# ADOLESCENTS' UNDERSTANDING OF

# LIMITS AND INFINITY

by

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# CONTENTS

## LIST OF TABLES

The first table in Chapter X is labelled TABLE X.1, the second is labelled TABLE X.2, etc. Exceptions to this are the tables, in Chapter Six, detailing responses to a particular question. Here the table for question Y is labelled TABLE QY.

"Finite to fail, infinite to venture." Emily Dickinson


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Finally I would like to thank Jean. Your support was essential. I will be spending a lot more time with you now.

# DECLARATION

I declare that the following thesis is entirely my own work. No part of this work has been published or presented for examination before now. The term *generic limit* was coined in conversation with Dr. D.O. Tall.

## SUMMARY

**AIM** To investigate mathematically able adolescents' conceptions of the basic notions behind the Calculus: infinity (including the infinitely large, the infinitely small and infinite aggregates); limits (of sequences, series and functions); and real numbers. To observe the effect, if any, on these conceptions, of a one year calculus course.

**EXPERIMENTS** Pilot interviews and questionnaires helped identify areas on which to focus the study. A questionnaire was administered to Lower Sixth Form students with O-level mathematics passes. The questionnaire was administered twice, once in September and again the following May. The A-level mathematicians had received instruction in most of the techniques of the Calculus by May.

Interviews, to clarify ambiguities, elicit reasoning behind the responses and probe typicality and atypicality, were conducted in the month following each administration.

A second questionnaire, an amended version of the first, was administered to a larger but similar audience. The responses were analysed in the light of hypotheses formulated in the analysis of data from the first sample.

**PRINCIPAL FINDINGS** Subjects have a concept of infinity. It exists mainly as a process, anything that *goes on and on*. It may exist as an object, as a large number or the cardinality of a set, but in these forms it is a vague and indeterminate form. The concept of infinity is inherently contradictory and labile.

Recurring decimals are perceived as dynamic, not static, entities and are not *proper* numbers. Similar attitudes exist towards infinitesimals when they are seen to exist. Subjects' conception of the continuum do not conform to classical or nonstandard paradigms.

Convergence / divergence properties are generally noted with infinite sequences and functions. With infinite series, however, convergence / divergence properties, when observed, are seen as secondary to the fact that any infinite series goes on indefinitely and is thus similar to any other infinite series.

The concept that *the limit is the same type of entitiy as the finite terms* is strong in subjects' thoughts. We coin the term *generic limit* for this phenomenon. The generic limit of 0.9, 0.99, ... is $0.\dot{9}$, not 1. Similarly the reasoning scheme that *whatever holds for the finite holds for the infinite* has widespread application. We coin the term *generic law* for this scheme.

Many of the phrases used in calculus courses (in particular *limit, tends to, approaches* and *converges*) have everyday meanings that conflict with their mathematical definitions.

Numeric/geometric, counting/measuring and static/dynamic contextual influences were observed in some areas.

The first year of a calculus course has a negligible effect on students conceptions of limits, infinity and real numbers.

**IMPLICATIONS FOR TEACHING** On introducing limits teachers should encourage full class discussion to ensure that potential cognitive obstacles are brought out into the open. Teachers should take great care that their use of language is understood. A-level courses should devote more of their time to studying the continuum. Nonstandard analysis is an unsuitable tool for introducing elementary calculus.

# CHAPTER ONE

# INTRODUCTION

In this chapter we present the aims and methods of this research. We then outline the content of subsequent chapters and state the main theses of the study.

Calculus is, for the majority of pupils at 16+, the beginning of higher mathematics. Extreme difficulties are often faced and often never overcome. This is not unnatural. What has taken the greatest mathematical minds centuries to perfect is unlikely to be taught without problems. The task for the teacher is to make the learning of calculus as problem free as possible. We take it as unnecessary for research to state that this will not consist of teachers proving theorems learnt in analysis courses at university.

Behind calculus are the concepts of a limit, of infinity and of real numbers. Although many students experience great difficulty with the algebraic manipulation involved in a first calculus course it is with these concepts that the real cognitive difficulties lie. They embody mathematics of a new type - no longer are finite deductions and equations sufficient.

Our work is to make clear the problems that students have with these concepts. To do this we do not examine students' mastery of the details taught in a first course but look behind the course at the intuitive ideas students have and how these are affected by a taught course.

## METHODOLOGY

Pilot studies and a review of relevant literature determined the concepts eventually examined. The period of the pilot studies was one of continuously formulating and testing hypotheses.

After the pilot studies our experiments partook of features of both cross-sectional and longitudinal methods. Control and experimental groups of Lower Sixth Form students with O-level mathematics passes completed a questionnaire. The control group were not doing A-level mathematics, the experimental group were doing A-level mathematics. The questionnaire was administered twice, once in September and again the following May. The A-level mathematicians had received instruction in most of the techniques of the Calculus by May.

Interviews, to clarify ambiguities, elicit reasoning behind the responses and probe typicality and atypicality, were conducted in the month following each administration of the questionnaire.

The data was then analysed and hypotheses formulated. In the June of the next year a second questionnaire, an amended version of the first, was administered to a larger but similar audience. The responses were analysed in the light of hypotheses formulated from the earlier data.

## DESCRIPTION OF THE CHAPTERS

Chapter Two sketches a history of the infinite in mathematics and philosophy, and a history of the calculus. The purpose of this is to provide mathematical, philosophical and pedagagic touchstones that can be referred to in subsequent chapters.

Chapter Three reviews the relevant cognitive research in this area that has been available to the author. Considering the extensive research that has gone on in many areas of mathematics education, research in this area is surprisingly scant.

Chapter Four describes the two main pilot studies in some detail (the first a series of interviews, the second a questionnaire). Both were important in the maturation of the theses arrived at and are, thus, integral to the complete study. Another two early experiments are described in less detail.

Chapter Five details the reason for the inclusion of items in the questionnaires and the wording, method of presentation and the samples used.

Chapter Six presents and analyses the results of the questionnaires. Conclusions are not attempted at this point but many of the theses are evaluated.

Chapter Seven details the purpose of the interviews, the expected behaviours to be examined, the method of interviewing and the rationale for the selection of subjects for interviews.

Chapter Eight takes up many of the points of Chapter Seven but this time from the point of view of individual subjects instead of overall group response.

Chapter Nine takes up the theses presented in Chapter One, this time evaluating them in terms of the evidence obtained from the questionnaires and interviews.

Chapter Ten outlines the major achievements of the study, considers the implications for teaching and suggests areas where further research would be useful.

## THESES

We present the main theses of this study in their barest outline (as *advance organisers*). We shall return to them in Chapter Nine and examine the evidence for and against them in the light of the complete study. The word *subject* clearly means someone who has taken part in the data collection. Our theses pertain to our subjects who are mathematically capable (i.e. have passed O-level mathematics) and are taking A-level courses in British schools. We believe that our findings relate to a much wider population but, by the nature of our samples, we cannot confirm this.

We use the terms *generic limit* and *generic law* in the remainder of this work. The generic law is the principle that *what holds for finite cases also holds for infinite cases*. By the generic law the limit of a convergent series of continuous functions is continuous. A less esoteric example exists in the case of $0.\dot{9}$ and 1. $0.\dot{9}$ is the limit of 0.9, 0.99, ... Each term is less than 1, thus $0.\dot{9}$ is less than 1. This example illustrates the idea of a generic limit. The generic limit of the above sequence is $0.\dot{9}$, not 1. $0.\dot{9}$ is the infinite term of the sequence but is qualitatively similar to the finite terms (it is made up of nines) whereas 1 is qualitatively different. In forming these terms we are not claiming that subjects consciously hold them, understand the word *limit* or that the two are inseparable (although the generic law is used to establish the generic limit the generic law can also be used in non generic limit contexts, e.g. in determining that the number of decimal numbers between 0 and 10 is larger than the number between 0 and 1).

We present the theses under eleven headings. This is done to simplify the communication and discussion of the theses. The theses themselves are interrelated and the ideas embodied in them should not be seen as peculiar to a single heading.

1) **SUBJECTS HAVE A CONCEPT OF INFINITY.**

i) This is manifested by subjects' cognizance of nonterminating processes (infinite subdivision of a line, infinite sequences and series, and, in general, infinite continuation of an operation.

ii) This is further manifested by subjects' cognizance of collections containing more than any given finite number of elements.

2) **INFINITY AS A PROCESS AND AS AN OBJECT.**

i) Infinity exists as a process, and as an object.

ii) Infinity means *going on and on* and, as such, is used as an evaluatory scheme for judging whether a question determines an infinite answer.

iii) Subjects reveal an understanding of infinity as an object in that they display a cognizance of a number at the end of the number line and of the cardinality of infinite sets.

3) **INFINITY AS A NUMBER**

i) Infinity as a number is an indeterminant form, a generalization of a large number.

ii) Although there is general recognition of infinity as the largest number cognitive belief in the existence of this number is low.

iii) Infinite numbers need not be numerically large. Recurring decimals and infinitesimals may also be granted the title *infinite numbers* because they go on and on.

iv) Subjects' concepts of infinity do not conform to infinite cardinal or ordinal paradigms.

## 4) INFINITESIMALS

i) Infinitesimals are not generally accepted but may be seen as useful fictions. When they are accepted they are seen as dynamic entities that exist in the process of a sequence of numbers, or a function, decreasing. Static infinitesimals do not conform to subjects' conceptions. A cognitive framework ripe for the introduction of the concepts of nonstandard analysis does not exist amongst subjects.

ii) Willingness to accept approximations is strong with small numbers.

## 5) INFINITE SEQUENCES AND SERIES.

i) Basic convergence/divergence properties of infinite sequences are generally noted though subjects often focus on mathematically unimportant features such as oscillations in evaluating convergence.

ii) The generic limit concept is dominant in subjects' conceptions of the limit of an infinite sequence. There is a small shift to the mathematicians' limit concept amongst A-level mathematicians.

iii) The convergence or divergence of an infinite series is not generally seen as its most important feature. Theoretical, physical and temporal problems of any infinite summation often override them as important features.

## 6) REAL NUMBERS

i) Subjects' ontological framework includes infinite recurring decimals but they are interpreted in a dynamic context and seen as qualitatively different from finite decimals. This leads to an inconsistent model and, ultimately, to cognitive conflict.

ii) Subjects' concepts of the continuum do not correspond to mature mathematicians' models of the continuum.

7) LANGUAGE

i) Phrases such as *gets to* and *goes on forever* suggest impossible situations.

ii) The phrases *tends to, approaches, converges* and *limit* have every-day connotations that affect subjects' mathematical interpretations.

8) REASONING

Reasoning schemes peculiar to problems dealing with limits and infinity are *infinity as a process* and *the generic law.* Both schemes have widespread application. Subjects may switch from one scheme to the other in response to similar questions.

10) CONTEXTS

Subjects' responses are affected by the context of a question. There are three notable divisions

i) Numeric and geometric. Subjects' sense of the existence of a limit of a convergent function, presented graphically, is stronger than their sense of the existence of a limit of a convergent numeric sequence. Also, generic limit ideas appear less pronounced in geometric contexts.

ii) Counting and measuring. A measuring context encourages subjects to ascribe a greater cardinality to the superset in cardinality questions.

iii) Static and dynamic. A dynamic interpretation of recurring decimals leads subjects to a view of the continuum which is often at odds with the static real complete continuum of higher mathematics. A dynamic interpretation of series often leads subjects to overlook the convergence and divergence of series and see them as similar because they both go on and on. Such interpretations also lead to physical and temporal factors affecting subjects considerations of series.

**11) SUBJECTS' CONCEPTIONS OF LIMITS AND INFINITY ARE CONTRADICTORY AND LABILE.**

i) Subjects' conceptions of limits and infinity are contradictory in that subjects are drawn to two opposing views, e.g. infinity is the largest number but you can't have a largest number, the limit of a sequence is the final number in the list but there is no final number, there are more natural than even numbers but there are the same (infinite) number of each.

ii) Subjects' responses are often not stated with great confidence and may be easily changed by context, reasoning and suggestion.

**11) THE EFFECT OF TEACHING.**

The first year of an A-level mathematics course which includes an introduction to all the basic ideas of calculus does not, generally, affect subjects' conceptions of limits, infinity or real numbers.

# CHAPTER TWO

## A BRIEF HISTORY OF THE INFINITE IN MATHEMATICS AND PHILOSOPHY AND OF THE CALCULUS

The substance of this section is worthy of many volumes. Our intention is, however, merely to establish mathematical, philosophical and pedagogic touchstones that can be referred to in the main body of the work. We thus limit the discussion to what we feel is relevant to adolescents' conception of infinity, limits and real numbers.

The infinite is a particularly interesting concept in which to look at the history of mathematics since, over the millennia, the three crises and debates in mathematics that assume particular importance have all been concerned with the infinite. They are Zeno's paradoxes, the introduction of infinitesimals, and the debate over the foundations of mathematics at the turn of the century.

The purpose of Zeno's arguments remain a matter of controversy. The effect of the paradoxes was to prevent Greek mathematical thought dealing with motion. The most famous example of this occurs with Archimedes who anticipated the early methods, and many of the early results, of infinitesimal calculus (though without a 17th century view of number) yet deliberately recast his proofs in a static form. Mathematicians today prefer static (arithmetized) forms of proof in calculus despite the fact that, as we shall see, students' view the limit concept in a dynamic (motion orientated) context. Historically the dynamic potential infinite has, especially since Arisitotle (who argued that it was the correct interpretation), vied with the actual infinite for the philosophical and mathematical correct interpretation. Only really in this century has the actual infinite won the debate.

The main forerunners to the calculus used methods that logically required the calculation of that most curious of ratios, 0/0. Interestingly both Leibniz and Newton used infinitesimals to overcome this problem (with different descriptions and notations, however) in their formulations of the calculus. The problems inherent in this approach were clear to contemporary mathematicians. Berkeley's famous rejection of infinitesimals as *ghosts of departed quantities* was justified at the time, even if the odd theory of sense perception that accompanied it was not. As Rotman has said, (1980, Chapter 4, p.9):

> In a sense, infinitesimals were Zeno's revenge, the price Renaissance mathematics paid for studying motion.

Both Newton and Leibniz vacillated in their interpretation of infinitesimals. Neither appears to have philosophical priority over the other, though Leibniz was a more public proponent. He formulated the law of continuity that:

> In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the terminus may also be included. (see Keisler, 1976, p.873)

In other words *what holds for infinitesimals holds with real number arguments.* This law is too vague for mathematics but it can be made precise and be shown to be true in nonstandard analysis. We must, however, posit the existence of infinitesimals. We must enlarge our ontology to include numbers with their properties. This is a problem for philosophers and mathematics teachers for not only are classical existence arguments required but also an examination of what Tall (1980c), calls *cognitive existence*, that is the extent to which a subject can believe in the reality of a posited entity. Tall (ibid.) found that University mathematics students warmed, with familiarity, to systems that included infinitesimals. The present work found, with some qualifications, that Sixth Form pupils did not accept infinitesimals.

Leibniz' disciple L'Hospital (1696) presented a partial axiomatization of the system;

1) We call *variables* those quantities that continually increase or decrease.

2) The infinitely small amount by which a variable continually increases or decreases is called its difference.

3) Any two quantities may be replaced by one another if they differ from each other by no more than an infinitely small amount.

A problem Leibniz did not correctly solve was how to deal with higher order infinitesimals. Cauchy formulated an acceptable account. Slightly amended, his account survived to this century:

If the limit of $v/u^m$ be finite and not zero, v is said to be an *infinitesimal of the mth order*, the standard being u. ( Lamb, 1897, p.61)

Nevertheless, the problems associated with infinitesimals eventually resulted in their rejection. Several unsuccessful attempts were made in the 18th century to put the calculus on a firm foundation (D'Alembert with limits and Lagrange with Taylor series expansion) but the first step to success only really came with Cauchy.

We must be careful with Cauchy's formulations. Boyer remarks that:

> With Cauchy, it may safely be said, the fundamental concepts of the calculus received a rigorous formulation. (1949, p.282)

Lakatos (1978), however, has shown that Cauchy did not possess the modern notion of the continuum (he accepted the post-Leibnizian ideas of an extended number system including infinitesimals). Cauchy placed limits at the heart of analysis but used infinitesimals to define limits:

> A variable is a quantity which is thought to receive successively different values...when the successive numerical values of a variable decrease indefinitely so as to become smaller than any given number, this variable becomes what is called an infinitesimal...when the successive values attributed to a variable approach, indefinitely, a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others. (quotes found in Robinson, 1966 and Rotman, 1980)

The last act in the establishment of a rigorous foundation for analysis was to establish a non-geometric definition of number, as Dedekind, Cantor and others did, and more or less reframe Cauchy's results in this context, as Weierstrass did:

> If Cauchy's ideas of a limit got rid of the ostensive reference to motion in the Newton-Leibniz formulation of Calculus, Dedekind wished to remove any ostention from mathematics' idea of the

continuum. (Rotman, 1980, Chapter 4, p.11)

It is important for mathematics teachers to realize that their sophisticated Weierstrassian concepts are at odds with dynamic continuum concepts. The latter, as we shall see, are often the pictures of the continuum held by pupils. As Tall (1975, p.3) points out, teachers so trained find they must reject the concepts of analysis when teaching calculus and, thus, may pass on consistency fears to their pupils. If a remedy for this situation can be found then it will be of benefit to millions. We must as teachers, get over the optimum understanding of the continuum relative to students' present and future learning needs to make the learning of the calculus a meaningful experience.

Another revolution occurred with Cantor's *discovery* or, some would prefer, *invention* of transfinite numbers. The paradise, as Hilbert called it, created by Cantor has become the establishment theory (accepted by all except a minority of constructivist mathematicians). For 80 years, until Robinson's formulation of nonstandard analysis in 1960, they became the paradigm of infinite numbers; so much so that infinitesimals could be rejected, not just in use but in also in theory, because they could not be obtained from dividing ordinal or cardinal numbers; so much so that considering nonstandard models in the Sixth Form or undergraduate class is considered unnecessarily radical.

With Cantor and the acceptance of the actual infinite came the third crisis in maths, that of its foundations. Put simply it is the problem of resolving the contradiction involved in calling anything of the form <x:F(x)>, a set. The accepted resolutions have been axiomatizations. These have left the mathematical world with many of its most important results relative to this or that axiom. An alternative is to accept a constructivist solution. The quotation below, our interviews would suggest, is much more likely to be accepted by students than arguments expounding the Axiom of Choice. This is, of course, not in itself an argument for constructivism in mathematics but should make us, as teachers, open to alternatives to a Cantorian universe:

> Generation of terms in accordance with a rule yields terms endlessly; it does not yield an endless extension. ... Laws of construction indefinitely can be included in finitist mathematics, since there is no need to interpret them as laws for the construction of an endless whole. Thus 'the class of integers' and 'the expansion of pi' will be unobjectionable if one takes them to refer to a law for constructing them indefinitely. (Ambrose, 1980, p.65)

## NONSTANDARD ANALYSIS

Remaining with classical set theory and logic one can construct a continuum very like that of Leibniz, with noncardinal infinite *numbers* and their reciprocals, infinitesimals. How is this done ?

One of the first results in Model Theory (the work of Skolem in 1933) was:

> If a first order theory of arithmetic (with identity) has its intended model, then it also has a model with the usual interpretation of identity that is not isomorphic to its intended model.

Such a model is called a *nonstandard model*. One can, in a variety of ways, construct a nonstandard model of the real number system which includes infinite and infinitesimal elements. Robinson first did this using a complicated type theory. Easier ultraproduct formulations have since been presented (see Luxemburg, 1973 and Stroyan, 1976 ) but these are still over technical to explain here. More relevant to the present work is the fact that several approaches have been presented that do not rely on hard theorems from mathematical logic and which could be presented to school students or undergraduates (see Henle & Kleinberg 1979, Keisler 1976a and 1976b, Tall 1980a and 1981). We turn briefly to these now, noting that they open up the possibility of using infinitesimals without fear and would seem to be approaches worth investigating. This idea was the main impetus to the present study. The main ideas of all formulations are the same: we embed the real number field R in a necessarily non-Archimedian ordered field R* (containing infinite and infinitesimal elements) and establish that a statement is true in R if and only if it is true in R*. Around each element of R* is a neighbourhood of infinitesimals. We now have infinitesimal, finite and infinite elements of R*. Each finite element of R* is either an element of R or of the form a+e where a is a real number and e is an infinitesimal. In this case we write a=st(a+e) and say that a *is the standard part of* a+e. Typically difficult definitions and theorems from ordinary analysis become remarkably easy, e.g. a function f is continuous in an open interval I of R if and only if f(st(x))=st(f(x)) for every $x \in R^*$ such that $st(x) \in I$. More pertinent to A-level mathematics is the *easy* defintion of the derivative of a function f, f'(x)=st((f(x+e)-f(x))/e).

Keisler's formulation, (1976a and 1976b), proceeds via an axiomatization of R* which includes an existence axiom:

> There exists a positive infinitesimal number

and a rather strange form of Leibniz' law of continuity:

> If two systems of formulas have exactly the same solutions in R, they have exactly the same solutions in R*.

Keisler utilizes an attractive feature of focusing on infinite and infinitesimal parts of a graph in R* by using *infinite telescopes* and *infinitesimal microscopes* (1976, p.28). An infinite telescope allows the examination of infinite portions of R* while an infinite microscope allows an infinitely small portion of R* to be examined.

Henle and Kleinberg (1979) do more or less the same thing but spend more time discussing language and logic and simply state that:

> *The same sentences are true in R* as are in R.*

Both texts acknowledge that R* is not unique but do not, understandably, go into great detail. Both texts then proceed to develop infinitesimal calculus in the style of a modern Leibniz and are arguably suitable for Sixth Form audiences.

Tall (1980a and 1981) differs from the rest in that one of the systems he develops is actually weaker, it will only handle analytic functions (which is enough for an A-level course). This system is considerably easier to construct than the others. Tall uses the well known fact (see E. Moise, 1963, Chapter 28) that the field of rational polynomial expressions with real coefficients form an ordered non-Archimedean field (an ordering is induced by defining $f>0$ if there exists $c\in R$ such that $f(x)>0$, for all $x>c$, and defining $f<g$ by, there exits a $c\in R$ such that $f(x)<g(x)$ for all $x\in R$ such that $0<x<c$). It is a simple matter to show that $f(x)=x$ takes the role of an infinitesimal in this system (it is interesting to note that Moise defines $f<g$ if there exists a $c\in R$ s.t. $f(x)<g(x)$, for every $x>c$ and thus $f(x)=x$ is infinitely large with respect to $g(x)=a$ but infinitely small with respect to $g(x)=x^2$). Having obtained infinitesimals Tall constructs his number system $R^*$ by means of power series, with real coefficients, on an infinitesimal. Tall then proceeds to define standard parts and develop calculus, like the others, in a Leibnizian fashion.

Keisler's approach has been tested at college level by Sullivan (1976). Such approaches gained some favour in America for Freshman courses, for a short time, but quickly fell from grace. No other pedagogic investigations, that we are aware of, have continued her study. Sullivan addressed herself to the questions:

> Will the student *buy* the idea of infinitely small ? Will the instructor need to have a background in nonstandard analysis ?

> Will the student acquire the basic calculus skills ? Will they really understand the fundamental concepts any differently ? How difficult will it be for them to make the transition into standard analysis courses if they want to study more mathematics?
>
> Is the nonstandard approach only suitable for gifted mathematics students ?

Sullivan used a control and an experimental group, both of 68 pre-university, mathematically able, college students from five colleges. She tested them after a one year course. She found the experimental group scored at least as well in all the tested areas (defining basic concepts, computing limits, producing proofs and applying basic concepts), were able to appreciate the standard methods and, in the opinion of their teachers, had a deeper understanding of calculus. She stresses, as all involved in such work have stressed, that this is not *calculus made easy*.

Sullivan's was an instructional investigation, not a psychological one. There is need in this area for both types. The present work was conceived as a instructional thesis but changed during investigations when the questions being asked changed from *Can this method bring improved results and understanding ?* to *What are students' intuitions of the basic ideas behind these methods ?* Sullivan claims the nonstandard analysis approach is closer to students' intuitions but does not investigate what these intuitions are. The results of the present work show that students' do not intuitively accept classical infinitesimals. Sullivan's claim need not be totally rejected, however. One can easily accept a statement even if it is not intuitive.

For example I intuitively believe that time is a continuous quantity, but if an eminent scientist told me on the basis of verifiable experiments that time is actually a discrete phenomena then I would accept this and find it an easy concept to handle in evaluating statements dealing with time. Thus, although infinitesimals may not be intuitive, once they are accepted, mathematics using them may be easier than mathematics without them. Moreover, one aspect of the process of education is to replace unfruitful intuitive beliefs by more coherent and useful ideas. In several uncontrolled experiments I have asked classes of Sixth Form mathematicians whether they accepted $\sqrt{-1}$ as a legitimate mathematical entity. All said 'No' before being taught complex numbers but the majority said 'Yes' after a month's exposure (and, as the present work shows, $\sqrt{-1}$ is seen as being as unbelievable as $1/\infty$ to the uninitiated).

While nonstandard analysis undoubtably has its advocates in elementary calculus and advanced analysis (complex analysis, measure theory, topology, etc. can all be developed in a nonstandard way) it also has its critics, and not just constructivists who disapprove of any mathematics dependent on the Axiom of Choice or similar tools. At a post graduate or research level it is difficult to see, apart from conservatism, why classical mathematicians should object to nonstandard analysis (it has been shown that anything true in a non standard space is true in its imbedded standard space, so if a result is true in a nonstandard space we have a standard result) but arguments against its introduction at a lower level can be suasive.

Schwarzenberger has considered the problem in two papers (1978 and 1980). In the 1980 paper Schwarzenberger attacks all, standard and nonstandard mathematicians, who would make calculus easy (by shortcuts) and defends all who would give a relational understanding of the calculus to their pupils. Calculus is not easy because R (and thus also R*) is simultaneously an ordered field, a complete ordered field, a metric space and a normed vector space. Attempts to make it easy, at a low or high level, omit one or more of R's aspects. Schwarzenberger's main criticism (in the 1978 paper) of a nonstandard analysis approach in schools in that unlike the reals there is not a unique model for R*:

> If it is objected that these disadvantages stem merely from the relative unfamiliarity of the hyperreals as compared with the reals, then it must be said clearly in reply that the familiarity and assurance with which we handle the real numbers stem largely from the uniqueness of R. Until mathematicians agree on a unique model for R* there can be little hope of making R* as familiar to pupils as R.

How many mathematicians are able to describe the construction of R by Cauchy sequences or Dedekind cuts ? I would wager very few. What we know is that we have seen this done (or know this can be done). It may be that we can work in nonstandard analysis without getting involved in the higher reaches of model and set theory that govern the structure of its models ? Moreover, as we shall see, students are very unfamiliar with the actual structure of R. This, however, is not an argument for introducing nonstandard numbers.

## OTHER STUDIES AND REPORTS ON CALCULUS

It is a great pity that there are so few investigations into the understanding of the basic concepts behind the calculus and of the cognitive effects of learning calculus. There is a wealth of articles giving armchair expositions of calculus topics to explore or approaches that can be taken, (see Brown, 1970 for a good example).

There have been several British theses. The only one to shed insight into areas concerned with here, however, is the thesis of Orton (1980a).

Orton examined students' understanding of the basic ideas of calculus in 110 pupils in Sixth Forms and colleges of Higher Education. Tasks were designed to test:

> The understanding of limits in a variety of mathematical situations independent of the calculus.
> The idea of integration as measuring area.
> Rates of change ... leading to differentiation
> A number of simple applications (Orton, 1980b)

Information was collected by interviews and was reclassified as items relating to a single aspect of elementary calculus. The 38 items resulting form an excellent calculus skills list and the mean scores certainly give us information on the ability of students to perform these skills. Orton classifies responses according to a Piagetian heirarchy to obtain a measure of cognitive demand of each item. Whatever one's reservations about the use of Piagetian heirarchies or about absolute measures of a correct response in the calculus his results give us details hitherto only surmized, not tested, by mathematics educators. They are that:

> Care needs to be taken that difficulties with algebra do not stand in the way of the development of students' understanding of calculus ... rates of change was poorly understood ... limits has been somewhat neglected as an idea to be developed throughout the main school mathematics programme ... some students had learned the rudiments of elementary calculus in an abbreviated and even an algorithmic way and may not have been taken back to reconsider any underlying mathematics (ibid.)

Such studies have great worth in establishing dimensions of difficulty but they do not get to the cognitive heart of the matter. To do this we must ask - *What are pupils intuitions of limits and infinity, how do they interact and develop and how can we use this knowledge to design better calculus courses ?*

Recent major British reports Mathematics Counts, (Cockcroft, 1982), and Mathematics in the Sixth Form, (HMI/DES 1982), discuss a range of social and curricular problems surrounding Sixth Form mathematics but do not enter debates on cognitive development in particular topics. Other reports, (Math. Assoc., 1982; SCUE/CNAA, 1978; SMP, 1980), go into details on the inclusion and structuring of topics within A-level syllabi but again do not cover cognitive aspects of A-level work. Content orientated reports (Math. Assoc., 1967), examiners' reports and groups such as SMP, MEI and Continuing

Mathematics go into details about errors (and ways to avoid them) and alternative approaches to topics but have again provided no cognitive or assessment work in the concepts behind the calculus.

Older reports, the Spens Report of 1938 and the Jeffery Report of 1944, as Orton (1985) has pointed out, argued that calculus should be taught from graphical origins and should reach a wider proportion of pupils. They do not, however, back their opinions with cognitive data.

These reports may be seen as a prelude to a lobby of mathematics educators motivating the introduction of calculus lower down the school in the 60's and 70's:

> The importance of the teaching of analysis in the secondary school continues to increase in many countries and nowhere has it decreased. One can thus say that there is a universal trend in reinforcing the teaching of analysis ... analysis could soon play the role in the fundamental mathematical education which has been attributed for a long time to geometry. (UNESCO 1972)

These words ring less true now, in Britain at least. This trend is going out of favour and is clearly at odds with the ideas of the 16+ system in that more problem solving and practical work on non calculus mathematics is stressed in 16+ criteria.

Whatever the stage of introducing the calculus in the future the words of the Mathematical Associations report (1951) are still valid:

> There is no part of mathematics for which the methods of approach and development are more important than the calculus, partly on

account of the novelty of the notation, but chiefly on account of intrinsic difficulties. These occur at the start and more acutely at the start than at any later stage. For this reason the early development must be gradual: any rushing of the introduction will lead to chaos.

The purpose of the preceding paragraphs was to emphasize what the reports left out - cognitive investigations. In our next section we examine those that have been carried out.

# CHAPTER THREE

# REVIEW OF

# PSYCHOLOGICAL LITERATURE

The previous chapter examined mathematical, philosophic and pedagogic thought relevant to this work. The present chapter reviews the cognitive research relevant to this study. The borderline between pedagogic and cognitive research is extremely fuzzy. Some authors' researches are re-examined from a cognitive viewpoint. Much of the present work's initial direction was as a direct result of influences reported in the following pages. For this reason work that came to the present author's attention towards the end of this research is reported under a separate section at the end of this chapter.

Piaget continues to influence cognitive debate in the Western world because, paraphrasing Whitehead on Plato, *Educational Psychology is still expanding his footnotes*. Another reason is that broad acceptance of Piaget's analysis firmly places one in a non behavioural school of thought. It is possible to call oneself a post-Piagetian while remaining highly critical of his use of the propositional calculus and group theory, of his stage theory and of his clinical method. Since the subjects in this study should be in the formal operational stage, it is useful to briefly consider criticisms of Piaget's stages. In Chapter Seven we consider his clinical method.

Ausubel & Ausubel (1966, p.405) sum up criticisms of Piaget's theory:

> They (American psychologists) argue that the transition between these stages occurs gradually rather than abruptly or discontinuously; that variability exists both between different cultures and within a given culture with respect to the age at which the transition takes place; that fluctuations occur over time in the level of cognitive functioning manifested by a given child; that the transition to the formal stage occurs at different ages both for different subject matter fields and for component subareas within a particular field; and that environmental as well as endogenous factors have a demonstrable influence on the rate of cognitive development.

Piaget's account of the modes of reasoning characteristic of each stage can also be criticized. Child (1973, p.129) observes that young children (preconceptual) do form and apply concepts. Of more relevance to this study is the nature of formal operational thought. In a study examining how intelligent adults test hypotheses Wason and Johnson-Laird (1972, p.188) found that:

> Highly intelligent adults fail to treat a rule as a rule, in the sense that they do not readily grasp all the consequences which follow from it. Indeed, those subjects who fail to gain any insight justify the reason for their selection in terms which, by any standard, are of a primitive kind.

Differentiating sharply between causal (practical) reasoning and logical reasoning (where truth and falsity are crucial) and noting that the framework in which problems are posed is an important factor, they go on to say (ibid., p.193):

> One answer would be that formal operational thought is less general than Piaget supposes, and that it may, in fact, be specific to a wide variety of tasks in which a causal and a logical analysis coincide. A rather different, and much more speculative answer would be that the novelty of our problem, when presented in abstract terms may induce a temporary regression to earlier modes of cognitive functioning ... The first answer is much more plausible. (ibid., p.193)

Related to the second, less plausible answer, is the very plausible thesis that:

> Generally mature students tend to function at a relatively concrete level when confronted with a particularly new subject area. (Ausubel & Ausubel, 1966, p.410)

More startling is the claim that few mature students actually function at the formal operational level. Ausubel, Novak & Hanesian (1968, p.238) report a study in which 15.6% of American junior-high school students, 13.2% of high school students and 22% of college students examined were at this stage of development.

The point of these reports for us is that although we shall broadly work in a post-Piagetian framework, we shall not relate our findings to his description of the formal operational stage.

## CONCEPTS

Cognitive science (or, rather, prescience for it is not yet in a state to be properly called a science) has not yet given a generally accepted definition of a concept. Concepts are generally recognized, however, as vehicles of thought. Following Child (1973, p.115) we note the following characteristics of concepts: Concepts are generalizations built up by abstraction; are dependent upon previous experience; have a symbolic function; form horizontal (e.g. different types of birds) and vertical (hierarchies) organizations; can function extentionally (public use) or intensionally (private use); can be irrational (e.g. superstitions or accepted dicta); and may be formed without our conscious awareness. We must further note that concepts may come immediately from sense experience (like *hot*) or may be built up from other concepts (like *function*).

The *meaningful*, as *opposed to rote acquisition of concepts* (see Ausubel, 1966, p.158) or the relational, as opposed to instrumental (see Skemp, 1976), may occur at two levels: through concept formation, where the criterial attributes of concepts are discovered inductively (either naturally or by experience conditions); or through concept assimilation, where the criterial attributes of concepts are presented through a medium of instruction. The role of language as an agent in the acquisition of concepts is much more prominant in concept assimilation. Ausubel holds that:

> When an individual *uses* language to acquire a concept, he is not merely labelling a newly learned generic idea; he is also using it in the *process* of concept attainment to acquire a concept that transcends by far ... the level of concept acquisition that can be achieved without the use of language. (Ausubel, 1966, p.165)

To Piaget the basic concepts that characterize the period of concrete operations are those of conservation, seriation, classification, number, space and time. With the onset of formal operational thought comes a fuller understanding of the concept of proportion (considered by Piaget as variation between two magnitudes) and an ability to conceive of infinite subdivision. With regard to the latter, Piaget found that in the pre-operational stage children could not continue subdivision very far. In the concrete operational stage they could continue a large but finite number of divisions. Only in the period of formal operations could they continue indefinitely. It is in this last stage, according to Piaget, that a child can imagine the limit of a shape as a point (NB Our subjects were more or less equally divided on this). He notes:

> Not until he reaches this stage does the child envisage the ultimate elements of continuity in this way. That is, as purely hypothetical points which can be neither seen nor touched but can be mentally separated and combined to the limits of infinity. (1956, p.145)

Useful as this first study was it is worth noting that Piaget thought he was investigating the child's conception of continuity (as evidence for his thesis that childrens' intrinsic geometry is first of all topological, then projective and then Euclidean). As Darke points out, however:

> Continuity may be founded upon limit in a formal exposition but from the point of view of both heuristics and the history of mathematics, continuity is not nec= sarily dependent upon limits. (1982, p.136)

Three post-Piagetian studies went further into these issues: Thomas (1975) and Orton (see Lovell 1975) on the concept of function and Taback (1975) on the concept of limit. Both studies on the concept of function were concerned with the modern notion of function and both shared the conclusion that although some aspects of the concept could be grasped by children in the period of concrete operations (interpreting arithmetic rules and using functions that produce straight line graphs, and thus involve a law of proportion), children must be in the period of formal operational thought before coming to terms with problems on domains, ranges and inversion. Moreover, children must be well established in this stage before they can tackle problems dealing with composition of functions and general function notation.

