{\rho_2}\rho_2)x = 0 \\ f_2 &= (b_{N_1}N_1 + b_{N_2}N_2 + b_{N_3}N_3 + b_\lambda\lambda + b_{\rho_1}\rho_1 + b_{\rho_2}\rho_2)y = 0 \\ f_3 &= (c_{N_1}N_1 + c_{N_2}N_2 + c_{N_3}N_3 + c_\lambda\lambda + c_{\rho_1}\rho_1 + c_{\rho_2}\rho_2)z = 0 \end{aligned}$$

where

$$N_1 = x^2 \qquad N_2 = y^2 \qquad N_3 = z^2$$

and

$$\lambda = R - R_c \qquad \rho_1 = r_1 - r_{1c} \qquad \rho_2 = r_2 - r_{2c}.$$

According to Golubitsky and Schaeffer [28] the coefficients are

$$
\begin{aligned}
a_{N_1} &= \frac{1}{6}\langle u_k^*, V_{kk}\rangle \\
a_{N_2} &= \frac{1}{2}\langle u_k^*, V_{kl}\rangle \\
a_{N_3} &= \frac{1}{2}\langle u_k^*, V_{km}\rangle \\
a_\lambda &= \langle u_k^*, (d\Phi_\lambda).u_k - d^2\Phi(u_k, L^{-1}E\Phi_\lambda)\rangle \\
a_{\rho_j} &= \langle u_k^*, (d\Phi_{\rho_j}).u_k - d^2\Phi(u_k, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2 \\
b_{N_1} &= \frac{1}{2}\langle u_l^*, V_{lk}\rangle \\
b_{N_2} &= \frac{1}{6}\langle u_l^*, V_{ll}\rangle \\
b_{N_3} &= \frac{1}{2}\langle u_l^*, V_{lm}\rangle \\
b_\lambda &= \langle u_l^*, (d\Phi_\lambda).u_l - d^2\Phi(u_l, L^{-1}E\Phi_\lambda)\rangle \\
b_{\rho_j} &= \langle u_l^*, (d\Phi_{\rho_j}).u_l - d^2\Phi(u_l, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2 \\
c_{N_1} &= \frac{1}{6}\langle u_m^*, V_{mk}\rangle \\
c_{N_2} &= \frac{1}{2}\langle u_m^*, V_{ml}\rangle \\
c_{N_3} &= \frac{1}{2}\langle u_m^*, V_{mm}\rangle \\
c_\lambda &= \langle u_m^*, (d\Phi_\lambda).u_m - d^2\Phi(u_m, L^{-1}E\Phi_\lambda)\rangle \\
c_{\rho_j} &= \langle u_m^*, (d\Phi_{\rho_j}).u_m - d^2\Phi(u_m, L^{-1}E\Phi_{\rho_j})\rangle, \quad j = 1,2.
\end{aligned}
$$

where E is the projection onto range L and, taking into account that the initial equations have only quadratic nonlinearity, we have that

$$
\begin{aligned}
V_{kk} &= -3d^2\Phi(u_k, w_{kk}) \\
V_{kl} &= -d^2\Phi(u_k, w_{ll}) - 2d^2\Phi(u_l, w_{kl}) \\
V_{km} &= -d^2\Phi(u_k, w_{mm}) - 2d^2\Phi(u_m, w_{km}) \\
V_{lk} &= -d^2\Phi(u_l, w_{kk}) - 2d^2\Phi(u_k, w_{kl}) \\
V_{ll} &= -3d^2\Phi(u_l, w_{ll}) \\
V_{lm} &= -d^2\Phi(u_l, w_{mm}) - 2d^2\Phi(u_m, w_{lm}) \\
V_{mk} &= -d^2\Phi(u_m, w_{kk}) - 2d^2\Phi(u_k, w_{km}) \\
V_{ml} &= -d^2\Phi(u_m, w_{ll}) - 2d^2\Phi(u_l, w_{lm}) \\
V_{mm} &= -3d^2\Phi(u_m, w_{mm})
\end{aligned}
$$

where

$$
\begin{aligned}
w_{kk} &= L^{-1}Ed^2\Phi(u_k, u_k) \\
w_{kl} &= L^{-1}Ed^2\Phi(u_k, u_l)
\end{aligned}
$$

$$\begin{aligned} w_{km} &= L^{-1}Ed^2\Phi(u_k, u_m) \\ w_{ll} &= L^{-1}Ed^2\Phi(u_l, u_l) \\ w_{lm} &= L^{-1}Ed^2\Phi(u_l, u_m) \\ w_{mm} &= L^{-1}Ed^2\Phi(u_m, u_m). \end{aligned}$$

4.8.1 Calculation of $a_{N_1}, b_{N_2}, c_{N_3}$

In this section we give a detailed description of the calculation of a_{N_1}. The coefficients b_{N_2} and c_{N_3} are obtained immediatly by substituting the mode numbers $\underline{k}$ by $\underline{l}$ and $\underline{m}$ respectively. The main technical difficulty is the calculation of w_{kk}. We reserve part (a) to deal with this problem. In part (b) we explain how to use Maple to do the rest of the work.

(a) Calculation of w_{kk}

The equation to solve in order to calculate w_{kk} is

$$L(w_{kk}) = Ed^2\Phi(u_k, u_k). \tag{4.21}$$

In order to simplify the notations we denote

$$\begin{aligned} A_k &= \sin(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3) \\ B_k &= \cos(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3) \\ C_k &= \sin(k_1\xi_1)\sin(k_2\xi_2)\sin(k_3\xi_3) \\ D_k &= \cos(k_1\xi_1)\cos(k_2\xi_2)\cos(k_3\xi_3) \\ E_k &= \sin(k_1\xi_1)\sin(k_2\xi_2)\cos(k_3\xi_3) \\ F_k &= \sin(k_1\xi_1)\cos(k_2\xi_2)\sin(k_3\xi_3). \end{aligned}$$

Recall from section 4.7 that

$$u_k = (a_k[1]A_k, a_k[2]A_{sk}, a_k[3]B_k, a_k[4]B_k, a_k[5]D_k)$$

where after calculating a_k explicitly we observed the equivariance under the group $\mathbf{S}_2$ as

$$a_{sk} = sa_k.$$

Now we have

$$\begin{aligned} d^2\Phi[1](u_k, u_k) &= -\frac{2a_k[1]}{\sigma}\left(\frac{k_1}{r_1}a_k[1]A_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}E_k - k_3a_k[3]B_kF_k\right) \\ d^2\Phi[2](u_k, u_k) &= -\frac{2a_k[2]}{\sigma}\left(\frac{k_2}{r_2}a_k[2]A_{sk}D_k - \frac{k_1}{r_1}a_k[1]A_kE_k - k_3a_k[3]B_kF_{sk}\right) \\ d^2\Phi[3](u_k, u_k) &= -\frac{2a_k[3]}{\sigma}\left(k_3a_k[3]B_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}F_{sk} - \frac{k_1}{r_1}a_k[1]A_kF_k\right) \\ d^2\Phi[4](u_k, u_k) &= -2a_k[4]\left(k_3a_k[3]B_kD_k - \frac{k_2}{r_2}a_k[2]A_{sk}F_{sk} - \frac{k_1}{r_1}a_k[1]A_kF_k\right) \\ d^2\Phi[5](u_k, u_k) &= 0. \end{aligned}$$

A direct calculation shows that

$$\langle u_j^*, d^2\Phi(u_k,u_k)\rangle = 0$$

for $j = k, l, m$, which means that $d^2\Phi(u_k,u_k) \in$ range L. Recalling that E in equation (4.21) denotes an orthogonal projection onto range L, this equation is the same as

$$L(w_{kk}) = d^2\Phi(u_k,u_k). \tag{4.22}$$

Also by direct calculation we have that $d^2\Phi(u_k,u_k)$ is S_2-equivariant. Thus, by lemma 12, the solution w_{kk} of system (4.22) must be such that

$$w_{sksk}(s\xi) = s w_{kk}(\xi). \tag{4.23}$$

Maple does not seem to solve PDEs and system (4.22) is very hard to solve without the help of some computer algebra. The approach we take here is to write w_{kk} as a polynomial function of $A_k, \ldots, F_k$. The coefficients in this polynomial are the solution of a system of linear algebraic equations. Denote

$$\begin{aligned}
X_k[1] &= (A_kD_k, A_{sk}E_k, B_kF_k, C_kF_{sk}) \\
X_k[2] &= (A_{sk}D_k, A_kE_k, B_kF_{sk}, C_kF_k) \\
X_k[3] &= (B_kD_k, A_{sk}F_{sk}, A_kF_k, C_kE_k) \\
X_k[4] &= (B_kD_k, A_{sk}F_{sk}, A_kF_k, C_kE_k) \\
X_k[5] &= (A_k^2, A_{sk}^2, B_k^2, C_k^2, D_k^2, E_k^2, F_k^2, F_{sk}^2)
\end{aligned}$$

and note that $d^2\Phi[j](u_k,u_k)$ is a linear combination of the components of $X_k[j]$. More precisely, by denoting

$$\begin{aligned}
b_k[1] &= \frac{2a_k[1]}{\sigma}\left(-\frac{k_1}{r_1}a_k[1], \frac{k_2}{r_2}a_k[2], k_3a_k[3], 0\right) \\
b_k[2] &= \frac{2a_k[2]}{\sigma}\left(-\frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], k_3a_k[3], 0\right) \\
b_k[3] &= \frac{2a_k[3]}{\sigma}\left(-k_3a_k[3], \frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], 0\right) \\
b_k[4] &= 2a_k[4]\left(-k_3a_k[3], \frac{k_2}{r_2}a_k[2], \frac{k_1}{r_1}a_k[1], 0\right) \\
b_k[5] &= (0,0,0,0,0,0,0,0),
\end{aligned}$$

we have that

$$d^2\Phi[j](u_k,u_k) = b_k[j].X_k[j].$$

Writing $w_{kk}[j]$ linearly on the components of $X_k[j]$ and substituting on (4.22) we get a system of linear algebraic equations on the coefficients. By doing this we arrived at the conclusion that the coordinates X_k are not convenient: the system we got is not determined and Maple cannot solve it. In order to change coordinates we introduce the matrices

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \quad \text{and} \quad P_2 = \begin{pmatrix} P_1 & P_1 \\ P_1 & -P_1 \end{pmatrix}$$

A better coordinate system is

$$\begin{aligned} Y_k[j] &= P_1 X_k[j] \quad \text{for } 1 \leq j \leq 4, \\ Y_k[5] &= P_2 X_k[5]. \end{aligned}$$

Note that this change of coordinates is orthogonal. In fact we have that

$$P_1^{-1} = \frac{1}{4} P_1 \quad \text{and} \quad P_2^{-1} = \frac{1}{8} P_2.$$

Let

$$\begin{aligned} d^2\Phi[j](u_k, u_k) &= q_k[j].Y_k[j] \\ w_{kk}[j] &= p_k[j].Y_k[j], \end{aligned} \tag{4.24}$$

where

$$q_k[j] = P_1^{-1} b_k[j]$$

and p_k are the coefficients that we want to calculate. As another symmetry observation we have that Y_k is S_2-equivariant as

$$\begin{aligned} Y_{sk}[1] &= Y_k[2] \\ Y_{sk}[3,2] &= Y_k[3,3] \\ Y_{sk}[4,2] &= Y_k[4,3] \\ Y_{sk}[5,2] &= -Y_k[5,8] \\ Y_{sk}[5,4] &= -Y_k[5,6], \end{aligned} \tag{4.25}$$

where $Y_k[i,j]$ represents the jth component of $Y_k[i]$. A simple reasoning says that w_{kk} written in the coordinates Y_k satisfies the equivariance (4.23) if and only if the coefficients p_k are constrained by the symmetries (4.25). For the same reason q_k satisfies the same symmetries.

Substituting (4.24) in system (4.22) we get a system of linear algebraic equations in the form

$$L_k p_k = q_k,$$

where q_k is already known and by permuting rows and columns, the matrix L_k may be written as

$$L_k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_k[1] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{sk}[1] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_k[2] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_k[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_k[4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_{sk}[4] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_k[5] \end{pmatrix},$$

where the matrices along the diagonal are

$$L_k[1] = \begin{pmatrix} -4\frac{k_2^2}{r_2^2} & 4\frac{k_2}{r_2} \\ \frac{k_2}{r_2} & 0 \end{pmatrix}$$

$$L_k[2] = \begin{pmatrix} -4\left(\frac{k_1^2}{r_1^2}+\frac{k_2^2}{r_2^2}\right) & 0 & -4\frac{k_1}{r_1} \\ 0 & -4\left(\frac{k_1^2}{r_1^2}+\frac{k_2^2}{r_2^2}\right) & -4\frac{k_2}{r_2} \\ -\frac{k_1}{r_1} & -\frac{k_2}{r_2} & 0 \end{pmatrix}$$

$$L_k[3] = \begin{pmatrix} -4k_3^2 & 1 & 4k_3 \\ R & -4k_3^2 & 0 \\ k_3 & 0 & 0 \end{pmatrix}$$

$$L_k[4] = \begin{pmatrix} -4\left(\frac{k_2^2}{r_2^2}+k_3^2\right) & 0 & 0 & 4\frac{k_2}{r_2} \\ 0 & -4\left(\frac{k_2^2}{r_2^2}+k_3^2\right) & 1 & 4k_3 \\ 0 & R & -4\left(\frac{k_2^2}{r_2^2}+k_3^2\right) & 0 \\ \frac{k_2}{r_2} & k_3 & 0 & 0 \end{pmatrix}$$

$$L_k[5] = \begin{pmatrix} -4\Lambda_k & 0 & 0 & 0 & -4\frac{k_1}{r_1} \\ 0 & -4\Lambda_k & 0 & 0 & -4\frac{k_2}{r_2} \\ 0 & 0 & -4\Lambda_k & 1 & -4k_3 \\ 0 & 0 & R & -4\Lambda_k & 0 \\ -\frac{k_1}{r_1} & -\frac{k_2}{r_2} & -k_3 & 0 & 0 \end{pmatrix}.$$

Letting c_k be the corresponding reordering of p_k we have that

$$\begin{aligned} c_k[0] &= p_k[5,1] \\ c_k[1] &= (p_k[2,1];p_k[5,2]) \\ c_k[2] &= (p_k[1,2],p_k[2,2],p_k[5,7]) \\ c_k[3] &= (p_k[3,1],p_k[4,1],p_k[5,3]) \\ c_k[4] &= (p_k[2,3],p_k[3,2],p_k[4,2],p_k[5,4]) \\ c_k[5] &= (p_k[1,4],p_k[2,4],p_k[3,4],p_k[4,4],p_k[5,5]), \end{aligned}$$

and we let d_k be the same reordering of q_k. Now systems of the form

$$L_{sk}[j]c_{sk}[j] = d_{sk}[j]$$

are redundant since $c_{sk}[j]$ can be obtained by conjugacy of $c_k[j]$. So we are restricted to solve the systems

$$L_k[j]c_k[j] \;=\; d_k[j]$$

for $1 \leq j \leq 5$, which may be easily solved by hand. We may also ask Maple to solve these systems without worying about any technical problem since the determinants are nonzero. All coefficients are now determined apart from $p_k[5,1]$ which can have any value. This is not surprising if we see that

$$Y_k[5,1] \;=\; 1.$$

Thus when we differentiate w_{kk} with respect to any ξ_j, the coefficient $p_k[5,1]$ will not appear. Finaly we have w_{kk} and we can proceed with the rest of the calculation of a_{N_1}.