Taback, in his study of children's concept of limit, investigated : rule of correspondence; convergence (divergence); neighbourhood; and limit point. The subjects were intellectually mature for their ages ( 8, 10 and 12 year olds) and the concepts were investigated at concrete and abstract levels. Taback found that eight year olds could do little more than follow a simple rule of correspondence; 10 year olds were similar, in performance, to 12 year olds; and the older children who understood convergence at a concrete level understood it at an abstract level. Only one 10 year old and eight 12 year olds (out of 25 from each age group) could conceive of infinitely many points in an open circle neighbourhood. Moreover, he found (1975, p.138):

> Even at the 12 year old age level only 20% of the subjects could conceptualize the infinite division of a line segment.

Useful as these studies were (especially in such a relatively unexplored field) they did not hit at the heart of the matter because they inherited from Piaget an attempt to classify concepts in a framework where the children's concepts were seen as hierarchical and internally consistent at each stage when many of the concepts held by the children were inherently contradictory. The three main reseachers in the 1970s, while working in a post-Piagetian framework, took the contradictory nature of subjects concepts of limits and infinity as fundamental to their analyses. They are Fischbein and colleagues in Israel, Tall in Britain and Cornu in France. We shall examine each of their contributions in turn.

## MODERN STUDIES

The examination of subjects' intuitions, in addition to their information processing abilities, played an important role in these studies. Like many psychological terms intuition is not easy to define. We accept the rather loose but useful characterizations of Fischbein et al. and Tall:

> We use the term intuition for direct, self evident forms of knowledge (Fischbein et al., 1979, p.5);

> The central property of intuition: the global amalgam of local mental processes using existing cognitive structure, as stimulated by a novel situation. (Tall, 1980d)

> Intuitive knowledge is determined by the confluence of two factors: level of confidence and obviousness (Fischbein et al., 1981, p.493)

Fischbein et al. developed a method of measuring the intuitive acceptance of a mathematical statement by asking a set of check questions after each mathematical question was put which probed the subjects' levels of confidence and obviousness. The present study is not concerned with analysing intuitions per se but in classifying subjects' intuitions of infinity to find what principles lay behind them. For example it is the case that the vast majority of intelligent people do not believe there is a largest number and this is, to them, a direct and self evident fact. This intuition, however, comes from their conception of the number system that includes the property that any number can be incremented. Clearly people do not subconsciously hold millions of propositions in their mind that they are waiting to affirm intuitively but rather these affirmations are deductions derived from deep rooted cognitive principles. What separates intuitions from beliefs gained by information processing is the length of these deductions, they are very short. These short deductions are often at variance with each other and with deductions obtained via information processing. Here lies the essence of cognitive conflict.We shall return to these issues later.

Fischbein et al. differentiate between intuitions and concepts. Intuitions are what we *really feel* (1979, p.33) whereas concepts are the result of *logical, explicit analysis*. Thus the concept of infinity may change under instruction but the intuition may remain stable. This is an interesting idea that we believe holds but the fuzzy boundary between a concept and an intuition is not clear and we shall not use this as a formal distinction between concepts and intuitions. We shall use the word concept to include both concepts and intuitions, as defined by Fischbein et al., but reserve intuition for the immediate, self evident form of knowledge.

The main paper on infinity by Fischbein et al. (1979) attempted to take Piaget's work further by using older subjects, asking questions based on denumerable and nondenumerable sets and by trying to determine the relationship between responses to questions on infinity and school attainment level of the subjects (their sample consisted of 470 children of both sexes and of all abilities between 10 and 15 years of age).

The main hypothesis developed in this paper is that our intuition of infinity is intrinsically contradictory because our logical schemes are naturally adapted to finite objects and events. Evidence for this thesis is offered in the form of large discrepancies in responses between infinitist reasoning (accepting infinite divisibility of a line and, in general, infinite continuation of an operation) and finitist reasoning (not accepting infinite continuation of an operation or using finite logical schemes, e.g. *the whole must be greater than the part*) in responses to questions:

> The lability of the intuition of infinity can be explained if admitting its intrinsic contradictory nature as a psychological reality. (1979, p.10)

Fischbein et al. note that both finitist and infinitist responses may be supported by concrete or abstract arguments. This prevents an easy dichotomy of responses. Moreover, intuition is very sensitive to context. They found that the intuition of infinity is relatively stable, with respect to age, from 12 years onward. The effect of teaching, they found, was varied, contributing to both finitist and infinitist responses. This further confirmed his main thesis:

> What explains the contradictory behaviour of the intuition of infinity is the fact that we tend to think on infinite sets of (sic) resorting to our usual logical schemes which are adapted to finite realities ... For nonstandard questions for which the pupils did not get specific information, we must expect high percentages of finitist (wrong) reactions even in spite of his more advanced general mathematical training (and sometimes as an indirect effect of just this mathematical training). (1979, p.37)

The work of Fischbein et al. was important and original (and we take up some of his points in the following chapters) but was restricted in not considering the related concept of limit, not examining these intuitions more fully in the context of arithmetic and in not examining the effect of language. For these we turn to the work of Tall and Cornu.

Tall has written more than any other author on students' concepts of limits and infinity. His subjects have generally been students at British universities and his interest is largely in their understanding of calculus. To appreciate his contributions we must start with his ideas on cognition.

Tall began his work in the mid 70s in a post-Piagetian position strongly influenced by Skemp's ideas on Schematic Learning and

Instrumental and Relational understanding. An early and retained interest was in cognitive conflict arising in calculus ideas, in particular with the concepts of limits and infinity ($1/3=0.\dot{3}$, $1/3 \times 3=1$, $0.\dot{3} \times 3=0.\dot{9}$ but $0.\dot{9} = 1$, *conflict*). Early models to account for this conflict were based on ideas from Catastrophe theory (Tall, 1977). This was in vogue at the time, especially at Warwick University. This was not awfully successful in that the sophisticated theory constructed was only loosely connected with the data obtained and was by no means tested by that data. It was, as is much educational theory, top heavy.

Tall soon dropped the Catastrophe theoretic framework in his work but retained the conflict aspects of this model in future papers. With Schwarzenberger (Schwarzenberger and Tall, 1978) he developed pragmatic ideas for a conflict free approach to the teaching of real numbers and limits. Here they noted that conflict may arise from the interference of everyday language meanings in a mathematical framework, from confusing ideas from separate but related areas of mathematics (e.g. sequences and series) and from students confusing ideas from their total mathematical experience.

Tall went on from here to develop, with Vinner (Tall and Vinner, 1981), a theory of conflict image and concept definition:

> We must formulate a distinction between the mathematical concepts as formally defined and the cognitive processes by which they are conceived. ... We use the term *concept image* to describe the total cognitive structure which is associated to the concept. ... We shall call the portion of the concept image which is activated

> at a particular time the *evoked concept image*. Only when conflicting aspects are evoked simultaneously need there be any sense of conflict or confusion. ... We shall regard the *concept definition* to be the form of words used to specify that concept. ... It may be the form of words the student uses for his explanation of the concept image he has. We shall call a part of the concept image or concept definition which may conflict with another part a *potential conflict factor* .. if they are evoked the factors concerned will then be called *cognitive conflict factors*.

Tall has argued that final year mathematics students at university with a clear acceptance of the concept definition of the actual infinite have a concept image of the potential infinite (Tall, 1980d). With other students of his, who would have met the conventional definition of the limit of a sequence, his investigations indicate that the concept image includes the fact that $S_n \rightarrow S$ precludes the possibility that $S_n=S$. Perhaps most important, in terms of practical curriculum factors, is his suggestion that students' concept image of a limit is of a dynamic process (related to the concept of a potential infinity) rather than a numerical quantity. Tall has not, unfortunately, presented a case study of the interactions between concept images and definitions.

Such a theory is difficult to verify in that it encourages analysis of facts already established rather than allowing predictions to be made but it does allow an explanation of facts to be placed in a context that permits classification and generalization (and it is this type of analysis that is often of most use in educational, as opposed to psychological, research).

The other main aspect of Tall's work is, like that of Fischbein et al., the cognitive aspects of students' mathematical intuitions. This is closely bound up with the study of students' conflicts. Intuitions are particularly important in the study of limits and infinity for Tall because of his work, described in the previous chapter , on nonstandard interpretations of infinity. Tall argues, students' intuition of infinity are often consonant with nonstandard infinite concepts rather than the standard Cantorian concepts, though neither is totally appropriate. We shall return to this point in our conclusions.

Tall describes intuition as:

> The global amalgam of local mental processes using existing cognitive structure,as stimulated by a novel situation. ... The concept of infinity varies from one individual to another and need not be globally coherent. (1980d)

Tall argues in (1980b) that viewing children's responses to questions on infinity through cardinal interpretations distorts our understanding of their conceptions. Moreover:

> ...different finite experiences (measuring as opposed to counting) can lead to different notions of infinity, giving a concept image containing potential conflicts. (1980d)

All of these aspects of Tall's work (the concept image / concept defin¯ition conflict model and the view that several legitimate interpretations of infinity hold and that children's intuition should not be weighed against any single one) have been incorporated into the present work along with his practice of examining infinity and limits together rather than isolating them.

We now turn to the third important field worker in this domain, Cornu, whose work is presented in a research report (Cornu, 1980) and in his Ph.D. thesis (Cornu, 1983).

Cornu, like Tall, is interested in students' problems with limits and infinity at the level of a first course in calculus and at the *classe de premiere*/university interface. Again, like Tall, he is interested in conflicts and the interference of intuitions (conceptions spontanes) and pupils' own conceptions (conceptions propres) with the taught concepts. His style of approach is continental in that he views the problem as one of dialectics - the continual synthesis of intuitive thesis and taught antithesis.

Cornu utilizes Tall and Vinner's concept image and concept definition but adds to this Bachelard's notion of *obstacles:*

> *Un obstacle est une connaisance: il fait partie de la connaissance de l'élève. Cette connaissance a en général été satisfaisante à une certaine époque, et pour résoudre certains problèmes. C'est précisément cet aspect satisfaisant qui a ancré la connaissance et en a fait un obstacle. Cette connaisance devient inadaptée, car on se trouve face à des problèmes nouveaux; mais cette inadéquation peut ne pas être apparente.*

> An obstacle is a piece of knowledge: it is part of the knowledge of the pupil. This knowledge was satisfactory at one time and resolved some problems. It is precisely this satisfactory aspect which fixed the knowledge and made it an obstacle. This knowledge becomes ill adapted for one faces new problems; but this inadequacy may be hidden (Cornu, 1983, p.30):

Cornu classifies obstacles as having their origin in: the cognitive maturity of the individual; methods of teaching; the personality of the individual; the social environment; technicalities e.g. number crunching; and in the nature of the mathematics being learnt.

Closely bound up with the notion of obstacle is that of *les erreurs*. Errors arise when the knowledge constituted in an obstacle ceases to apply to a problem. Exercises often hinder development in that only the aspect of a concept constituted in an obstacle may be used. For example students may, and often do, have as their concept of convergent sequences, monotone bounded sequences:

> *Il s'agit d'une connaissance, partiellement eronée, qui constitute typiquement un obstacle. Cette connaissance va conduire à des succès partiels, mais aussi à des erreurs caractérisées, et ce sont bien souvent ces erreurs qui permettront de déceler la présence de cet obstacle.*

> It is a partially erroneous knowledge that typically constitutes an obstacle. This knowledge is going to lead to partial success but also characteristic errors and it is often these errors that

will allow us to unlock the presence of this obstacle. (ibid.,p.33)

Errors should not be seen as arising simply from ignorance for they are often the logical consequences of the subjects knowledge (as realized by mathematics educationalists throughout the world).

Cornu isolates various obstacles in the concept of limit (ibid., pp.151-154):

<u>The metaphysical aspect of limit.</u>

*L'infini intervient, et il est entouré de mystère. L'élève a du mal à "y croire".*

The infinite intervenes and it is surrounded in mystery. The pupil has difficulty in believing it.

<u>The infinitely small and the infinitely large.</u>

Again pupils have difficulty in believing in them.

<u>The limit attained.</u>

Students have cognitive traumas over whether limits are attained or not. Some use different expressions for the limit attained as opposed to the limit not attained.

<u>Passage from the finite to the infinite.</u>

*Par example, dans l'activité sur la tangente, "la règle va tomber".*

For example, around the tangent, "the ruler will fall".

This example refers to using a ruler in drawing approximating secants and around the tangent itself the ruler will fall off the curve.

Obstacles in the limit notion may also exist in incomplete understanding of other mathematical ideas.

From his analysis Cornu suggests teachers lead classes to explore and discuss their own ideas on being introduced to limits:

> *Pour qu'un obstacle puise être franchi, il faut qu'il y ait apparition d'un conflit, et prise de conscience de ce conflit. A partir de ce conflit, s'instaurera chez l'élève une dialectique entre la "problème" et sa connaisance, et cette dialectique pourra donner naissance à une connaissance nouvelle, par laquelle l'obstacle aura été franchi.*
>
> In order for an obstacle to be overcome there must be an appearance of conflict and an awareness of this conflict. From this conflict there will be installed in the pupil a dialectic between the problem and his or her knowledge and this dialectic can give birth to new knowledge through which the obstacle will be overcome. (ibid., p.34)

Cornu's work is exploratory and thus may be partially criticized by psychometricians as lacking complete rigour. He admits in his conclusions that his work raises more questions that it answers. The present work has incorporated Cornu's tool of using subjects' everyday phrases to examine obstacles bound up with the words 'limit', 'tend to', etc.

## RECENT STUDIES

Since starting the present work several new studies have been published. We present these together here, rather than in the prece^ding paragraphs, to stress that they did not have a formative influence on the design of the present work.

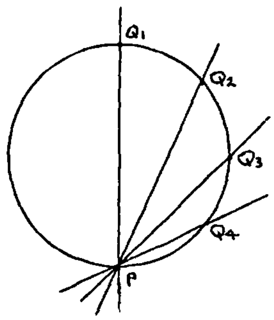
It is undeniably the case that there has been a shift in British mathematics education research away from Piagetian heirarchies and factor analytic methods that Orton employed in his thesis. Bishop (1972), as early as 1972, made reference to the former.

In two papers (1983a,b) Orton re-examined his thesis data in the light of Donaldson's three types of error:

> Structural errors were described as those "which arose from some failure to appreciate the relationships involved in the problem or to grasp some principle essential to solution". Arbitrary errors were said to be those in which the subject behaved arbitrarily and failed to take account of the constraints laid down in what was given. Executive errors were those which involved failure to carry out manipulations, though the principles involved may have been understood. (Orton, 1983a, p.4)

Orton found very few instances of arbitrary errors, executive errors in about half the items and structural errors in almost every item. This is consistent with the view that students acquire adequate skills in calculus without understanding the principles behind it:

An example of a structural error is




As Q gets closer and closer to P what happens to the secant.

> 43 students were unable to state that the secant eventually became a tangent ... Typical unsatisfactory responses included: "The line gets shorter"; "It becomes a point"; "The area gets smaller"; "It disappears" (1983b, p.237).

Useful as such results are we must treat them some with care for Orton's interpretation of an error is that which does not conform to limit methods employed by most A-level teachers. In the present study we attempt to reserve the word *error* or *incorrect* only for the case of errors in finite calculations. This allows us freedom to interpret student models of infinite processes as models in their own right without necessarily comparing them to formally correct models. For example in examining responses to a question asking subjects to find the limit of 3n/(n+1) Orton states:

> Over half of the students responded incorrectly to part (d), including a large number who said '  ' ... Errors made on this item appeared to be largely structural, revealing an absence of real understanding. (1983a, p.6)

While '  ' is clearly an error by formal mathematical criteria we disagree that it necessarily reveals an absence of real understanding. From the student point of view, as n goes on and on, the limit goes on and on. Many subjects view infinity as meaning *goes on and on*. Thus the limit is infinity. This reveals an absence of formal understanding but not an absence of understanding.

Of course Orton is formally right in viewing this as an error. We stress our divergence from his analysis to emphasize that our concern in this research is in understanding students' ideas on infinity. Comparison with mathematical correctness is thus proper for him but not suitable for us.

The thesis of Robert (Paris, 1982) only came to the attention of the author through personal correspondence with Tall towards the end of this work. We can thus only report secondhand knowledge. The results are, however, of great interest.

Robert examined the concept of the limit of numerical sequences via a questionnaire given to 1253 French students in higher education. Responses to the request for a definition of a convergent sequence allowed a classification of student models:

Primitive models She produces three types.

stationary: *The final terms always have the same value.*

barrier: *The values cannot pass 1*

monotonic: *A sequence is convergent if it is increasing and bounded below (or decreasing and bounded below).*

Dynamic models have a sense of motion implied by phrases such as *tends to*.

Static models involve a reformulation of the standard definition such as, *All intervals contain all the $u_n$ except a finite number.*

Models were also found which were mixed or unclassifiable. After tracing these models through higher education she concludes:

> It is best ... sometime after the course, to make the students conscious of their mental images and try to rectify them in a mathematical way by reflecting on their erroneous mental images.

Tirosh was Fischbein's pupil and a joint author of the two articles on the intuition of infinity reviewed above (Fischbein et al., 1979 and 1981). In a recent report (Tirosh, 1985) she presents results of research whose objectives were:

1. To identify the inner conflicts in the intuitive understanding of the notion of actual infinity.
2. To try to improve the high school students' intuitive understanding of the notion related to actual infinity through systematic instruction.

Tirosh used twenty lessons with 158 academically able fifteen year olds in experimental classes to teach set theory up to and including non-denumerable cardinals. She collected the data using pre-test and post-test questionnaires composed of 16 questions on the equivalence of sets. Subjects were to justify their responses by mathematical arguments. Comparing results with 122 similar students in a control group she found the majority of subjects in the pre-test claimed that both sets had the same number of elements:

> *only one kind of infinity exists, therefore all the infinite sets have the same number of elements.* This idea of equivalence corresponds to the primary intuitive understanding of the infinite as an endless process. ... Students justified the "non-equivalence claims" by three main arguments: *A set contains more elements than its proper subset; A non bounded set contains more elements than a bounded set; A two dimensional set contains more elements than a linear set (ibid., p.504).*

She regards the conlict between the equivalence and non-equivalence claims as a basic difficulty in the intuitive understanding of the actual infinite. 84% of subjects were inconsistent and only 5.7% were aware of the deep contradiction between these two claims.

These results are very close to those obtained from the cardinality questions in the present study. The second part of Tirosh's work is of less concern to us as our mathematical interest is calculus and not transfinite set theory. She concludes:

> by using suitable teaching methods, including an active didactical approach towards the intuitive tendencies of the student, it is possible to improve the students' intuitive understanding ... students' awareness of the inner conflicts in their intuitive ways of thinking produced in them a much deeper understanding of the need and importance of formal mathematical proof.

# CHAPTER FOUR

# PILOT STUDIES

This chapter reports on preliminary studies and their use in evaluating early hypotheses and questionnaire items. The studies are also interesting in their own right and may be used as secondary data sources supplementing the primary sources - the questionnaires and interviews described in Chapters Five to Eight.

Pilot runs of performance tests are often necessary to remove items that have a low index of discrimination and to check the reliability and validity of a test. Our study, however, was not intended as a performance or intelligence test. Moreover, subjects' conceptions of limits and infinity are open to considerable variation. For these reasons formal checks on reliability and validity were not conducted. The experimental method of this study is described in Chapters Five and Seven.

The purpose of pilot investigations in this study was to break the conceptual ground. Our task at the outset of this study was twofold: to posit adolescents' concepts on the basis of mathematical concepts, previous research and from experience working with adolescents (and to test these hypotheses) but, at the same time, to keep an open mind and allow revisions of assumptions to be made at any stage. These aims should, of course, be present in every stage of cognitive research but are especially relevant in pilot studies where many exploratory hypotheses may be investigated before more rigorous data collection techniques are used.

Data collection began by asking pupils, friends and colleagues all sorts of questions both in a formal and in an informal manner. While not suitable for presentation here they nevertheless resulted in a feel for the area to be charted and were most useful.

The first pilot test took the form of structured interviews conducted at Morecambe High School (MHS) in January, 1982. Ten mathematically competent pupils, a girl and a boy from each of the Third, Fourth, Fifth, Lower Sixth and Upper Sixth Years were arbitrarily chosen from O and A-level mathematics classes (subjects are referred to as 3B - Third Year boy, L6G - Lower Sixth girl, etc. in the following). For the benefit of readers who are not familiar with the British education system, Third Year pupils must be 13 years of age in the September that the school year starts. Succession to the subsequent year each September is automatic. Pupils may leave school at the end of their Fifth Year.

The school follows SMP mathematics (a large and established modern mathematics programme). Each subject was asked the 18 questions over four separate sessions lasting from 10 to 20 minutes. The questions were presented on cards and accompanied by a uniform explanation. If this was not understood then various alternative explanations were offered. The order of presentation of the four sections was different for each subject in order to prevent replies to initial questions affecting replies to later questions in a uniform manner. Each interview was recorded and transcribed. We adopt a casual presentation of the results of the pretests as they were exploratory studies.

We hypothesized, at the time, that four concepts were possible: the potential infinity of Aristotle; the actual infinity of Cantor; the actual infinity of Robinson; and practical (as opposed to philosophical) finitism. We included items to examine this hypothesis. Many of the questions (1-4 and 12-18) were taken from, or complemented, questions from Fischbein et al., (1979). There is an advantage in using other workers' question in that results can be compared. There is a disadvantage in that a similar analysis may be encouraged. Questions 5-8 and 10-11 asked straightforward questions on infinite sequences and series. Question 9 looked at recurring decimals and question 9a checked that subjects could compare finite decimals.

## ITEMS IN THE FIRST PILOT STUDY

The question in each of 1-4 below was: *Is there a smallest line ?*

1) Figure 1 (not shown here) shows a line, under this a line half the first line's length, under this a line half the second line's length, etc.

2) Figure 2 (not shown here) shows a line, under this a line a tenth of the first line's length, under this a line a tenth of the second line's length, etc.

3) Half of an 8cm line is a 4cm line.
   Half of a 4cm line is a 2cm line... etc.

4) One tenth of a 10cm line is a 1cm line.
   One tenth of a 1cm line is a 0.1cm line... etc.

5) Can you add together 1+1+1+...
   and go on forever and get an answer ?

6) Can you add together 0.1+0.01+0.001+...
   and go on forever and get an answer ?

7) Can you add together 1/2+1/4+1/8...
   and go on forever and get an answer ?

8) Can you add together 1/2+1/3+1/4...
   and go on forever and get an answer ?

9) Is $0.\dot{9}$ smaller, equal of bigger than 1, or can't we compare them ?

9a) Is 0.1010 less than 0.1001 ?

10) Consider the pattern 0.1, 0.01, 0.001,... Will we ever get to 0 ?

11) Consider the pairs of numbers $\begin{cases}0.1\\0.09\end{cases}$ $\begin{cases}0.01\\0.009\end{cases}$ $\begin{cases}0.001\\0.00009\end{cases}$ etc.

Will there ever be no difference between the pairs of numbers ?

12) 1, 2, 3, 4, 5, ...

2, 4, 6, 8, 10, ... Are there more numbers in the first row than there are in the second row ?

13) Consider the number of points on the line

and the number of numbers 1, 2, 3, 4, ...

Are there: i) More points than numbers ?

ii) More numbers than points ?

iii) The same amount of each ?

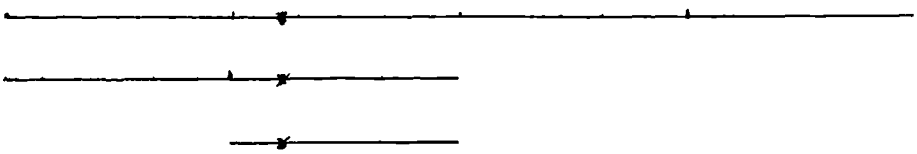
or iv) Can't you compare them ?

14) Consider the number of points on the line

and on the line

Are there more points on the first line ?

15) A point is marked anywhere on a line. Repeated halves of each line are shown so that the point is always on the line.



Will there always be a line with the point at its very end ?

16) Consider a line 5cm long and a square of side 5cm (these were drawn). Is there a point on the line for each point in the square ?

17) Consider a rectangle like the one on the right (not shown here). We make new rectangles by increasing the length and decreasing the width in a way that keeps the perimeter the same.

What happens to the areas as the process continues ?

18) Consider the pattern (a sequences of regular polygons with increasing sides was displayed).

If the process continues long enough, will we get a circle ?

## ANALYSIS OF RESPONSES

**Questions 1-4** considered the infinite divisibility of a line. As we have seen (p.31), Piaget claimed that with the advent of formal operational thought unlimited subdivision no longer presents difficulties. Our findings, in contrast to Fischbein et al. (1979, p.11) who found 55% of their subjects took a finitist position, agreed with Piaget's. Our question *Is there a smallest line ?* was intended to separate the potential infinitists from the actual infinitists. All subjects except 3G, 3B and L6G thought there was not a smallest line. The idea of a potential infinity dominated the reasoning of the other seven subjects.

An interesting finding was the use of *fixing a point*. This was unexpected and may be seen as using a finite scheme to interpret an infinite phenomenon. This was to recur in replies to other questions:

4G I don't think there'd be a shortest line unless you say *I'm stopping here'.*

5G If you give us a fixed point to stop, at that point then you will have a smallest line. But if you just carry on then you will have a small line but not the smallest.

The three who thought there was a shortest line presented finitist and infinitist reasons:

3G It gets too small to bother about.

3B Down to the smallest line you could have a line two atoms long.

L6G There must be a point at which you can't halve it any more. It'll be at infinity.

The replies to question 2 corresponded, for all subjects, to the replies for question 1. Of the three who thought there would be a smallest line in question 1, 3G thought that the smallest line would be 10 times smaller in question 2, 3B was unsure and L6G thought we would reach the smallest point quicker in question 2. We thought that questions 2 and 4 suggested a time factor. While it was expected that subjects would see temporal aspects to infinite processes, where pure mathematics sees none, it was felt that future questions should not suggest this.

Questions 3 and 4 were included to examine the effect that an arithmetic, as opposed to a geometric, context had on subjects' conceptions. There appeared to be no general effect. The replies of the seven subjects who said 'No' in questions 1 and 2 remained the same. Of the others 3B remained finitist and L6G remained infinitist. Only 3G displayed a change in her thoughts. Whereas the geometric line in questions 1 and 2 got *too small to bother about*, the length of the arithmetic line in questions 3 and 4 did matter:

Question 3 presented.

3G Well, it will carry on until you've got millions of numbers after the point. There's no stop really because numbers go on forever.

Questions 4 presented.

3G No. It will always go on, like always build up the noughts between the point and the 1.

INT Would this, if you drew it, not go down to a point ?

(NB INT refers to the interviewer, the author)

3G If it was sort of on a measurement it would go down to a point. But, I mean, they are numbers, it could go on forever.

We cannot generalize from one subject out of ten but there may be an effect of context amongst a proportion of the student population (that this proportion may be small must not make us blind to it). Clearly there is a limit after which further drawing becomes *pointless*, whereas the difference between 0.00001 and 0.000001 is easily seen. Most subjects appear to arithmetize geometric questions, but how general is this ? Is the effect of context displayed by 3G above due to her age, sex or ability ? Such questions must be kept in mind in future investigations.

**Questions 5 to 8** concerned infinite summation. There were two main categories of answers to question 5: infinity means *going on and on* and so there is no definite answer; and *infinity is the answer*. Typical of responses for the former were:

4B Well, you can go on forever but there's no limit to what you can get.

5B No, because you can go on forever. It's infinite.

U6G It goes on to infinity, doesn't it ?

INT Would you get an answer ?

U6G No. It would just carry on and on.

Subjects easily jump from one view to another, however. The subjects quoted below initially accepted infinity as an answer but, when pressed, agreed with those just quoted. It is interesting to note that similar replies span the five year age range of the subjects:

3B Well, if you go on until infinity adding 1+1+1+1 then your answer will be infinite, if you go on forever, which is infinity. So you've got an infinite answer.

INT An infinite answer, or can you (subject interupts).

3B Well, if you go on to infinity it's never ending so I suppose you wouldn't get an answer.

U6B You get the answer of infinity. You can go on forever and you get the answer of infinity. It'll just continue and continue. You couldn't write it down as a number like 1 or any other number, you just continue going on.

INT Is infinity a number ?

U6B No, it's more of an idea. It's what somebody's defined as something. It's not actually a number. You can say you go towards infinity or away from infinity but you can't actually say you get there in the form of a number.

Notice the dynamic wording: infinity is something that goes on or something we can go towards. It is not a static or uniquely fixed entity in these responses.

Question 6 presented a convergent series whose sum is $0.\dot{1}$. This was answered correctly by L6G, U6B and U6G. We ascribe this to Sixth Form training. Recurring decimals presented a problem to the others:

3B Well, if you went on forever this time you'd get 0.1111111 stretching off into infinity. It would be *nought point* and then a whole string of ones stretching off into infinity.

INT So would you get an answer ?

3B No, because it would be never ending. It would be going on so far that you'd never get an answer.

INT What about when we say $0.\dot{3}$ ? Isn't that going on forever ?

3B Well, when you put recurring, well that's just a way of simplifying it but actually very complicated.

Except for 3G, who was confused by the question and replied that we would get an answer at each stage, this was the view of the rest. Clearly real numbers are not an easy concept for pupils.

Questions 7 and 8 looked at the same problem with fractional terms. Subjects focussed their attentions on the numeric difficulties of the question (finding common denominators of the partial sums) which took their minds of the main problem. When these problems were overcome subjects saw the questions as identical in principal to questions 5 and 6. None of the subjects noted the convergence of question 7 and the divergence of question 8.

**Questions 9 to 11** were concerned with infinite decimals. Question 9a was inserted to check that subjects were competent in the finite theory of decimals. All answered this correctly. Answers to the other questions were uniform. All answered questions 10 and 11, on infinite sequences, negatively (*it will never get to 0, there will never be no difference*). This is perfectly reasonable. Only a strict finitist or a mathematician interpreting the question as *Is the limit 0 ?* would reply 'Yes'. Subjects' rationales were, a *tenth of something can't be nothing* and *there'll always be a difference.*

**Questions 12 to 14** concerned cardinality questions. Our interest was not primarily in whether or not subjects' intuitions accorded with Cantorian results, but rather in the processes invoked by subjects in answering these questions.

Question 12 asked if the cardinality of the natural numbers was greater than that of the even numbers. Fischbein et al. (1979, P.18) found that 81% of high ability, 78% of middle ability and 49% of low ability subjects responded 'Yes'. Only two of our subjects responded 'Yes', five responded 'No' and three were unsure. The question of Fischbein et al. may have been misleading. They asked, *Which of the two sets contains more elements ?* This implies that one set is bigger.

Of our subjects 3G, 3B, 4B, L6B and U6B responded 'No' (there are not more in the first row). 3B and 4B used one-one correspondence:

> 3B No, if you carry on until infinity with both rows, then for each number on the top line you'll have a number on the bottom line, even if you go on to infinity and get really big numbers.

3G and L6B used the finite scheme of using a fixed point referred to above:

> 3G Well, if the numbers did stop I suppose there'd be more in that one than the bottom one. But they don't stop.

5B and L6G both thought there are more natural numbers and both referred to a qualitative change in the infinite case:

> 5B If infinity is somewhere, there's not going to be as many there as there are there because you're missing out on that one each time.

Questions 13 and 14 revealed a rich variety of ideas. Several subjects gave conflicting interpretations simultaneously. Subjects experienced problems with the concept of a point. In question 13 3B, 5G and 5B felt that a number was essential to define a point. We felt that future questions should put numbers on lines so that this desire to arithmetise a line did not interfere with the investigation of subjects' concepts of infinity. Subjects were less willing in question 13 than they were in question 12 to give yes or no replies. 3B, 4B, L6G, L6B and U6G thought the sets could not be compared. 4G, 5B and U6B, however, thought there were the same (infinity) in each.

In question 14 the size of the point was seen as a crucial factor. This spanned the age range:

> 3B It depends on how the points have been spaced.
>
> 4G It depends on how big the point is.
>
> 4B It depends on how wide the points are, I suppose.
>
> 5B If each point occupies the same space, then there's more space on the top line.
>
> L6G The first line's bigger than the second.
>
> U6B It depends what you take to be a point in the first place.

We were surprised that only 4B, 5B and L6G replied 'More' (on the longer line) in question 14. We expected the generic law to be dominant here.

**Questions** 15-18 were the least productive of the questions. All subjects found question 15 difficult to understand. We find it very surprising that Fischbein et al. (1979) did not comment on this. The question is really asking if any point on the real line can be uniquely defined by an infinite converging sequence. The question appears to be too sophisticated for school students. The amount of explanation required to get over the idea resulted in subjects being led to an extent that responses were felt to be of little use.

Question 16 compared the cardinality of a line and a square by asking if one-one correspondence was possible. In retrospect it was felt that specifying the length emphasised the physical nature of the line and square and that this should not be done in future. 3G and 3B were so confused by initial questioning on the nature of lines and points, that the question was not put. 4G, 4B and L6G responded that correspondence was not possible. 4B refered to the size of the points. L6G claimed a square must have more. The remainder responded 'Yes' but found it difficult to say why.

We found question 17 of little use in illuminating subjects' concepts of limits and infinity. The question was intended to examine the idea of conservation in limiting processes. Most of the subjects wrongly thought that the area would remain the same. Only 3B and 4G thought it would get smaller. They did not appear to consider the limiting case, however. Subjects appeared to only consider the initial cases. It was decided that it was not a suitable question with which to analyze subjects' concepts of limits.

Question 18 displayed a sequence of regular polygons with the question *If the pattern continues long enough, will we ever get a circle ?* Interpretations were divided between accepting approximations and viewing the question from the point of pure mathematics. 3G, 4B and L6G accepted physical approximations:

3G It'll turn out to look near enough a circle.

4B Eventually they become like a circle, or certainly to the human eye.

5G, L6B, U6G and U6B considered that, theoretically, it would not whereas 3B, 4G and 5B stated both interpretations:

5G A straight line will never go to make up a circle.

L6B It would look like a circle but mathematically it wouldn't be.

U6B It would look very much like a circle but you would still not have one continuous side.

Questions 15-18 were accompanied by computer graphics which illustrated the questions. This was judged, subjectively, to have a neutral effect on responses. Moreover, it was suspected that some subjects, in a larger sample, would realize the discrete nature of computer graphics and respond accordingly to questions we would wish them to consider in a continuous context. We thus omitted computer simulations in further investigations.

## DISCUSSION

The intention of this first study was, as mentioned on p.52 above, to evaluate the hypothesis that four concepts of infinity are possible, to evaluate items on limits and infinity for their power to reveal adolescents' conceptions and to *get a feel* for the area to be covered. By the end of the study it was clear that the four concepts of infinity hypothesis was a projection of what might be and **had little basis in the actual concepts of adolescents.** There was some use in investigating this, however. The dominant conception of infinity was that of the potential infinity. This was due to viewing infinity as a process, something that goes on indefinitely. Finitism existed in our subjects but stemmed not from an inability to conceive of infinity but rather from approximating in a physical world setting (where theoretical mathematical limits are unimportant). Robinson-like and Cantorian concepts, it appeared, found no real analogues in adolescents' thoughts.

Our evaluation of the utility of the items was as follows:

**Questions 1 to 4** Repeated subdivision of a line. Mathematically able adolescents can conceive of the infinite subdivision of a line. We felt that investigating new ground was more useful than reworking established results. It was thus decided not to include these questions in further studies. Moreover, we felt that further work on this question would require a deeper investigation into subjects' conceptions of lines and points. While such a study is relevant to the present study it was felt that broadening the present one to include adolescents' conceptions of lines and points was unwise.

In retrospect we feel that adolescents' understanding of the nature of repeated subdivision of a line should have been investigated further. Although Piaget (1956) found that adolescents in the stage of formal operational thought could conceive of the infinite subdivision of a line, Fischbein et al. (1979) found 55% of their subjects took finitist positions on this question (as we have seen above, p.55). It must be noted, however, that the percentages of Fischbein et al. were obtained from subjects of high, middle and low ability in mathematics. 64% of their subjects from the high ability group acknowledged the infinite nature of the process of subdivision. As our subjects were able (in that they had obtained O-level mathematics or were, as it emerged, to obtain O-level mathematics) our discrepency with Fischbein et al. is somewhat less than first appears. Nevertheless, this could have been examined more closely in subsequent studies.

Concern over this omission caused us to perform a late test. In June 1985, at Morecambe High School, five Fourth Year classes were visited by the author and the following question was written on the blackboard:

> Consider a line ________________ , halve it ________ , halve it again ____ , and again __ and continue.
> Will you ever reach a situation where it is impossible to continue halving it ? Explain your answer.

The Fourth Year was selected because pupils in it represented young adolescents (almost all of the subjects were 15 years of age at the time of the test). A day was selected when about half of the year was out on a trip. This enabled us to generate a good atmosphere with small classes. We particularly wanted to contrast middle and lower ability pupils with able pupils. Thus out of 10 sets we selected sets 1, 4, 5, 6 and 7 (set 1 being the most able set in mathematics). The more able sets had larger class sizes. Thus about 60% of the Fourth Year were in sets 1 to 5. 61 pupils were asked the question. Three of these gave silly (joke) responses. Of the remainder all but four in set 7 attempted to explain their answers. The responses were:

TABLE 4.1

| Set 1 | | Set 4 | | Set 5 | | Set 6 | | Set 7 | |
|---|---|---|---|---|---|---|---|---|---|
| No | Yes | No | Yes | No | Yes | No | Yes | No | Yes |
| 10 | 2 | 9 | 0 | 11 | 5 | 5 | 4 | 4 | 8 |

The sample was small and the test isolated in that no other question was asked. We thus cannot attach too much weight to the results. The results indicate, however, that recognition of the infinite nature of repeated subdivision of the line is related to the mathematical ability of pupils (as the 1979 study of Fischbein et al. indicated). This is not inconsistent with Piaget's claim, however, as he claims only that children at the stage of formal operational thought can conceive of unlimited subdivision.

The 'No' responses were accompanied by explanations that *there is always something left to halve* or *half of something can never be nothing*. In about half the cases the 'Yes' responses reflected a practical appreciation of the problem, e.g. *it will get so small that you couldn't see it*. This was the explanation of the two pupils in set 1 who responded 'Yes'. It cannot be assumed that such responses indicate a failure to appreciate the the infinite nature of the problem. Further probing is required in such cases. The remainder of the 'Yes' responses appeared to suggest that the infinite nature of the problem was not seen, e.g. *it will fade away to nothing* and *you'll eventually halve down to nothing.*

These results, in conjunction with the pilot study, indicate to us that mathematically able adolescents can conceive the infinite nature of repeated subdivision of a line.

**Questions 5 to 8** Infinite series

Questions 5 and 6 were useful and should be used in further investigations. The fractions in questions 7 and, especially, 8 distracted subjects from the main aim of considering convergence and divergence. These questions should be omitted in further investigations.

**Questions 9 to 11** Decimals and decimal sequences.

Questions 9, 9a and 10 were useful and should be used in further investigations. Question 11 would not be be used again as it was seen as (and is) identical to question 10.

**Question 12 to 14** Comparing cardinalities

These questions were useful but the four options presented in question 13 should be employed each time so as to avoid leading subjects to any one answer. The lines in questions 13 and 14 should be marked so that problems concerning assigning numbers to points or on the size of points do not arise; and two further questions using two dimensional sets of points should be included, one to compare the points on a line with the points on a square constructed on the line and one comparing the number of points in a square with those in an enclosing circle.

**Questions 15 to 18** As we have mentioned, questions 15 to 17 did not assist the isolation or analysis of concepts of interest and were to be omitted from further investigations. Question 18 was useful in revealing subjects' limit concepts. We were interested in what effects similar sequences of shapes had, however, and resolved to use another shape in future studies.

An area that was seen to be mistakenly under-examined by this study was adolescents' understanding of real numbers, especially recurring decimals, infinity as a number and infinitesimals. The role of language in affecting conceptions was also seen as an important factor that further investigations should address themselves to.

Finally larger samples were seen as essential. This first study caused us to agree with Fischbein et al. (1979, p.32) and Tall (1980b, p.282) that adolescents' conceptions (intuitions) of infinity are very sensitive to changes of wording, the context of a question and the mood of the subject. Nevertheless larger samples would give us data that permitted a more detailed analysis. We shall discuss the questions of item design in more detail, in relation to the main studies, in the next chapter.

## THE SECOND PILOT STUDY

The second formal pilot study took the form of a questionnaire administered to the Lower Sixth A-level mathematicians at Morecambe High School in November, 1982. Subjects had recently covered an introduction to limits and differentiation in their mathematical studies. The questionnaire was administered in a mathematics lecture. The author read out and explained each question. Subjects did not respond until this had been done. In question 3 subjects were asked only to put 'don't know' if they were very unsure. 30 students were present (20 female). 23 had SMP O-level mathematics passes, the remainder had traditional syllabi O-level passes. All the subjects were following an SMP A-level mathematics course which had, by the time of the questionnaire, covered an introduction to limits of sequences and differentiation.

The questionnaire was inspired by gaps left in the first pilot study and by a study of the work of Cornu (1980). The overall aim was to investigate the importance of language, especially of the phrases *tends to, approaches* and *limit.* We display and comment on the results below.

TABLE 4.2

1) A car has a maximum speed of 120 mph. It starts and speeds up without stopping. Does the speed tend to:

| | i) 100 mph ? | ii) 120 mph ? | iii) 150 mph ? |
|---|---|---|---|
| Yes | 14 | 30 | 6 |
| No | 16 | 0 | 24 |

In retrospect the physical context generated by the question was seen as obscuring perceptions of the concepts in that it invited a non mathematical use of *tend to*. It is then, difficult to say what the uniform divide on i) represents. There will be a time, in the everyday sense of *tend to*, at which the speed will tend to 100 mph. Is this what the 'Yes' responses mean or is there some other rationale ? This problem of interpretation is, of course, partly a fault of questionnaires in that they do not allow us to probe the intention of responses. The problem of interpretation is also due to the physical context, however, in that we are interested in examining the interference caused by everyday meanings in pure mathematical contexts. Nevertheless, it is interesting to note that the responses do generally concur with formal mathematical correctness.

**TABLE 4.3**

| 2) Is 1 bigger than 0.9999 (recurring) ? | Yes | No |
|---|---|---|
| | 29 | 1 |

As was expected, generic limit ideas were very strong. It was decided that future studies should attempt to probe subjects' rationales for both responses and should retest considerably later in the course to see if their ideas change.

**TABLE 4.4**

3) What do you think of the following sentences ?

| | True | False | Unsure |
|---|---|---|---|
| 3+h tends to 5 as h tends to 0 | 6 | 23 | 1 |
| 3+h approaches 5 as h approaches 0 | 12 | 17 | 1 |
| 3+h tends to 2 as h tends to 0 | 12 | 15 | 3 |
| 3+h approaches 2 as h approaches 0 | 19 | 10 | 1 |
| 3+h tends to 3 as h tends to 0 | 27 | 2 | 1 |
| 3+h approaches 3 as h approaches 0 | 26 | 4 | 0 |

The question was intended to examine students' conceptions of the phrases *tends to* and *approaches* in a numeric context. In particular to see if an implied sequence tends to or approaches a number above (5) or below (2) the true limit (3).

There appeared to be a difference in connotation between *tends to* and *approaches (approaches* being more acceptable in both the above (5) and below (2) cases). Also *tends to* and *approaches* 2 was more readily accepted than was *tends to* and *approaches* 5. This, presumably, is because the implied sequence is decreasing or going towards 2 and away from 5. There was strong agreement that both phrases were correctly applied to 3.

We felt that future studies should examine the words *limit* and *converges* as well and make the sequence explicit or present a graph of a function. We also felt that although looking at the above (5) and below (2) cases was interesting in terms of the interference of everyday concepts in mathematics, it was more pertinent to the main aim to study conceptions of what goes on around 3.

TABLE 4.5

| 4) Complete the following sentences: | Responses | |
|---|---|---|
| | 1 | others |
| $1+h$ tends to_____ as h tends to 0. | 26 | 0,-1,2,2 |
| $1+2h$ tends to_____ as h tends to 0. | 27 | 2,3,3 |
| $1+h^2$ tends to_____ as h tends to 0. | 28 | 0,2 |

We expected and obtained a very high percentage of correct answers to this question. It was intended to show that despite varying student conceptions, subjects nevertheless could write down correct responses to standard questions which do not really probe generic, or other, ideas.

5) In the graphs below can we say that *the curve tends to 0* as x gets larger and larger? Assume the pattern continues.

6) In the graphs below can we say that *the curve has 0 as a limit* as x gets larger and larger? Assume the pattern continues.

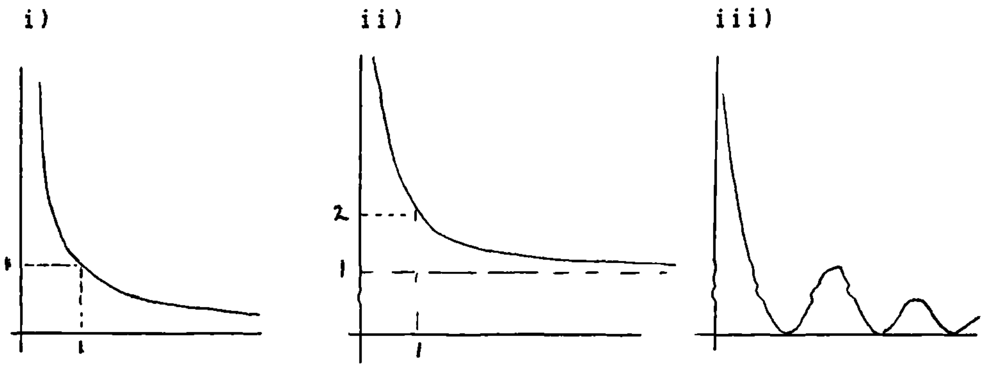
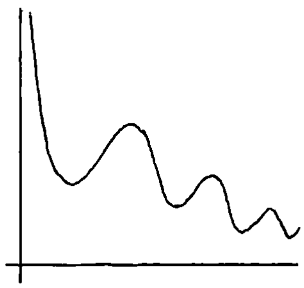



TABLE 4.6

| | i) Yes | i) No | i) ? | ii) Yes | ii) No | ii) ? | iii) Yes | iii) No | iii) ? |
|---|---|---|---|---|---|---|---|---|---|
| 5) | 26 | 4 | 0 | 7 | 22 | 1 | 14 | 12 | 4 |
| 6) | 17 | 13 | 0 | 4 | 25 | 1 | 22 | 7 | 1 |

The items generated a number of questions that would have to be studied in future investigations. Why does i) tend to but not have a limit 0 ? Is *limit* a stronger concept ? Does *limit* suggest that the curve will reach it whereas *tends to* does not ? Why does iii) reverse the trends in i) ? Is this because it touches 0 or because of the oscillations (it tends to 0 and then tends away from 0) ? What response would the curve on the right give ? Future studies should present sufficient curves to answer these questions.



7) Consider the pattern of numbers 0.9, 0.99, 0.999,...

Which of the following sentences are true of this pattern ?

**TABLE 4.7**

| | Yes | No |
|---|---|---|
| i) It tends to 0.999..(recurring) | 20 | 10 |
| ii) It tends to 1 | 22 | 8 |
| iii) It approaches 0.999..(recurring) | 19 | 11 |
| iv) It approaches 1 | 25 | 5 |
| v) Its limit is 0.999..(recurring) | 20 | 10 |
| vi) Its limit is 1 | 22 | 8 |

The responses show little difference between $0.\dot{9}$ and 1. Moreover, it was not the case that subjects were simply giving the same response to both $0.\dot{9}$ and 1 as the table below shows.

**TABLE 4.8**

| i/ii | iii/iv | v/vi |
|---|---|---|
| Yes/Yes - 12 | Yes/Yes - 14 | Yes/Yes - 1 |
| Yes/No - 8 | Yes/No - 5 | Yes/No - 19 |
| No/Yes - 10 | No/Yes - 11 | No/Yes - 7 |
| No/No - 0 | No/No - 0 | No/No - 3 |

It was seen as important to follow up these questions in future studies to gain an understanding of the rationales behind such responses. Indeed it was felt with all the questions that future studies must have follow up interviews to probe typicality and extreme responses. Moreover, it was felt that the three phrases *tends to*, *approaches* and *limit* should be supplemented by using *converges* and using as many of these phrases as possible in each question.

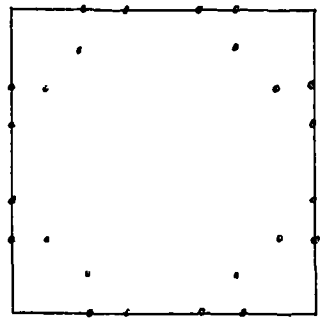
# OTHER EARLY STUDIES

Two further early studies are worthy of comment. In April 1982 five mathematically able First year pupils (2 girls and 3 boys) from Morecambe High School were individually interviewed for five to ten minutes. The interviews were recorded and transcribed. Portions of the girls' responses were, unfortunately, corrupted by a faulty tape.

The first question was: Is there a number at the end of 1, 2, 3, ... ? All responded 'No', the numbers go on and on'. We were interested to see if they viewed infinity as the largest number. Although the sample size is too small generalize it was nevertheless clear that viewing infinity in this way was not an immediate conception. We then asked them if these numbers could be collected together to form a single set. All understood the question and all responded 'Yes'. The purpose of this question was merely to see if infinite collections were possible. It appeared they were.

The idea of infinite collections was taken further with the next questions. We drew a line, pointed to the half way point, pointed to the two half way points on the left and right segments and continued the process several times. Indicating that the process continued we asked if the total number of half way points could be collected. We were interested in observing if a more complicated infinite collection affected responses. It did. One response was lost (due to the faulty tape), two responded 'Yes' and two responded 'No'. One of those responding 'No' said that the different distances involved caused him to say 'No'.

The last question on infinite collections posed a similar problem



in two dimensions. A series of diagrams was presented in which the half way points idea was extended to points in the square. The diagram on the right resulted and subjects were asked to imagine the pattern continuing. Again one response was lost, two responded 'Yes' and two responded 'No'. Considering both questions the responses were NN, YY, NY and YN. The subject who made the last response said it would be 'Yes' if the points were joined up. Neither question was felt to be very successful in that, despite probing, it was difficult to determine whether subjects were imagining infinite collections or simply very large finite collections.

The remaining questions looked at infinite aspects of real numbers. The partial sums of 0.1+0.01+... were obtained and subjects were asked if there was a final answer. All responded 'No, the numbers go on forever'. Next $0.\dot{3}$ was considered. All knew that $1/3=0.\dot{3}$ . When asked if there was any problem in saying this only one said 'Yes', saying that it never quite got there. Finally we were curious as to whether infinite sums posed problems when $0.\dot{3}$ was added to 0.3. All responded that $0.\dot{3}+0.\dot{3}=0.\dot{6}$ and there was no problem with this.

Despite the fact that all subjects were clearly at ease during the interviews it was difficult to get more than yes or no responses from them. This was considered to be due to their age and mathematical immaturity. It was felt that considering a wide age band in future studies would widen the study at the expense of detail.

Another very small scale experiment was conducted after Prof. Schwarzenberger, of Warwick University, suggested that it would be useful to get an A-level mathematics class to keep individual diaries detailing problems with calculus and infinity. This appeared to be a potentially fruitful method of data collection and in December 1982 a Lower Sixth mathematics group were asked to:

> Write a page on what you found difficult about calculus. Comment on what it is (or does). Which approach was easier - the one with limits or the one with infinitesimally small numbers ? Do you believe in infinitesimally small numbers ? What does "limit" mean in mathematics ?

Seven out of a class of ten responded in essay form. We reproduce the essential parts of each essay below. The initials refer to subjects' initials. Three dots indicates that a sentence has been omitted.

SD Calculus is the study of functions and derived functions. ... I must admit I find it difficult to understand either method, particularly the one concerning limits. The derived function is the gradient function. If a graph of a function is drawn then the graph of the derived function can be drawn from it. This I found easy to do. ... In mathematics the limit is the furthest extent that something will go towards. ... I can understand the idea of infinitesimally small numbers. ...

JM Calculus generally is a particular method of calculating or reasoning. Differential calculus is the study of rates of change. A limit in Maths is the quantity which a function can be made to approach as closely as is wished, but it can never be reached. ... ... I can believe in an infinitesimally small number to a certain extent, but, however small a number becomes, won't there always be one just a tiny bit smaller ? ...

SC ... ... ... The approach with infinitesimally small numbers was easier to grasp since when talking about a 'limit' or tending towards it, you can never actually reach it, thus in maths it is a kind of hypothesis, meaning the destination which the numbers are aiming for but never actually get there.

SM ... It was "sickening" to find out about the formula for deriving functions after having slogged through both these very difficult methods. I found the method of limits easier than the $\delta x$ and $\delta y$ method. However, the most difficult thing was finding derivatives using graphs. ... ... I do not believe in infinitesimally small numbers because whatever number a person says, I can quote a smaller number. ... However, I think there is a point on the number scale beyond which numbers are of no use. In maths the word "limit" means the furthest you can go. If a sequence reaches a limit, it cannot proceed any farther.

LN ... ... I didn't find either approach easier than the other, but for some reason preferred the one with the limits. I do not believe in infinitesimally small numbers but I've learnt to work with them because it is necessary to do so. Anybody with the slightest bit of logic in them must realize that it is impossible to have a number which is smaller than any other number, but which is not zero. The limit of a graph is that point or line which all the other points lead to but never actually reach.

KR ... ... The easiest approach to calculus for me was the one with the infinitesimally small numbers, though I do not believe in them. The word 'limit' in mathematics means a restriction at one end at a range of numbers ... ...

AG ... A limit in maths is where a set of numbers approach one number until they eventually reach that number. I don't mind which way is used and I can agree that there is one infinitesimally small number. ... ...

It should be noted that the author taught the group and used the SMP approach to differentiation (since subjects were to sit an SMP A-level paper). It was only after this had been done that one lesson was set aside to talk of other methods, in particular infinitesimal ones. We were careful not to over explain limit ideas but to follow SMP ideas. Thus, at no point were subjects told either that a limit could be reached or that it could not be reached. Subjects then, are displaying their own interpretations of standardly taught concepts.

Again the sample is too small to generalize. It is, however, interesting to note the many differences between subjects of similar ability so early on in a course for which they all received the same instruction. Moreover, differences in prior instruction were slight: SC did a traditional O-level with some calculus, KR did a traditional

O-level with no calulus and the rest did SMP O-level with no calculus.

Neither limit nor infinitesimal methods were seen as easier by all the subjects and although some reacted hostiley towards infinitesimals at least one, LN, felt she had to come to terms with them. This must cause us to question the claim that an infinitesimal approach is a more intuitive approach to students (Marchi, 1980). We must not make too much of this, however, for the teaching programme was not structured in order to be assessed. Had we been evaluating a programme of instruction then all ideas presented in the classroom would have to be thoroughly examined. Regardless of teaching programmes these remarks left us with a resolve not to omit an examination of students' conceptions of infinitesimals from future studies.

Limit clearly had many meanings: furthest extent; restriction; approach and eventually equal; and approach and never equal. This experiment was carried out before the second pilot study and this diversity of interpretation as to what a limit is was one reason for the various questions on limits in that second pilot study.

Data collection of this kind is certainly interesting. There are several reasons, however, for not employing it as a main data source: i) As a long term scheme it would be a burden on subjects; ii) By their continued reflection on the concepts our subjects could easily become atypical subjects; iii) The study would become much more an examination of a style of instruction, this would be interesting but was not our intention; iv) We would either have to tell the subjects what to write about and so bias their perceived problems or give them a freedom to write about whatever they like and in doing so risk not capturing the ideas we are mainly interested in. With regard to last point the parts omitted from the essays were *I find maxima and minima difficult. Sketching graphs of complicated functions is hard.*, etc. While this is useful for the teacher to know it is not particularly illuminating from the point of view of this research).

# CHAPTER FIVE

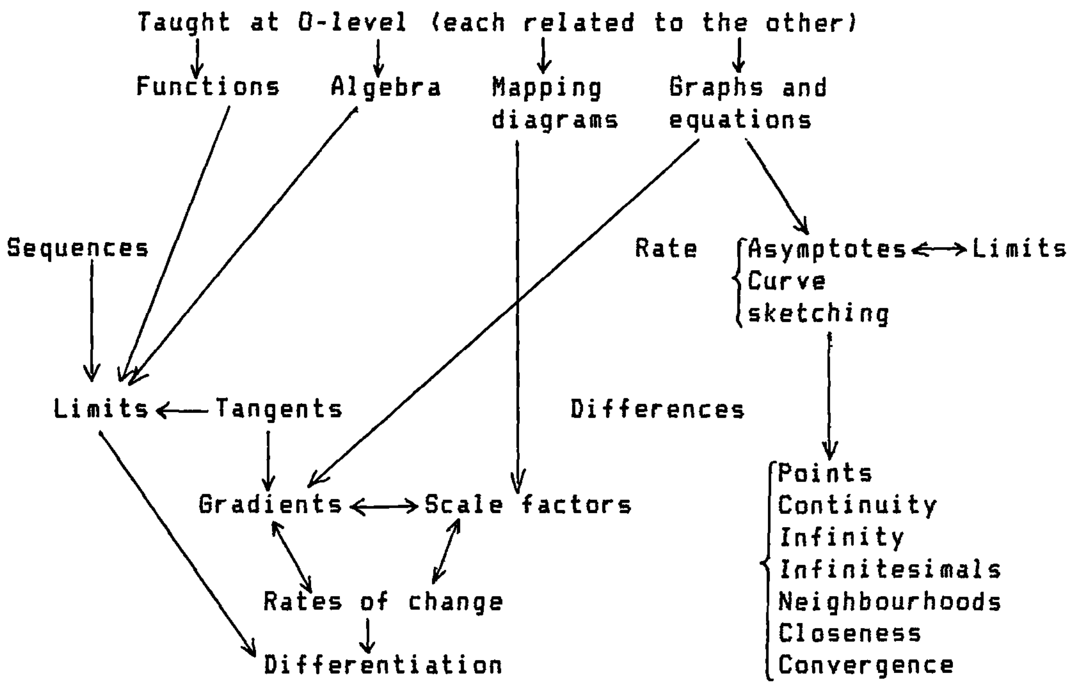
# THE QUESTIONNAIRES

The method of investigating adolescents' understanding of limits, infinity and related concepts was a written questionnaire followed by selective interviews and then a larger sample questionnaire. In this chapter we present details of the questionnaires, leaving the analysis of the results obtained until the next chapter.

## THE CONCEPTS STUDIED

Before going into the details of the questionnaires we must examine what concepts are to be studied and why they are to be studied.

Initially all taught concepts related to differentiation were to be investigated. Differentiation was chosen as being the usual first step in a calculus course. Integration, it was felt, would make the study too large and unmanageable . The following schematic diagram of the conceptual hierarchies was produced.




(The concepts in the block at the bottom right are all related to each other. None are taught and all are closely bound up with the real number concept.)

The concepts were broken down into three categories: a) those taught prior to, and made use of in, a calculus course - functions, graphs and equations, gradients, rates of change, tangents, decimals and algebraic manipulation ; b) those that would not be taught explicitly but would be present in a calculus course - infinity, infintesimals, continuity, closeness and neighbourhoods ; c) new concepts that would be formally taught in a calculus course - limits, differentiability and convergence.

Further work suggested that the concepts in (a) enlarged the proposed study beyond what was feasible (we could, after all, investigate adolescents' concepts of mathematics !). It was thus proposed to consider these concepts only when they directly impinged themselves on students' concepts of limits and infinity. Continuity, closeness and neighbourhoods are important topological notions in real analysis (which was their initial reason for being included) but tying these advanced concepts to the concepts within adolescents' cognitive experience seemed futile (Darke (1982) in reviewing research on Piaget's topological primacy thesis concluded that the evidence for this thesis was scant and complicated by attempts to fit results to neat theories, i.e. structuralism). Continuity and neighbourhoods were thus dropped as too advanced concepts. Closeness was retained (as the least advanced and thus most accessible of these concepts) but, as we shall see, did not lead to any constructive results. Differentiability was omitted when it was decided to compare responses between a group studying calculus and a group not studying it. Clearly we could not give questions on differentiation to the latter group. The real number concept was added to the set of concepts to be investigated because the completeness of the reals rests on limit ideas and this is relevant in students' understanding of limits and infinity, e.g. Is $0.\dot{9}<1$ ?

The concepts studied are thus infinity (including the infinitely large, the infinitely small and infinite aggregates), limits (of sequences, series and functions), convergence and real numbers.

## QUESTIONNAIRE DESIGN

Written questionaires are the quickest way to obtain responses from a large number of subjects but they must ask unambiguous questions which permit an analysis of the data in line with the concepts under investigation. Care must be taken in analysing data obtained from questionnaires. A given response may be made for a variety of reasons and the analyst will not have recourse to probe as s/he would in an interview situation. The possible misinterpretations made by the analyst clearly vary from question to question. However, even though we may not know why a mistake on a question such as *What is 1/0.01 ?* is made, it does tell us that the subject is not fully competent with all operations on real numbers. A reply of 'infinity' or 'undefined' to the question *What is 1/0 ?*, however, may be one of several the candidate may offer and does not reveal uncertainties or qualifications that the subject may make clear in an interview. It was, nevertheless, felt that a written test would give us knowledge of the subjects' unqualified, global beliefs/knowledge (e.g. subjects clearly believe that there is not a biggest number). Interviews were to provide *flesh* to this data (e.g. a typical qualification was *Well, I suppose you could say infinity is the biggest number, but its not really a number*). We should not be too dismissive of these global results obtained by the written tests. The fact that we now know that

British adolescents do not immediately acknowledge the existence of a largest number is added to mathematics education's knowledge. On the other hand, however, it was felt that over elaboration (especially statistical) of the expected data would not be appropriate.

It should be noted that the questionnaires are questionnaires and not performance or intelligence tests and that students' conceptions and intuitions about these concepts are open to much variation. Errors of measurement of a subjects' *true score* are thus virtually impossible to assess and checks on reliability are irrelevant. For the same reason the only form of test validity suitable here is content validity. This was checked by the judgement of the supervisor and by making every effort to ensure that the data collected was *dependable* in the sense of Diesling (1971):

> The dependability of a source of evidence is the extent to which its output can be taken at face value relative to other sources of evidence, in the process of interpreting manifold evidence ... none is ever completely free from the need for cross-checking and reinterpretation.

The experiments partook of features of both cross-sectional and longitudinal methods. Two questionnaires were administered. These were almost identical (the second clearing up some ambiguities of, and eliminating questions which were not useful, in the first). These are contained in Appendix A. The first questionnaire was administered twice (at the beginning and at the end of a school year). The second was administered at the end of the following school year. For ease of reference we shall call the first questionnaire *Questionnaire 1* and the second *Questionnaire 2*. When we wish to distinguish between the first administration of Questionnaire 1 from the second we shall refer to *Questionnaire 1.1* and *Questionnaire 1.2*. Questionnaire 1.1 and Questionnaire 1.2 were followed by selective interviews. Questionnaire 2 was not.

This study is neither purely cognitive nor purely concerned with teaching. We were/are interested in all adolescents conceptions of limits and infinity and also in whether a first course in calculus affects these conceptions. Filtering data through the experience of a course of instruction may provide useful information for such a course but an analysis of taught concepts is not the goal of this study. Nevertheless, given that some subjects were going to have a period of instruction an experimental group (doing A-level mathematics) and a control group (similar in as many respects as possible, in particular, having passed O-level mathematics) was deemed necessary. Experimental and control groups were used in both administrations of both questionnaires. To probe typicality and possibly to isolate extreme naivity and sophistication, a small group of Fourth Year O-level mathematics pupils from the same school and a small group of First Year university mathematics education students were also given one administration of Questionnaire 1.

The timing and the number of administrations of Questionnaire 1 was given careful consideration. A greater number of administrations was initially desired. Administering the questionnaires in parts over a period of time (to keep them short) was also desired. Both of these features, on reflection and in the opinion of more experienced researchers, would have made the questionnaires intrusive and created atypical subjects (i.e. they would think about the concepts more than their peers not doing it).

September and May were chosen as times for the administrations, giving subjects eight months to *forget* their previous answers. To ensure that both administrations of Questionnaire 1 measured the same behaviours no revision of Questionnaire 1.1 was made in Questionnaire 1.2.

With the exception of the few done by university students all administrations were supervised and took about 45 minutes. Questionnaire 1.1 and Questionnaire 1.2 were supervised by the author and each question was read out in a uniform manner. A friendly atmosphere was established. The following opening remarks were made:

> I am interested in your immediate responses to the following questions. Do not worry about getting them wrong, you will not be assessed on them. They are important, however, so please take them seriously.
>
> In many of the questions you will be asked to circle Yes / think so / ? / think not / No. Try to use '?' only when you are very uncertain. I am only interested in your immediate responses. If a question seems similar to a previous one don't go back and try to make the two answers *fit*. Treat each question as an isolated queston.
>
> Many people feel they should *chop about* in multiple choice questions. They feel that if they pick the first box every time, then they have done something wrong. Please don't think this. If

you find there is a pattern to your answers please don't consciously try to continue it or break it. Please don't read ahead. Please do not write an answer until asked to do so. I will read each question out.

Subjects in Questionnaire 2 were supervised by the Head of Mathematics of the volunteer schools taking part. These subjects worked individually at their own speed but did not consult each other. The teachers clarified points as they arose. The title page made the essential reassuring points made verbally to those taking Questionnaire 1.

None of the many subjects who took Questionnaire 1, who were asked by the author, thought that the questionnaire was too long or too difficult. Impressions from the schools taking Questionnaire 2 revealed only one school where some subjects (all of whom were in the control group) thought some questions off putting. In the subjective, but honest, opinion of the author, however, the questionnaires were, by the vast majority, completed without undue worry or exhaustion.

## THE ITEMS

In Questionnaire 1 subjects were instructed to respond to the Yes/No questions by a mark on a five point scale (Yes / think so / ? / think not / No). As we were investigating immediate conceptions (intuitions) it was felt that such a scale would allow strength of conviction to be recorded. It emerged, however, from the analysis of the data and the interviews that this was an over subjective and poorly controlled factor. The scale was thus compressed to three points, Yes / ? / No. The three point scale was employed in Questionnaire 2. A point of interest for those concerned with gender differences in mathematics is that it was primarily girls who used the 'think so' and 'think not' categories.

To allow some comparison with previous research (in particular that of Tall, Fischbein and Cornu) items from other questionnaires were used along with specially designed items. This had the extra advantage of providing a partial check on the dependability of the data collected.

Each item was thoroughly examined to determine what aspect(s) of subjects' understanding it was testing. Items were initially designed to examine knowledge, comprehension, application and analysis (in the sense of Bloom's taxonomy (Bloom, 1956)) of the concepts isolated for study. After much work it became clear that this approach was forcing an unsuitable tool on the study and it was dropped. This initial method did, however, (and this is an important factor) focus our attention on the importance of each item. Moreover, it helped to ensure that the questions included, as far as possible, covered the concepts we had decided to study, were relevant, were sufficient for analysis and that the rationale for the inclusion of each item was clear.

No formal method of item analysis was utilised because we were not looking for items that would discriminate between good and poor performers (our questions did not, in the main, have correct answers in terms of school mathematics).

A trial run of Questionnaire 1 was conducted on two O-level mathematics Fourth Year pupils and one A-level mathematics Lower Sixth student. None of the subjects were to take part in the main questionnaire. Apart from clearing up ambiguities a trial analysis of this data convinced us that Questionnaire 1 was sufficient for our purposes.

Henceforth we shall use the notation Q1.1 and Q2.1 to refer to question one in Questionnaire 1 and Questionnaire 2 respectively. Questions were not numbered in Questionnaire 2 but are here for ease of reference.

It is usual for a questionnaire to begin with several questions that will not be analysed but give subjects a chance to *warm up*. In Questionnaire 1 questions one and two were used for this purpose. Question two was also to be used to help categorize subjects as self assured or not. In Questionnaire 2 this *prelude* phase was obtained by the requests for personal details on the title page. Questions of a particular type (on infinity as a number, on cardinality, on series, etc.) were sometimes grouped together so that subjects would apply the same criteria to all questions and sometimes separated by different types of questions so that comparison with another of the same type was not immediate. The rationale for the layout was very subjective.

Our first questions (Q1.3, Q1.4, Q2.1 and Q2.2) were very simple: *Is there a largest number?* and *Is there a smallest number, greater than 0?* Responses would tell us what subjects' immediate conceptions on infinity as the number at the end of the number line and infinitesimals were. We appreciated that some subjects would have finitist conceptions but we would have to wait for the interviews to examine this (*greater than 0* was not included in Questionnaire 1 but was read out, in each adminstration, by the supervisor).

Q1.7 (Q2.3) *What is 1/0 ?* was included for comparison with Q1.3. 'Infinity' was expected to be the response of many of the subjects and this, coupled with the expected 'No' response to Q1.3, would demonstrate that infinity was not generally seen as a specific entity but as a vague generalization for a large number or as a process (0 keeps going into 1 with remainder).

Q.1.14 (Q2.9) *What is* $1/(1-0.\dot{9})$ *?* was included for comparison with 1/0 and was specifically inserted after *Is* $0.\dot{9}<1$ *?* The latter was included to examine subjects' conceptions of real numbers (specifically, their conceptions of infinite recurring decimals). We were sure that the vast majority of subjects would reply 'Yes'. However, we did not know if the first year of a calculus course would effect this response. We were thus particularly interested in obtaining data on this from Questionnaire 1.2. Now if $0.\dot{9}<1$ then $1-0.\dot{9}$ will be an infinitesimal of sorts. We were interested in finding out whether the reciprocal of this was conceived of as different from 1/0 (perhaps 1/0 would be undefined but $1/(1-0.\dot{9})$ would be infinity).

Q1.22i,ii (Q2.11,12) asked subjects to imagine infinity as an enormous number. By asking *Is* $\infty+1>\infty$ *?* and *Does* $1/\infty=0$ *?* we sought to examine the arithmetic properties ascribed to infinity as a number. Similarly with Q1.23i,ii (Q2.14,15) *Does* $2+s=2$ *?* and *Does* $2xs=s$ *?* we sought to examine the arithmetic properties ascribed to a hypothetical infinitesimal, s. Both of these questions were motivated by a desire to know how ideas that could arise in a nonstandard elementary calculus course would be received. Q2.13 and Q2.16 *Is this how you think of infinity ?* and *Can you believe in such a number ?* were unfortunately not included in Questionnaire 1. We found ourselves wishing we had included questions along these lines when we started analysing the data from Questionnaire 1 and thus included them in Questionnaire 2. This was especially important in Questionnaire 2 as we would not have recourse to interviews to clear matters like this up.

Q1.24,25 and 26 (Q2.17,18 and 19) asked, respectively, *Can you add 1+1+1+... and get an answer ?*, *Can you add 0.1+0.01+... and get an answer ?* and *Can 1/9 be defined as 0.1+0.01+... ?* They were the only questions on series included in both questionnaires. As we have seen, p.59, the pilot studies indicated that sophisticated questions on series were beyond the immediate grasp of most students. We thus kept the questions very simple (avoiding fractions) and included one divergent and one convergent series. Interviews indicated that the phrase *and goes on forever* suggested an impossible situation. To minimize unwanted suggestions we avoided this in Questionnaire 2. By the time we started analysing the data we were sorry we had not put in questions that would examine whether the mathematicians recognized convergent series and thus after some trials (described in Chapter Six) inserted Q2.50, which asked subjects to place five given series into two groups of their own choice. This was separated from the questions above to minimize the transfer of cognitive problems generated by these questions (it was clear from the interviews that subjects experienced cognitive conflict when claiming that 0.1+0.01+... could not be summed but did define 1/9. We wished to put this conflict behind them before asking them further questions on series). Q1.25 (Q2.19) was included to see, regardless of its legitimacy, if a series could be used in defining a real number. We did not clearly know what to expect but, as we shall see in Chapter Six, this proved a very interesting item.

Before considering subjects' conceptions of cardinality problems it seemed essential that we establish whether or not they could conceive of infinite collections. This was done by the first two parts of Q1.12 (Q2.6,7) which asked if N and/or the decimals numbers between 0 and 1 could be regarded as single sets. We included both to see if there was any difference between discrete and continuous sets. We were not primarily interested in whether or not subjects had Cantorian ideas but, rather, in examining the reasoning they employed. Cardinality problems themselves were covered by Q1.9, 12 (part 3), 15, 20 and 34iii (Q2.4, 8, 10, 20 and 67 respectively). The questions cover, respectively, comparison of: discrete sets, both unbounded (the natural numbers and the even numbers); an unbounded discrete set with a bounded continuous set (the natural numbers and the real interval [0,1]) ; a bounded and continuous one dimensional subset with a bounded two dimension superset ([0,1] and [0,1]x[0,1]); a bounded and continuous one dimensional subset with a bounded one dimensional superset ([0,1] and [0,10]); and two bounded two dimensional continous sets, one a subset of the other (a circle containing a square). Other permutations were open, e.g. an unbounded discrete subset of unbounded continuous set, but these five were considered sufficient for our expected analysis. We tried to ensure that the options covered all possible responses and brought the format of Q1.34iii in line with the format of the other questions in Questionnaire 2 (that is we gave the options: more in one / more in the other / same in both / can't compare).