(b) The rest of the job with Maple

Now, Maple has no problem in doing the rest of the job. Recall that what is left is the computation of

$$V_{kk} \;=\; -3d^2\Phi(u_k, w_{kk}),$$

where u_k is one of the eigenfunctions generating $\ker L$ and w_{kk} is what we have just calculated. Given the values of u_k and w_{kk}, that we know at this point, Maple computes V_{kk} according to the formula above. Then we have one of the coefficients of the reduced equations by computing

$$\begin{aligned} a_{N_1} &= \frac{1}{6}\langle u_k^*, V_{kk}\rangle \\ &= \frac{1}{6}\int_0^\pi\int_0^\pi\int_0^\pi u_k^*.V_{kk}\,dx\,dy\,dz, \end{aligned}$$

where the dot is the usual dot product in $\mathbb{R}^5$ and we recall that u_k^* is one of the eigenfuntions generating $\ker L^*$. Two of the remaining coefficients of the bifurcation equations are obtained by substituting the mode numbers $\underline{k}$ in the formuli above by $\underline{l}$ and $\underline{m}$ respectively. The approximate results are as follows

	Critical point 1	Critical point 2
a_{N_1}	$-\frac{2.712}{\sigma}\pi^3$	$-\frac{2.816}{\sigma}\pi^3$
b_{N_2}	$-\frac{1.529\sigma-0.104}{\sigma^2}\pi^3$	$-\frac{1.451\sigma-0.876}{\sigma^2}\pi^3$
c_{N_3}	$-\frac{4.148}{\sigma}\pi^3$	$-\frac{2.816}{\sigma}\pi^3$

4.8.2 Calculation of $a_{N_2}, a_{N_3}, b_{N_1}, b_{N_3}, c_{N_1}, b_{N_2}$

This section gives a detailed explanation of how to obtain a_{N_2}. The other five coefficients are obtained automaticaly by substitution of mode numbers. Again we reserve part (a) to deal with the main technical difficulty: the calculation of w_{kl}. In part (b) we show how Maple can do the rest of the work.

(a) Calculation of w_{kl}

Now we want w_{kl} satisfying the equation

$$L(w_{kl}) \;=\; Ed^2\Phi(u_k, u_l). \tag{4.26}$$

Recall that

$$\begin{aligned} u_k &= (a_k[1]A_k, a_k[2]A_{sk}, a_k[3]B_k, a_k[4]B_k, a_k[5]D_k) \\ u_l &= (a_l[1]A_l, a_l[2]A_{sl}, a_l[3]B_l, a_l[4]B_l, a_l[5]D_l), \end{aligned}$$

where a_k and a_l are S_2-equivariant. By analogy with the previous section we denote

$$\begin{aligned} X_{kl}[1] &= (A_lD_k, A_{sl}E_k, B_lF_k, C_lF_{sk}, A_kD_l, A_{sk}E_l, B_kF_l, C_kF_{sl}) \\ X_{kl}[2] &= (A_{sl}D_k, A_lE_k, B_lF_{sk}, C_lF_k, A_{sk}D_l, A_kE_l, B_kF_{sl}, C_kF_l) \\ X_{kl}[3] &= (B_lD_k, A_{sl}F_{sk}, A_lF_k, C_lE_k, B_kD_l, A_{sk}F_{sl}, A_kF_l, C_kE_l) \\ X_{kl}[4] &= (B_lD_k, A_{sl}F_{sk}, A_lF_k, C_lE_k, B_kD_l, A_{sk}F_{sl}, A_kF_l, C_kE_l) \\ X_{kl}[5] &= (A_lA_k, A_{sl}A_{sk}, B_lB_k, C_lC_k, D_lD_k, E_lE_k, F_lF_k, F_{sl}F_{sk}). \end{aligned}$$

Denote also

$$\begin{aligned} a_{kl}[1] &= \frac{1}{\sigma}\left(-\frac{k_1}{r_1}a_k[1]a_l[1], \frac{k_2}{r_2}a_k[1]a_l[2], k_3a_k[1]a_l[3], 0\right) \\ a_{kl}[2] &= \frac{1}{\sigma}\left(-\frac{k_2}{r_2}a_k[2]a_l[2], \frac{k_1}{r_1}a_k[2]a_l[1], k_3a_k[2]a_l[3], 0\right) \\ a_{kl}[3] &= \frac{1}{\sigma}\left(-k_3a_k[3]a_l[3], \frac{k_2}{r_2}a_k[3]a_l[2], \frac{k_1}{r_1}a_k[3]a_l[1], 0\right) \\ a_{kl}[4] &= \left(-k_3a_k[4]a_l[3], \frac{k_2}{r_2}a_k[4]a_l[2], \frac{k_1}{r_1}a_k[4]a_l[1], 0\right) \\ a_{kl}[5] &= (0,0,0,0), \end{aligned}$$

and finally define

$$b_{kl}[j] \;=\; (a_{kl}[j], a_{lk}[j]).$$

Now we have that $d^2\Phi[j](u_k, u_l)$ is a linear combination of the components of $X_{kl}[j]$ as

$$d^2\Phi[j](u_k, u_l) \;=\; b_{kl}[j].X_{kl}[j].$$

A direct calculation shows that the result of this dot product is in range L and then equation (4.26) becomes

$$L(w_{kl}) \;=\; d^2\Phi(u_k, u_l). \tag{4.27}$$

It can also be checked directly that $d^2\Phi(u_k, u_l)$ is S_2-equivariant and by lemma 12, the solution w_{kl} of system (4.27) must satisfy the equivariance

$$w_{sksl}(s\xi) \;=\; sw_{kl}(\xi). \tag{4.28}$$

By writing w_{kl} as a polynomial function of $A_k, A_l \ldots, F_k, F_l$ we reduce the problem to a system of linear algebraic equations. Again we checked that X_{kl} is not a convenient coordinate system and we apply the change of coordinates P_2 defined in the previous section. Thus we denote

$$Y_{kl}[j] = P_2 X_{kl}[j].$$

and let

$$\begin{aligned} d^2\Phi[j](u_k, u_l) &= q_{kl}[j].Y_{kl}[j] \\ w_{kl}[j] &= p_{kl}[j].Y_{kl}[j], \end{aligned} \tag{4.29}$$

where

$$q_{kl}[j] = P_2^{-1} b_{kl}[j]$$

and p_{kl} are the coefficients that we want to calculate. Now we have that Y_{kl} satisfies the equivariance

$$\begin{aligned} Y_{sksl}[1] &= Y_{kl}[2] \\ Y_{sksl}[3,2] &= Y_{kl}[3,3] \\ Y_{sksl}[3,6] &= Y_{kl}[3,7] \\ Y_{sksl}[4,2] &= Y_{kl}[4,3] \\ Y_{sksl}[4,6] &= Y_{kl}[4,7] \\ Y_{sksl}[5,2] &= -Y_{kl}[5,8] \\ Y_{sksl}[5,4] &= -Y_{kl}[5,6]. \end{aligned} \tag{4.30}$$

The S_2-equivariance of $d^2\Phi(u_k, u_l)$ and w_{kl} imposes the symmetry constraints (4.30) on the coefficients q_{kl} and p_{kl} respectively.

Substituting (4.29) in system (4.27) we get a system of linear algebraic equations as

$$L_{kl} p_{kl} = q_{kl},$$

where q_{kl} is already known and by permuting rows and columns, the matrix L_{kl} may be written as

$$L_{kl} = \begin{pmatrix} L_{kl}[1] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{sksl}[1] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & L_{kl}[2] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{sksl}[2] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_k[3] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L_{sk}[4] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & L_k[5] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_k[6] \end{pmatrix},$$

and the matrices along the diagonal are of the form

$$L_{kl}[j] = \begin{pmatrix} -d & 0 & 0 & 0 & a \\ 0 & -d & 0 & 0 & b \\ 0 & 0 & -d & 1 & c \\ 0 & 0 & R & -d & 0 \\ a & b & c & 0 & 0 \end{pmatrix},$$

where $d = a^2 + b^2 + c^2$ and

- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = \frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = -k_3 + l_3$ if $j = 1$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = \frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = k_3 + l_3$ if $j = 2$.
- $a = -\frac{k_1}{r_1} - \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} - \frac{l_2}{r_2}$ and $c = k_3 - l_3$ if $j = 3$.
- $a = -\frac{k_1}{r_1} - \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} - \frac{l_2}{r_2}$ and $c = -k_3 - l_3$ if $j = 4$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = k_3 + l_3$ if $j = 5$.
- $a = -\frac{k_1}{r_1} + \frac{l_1}{r_1}$, $b = -\frac{k_2}{r_2} + \frac{l_2}{r_2}$ and $c = -k_3 + l_3$ if $j = 6$.

Letting c_k be the reordering of p_k corresponding to the permutation of columns of L_{kl} we have that

$$\begin{aligned} c_{kl}[1] &= (p_{kl}[1,7], p_{kl}[2,1], p_{kl}[3,7], p_{kl}[4,7], p_{kl}[5,2]) \\ c_{kl}[2] &= (p_{kl}[1,5], p_{kl}[2,3], p_{kl}[3,2], p_{kl}[4,2], p_{kl}[5,4]) \\ c_{kl}[3] &= (p_{kl}[1,2], p_{kl}[2,2], p_{kl}[3,5], p_{kl}[4,5], p_{kl}[5,7]) \\ c_{kl}[4] &= (p_{kl}[1,4], p_{kl}[2,4], p_{kl}[3,4], p_{kl}[4,4], p_{kl}[5,5]) \\ c_{kl}[5] &= (p_{kl}[1,6], p_{kl}[2,6], p_{kl}[3,1], p_{kl}[4,1], p_{kl}[5,3]) \\ c_{kl}[6] &= (p_{kl}[1,8], p_{kl}[2,8], p_{kl}[3,8], p_{kl}[4,8], p_{kl}[5,1]), \end{aligned}$$

and we let d_k be the same reordering of q_k. By eliminating the redundant blocks of the form $L_{sksl}[j]$ we are restricted to solve the systems

$$L_{kl}[j]c_{kl}[j] = d_{kl}[j]$$

for $1 \le j \le 6$, which is a reasonably easy calculation with or without computer algebra.

(b) The rest of the job with Maple

Now, Maple has no problem in doing the rest of the job. Recall that what is left is the computation of

$$V_{kl} = -d^2\Phi(u_k, w_{ll}) - 2d^2\Phi(u_l, w_{kl}),$$

where u_k, u_l are two of the eigenfuntions generating ker L, w_{ll} was computed in section 4.8.1 and w_{kl} is what we have just calculated. Given the values of u_k, u_l, w_{ll} and w_{kl}, that we know at this point, Maple computes V_{kl} according to the formula above. Then we have one more coefficient of the reduced equations by computing

$$a_{N_2} = \frac{1}{2}\langle u_k^*, V_{kl}\rangle,$$

where we recall that u_k^* is one of the eigenfuntions generating ker L^*. Five of the remaining coefficients of the bifurcation equations are obtained by substituting the mode numbers $\underline{k}$ in the formuli above by $\underline{l}$ and $\underline{m}$ respectively. The results are as follows

	Critical point 1	Critical point 2
a_{N_2}	$-\frac{9.676\sigma-0.142}{\sigma^2}\pi^3$	$-\frac{9.689\sigma-0.119}{\sigma^2}\pi^3$
a_{N_3}	$-\frac{7.693\sigma+3.474}{\sigma^2}\pi^3$	$-\frac{5.902\sigma+0.540}{\sigma^2}\pi^3$
b_{N_1}	$-\frac{8.733\sigma+0.692}{\sigma^2}\pi^3$	$-\frac{8.946\sigma+0.573}{\sigma^2}\pi^3$
b_{N_3}	$-\frac{5.094\sigma+0.386}{\sigma^2}\pi^3$	$-\frac{3.713\sigma-1.092}{\sigma^2}\pi^3$
c_{N_1}	$-\frac{6.366\sigma-1.824}{\sigma^2}\pi^3$	$-\frac{5.902\sigma+0.540}{\sigma^2}\pi^3$
c_{N_2}	$-\frac{5.094\sigma+0.386}{\sigma^2}\pi^3$	$-\frac{4.226\sigma+1.848}{\sigma^2}\pi^3$

4.8.3 Calculation of $a_\lambda, b_\lambda, c_\lambda$

In this section we describe how to obtain a_λ. Then the coefficients b_λ and c_λ come immediately by substituting the mode numbers $\underline{k}$ by $\underline{l}$ and $\underline{m}$ respectively. The calculation of a_λ is very simple comparing with the complication of the previous sections. Recall that the formula is

$$a_\lambda = \langle u_k^*, (d\Phi_\lambda).u_k - d^2\Phi(u_k, L^{-1}E\Phi_\lambda)\rangle.$$

By differentiating Φ with respect to λ we get

$$\Phi_\lambda = (0,0,0,v_3,0)^t.$$

Since Φ_λ is a linear operator, its linearization evaluated at u_k is

$$(d\Phi_\lambda).u_k = (0,0,0,u_k[3],0)^t.$$

Note that Φ_λ is odd, which implies that when $u = 0$ we have

$$\Phi_\lambda = 0.$$

All together, this results say that

$$\begin{aligned} a_\lambda &= \langle u_k^*, (d\Phi_\lambda).u_k\rangle \\ &= \int_0^\pi\int_0^\pi\int_0^\pi u_k^*[4]u_k[3]dxdydz. \end{aligned}$$

The result evaluated by Maple is approximatly

	Critical point 1	Critical point 2
a_λ	$2.752\pi^3$	$2.790\pi^3$
b_λ	$1.622\pi^3$	$1.585\pi^3$
c_λ	$3.244\pi^3$	$2.790\pi^3$

4.8.4 Calculation of $a_{\rho_1}, a_{\rho_2}, b_{\rho_1}, b_{\rho_2}, c_{\rho_1}, c_{\rho_2}$

In this section we describe how to calculate a_{ρ_1} and the remaining five coefficients are obtained by analogy. Recall that the formula is

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k - d^2\Phi(u_k, L^{-1}E\Phi_{\rho_1})\rangle.$$

Differentiating Φ with respect to the parameter ρ_1 we get

$$\begin{aligned}
\Phi_{\rho_1}[1] &= -\frac{2}{r_1^3}\frac{\partial^2 v_1}{\partial\xi_1^2} + \frac{1}{r_1^2}\frac{\partial p}{\partial\xi_1} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_1}{\partial\xi_1}\\
\Phi_{\rho_1}[2] &= -\frac{2}{r_1^3}\frac{\partial^2 v_2}{\partial\xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_2}{\partial\xi_1}\\
\Phi_{\rho_1}[3] &= -\frac{2}{r_1^3}\frac{\partial^2 v_3}{\partial\xi_1^2} + \frac{1}{\sigma r_1^2}v_1\frac{\partial v_3}{\partial\xi_1}\\
\Phi_{\rho_1}[4] &= -\frac{2}{r_1^3}\frac{\partial^2 \Theta}{\partial\xi_1^2} + \frac{1}{r_1^2}v_1\frac{\partial \Theta}{\partial\xi_1}\\
\Phi_{\rho_1}[5] &= -\frac{1}{r_1^2}\frac{\partial v_1}{\partial\xi_1}.
\end{aligned}$$

By linearizing this operator about $u = 0$ and evaluating at u_k we get

$$\begin{aligned}
(d\Phi_{\rho_1}[1]).u_k &= \left(\frac{2}{r_1^3}k_1^2a_k[1] - \frac{1}{r_1^2}k_1a_k[5]\right)A_k\\
(d\Phi_{\rho_1}[2]).u_k &= \frac{2}{r_1^3}k_1^2a_k[2]A_{sk}\\
(d\Phi_{\rho_1}[3]).u_k &= \frac{2}{r_1^3}k_1^2a_k[3]B_k\\
(d\Phi_{\rho_1}[4]).u_k &= \frac{2}{r_1^3}k_1^2a_k[4]B_k\\
(d\Phi_{\rho_1}[5]).u_k &= -\frac{1}{r_1^2}k_1a_k[1]D_k.
\end{aligned}$$

By inspection on the operator Φ_{ρ_1} we see that $u = 0$ then

$$\Phi_{\rho_1} = 0.$$

All together, the results above say that

$$a_{\rho_1} = \langle u_k^*, (d\Phi_{\rho_1}).u_k\rangle.$$

The result evaluated by Maple for the six coefficients is approximatly

	Critical point 1	Critical point 2
a_{ρ_1}	0	0
a_{ρ_2}	$8.249\pi^3$	$8.666\pi^3$
b_{ρ_1}	$0.842\pi^3$	$0.567\pi^3$
b_{ρ_2}	$4.373\pi^3$	$4.534\pi^3$
c_{ρ_1}	$6.732\pi^3$	$4.333\pi^3$
c_{ρ_2}	0	0

4.8.5 Normal Form and Bifurcation Diagrams

Given a set of equations $f(x,\lambda) = 0$, where f is obtained by a Liapunov-Schmidt reduction we can find the topology of the bifurcation diagram, but not the stability. The corresponding differential equation is one of

(a) $\dot{x} + f(x,\lambda) = 0$.