The remainder of the questions consider subjects' conceptions of limits and the effect of language, in particular the effect of the phrases *tends to*, *limit*, *converges* and *approaches*. Taback (1975) used some game-like questions but found that non-mathematical contexts may influence subjects responses by encouraging subjects to use everyday meanings of *limit*. We believed everyday meanings would enter regardless of context but that it would be wise not to encourage this. Thus, apart from asking subjects to write sentences using the four phrases in Questionnaire 1.1, we restricted our questions to mathematical contexts. Questions were designed to examine subjects' conceptions in both arithmetic and geometric settings. Many were suggested by the work of Cornu (as we have mentioned in our report of the second pilot study). Questions on the four phrases represent about half the questions on the questionnaires but subjects spent considerably less than half the time on them as they were grouped to enable them to answer quickly.

Q1.35i, ii, iii (Q2.21, 22, 23) ask subjects to complete

*1+h tends to_______ as h tends to 0.*

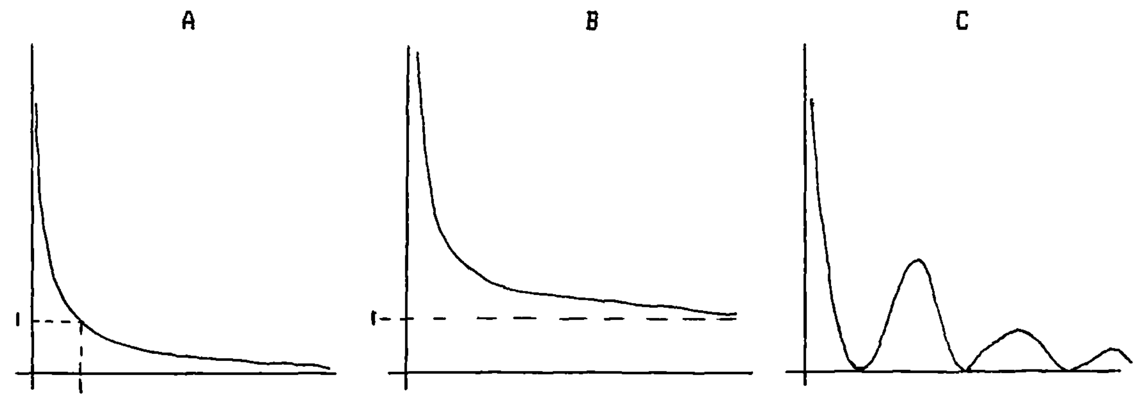
*The limit of* $(2+h)^2$*, as h tends to 0 is _______.*

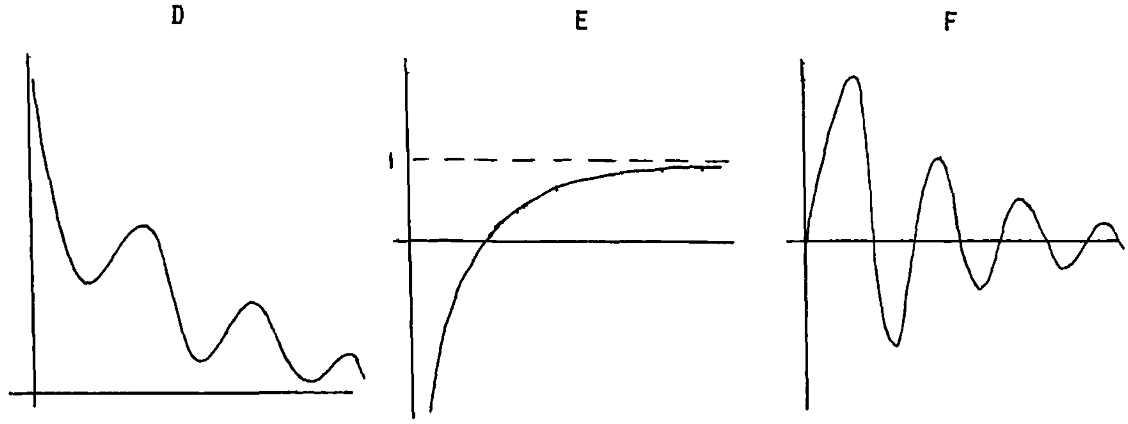
*1, 1/2, 1/4, ... converges to _______.*

These were inserted because they reflect the kind of questions sometimes given to A-level mathematicians but can nevertheless be given to subjects not studying A-level mathematics. We suspected that the A-level mathematicians would mainly give formally correct answers but that the control group would not. If this pattern emerged here but not elsewhere, then this would lend support to the thesis that A-level mathematics courses produce students who can give formally correct answers to standard questions without advancing their basic conceptions of limits and infinity.

Q1.29i,..., viii (Q2.53,..., 60) considered the sequence 0.9, 0.99, 0.999, ... Subjects were asked if the four phrases were applicable to $0.\dot{9}$ and/or 1. The rationale for inclusion was to examine the effect of the four phrases in the context of an arithmetic convergent sequence and to examine (by comparing responses for $0.\dot{9}$ with those for 1) the extent of generic limit concepts. The latter was examined in a geometric setting by Q1.27 (Q2.51), which presented a sequence of jagged function decreasing in height. To check that subjects' interpretations of the responses were consistent we included another question (Q2.48) in Questionnaire 2 presenting a converging sequence of nested triangles . As we would not have recourse to interviews to clarify matters and because we found that we had wanted this information in Questionnaire 1 we added the extra questions (Q2.49 and Q2.52) asking subjects whether they imagined the situation theoretically or in terms of drawing.

Q1.30A,..., 33F (Q2.24,..., 47) were included to examine subjects' interpretations of the four phrases with regard to functions presented geometrically.






We were interested in whether subjects focussed on the features mathematicians focus on. In particular we wanted to know whether touching 0 and whether being strictly monotone were essential features for any or all the phrases to hold true. Two of the functions do not approach 0 but one of these went through 0. Three of the functions were not monotone and of these, two repeatedly touched zero (one going through and one just touching).

Q1.36i,..,iv,..,37iv (Q2.68,..,72) were intended to examine the same phenomena but in an arithmetic setting. We did not know if a geometric or arithmetic setting would make any difference but it is certainly of interest to find out. We intended to examine the effect of all four phrases but this appeared to make the questionnaire too long. We compromised and included questions on what appeared to be the two phrases most commonly used in A-level courses *limit* and *converges* (and, as it emerged, the two most difficult for the subjects to understand).

Analysis of Questionnaire 1 and interviews led us to believe that subjects had a very different classification of types of numbers ($0.\dot{9}$, for example, was not, somehow, proper). To examine this we designed and tested a further question asking subjects to indicate, on a five point scale, which of 9, -9, 1/9, 0.9, $\sqrt{2}$, $0.\dot{9}$, $\pi$, $\infty$, $1/\infty$, 1/0, $1/0.\dot{9}$, $1/(1-0.\dot{9})$, sin 32° and $\sqrt{-1}$ were proper numbers. To gain more data we included Q2.61,..,66 in Questionnaire 1. We reduced the original set of numbers given because, again, we were worried about the overall length of the questionnaire.

**Several questions from Questionnaire 1 were not inserted in Questionnaire 2.** Q1.5 *Write down a number between 2.105931 and 2.10604* was included to check subjects' facility with decimals. With few exceptions subjects answered correctly. The only one who did not, and who was interviewed, immediately corrected his answer in the following interview. We saw no need for further testing in Questionnaire 2 and omitted the question there.

Q1.6 and Q1.13 *What is 1/0.001 ?* and *What is 1/(1-0.99) ?* were included to prepare subjects for Q1.7 and Q1.14 *What is 1/0 ?* and *What is* $1/(1-0.\dot{9})$ *?* Few in the group doing A-level mathematics got these wrong but an average of 43% got these wrong in the control group (mainly those with grade C at O-level, but several with grade B as well). While this is worrying in terms of standards at O-level and interesting in that we cannot assume that 1/0 will be understood by all subjects, the questions did not seem sufficiently useful as warm up questions to justify their extension of the length of the questionnaire and were thus omitted from Questionnaire 2.

Q.1.11 *Is there a number smaller than* $1-0.\dot{9}$ *?* was intended to examine if subjects, claiming $0.\dot{9}<1$, would view $1-0.\dot{9}$ as the smallest non zero number. Interviews revealed that a number of subjects were very confused by this question. We did not want questions that gave unclear responses and thus omitted it from Questionnaire 2.

Subjects (with one exception in the control group in Questionnaire 1.2) were unanimous that the sequence 0.1, 0.01, ... in Q1.21 did not get to 0. This is perfectly reasonable - it will not get to 0! The question is really covered by Q1.36i and Q1.37i, where the same sequence is presented with the questions *Does the sequence have a limit ?* and *Does the sequence converge ?*, and was thus not included in Questionnaire 2.

Q1.34i, ii were suggested by Orton's study of functions (see Lovell 1975). Q34i *Can I get to every point on the circumference this way ?* was included to check that continuity was observed by the subjects (It generally was in both groups in that the overall response was 'Yes'). Q34ii *Suppose two points are very close on the square. Will the corresponding points on the circle be very close ?* was to investigate closeness. It was thought that subjects may have ideas corresponding to topological ideas of neighbourhoods. The main response in both groups was 'it depends' and this was not elaborated on in interviews (indeed could not be in the sense of subjects saying *It's all just so relative*). Our initial reservations about examining topological notions were confirmed and we did not pursue these questions very far in the interviews. The questions were omitted from Questionnaire 2.

Q1.28 asked subjects to write four sentences, one each using the phrases *tends to, converges , approaches* and *limit*. They were told that the context need not be mathematical. One administration was considered sufficient for the purpose of gleaning their usual everyday connotations. It was thus given only to those doing Questionnaire 1.1. In Questionnaire 1.2 subjects were asked to write one sentence using the word *limit* but not in the sense of speed limit (almost all had used this in Questionnaire 1.1). We left space for the subjects taking Questionnaire 2 to comment along these lines. This was the only optional question (we put it at the end in case time was running out).

Q1.8 *Sketch the curve $y=1/x$* was omitted from Questionnaire 2 because subjects took so long completing it (in trial runs of the questionnaire this was noticed, but a large number of students doing it took much longer than the three who took the trial run). It was initially included to see if subjects who may not consciously see 1/0 as infinite or indeterminate would, in practice, see this. The results were not without interest, however: of the 27 subjects in each group, 20 in the experimental group and 9 in the control group sketched the graph correctly in Questionnaire 1.1. In Questionnaire 1.2, 25 in the experimental group and 11 in the control group sketched the graph correctly. However, only three in the control group (in each administration) gave 'infinity' or 'indeterminate' as responses to *What is 1/0?*

Q1.16 to 19 formed a block in which conflict was purposely induced. *Does $0.\dot{3}=1/3$ ?* *What is $0.\dot{3}\times 2$ ?* *Does $0.\dot{3}\times 3=0.\dot{9}$ ?* *Does $0.\dot{9}=1$ ?* The result was that although very few in either group gave answers other than 'Yes', '$0.\dot{6}$' and 'Yes' to the first three questions, this did not affect their intuition that $0.\dot{9} < 1$. This is interesting and we shall look at it again in later chapters but the conflict here must be examined closer in an interview situation. Because subjects taking Questionnaire 2 were not to be interviewed and because we were attempting to keep the length of the questionnaire within reasonable bounds, we did not include this block of questions in Questionnaire 2.

## THE SAMPLES

Many students took part in our tests. There were those who took part in the pilot studies and the small group of Fourth Year pupils and the university students used to probe extreme responses. We shall regard as our samples, however, those who took part in Questionnaire 1 and Questionnaire 2. Only one student, in Questionnaire 1, was also used in the pilot studies. It should be noted that neither sample was randomly, normally or otherwise distributed (we got what we could !). We wanted subjects capable of understanding the concepts of calculus and took for our criteria for this a pass (A, B or C) at O-level mathematics.

The sample for Questionnaire 1 was made up of 27 pupils doing SMP A-level mathematics and 27 similar Lower Sixth pupils *not doing* A-level mathematics. The subjects all went to the school the author teaches in, a large comprehensive in Morecambe, a resort area in the North West of England. Most of the subjects did SMP O-level mathematics at this school though some came into the Sixth Form from other schools and had done other boards at O-level. The sample reflects a wide variety of social bckgrounds.

We are aware that sampling in one's own school has pitfalls both in terms of possibly introducing students to ideas that are to be examined and in terms of subjects' emotional reaction to the supervisor. On the first point every effort was made to ensure that this did not happen. A talk on the work was delivered to the Mathematics Department, prior to the first administration of Questionnaire 1 and all teachers taking A-level groups (six in all, including the author) agreed to try and avoid introducing topics that would prejudice the responses (if at all possible - there was one noticeable slip where a teacher got involved in discussing $0.\dot{9}$). On the second point there is little one can do. The author is, however, neither disliked nor the most popular teacher in the school so, it is hoped, extreme reactions that may bias the data rarely arose. There is a positive side to research in one's own school in that the researcher is aware of all the factors likely to affect the results.

There were initially (Questionnaire 1.1) 31 subjects in both the experimental and control groups. This seemed satisfactory as 30 is the generally accepted cut off point between small and large samples. None of the subjects volunteered (Questionnaire 1.1 was sprung on them during their first week and it appeared to them as just part of the proceedings). They appeared quite happy to oblige and on being given the choice to leave or not, none left.

A larger number (than 31) of the non A-level mathematics students actually sat Questionnaire 1.1 but several had not passed O-level mathematics and several more were going to do a non-exam 'Mathematics for Sixth Form Scientists' course that included calculus. Both of these groups were excluded in our data. By the time it came to Questionnaire 1.2 several of the original sample had left school and several others were on long term illness. Questionnaire 1.2 was given during a General Studies period. The total who completed both questionnaires was 27 in each group. The details are displayed at the end of this chapter.

The sample in Questionnaire 2 was made up of 190 pupils from six English schools. As the experimental group in the Questionnaire 1 was doing SMP A-level it was felt that the sample in Questionnaire 2 should too. It was impossible to control all variables but SMP A-level was one we felt we should not compromise on. We outline the essential features of SMP in Appendix B. Our basic reasons for insisting on using subjects following this course, however, are: i) SMP A-level more or less follows the SMP books (1 to 3) whereas other exams follow a wide variety of books. We thus have a very good idea of what is being covered in the course; ii) Traditional A-levels often put more emphasis on formal limit ideas with sequences and series whereas SMP has a slower spiral development of the concepts.

Several large comprehensives doing SMP were initially approached and a sample reflecting the national population on sex, type of school and O-level grade was aimed at. Not one replied, however. We thus sought the advice of J. Hersee, Executive Director of SMP. He generously offered to find volunteer schools. His comments on the typicality of SMP and the possibility of finding an average sample, moreover, gave us food for thought:

> I don't know whether such a sample exists ! There are those who assert that those who enter for SMP A-level are more able than those who enter for other A-levels; others hold the opposite view ! You may think that's a trivial point, but it has significant consequences. I know of teachers who feel that SMP A-level is not designed for average and below candidates and who, therefore, enter their top set for SMP A-level and their other candidates for another examination -AEB perhaps. So what I'm saying is that, apart from the difficulty of finding a

representative sample, the whole population from which you are choosing may be biased in a number of ways.

We obtained assistance, in the end, from five independent schools and one comprehensive school in the south and midlands of England. One was a girls' school and two were boys' schools that admitted girls in the Sixth Form. They were not randomly picked (one may assume that their Heads of Mathematics are involved in national schemes and thus know Mr. Hersee) but we had no hand in choosing them and thus did not enforce a bias. Independent schools tend to do Additional O-level more than comprehensives (they do the exams, it does not follow that the pupils are more able). We test for bias introduced here in Chapter Six. The schools do SMP O-level but also take in Sixth formers from other schools. Pupils often do not know what Examination Board they have done. Rather than burden our volunteer Heads of Department with a request for these details (and possibly put them off) we decided we would not gather this information. One of the schools informed us that they randomly picked their sample. Another informed us that they asked for volunteers amongst the non mathematicians. We can assume that some, at least, were volunteers. While this is generally not healthy in a questionnaire, we cannot think of any aspect of this study where a volunteer would answer differently than a nominee. We display details of both samples below. MHS refers to the author's school. The larger sample is called MAIN. N refers to the group not doing A-level mathematics. M refers to the group doing A-level mathematics. We shall use these abbreviations in the remainder of the work.

MHS sample 27 in each group

TABLE 5.1

| | | N | M | | | N | M |
|---|---|---|---|---|---|---|---|
| SMP O-level | | 21 | 19 | Male | | 10 | 17 |
| Trad. O-level | | 6 | 8 | Female | | 19 | 8 |
| O-level grade | A | 4 | 10 | | | | |
| | B | 11 | 15 | | | | |
| | C | 12 | 2 | | | | |

MAIN sample

| | | N | M | | | N | M |
|---|---|---|---|---|---|---|---|
| Total | | 76 | 114 | Male | | 33 | 79 |
| | | | | Female | | 43 | 35 |
| O-level grade | A | 4 | 10 | A/O grade | A | 1 | 20 |
| | B | 11 | 15 | | B | 4 | 22 |
| | C | 12 | 2 | | C | 11 | 26 |

Numbers from each school (abbreviations A, O, H, E, B, W used henceforth).

| | A | O | H | E | B | W |
|---|---|---|---|---|---|---|
| N | 18 | 18 | 21 | 9 | 2 | 8 |
| M | 34 | 27 | 17 | 16 | 5 | 15 |

# CHAPTER SIX

# ANALYSIS OF QUESTIONNAIRE RESULTS

The results of Questionnaire 1.1, Questionnaire 1.2 and Questionnaire 2 are presented. We comment on each result in turn using elementary descriptive and inferential statistics. The order of presentation of the questionnaires is not followed. The order of presentation here groups similar questions together. The question numbering in this chapter shall be used as a reference in the following chapters.

For ease of reference we repeat below the abbreviations and conventions we shall use in this and subsequent chapters.

| | |
|---|---|
| MHS | Morecambe High School sample |
| MAIN | Larger (six school) sample |
| M | A-level mathematics group |
| N | Non A-level mathematics group |
| 1, 2 | First or second administration of Questionnaire 1 |
| Q1 | Question 1 (Q2, Q3, ... likewise) |

Other notations will be explained as they arise. Unless otherwise stated the sample size for all MHS administrations is 27 (in each group), 76 for MAIN N and 114 for MAIN M. This will not be restated in each of the many tables presented in this chapter. Unless otherwise stated the tables display rounded integer percentages. This facilitates ease of reading. Actual numbers of responses can be accurately worked out in all but a few ambiguous cases in MAIN M. Thus responses of 36 and 37 (out of 114) gives 31.6% and 32.46% respectively, both of which round to 32%. We have marked these on the table as 32> and 32< respectively. In some tables, some columns add up to 99% or 101%. This is due to rounding errors. The tables are self contained, however, in that actual numbers for responses and thus decimal percentages can be obtained from the tables themselves.

As has been mentioned, the distribution of questions on the questionnaires was purposely designed so that similar questions were sometimes together and sometimes separated by dissimilar questions. We present them here with similar questions together, always. Unless otherwise stated the question numbering of this chapter will serve to reference questions in subsequent chapters.

We avoid advanced statistical techniques that depend on assumptions we cannot (or have not) ascertained. In many cases simple descriptive statistics suffice. In other cases hypothesis testing based on basic probability theory or chi-squared tests is used. When a large collection of data, such as we have here, is analysed significant results can appear at random (1 in every 20 times on average at a 5% significance level). In an attempt to avoid this we made numeric hypotheses concerning the MAIN sample before the data was collected. This provides an extra check against the introduction of random significant results. These numeric hypotheses are often fairly arbitrary, however, e.g. in Q1 we hypothesised that more than 90% would respond 'No' in both groups. 90% is arbitrary (why not 87% or 92% ?) but is a numeric way of saying *the 'No' response will be very strong in both groups.*

Although the chi-squared statistic is a very simple one to work out there are many ways of doing this (giving slightly different results). Consider, for our example, Q1: *Is there a largest number ?* Y/?/N

The table below shows the percentage scores of the MAIN group with the actual numbers in brackets (chi-squared tests are not carried out on the MHS sample because of the smaller numbers and because they were instrumental in determining our numeric hypotheses - though not, it should be noted, simply by tranferring the percentage responses to the MAIN group but using this as a guide with the protocol data).

| MAIN | N | M |
|---|---|---|
| Y | 9 (7) | 15 (17) |
| ? | 1 (1) | 1 (1) |
| N | 89 (68) | 84 (96) |

Before considering chi-squared tests let us clarify what percentages are presented. We present percentage of column rather than of row or of total. We do not present percentage of total since these would give us information on all responses and thus would not clearly show the relationship between variables. We do not present percentage of rows because this would in effect make the response the independent variable.

The '?' row contains all responses left blank or containing a question mark. The chi-squared test is not reliable if entries of very small value are used, thus when the '?' total is small we shall ignore these values. Blank or '?' responses can be very problematic, however, and we must not always ignore them (this is especially true in this study as we obtained more blank responses than we expected and made no prior hypotheses concerning them). In statistical folklore there is a rule of thumb that expected (not observed) values must be at least 10 for 2 by 2 tables and about 5 (certainly not less than 1) for larger tables. We shall work more or less to this but include subjective evaluations. For example if we obtain blank responses of 16 and 3 we must examine whether these are truly neutral responses or if they are characteristic of another train of thought. We proscribe no general rules here but consider each case as it arises. Similar points are applicable to larger tables that contain small value cells. Statistics is a tool we must not become slaves to.

We are now in a position to perform a chi-squared test using the numbers 7, 17, 68 and 96. These are our observed values. We have a choice of expected values depending on the hypothesis being tested. We consider two examples:

H0: There is no difference between the two variables.

H1: There is a difference (no direction given, thus the need for initial hypotheses).

In this case we work out our expected value as :

column marginal X row marginal / total

H0: 90% (for example) will say 'No'.

H1: Other than 90% will say 'No'.

In this case our expected values are calculated as 0.1x76=7.6, 0.1x114=11.4, 0.9x76=68.4 and 0.9x114=102.6. There is an obvious problem here in that we are merely using 90% an a numerical indicator of a *very strong 'No' response* and a less strong response will refute the hypothesis. This is a problem best dealt with by examining individual cases as they arise.

We shall generally regard a result as significant if $P<0.05$ and very significant if $P<0.01$. This must never prevent further examination of the results. A problem frequently encountered in the following pages is that there is often little difference between the N and M groups. $X^2$ tests thus do not refute the hypothesis. We must never assume that this proves the hypothesis. Given prior hypotheses of the expected results this may, however, give us confidence that our interpretations do accord with reality.

Please note that we use Yates' continuity correction for 2 by 2 tables. An account of this can be found in almost any elementary text on statistics.

Towards the end of this chapter we develop our own method for quickly classifying the results of a large number of tables. We leave an exposition of this method until such a time as it is useful.

Q1 Is there a largest number ? Y/?/N

TABLE Q1

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 0 | 4 | 4 | 7 |
| ? | 0 | 4 | 0 | 0 |
| N | 100 | 93 | 96 | 93 |

| MAIN | N | M |
|---|---|---|
| Y | 9 | 15 |
| ? | 1 | 1 |
| N | 89 | 84 |

As Table Q1 shows both groups reject the existence of a largest number. We hypothesized that about 90% in both groups would say 'No'. This was slightly higher than the obtained figure but under this hypothesis we get $X_1^2=2.65$ which does not negate our hypothesis $(0.1<P<0.15)$. We further assumed that there would be no difference between the groups. Under this hypothesis we get $X_1^2=0.86$, which again does not refute our hypothesis $(0.3<P<0.35)$. Interviewees' most common response was that as numbers go on and on it was impossible to have a largest number. The only subject interviewed who responded 'Yes' claimed that infinity was the largest number.

As only subjects from the MHS sample were interviewed it is hard to say what the slightly larger proportion in the MHS sample indicates. One possibility, always open in the following pages, is that the small MHS sample size is less reliable. Another possibility is that one school in the MAIN sample biassed the results. This was investigated and the following distribution of 'Yes' responses to Q1 was found:

TABLE Q1.1

| | A | O | H | E | B | W |
|---|---|---|---|---|---|---|
| N | 2 | 1 | 3 | 1 | 0 | 0 |
| M | 3 | 5 | 2 | 5 | 1 | 1 |

Given the relative size of the schools this is a fairly even distribution as the following table of expected values (based on simple ratios) shows:

TABLE Q1.2

| | A | O | H | E | B | W |
|---|---|---|---|---|---|---|
| N | 1.7 | 1.7 | 1.9 | 0.8 | 0.2 | 0.7 |
| M | 5.1 | 4.0 | 2.5 | 2.3 | 0.7 | 2.2 |

Moreover, the question is quite straightforward making misinterpretation unlikely. Could it then be finitism behind the 9% and 15% ? (Sinclair computers do claim that an integer is a number between -32768 and +32768). Interviews revealed practical finitism (in the sense of *well it's good enough*), especially with small numbers, but no evidence of theoretical finitism (in the sense of not believing or not being able to conceive of numbers beyond some number) was evident in the protocols.

Yet another possibility is that the 'Yes' responses here do, largely, think of infinity as the largest number. We would expect, then, the majority of those responding 'Yes' here to be in the 'Yes' cells in Table Q7 (Q7 asks subjects to say whether they think of infinity as an *enormous number* or not). This could be read by them as either the number at the end of the number line or as a very large finite number . However, only 1 of the 7 in the N group and 8 of the 17 in the M group who responded 'Yes' above, responded 'Yes' in Q7. Our suggestion that subjects responding 'Yes' to Q1 think of infinity as the largest number is thus neither confirmed nor refuted by our investigations so far. We shall return to this question again:

Q2 Is there a smallest number, greater than 0 ? Y/?/N

TABLE Q2

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 15 | 4 | 15 | 19 |
| ? | 0 | 4 | 0 | 0 |
| N | 85 | 96 | 85 | 81 |

| MAIN | N | M |
|---|---|---|
| Y | 25 | 22 |
| ? | 1 | 2 |
| N | 74 | 76 |

As Table Q2 shows both groups reject the existence of a smallest number. We hypothesized that there would be no significant difference between the groups and that about 90% in both groups would say 'No'. Under the first hypothesis we obtained $X^2_1=0.09$ ($0.75<P<0.8$), which does not refute our hypothesis. Interviewees' most common response was that any number could be divided (halving or dividing by 10 being common examples). The 90% hypothesis was clearly too bold and, indeed, under this hypothesis we get $X^2_1=35$ ($P<0.001$) and the hypothesis must be rejected. 90% was, however, a relatively arbitrary numeric version of *the 'No' responses will be strong*. We can see that 75% of the total sample (a large number) thought there is no smallest number.

As in Q1 it is difficult to explain the 'Yes' responses. Some of the MHS subjects thought it was 0 despite the fact that the questions were read out on both occasions and, in particular, 'greater than 0' was stressed. In Questionnaire 2, 'greater than 0' was initially omitted but later hand written on each copy, thus accentuating it. It is possible, however, that some thought of it as 0. Apart from simple guesses or misreading of the question there appear three possible reasons for 'Yes' responses: finitism; belief in infinitesimals; and regurgitation of received knowledge.

Finitism, as we have suggested above, is practical finitism - 'to all intents and purposes 0.00000001 is as small as you can get' and does appear stronger with small numbers than large numbers. We believe that two or three of the 'Yes' responses have this as a reason. Belief in infinitesimals (or an infinitesimal) would suggest strong correlation with the 'Yes' responses to Q10 *Can you believe in an infinitesimally small number ?* (this time the wording is not ambiguous as it was in the case of 'an enormous number'). Of those who responded 'Yes' above, 10 out of the 19 in the N group and 13 out of the 25 in the M group claimed they could believe in such a number as described in Q10. It would appear (the evidence suggests this, it is not conclusive) that a small (10%-15%) of subjects from both groups believe in infinitesimal numbers. We shall leave further investigations here until we examine Q8 and Q9, which examine infinitesimal arithmetic.

Finally we must consider whether this belief is simply regurgitating received views. If this is so, then it is likely that one school is effecting the response. The number of 'Yes' responses here, unlike the numbers in Q1, just allow a chi-squared test on five of the schools (A, O, H, E and W). Tabulating the two responses against the schools and taking as the null hypothesis that there is no difference between the schools we obtain $X^2_4$ =5.3 (0.15<P<0.2). This does not refute the hypothesis that the responses do not come from the instruction of a particular teacher or school. It thus remains an open question then, whether regurgitation of received views affects responses here.

Q3 What is 1/0 ?

**TABLE Q3**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Infinity | 7 | 7 | 41 | 74 |
| Indet | 4 | 4 | 4 | 19 |
| 0 | 63 | 70 | 52 | 7 |
| 1 | 26 | 19 | 4 | 0 |
| ? | 0 | 0 | 0 | 0 |

| MAIN | N | M |
|---|---|---|
| Inf | 38 | 76 |
| Indet | 4 | 16 |
| 0 | 47 | 4< |
| 1 | 8 | 2 |
| ? | 3 | 2 |

NB 'indet' stand for 'indeterminate' in all tables in this chapter.

Q4 What is $1/(1-0.\dot{9})$ ?

TABLE Q4

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Infinity | 41 | 33 | 52 | 78 |
| Indet | 11 | 0 | 0 | 4 |
| Wrong | 33 | 37 | 33 | 15 |
| ? | 15 | 30 | 15 | 4 |

| MAIN | N | M |
|---|---|---|
| Inf | 25 | 75 |
| Indet | 1 | 5 |
| Wrong | 53 | 16 |
| ? | 21 | 4< |

These two questions resulted in many incorrect answers. As we have mentioned we have tried to avoid labelling nonstandard intuitions of limits and infinity as incorrect. Nevertheless, responses of 0, 1, 100, etc. here are clearly wrong (we have not recorded frequencies of each incorrect response in Q4 as there were many different ones). There was a much higher proportion of such responses in the N group. This resulted in a significant difference between the groups: $X^2_2$=59.2 (P<0.001) in the case of Q3 when Table Q3 was collapsed to infinity or indeterminate and wrong; $X^2_3$ =54.1 (P<0.001) in the case of Q4 when Table Q4 was collapsed to infinity or indeterminate, wrong and '?'.

We expected that the data from the MAIN sample would roughly mirror that of the MHS sample, which it does very closely in the case of the M group (remembering that MAIN M should be compared to MHS M2), but a notable difference occurs in Q3 with the two N groups with the response 'infinity'. It was suspected that a large number of the 29 'infinity' responses in MAIN N had done Additional Mathematics at O-level but of the 19 in this group who did Additional Mathematics only five responded 'infinity' (in fact only 26% of those who had done Additional Mathematics responded 'infinity', compared to 42% of those who had not done Additional Mathematics). The reason for this discrepency remains, it must be confessed, a mystery.

The data, coupled with reasons offered in interviews, offers some very interesting details. Note the increase in the 'infinity' response for the MHS M group in both questions. This together with the very close correlation with the MAIN M group indicates that an A-level course does force adolescents to consider infinity (asymptotes were mentioned in the interviews). Interviews revealed that this response was very close to the 'indeterminate' response (it is possible to divide 1 by 0 forever). The numerous responses of '0' in Q3 was explained as misreading the question as 0/1 (several of those interviewed immediately changed their minds to 'infinity' on seeing their mistake). The large number of wrong responses in Q4 arise from the complexity of the question: $0.\dot{9}$ is a difficult concept, $1-0.\dot{9}$ is more difficult, $1/(1-0.\dot{9})$ is even more difficult. It is very easy to lose your way and many did. The wrong responses varied from 0 to 0.1 to 1 to 1.1 to 10 to 100. We shall take up this descriptive analysis again in the protocal data chapter.

**Questions 5, 6 and 7**

Infinity, $\infty$, means different things to different people. Suppose, for the sake of argument, it exists as an enormous number. Then:

Q5 Is $\infty + 1 > \infty$ ? Y/?/N

**TABLE Q5**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 70 | 63 | 85 | 78 |
| ? | 7 | 4 | 4 | 0 |
| N | 22 | 33 | 11 | 22 |

| MAIN | N | M |
|---|---|---|
| Y | 66 | 57 |
| ? | 1 | 2 |
| N | 33 | 41 |

Q6 Is $1/\infty = 0$ ? Y/?/N

TABLE Q6

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 0 | 15 | 11 | 19 |
| ? | 4 | 4 | 0 | 0 |
| N | 96 | 78 | 89 | 81 |

| MAIN | N | M |
|---|---|---|
| Y | 30 | 38 |
| ? | 7 | 4< |
| N | 62 | 58 |

Q7 Is this how you think of infinity ? Y/?/N

TABLE Q7 (MAIN only)

| | N | M |
|---|---|---|
| Y | 30 | 32< |
| ? | 16 | 12 |
| N | 54 | 55 |

**Q5** We have seen in Q1 that subjects generally do not believe in a largest number. Students are often, however, asked to accept the existence of numbers they initially find unbelievable or unacceptable: fractions, recurring decimals, negative and complex numbers. We shall be examining adolescents general mathematical ontological framework later. In Q5 we are asking them to accept as a premise that infinity exists as an enormous number. As we observed in our discussion of Q1 the wording is, unfortunately, ambiguous. Is it the number at the end of the number line, a one point compactification of R, or a huge but finite number ? We wanted them to imagine the former, which, interviews revealed, many did. It is arguably better, however, to let them find their own level and answer as they see fit. Interviews revealed two basic rationales: i) 'Yes', because any number can be incremented. Subjects here are focussing on elementary arithmetic operations. ii) 'No', because $1 + \infty$ is still infinity.

We hypothesized that there would be no difference between the groups, for this type of question is not usually discussed in mathematics classes. We obtained $X_1^2 = 1.35$ $(0.2<P<0.25)$ which did not refute this. We further hypothesized that about 70% in both groups would respond 'Yes' (this being a numeric version of 'fairly strong'). This was rejected, $X^2_1$ =7.15 (P<0.01), but a bias to the 'Yes' response, nevertheless, can be seen. We rely on protocols to clarify subjects' thoughts here and thus defer further discussion until Chapter Eight.

Q6 Q3 revealed that A-level mathematicians generally considered 1/0 to be infinity (or, as we mentioned, *infinite*, which has different connotations, it is not necessarily a number, merely something that goes on and on). We might well expect the A-level mathematicians to respond 'Yes' here. This is not what we were led to believe, however. On the one hand the MHS sample indicated a strong 'No' response on their questionnaires. On the other hand the interviews strongly suggested the belief that 1/x cannot equal 0, **for any number**. This is further evidence for the claim we made, in the discussion following Q3, that *infinity* was generally not meant as the unique number at the end of the number line but as a process - *it is infinite, it goes on and on*. We thus hypothesized that the 'No' response would be fairly strong, numerically putting this at 70%. This was refuted by the data, $X^2_1$ =4.84 (0.01<P<0.05). If we weaken this numeric assumption to 65% 'No' we do not refute the assumption (0.1<P<0.15), but this is dangerously close to a random distribution. Thus, although there appears to be a trend we cannot claim evidence for it from the figures obtained. Although this type of question is more likely to be considered by students doing an A-level mathematics course than was Q5 (1/0 does arise in asymptotes and the scheme a/b=c --> a/c=b is, we believe, firmly embedded in most A-level mathematicians' minds) it is sufficiently novel as a direct question for us to assume that there would be no significant difference between the groups. $X_1^2$ =0.54 (P>0.4) does not refute this assumption.

Q7 With Q7 we must again be careful with our interpretation. Subjects responded on their interpretation of the question, not neccesarily on that assumed by the author or reader. Bearing this in mind and being led to believe that infinity was seen more as a process than a number, we thought the responses in both groups would be similar and largely 'No'. Under the hypothesis that there would be no difference between the groups we obtained $X_2^2$ =0.0008 (P>0.95). This clearly does not refute our hypothesis. The 'No' response is not very strong, however. If the '?' responses lent towards the 'Yes' response then the division in each group would be roughly equal. Considering the lability of adolescents' concepts of infinity and the fact that one context will evoke one aspect of their concepts and not another, a more or less random response is quite compatible with Table Q7. We leave further investigation here until the protocols have been examined.

The association between the three questions does not appear to shed any further light on the subject. Examining all possible responses to questions 5, 6 and 7 respectively we obtain (the figures represent actual responses, not percentages):

TABLE Q7.1

| | YYY | YYN | YNY | NYY | YNN | NYN | NNY | NNN |
|---|---|---|---|---|---|---|---|---|
| N | 7 | 3 | 8 | 4 | 19 | 6 | 2 | 9 |
| M | 15 | 7 | 5 | 9 | 32 | 7 | 7 | 13 |

From the responses to each question separately the 'YN*' ('*' indicating Y or N) must be dominant, as it is. There appears to be no particular trend, however, except that the two groups are roughly similar. Under the assumption that there is no difference between the groups we obtain $X_9^2=5.39$, $(P\approx 0.6)$ which does not refute the assumption. If there is agreement between the groups we believe this would be due to general agreement on individual questions rather than consistency over the three questions taken together.

**Questions 8, 9 and 10** asked similar questions only this time assuming the existence of infinitesimals.

Suppose, for the sake of argument, that there is a number smaller than any other number but bigger than zero. Call it s. Then:

Q8 Does 2+s=2 ? Y/?/N

**TABLE Q8**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 0 | 0 | 0 | 11 |
| ? | 0 | 0 | 0 | 0 |
| N | 100 | 100 | 100 | 89 |

| MAIN | N | M |
|---|---|---|
| Y | 13 | 22 |
| ? | 3 | 0 |
| N | 84 | 78 |

Q9 Does 2xs=s ? Y/?/N

**TABLE Q9**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 0 | 0 | 4 | 4 |
| ? | 0 | 0 | 0 | 0 |
| N | 100 | 100 | 96 | 96 |

| MAIN | N | M |
|---|---|---|
| Y | 8 | 18< |
| ? | 3 | 1 |
| N | 89 | 81 |

Q10 Can you believe in such a number ? Y/?/N

**TABLE Q10** (MAIN only)

| | N | M |
|---|---|---|
| Y | 45 | 35 |
| ? | 7 | 4< |
| N | 49 | 61> |

We have seen, in Q2, that subjects generally (not totally) reject the existence of a smallest number. In these questions we have, nevertheless, asked them to assume the existence of an infinitesimal. Again, as for the infinite number in questions 5 to 7, there is no assurance that they will assume this to be a Leibniz or Robinson-like infinitesimal: they may work in a realm of practical finitism (the response *it is to all intents and purposes 0* occurred in the interviews).

We hypothesized that there would be no difference between the groups in all of these questions, in the belief that *infinitesimal* calculus was not taught nor reinforced by modern cakulus courses. This was not refuted by the chi-squared values for questions 8 and 10 which gave $X_1^2=1.58$ $(0.2<P<0.25)$ and $X_1^2=1.79$ $(0.15<P<0.2)$ respectively. Q9 gave $X_1^2=3.17$ $(0.05<P<0.1)$, but an examination of the table shows the difference to be slight. We further hypothesized that questions 8 and 9 would be strongly biased to a 'No' response (putting this numerically at 80%) and that Q10 would be split 40%/60%, Yes/No. This was not refuted for questions 8 and 10, $X_1^2=1.73$ $(0.15<P<0.2)$ and $X_1^2=2.33$ $(0.1<P<0.15)$ respectively. For Q8 the 'No' response is certainly strong and we feel our hypothesis is supported. For Q10 the figures could have been obtained by random selection. Moreover, given that subjects' interpretations are not always clear from the questionnaire data alone, we feel that judgement here must be deferred until after the protocols have been examined.

For Q9 $X_1^2=5.79$ $(0.01<P<0.02)$ and the 80% hypothesis must be rejected. Examining the table we see, however, that the discrepancy occurs only because the N group's rejection of '2xs=s' is stronger than that of the M group. With an 85% hypothesis we obtain $X_1^2=2.35$ $(0.05<P<0.1)$. Thus it appears that our numeric assumption and not our assumption of a strong response is all that is questionable.

There is a hard core who accept both statements: five of the N group and 17 of the M group responded 'Yes' to both questions 8 and 9. For those from the M group we were curious if this could be put down to a particular school. Inspection revealed that this was not, however, the case.

The principle reason for the responses to questions 8 and 9 are, as for questions 6 and 7, the cognitive hold of the fundamental principles of arithmetic - if s is a number but not 0, then, by all that is taught in lower school mathematics, 2+s cannot equal 2, nor can 2xs=s. The implications for teaching are clear. If infinitesimal calculus is ever to be taught then we must be very clear that taking standard parts (saying st(2+s)=2) is an procedure outside of standard arithmetic.

Q10, which was not given in the MHS questionnaire, appears at odds with Q2, which asked *Is there a smallest number greater than 0*. In the N group 10 of the 19 whose responded 'Yes' in Q2 responded 'Yes' in Q10. That is 24 responded 'No' in Q2 but 'Yes' in Q10. In the M group it was 13 out of 25 (the same proportion in both groups). This leaves 27 who responded 'No' to Q2 but 'Yes' to Q10. This is very strange. Is it simply the lability of the intuition of infinity or has considering the difference between 1 and 0.9 convinced the subjects of the existence of infinitesimals ? While both may account for some of the replies we conjecture that the main reason lies in degrees of belief. Q1 asks *is there* while Q10 asks *can you believe*. Interviews confirmed that subjects can accept useful fictions (which is how Archimedes and Leibniz thought of infinitesimals). A protocol response was:

I can believe in something infinitely small, just something to say *it's extremely small*, like infinity is useful for something that is extremely large. Just a sort of expression.

Q11 Is 0.9 < 1 ? Y/?/N

TABLE Q11

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 100 | 100 | 100 | 74 |
| ? | 0 | 0 | 0 | 4 |
| N | 0 | 0 | 0 | 22 |

| MAIN | N | M |
|---|---|---|
| Y | 89 | 90 |
| ? | 5 | 1 |
| N | 5 | 9 |

It has been known for sometime that students do not accept that $0.\dot{9}=1$ (Schwarzenberger and Tall, 1978). There are many reasons for this: i) 0.9<1, 0.99<1, etc and thus by the Generic Law $0.\dot{9}<1$ ii) $0.\dot{9}$ may equal 1 at infinity but as infinity doesn't exist $0.\dot{9}$ does not equal 1 iii) The difference between $0.\dot{9}$ and 1 is the smallest number (despite the fact that there is no smallest number) iv) $0.\dot{9}$ gets close to 1 (dynamic conception) but never *reaches* 1. We shall have more to say on these ideas later but leave this until we have considered the other questions.

We hypothesized that there would be no difference between the groups and that the 'No' response would be very strong (numerically 90%). We obtained $X_1^2=0.29$ $(0.55<P<0.6)$ for the *no difference* claim and $X_1^2=1.34$ $(0.2<P<0.25)$ for the 90% claim. Neither of these values refutes our hypotheses and, although this does not confirm our hypotheses, we have a very high degree of confidence in them.

## SERIES QUESTIONS

Q12 Can you add 1 + 1 + 1 + ..... (the dots indicate continuation) and get an answer ? Y/?/N

TABLE Q12

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 22 | 11 | 37 | 22 |
| ? | 0 | 0 | 0 | 0 |
| N | 78 | 89 | 63 | 78 |

| MAIN | N | M |
|---|---|---|
| Y | 37 | 25< |
| ? | 3 | 4> |
| N | 60 | 71 |

Q13 Can you add 0.1 + 0.01 + 0.001 + ..... and get an answer ? Y/?/N

TABLE Q13

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 22 | 15 | 56 | 37 |
| ? | 0 | 4 | 4 | 0 |
| N | 78 | 81 | 41 | 63 |

| MAIN | N | M |
|---|---|---|
| Y | 42 | 46< |
| ? | 3 | 3 |
| N | 55 | 51 |

Q14 Just as we often write $1/3=0.\dot{3}$, we can write $1/9=0.\dot{1}$ Can 1/9 be defined as 0.1+0.01+0.001+..... ? Y/?/N

TABLE Q14

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 59 | 48 | 93 | 100 |
| ? | 4 | 19 | 0 | 0 |
| N | 37 | 33 | 7 | 0 |

| MAIN | N | M |
|---|---|---|
| Y | 57 | 89> |
| ? | 21 | 3 |
| N | 22 | 9 |

This group of questions gave us a great surprise when we first obtained the data from the MHS sample. We expected subjects not to focus strongly on the difference between convergent and divergent series but how was it that the definition of 1/9 in Q14 was accepted by the M group when subjects were more or less equally divided as to whether the series in Q13 was legitimate or not? Interviews revealed that $0.\dot{1}$ wasn't seen as a proper number. Thus using a suspicious series to define an improper number was, in itself, acceptable. This, as we can see, was stronger in the M group. We shall be considering subjects' conceptions of proper numbers shortly.

We hypothesized that subjects in both groups would be quite strong in their rejection of the series in Q12 (numerically 70%). We assumed that some would reject it simply because it is an infinite summation but that a smaller percentage would reject it because it is unbounded (divergent). We thought that acceptance would arise from seeing an answer at each stage, viewing infinity as an answer and simply from not appreciating the complexity of the question. We obtained $X_1^2=2.21$ $(0.1<P<0.15)$ for the hypothesis that there would be no difference between the groups. Although this is not rejected by the data the results are not particularly strong in the N group and merit further investigation, which we carry out in Chapter Eight. It does seem reasonable, however, that individuals in the M group should be mathematically more mature, see the divergence of the series and thus push up the 'No' response of the M group proportionally higher than that of the N group. For the 70% hypothesis we obtained $X_1^2=2.46$ $(0.1<P<0.15)$ which again does not reject the hypothesis. Our observations immediately above apply here and this, too, must be investigated further through the protocols.

*It was felt that many would continue to reject the series in Q13* simply because it was an infinite summation, that some would continue to accept it (for the reasons stated above) but that some, stronger in the M group who will have considered the matters like this in their

A-level course, would focus on the convergence of this series. We thus hypothesized that the M group would be split, roughly 50%/50%,but that the percentage reduction in 'No' responses in the N group would be less marked (roughly 40%/60%, Y/N). Such a hypothesis is difficult to test using the chi-squared statistic and indeed although this hypothesis is not rejected $X_1^2$=0.47 (0.5<P<0.55), the hypothesis that there is no difference is also not rejected, $X_1^2$=0.20 (0.65<P<0.7).

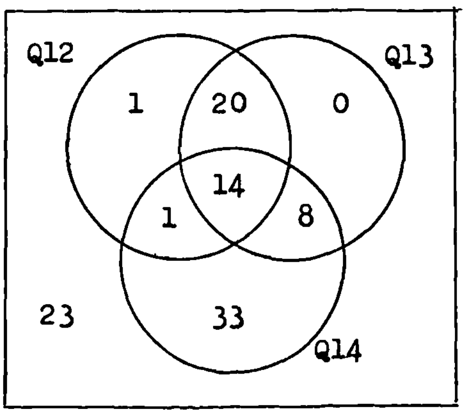
However, another way of stating this assumption is that there will be a difference in the M group but not in the N group when we test the data for each group over the two questions. We get $X_1^2$=0.34 (0.55<P<0.6) for the N group, which does not reject the hypothesis, and $X_1^2$=8.59 (P<0.01) for the M group which, rejects the hypothesis. Again we cannot use the $X^2$ statistic to prove results but this is consistent with our beliefs. We shall investigate the question further in the protocols.

In Q14 we see an acceptance (particularly strong in the M group) that the series in Q13 can be used to define 1/9, or rather $0.\dot{1}$. The reason for this is that although the series does not have an answer it can be called $0.\dot{1}$ because $0.\dot{1}$ does not represent a definite *number*. This view is stronger in the M group because they have met concepts like this (and been confused by them) in their A-level course. We hypothesized that there would be a difference between the groups. The *no difference hypothesis* was clearly rejected, $X_1^2$=9.5 (P<0.005). Taking the strong '?' response into account we obtain $X_2^2$=27.6 (P<0.001). Initial testing, then, is compatible with our beliefs. We further hypothesized that 'Yes' responses would be very strong in the M group (numerically 90%) but weaker in the N group (numerically 60%).

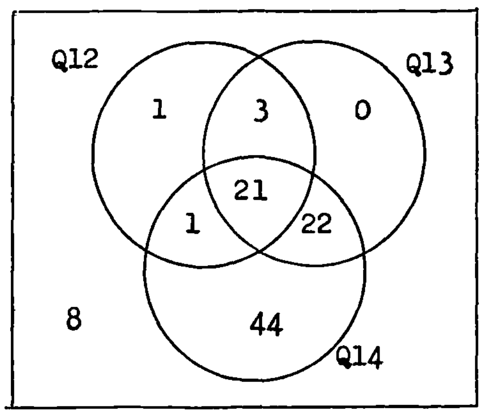
The number of '?' responses makes this difficult to verify but the data is certainly compatible with this.

The Venn diagrams below illustrate the distribution of *formally correct* responses to these three questions (i.e. *No, Yes, Yes* respectively). The numbers represent percentages. Note the overall more *conventionally correct* answers of the M group.

**N**




**M**




**Q15** The following question was not on the MHS questionnaire. It was given to 26 Lower and 26 Upper Sixth A-level mathematicains from MHS in October 1984. The majority of the Upper Sixth pupils were in the M group in the MHS questionnaire. Having observed that students experience great problems with series we were interested in determining whether they could nevertheless note convergence and divergence. Subjects performed the test in mathematics classes at Morecambe High School. They were given five minutes.

Q15 (Questionnaire 1 wording)

*If you were given a box of large and small, blue and red balls and I asked you to sort them out into two groups you might sort them into large and small groups or you might sort them into red and blue groups. I'd like you to sort out the 'sums' below into two* *groups in a similar way e.g. you might put numbers 1,2,3,4,5 and 6 in group A and 7,8 and 9 in group B. Please do this according to your own rule.*

1) $1 + 1 + 1 + 1 + \ldots$ 2) $\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \ldots$

3) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$ 4) $0.3+0.03+0.003+ \ldots$

5) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ 6) $0.1+0.1+0.1+0.1+ \ldots$

7) $1.1+1.01+1.001+ \ldots$ 8) $0.1+0.01+0.001+ \ldots$

9) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$

We designed the question so that subjects could focus on the terms being: the same (1, 3 and 6) or different; fractional (2, 3, 5 and 9) or not; decimal (4, 6, 7 and 8) or not; and on the series being convergent (4, 5, 8 and 9) or divergent.

It is possible that subjects may notice convergence/divergence but nevertheless regard the terms being fractional or not as a more important property. We assume, however, that any student capable of recognizing convergence/divergence will recognize that this is the more important property. In the Lower Sixth 12 subjects focussed on the terms being the same, 11 on the terms being fractional, two appeared to have no rationale and only one isolated the convergent series. In the Upper Sixth 13 subjects focussed on the terms being the same, four appeared to have no rationale, six grouped 1, 2, 3 and 6 together (this is almost the formally correct response), one appeared to note convergence except that the eighth series was not placed in a group, and two subjects appeared to note convergence. There thus seems to be a small shift to recognition of convergence/divergence in the Upper Sixth. The sample, however, is very small and we are thus merely conjecturing. We decided that such a question was useful and included a similar one in Questionnaire 2. We reduced the number of series because of worries of making the questionnaire too long but kept the same four divisions (terms the same, fractional, decimal and series convergent).

Q15 (Questionnaire 2 wording)

*Two of these 'sums' don't belong to the rest. Put the letters of the odd ones out in the boxes (the dots indicate that the process continues).*

*A)* $0.1 + 0.1 + 0.1 + .....$ *B)* $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + .....$

*C)* $1 + 2 + 3 + 4 + .....$ *D)* $0.1+0.01+0.001+ .....$

*E)* $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + .....$

**TABLE Q15** (actual number in brackets)

| MAIN | N | M |
|---|---|---|
| AE | 46 (35) | 60 (68) |
| BD | 12 (9) | 34 (39) |
| BE or CE | 8 (6) | 5 (6) |
| others | 34 (26) | 1 (1) |

As can be seen, the majority in both groups focus on the terms being the same. We hypothesised that there would be a small number of A-level mathematicians who would recognize convergence/divergence and that there would be a difference between the groups. Under a *no difference hypothesis* (with independent variables AE, BD, and all others) we obtain $X_2^2=41.53$ (P<0.001), which clearly refutes the *no difference hypothesis.*

There are 10 possible responses to this question. The 9 BD responses in the N group could thus, conceivably, have occurred by chance. However, it is extremely unlikely that the 39 BD responses in the $\bar{M}$ group did. It would appear, then, that the first year of an A-level mathematics course does communicate someconventional understanding of the nature of convergent series to students. We must wait until we examine the protocols before we can determine what kind of understanding this is.

Q16 Like Q15 this question was not on Questionnaire 1. It was initially given to 25 Lower Sixth and 31 Upper Sixth A-level mathematicians in Morecambe High School. Most of the Upper Sixth group were in the Lower Sixth MHS Questionnaire 1 sample. The question was administered in the autumn of 1984 (one week after Q15) and was given because Questionnaire 1 and subsequent interviews suggested that subjects did not consider $0.\dot{9}$ a *proper* number. The author and two colleagues administered the question during mathematics lessons. Subjects were given about five minutes to answer. The question was subsequently included in Questionnaire 2. We used a five point scale with both the MHS and MAIN sample but when it came to analyse the results it was felt that the scale was not dependable (due to subjects' personalities rather than their mathematical confident). We thus collapsed it to the three point scale displayed in the table. The question in each administration was:

> *Use the five possible answers (yes / think so / ? / think not / no) to indicate whether you think the following are proper numbers. For example, you may think $0.\dot{9}$ is a proper number but not be completely sure, then put 'think so'.*

The MHS question presented subjects with 14 numbers: 9, -9, 1/9, 0.9, $\sqrt{2}$, $0.\dot{9}$, $\pi$ , $\infty$ , $1/\infty$, 1/0, $1/0.\dot{9}$, $1/(1-0.\dot{9})$, sin 32° and $\sqrt{-1}$. These span a wide range of mathematically well defined and undefined numbers. Again with Questionnaire 2 we were concerned with the length and time of the questionnaire. We thus reduced the number of numbers. 9, -9, 1/9, 0.9, $\pi$ and sin 32° gained very high acceptance in the MHS test and were thus omitted as they are not central to our interest. 1/0 and $1/(1-0.\dot{9})$ had already been examined and were thus omitted. The omission of $1/0.\dot{9}$ was, in retrospect, an oversight. $1-0.\dot{9}$ was included in Questionnaire 2 as interviews revealed that subjects had problems conceiving of *nought point nought recurring one* (as several had phrased it). It was overlooked in the MHS question. The results are presented below.

TABLE Q16

| | MHS | | | | | | MAIN | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | L6(N=25) | | | U6(N=31) | | | N | | | M | | |
| | Y | ? | N | Y | ? | N | Y | ? | N | Y | ? | N |
| $0.\dot{9}$ | 64 | 4 | 32 | 87 | 6 | 6 | 54 | 4 | 42 | 72 | 1 | 27 |
| $\infty$ | 32 | 4 | 64 | 26 | 13 | 61 | 11 | 13 | 76 | 21 | 3 | 76 |
| $\sqrt{2}$ | 84 | 0 | 16 | 81 | 13 | 6 | 57 | 5 | 38 | 75> | 3 | 23 |
| $1/\infty$ | 36 | 4 | 60 | 23 | 10 | 67 | 16 | 13 | 71 | 35 | 3 | 62 |
| $\sqrt{-1}$ | 28 | 12 | 60 | 19 | 6 | 74 | 39 | 12 | 49 | 25< | 3 | 72 |
| $1-0.\dot{9}$ | | | | | | | 54 | 7 | 39 | 59 | 4> | 38 |

As this question was not included in Questionnaire 1 and thus was not examined in the protocols we were less confident in making predictions concerning expected outcomes. We thus do not perform chi-squared tests on numeric hypotheses as we have done for other questions. Our main concern was to compare the *properness* of infinity with $0.\dot{9}$ (which several subjects claimed in interviews was not proper) and to compare these, across groups, with several other numbers that may be deemed improper. The word *proper* was used by subjects in the interviews but. It nevertheless caused some confusion (what do you mean by *proper* Sir ?).

There were a high number of '?' and blank responses in the M group. The only cases that suited a 3 by 2 table, however, were $\infty$ and $1/\infty$ (and these *just* - the lowest expected value in these tables being 5.2). The difficulty of interpreting these makes us very dubious of the value of the tests as mentioned in the early part of this chapter, regarding these as neutral is not neccesarily correct. We give the 2 by 2 $X^2$ values to emphasize this point . We take the numbers in order of their perceived *properness* by the subjects.

$\sqrt{2}$ We posited that both groups would respond 'Yes' but only the M group displayed a strong 'Yes' response. $X_1^2=5.12$ $(0.02<P<0.03)$ accordingly rejects the *no difference* hypothesis at a 5% significance level. It is clearly a sign of ignorance that an irrational number (and thus an infinite decimal) is slightly more acceptable than $0.\dot{9}$ for $\sqrt{2}$ is a disguised infinite decimal.

$0.\dot{9}$ We posited that both groups would respond 'Yes' but again only the M group displayed a strong 'Yes' response. $X_1^2=4.61$ $(0.3<P<0.4)$ again rejects the *no difference* hypothesis. Note that in both cases the M group is more sure that $0.\dot{9}$ is a proper number.

$1-0.\dot{9}$ Interviews concerned with $1/(1-0.\dot{9})$ indicated that subjects were easily aware of the problems, under their interpretation of $0.\dot{9}$. We posited that both groups would respond 'Yes' but if this is the case then it is very weak. $X_1^2=0.07$ $(0.75<P<0.8)$ does not refute the *no difference* hypothesis between groups. An interesting difference is clear when we compare responses to this with those for $0.\dot{9}$. We believe that if these A-level mathematicians were independently asked if R was closed under subtraction, then more than 90% would say 'Yes'. The potential conflict in 0.9 is very great.

$\infty$ and $1/\infty$ were both seen as improper by the majority in both groups, as we expected. The large number of '?' responses makes an analysis of difference difficult to evaluate. For the *no difference* hypothesis for $1/\infty$ , we obtain $X_1^2=1.92$ $(0.15<P<0.2)$, which does not reject the hypothesis, ignoring the '?' responses. Regarding the '?' responses as a separate neutral category gives us $X_2^2=10.4$ $(P<0.01)$, which rejects the hypothesis. For we obtain $X_1^2=5.53$ $(0.1<P<0.02)$, ignoring the '?' responses, and $X_2^2=14.12$ $(P<0.001)$, including the '?' responses. Both tests here reject the hypothesis. Curiously, but consistent with the discrepency between Q1 and Q2 (*Is there a largest/smallest number ?*), $\infty$ was seen as less proper than was $1/\infty$ . As we shall see in Chapter Eight, $1/\infty$ can be assigned the meaning *the number continues getting smaller*. Perhaps this is what the 'Yes' responses here meant. One would expect in this case, however, $\infty$ to mean *the number continues getting bigger*. We believe responses to these questions to be particularly labile and thus leave further analysis to the qualitative approach possible in the interviews.

$\sqrt{-1}$ The remarkable fact here is that the N group was more sure of its properness than was the M group. We are not aware of research into adolescents understanding of imaginary numbers but assume that this is because the majority in the M group have not met it (it does not occur until the Upper 6th in the SMP course but we assumed that some teachers would have mentioned complex numbers and that some subjects would have read about them). We further assume that the N group would have thought less about the consequences of taking the square root of a negative number. The point of including it was to compare a very strange number's properness with the properness of $\infty$ and $1/\infty$ . The fact that $\sqrt{-1}$ compares in degrees of properness with $1/\infty$ suggests that infinitesimals could be introduced as $\sqrt{-1}$ is, but we must remember that $\sqrt{-1}$ can be shown to be a mathematically consistent concept whereas problems occur here with $1/\infty$ . We predicted that the M group would respond 'Yes' and the N group 'No'. What this shows is that researchers must be very clear about all aspects of concepts being compared before making comparisons.

## CARDINALITY QUESTIONS

Q17 Can we think of 1, 2, 3,... as a single set ? Y/?/N

**TABLE Q17**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 100 | 100 | 89 | 96 |
| ? | 0 | 0 | 0 | 0 |
| N | 0 | 0 | 11 | 4 |

| MAIN | N | M |
|---|---|---|
| Y | 71 | 82< |
| ? | 16 | 5 |
| N | 13 | 12 |

Q18 Can we think of all the decimal numbers between 0 and 1 as a single set ? Y/?/N

**TABLE Q18**

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 93 | 93 | 93 | 89 |
| ? | 0 | 4 | 0 | 0 |
| N | 7 | 4 | 7 | 11 |

| MAIN | N | M |
|---|---|---|
| Y | 59 | 75> |
| ? | 14 | 6 |
| N | 26 | 19 |

These two questions were meant to preface our examination of cardinal concepts (surely if we are to analyse subjects concepts of infinite cardinals we must be sure that they can imagine them). We expected that both groups would respond strongly 'Yes' (numerically

80%). The MHS sample, the interviews and teaching experience all indicated this was so. The MAIN N results, however, were weaker than we imagined. This caused us some concern. Could it be that a significant number of our MAIN control group could not understand the idea of infinite collections? Reflection convinced us that this was not so: a definite tendency to the 'Yes' response is visible; the First Year pupils in the pilot test, reported on p.74, could all appreciate infinite collections; and the MHS sample clearly accepted the concept. We felt, in retrospect, that the wording was the problem here. In Questionnaire 1 (and with the First Year pupils) the author made it clear by paraphrasing *single set* as *group them together as one thing*. Although we cannot be completely certain it seems likely that it was the phrase *single set* rather than the concept that created the problem.

Uncertainty on how to interpret the '?' responses in the N group makes it difficult to state what degree of difference there is between the groups. Certainly it appears that the N group is less sure of the legitimacy of infinite collections. Again we believe the main problem here was the wording of the question.

**Q19 to Q23** These five questions asked subjects to compare two infinite sets. It should be noted that we did not expect either group to give the correct answers, in terms of transfinite arithmetic. Our interests lay in discovering if all infinite aggregates were seen as having the same number of elements (an infinite amount) or if the generic law was most prominent (leading to subsets having a smaller number of elements) or if comparing infinite quantities was seen as an impossibility (can't compare). We were on the lookout for differences between the two groups with respect to these categories of responses but did not expect them . The variety of rationales for responses was looked into particularly closely in the interviews. Here we note general trends.

As Tables 19 to 23 show, there is general agreement between the two groups. The M group, however, seems more consistent in that only 13 responded *same in each* in three or more of the questions whereas 33 in the N group did. Moreover, only 17 in the N group responded *can't compare* in three or more questions whereas 35 of the M group did. Perhaps such consistency is a feature of a mathematical frame of mind. The five questions, as we observed in the previous chapter, were purposely separated by other questions. Thus, consistent responses probably indicate a consistency in a subject's mind as opposed to a subject simply repeating a prior response. We made no numeric hypotheses in these questions, believing the responses would be approximately random with one or two *silly* responses (e.g. more in the subset). We thought the semi-randomness would refute a 'no difference' between groups hypothesis in some cases (we were not sure which). Moreover, we thought that there would be a slight tendency to *can't compare* in the M group (we felt that one effect of an A-level course would be that there are not always easy answers to non finite questions and that this would lead to a slight increase in this response). Although there is a slight tendency to *can't compare* in the M group it is not significant.We examine each question in turn.

Q19 Consider the two sequences of numbers 1,2,3,4,... and 2,4,6,8,...

Are there: (questions as below)

TABLE Q19

| | MHS | | | | MAIN | |
|---|---|---|---|---|---|---|
| | N | | M | | | |
| | 1 | 2 | 1 | 2 | N | M |
| i) more in first row | 41 | 19 | 11 | 11 | 18 | 15 |
| ii) more in second row | 0 | 0 | 0 | 0 | 2 | 2 |
| iii) same in both | 30 | 41 | 52 | 30 | 42 | 47 |
| iv) can't compare | 30 | 41 | 37 | 56 | 34 | 35 |
| ? | 0 | 0 | 0 | 1 | 2 | 1 |

Under the *no difference* hypothesis, and ignoring the responses 'more in the second row' and '?', we obtain $X^2_3$=0.60 (0.65<P<0.7). This does not refute the hypothesis and indeed, the reponses for the two groups are remarkably close. Although generally compatible with the MHS results we are surprised that so few claim 'more in the first row'. This result is at odds with that of Fischbein et al (1979). They found the majority of their subjects claimed the set of natural numbers was bigger (71% overall and 81% in the high ability group). We suspect that wording of Fischbein et al. was leading, *Which of the two sets contains more elements ?* This rather implies that one set does have more elements, in which case the answer is obviously the set of whole numbers.

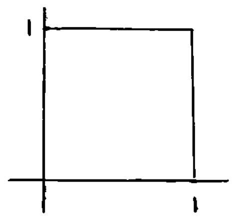
Q20 Consider all the whole numbers 1,2,3,4,... and all the decimal numbers between 0 and 1. Are there: (questions as below)

TABLE Q20

| | MHS | | | | MAIN | |
|---|---|---|---|---|---|---|
| | N | | M | | | |
| | 1 | 2 | 1 | 2 | N | M |
| i) more whole numbers | 4 | 0 | 4 | 4 | 4 | 3 |
| ii) more decimal numbers | 19 | 11 | 15 | 4 | 42 | 24 |
| iii) same number of each | 19 | 33 | 41 | 33 | 20 | 34 |
| iv) can't compare | 56 | 56 | 41 | 59 | 34 | 37 |
| ? | 4 | 0 | 4 | 0 | 1 | 3 |

Under the *no difference* hypothesis and ignoring the responses 'more whole numbers' and '?', we obtain $X^2_2=8.4$ $(0.01<P<0.02)$. This refutes the hypothesis. The result is very curious indeed - it is caused by the N group having more *conventionallycorrect* answers. The question caused the most confusion in the interviews. It is also worrying in terms of A-level mathematicians' conceptions of the real number line (we do not expect the completeness of R to be comprehended but surely the denseness of R should be reinforced by A-level work - it would appear not). The confusion is understandable, neither is a subset of the other, as in the other questions. We must keep this result in mind when we come to examine the protocols in Chapter Eight.

Q21 Consider all the decimal numbers between 0 and 1 and all the coordinate points in the square below.




Are there: (questions as below)

TABLE Q21

| | MHS N 1 | MHS N 2 | MHS M 1 | MHS M 2 | MAIN N | MAIN M |
|---|---|---|---|---|---|---|
| i) more points | 41 | 7 | 37 | 19 | 16 | 24 |
| ii) more numbers | 7 | 15 | 0 | 7 | 13 | 8 |
| iii) same number ofeach | 33 | 44 | 44 | 33 | 57 | 46 |
| iv) can't compare | 19 | 33 | 19 | 37 | 12 | 21 |
| ? | 0 | 0 | 0 | 4 | 3 | 2 |

Under the *no difference* hypothesis and ignoring the '?' responses we obtain $X^2_3=5.98$ $(0.1<P<0.15)$. This does not refute the hypothesis. Again we are surprised by the result. We would expect, apart from the MHS result, that this was an obvious case for the generic law since the question can be interpreted as comparing the points on a line with the points on a square constructed on the line, but relatively few claim there are more points. A reason for this may be that subjects are thinking of numbers theoretically but of points in terms of drawing. However, this was not displayed in the interviews. Again the result is at odds with that of Fischbein et al. At least 65% in every one of their categories claimed *It is not possible to find a point of correspondence on the segment for each point on the square.* However, the wording here is different. To the mathematician a one-to-one correspondence implies equal cardinality but we cannot assume that adolescents will see this. The questions, then, are different.

Q22 Consider all the decimal numbers between 0 and 1 and all the decimal numbers between 0 and 10.

Are there: (questions as below)

TABLE Q22

| | MHS | | | | MAIN | |
|---|---|---|---|---|---|---|
| | N | | M | | | |
| | 1 | 2 | 1 | 2 | N | M |
| i) more between 0 and 1 | 4 | 4 | 0 | 0 | 0 | 1 |
| ii) more between 0 and 10 | 67 | 52 | 44 | 19 | 51 | 35 |
| iii) same number of each | 7 | 22 | 19 | 26 | 20 | 29 |
| iv) can't compare | 22 | 22 | 37 | 56 | 28 | 34 |
| ? | 0 | 0 | 0 | 0 | 1 | 1 |

Under the *no difference* hypothesis and ignoring the responses 'more between 0 and 1' and '?', we obtain $X^2_2=5.04$ $(0.05<P<0.1)$. This does not refute the hypothesis. Despite this there appears to be an increased use of the generic law in the N group. The two sets are of the same type here (bounded, one dimensional and continuous) and the use of the generic law is justifiable in this case. This does not appear to convince the M group which is more or less equally divided bewteen the three intelligent answers. The result of the chi-squared test, however, prevents us attaching too much weight to this observation.

Q23 Consider the circle and square below.

Are there: (questions as below)

NB In Questionnaire 1 this read

*Are there more coordinate points in the circle ?*

Yes/?/No'. This was considered slightly leading.

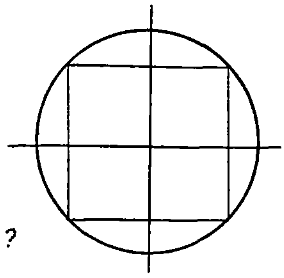


**TABLE Q23**

| | MHS | | | | MAIN | |
|---|---|---|---|---|---|---|
| | N | | M | | | |
| | 1 | 2 | 1 | 2 | N | M |
| i) more in the circle | 63 | 48 | 70 | 30 | 50 | 40 |
| ii) more in the square | | | | | 5 | 1 |
| iii) same in each | 22 | 33 | 30 | 67 | 18 | 29 |
| iv) can't compare | | | | | 25 | 29 |
| ? | 15 | 15 | 0 | 4 | 1 | 1 |

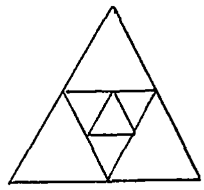
Under the *no difference* hypothesis and ignoring the responses 'more in the square' and '?', we obtain $X^2 =$ 3.19 (0.2<P<0.25). This does not refute the hypothesis. We must be very careful about our interpretation of the $X^2$ results here (as always) for comparing the figures here with those of Q22 there is, really, very little difference. The N group is quite consistent while the M group appears to make only a slightly increased use of the generic law, but the shift only involves 6 out of 114 subjects.

As in Q22 the two sets here are of the same type and again the response 'more in the superset' increases. It is not simply that the same subjects responding 'more' in both questions. 26 (of the 38 for each question) of the N group and 30 (of the 37 in Q22 and 46 in Q23) of the M group responded 'more in the superset' to both questions. Unfortunately this puzzle was not examined in the interviews and we cannot rationally explain it without further research.

The last remark must make us wonder if the results are truly random (an examination of the responses to all five questions revealed only three total responses where agreement was made by more than six subjects: 33333, two in the N group and 18 in the M group; 44344, four in the N group and seven in the M group; 44444, three in the N group and seven in the M group). Several features persuade us, however, that the results are not random : there are very few *silly*(e.g. 'more in the superset') responses; the consistency of the 56 (out of 190) subjects above; and the rationales, that were intelligent rationales, given in interviews. Rather than being simple guesses we believe the results to be the outcome of a path dependent logic (described in Appendix C) where subjects confronted with a number of possible choices, each roughly equally reasonable to them, will this time pick one choice, another time another choice. To test this theory in this case we would need to present the questions without giving options (many subjects would be utterly confused by this and a suitable non leading but explanatory wording would present difficulties). We overcome this partially in the interviews, which will shed more light on the rationales, but subjects had, by then, seen the questions with the options.

## QUESTIONS ON THE LIMIT OF A GEOMETRIC SEQUENCE

The questions below were designed to examine generic (or non generic) limit concepts in a geometric setting. Questionnaire 1 only had Q25. This was a mistake. Having two similar questions enables us to examine the consistency of the concept.



Q24a Consider the triangles below.

Is the limit (questions as below)

**TABLE Q24a** (MAIN only)

| | N | M |
|---|---|---|
| a triangle | 50 | 50 |
| don't know | 7 | 4< |
| a point | 43 | 44 |

Q24b Did you answer the last question theoretically or in terms of actually drawing the triangles ?

**TABLE Q24b** (MAIN only)

| | N | M |
|---|---|---|
| theoretically | 79 | 84 |
| ? | 7 | 6 |
| by drawing | 14 | 10 |

Q25a Consider this sequence of graphs. We have only shown the first three but imagine the process continuing. Is the limit (questions as below)

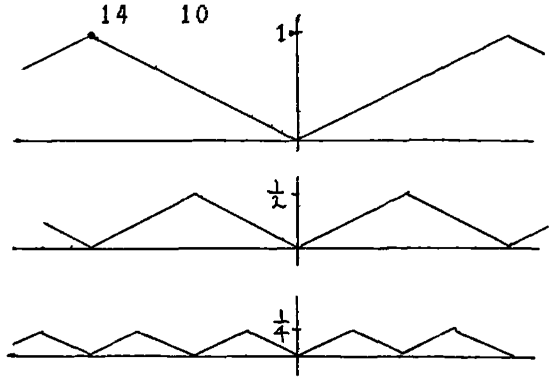



**TABLE Q25a**

| | MHS N 1 | MHS N 2 | MHS M 1 | MHS M 2 | MAIN N | MAIN M |
|---|---|---|---|---|---|---|
| perfectly straight | 19 | 37 | 11 | 44 | 39 | 50 |
| ? | 0 | 0 | 0 | 4 | 1 | 0 |
| slightly jagged | 81 | 63 | 89 | 52 | 59 | 50 |

Q25b Did you answer the last question theoretically or in terms of actually drawing the triangles ?