(b) $\dot{x} = f(x,\lambda)$.

In order to get information about stability we must know if our mapping f satisfies (a) or (b). In previous sections a 3rd order truncation of f was calculated for the (0,1,1)-(1,1,1)-(2,0,1) mode interaction on the Bénard problem. The result is a $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant as

$$\begin{aligned} f_1(x,y,z,\lambda,\rho_1,\rho_2) &= (a_{N_1}N_1 + a_{N_2}N_2 + a_{N_3}N_3 + a_\lambda\lambda + a_{\rho_1}\rho_1 + a_{\rho_2}\rho_2)x \\ f_2(x,y,z,\lambda,\rho_1,\rho_2) &= (b_{N_1}N_1 + b_{N_2}N_2 + b_{N_3}N_3 + b_\lambda\lambda + b_{\rho_1}\rho_1 + b_{\rho_2}\rho_2)y \\ f_3(x,y,z,\lambda,\rho_1,\rho_2) &= (c_{N_1}N_1 + c_{N_2}N_2 + c_{N_3}N_3 + c_\lambda\lambda + c_{\rho_1}\rho_1 + c_{\rho_2}\rho_2)z \end{aligned}$$

where $N_1 = x^2$, $N_2 = y^2$, $N_3 = z^2$. The bifurcation parameter λ is the Rayleigh number with the origin shifted and ρ_1, ρ_2 are the unfolding parameters that have to do with the lengths of the domain. The coefficients a, b, c are computed in previous sections by a Liapunov-Schmidt reduction.

When $\rho_1 = \rho_2 = 0$, the linearization of f about the origin is

$$df = \begin{pmatrix} a_\lambda\lambda & 0 & 0 \\ 0 & b_\lambda\lambda & 0 \\ 0 & 0 & c_\lambda\lambda \end{pmatrix},$$

which has three negative real eigenvalues if $\lambda < 0$ since $a_\lambda, b_\lambda, c_\lambda$ are positive numbers as in section 4.8.3. Thus an equation in the form (a) for this particular f leads to instability of the solution $(x,y,z) = (0,0,0)$. This means instability of the translation-invariant solution $u = 0$ to the Bénard problem when the Rayleigh number is below the critical value, which does not make sense physically. In the same way, we see that an equation in the form (b) gives the correct stability.

In appendix A we analyse a normal form with $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry. The stabilities are given under the assumption that the mapping f satisfies the differential equation (a). In order to apply the results obtained to this particular problem it is enough to change the signs of all coefficients in f and reduce the result to the equivalent normal form.

Some of the coefficients depend of the Prandtl number σ. From now on we assume that the fluid inside the box is water, which corresponds to

$$\sigma \approx 7.03.$$

Substituting this number in the coefficients calculated in previous sections and reversing sign we get

	Critical point 1	Critical point 2
a_{N_1}	11.962	12.421
a_{N_2}	42.589	42.657
a_{N_3}	36.111	26.371
a_λ	−85.328	−86.515
a_{ρ_1}	0	0
a_{ρ_2}	−255.76	−268.71
b_{N_1}	38.951	39.893
b_{N_2}	6.6793	6.3503
b_{N_3}	22.711	15.692
b_λ	−50.294	−49.144
b_{ρ_1}	−26.093	−17.573
b_{ρ_2}	−135.59	−140.58
c_{N_1}	26.935	26.371
c_{N_2}	22.711	19.799
c_{N_3}	18.297	12.421
c_λ	−100.59	−86.515
c_{ρ_1}	−208.75	−134.35
c_{ρ_2}	0	0

In order to compute the normal form for f we apply a $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalent transformation as

$$H(x,y,z,\lambda,\rho_1,\rho_2) = Sf(X,\Lambda,\rho_1,\rho_2) \tag{4.31}$$

where

$$\begin{aligned}
X(x,y,z,\lambda,\rho_1,\rho_2) &= (Ax, By, Cz)\\
\Lambda(\lambda,\rho_1,\rho_2) &= \lambda - \frac{a_{\rho_1}}{a_\lambda}\rho_1 - \frac{a_{\rho_2}}{a_\lambda}\rho_2\\
S(x,y,z,\lambda,\rho_1,\rho_2) &= \begin{pmatrix} D & 0 & 0\\ 0 & E & 0\\ 0 & 0 & F\end{pmatrix}
\end{aligned}$$

and A, B, C, D, E, F are positive constants. Substituting X, Λ and S in (4.31) leads to

$$\begin{aligned}
H_1(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (a_{N_1}A^3Dx^2 + a_{N_2}AB^2Dy^2 + a_{N_3}AC^2Dz^2 + a_\lambda AD\lambda)x\\
H_2(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (b_{N_1}A^2BEx^2 + b_{N_2}B^3Ey^2 + b_{N_3}BC^2Ez^2 + b_\lambda BE\lambda + \tilde{r}_1)y\\
H_3(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (c_{N_1}A^2CFx^2 + c_{N_2}B^2CFy^2 + c_{N_3}C^3Fz^2 + c_\lambda CF\lambda + \tilde{r}_2)z,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{r}_1 &= BE\left(b_{\rho_1} - \frac{a_{\rho_1}b_\lambda}{a_\lambda}\right)\rho_1 + BE\left(b_{\rho_2} - \frac{a_{\rho_2}b_\lambda}{a_\lambda}\right)\rho_2\\
\tilde{r}_2 &= CF\left(c_{\rho_1} - \frac{a_{\rho_1}c_\lambda}{a_\lambda}\right)\rho_1 + CF\left(c_{\rho_2} - \frac{a_{\rho_2}c_\lambda}{a_\lambda}\right)\rho_2.
\end{aligned}$$

By imposing the conditions

$$\begin{aligned} |a_{N_1}|A^3D &= 1 \\ |b_{N_2}|B^3E &= 1 \\ |c_{N_3}|C^3F &= 1 \\ |a_\lambda|AD &= 1 \\ |b_\lambda|BE &= 1 \\ |c_\lambda|CF &= 1 \end{aligned}$$

we get H in the desired form

$$\begin{aligned} H_1(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (\epsilon_1 x^2 + n_1 y^2 + n_2 z^2 + \epsilon_2\lambda)x \\ H_2(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (n_3 x^2 + \epsilon_3 y^2 + n_4 z^2 + \epsilon_4\lambda + \tilde{r}_1)y \\ H_3(x,y,z,\lambda,\tilde{r}_1,\tilde{r}_2) &= (n_5 x^2 + n_6 y^2 + \epsilon_5 z^2 + \epsilon_6\lambda + \tilde{r}_2)z, \end{aligned}$$

where

$$\begin{aligned} \epsilon_1 &= \operatorname{sgn}(a_{N_1}) \\ \epsilon_2 &= \operatorname{sgn}(a_\lambda) \\ \epsilon_3 &= \operatorname{sgn}(b_{N_2}) \\ \epsilon_4 &= \operatorname{sgn}(b_\lambda) \\ \epsilon_5 &= \operatorname{sgn}(c_{N_3}) \\ \epsilon_6 &= \operatorname{sgn}(c_\lambda) \end{aligned}$$

and

$$\begin{aligned} n_1 &= \left|\frac{b_\lambda}{b_{N_2}a_\lambda}\right| a_{N_2} \\ n_2 &= \left|\frac{c_\lambda}{c_{N_3}a_\lambda}\right| a_{N_3} \\ n_3 &= \left|\frac{a_\lambda}{a_{N_1}b_\lambda}\right| b_{N_1} \\ n_4 &= \left|\frac{c_\lambda}{c_{N_3}b_\lambda}\right| b_{N_3} \\ n_5 &= \left|\frac{a_\lambda}{a_{N_1}c_\lambda}\right| c_{N_1} \\ n_6 &= \left|\frac{b_\lambda}{b_{N_2}c_\lambda}\right| c_{N_2}. \end{aligned}$$

The unfolding parameters are

$$\begin{aligned} \tilde{r}_1 &= \frac{1}{|b_\lambda|}\left(b_{\rho_1} - \frac{a_{\rho_1}b_\lambda}{a_\lambda}\right)\rho_1 + \frac{1}{|b_\lambda|}\left(b_{\rho_2} - \frac{a_{\rho_2}b_\lambda}{a_\lambda}\right)\rho_2 \\ \tilde{r}_2 &= \frac{1}{|c_\lambda|}\left(c_{\rho_1} - \frac{a_{\rho_1}c_\lambda}{a_\lambda}\right)\rho_1 + \frac{1}{|c_\lambda|}\left(c_{\rho_2} - \frac{a_{\rho_2}c_\lambda}{a_\lambda}\right)\rho_2. \end{aligned}$$

By substititing the derivatives of a, b, c given above in the formulae for ϵ_j and n_j we get

$$\begin{aligned} \epsilon_1 = \epsilon_3 = \epsilon_5 &= 1 \\ \epsilon_2 = \epsilon_4 = \epsilon_6 &= -1 \end{aligned}$$

and

	Critical point 1	Critical point 2
n_1	3.758	3.816
n_2	2.327	2.123
n_3	5.524	5.654
n_4	2.482	2.224
n_5	1.910	2.123
n_6	1.700	1.771

Now the unfolding parameters are approximatly

	Critical point 1	Critical point 2
$\tilde{r}_1$	$-0.519\rho_1 + 0.302\rho_2$	$-0.358\rho_1 + 0.245\rho_2$
$\tilde{r}_2$	$-2.075\rho_1 + 2.997\rho_2$	$-1.553\rho_1 + 3.106\rho_2$

Taking these numbers to appendix A, the bifurcation diagrams can be drawn directly by reading the tables. The nondegeneracy conditions are satisfied, since by using the notation in the appendix we have

	Critical point 1	Critical point 2
(a)	−2.758	−2.816
(b)	−1.327	−1.123
(c)	−4.524	−4.654
(d)	−1.482	−1.224
(e)	−0.910	−1.123
(f)	−0.700	−0.771
(g)	−19.76	−20.57
(h)	−3.444	−3.508
(i)	−3.220	−2.939
(j)	−6.802	−6.354
(k)	6.144	5.341
(l)	3.980	3.369
(m)	11.24	10.26

and they are all nonzero. In appendix A we use the signs if these constants to get all the information about existence and criticality of bifurcating branches. Since the signs are the same for the two critical points they lead to the same bifurcation diagrams. From now on we concentrate on point 1. Before drawing the bifurcation diagrams we make a convention about the labels of possible branches.

(0) Translation-invariant solution $u = 0$.

(1) Pure mode (0,1,1).

(2) Pure mode (1,1,1).

(3) Pure mode (2,0,1).

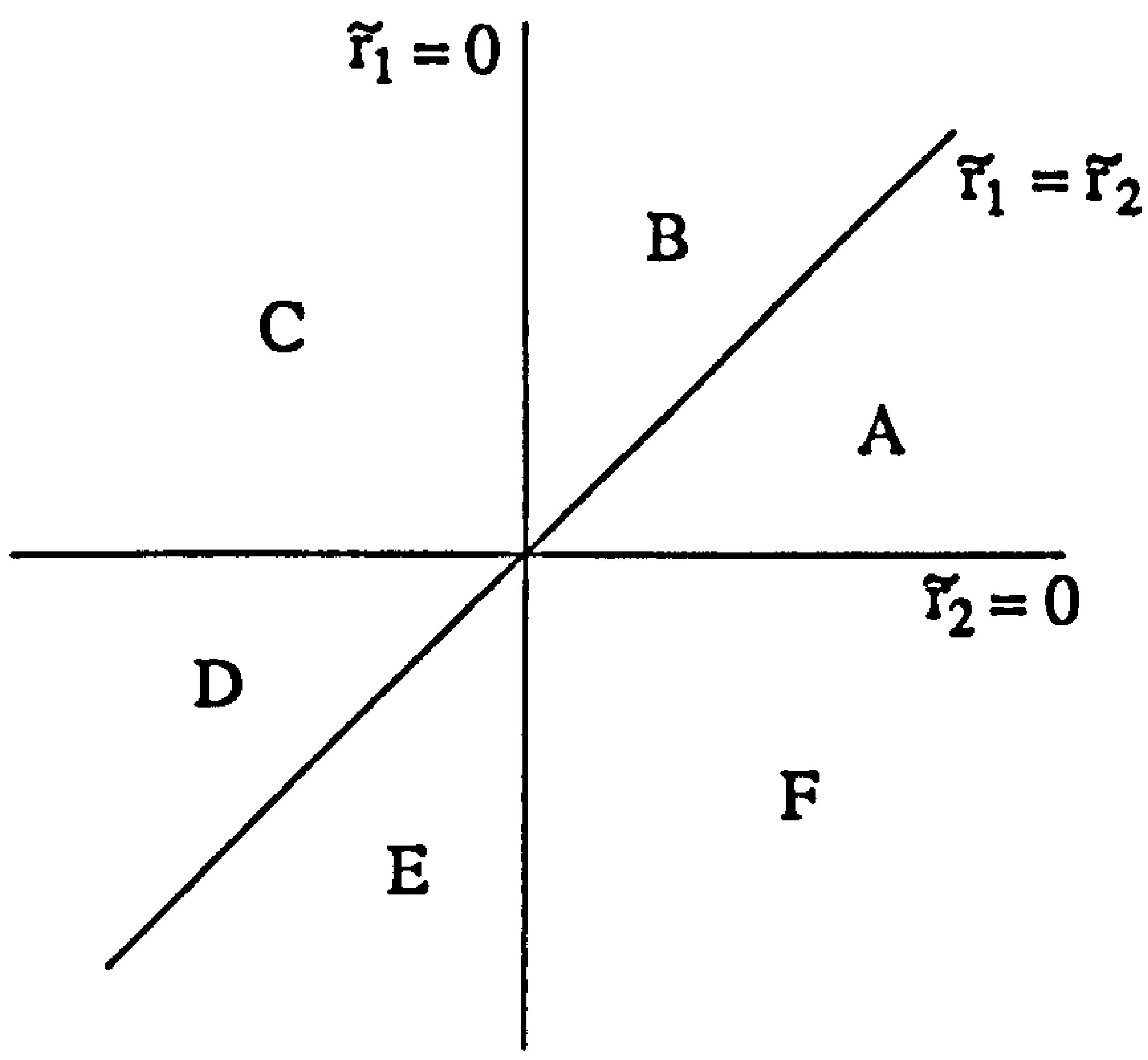
(4) Mixed mode (0,1,1)-(1,1,1).

(5) Mixed mode (0,1,1)-(2,0,1).

(6) Mixed mode (1,1,1)-(2,0,1).

(7) Mixed mode (0,1,1)-(1,1,1)-(2,0,1).

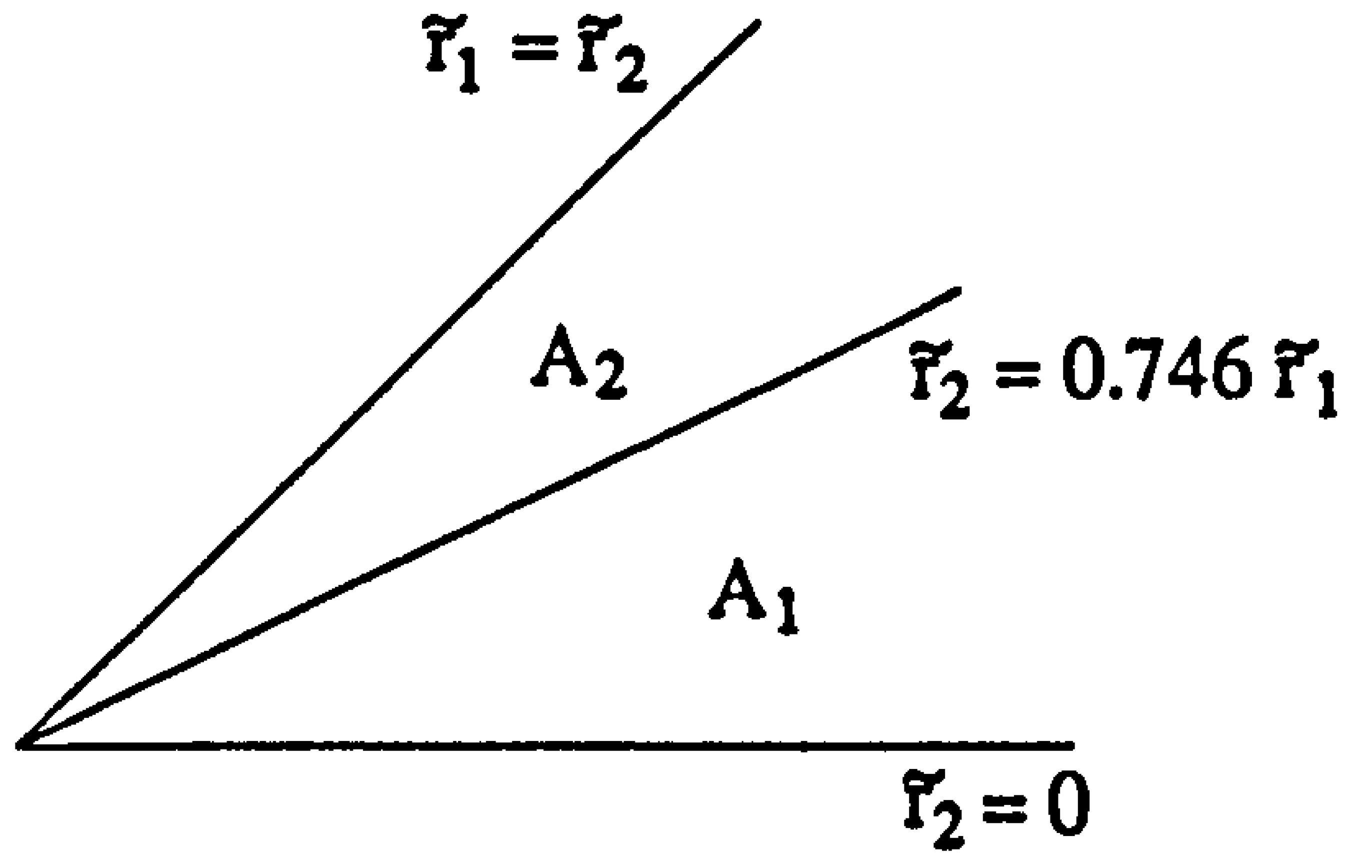
Now the unfolding parameter space is divided into six distinct regions according to the order of primary bifurcations when the parameter is increased. This division is the following

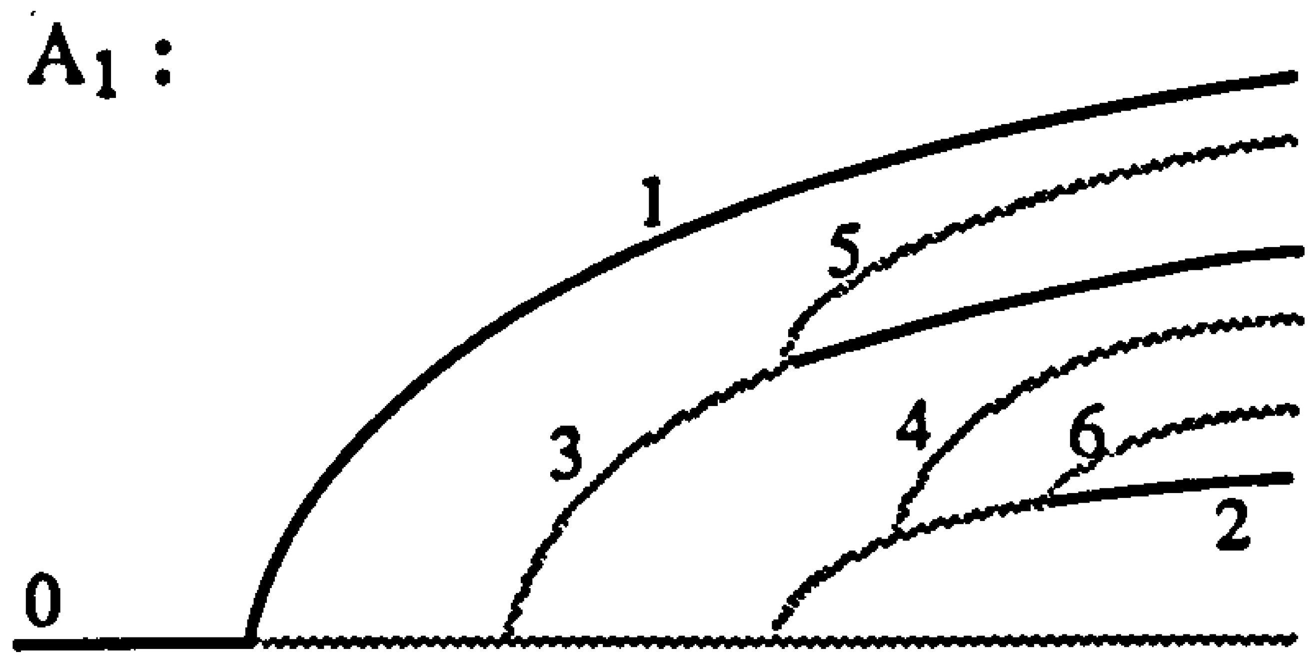


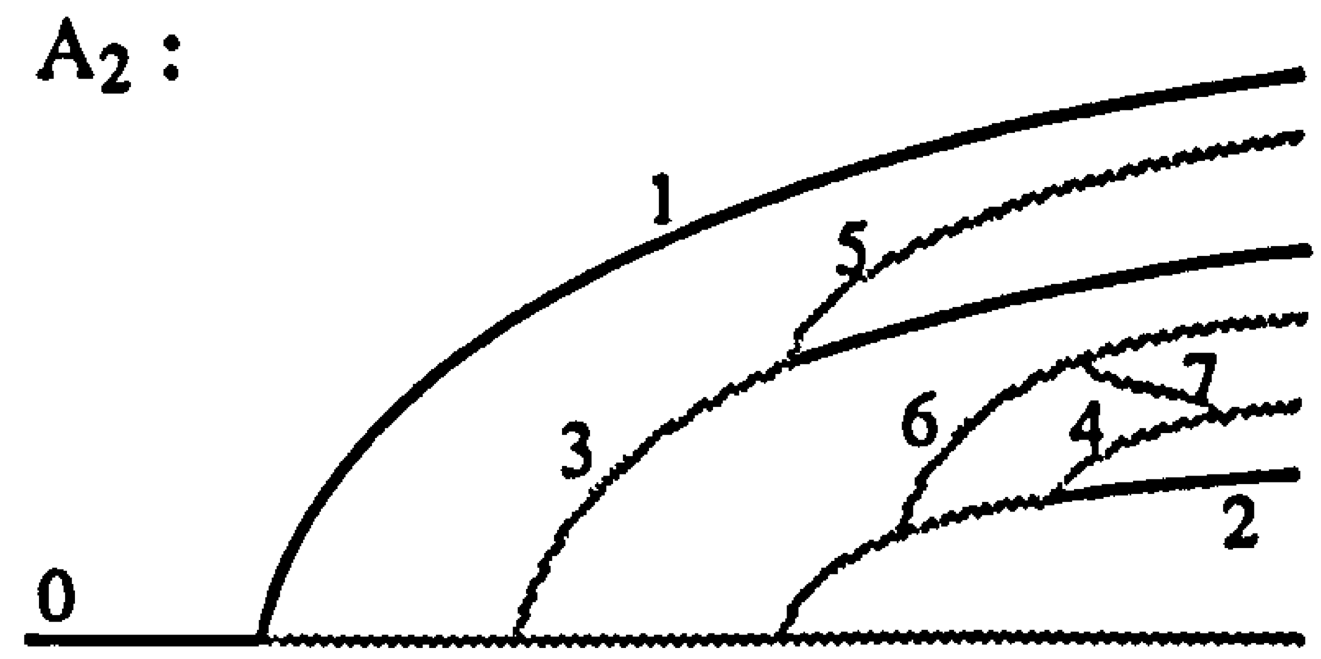
Then some of these regions need a subdivision according to the order of secondary bifurcations. This will be illustrated when appropriate and we proceed by drawing the bifurcation diagrams.

Region A

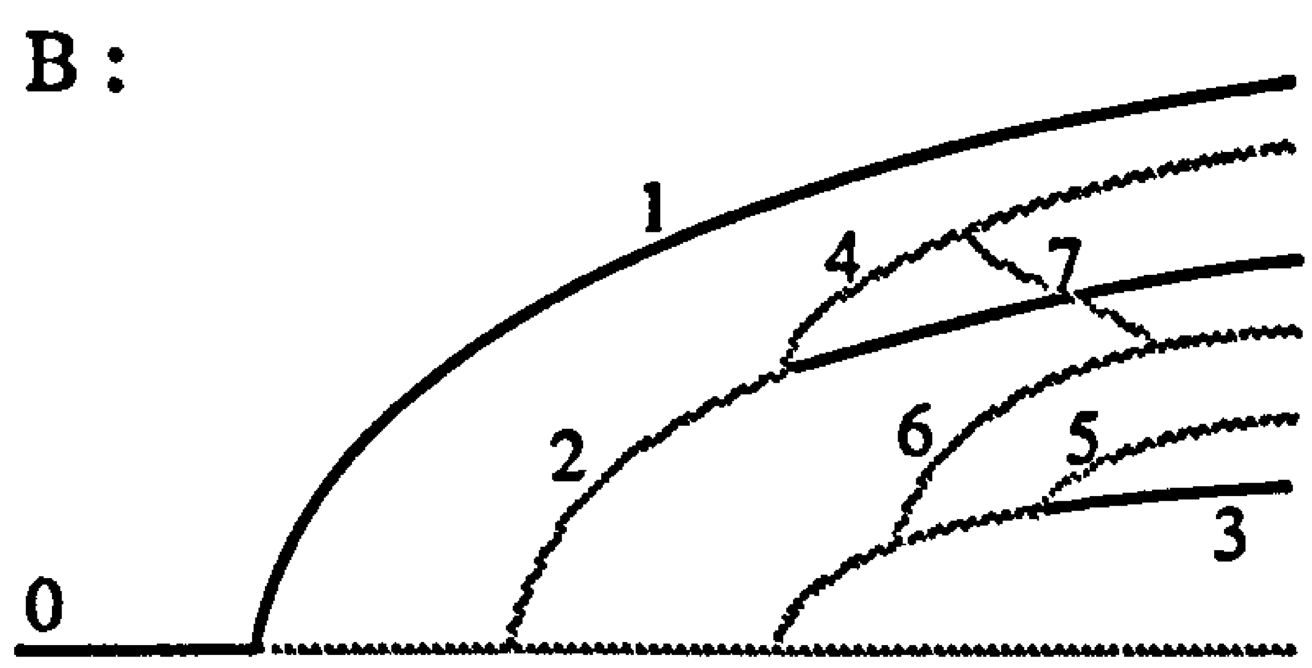
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagrams corresponding to each subregion.