**TABLE Q25b** (MAIN only)

| | N | M |
|---|---|---|
| theoretically | 84 | 89> |
| ? | 5 | 4< |
| by drawing | 11 | 7 |

As we have said, we believe the generic limit concept to be dominant in adolescents' ideas on limits in an arithmetic context but that a small group of A-level mathematicians would be moving towards a more standard mathematicians' limit concept. In a geometric context we believed these and other, different limit ideas would be present. We did not expect to isolate all of subjects' ideas but one of the ideas we believed was present was that approximation was more widely utilized in geometric contexts even though this was not seen as approximation. This was suggested to us not by the MHS responses to Q25a (which were inadequately followed up in the interviews) but by responses and interviews concerning the limit of $y=1/x$, presented graphically. Belief in the limit being 0 in that question appeared stronger than belief that the limit of 0.1, 0.01, ... is 0. Evidence in the protocols suggested that subjects were viewing the questions in a theoretical light.

Our prior hypotheses were: There would be no significant difference between the groups for any of the questions above; the non '?' responses for Q24a and Q25a would be evenly divided; and more than 80% of subjects would consider that they were answering from a theoretical position. Although our numeric hypotheses were very accurate we should point out that we were less certain here that responses would be in line with our predictions than we were in most of the other questions. Chi-squared tests for these hypotheses gave:

Q24a Ho: No Difference $X_1^2=0.56$ ($P=0.7$). Not rejected.

Q24a Ho: Even division of non '?' responses $X_1^2=0.56$ ($0.45<P<0.5$). Not rejected.

Q24b Ho: No Difference $X_1^2=0.64$ ($0.4<P<0.45$). Not rejected.

Q24b Ho: 80% 'theoretical' $X_1^2=6.37$ ($0.02<P<0.05$). Rejected at 5% level.

Q25a Ho: No Difference $X_1^2$=1.44 (0.2<P<0.25). Not rejected.

Q25a Ho: Even division of non '?' responses $X_1^2$=2.62 (0.1<P<0.15). Not rejected.

Q25b Ho: No Difference $X_1^2$=0.37 (0.5<P<0.55). Not rejected

Q25b Ho: 80% 'theoretical' $X_1^2$=13.2 (P<0.001). Rejected.

As always, non rejection of the hypothesis must not be taken as acceptance of the hypothesis. It does appear, however, that generic limit concepts are not dominant in geometric contexts. Note that the two 80% hypotheses were rejected because the figures were greater than 80%. We will further examine the effect of context later in this chapter when we look at the questions based on the four phrases. We expected those who claimed they answered in terms of drawing to put 'point' and 'straight line'. This was the case in Q24 but not Q25 (of the 11 responding 'drawing' in each group for Q24, 9 in N put 'point' and 8 in M. In Q25 8 in each group responded 'drawing' but, of these, only 3 in the N group and 5 in the M group put 'straight line'). There thus appears to be a difference between the two questions in the minds of the pupils. We have not, however, isolated what this difference is. Table 6.1 below shows that most subjects in both groups are consistent over the two questions. The percentages do not add up to 100 as we have omitted blank responses.

TABLE 6.1

| N | straight line | jagged line |
|---|---|---|
| point | 29 | 14 |
| triangle | 11 | 39 |

| M | straight line | jagged line |
|---|---|---|
| point | 35 | 9 |
| triangle | 11 | 39 |

We were very interested in how those responding in a non generic fashion here (point or straight line) compared to those responding in a non generic fashion to similar questions in an arithmetic context. A subject completely out of the generic limit phase would respond 'No' to Q11 *Is* $0.\dot{9}$ *< 1 ?* and 'Yes' to all parts of Q27 (Consider the sequence 0.9, 0.99, ...), in particular *Is the limit* $0.\dot{9}$ *?* and *Is the limit 1 ?*. According to this criterion no subjects attained the mathematicians' concept image (only 4 out of 76 in the N group and 10 out of 114 in the M group responded 'No' to Q11 and none of these answered 'Yes' to the limits $0.\dot{9}$ and 1 above). Let us, then, see if any subjects are moving away from the generic limit concept in arithmetic and geometric contexts. It would seem reasonable to claim a movement away if, as well as 'point' and 'straight line' responses here, they also gave 'Yes' responses to *Is the limit of 0.9, 0.99, ... 1 ?* and *Does the limit of 0.1, 0.01, ... exist ?* (as we have noted, subjects, on the whole, do not see *nought point nought recurring one* as an acceptable number and will generally see this limit, if it exists, as 0). Table 6.2 below is interesting. The rows represent responses to 'limit of 0.1, 0.01, .. exists' and the columns responses to 'limit of 0.9, 0.99, .. is 1' of subjects who answered both 'point' and 'straight line' to Q24b and Q25b respectively (the percentages are thus out of totals of 22 in the N group and 40 in the M group). We predicted that the 'Yes-Yes' response would be significant only in the M group (where a movement away was occurring).

**TABLE 6.2**

| N | Yes | No |
|---|---|---|
| Yes | 0 | 23 |
| No | 0 | 64 |

| M | Yes | No |
|---|---|---|
| Yes | 33 | 8 |
| No | 23 | 28 |

This is not conclusive evidence but does point to a slight movement away from generic limit concepts in A-level mathematicians. Although this idea is taken up in the interviews it is an area that needs further research (a very important area too).

**Open questions using the phrases *tends to, limit* and *converges.***

Q26 Complete the following:

a) 1+h tends to ______ as h tends to 0.

b) The limit of $(2+h)^2$ as h tends to 0 is ______.

c) $1, \frac{1}{2}, \frac{1}{4}$, ... converges to ______ .

**TABLE Q26a**

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| 1 | 59 | 78 | 74 | 100 |
| decrease | 7 | 7 | 11 | 0 |
| infinity | 4 | 0 | 4 | 0 |
| indet | 0 | 0 | 4 | 0 |
| wrong | 7 | 7 | 0 | 0 |
| ? | 22 | 7 | 7 | 0 |

| MAIN | N | M |
|---|---|---|
| 1 | 49 | 99 |
| dec | 5 | 0 |
| inf | 5 | 0 |
| indet | 0 | 0 |
| wrong | 8 | 1 |
| 0 | 33 | 0 |

**TABLE Q26b**

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| 4 | 67 | 59 | 85 | 93 |
| infinity | 0 | 0 | 7 | 0 |
| indet | 0 | 0 | 4 | 4 |
| wrong | 0 | 15 | 0 | 4 |
| ? | 33 | 26 | 7 | 0 |

| MAIN | N | M |
|---|---|---|
| 4 | 39 | 95 |
| inf | 5 | 0 |
| indet | 1 | 2 |
| wrong | 14 | 4> |
| ? | 39 | 0 |

**TABLE Q26c**

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| 0 | 48 | 41 | 41 | 70 |
| 1/∞ | 22 | 19 | 44 | 22 |
| infinity | 15 | 7 | 11 | 0 |
| 2 | 4 | 0 | 0 | 0 |
| wrong | 0 | 7 | 4 | 7 |
| ? | 11 | 26 | 0 | 0 |

| MAIN | N | M |
|---|---|---|
| 0 | 24 | 32> |
| 1/∞ | 9 | 24 |
| inf | 18 | 5 |
| 2 | 7 | 34 |
| wrong | 8 | 4> |
| ? | 34 | 2 |

The questions were included to show that despite the fact that A-level courses do little to advance students' basic intuitions and conceptions of infinity, limits and real numbers*, these courses do produce students who can give the formally correct answers to standard questions in this area. They were chosen as typical of the kind of question a non A-level mathematician can understand. They are not exhaustive. $X^2$ tests were not performed as there are so many low response cells and, quite frankly, the numbers speak for themselves.

a) We expected the M group to be mainly correct, the N group to be largely correct with several '?' responses. The response 'decrease', which surprised us,initially, in the MHS sample, is quite an intelligent response; it *does* decrease. The response 'infinity' is understandable too; it is infinite in that it goes on and on.

b) As for the comments in a).

c) The response '2' is, presumably, the sum of the series. Putting this (as a misinterpretation that is, however, a correct misinterpretation) with the formally correct responses and $1/\infty$ (also arguably correct) we get a *correct* response of 40% in the N group and 90% in the M group. Note that 'converges' causes the most problems. This is a feature of the remaining questions.

## WORDS

The number of tables in this section suggests that it accounts for about half of the study. As we have mentioned in the previous chapter, however, each question was answered fairly quickly.

We do not attempt to give a theory of language and mathematics. Our aim is threefold: we want to determine the dominant everyday connotations of the phrases *limit, tends to, converges* and *approaches* (in Linguistics a noun phrase or a verb phrase may be a single word); we want to examine the interpretations given to these phrases in mathematical contexts (geometric and arithmetic); and we want to see what *obstacles*, to use Bachelard's and Cornu's expression, these interpretations present by examining responses to questions using these phrases where an irrelevant mathematical feature, in terms of limits, is presented (e.g. comparing a monotone decreasing sequence with a oscillating but nevertheless decreasing sequence). These aims are relevant to our study in that these phrases are constantly used by tertiary teachers and texts to describe/explain infinite and limiting processes.

We begin by examining the responses to the sentence questions. Subjects were asked to write four sentences, one each using each of the phrases. This was initially to be given in the first administration of Questionnaire 1 only (to obtain the dominant everyday meanings before A-level mathematicians were formally introduced to them). 'Speed limit' was used so often, however, that we asked subjects to write another sentence using limit, but not speed limit, in the second administration Questionnaire 1. Space was also given at the end of Questionnaire 2 for subjects to relate any confusion they found with the phrases. This was the only optional part of the questionnaire. We look at these remarks after examining the sentences given in Questionnaire 1.

The responses are very similar to those noted in the second pilot study. We look at each phrase in turn and describe the interpretations starting with the most common. Frequency counts seem unnecessary for such a descriptive task, we merely relate relative weightings.

LIMIT 'Speed limit' was by far the most common example. In most cases this is a convential law: the legal limit it is forbidden to exceed. Most people do exceed it sometimes, however. In a graph drawing question (on drawing y=1/x with positive x increasing, say) this concept image of a limit would suggest *Well, you can get to 0 if you like but the rule is DON'T* . In contrast the mathematician regards the rule as a neccesary feature of the curve. A typical mathematical response may be *It is 1/x. You can't just suddenly jump up or down !*.

After speed limit came physical limits and mental limits. Physical limits are boundaries that are technically highly unlikely to be passed such as limit of the amount of alcohol one can consume or the height one can jump. They need not be concerned with humans. Planes have a limit (ceiling), radar has its limit of detection and there are physical limits to cars' speeds. These limits are usually just on the boundary. The limit to the speed humans can run the mile in is topical. 1 minute would not be considered a time (and hence a speed) limit but 3.5 minutes may be. In mathematical situations this can be thought of in two ways, with, say, the sequence 0.9, 0.99, ... The limit may be the boundary, $0.\dot{9}$, or just past the boundary, 1. We must take care, with respect to the limit being 1, that we don't regard such a statement, by itself, as providing evidence that a subject is beyond the influence of the generic limit concept.

Mental limits have no mathematical analogue. They are the limits of people's patience, nerves or intellectual abilities. They can also be what people drive themselves to, their breaking points. Fewer instances occurred of: conventional limits (social customs); restrictions (*you must limit your salt intake*); and limit as a special word (*you are the limit*).

**APPROACHES** About 7/8 of the M group and 2/3 of the N group used *approach* in the sense of 'drawing nearer': *the train approaches the station: the car approaches the traffic lights; winter approaches; the dog approaches the cat.* In the first three examples the object being approached will, eventually, be reached though it has not been at the time the sentence is uttered. This temporal aspect can be transferred into mathematical contexts. Mathematicians do not view a convergent series in a temporal light but subjects may: 0.9+0.09+... approaches 1, but it will never get there. The 'dog approaches the cat' example has a connotation implying that it may not reach it. If a rogue dog is to be moved away from children then one will, presumably, approach it, but one would have to be desperate to touch it. A safe distance would be a 'limit'. In this sense y=1+1/x approaches 0 as positive x increases.

The remainder used three other meanings of approaches. A method of doing something: *different approaches to mathematics; several approaches to the question of abortion.* A route or way into something (note the indefinite article): *there are three approaches to Morecambe; several approaches to my house.* Resembling: *Racism approaches Facism; his behaviour approaches the ridiculous.*

**CONVERGES** Converges has fewer everyday meanings and was mainly used in three common examples: *the light rays converge; the roads converge; the lines converge.* In each instance two continuous objects come nearer and in most cases, touch. If these are subjects' dominant or only concept image of converge then it is difficult to see how they will make sense of a sequence converging to a number. Graphs will make more sense but if y=1/x converges to 0, then we must think of 0 as the line y=0 and not a number.

The remainder used examples where individual (discrete) objects come into contact or close proximity: *the cars converged; the footballers converged on the ball; the crowd converged on the politician*. Interesting isolated examples were: *my thoughts converge to Christian thought; a straight line converges the farther away you look; two lines converge to a point; two objects which converge eventually meet*.

**TENDS TO** With eight exceptions all examples were of personal inclination (*she tends to drink a lot; he tends to wear jeans*) or of general trends (*holiday weather tends to be bad; eggs tend to break when dropped*). These two senses have considerable overlap (*chemistry tends to be hard; I tend to eat breakfast at 8.00*). As a general trend *tends to* may be used in a mathematics class but would be more suited to comparing bar charts (*the frequencies tend to be low in the early graphs*) than discussing the behaviour of algebraic curves.

Apart from caring (*the nurse tends to the patient*) the remainder used mathematical examples: *1/9 tends to 0.1; 1, repeatedly divided by 10, tends to 0; a sequence may, eventually, tend to a limit*.

Clearly all of these aspects of these phrases do not act simultaneously in an evoked concept image in a mathematics context but they all contribute to the total concept image.

The MAIN sample responses were classified by first reading them all and then putting them into piles according to the dominant response. This is a rather rough and ready response analysis technique but is useful in that definite types of responses were easily isolated. There is much overlap between the types of response identified but little difficulty in sorting questionnaire paper into appropriate piles was experienced. The percentages are rough guides (rounded to the nearest 5%). The types of response that emerged are:

No response, 20% - blank spaces or just the word *No*.

All the same, 20% - some put *all are confusing*, others put *I can't see the difference* (some qualifying this with *but I suppose there must be*). A typical comment was:

> All of them. Approaches - does it actually reach 0 ?
> Has a limit ? What sort of limit ?
> Tends to ? Absolutely no idea what is difference (sic) between tends to, approaches and converges.

*Converges* and *approaches* seen as the same or equally confusing, 15%.

*Converges, approaches* and *tends* to are all the same (or, fewer, are all confusing), 10% - this means *limit* is seen as somehow different.

*Converges* seen as confusing, 10%.

*Tends to* and *approaches* seen as the same (Sometimes qualified with *both are vague*), 10%.

Others, 15% - e.g. isolating *converges* and *tends to*. Most combinations not mentioned above were included here.

Four of the subjects gave extended responses that are worth including in full; not for their typicality but for their range of impressions. The first three below were doing A-level mathematics, the final one was not:

i) Yes. The similarity in the meanings of the phrases is itself confusing. Further, the term 'converges to' can mean many different things and depending upon which definition or meaning is put into practice, the answer to any question can differ. The actual definition of 'to converge' was clear enough but in here, it is more difficult to decide what the answer should be. The term 'tends to' is slightly confusing and apparently exactly similar to 'has as a limit'. The phrase 'approaches' is the source of confusion, as to whether a number which the function approaches more closely as x increases but which it can never reach are in a supposed infinite limit. Can be supposed to be approached by the function. It seems that these terms in normal everyday mathematics have little notion of their significance or meaning in fact.

ii) I think all these words are ever so confusing and it makes me even more confused when I try to understand and use them properly. I don't think it's my personal problem though because different books and teachers use different words (terms). Especially those 'approaches' and 'tends'. Even in this test, I got so confused that I probably ticked all the wrong ones.

iii) To say that a number such as infinity 'tends to a limit' would be impossible to do because as far as theory can see, the number infinity goes on forever. However, in my mind, by looking at the number there must be a limit since 'for ever and ever' must have an end eventually. These two ideas conflict and cause a problem in answering the questionnaire with a definite 'Yes' or 'No'.

iv) The mathematical terminology is confusing. If I had initially been able to understand this, I would be a mathematician !

Infinity is something more easily related to concepts (i.e. God etc.) than actual mathematical figures.

Infinity is to do with time and space. I find it hard to relate the concepts to maths and mathematical equations. To relate infinity to maths I lack a basic understanding of maths and therefore the questions were sometimes unclear and difficult.

## QUESTIONS ON THE FOUR PHRASES

There is a problem with the blank, '?', responses in the remaining tables - there are a lot of them and there are, with one exception, a greater percentage in the N group (in some cases, a much greater percentage). We perform two $X^2$ tests for each one under the null hypothesis that there is no difference in the groups. One, DF=2, will utilize the '?' response. The other, DF=1, will ignore them. We could distribute them proportionally amongst the 'Yes' and 'No' responses or we could consider the worst case where all '?' responses are grouped with the smaller of the 'Yes' or 'No' responses. Both approaches, however, have many dangers of unnatural biassing. If both $X^2$ tests do not refute the null hypothesis (or if both do refute it), then we can be fairly confident that the hypothesis is not refuted (or that it is). If the DF=2 test refutes the hypothesis but the DF=1 test does not, then we shall consider the case further. In most cases where this occurs this is due to the extra '?' responses in the N group. We shall comment on these cases as they occur. There were no cases where the DF=1 test refuted the null hypothesis but the DF=2 case did not.

Both to isolate a tendency to either pole, 'Yes' or 'No', and to obtain an overall picture of all the following tables a method that gives numeric cut off points would be useful. There are many ways to do this. The following is rather arbitrary in determining its cut off points but does give us an easy to use scale by which to classify the tendency to a pole ('Yes' or 'No') as strong or not.

We use the raw data (frequencies in each group) and not the percentages. This is because the sample size is important in the following. We ignore '?' responses, they are spoilt for the purposes here. With n as the resulting non '?' sample (e.g. if there are 8 '?' responses then n for the N group is 76-8=68) we examine binomial models (because the situation now is a Bernoulli trial) via their Normal approximations $\mathcal{N}(np, np(1-p))$. With p taking values 0.5, 0.6, 0.7, 0.8, 0.9 (this is the arbitrary feature of the procedure) we calculate the 5% critical percentage points of the extreme tail only, x(p), by the following formula

$$x(p) = 100(1.645\sqrt{np(1-p)} + np)/n$$

Where the inner bracket is derived from

$$\Phi(w-np) = 0.95 \;\; \text{-->} \;\; w = 1.645\sqrt{np(1-p)} + np$$

(1.645 is the 5% critical region cut off point taken from Normal Distribution tables).

With t representing the largest value in the table we adopt the following ranking (called here S0, S1, .. ,S5 'S' for *significance*):

| | | |
|---|---|---|
| S0 | $0 \le t < x(0.5)$ | No tendency to either pole. |
| S1 | $x(0.5) \le t < x(0.6)$ | Slight tendency to 'Yes'/'No' |
| S2 | $x(0.6) \le t < x(0.7)$ | Tendency to 'Yes'/'No' |
| S3 | $x(0.7) \le t < x(0.8)$ | Marked tendency to 'Yes'/'No' |
| S4 | $x(0.8) \le t < x(0.9)$ | Strong tendency to 'Yes'/'No' |
| S5 | $x(0.9) \le t$ | Very strong tendency to 'Yes'/'No' |

As a very rough guide S1 is above 59%, S2 is above 69%, etc. We present tables in the following format.

**TABLE Q27a**

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 81 | 74 | 89 | 89 |
| ? | 0 | 7 | 0 | 0 |
| N | 19 | 19 | 11 | 11 |

| MAIN | N | M |
|---|---|---|
| Y | 66 | 74 |
| ? | 16 | 4> |
| N | 18 | 23 |

N group: tendency to 'Yes', S2
M group: tendency to 'Yes', S2
$X^2_1=0.01$, $(0.9<P<0.95)$ Do not reject Ho
$X^2_2=8.99$, $(P<0.01)$ Reject Ho

The table follows the format of earlier tables. The comments below the table summarize the *tendency to a pole* and the results of the $X^2$ tests under the hypothesis that there is no difference between the groups.

We now examine each question in turn, commenting after the results for each part of the question have been tabulated.

Q27 Consider the sequence 0.9, 0.99, 0.999, 0.9999 ...
Which of the following sentences are true of this sequence ?

**TABLE Q27a** It tends to $0.\dot{9}$

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 81 | 74 | 89 | 89 |
| ? | 0 | 7 | 0 | 0 |
| N | 19 | 19 | 11 | 11 |

| MAIN | N | M |
|---|---|---|
| Y | 66 | 74 |
| ? | 16 | 4> |
| N | 18 | 23 |

N group: tendency to 'Yes', S2
M group: tendency to 'Yes', S2
$X^2_1=0.01$, $(0.9<P<0.95)$ Do not reject Ho
$X^2_2=8.99$, $(P<0.01)$ Reject Ho
We hypothesised that both groups would be about 80% 'Yes'.

**TABLE Q27b** It approaches $0.\dot{9}$

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 85 | 59 | 70 | 85 |
| ? | 0 | 11 | 0 | 0 |
| N | 15 | 30 | 30 | 15 |

| MAIN | N | M |
|---|---|---|
| Y | 54 | 60 |
| ? | 12 | 11< |
| N | 34 | 29 |

N group: slight tendency to 'Yes', S1
M group: slight tendency to 'Yes', S1

$X^2_1=0.4$, $(0.5<P<0.55)$ Do not reject Ho
$X^2_2=0.67$, $(0.4<P<0.45)$ Do not reject Ho
We hypothesised that both groups would be about 80% 'Yes'.

**TABLE Q27c** It converges to $0.\dot{9}$

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 52 | 48 | 67 | 70 |
| ? | 4 | 19 | 0 | 4 |
| N | 44 | 33 | 33 | 26 |

| MAIN | N | M |
|---|---|---|
| Y | 43 | 41 |
| ? | 18 | 20 |
| N | 38 | 39> |

N group: no tendency, S0
M group: no tendency, S0
$X^2_1=0.001$, $(P>0.95)$ Do not reject Ho
$X^2_2=0.13$, $(0.7<P<0.75)$ Do not reject Ho
We hypothesised that both groups would be about 80% 'Yes'.

**TABLE Q27d** Its limit is $0.\dot{9}$

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 85 | 74 | 93 | 67 |
| ? | 4 | 7 | 0 | 4 |
| N | 11 | 19 | 7 | 30 |

| MAIN | N | M |
|---|---|---|
| Y | 59 | 62 |
| ? | 21 | 7 |
| N | 20 | 31 |

N group: tendency to 'Yes', S2
M group: slight tendency to 'Yes', S1
$X^2_1=0.82$, $(0.35<p<0.4)$ Do not reject Ho
$X^2_2=9.26$, $(P<0.01)$ Reject Ho
We hypothesised that both groups would be about 80% 'Yes'.

TABLE Q27e Tends to 1

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 37 | 52 | 52 | 74 |
| ? | 7 | 0 | 0 | 0 |
| N | 56 | 48 | 48 | 26 |

| MAIN | N | M |
|---|---|---|
| Y | 54 | 75> |
| ? | 18 | 5 |
| N | 28 | 20 |

N group: slight tendency to 'Yes', S1
M group: marked tendency yo 'Yes', S3
$X_1^2$=2.62, (0.1<P<0.15) Do not reject Ho
$X_2^2$=11.5, (P<0.001) Reject Ho
We hypothesised that the M group would be about 50% 'Yes' and that the N group would be about 80% 'Yes'.

TABLE Q27f Approaches 1

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 81 | 56 | 78 | 96 |
| ? | 0 | 7 | 0 | 0 |
| N | 19 | 37 | 22 | 4 |

| MAIN | N | M |
|---|---|---|
| Y | 71 | 81 |
| ? | 9 | 4< |
| N | 20 | 15 |

N group: tendency to 'Yes', S2
M group: marked tendency to 'Yes', S3
$X_1^2$=0.7, (0.35<P<0.4) Do not reject Ho
$X_2^2$=2.86, (0.05<P<0.1) Do not reject Ho
We hypothesised that the M group would be about 60% 'Yes' and that the N group would be about 80% 'Yes'.

TABLE Q27g Converges to 1

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 33 | 26 | 26 | 56 |
| ? | 4 | 19 | 0 | 4 |
| N | 63 | 56 | 74 | 41 |

| MAIN | N | M |
|---|---|---|
| Y | 21 | 22 |
| ? | 22 | 15 |
| N | 57 | 63 |

N group: tendency to 'No', S2
M group: tendency to 'No', S2
$X_1^2$=0.001, (P>0.95) Do not reject Ho
$X_2^2$=1.76, (0.15<P<0.2) Do not reject Ho
We hypothesised that the M group would be about 50% 'Yes' and that the N group would be about 50% 'Yes'.

TABLE Q27h Its limit is 1

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 11 | 19 | 19 | 52 |
| ? | 4 | 11 | 0 | 4 |
| N | 85 | 70 | 81 | 44 |

| MAIN | N | M |
|---|---|---|
| Y | 24 | 36 |
| ? | 15 | 7 |
| N | 62 | 57 |

N group: tendency to 'No', S2
M group: slight tendency to 'No', S1
$X_1^2$=1.7, (0.15<P<0.2) Do not reject Ho
$X_2^2$=4.93, (0.02<P<0.05) Do not reject Ho at the 1% level but reject Ho at the 5% level
We hypothesised that the M group would be about 50% 'Yes' and that the N group would be less than 20% 'Yes'.

In the subjects' minds the sequence does *tend to* $0.\dot{9}$ and there is an indication of agreement between groups. *Limit* is similar but the certainty of the M group is less. With both *approaches* and *converges* there is strong agreement between groups but no significant tendency (absolutely none with *converges*). The MAIN results are compatible with the MHS results for *tends to* and *limit* but are weaker than expected with *approaches* and *converges*. We had only expected a difference between groups with *converges* ,as the most confusing phrase, but the agreement between groups was closest here (perhaps the confusion created a random response).

With $0.\dot{9}$ replaced by 1 the results dichotomize into 'Yes' for *tends to* and *approaches* (because of their vagueness) and 'No' for *converges* and *limit*. This was generally expected though the M group was expected to be split, in roughly even proportions, for *converges* and *limit*. Except for *tends to* there is general agreement between groups. The only phrase to generate a reverse shift from 'Yes' to 'No' was *limit*.

The responses are compatible with the claim that the generic limit concept is dominant in adolescent thought. Observe the N, S2 and M, S1 'Yes' responses for *the limit is* $0.\dot{9}$ and the N, S2 and M, S1 'No' responses for the limit is 1. This is close to what we expected on the assumption that the generic limit concept would generate a greater 'Yes' response for $0.\dot{9}$.

Q28 For each of the sequences below say whether it has a limit.

TABLE Q28a 1, 0.1, 0.01, 0.001, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 0 | 0 | 4 | 44 |
| ? | 4 | 0 | 0 | 0 |
| N | 96 | 100 | 96 | 56 |

| MAIN | N | M |
|---|---|---|
| Y | 4 | 43 |
| ? | 5 | 2 |
| N | 91 | 55 |

N group: very strong tendency to 'No', S5
M group: no tendency either way, S0
$X_1^2=31.9$, (P<0.001) Reject Ho
$X_2^2=35.4$, (P<0.001) Reject Ho
We hypothesised that the M group would be about 50% 'Yes' and that the N group would be about 80% 'No'.

TABLE Q28b 1, 0, 0.1, 0, 0.01, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 7 | 15 | 7 | 33 |
| ? | 0 | 7 | 0 | 0 |
| N | 96 | 78 | 96 | 67 |

| MAIN | N | M |
|---|---|---|
| Y | 12 | 33 |
| ? | 13 | 4> |
| N | 75 | 63 |

N group: marked tendency to 'No', S3
M group: slight tendency to 'No', S1
$X_1^2=8.18$, (P<0.01) Reject Ho
$X_2^2=15.2$, (P<0.001) reject Ho
We hypothesised that the M group would be about 70% 'No' and that the N group would be about 80% 'No'.

TABLE Q28c 1, 0.1, 1, 0.01, 1, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 7 | 11 | 7 | 26 |
| ? | 0 | 0 | 0 | 0 |
| N | 93 | 89 | 93 | 74 |

| MAIN | N | M |
|---|---|---|
| Y | 12 | 18< |
| ? | 8 | 3 |
| N | 80 | 79 |

N group: marked tendency to 'No', S3
M group: marked tendency to 'No', S3
$X_1^2=0.74$, (0.35<P<0.4) Do not reject Ho
$X_2^2=3.93$, (0.1<P<0.15) Do no reject Ho
We hypothesised that both groups would be about 80% 'No'.

TABLE Q28d 1, 1, 1, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 52 | 52 | 85 | 74 |
| ? | 4 | 0 | 0 | 0 |
| N | 44 | 48 | 15 | 26 |

| MAIN | N | M |
|---|---|---|
| Y | 24 | 54 |
| ? | 3 | 3 |
| N | 74 | 44 |

N group: tendency to 'No', S2
M group: no tendency either way, S0
$X_1^2=15.8$, (P<0.001) Reject Ho
$X_2^2=17.0$, (P<0.001) Reject Ho
We hypothesised that the M group would be about 80% 'Yes' and that the N group would be about 50% 'Yes'.

In the first sequence we see a very strong 'No' response in the N group and an even split in the M group (and thus a significant difference between groups). This is precisely what we expected. The rationale we posited behind this response was that the intuitive rationale would be: **the sequence will *never get to 0* and thus it has no limit.** We believed that this rationale would be partially overcome by some subjects doing A level mathematics. Looking at the *limit* responses in Q27 in the light of Q28a, compatibility with the following rationale can be noticed: The sequence 0.9, 0.99, ... has a limit $0.\dot{9}$ but not 1 (and $0.\dot{9}$ is a *proper number* and not equal to 1) but the sequence 0.1, 0.01, ... does not have a limit. It would if 0.01 was a proper number but it is not. The only candidate for a limit here is 0. The sequence will never get to 0, however, just as it will never get to 1 in the other case. There is then, no limit to the sequence 0.1, 0.01, ... although there is to 0.9, 0.99, ... We rely on interviews to probe deeper here but, regardless of any interpretation, the concept of a bounded monotone sequence cannot be said to have been understood by A-level mathematicians !

The sequences b) and c) display are mathematically minor and major (respectively) variations of the first sequence. Again the results are very close to the MHS results and our expectations (80% 'No' for both in the N group and 70% and 80% 'No', respectively, in the M group). The difference between groups shown is also compatable with this. Our main projected rationale of the subjects was:

**a) doesn't have a limit and so neither does b) or c)**

For those in the M group who responded 'Yes' to a) we projected a few would view the fluctuation in b) as preventing the sequence having a limit and a few more (holding the mathematically correct line), who were not deceived by b), perceiving the non converging values of c) as preventing the sequence having a limit. These projected cognitive reasons can be more closely examined. If the view *a) doesn't have a limit and so neither does b) or c)* is dominant then we would expect the majority of those saying 'No' to b) and c) to also say 'No' to a). All 52 in the N group did and 51 out of 68 in the M group did. Moreover, in our projected rationale in the M group, we would expect the majority of those saying 'Yes' in b) to also say 'Yes' in a). 28 out of 38 did. Interesting as such figures are they should be seen strictly as merely supporting a belief rather than testing an hypothesis.

The difference between the MAIN and the MHS results (and our expectations) for d), with a constant term, shows how destabilizing some mathematically irrelevant features are.

**The next set of questions** are identical except that *limit* is replaced by *converges*. The tables are seen as less important because *converges* caused so much confusion.

We expected *converges* to cause much confusion and for both groups to be roughly evenly split between 'Yes' and 'No' but for there to be a slight tendency towards correct answers in the M group. We interpret the very high '?' response in both groups as confirming our belief that *converges* causes cognitive conflict (or confusion). This makes an analysis of the data very difficult.

Q29 For each of the sequences below say whether it converges

TABLE Q29a 1, 0.1, 0.01, 0.001, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 33 | 26 | 56 | 70 |
| ? | 7 | 22 | 0 | 0 |
| N | 59 | 52 | 44 | 30 |

| MAIN | N | M |
|---|---|---|
| Y | 41 | 47 |
| ? | 20 | 10 |
| N | 39 | 43 |

N group: no tendency either way, S0
M group: no tendency either way, S0
$X_1^2$=0.001, (P>0.94) Do not reject Ho
$X_2^2$=3.97, (0.1<P<0.15) Do not reject Ho
We hypothesised that both groups would be evenly divided with a slight tendency for the M group to respond correctly.

TABLE Q29b 1, 0, 0.1, 0, 0.01, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 37 | 30 | 56 | 52 |
| ? | 7 | 26 | 0 | 0 |
| N | 56 | 44 | 44 | 48 |

| MAIN | N | M |
|---|---|---|
| Y | 28 | 52 |
| ? | 22 | 11< |
| N | 50 | 37 |

N group: slight tendency to 'No', S1
M group: slight tendency to 'Yes', S1
$X_1^2$=6.87, (P<0.01) Reject Ho
$X_2^2$=11.6, (P<0.01) Reject Ho
We hypothesised that both groups would be evenly divided with a slight tendency for the M group to respond correctly.

TABLE Q29c 1, 0.1, 1, 0.01, 1, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 33 | 30 | 33 | 41 |
| ? | 7 | 22 | 11 | 0 |
| N | 59 | 48 | 56 | 59 |

| MAIN | N | M |
|---|---|---|
| Y | 28 | 28 |
| ? | 22 | 12 |
| N | 50 | 60 |

N group: slight tendency to 'No', S1
M group: slight tendency to 'No', S1
$X_1^2$=0.08, (0.75<P<0.8) Do not reject Ho
$X_2^2$=3.61, (0.15<P<0.2) Do no reject Ho
We hypothesised that both groups would be evenly divided with a slight tendency for the M group to respond correctly.

TABLE Q29d 1, 1, 1, ...

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 37 | 19 | 56 | 30 |
| ? | 19 | 22 | 0 | 4 |
| N | 44 | 59 | 44 | 67 |

| MAIN | N | M |
|---|---|---|
| Y | 14 | 27 |
| ? | 17 | 11> |
| N | 68 | 62 |

N group: marked tendency to 'No', S3
M group: tendency to 'No', S2
$X_1^2$=2.78, (0.09<P<0.1) Do not reject Ho
$X_2^2$=5.1, (0.05<P<0.1) Do not reject Ho
We hypothesised that both groups would be evenly divided with a slight tendency for the M group to respond correctly.

Apart from the 'No' response to d), there is no significant leaning to either pole in the others. The percentage of correct answers amongst the mathematicians is also higher in every case, though only in b) does this refute the 'No difference' hypothesis.

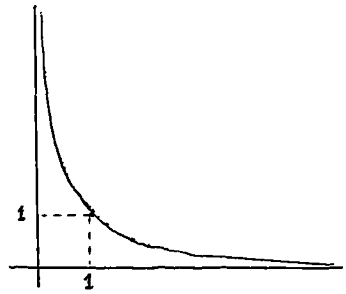
These responses, considered with the protocol data, convince us that *converges* is not understood in the context of limits of sequences of real numbers.

## THE FOUR PHRASES APPLIED TO FUNCTIONS

The remaining six questions all asked if any of the four phrases could be applied to functions presented as geometric curves.

Q30 We predicted that the 'Yes' response would be very strong in both groups with the phrases *tends to* and *approaches*, which it is in the M group and still is, though less so, in the N group. The stronger belief in the M group results in a refutation of the *no difference* hypothesis in these cases. We expected a strong 'Yes' response in the M group and an even split in the N group for *converges* and *limit*. While the even split emerged in the N group it also did in the M group. This is compatible with other data that suggests that *converges* and *limit* are confusing or less applicable to real sequences and functions or are stronger (more strictly defined) concepts. We shall consider the evidence for these possibilities in Chapter Eight and Chapter Nine.