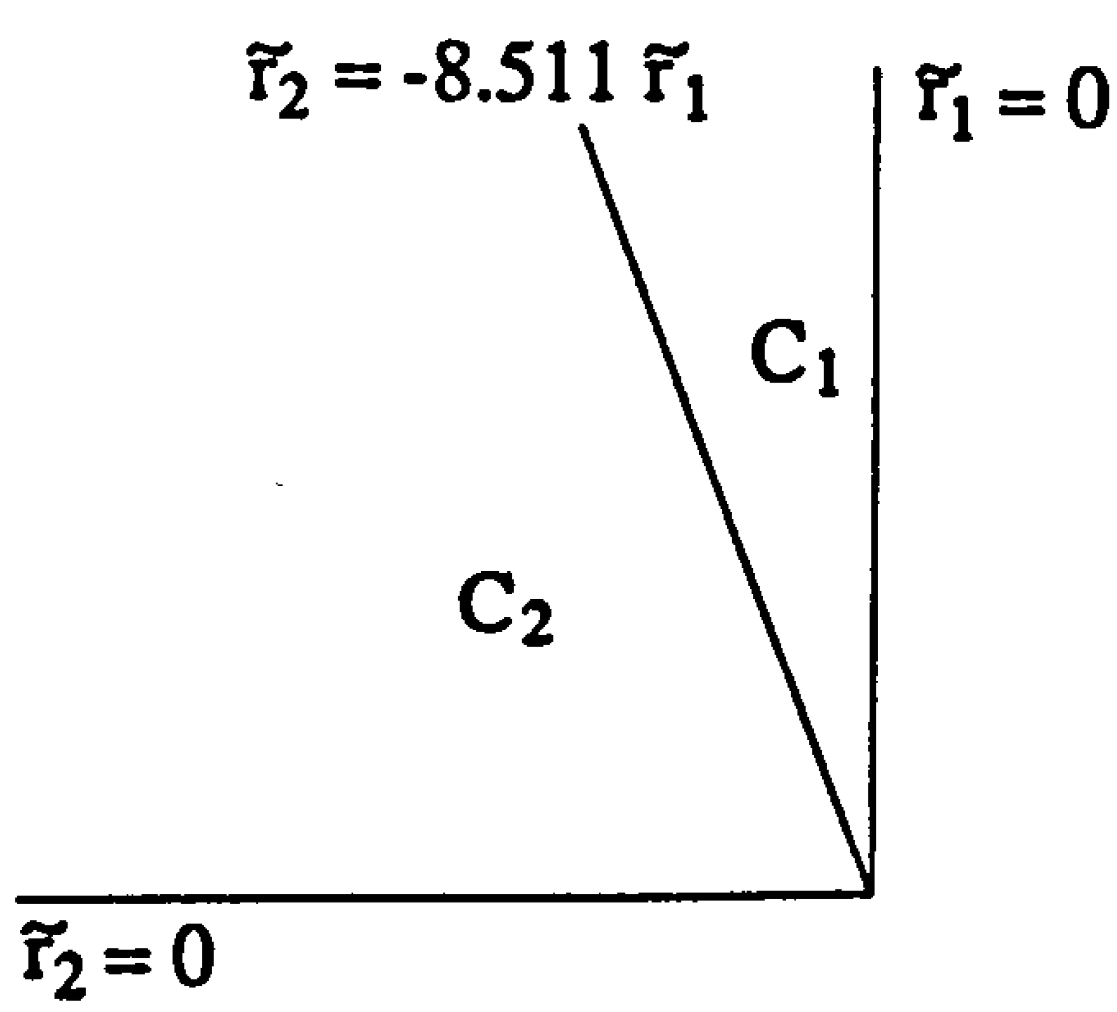


Region B

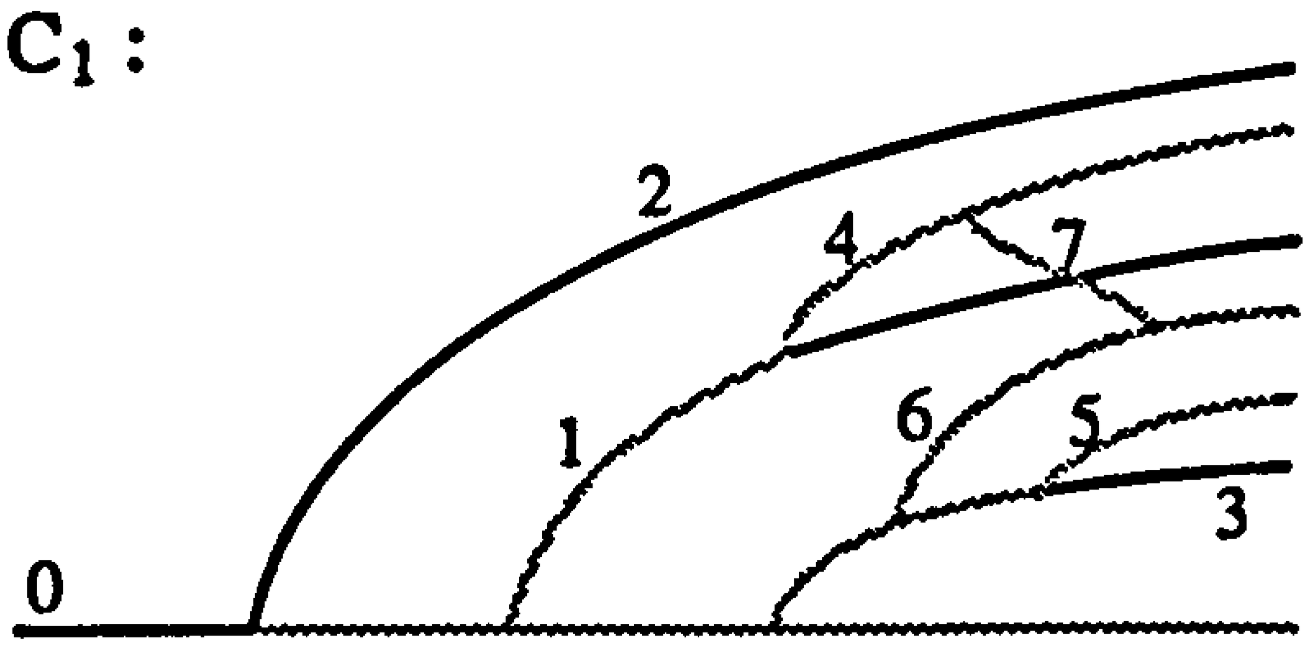


Region C

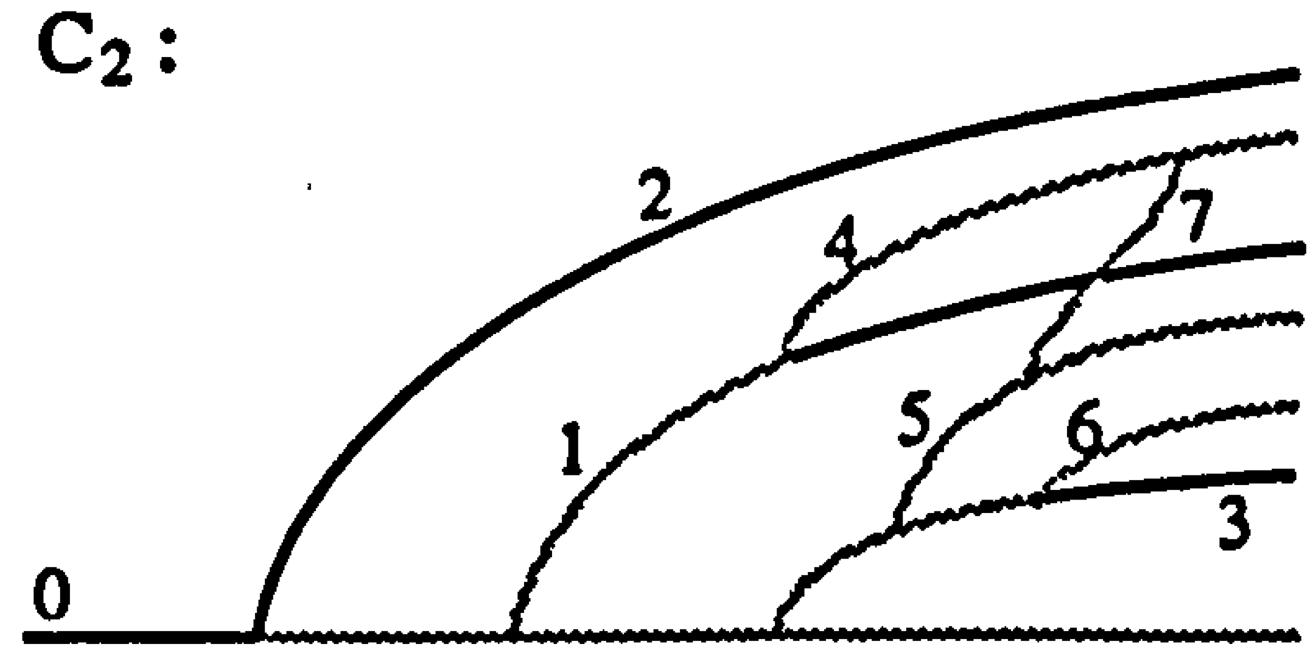
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



C_1 :

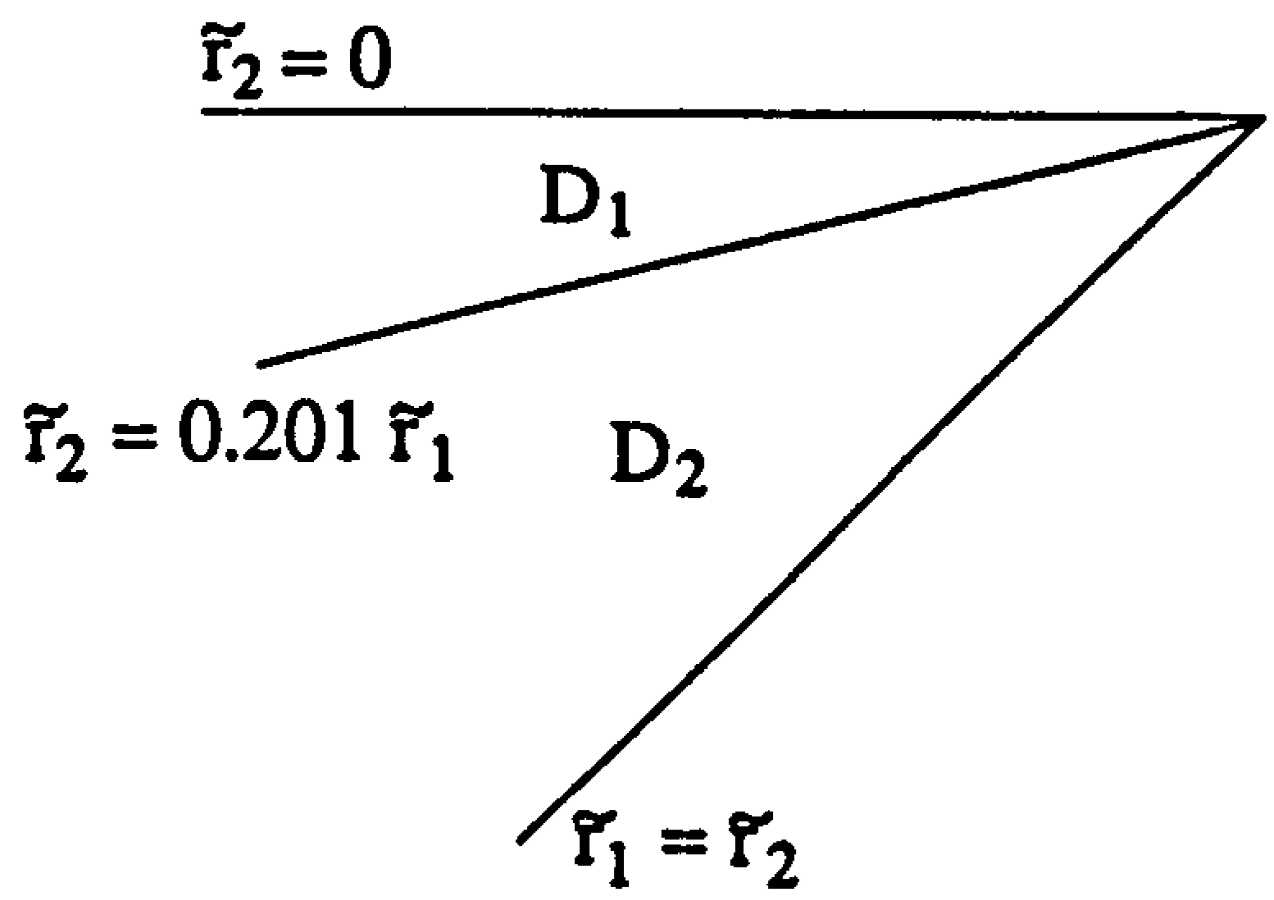


C_2 :

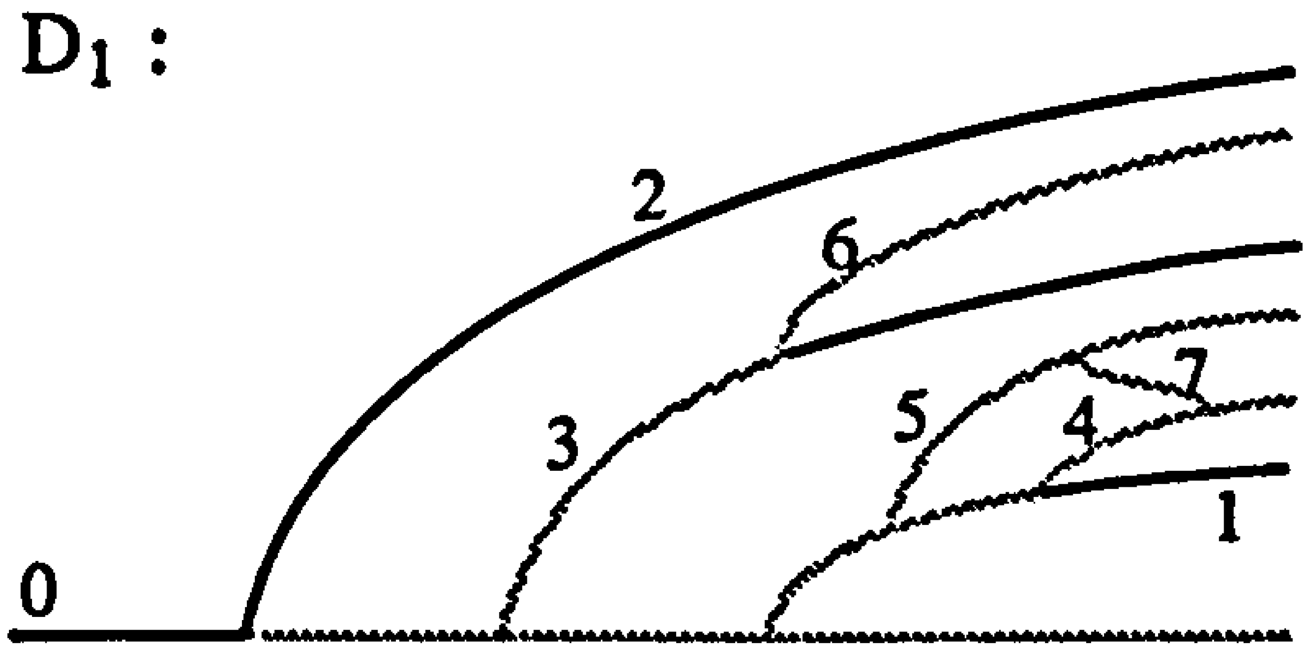


Region D

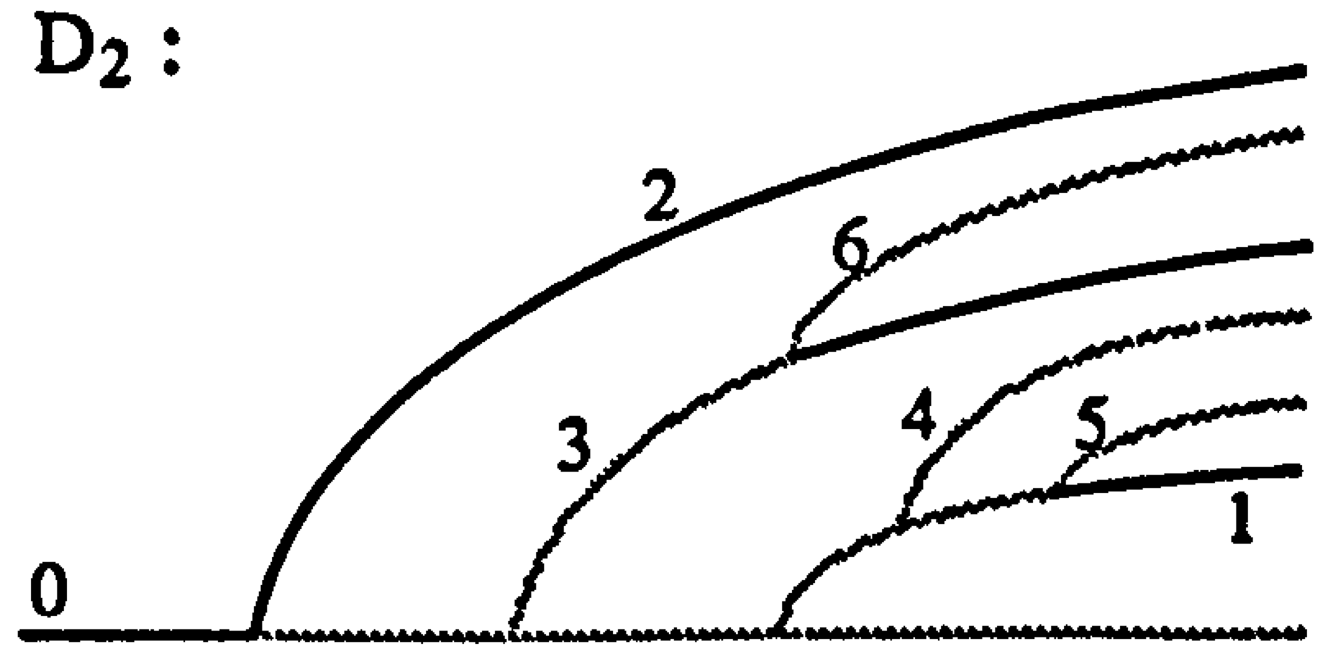
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



D_1 :

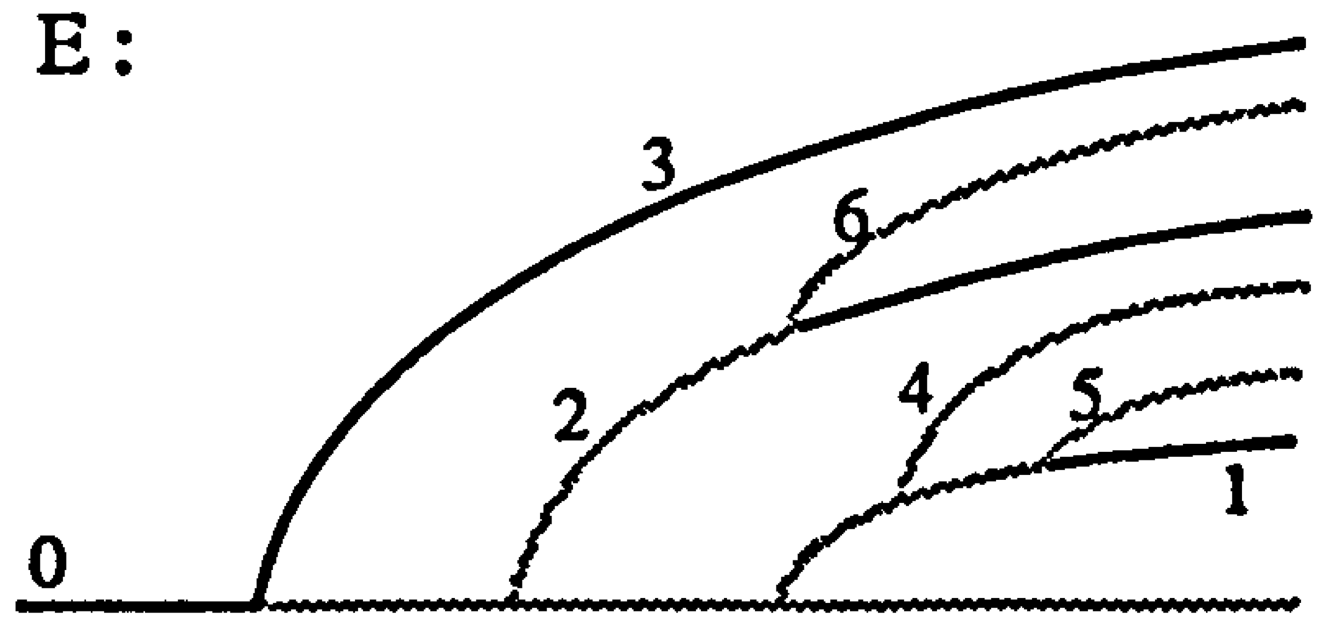


D_2 :