Comparing *limit* with its arithmetic counterpart, Q28a, we see a change in the N group from a very strong 'No' response in an




Q30 Can we say the curve (questions as below) as x gets larger and larger

**TABLE Q30a** tends to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 78 | 89 | 100 | 100 |
| ? | 4 | 0 | 0 | 0 |
| N | 19 | 11 | 0 | 0 |

| MAIN | N | M |
|---|---|---|
| Y | 70 | 92 |
| ? | 11 | 4> |
| N | 20 | 4< |

N group: tendency to 'Yes', S2
M group: very strong tendency to 'Yes', S5
$X_1^2=11.2$, ($P<0.001$) Reject Ho
$X_2^2=16.5$, ($P<0.001$) Reject Ho
We hypothesised that both groups would respond about 90% 'Yes'.

**TABLE Q30b** has 0 as a limit

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 56 | 52 | 48 | 85 |
| ? | 0 | 4 | 0 | 0 |
| N | 44 | 44 | 52 | 15 |

| MAIN | N | M |
|---|---|---|
| Y | 34 | 46< |
| ? | 13 | 9 |
| N | 53 | 45 |

N group: slight tendency to 'No', S1
M group: no tendency either way, S0
$X_1^2=1.73$, ($0.15<P<0.2$) Do not reject Ho
$X_2^2=3.08$, ($0.2<P<0.25$) Do not reject Ho
We hypothesised that the N group would be evenly divided but that the M group would be about 80% 'Yes'.

**TABLE Q30c** converges to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 74 | 56 | 70 | 70 |
| ? | 0 | 11 | 0 | 0 |
| N | 26 | 33 | 30 | 30 |

| MAIN | N | M |
|---|---|---|
| Y | 43 | 34 |
| ? | 20 | 18< |
| N | 37 | 47 |

N group: no tendency either way, S0
M group: no tendency either way, S0
$X_1^2=1.73$, ($0.15<P<0.2$) Do not reject Ho
$X_2^2=2.23$, ($0.3<P<0.35$) Do not reject Ho
We hypothesised that the N group would respond about 60% 'Yes' and that the N group would respond about 80% 'Yes'.

**TABLE Q30d** approaches 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 93 | 93 | 93 | 100 |
| ? | 0 | 4 | 0 | 0 |
| N | 7 | 4 | 7 | 0 |

| MAIN | N | M |
|---|---|---|
| Y | 71 | 87 |
| ? | 9 | 4< |
| N | 20 | 9 |

N group: tendency to 'Yes', S2
M group: srong tendency to 'Yes', S4
$X_1^2=4.53$, ($0.02<P<0.05$) Reject Ho at 5% sig. level
$X_2^2=7.26$, ($0.02<P<0.05$) Reject Ho at 5% sig. level
We hypothesised that both groups would respond about 90% 'Yes'.

arithmetic context to a very weak 'No'response in a geometric context. The M group, however, remain evenly split in both contexts. We shall have opportunity again to examine the effect of context in Q33, which presents an oscillating function which converges to 0 from above. Comparing *converges* with its arithmetic counterpart, Q29a, we see basic agreement, between groups, of no significant tendency to either pole.

**Q31** The correct answer to each question here is 'No'. We thought that this would be easily recognized and be very strong in the M group, strong (but less so) in the N group and that both groups would drop, noticeably, with *approaches*, as there is a sense (the *dog approaching the cat* sense) in which the curve approaches 0. This is generally borne out by the results. With the exception of the N group, with *approaches*, all tables have larger entries in the 'No' row (and this is significant in most cases). Moreover the $X^2$ tests reveal that the 'No' responses are significantly larger with the M group. The last table reveals the extent to which we underestimated the strength with which the curve approaches 0.

**Q32** We expected a very strong 'No' response in both groups. The results in the M group give this. The results in the N group also give this but are uniformly less strong. This caused the $X^2$ tests to register a difference between groups.

Q31 Can we say the curve (questions as below) as x gets larger and larger ?

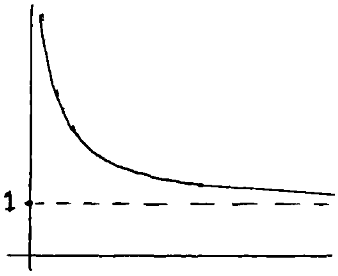



TABLE Q31a tends to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 19 | 11 | 7 | 0 |
| ? | 4 | 0 | 0 | 0 |
| N | 78 | 89 | 93 | 100 |

| MAIN | N | M |
|---|---|---|
| Y | 37 | 6 |
| ? | 9 | 4> |
| N | 54 | 90 |

N group: no tendency either way, S0
M group: strong tendency to 'No', S4
$X^2_1=28.1$, $(P<0.001)$ Reject Ho
$X^2_2=33.9$, $(P<0.001)$ Reject Ho
We hypothesised that the N group would be about 80% 'No' and the M group would be about 90% 'No'

TABLE Q31b has 0 as a limit

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 11 | 15 | 0 | 4 |
| ? | 4 | 4 | 0 | 0 |
| N | 85 | 81 | 100 | 96 |

| MAIN | N | M |
|---|---|---|
| Y | 12 | 5 |
| ? | 14 | 5 |
| N | 74 | 89< |

N group: marked tendency to 'No', S3
M group: strong tendency to 'No', S4
$X^2_1=2.55$, $(0.1<P<0.15)$ Do not reject Ho
$X^2_2=8.19$, $(0.01<P<0.02)$ Reject Ho at the 2% sig. level
We hypothesised that the N group would be about 80% 'No' and the M group would be about 90% 'No'.

TABLE Q31c converges to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 19 | 11 | 26 | 7 |
| ? | 4 | 11 | 4 | 0 |
| N | 78 | 78 | 70 | 93 |

| MAIN | N | M |
|---|---|---|
| Y | 17 | 4< |
| ? | 21 | 7 |
| N | 62 | 89> |

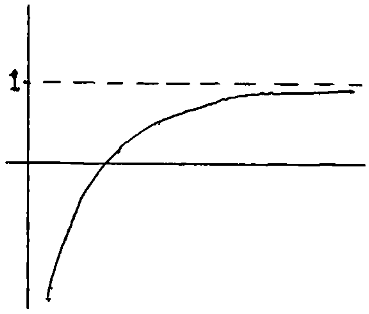
N group: tendency to 'No', S2
M group: very strong tendency to 'No', S5
$X^2_1=9.7$, $(P<0.01)$ Reject Ho
$X^2_2=19.1$, $(P<0.001)$ Reject Ho
We hypothesised that the N group would be about 80% 'No' and the M group would be about 90% 'No'.

TABLE Q31d approaches 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 41 | 33 | 59 | 19 |
| ? | 0 | 4 | 0 | 0 |
| N | 59 | 63 | 41 | 81 |

| MAIN | N | M |
|---|---|---|
| Y | 47 | 32> |
| ? | 14 | 5 |
| N | 37 | 63 |

N group: no tendency either way, S0
M group: slight tendency to 'No', S1
$X^2_1=8.3$, $(P<0.01)$ Reject Ho
$X^2_2=13.8$, $(P<0.001)$ Reject Ho We hypothesised that the N group would be about 70% 'No' and the M group would be about 80% 'No'.




Q32 Can we say the curve (questions as below) as x gets larger and larger ?

**TABLE Q32a** tends to 0

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 0 | 4 | 4 | 0 |
| ? | 4 | 0 | 0 | 0 |
| N | 96 | 96 | 96 | 100 |

| MAIN | N | M |
|---|---|---|
| Y | 14 | 4< |
| ? | 11 | 5 |
| N | 75 | 90 |

N group: marked tendency to 'No',S3
M group: very strong tendency to 'No', S5
$X_1^2=5.41$, ($P=0.02$) Reject Ho at the 5% sig. level
$X_2^2=8.5$, ($0.01<P<0.02$) Reject Ho at the 2% sig. level
We hypothesised that both groups would be about 90% 'No'.

**TABLE Q32b** has 0 as a limit

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 0 | 4 | 4 | 4 |
| ? | 0 | 0 | 0 | 0 |
| N | 100 | 96 | 96 | 96 |

| MAIN | N | M |
|---|---|---|
| Y | 11 | 5 |
| ? | 12 | 5 |
| N | 78 | 89< |

N group: strong tendency to 'No', S4
M group: strong tendency to 'No', S4
$X_1^2=1.5$, ($0.2<P<0.25$) Do not reject Ho
$X_2^2=4.97$, ($0.02<P<0.05$) Reject Ho at the 5% sig. level
We hypothesised that both groups would be about 90% 'No'.

**TABLE Q32c** converges to 0

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 19 | 0 | 4 | 0 |
| ? | 0 | 11 | 0 | 0 |
| N | 81 | 89 | 96 | 100 |

| MAIN | N | M |
|---|---|---|
| Y | 18 | 4 |
| ? | 11 | 10 |
| N | 71 | 86 |

N group: marked tendency to 'No', S3
M group: very strong tendency to 'No', S5
$X_1^2=8.74$, ($P<0.01$) Reject Ho
$X_2^2=10.3$, ($P<0.01$) Reject Ho
We hypothesised that both groups would be about 90% 'No'.

**TABLE Q32d** approaches 0

| | N | | M | |
|---|---|---|---|---|
| MHS | 1 | 2 | 1 | 2 |
| Y | 4 | 4 | 4 | 0 |
| ? | 0 | 4 | 0 | 0 |
| N | 96 | 93 | 96 | 100 |

| MAIN | N | M |
|---|---|---|
| Y | 26 | 14 |
| ? | 13 | 4< |
| N | 61 | 82> |

N group: slight tendency to 'No', S1
M group: marked tendency to 'No', S3
$X_1^2=5.22$, ($0.02<P<0.05$) Reject Ho at the 5% sig. level
$X_2^2=10.8$, ($P<0.01$) Reject Ho
We hypothesised that both groups would be about 90% 'No'.

The only real difference is with *approaches*. The increased 'No' is, we posit, due to the *dog and cat* example - the curve does not approach 0 because it goes past it. If subjects are answering consistently then we would expect the vast majority of those responding 'No' to Q29 would also respond 'No' in Q30 (provided aspects we have not considered are not affecting subjects'responses). We obtain:

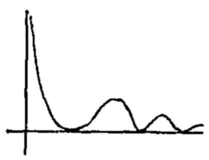
88% of the N group and 98% of the M group who responded 'No' for *tends to 0* in Q31 responded 'No' for this in Q32.

89% of the N group and 96% of the M group who responded 'No' for *has 0 as a limit* in Q31 responded 'No' for this in Q32. 83% of the N group and 93% of the M group who responded 'No' for *converges to 0* in Q31 responded 'No' for this in Q32.

For each of Q31 and Q32 the vast majority in each group responding 'No' to any one of *tends to, limit,* and *converges* , responded 'No' to the other two phrases as well. We see, then, considerable uniformity of mathematically correct response when a function does not tend to a given limit. The exception being with *approaches*, in which everyday meanings are believed to affect responses considerably.

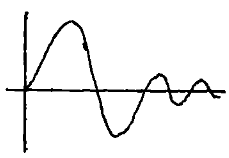
**Q33, Q34 and Q35** form a group with fluctuations which:

touch y=0



do not touch y=0

go through y=0



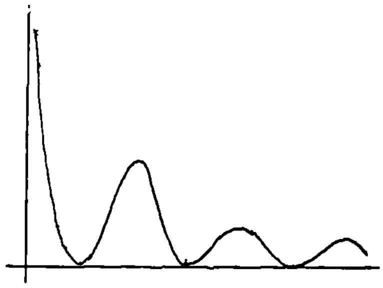
These fluctuations, as can be seen, are mathematically entirely irrelevant with regard to the limits of the functions. We are interested in observing changes of response between the three questions and comparison of responses with Q28b and Q29b (which are similar but set in an arithmetic, as opposed to geometric, context).

Q33 The responses indicate no tendency to either pole with *tends to* and no significant difference between the groups. We expected a fairly strong 'Yes' response in the N group and a strong 'Yes' response in the M group. We did expect and were aware, from interviews, that some subjects would focus on the facts that the function touches 0, and that the function tends to 0 and then tends away, and see these as preventing it from tending to 0. However, we did not think that they would be so strong.

In retrospect we feel we should have included the question *Does 1, 0, 0.1, 0, 0.01, ... tend to 0 ?* We may then have been able to judge whether there is an effect of context in questions with fluctuations with the phrase *tends to*. We suspect that there is and that there would be a significantly higher 'Yes' response in the arithmetic question.

The remarks made in the first paragraph above apply equally well to *converges*. We must remember, however, that this word causes a great deal of confusion.

With *limit* and *approaches* there is a tendency towards 'Yes' but not strong except for the M group with *limit*. As with the others the hypothesis that there is no difference between the groups is not refuted.



Q33 Can we say the curve
(questions as below) as
x gets larger and larger ?

TABLE Q33a    tends to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 67 | 56 | 74 | 70 |
| ? | 4 | 0 | 0 | 0 |
| N | 30 | 44 | 26 | 30 |

| MAIN | N | M |
|---|---|---|
| Y | 45 | 51 |
| ? | 17 | 8 |
| N | 38 | 36 |

N group: no tendency either way, S0
M group: slight tendency to 'Yes', S1
$X_1^2$=0.17, (0.7<P<0.75)   Do not reject Ho
$X_2^2$=3.59, (0.15<P<0.2)   Do not reject Ho
For all four parts of this questions, we hypothesised that about 70% of the N group and about 80% of the M group would respond 'Yes'.

TABLE Q33b    has 0 as a limit

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 56 | 63 | 85 | 67 |
| ? | 0 | 0 | 7 | 0 |
| N | 44 | 37 | 7 | 33 |

| MAIN | N | M |
|---|---|---|
| Y | 63 | 75> |
| ? | 8 | 4< |
| N | 29 | 21 |

N group: slight tendency to 'Yes', S1
M group: marked tendency to 'Yes', S3
$X_1^2$=1.5, (0.2<P<0.25)   Do not reject Ho
$X_2^2$=2.99, (0.2<P<0.25)   Do not reject Ho
Hypothesis as above.

TABLE Q33c    converges to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 63 | 56 | 70 | 63 |
| ? | 0 | 11 | 4 | 0 |
| N | 37 | 33 | 26 | 37 |

| MAIN | N | M |
|---|---|---|
| Y | 41 | 50 |
| ? | 13 | 11> |
| N | 46 | 39< |

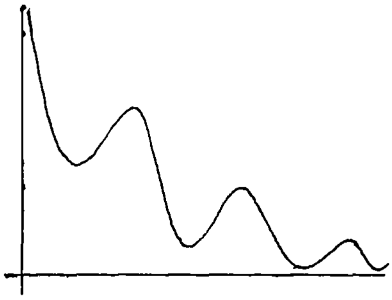
N group: no tendency either way, S0
M group: no tendency either way, S0
$X_1^2$=0.94, (0.3<P<0.35)   Do not reject Ho
$X_2^2$=1.58, (0.45<P<0.5)   Do not reject Ho
Hypothesis as above.

TABLE Q33d    approaches 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 63 | 59 | 70 | 74 |
| ? | 0 | 4 | 0 | 0 |
| N | 37 | 37 | 30 | 26 |

| MAIN | N | M |
|---|---|---|
| Y | 62 | 54< |
| ? | 13 | 11> |
| N | 25 | 35 |

N group: tendency to 'Yes', S2
M group: slight tendency to 'Yes', S1
$X_1^2$=1.5, (0.2<P<0.25)   Do not reject Ho
$X_2^2$=2.21, (0.3<P<0.35)   Do not reject Ho
Hypothesis as above.



Q34 Can we say the curve (questions as below) as x gets larger and larger ?

**TABLE Q34a** tends to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 67 | 63 | 74 | 52 |
| ? | 4 | 0 | 4 | 11 |
| N | 30 | 37 | 22 | 37 |

| MAIN | N | M |
|---|---|---|
| Y | 53 | 74 |
| ? | 17 | 10 |
| N | 30 | 17 |

N group: slight tendency to 'Yes', S1
M group: marked tendency to 'Yes', S3
$X_1^2=5.83$, $(0.05<P<0.1)$ Do not reject Ho
$X_2^2=8.92$, $(0.01<P<0.02)$ Reject Ho at the 2% sig. level
We hypothesised that about 60% of both groups would respond 'Yes'.

**TABLE Q34b** has 0 as a limit

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 30 | 44 | 52 | 52 |
| ? | 0 | 0 | 4 | 7 |
| N | 70 | 56 | 44 | 41 |

| MAIN | N | M |
|---|---|---|
| Y | 25 | 39> |
| ? | 30 | 14 |
| N | 45 | 47 |

N group: slight tendency to 'No', S1
M group: no tendency either way, S0
$X_1^2=0.8$, $(0.35<P<0.4)$ Do not reject Ho
$X_2^2=8.46$, $(0.01<P<0.02)$ Reject Ho at the 2% sig. level
We hypothesised that about 50% of both groups would respond 'Yes'.

**TABLE Q34c** converges to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 56 | 52 | 70 | 52 |
| ? | 0 | 15 | 4 | 4 |
| N | 44 | 33 | 26 | 44 |

| MAIN | N | M |
|---|---|---|
| Y | 38 | 36 |
| ? | 26 | 14 |
| N | 36 | 50 |

N group: no tendency either way, S0
M group: no tendency either way, S0
$X_1^2=1.05$, $(0.3<P<0.35)$ Do not reject Ho
$X_2^2=5.85$, $(0.05<P<0.1)$ Do not reject Ho
We hypothesised that about 60% of both groups would respond 'Yes'.

**TABLE Q34d** approaches 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 78 | 78 | 78 | 63 |
| ? | 0 | 4 | 0 | 4 |
| N | 22 | 19 | 22 | 33 |

| MAIN | N | M |
|---|---|---|
| Y | 68 | 73 |
| ? | 17 | 11> |
| N | 14 | 17 |

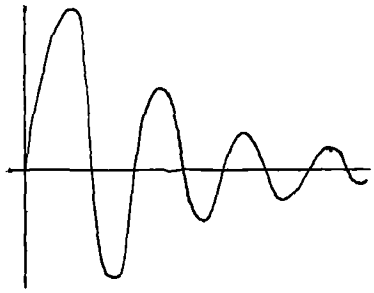
N group: marked tendency to 'Yes', S3
M group: marked tendency to 'Yes', S3
$X_1^2=0.0004$, $(P>0.95)$ Do not reject Ho
$X_2^2=1.76$, $(0.4<P<0.45)$ Do not reject Ho
We hypothesised that about 70% of both groups would respond 'Yes'.

Comparing *limit* with Q28b, its arithmetic counterpart, we see a totally reversed bias: the limit does not exist in an arithmetic context but it does in a geometric context. It would appear that context makes a considerable difference. Comparing *converges* with Q27b, its arithmetic counterpart, we see no tendency to either pole in either group or in either context.

Q34 The function *tends to* 0. This is strong in the M group but weak in the N group. As a result the $X^2$ test indicates a difference between groups. We expected a moderately strong 'Yes' response in both groups. The 'Yes' responses for the M group are consistent with those for Q33a in that 57, of the 58, of those responding 'Yes' in Q33 also responded 'Yes' here. This consistency was less noticeable in the N group (21 out of 34). The responses for *approaches* are similar to those for *tends to* but the 'Yes' response is significantly stronger (as expected). Again there is a consistency in the responses, compared to Q33d, in the M group in that 57 out of 62, only 38 of the 57 above, who responded 'Yes' there also responded 'Yes' here. Again this consistency is less noticeable in the N group (38 out of 47).

The responses for *limit* and *converges* are not significant for either pole (the '?' response is noticeably large giving strong agreement between the groups only when these responses are ignored, DF=1).

Q35 The function here *tends to* and *approaches* 0. This is only moderately strong. This was as expected. 'Yes' for *converges* is quite strong here for both groups (we expected it to be strong in both



Q35 Can we say the curve (questions as below) as x gets larger and larger ?

**TABLE Q35a** tends to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 63 | 67 | 81 | 70 |
| ? | 4 | 0 | 0 | 4 |
| N | 33 | 33 | 19 | 26 |

| MAIN | N | M |
|---|---|---|
| Y | 63 | 68> |
| ? | 12 | 6 |
| N | 25 | 26 |

N group: tendency to 'Yes', S2
M group: tendency to 'Yes', S2
$X_1^2=0.02$, $(0.85<P<0.9)$ Do not reject Ho
$X_2^2=1.92$, $(0.35<P<0.4)$ Do not reject Ho
We hypothesised that about 80% of both groups would respond 'Yes'.

**TABLE Q35b** has 0 as a limit

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 22 | 33 | 63 | 52 |
| ? | 0 | 0 | 0 | 0 |
| N | 70 | 67 | 37 | 48 |

| MAIN | N | M |
|---|---|---|
| Y | 28 | 42 |
| ? | 20 | 8 |
| N | 53 | 50 |

N group: slight tendency to 'No', S1
M group: no tendency either way, S0
$X_1^2=1.59$, $(0.2<P<0.25)$ Do not reject Ho
$X_2^2=7.75$, $(0.02<P<0.05)$ Reject Ho at the 5% sig. level
We hypothesised that about 40% of the N group and about 70% of the M group would respond 'Yes'.

**TABLE Q35c** converges to 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 63 | 56 | 81 | 93 |
| ? | 0 | 11 | 0 | 0 |
| N | 37 | 30 | 15 | 2 |

| MAIN | N | M |
|---|---|---|
| Y | 66 | 71 |
| ? | 11 | 8 |
| N | 24 | 21 |

N group: tendency to 'Yes', S2
M group: tendency to 'Yes', S2
$X_1^2=0.13$, $(0.7<P<0.75)$ Do not reject Ho
$X_2^2=0.68$, $(0.7<P<0.75)$ Do not reject Ho
We hypothesised that about 60% of the N group and about 90% of the M group would respond 'Yes'.

**TABLE Q35d** approaches 0

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 74 | 63 | 74 | 70 |
| ? | 0 | 4 | 0 | 0 |
| N | 26 | 33 | 26 | 30 |

| MAIN | N | M |
|---|---|---|
| Y | 58 | 62 |
| ? | 20 | 9 |
| N | 22 | 29 |

N group: tendency to 'Yes', S2
M group: tendency to 'Yes', S2
$X_1^2=0.12$, $(0.7<P<0.75)$ Do not reject Ho
$X_2^2=5.06$, $(0.05<P<0.1)$ Do not reject Ho
We hypothesised that about 80% of both groups would respond 'Yes'.

groups but significantly weaker in the N group). How much this agreement with expectations is worth is debatable, however, since the agreement with expectations for *converges* with the other two fluctuating functions was not the case. *Limit* is not significantly biased to either pole. What this data shows is the dramatic cognitive affect that seemingly irrelevant alterations can have.

Looking globally at all the questions on the four phrases we can see general agreement between groups for *approaches* and *converges*. As we have seen, however, *converges* is difficult to analyse because of the confusion the phrase generates. There appears no clear pattern to agreement between groups with *tends to* and *limit*.

Let us temporarily speak of conventionally correct and incorrect answers. Subjects rarely got the wrong answer and more often than not got the right answer. If we do a count we obtain (*correct* here is S2 or above, *no tendency* is S0 or S1, *incorrect* is S2 or above for the mathematically incorrect answer)

**TABLE 6.3**

| | N | M | Total |
|---|---|---|---|
| correct | 15 | 18 | 33 |
| no tendency | 15 | 13 | 28 |
| incorrect | 2 | 1 | 3 |

The three wrong responses were all concerned with sequences (two with *converges* and one with *limit*). The functions where 'No' was the correct response were correct 5 out of 8 times with the N group and 7 out of 8 times with the N group. There is, however, a sense in which it is easier for subjects to say when these phrases don't apply. The fluctuating curve that passes though y=0 was correct 6 out of 8 times (not with *approaches* with either group).

# CHAPTER SEVEN

# THE INTERVIEWS

The method of investigating adolescents' understanding of limits, infinity and related concepts was by written questionnaire and selective clinical interview. In this chapter we present our overall approach to the interview and details worthy of note concerning the particular interviews we undertook. We leave the analysis of the protocols obtained until the next chapter.

In education research, as in most areas, research does not proceed via a steady flow of corroborating hypotheses but by asking questions and conjecturing and refuting tentative answers. We began this study with rather naive and fixed ideas on the expected results (believing that we would find some pupils with Aristotelian potential infinite intuitions, others with more Cantorian intuitions, others with intuitions akin to nonstandard analysis interpretations and most with a mixture of ideas). The pretests forced us to think again ! Students' had ideas that we never dreamt of. We must now agree with Wason and Johnson-Laird (1972):

> There is another advantage to the experimental approach: totally unanticipated phenomena are discovered by it. We almost feel inclined to say that we consider an experiment a failure when it fails to surprise us.

Moreover, strong post positivist ideas on the nature of scientific inquiry were gradually eased to one side as the need for openness in our investigations became apparent. For these reasons interviews were necessary. Talking to the subjects allowed us to be led by their ideas rather than our own. The importance of interviews is clearly described by Gay (1980):

> In contrast to the questionnaire, the interview is flexible; the interviewer can adapt the situation to each subject. By establishing rapport the interviewer can often obtain data that

the subjects would not give on a questionnaire. The interview may also result in more accurate and honest responses ... the interviewer can follow up on incomplete or unclear responses by asking additional probing questions. Reasons for particular responses can also be determined.

The type and form of an interview must be carefully considered before conducting it. There are several established types of interview techniques, see Knight (1982). The main ones used in mathematics education research are: (a) The structured individual interview where each subject is given the same set of questions. This method facilitates formal and statistical analysis. It is unsuitable for us because it would restrict our field of investigation and our questionnaire results have already collected such data (though a questionnaire is an inferior tool). (b) The Clinical (or Piagetian) interview technique is an unstructured and open ended method which, although it does begin with set questions, is free to depart from any set course in order to follow the subject's thought processes. From what has been said on allowing ourselves to be open to unanticipated results it can be appreciated that this method was the one chosen for our study.

It must be admitted that many mathematics educationalists are sceptical of this method (it clearly cannot be standardized) while others feel it suitable only for pilot studies. To this we reply that our work is not in one of the main areas of mathematics education research and may thus be seen as a pilot study itself. We have, moreover, used interviews as a supplement to the questionnaires.

Ginsburg (1981) argues that investigations into mathematical thinking involve three aims:

> The discovery of cognitive activities (structures, processes, thought patterns, etc.), the identification of cognitive activities and the evaluation of levels of competence. In any given study, the distinction among these aims may be blurred, and more than one aim may be involved ... each requires a distinctive type of clinical method.

The aims of this study, accepting Ginsburg's classification, are discovery and identification. In a pragmatic way we decided to utilize Ginsburg's suggestions on interview structure for these activities (Ginsburg, op.cit.):

> When the discovery function is stressed, the clinical interview procedure begins with (a) a task, which is (b) open ended. The examiner then asks further questions in (c) a contingent manner, and requests a good deal of reflection on the part of the subject. ... For the purposes of identifying and describing structure, the clinical interview involves three especially relevant sub-goals. First, the clinical interview is intended to facilitate rich verbalization which may shed light on underlying processes. .. (second) attempts to check verbal reports and clarify ambiguous statements. Third, the method uses procedures aimed at testing alternative hypotheses concerning underlying processes.

There remain, however, many problems with this method. As Posner notes (1978):

> The problem with this method is the interviewer's limited memory capacity. It is difficult to remember, to code and to review just what parts of a substantial body of knowledge has been included in an interview.

The problem has not been resolved in this study. Awareness of it may, however, minimize its effects.

Another problem is that pupils are not accustomed to thinking aloud. As Krutetskii writes (1976):

> Sometimes the pupil might think he is being asked to give an observation and description of his own mental processes ... The very purpose of observing, as is known, can completely distort the picture of thought.

Kruteteskii's attempt to overcome this problem was utilized here:

> First it was explained to the examinee just what was required of him: that he not tell about how he was thinking but that he simply think aloud ... An instruction went like this: "Think aloud ... I am interested not in your final decision, not in the time it takes, but in the process itself ... pretend there is no one here but yourself.

There is also a very practical problem pinpointed by Piaget (1951):

> It is hard not to talk too much when questioning .. to be suggestive .. to find a middle course between systemization due to preconceived ideas and incoherence due to the absence of any directing hypothesis.

As well as asking leading questions it is also easy to interpret responses to fit one's own theory. Again awareness of this possibility and an honest desire to avoid it are the best tools to fight against this. Finally the responses may be affected by the subject's reaction to the interviewer. This was especially acute in this study, as we have mentioned, as the subjects were students in the same school that the author (the interviewer) taught in. The problems of teaching the concepts surrounding those being examined and of students having a strong (good or bad) personal reaction to the interviewer were thus very real. Given that the author is, and was, a full time school teacher and time off was not possible there was no choice over this. It will have to suffice to say that the problems were recognized and every attempt was made to negate them.

## DETAILS OF THE INTERVIEWS

Questionnaire 1 was first administered on 8/9/83 and we hoped to conduct the interviews as quickly as possible after this so that subjects would remember their reasons for their responses. Interviewees were to be selected mainly as representatives of typical responses but clearing up ambiguities in written responses, probing naive and sophisticated answers and following up unexpected replies were also to act as selection factors. Collating the data proved more burdensome than was expected. We were able to work largely to the above criteria but the timing was somewhat later than desired. The interviews were conducted between 30/9/83 and 19/10/83. Twelve subjects were initially asked to oblige and although three of these declined, two other subjects later agreed to help. Details of those interviewed are displayed below. We decided beforehand that a weighting of two to one, concentrating on those doing A-level mathematics, would allow the best evaluation to be made of the effect of a first course in calculus. We ended up with seven who were doing A-level mathematics and four who were not. Questionnaire 1 was administered for a second time on 4/5/84 and the second set of interviews conducted between 16/5/84 and 13/6/84. The general rationale for selection was to use subjects who had been interviewed before so that rationales for variations in responses could be compared with available data. One subject (non A-level mathematics) had replied *I don't know why* to a great many questions in the first interview and was thus not interviewed a second time. Another (again non A-level mathematics) appeared willing but repeatedly failed to turn up. It was felt best not to push an unwilling subject. Two other subjects, both doing A-level mathematics, were chosen. Neither had been interviewed the first time. Both had proven themselves very able at A-level mathematics. As several subjects in the A-level mathematics group were clearly weak (re the A-level course) it was felt that strong subjects would give us a better overview. The second set of interviews thus had nine mathematicians and two non mathematicians.

All interviews were conducted in the school. Busy days for the students were avoided so that they were not too tired. The interviews were held in a small room. The interviewer sat beside the subject with the questionnaire and a tape recorder in front of them. A relaxed atmosphere was created and the interviews started with jokes and questions on how they were getting on. We went into each interview with specific points to to be covered but with the intention of following up points raised in the session. A useful technique was intimated, prior to the first interview, by the District's Educational Psychologist. His advice was that the interviewer should often simply repeat what the subject said. This, it was claimed, gave the interviewee the feeling that the interviewer was on the same wavelength as the subject and would encourage them to elaborate. This was usefully employed. A common dialogue, then, would often have the following format:

SUBJECT   I don't think of infinity as a number.

INTERVIEWER   Infinity isn't a number.

SUBJECT   No it's more of an idea.

The interviews lasted between 10 and 40 minutes with an average time of about 20 minutes. The recordings were later transcribed. The transcipts do not form part of the present work.

## DETAILS OF THE INTERVIEWEES

We refer to interviewees by their initials. We give their sex, whether they are doing A-level mathematics (group N and M as in Chapter Six), their O-level mathematics grade, the times of the interviews and the reasons for initial selection. The areas covered in interviews are displayed in Table 7.1. The question numbers are those on Questionnaire 1.

GA male M B 11/10/83 and 13/6/84

Typical responses but reasons for No,No,Yes for questions 24,25 and 26 required.

PB male M A 7/10/83 and 6/6/84

Typical in most of his responses but put 'No' for most of the graph questions (questions 30-33). Expected sophisticated responses.

JC male N B 18/10/83 and 8/6/84 Typical responses.

CE male N B 19/10/83 and 6/6/84

Typical responses but changed his answers a number of times, especially on cardinality problems.

JH male N B 5/10/83 and 18/5/84

Typical responses but believed in a smallest number.

VM female M B 30/9/83 and 16/5/84

Atypical responses to the cardinality questions 9 and 12 and 'Yes' for all of the series questions (24,25 and 26). Atypical response to Q27 (sequence of jagged lines).

PP male M B 4/10/83 and 18/5/84

Atypical responses to many questions (3,4,9,11,24 and 35). Strong sense of actual infinite suspected.

LS female M A 3/10/83 and 17/5/84

Typical but conceived of a smallest number.

SW female M B 3/10/83 and 8/6/84

The nearest to a perfectly typical overall response in the mathematics group.

GH female N C 14/10/83

The only subject to have failed O-level mathematics the first time (passed in retake year). Selected as an representative of those weak at mathematics.

MW male N B 12/10/83

Typical responses but unsure about the meaning of 'converges'.

DG male M A 22/5/84

Selected as a very able A-level mathematician.

DL male M B 22/5/84

Selected as a very able A-level mathematician (despite the B at O-level).

The table below displays the items/concepts discussed in each interview. The key to the column headings is: SIZ - size of numbers, whether largest and/or smallest nubers exist; INF - infinity, discussion on the nature of infinity; 1/0 - discussion on the nature of 1/0, $1/(1-0.\dot{9})$, etc.; $0.\dot{9}$ - discussion on the nature of $0.\dot{9}$ ; CAR - discussion of some or all of the cardinality questions; SER - discussion of the questions on series; HYP - discussion on questions 22 and 23 where infinity and infinitesimals are hypothesized; /\/\/\ - discussion on the limit of the sequence of jagged lines;

SEQ - discussion of the four phrases with respect to the sequence 0.9, 0.99,...; GRA - discussion on the four phrases with respect to the graphs; WOR - discussion on the remaining questions using the four phrases (questions 35, 36 and 37). The table does not indicate how long was spent on each question. Sometimes a great deal of time was spent, other times little time was spent. Moreover the closeness of many of the concepts means that some divisions are artificial, e.g. infinity is clearly discussed when 1/0 is. The rows refer to the subjects and the interviews (one or two, where applicable).

**TABLE 7.1** CONCEPTS DISCUSSED IN INTERVIEWS

| | SIZ | INF | 1/0 | $0.\dot{9}$ | CAR | SER | HYP | ΛΛΛ | SEQ | GRA | WOR |
|---|---|---|---|---|---|---|---|---|---|---|---|
| GA1 | X | X | | | | X | X | | | X | X |
| 2 | X | X | X | X | X | X | X | X | | X | X |
| PB1 | | | | X | X | | | X | | X | X |
| 2 | X | X | | | X | X | X | | X | X | |
| JC1 | X | | X | | X | X | X | X | X | | X |
| 2 | X | | | | | X | | | X | | X |
| CE1 | | | X | X | X | | | | X | | X |
| 2 | | | | | X | X | | | X | | |
| JH1 | | | | X | X | | | | X | X | X |
| 2 | X | X | X | | X | X | X | | X | | X |
| VM1 | | X | | X | X | X | | X | X | | |
| 2 | | | X | | X | X | | X | | X | |
| PP1 | X | | X | X | X | X | X | | | | |
| 2 | X | X | X | | X | | | | X | X | X |
| LS1 | X | | | X | X | | X | X | | X | X |
| 2 | X | X | X | X | X | | X | | X | X | X |
| SW1 | | X | X | X | X | | | X | X | | |
| 2 | | X | X | | X | X | | | X | | X |
| GH | X | | X | | | X | X | X | | | X |
| MW | | | | | X | | X | | X | | X |
| DG | X | X | X | X | X | X | | X | X | | X |

# CHAPTER EIGHT

# ANALYSIS OF PROTOCOL DATA

We discussed the place of the interview in research, the structure of the interview and the reasons for selection of subjects in the previous chapter. In this chapter we present selected protocols.

We have three aims: evaluating the theses outlined in Chapter One; clarifying questions left unanswered by the questionnaire data; and providing a clear exposition of subjects' thought processes. The first aim requires an attempt to falsify the theses; the second aim requires a close examination of the protocols in the light of the questionnaire data; the third aim permits us to quote selected protocols that need not be typical but reflect an aspect of adolescent thought.

Subjects are identified by their initials, group (M or N) and whether the interview was the subject's first (October) or second (May) one. Thus PPM1 indicates the subject PP (as in the previous chapter) is in the M group and it is the first interview. The questions are numbered as in Chapter Six. We use INT for interviewer and SUB for subject throughout this chapter.

# INFINITY

Sixth Formers have a concept of infinity. It emerges in terms of non terminating processes, in terms of aggregates containing more than any given number of elements and as a generalization of a large number.

## NON TERMINATING PROCESSES.

All of the subjects except PPM used language that suggested they understood the non terminating nature of an infinite operation. The following are taken from the first interview with each subject.

**NB To save the reader continuous reference to Chapter Six we present an abridged version of the question after the question number. Exceptions to this occur when the question has recently been presented.**

> GAM1 (Q13, 0.1+0.01+...) You're still going to have a number, so you're still going to be adding something else onto it, continually.