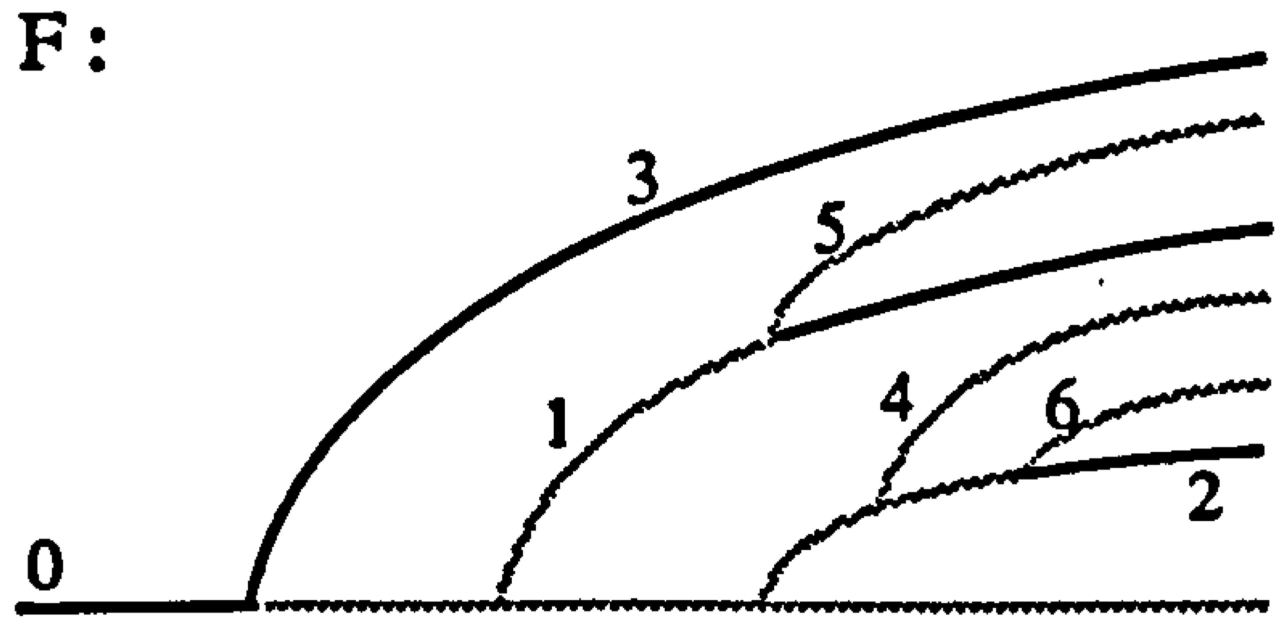
Region E

E :



Region F

F:



4.9 Interaction of Two Modes in a Box with Square Cross Section

Now we assume that $r = r_c$ is such that the two modes $\underline{k}, \underline{l} \in \mathbb{N}^3$ bifurcate simultaneously when the Rayleigh number is increased across a critical value $R = R_c$. By imposing the extra condition $r_1 = r_2$ we have that ker L is either 2, 3 or 4-dimensional depending on the mode numbers as

1. $V_k \oplus V_l$ if $k_1 = k_2$ and $l_1 = l_2$
2. $V_k \oplus V_l \oplus V_{sl}$ if $k_1 = k_2$ and $l_1 \neq l_2$
3. $V_k \oplus V_{sk} \oplus V_l \oplus V_{sl}$ if $k_1 \neq k_2$ and $l_1 \neq l_2$

and ker L^* is

1. $V_k^* \oplus V_l^*$ if $k_1 = k_2$ and $l_1 = l_2$
2. $V_k^* \oplus V_l^* \oplus V_{sl}^*$ if $k_1 = k_2$ and $l_1 \neq l_2$
3. $V_k^* \oplus V_{sk}^* \oplus V_l^* \oplus V_{sl}^*$ if $k_1 \neq k_2$ and $l_1 \neq l_2$

where V_k and V_k^* are defined in section 4.7.1 for any given set of mode numbers $\underline{k}$. For the particular case of the Bénard convection we did not find any mode interaction of type 1. We performed a Liapunov-Schmidt reduction for a mode interaction of type 2 which gives a 3-dimensional system of ODEs. For the particular modes chosen, a 3rd order truncation is enough to determine the bifurcation diagrams and the analysis described in section 4.8 applies directly. Further analysis of the points of type 3 is left for further work. The reason for not doing it here is that an expansion of the reduced equations up to 3rd order would not be enough to catch all the features of the bifurcation diagrams and we are not technicaly prepared to go up to higher order yet.

Now we consider a bifurcation of type 2 with mode numbers

$$\begin{aligned} \underline{k} &= (1,1,1) \\ \underline{l} &= (0,1,1). \end{aligned}$$

A calculation shows that these modes bifurcate simultaneously when the unfolding parameters are

$$r_{1c} = r_{2c} \approx 1.687$$

and the bifurcation parameter is

$$R_c \approx 7.024.$$

As in section 4.5.2, the bifurcation equations obtained by a Liapunov-Schmidt reduction heve $\mathbf{Z}_2 \oplus \mathbf{D}_4$-symmetry. Thus, up to 3rd order they are of the form

$$\begin{aligned}
f_1 &= [a_{N_1}N_1 + a_{N_2}N_2 + a_\lambda\lambda + a_{\rho_1+\rho_2}(\rho_1+\rho_2)]\,x \\
f_2 &= [b_{N_1}N_1 + b_{N_2}N_2 + c\delta + b_\lambda\lambda + b_{\rho_1+\rho_2}(\rho_1+\rho_2) + b_{\rho_1-\rho_2}(\rho_1-\rho_2)]\,y_1 \\
f_3 &= [b_{N_1}N_1 + b_{N_2}N_2 - c\delta + b_\lambda\lambda + b_{\rho_1+\rho_2}(\rho_1+\rho_2) - b_{\rho_1-\rho_2}(\rho_1-\rho_2)]\,y_2
\end{aligned}$$

where

$$N_1 = x^2 \qquad N_2 = y_1^2 + y_2^2 \qquad \delta = y_2^2 - y_1^2$$

and

$$\lambda = R - R_c \qquad \rho_1 = r_1 - r_{1c} \qquad \rho_2 = r_2 - r_{2c}.$$

The coefficients in the formulae for f are computed as in section 4.5.2 and the result for $\sigma = 7.03$ (Prandtl number for the water) is

$$\begin{aligned}
a_{N_1} &\approx 10.715 \\
a_{N_2} &\approx 35.738 \\
a_\lambda &\approx -59.780 \\
a_{\rho_1+\rho_2} &\approx -55.835 \\
b_{N_1} &\approx 46.241 \\
b_{N_2} &\approx 15.643 \\
b_\lambda &\approx -80.288 \\
b_{\rho_1+\rho_2} &\approx -96.732 \\
b_{\rho_1-\rho_2} &\approx 96.732 \\
c &\approx 5.6526
\end{aligned}$$

In order to compute the normal form and bifurcation diagrams we apply a $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivariant transformation

$$H(x, y_1, y_2, \lambda, \rho_1+\rho_2, \rho_1-\rho_2) = Sf(X, \Lambda, \rho_1+\rho_2, \rho_1-\rho_2) \tag{4.32}$$

where

$$
\begin{aligned}
&X(x,y_1,y_2,\lambda,\rho_1+\rho_2,\rho_1-\rho_2) = (Ax, By_1, By_2)\\
&\Lambda(\lambda) = \lambda - \frac{a_{\rho_1+\rho_2}}{a_\lambda}(\rho_1+\rho_2)\\
&S(x,y_1,y_2,\lambda,\rho_1+\rho_2,\rho_1-\rho_2) = \begin{pmatrix} C & 0 & 0\\ 0 & D & 0\\ 0 & 0 & D\end{pmatrix}
\end{aligned}
$$

and A, B, C, D are positive constants. By substituting X, Λ and S in (4.32) we get

$$
\begin{aligned}
H_1(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= [a_{N_1}A^3Cx^2 + a_{N_2}AB^2C(y_1^2+y_2^2) + a_\lambda AC\lambda]x\\
H_2(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= [b_{N_1}A^2BDx^2 + b_{N_2}B^3D(y_1^2+y_2^2) + cB^3D(y_2^2-y_1^2)\\
&\quad +b_\lambda BD\lambda + \tilde\rho_1 + \tilde\rho_2]y_1\\
H_3(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= [b_{N_1}A^2BDx^2 + b_{N_2}B^3D(y_1^2+y_2^2) - cB^3D(y_2^2-y_1^2)\\
&\quad +b_\lambda BD\lambda + \tilde\rho_1 - \tilde\rho_2]y_2.
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde\rho_1 &= BD\left(b_{\rho_1+\rho_2} - \frac{a_{\rho_1+\rho_2}b_\lambda}{a_\lambda}\right)(\rho_1+\rho_2)\\
\tilde\rho_2 &= BDb_{\rho_1-\rho_2}(\rho_1-\rho_2).
\end{aligned}
$$

By imposing the conditions

$$
\begin{aligned}
|a_{N_1}|A^3C &= 1\\
|a_\lambda|AC &= 1\\
|c|B^3D &= 1\\
|b_\lambda|BD &= 1
\end{aligned}
$$

we get H in the desired form

$$
\begin{aligned}
H_1(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= \left[\epsilon_1x^2 + n_1(y_1^2+y_2^2) + \epsilon_2\lambda\right]x\\
H_2(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= \left[n_2x^2 + n_3(y_1^2+y_2^2) + \epsilon_3(y_2^2-y_1^2) + \epsilon_4\lambda + \tilde\rho_1 + \tilde\rho_2\right]y_1\\
H_3(x,y_1,y_2,\lambda,\tilde\rho_1,\tilde\rho_2) &= \left[n_2x^2 + n_3(y_1^2+y_2^2) - \epsilon_3(y_2^2-y_1^2) + \epsilon_4\lambda + \tilde\rho_1 - \tilde\rho_2\right]y_2
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon_1 &= \mathrm{sgn}(a_{N_1})\\
\epsilon_2 &= \mathrm{sgn}(a_\lambda)\\
\epsilon_3 &= \mathrm{sgn}(c)\\
\epsilon_4 &= \mathrm{sgn}(b_\lambda)
\end{aligned}
$$

and

$$\begin{aligned} n_1 &= \left|\frac{b_\lambda}{ca_\lambda}\right| a_{N_2} \\ n_2 &= \left|\frac{a_\lambda}{a_{N_1} b_\lambda}\right| b_{N_1} \\ n_3 &= \left|\frac{1}{c}\right| b_{N_2}. \end{aligned}$$

The unfolding parameters are

$$\begin{aligned} \tilde{\rho}_1 &= \frac{1}{|b_\lambda|}\left(b_{\rho_1+\rho_2} - \frac{a_{\rho_1+\rho_2} b_\lambda}{a_\lambda}\right)(\rho_1+\rho_2) \\ \tilde{\rho}_2 &= \frac{1}{|b_\lambda|} b_{\rho_1-\rho_2}(\rho_1-\rho_2). \end{aligned}$$

By substituting the derivatives of a, b, c given above in the formulae for ϵ_j and n_j we get

$$\begin{aligned} \epsilon_1 = \epsilon_3 &= 1 \\ \epsilon_2 = \epsilon_4 &= -1 \end{aligned}$$

and

$$\begin{aligned} n_1 &\approx 8.491 \\ n_2 &\approx 3.213 \\ n_3 &\approx 2.767. \end{aligned}$$

Now the unfolding parameters are

$$\begin{aligned} \tilde{\rho}_1 &\approx 0.663(\rho_1+\rho_2) \\ \tilde{\rho}_2 &\approx 1.205(\rho_1-\rho_2). \end{aligned}$$

A complete stability analysis of steady solutions of the system

$$\begin{aligned} \dot{x} &= H_1(x, y_1, y_2, \lambda, \tilde{\rho}_1, \tilde{\rho}_2) \\ \dot{y}_1 &= H_2(x, y_1, y_2, \lambda, \tilde{\rho}_1, \tilde{\rho}_2) \\ \dot{y}_2 &= H_3(x, y_1, y_2, \lambda, \tilde{\rho}_1, \tilde{\rho}_2) \end{aligned}$$

is performed in appendix B. The results are condensed into tables giving the branches of solutions and their criticality with respect to the bifurcation parameter λ if the system is nondegenerate. Our particular mapping H satisfies the nondegeneracy conditions since we have

$$\begin{aligned} (a) &\approx -2.213 \\ (b) &\approx 2.767 \\ (c) &\approx 1.767 \\ (d) &\approx -5.724 \\ (e) &\approx -6.724 \\ (f) &\approx -24.52 \\ (g) &\approx -26.28 \end{aligned}$$

and they are all nonzero. We proceed by drawing the bifurcation diagrams for the case when the mapping H is $\mathbf{Z}_2 \oplus \mathbf{D}_4$-equivariant. This happens when $\tilde{\rho}_2 = 0$. Then we show the effects of breaking the $\mathbf{S}_2$ symmetry with the parameter $\tilde{\rho}_2$. Before drawing the bifurcation diagrams we make a convention about the labels of possible branches.

(0) Translation-invariant solution $u = 0$.

(1) Pure mode (1,1,1).

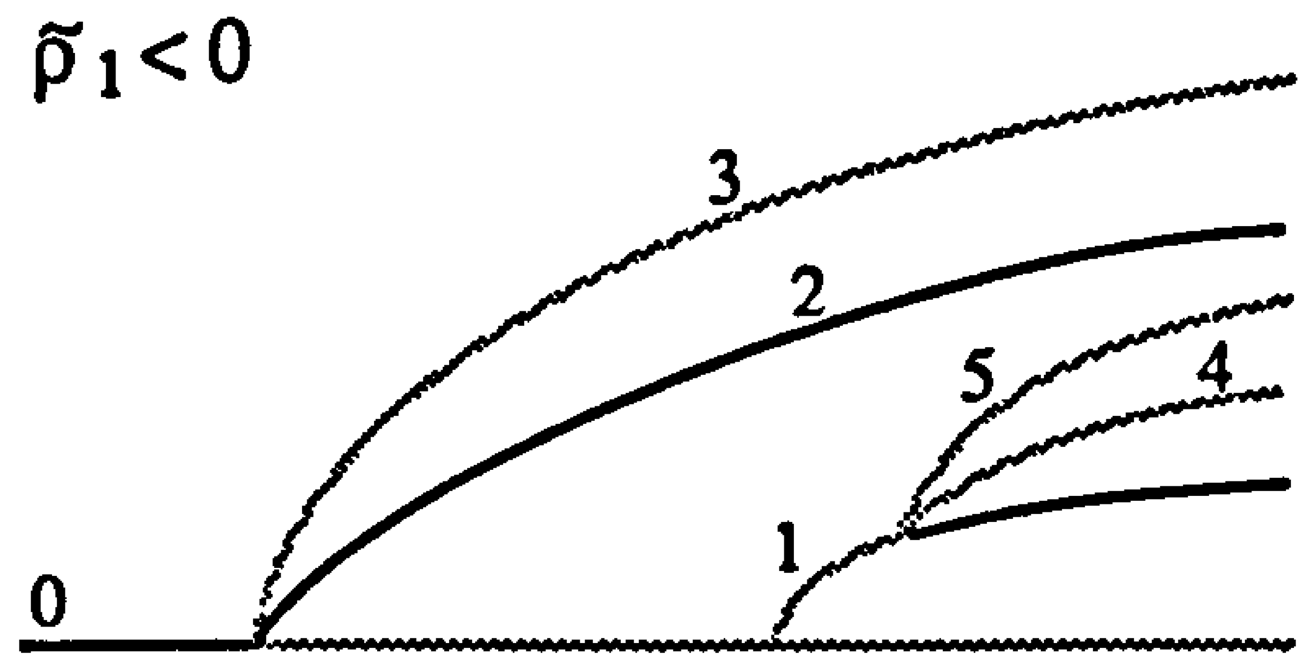
(2) Pure modes (0,1,1) and (1,0,1).

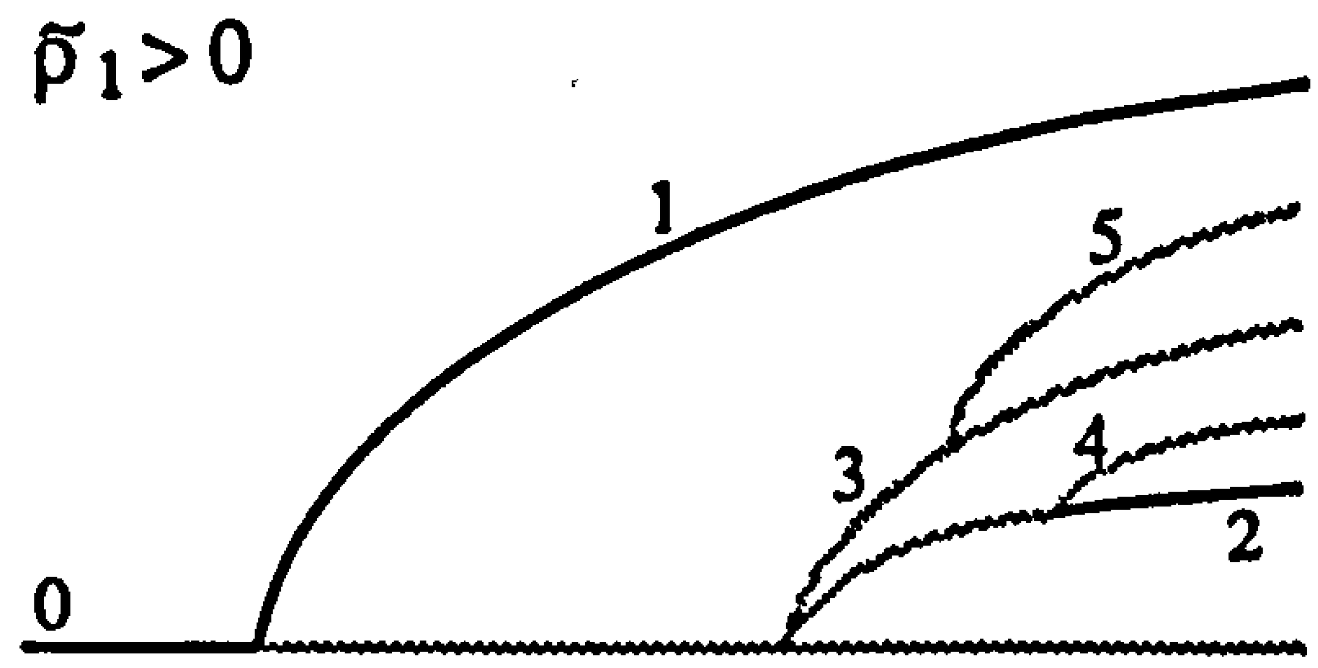
(3) Mixed mode (0,1,1)-(1,0,1).

(4) Mixed modes (1,1,1)-(0,1,1) and (1,1,1)-(1,0,1).

(5) Mixed mode (1,1,1)-(0,1,1)-(1,0,1).

Now the bifurcation diagrams depend on the sign of $\tilde{\rho}_1$ as





We proceed by breaking the $\mathbf{S}_2$ symmetry with the parameter $\tilde{\rho}_2$. Now the modes (0,1,1) and (1,0,1) are no longer conjugate. In order to make the bifurcation diagrams clear we need new labels

(2.1) Pure mode (0,1,1).

(2.2) Pure mode (1,0,1).

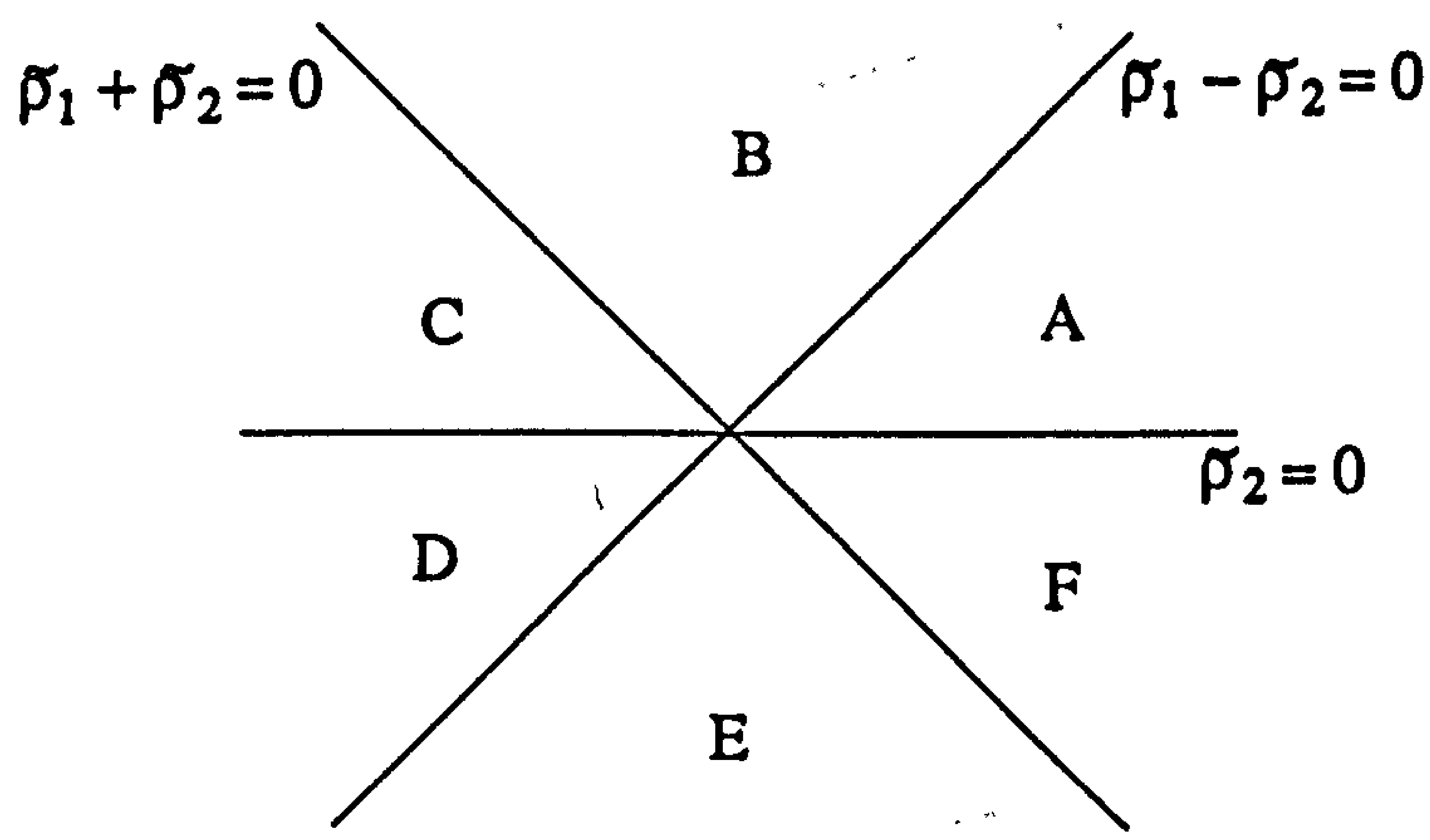
(4.1) Mixed mode (1,1,1)-(0,1,1).

(4.2) Mixed mode (1,1,1)-(1,0,1).

(6) Mixed mode (1,0,1)-(0,1,1).

(7) Mixed mode (1,1,1)-(0,1,1)-(1,0,1).

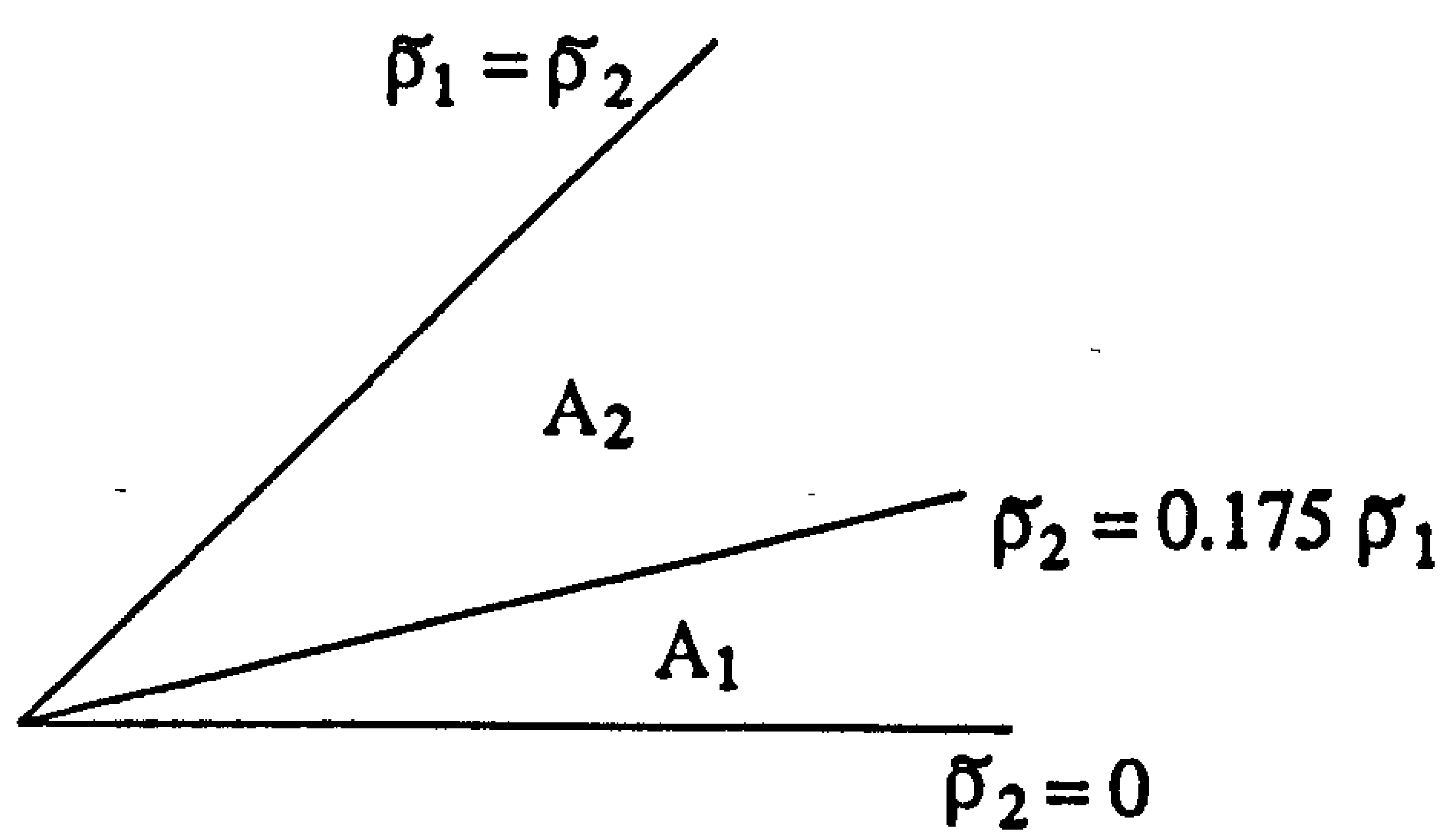
The unfolding parameter space is divided into six distinct regions according to the order of primary bifurcations when the bifurcation parameter is increased. This division is the following



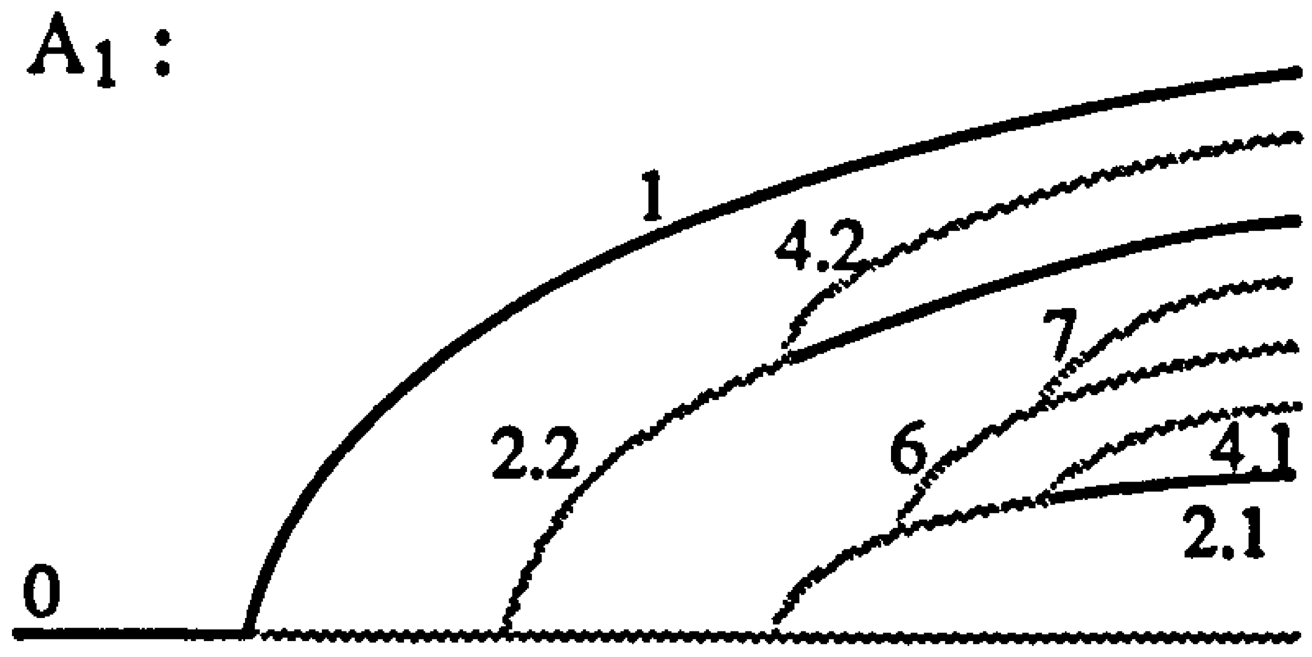
Then some of these regions need a subdivision according to the order of secondary bifurcations. This will be illustrated when appropriate and we proceed by drawing the bifurcation diagrams.

Region A

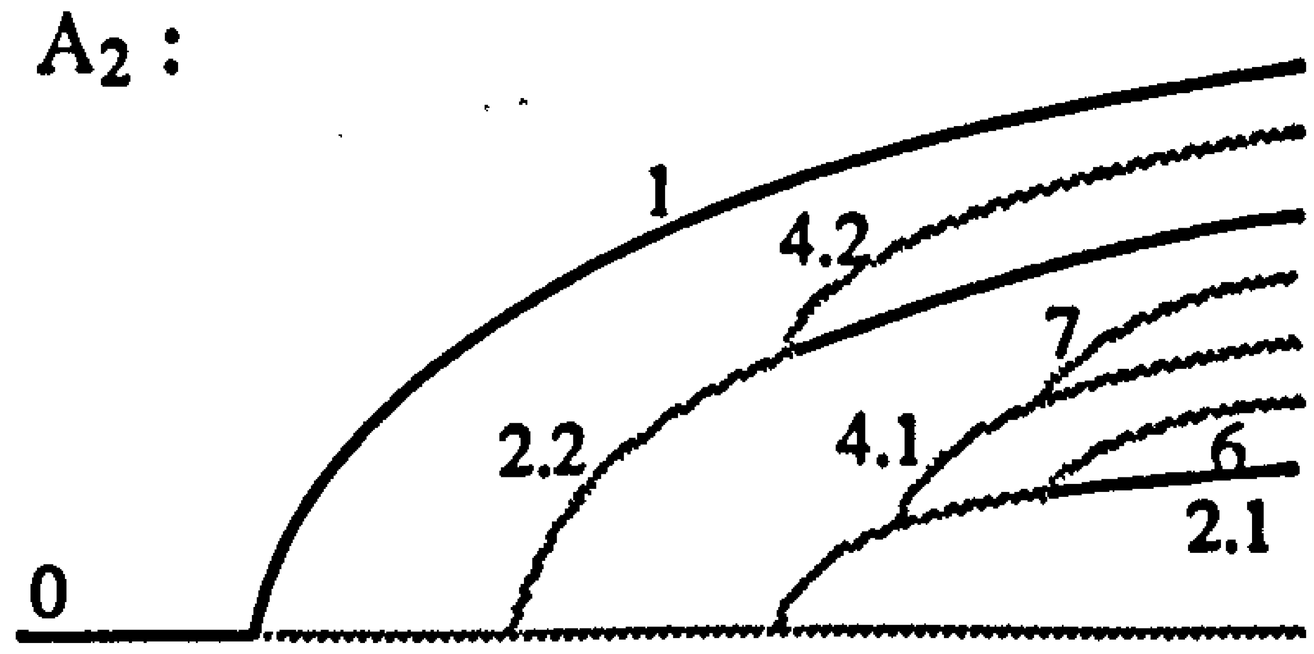
This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.



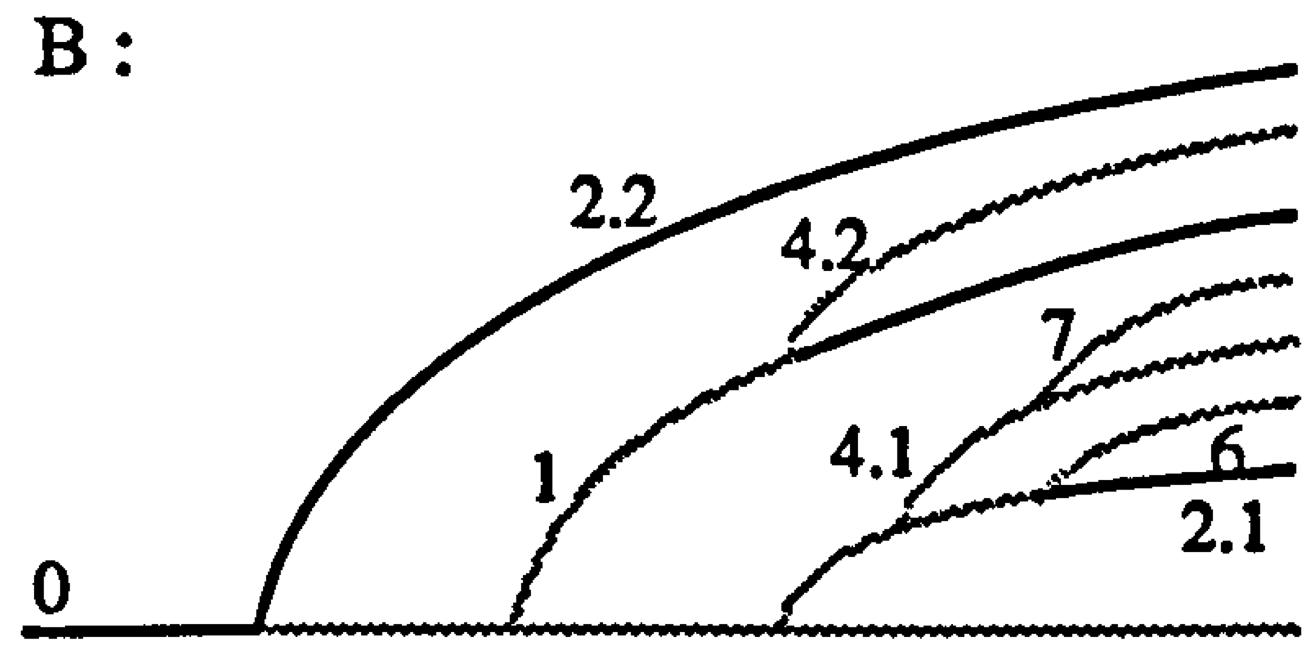
A_1 :



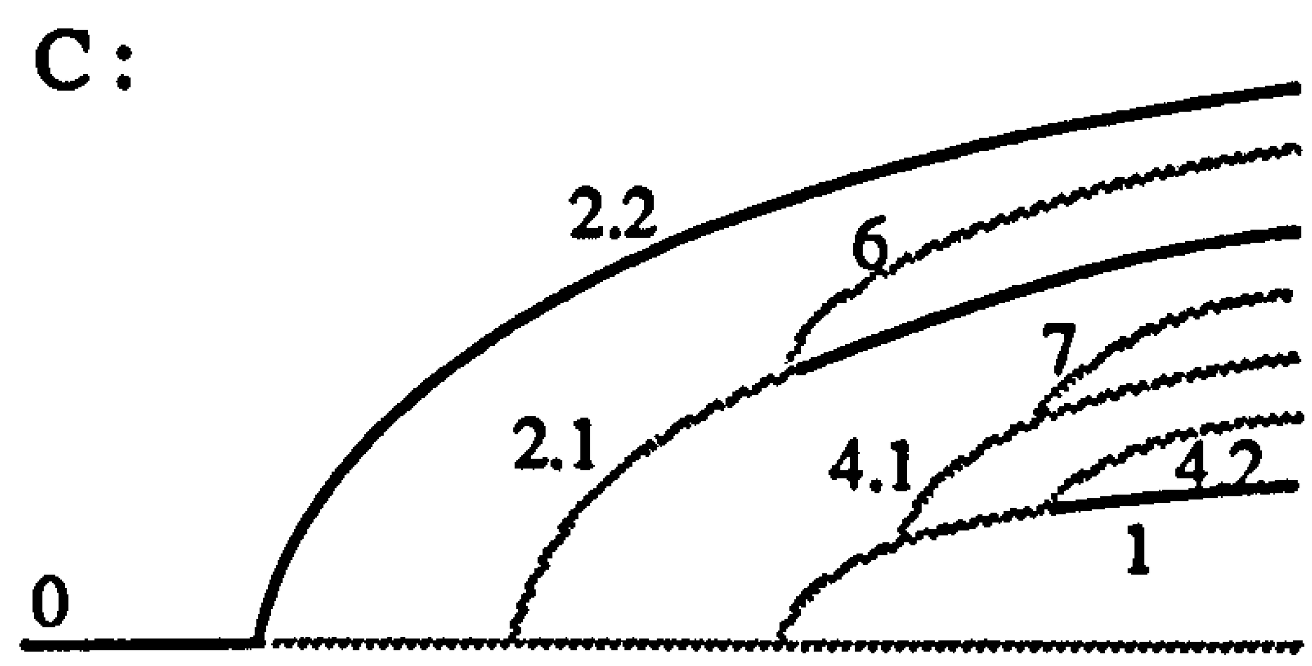
A_2 :



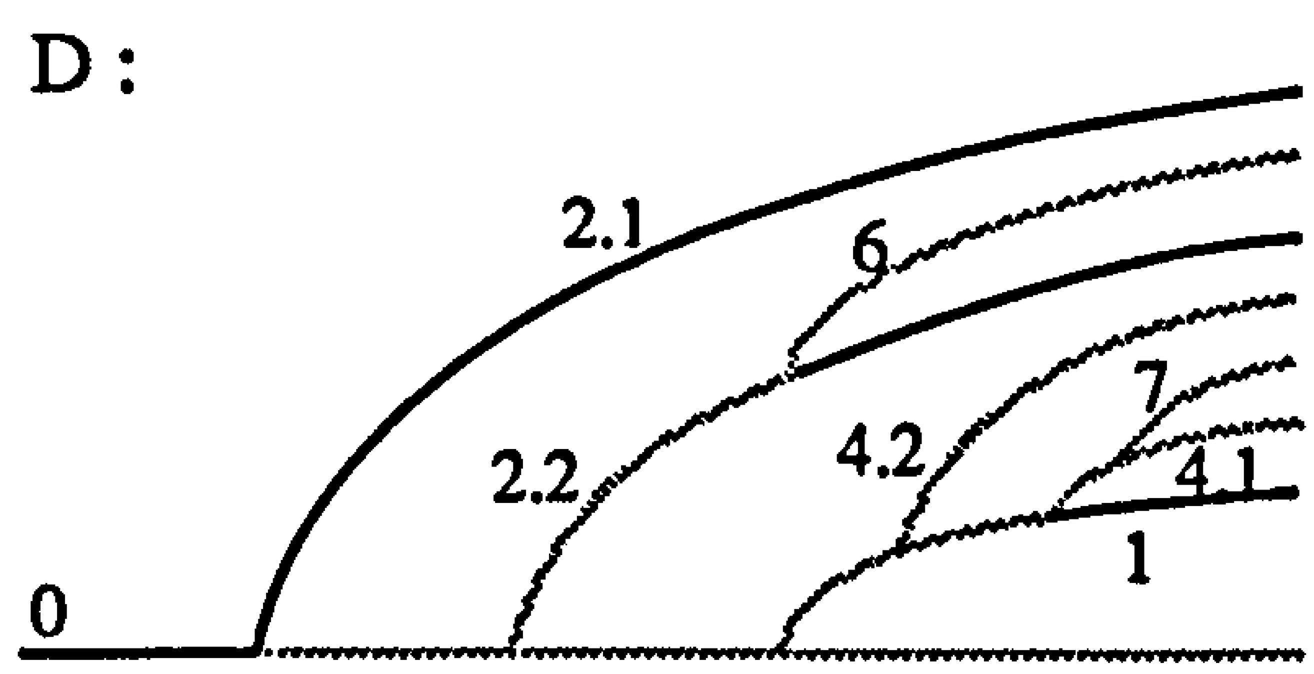
Region B



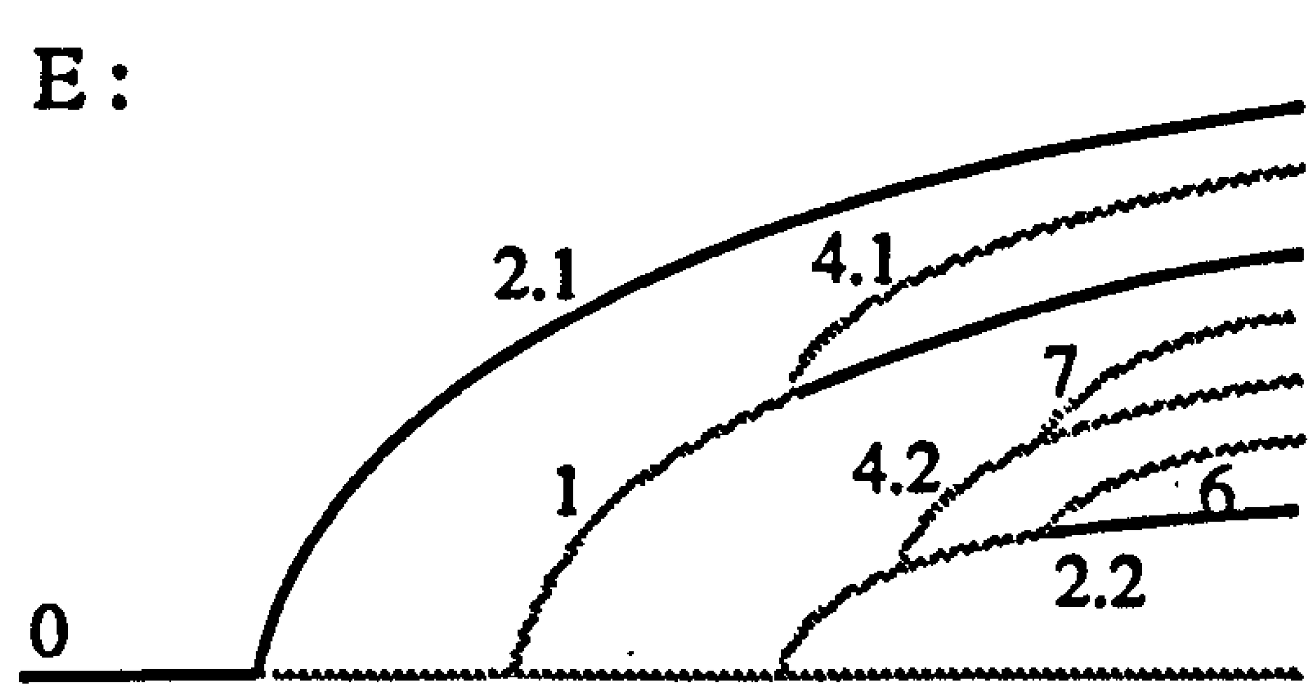
Region C



Region D

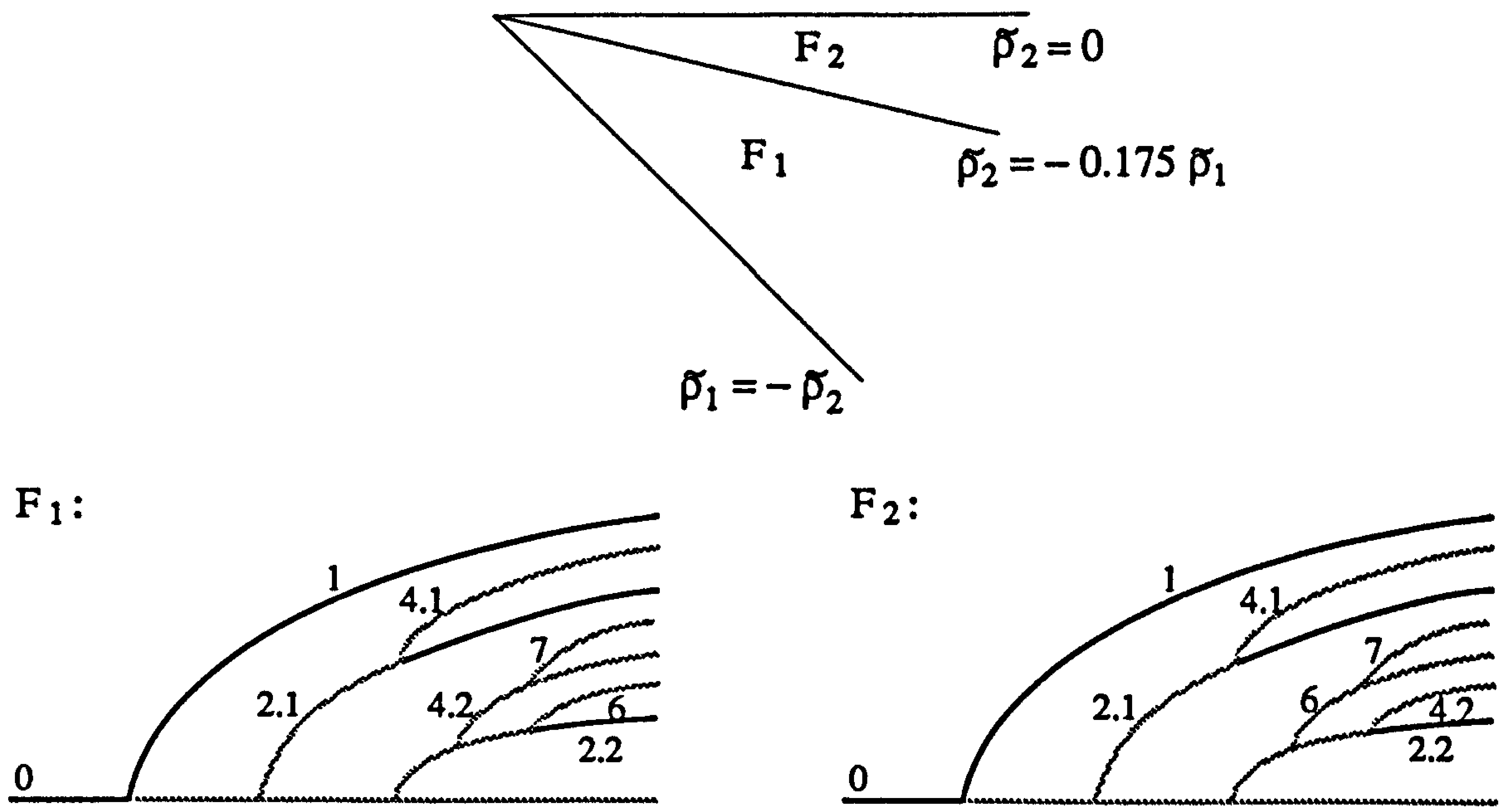


Region E



Region F

This region is divided into two subregions according to the order of bifurcations from the primary branches. We show this division and then proceed by drawing the bifurcation diagram corresponding to each subregion.


F_2
$\tilde{\beta}_2 = 0$
F_1
$\tilde{\beta}_2 = -0.175\,\tilde{\beta}_1$
$\tilde{\beta}_1 = -\tilde{\beta}_2$
F_1:
1
4.1
7
2.1
4.2
6
2.2
0
F_2:
1
4.1
7
2.1
6
4.2
2.2
0