> PBM1 (Q19, comparing the cardinality of N with that of the even numbers) Well, I thought they both go on indefinitely. I wouldn't think you can really compare them.

> JCN1 (Q25a, sequence of jagged functions) There'd always be a slight wave. You can go on to infinity going 1/32, 1/64.

> CEN1 (Q19) This sequence will never end.

JHM1 (Q11, $0.\dot{9}<1$ ?) I don't think 0.9 is 1 because however when you go on you're always one little bit off.

VMM1 (Q20, comparing the the cardinalities of N and $R_{(0,1)}$) 'cos it's easier to think of whole numbers going on for ever and ever and ever and ever rather than to get all the decimal numbers between 0 and 1.

LSM1 (Q25a) It'll never get down to 0. However far you divide a fraction by 2, keep on dividing by 2, it's never ever going to reach 0.

SWM1 (Q20) I thought if you went to thousands of millions of decimal places you'd get as many numbers as whole numbers but you can go on and on. I couldn't compare them.

GHN (Q12, 1+1+1+...) You can carry it on 'cos there isn't, I don't know what I put. How big's the biggest number ? You can carry on and on can't you ?

MWN (Q28, Is there a limit ? - applied to numeric sequences) Those two (first two sequences) are going to that infinite value aren't they. Its going on to some infinite number. It won't actually reach that number.

DGM (Q3, What is 1/0 ?) If you think of it (infinity) as the highest number you can get then you can add one to it and get a higher number, so there's no numeric answer.

DLM (Q13, 0.1+0.01+...) As you keep going higher and higher you're just evening them out each time, then you can't get an answer. You just keep going forever.

The subject PP gave interesting responses. As we shall see below he was the only one to clearly regard infinity as the largest number. During the interviews we thought the subject's concept image of the real line was very similar to an intuitive image of the nonstandard number line. On re-examining the transcript a finitist interpretation to some of his thoughts was possible, however. For example, after a long pause he says (of $1/\infty$):

Would it not be 0.00, lots of noughts, with a number on the end ?

a *number on the end*. Is this a finitist statement ? Later in this first interview he claims:

$0.\dot{9}$ is 0.9 carrying on forever, carrying on for a long time.
So it must be less than 1.

*for a long time*. Again, is this a finitist statement ? In the second interview similar statements are made. We initially thought the following claim indicated recognition of the non terminating nature of infinite processes (which it may). Explaining why the limit of the curve in Q30 (y=1/x) is not 0, he says:

'cos it never actually reaches zero. It'll get very close to it but it will never actually reach it.

It is possible to give this a finitist interpretation. 1/1000000 is very close to 0, though it does not reach it.

We suspect this subject to be thinking in both a finitist and an infinitist manner. This is quite possible. Tall (1980b) develops the idea of an infinite measuring number by extending the idea of finite measuring numbers. Our subject could be in the transition stage of this development. Unfortunately we cannot substantiate this conjecture.

A close inspection of all the protocols, with the specific aim of finding instances of non recognition of the non terminating nature of infinite operations, revealed no such instances, apart from PP above. Three subjects, however, appeared to have quasi finitist ideas mixed in with their non terminating view of infinite processes:

GHN (Q2, Is there a smallest number ? Subject responded 'No')
They keep going down in points and they get smaller and smaller. But I think you have to stop sometime.

INT Why ?

SUB Well, you couldn't get any smaller. Well they'd be tiny. So small you couldn't measure it.

INT What would that be ?

SUB I'm not sure but I know you couldn't measure it. Your brain wouldn't be able to and even the most advanced computer wouldn't be able to measure it.

SUB Well it will get really small but it won't actually get to 0.

INT But they'll be a time at which we won't be able to go any farther will there ?

SUB Yeh, but it won't be zero will it ?

As with PPM, the subject here appears to display aspects of finitist thought even though elsewhere she displays an understanding of the non terminating nature of infinite processes.

The subject below changed his mind during the second interview. He had responded 'No' to Q13 (0.1+0.01+...) on both administrations of Questionnaire 1. On being asked why there was a long pause, then:

JHM2 I was thinking eventually you will get to the end of your infinity of noughts and they will add up.

INT And what will your answer be ?

SUB A row of noughts.

INT $0.\dot{1}$

SUB Yes.

INT So you're now saying we can get to $0.\dot{1}$ ?

SUB I only think theoretically we can get to it.

At the end of the interview the subject asked to return to this point. He stated that he would say 'Yes' now:

SUB I think maybe it's because you will reach your endpoint, an infinity of noughts, and then you can add your ones up. You'll have an infinity of numbers of ones.

INT Say that again.

SUB You'll eventually add one nought to it and it won't get any smaller as when you add one to infinity it won't get any bigger, and your ones will add up and you'll get an infinity of number of ones.

Although this has some resemblance to the former subject's protocol (GHN) in that there is a terminus, we do not believe the subject sees this as a finite terminus. We interpret his remarks as a move towards the mathematicians' limit concept. Unfortunately we had not asked this question in the first interview and so we are unable to judge whether this was the result of the calculus course.

With the next (and last) of the possible candidates, for not recognizing the non terminating nature of infinite operations, it is difficult to determine whether finitist thought is present. This is because many prior remarks by this subject display a recognition of the non terminating nature of infinite operations. Nevertheless, it does appear that the subject embraces a practical finitism akin to GHN, above. The subject is responding to the question *Does 0.1, 0.01, ... get to 0 ?* :

GAM1 Well its limit is 0, but I wasn't quite sure whether it got to 0. I think effectively it will be at zero, won't it ?

INT How ?

SUB Well it's getting smaller and smaller, isn't it ? So eventually it will be at 0, it will be so small it will be at 0. So really I suppose, I'm not quite sure.

INT What do you mean by effectively ?

SUB I think it's so close it doesn't make any difference.

INT Not even in theory ?

SUB No, not even in theory.

We conjecture, as with PPM, that the subject is in a transition stage utilizing concepts from both finitist and infinitist thought.

We believe the counter examples do not refute the claim that subjects recognize the non terminating nature of infinite operations. They do suggest, however, that other concepts, including finitist ones, are brought into play.

We now turn to the second part of our thesis that adolescents do have a concept of infinity, that they can conceive of aggregates containing more than any given finite number of elements.

## INFINITE AGGREGATES

The second argument to our opening thesis is that subjects do have a concept of infinity in that they can conceive of aggregates containing more than any given finite number of elements. We now examine subjects' protocols to the cardinality questions.

As expected, subjects had no idea of standard cardinality concepts as perceived by mathematicians. The responses to questions 19 to 23, as seen in Chapter Six, clearly show this. Interviews showed, however, that all subjects interviewed could perceive of infinite collections. Although our purpose here is merely to show this, attempts to compare infinite sets utilized three main forms of argument. We thus illustrate each argument with selected protocols. These extracts are by no means exhaustive. Moreover, subjects, as we shall see later, moved from one rationale to another in response to the questions. Our extracts come from every subject except GHN, who was not questioned on her cardinality responses.

**Same number of each** - because both are infinite.

GAM2 (Q19, comparing the cardinality of N with that of the even numbers) INT Why the same number of numbers in both ?
SUB Mmm...Well, in both cases you're going to, you can't stop. You're always going to add one on in that case and two on in this case. So effectively you're going on to infinity, and infinity equals infinity. So that's probably why the same in both.

CEN1 (Q19) I suppose I put *the same in both* because the definition of *same* there is an *endless number* really. This sequence will never end, neither will this one. Therefore, I suppose, you could say there are the same in that both stretch to infinity.

PPM2 (Q22, comparing the cardinalities of $R_{(0,1)}$ and $R_{(0,10)}$) Well there's going to be an infinite number between 0 and 1 and there's going to be an infinite number between 0 and 10, so they're the same.

LSM1 (Q19) If this one goes on forever, so must this one. So there's the same number in both.

**Can't compare** Some held that they could not compare cardinalities because they were not sure. Others held that the sets could not be compared because infinity did not allow comparisons. As can be seen from the responses below, the boundary between *can't compare* and *same in each* is very fuzzy:

PBM2 (Q20, comparing the cardinalities od N and $R_{(0,1)}$. Subject put 'the same in each').

INT Why the same number of each ?

SUB ...not sure...Both have an infinite number of numbers in them ...I'll change my mind. You can't really compare these because both will go on to infinity. For every number there is between 0 and 1 there will be a whole number. You could pair them off and keep on going forever. Neither of them actually finishes or ends anywhere.

INT (questions the subject's use of pairing).

SUB You can't really compare the two, there's so many numbers. There's not an actual definite number, so you can't say it has more than another. They've both got an infinite number.

JCN1 (Q20) Well there's an infinite number of whole numbers and there's an infinite number of numbers between 0 and 1. But I'm not sure of this 'cos you don't know how much infinity is. So you don't know if there's the same number of each.

DGM (Q19, comparing the cardinality of N with that of the even numbers) Because again both series is infinite.

INT And you can't compare one infinity with another ?

SUB Yeh.

INT If there's an infinite number of numbers there and there, are they the same infinity ?

SUB Yeh, I suppose it is, so I should have put iii. You can't really compare them.

VMM1 (Q22) Sounds like you have 10 times more decimal numbers but you haven't really 'cos they go on to infinity so you can't really count them .. if it wasn't infinity there'd be 10 times more numbers between 0 and 10 than 0 and 1, but since it's infinity you can't say how many there is.

SWM1 (Q20) I thought if you went to thousands of millions of decimal places you'd not get as many numbers as whole numbers, but you can go on and on. I couldn't really compare them.

**The generic law** Although none of the subjects outrightly quoted the generic law as a rule, many, as we shall see, used it as a premise:

PBM2 (Q21, comparing the carinalities of $R_{(0,1)}$ and the unit square) For every coordinate between 0 and 1. Say that was infinity, then for all the points in the square it would be infinity squared because there are that many points going up as well as along.

INT And infinity squared is bigger than infinity ?

SUB Yes.

INT And bigger than the infinity between 0 and 10 ?

SUB Well thats another of these that's difficult to compare because you haven't got the same starting point 'cos this is to do with area and that has to do with a single length. Between a known length and a known squared length it seems more obvious that there's going to be more points on the whole square than there are on the whole line, while they've both got an infinite number of points in them.

INT What about if we quartered the square ?

SUB Well,there should be a quarter the number of points. If it was infinity squared then that would be a quarter of infinity squared.

MWN (Q19 Subject put 'more in first row') Well it seems on first looking at it that there's twice as many but when you try and complicate it because you don't know where the sequence ends (sic). You can't think of it. You can't sort of define it...I'd probably put same in both now or can't compare.

INT Same in both. Could we call it infinity in both ?

SUB Yeh, if you think of infinity in terms of it never ends. But if it never ends you can't really comapre, can you...It's hard to say which one to try to compare them.

JHM2 (Q19) Because you're dealing with a greater set of numbers. If you do reach infinity you're bound to have more of those than even numbers.

INT Why ?

SUB Because there are only half as many even numbers as there are numbers.

The generic law may disguise finitist thinking as the following shows. The subject put 'No' in the first administration of Questionnaire 1 and 'Yes' the second time. However, his ability to perceive of infinite aggregates is clearly evident:

(Q23, com^paring the cardinalities of a circle and enclosed square) Well, the first time I probably imagined there being a certain amount, maybe a defined value the size of a pen or something. The second time I thought theoretically you could get any number of points there and any number of points in any of them 'cos it's infinity.

## INFINITY AS A PROCESS AND AS AN OBJECT

Our second thesis is that infinity exists, to subjects, as both a process and an object.

In traditional grammar an object is a substantive that receives the action of a verb. We adopt a looser definition and view an object as a single entity that can be referred to in speech. The protocols above show that subjects are capable of perceiving the cardinality of an infinite set as an object in that they can compare the cardinality of one infinite set with the cardinality of another. Even when subjects say *can't compare* they use language that indicates that they are considering single entities that can be referred to, e.g.

PBM2 (Q20, comparing the cardinalities od N and $R_{(0,1)}$ )
They've both got an infinite number.

As we shall see shortly, although most subjects reject the concept of a largest number they can nevertheless conceive of such a thing as an object.

Regardless of how they perceived infinity as an object most subjects also saw infinity as a process (*also* in the sense of being intrinsically tied in with, not in a separate way). Forming questions to test this is difficult because subjects do not usually theorize about concepts of infinity as mathematics educators may. Thus the question *Is infinity a number, idea or process ?* is not the kind of question that will lead to a meaningful answer. The best way to examine subjects' ideas on infinity as a process is to examine their responses to questions in interviews. This is problematic too, however, since the questionnaire was number based and subjects geared their answers to that. The clearest statement of infinity as a process was:

VMM1 INT What is infinity ?
SUB Something that goes on forever.
INT What, like 1111111... ?
SUB Doesn't have to be a number...
INT Go on.
SUB The answer to that is infinity but it's not what I think of as infinity ...1/0 you can't work out but infinity, I think, is doing something where you get a continual answer, it goes on forever.

*Infinity as a process* is the claim that *infinity means 'going on and on'* and is, we hold, the schema behind subjects' recognition of the non terminating nature of infinite operations. As with recognition of the non terminating nature of infinite operations most of the subjects used language to suggest this:

GAM2 (Q19, comparing the cardinality of N with that of the even numbers) You're **always** going to add one in that case and two in this case, so effectively you're going on to infinity.

PBM2 (Q3, What is 1/0 ?) There isn't actually a largest number, but because there isn't a largest number you can say infinity which means the number just continues going on.

JCN2 (Q26c, 1, 1/2,... converges to __. Subject put '$1/\infty$' in the the first administration of Questionnaire 1 and gave no response the second time) You can't get smaller than $1/\infty$ . Infinity is going on forever, so it just carries on getting smaller.

CEN1 (Q26c Subject put '$1/\infty$'. He is asked if infinity is a large number). A never ending rainbow if you like. I suppose that's what I think of infinity as. I don't think of it as any specific thing. It's just something which you'll never get to.

JHM2 (Q3, Subject put '$\infty$' in the first administration of Questionnaire 1 and 'impossible' the second time)

I must have thought that if you divide something by 0 you can just keep going and going and going.

VMM1 (Infinity is) something that goes on forever.

LSM1 (Q2, Is there a smallest number ?)
No, 'cos infinity goes on forever.

GHN (Infinity) Well it's something that doesn't end.

MWN (Q19) I'd probably put the same in both now or can't compare.
INT Same in both. Could we call it infinity in both ?
SUB Yeh, if you think of infinity in terms of it never ends. But if it never ends you can't really compare, can you.

The subjects who did not verbalise that infinity means *going on and on* were all in the M group. PPM, as we have seen, saw infinity as the largest number. SWM also saw infinity as a number but was less sure of herself than PP:

SWM1 (Q3 Subject answered '$\infty$')
INT When you write infinity, what do you mean by it ?
SUB Something very big. Can't define it - wouldn't know how to define it.
INT Would it be a number or an idea or something else ?
SUB It would be a number.

In the second interview she said (to the same question)

> There's no answer. Well, to me there's no answer.

DGM and DLM (recall they were selected for the May/June interviews because they were particularly able mathematicians) were both dubious of the legitimate mathematical status of infinity. Their responses to Q3 (What is 1/0 ?) were identical:'$\infty$' in the first administration of Questionnaire 1 and '*undefined*' the second time:

> DGM Well again, it's the - if you think of it as the highest number you can get then you can add one to it and get a higher number. So there's no numeric answer to it.

> DLM Well first I thought you can get any amount of noughts into 1 so it's $\infty$ , but then I probably thought since you can put any amount you can't really put a number to it so I put undefined.

Notice that both of these subjects do recognize infinity as a process but see the logical inconsistency of the schema.

We must stress that subjects who claimed 'infinity is *going on and on*' had other concepts of infinity as well. As we have said, we shall see this below when we consider infinity as a number. All we claim is that *infinity as a process* is a widespread schema used by adolescents. As a schema it can be used to answer questions about infinite sequences or collections. We have seen this above with the protocols on the cardinality questions.

Moreover *infinity as a process* can be seen as a major rationale for the description of various types of numbers as infinity. We select three examples of large, small and recurring *infinities* to illustrate the point:

CEN1 (from Q20, comparing the cardinalities of N and $R_{(0,1)}$. Subject said 'can't compare' and explained why).

INT Would you still agree ?

SUB Well, no. Maybe given time to think about it no I wouldn't, 'cos this again carries on endlessly. But even though these are two specific numbers (meaning 0 and 1), the number of numbers you can have between them also carries on endlessly. So there's an infinite number of numbers in that and that.

INT So what would you say now ?

SUB Well, I suppose, the same number of each but you can't sort of say a specific number. It's just a massive number. Well, it is just infinity in each set.

GAM2 (Q25a, sequence of jagged functions) Well it's like infinity. Same sort of principle. It's something that gets smaller and smaller.

GHN (Q3 Subject put 'infinity')

INT Is infinity a number ?

SUB I don't know. Well, infinity with numbers it's like $3.\dot{3}$ and it carries on.

## INFINITY AS A NUMBER

**Infinity as a large number** As replies to Q1 (Is there a largest number ?) clearly show, subjects do not believe in a largest number. This is stable over the groups and (with the MHS sample) over time. The replies to Q3 (What is 1/0 ?) initially appear to conflict with this but, as we shall see, the strong *infinity* response from the M group does not indicate a numeric response but a generalization of a numeric response. The only subject from the MHS sample to respond 'Yes' both times claimed:

PPM1 I was thinking of infinity.Infinity is the largest number.

Several subjects thought of the largest number as an abstraction that doesn't really exist:

MWN INT What is infinity ?
SUB I've always thought of it in terms of the largest number. There isn't just one thing but you think of it as the largest number to simplify.

PBM2 (Q3) INT What do you mean by infinity there ?
SUB A number. There isn't actually a largest number but because there isn't a largest number you can say infinity which means the number just continues going on.

Similarly, with a slight change of terminology, some subjects claimed that infinity is a generalization of a large number:

LSM2 (Q5, Is $\infty+1>\infty$ ? Subject said 'No'.) 'cos infinity you're just generalizing,to me, a whole mass of numbers somewhere over there.

GAM1 (Q3) INT What is infinity ?
SUB Well it's *not* a *number*, is it ? It's just something that's extremely large, so large you can't put a number to it, just call it infinity.

This was stable over time with the last subject:

GAM2 (Q1) It's something that's *extremely* large. *I don't* think of it as a specific number.

It would be a mistake to see different replies as necessarily indicating different cognitive processes. For example, the replies of 'undefined' and 'infinity' to Q3 can be seen as very similar responses in subjects' conceptions, as we saw with *DGM* and *DLM* above.

The lability of this dichotomy is further illustrated by the following subjects (note that the Mr X referred to by both, but credited with different interpretations of 1/0, is the same teacher and was one of the MHS staff aware of this research):

*JHM2* (Q3 Subject *changed* his *mind from* 'infinity' to 'impossible')

SUB That's from the A-level course

INT Has Mr X said that ?

SUB Yes

INT Do you believe him ?

SUB Yeh,and the computer gives me an error as well.

INT It couldn't be infinity ?

SUB No. I don't think so really.

INT How come ?

SUB Well you can't really divide anything with 0. I don't know. I can't explain really why...(encouraged but not prompted)...Well I think there I must have thought that if you divide something by 0 you can just keep going and going and going.

INT And now you don't think you can ?

SUB Well, mainly because of what people told me. I don't know really.

VMM1 (Q3 She put '0' but saw her mistake)

INT Do you still agree ?

SUB No, it's infinity.

... later ...

SUB Like 1/0 wouldn't give you an answer, would it, so you can't say it's infinity.

VMM2 (Q3 Subject put 'infinity' this time)

SUB 'cos I've learnt that 1/0 is infinity and I didn't know that before.

INT Who told you that ?

SUB Mr X, to do with asympt‾otes on a graph.

A cogent reason for not seeing infinity as a number is the fact that any number can be incremented. Several subjects displayed this. The following were responses to *Why isn't there a largest number ?:*

DGM Because if you think of any number you can add one to it.

PBM2 You can keep on adding one to any number you got.

GAM1 Well,if you think of a very large number that comes into your mind with so many noughts,you can always think of one number higher.

The above merely notes trends in adolescent thought on infinity as a large number. We are not in a position to quantify the relative strengths of these trends. What emerges is a recognition that infinity is the largest number but such a number does not really exist. Rather, it is an indeterminate form, a generalization of a large number.

**Small numbers** The paragraph above applies equally to the infinitely small (the derivation of both from infinite processes tying them together in the subjects' minds, as we shall see shortly). As replies to Q2 (Is there a largest number ?) show, subjects do not believe in a smallest number. This was stable over the groups and (with the MHS sample) over time. The responses to Q8 and Q9 (concerning an assumed infinitesimal) are, as with Q5 and Q6 (concerning an assumed infinite number), we hypothesise, the result of centring on number properties.

They show, at the very least, that nonstandard numbers would have to be introduced very carefully. Q10 (Can you believe in an infinitesimal number ?) was, unfortunately, not included in Questionnaire 1. They appear to conflict with the responses to Q2. We did, however, put the question to several subjects in the interviews.

Two of the subjects interviewed responded 'Yes' to Q2. PPM (who was mentioned above as viewing infinity as the largest number). He appears to be consistent as he viewed $1/\infty$ as the smallest number. The other, LSM1, admitted she was thinking of 0 and changed her mind:

INT Would it be something like nought point nought recurring one?
SUB It can't be 'cos it just goes on to infinity.
INT Is there an infinitesimally small number ?
SUB No, 'cos infinity goes on forever.

We expected the main reason why there is no smallest number to be that any number can be halved, tenthed, etc. This is the dual of the belief that there is no largest number because any number can be incremented:

GAM1 (Q1 and Q2) Well, if you think of a very large number that comes into your mind with so many noughts, you can always think of one number higher, higher than that. So there really isn't a largest number. And the same will be true of the smallest number.

DLM If you choose a number you can always find one that's bigger or find one that's smaller.

This was not the most quoted reason, however. Of the other subjects quizzed on their responses to Q2 two, PBM2 and DGM, considered negative numbers; one GHN (see above, p.190) had quasi finitist ideas; while the others used, as LSM above, the schema of *infinity as a process*. This is very similar to the idea that any number can be halved, etc.:

JCN1 Well, you can go on putting 0.000 as long as you like.

JHM2 Because you can have an infinity of noughts before you can have a one. So, since you can't reach infinity, you can't reach the smallest number.

While there is clearly some truth in our hypothesis there are also clearly other factors affecting subjects' intuitions of the infinitesimally small.

As with infinitely large numbers, infinitely small numbers were seen as abstractions that don't really exist and as generalizations of small numbers. In this sense they could be seen as the classical useful fictions of infinitesimal calculus. Only one subject stated this:

GAM2 (Q8, Does 2+s=2 ?) INT Can you believe in a number like s there ?

SUB I can believe in something infinitely small, just something to say it's extremely small, like infinity is useful for something that's extremely large. Just a sort of expression.

Several subjects used $1/\infty$ as an expression for the smallest number. This was unexpected and arose several times in both the questionnaire and interview responses. $1/\infty$ can be seen in two ways, as an ideal element and as a dynamic infinitesimal. Although we do not claim that infinitesimals as useful fictions or infinitesimals as dynamic entities are the principle conceptions adolescents have of infinitesimals we do believe these conceptions are present in many aspects of their thoughts. The protocols below arise from Q26c. The subjects below claimed 1, 1/2, 1/4, ... converged to $1/\infty$:

> CEN1 That's because this denominator will never come to an end, it just keeps on going 1/16, 1/32. So infinity there can represent any number depending on where you draw the line ... I suppose that's why I put $1/\infty$, it could be any number that can be divided by 2.

> JCN2 ...something as small as possible.
>
> INT *The smallest number ?*
>
> SUB Well no, I don't think there is a smallest number.
>
> INT There seems to be a contradiction there.
>
> SUB You could say the smallest number is $1/\infty$ which is ..(long pause).. you can't get any smaller than $1/\infty$ ..(pause).. infinity is going on forever, so it just carries on getting smaller.

> PBM2 So if you were to take it as meaning getting there it wouldn't actually get to 1, oh, get to 0. It would get close to it but it wouldn't actually get there. Whereas if you were to

take converges to mean actually getting there I would have thought it would be $1/\infty$.

INT $1/\infty$ means a definite thing to you ?

SUB It means 1 over the largest number, well, if there were a largest number. You know, carrying on.

SWM2 (Subject put '$1/\infty$' in the first administration of Questionnaire 1 and '0' in the second time)

INT Why 0 and why the change of mind ?

SUB Converged to 0 because the number underneath gets bigger so that's more. It gets closer to 0. I don't think it actually gets to 0. So I'd agree with the first one more.

INT $1/\infty$ meaning ?

SUB Something very small.

INT Why do you think you put 0 ?

SUB I think I probably paid more attention to the converges, to the general sort of limit rather than, that I thought, you know, what was it, taking it as a more definite thing. $1/\infty$, I don't know what it is but it sounds more definite than 0. You have to round it up or down somehow.

These remarks bring up the point of effective, as opposed to actual, infinitely small numbers. Approximation, we believe, is more readily accepted with small numbers than with large numbers:

GAM2 (Q9, Does 2xs=s ? The subject was the only one to say 'Yes')

SUB I think I thought of it as 0 at the time. It's effectively close to 0 and 2x0=0.

LSM1 (Q8) SUB Well, the number would be so fantastically small that added to 2 it would make very little difference. Well,considering the difference between 2 and the number, so you can forget about it 'cos it's so small.

GHN (Q2, Is there a smallest number ?) SUB They keep going down in points and they get smaller and smaller, but I think you have to stop somewhere.

INT Why ?

SUB Well, you couldn't get any smaller. Well, they'd be tiny, so small you couldn't measure it.

Again this is not a general feature of adolescent thought but merely one of many factors at work.

**Infinite Numbers of All Sizes** We believe that mathematicians, in their intuitive and non analytic moments, tend to think of infinity as a large number and classify the infinitesimals and infinite decimals as different from infinity itself. This has more or less been passed down to their pupils but not completely. It was with some surprize that we came to understand that all three categories could be taken as *infinity*:

GAM2 (Q9) Interesting, because in this respect I've thought of it as an extremely large number, but in this respect I've thought of it as something extremely small.

PBM2 (Q3, What is 1/0 ?) ...like a recurring number will have an infinitesimal (he meant infinite) number of numbers in it..say 1/3 in decimals. Well, that will carry on going. You can't say how many threes there will be in it, so you just use infinity for that.

JCN1 INT So what does infinity mean to you ?

SUB Something with no limit.

INT What kind of something ?

SUB You can have anything really.

INT Is $0.\dot{3}$ an infinite number ?

SUB Yeh.

INT What about the idea of infinity being something very big ?

SUB Well,it can be very small as well.

Then, later in the same interview (Q8):

SUB It must be more than 0, infinity must be greater than 0 because you can get 0.000...1 . It would be a very big number.

LSM2 (Q4, What is $1/(1-0.\dot{9})$ ?)

INT What do you mean by infinity here ?

SUB A number way, way too small to be calculated.

INT Too small to be calculated ?

SUB Be too large wouldn't it.

INT Very small and very large both mean infinity to you ?

SUB Yeh.

SWM1 INT Is there a number smaller than $1-0.\dot{9}$ ?

SUB I was thinking it equals point, a lot of noughts, and a one at the end.

INT What kind of number is that ?

SUB I suppose an infinite number.

GHN (Q3 Subject put 'infinity')

INT Is it a number ?

SUB I don't know. Well, infinity with numbers it's like $3.\dot{3}$ and it carries on.

INT So that's an infinite number ?

SUB Yeh, I think so.

MWN (Q28, For each of the following sequences say whether it has a limit. The subject here is explaining why he said 'no limit' for the first two sequences: 1,0.1,0.01,.. and 1,0,0.1,0,0.01,..).

SUB Yeh, because those two are going to that infinite value aren't they ? It's going to carry on to some infinite number.

We do not claim that these observations reflect the general body of adolescent thought but they clearly indicate that students may see infinite numbers as not only those beyond finite magnitude but as any number generated by a non terminating process.

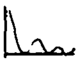
## SEQUENCES AND FUNCTIONS

Protocols indicate that subjects generally recognize the convergence or divergence of monotone infinite sequences and functions. Sequences and functions which oscillate are, however, often misinterpreted.

We must be very careful not to confuse the first claim with the proposition that subjects possess a mathematicians' concept of monotone convergence. As we have seen in Chapter Six and will see below, subjects possess mathematically incorrect generic limit concepts and are led to conventionally incorrect responses by the everyday connotations of mathematical language. We regard an adolescent who registers the general trend to a limit (or to no limit in the case of divergence) as recognising convergence (or divergence) regardless of whether or not they possess generic limit concepts or give conventionally incorrect responses due to language, e.g. *A sequence of numbers cannot converge.*

A very close examination of the protocols revealed **no instances** of failure to notice monotone convergence or divergence of sequences or functions.

Oscillations, however, did affect subjects' responses. Although the overall pattern of responses to the questions with oscillations (33, 34, 35, 28, 29b and 29c) is the same as the overall pattern to the questions without oscillations (30, 31, 32, 28a and 29a), a number of subjects in the interviews centred on aspects of the curves and sequences that mathematicians would not see as important characteristics and vice versa. For example several subjects saw the fact that the curve in Q33 touched the x-axis and that the curve in Q34 did not, as an important feature with regard to convergence but gave the same responses for the sequences 28b and 28c (and 29b and 29c) despite the fact that b (1, 0, 1, 0.1, ...) is convergent and c (1, 0.1, 1, 0.01, ...) is divergent. Answers here are open to a great deal of variation and we present no unifying thesis. However, we feel that the deviations from the mathematicians' thoughts are worthy of note:

JHM1 (Q33,  ) I'd say its limit is but I wouldn't say it tended to 0.

INT How come ?

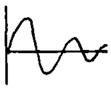
SUB Because its limit is 0 .

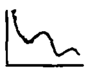
INT So if something's limit is 0 it can't tend to it ?

SUB No tends to is just getting close to. Limit is what it actually is.

INT Is that because it touches 0 ?

SUB Yeh, because it actually touches.

VMM2 (Q35  Subject responded 'Yes' in the first administration of Questionnaire 1, 'No' the second time)
I think of tending to 0 more like that (Q30 ), just one side. I don't know why but .. I thought that when it's tending to , I didn't think that was tending to 'cos it went into minuses as well , so it passed through 0 ,so .. it doesn't tends to 0 then.

LSM1 (Q34  Subject said 'No'). As it approaches it goes away again.

DGM (Q34) I thought it converges because the curve gets closer to a straight line all along but it doesn't approach. I don't Know . I think it's probably because, I put that, because it's approaching there (he points to a decreasing séction) but it's not approaching there (he points to an increasing section).

INT Is that the same for Q35 ?

SUB Yeh, it's approaching the line there but there it's going further away from 0.

DLM (Q28 Subject is asked why 'a' has a limit but 'b' does not).

SUB Because somehow you're alternating between 0 and another number and there's always going to be another number, so there it sort of confuses the issue somehow.

While not an oscillation, the sequence 1, 1, 1,... of Q28d and Q29d presented peculiarities of a similar kind. We believe that subjects' dynamic conceptions find this seemingly static sequence awkward to handle:

GAM2 (Q28d) INT How come the line of ones doesn't have a limit ?

SUB Well it's already at 1 isn't it. It's going to stay at one.

INT And that means its limit isn't 1 ?

SUB Well limit as I see it, when it approaches, when it goes towards 0 (corrects) .. Well this one is already at 1.

SWM2 (Q28d) If something is always there I don't think it has a limit because I think you have to move towards a limit or something like that. If it's all the same number I don't think they .. No, it has to have motion towards it somehow. I don't know why.

CEN2 (Q28d Subject said 'No' both times)

SUB Yes because this sequence of numbers will inevitably be 1 by the nature of the sequence. It can't go over 1 or below 1. It just stays a constant 1, therefore its limit is 1, whereas this is constantly changing, this number here is constantly changing therefore it doesn't have a definite limit.

PPM2 (Q28d) Well I thought if it has a limit it converges onto 1 at a glance, but it keeps on going as 1 all the time, then it won't have a limit it can tend to.

**The existence of limits of sequences and functions.**

It is useful to consider the following hypothesis at this point in our analysis of the protocols: subjects' sense of the existence of a limit of a convergent function, presented graphically, is stronger than their sense of a limit of a convergent numeric sequence.

Questionnaire responses suggest this is the case. We base this on a comparison of Q30 (with *limit*) and Q28a and also on a comparison of Q33 (with *limit*) and Q28b. There is a tendency for subjects to claim there is a limit to the curve but not to the sequence. We shall take this point up again in the next chapter. Protocols, however, do not support this. Unfortunately we did not question subjects on this apparent phenomenon in the interviews. Four subjects were asked about both responses but three of these provide evidence against the thesis and the conceptions of the fourth are difficult to interpret. We turn now to consider their protocols.

GAM1, despite generic limit responses in the first administration of Questionnaire 1, appeared to develop mature limit notions by the time of the first interview:

> (Q28a) I wasn't sure what limit meant at the time but I would say the limit was 0.
>
> (Q30) Definitely has limit 0.

JHM1 stated that only the oscillating curve had a limit in the first administration of Questionnaire 1 but changed his mind during the interview so that numeric and geometric cases were similar:

> (Q28b Subject changed his mind, limit now 0) 'cos it hits 0.
>
> (Q30) INT Why does the curve tend to 0 but not have a limit 0 ?
>
> SUB Because it doesn't actually reach 0.
>
> (Q33) I'd say its limit is but I wouldn't say it tended to 0.

PPM1 said both curves had a limit and both sequences did not have limits in the first administration of Questionnaire 1 but he was not questioned on this. He did a complete volte face in the second interview, viewing only the numeric sequences as having limits. This appears to have been prompted by the recognition of $1 \times 10^{-\infty}$ as a number.

Q28a (Subject wrote $1 \times 10^{-\infty}$ on the paper)

INT That exists as a number, does it ?

SUB I think so but as a very, incredibly small number.

Q30 INT Why not limit 0 ?

SUB 'cos it never actually reaches 0.

The fourth subject interviewed on these question was LSM. In the first administration of Questionnaire 1 she considered only the oscillating curves which touched the x-axis as having a limit (Q33 and Q35). Neither monotone case (Q28a or Q30) had a limit because neither reached 0. She appeared to be accepting finite approximations in the case of the oscillating curve in Q33 but was unsure on reflection:

INT Limit 0 ?

SUB Yes, think so.

INT Because it touches it ?

SUB Mmm, but that wouldn't agree with the jagged line principle really if you'll get to the thickness of that line along there.

In the second administration of Questionnaire 1 this subject said 'Yes' for both curves and 'No' for both sequences. Again she appeared to be accepting approximation only in the geometric context:

(Q29b 'No') 'cos that one gets closer and then goes away again.

(Q33) 'cos you get to a point where the bumps are so infinitesimally small, you're going to call it 0 for convenience.

This appears to support the hypothesis but the discussion on this point in her interview did not go in to this in depth. We must, then, conclude that the protocols do not support the hypothesis and that further research is needed. A conjecture for such research (suggested by the protocol of LSM above) is that: approximation is more readily accepted in a geometric context.

## GENERIC LIMIT

We posit three kinds of limits: finite or real world limits ; mathematical limits ; and generic limits.

Mathematicians have learnt to think of limits in a formal, well defined way (be they of sequences, series or functions). Young children, as Piaget (1956) and Taback (1975) have shown, think of finite or real world limits. It is a major hypothesis of this study that adolescents predominantly think in terms of generic limits.

It is difficult to keep separate all the influences at work when examining subjects' responses re generic limits. We have seen that subjects may interpret the words mathematicians use to describe limits in an extramathematical manner. This is obviously a factor to bear in mind when considering responses. Another factor to keep in mind is the fact that the great majority of subjects see $0.\dot{9}$ as strictly less than 1 (indeed this can be seen as partially determining and partially being determined by generic limit ideas). It is thus internally consistent for subjects to say that the limit of a sequence is $0.\dot{9}$ and not 1 since the two numbers are not identical to them.

Our hypotheses concerning generic limits are: generic limit concepts are dominant in adolescent conceptions of limit; generic limit contexts are slightly stronger in arithmetic contexts than they are in geometric contexts; there is a slight movement away from generic limit and towards the mathematicians' limit concept in the M group. We shall examine each of the interviewees' responses in turn and evaluate the evidence the protocols provide for and against these theses. A fuller evaluation will, as with other theses, be made in the next chapter where we will consider questionnaire and protocol data together.

We examine responses to questions 11, 25a, 27d, 27h, 28a and 30b. As we have already seen and will see again soon, subjects interpret *tends to* and *approaches* as indicating vague, approximate trends. Moreover *converges* was largely seen as an inappropriate and confusing phrase to describe limiting phenomena. We thus only look at responses using the phrase *limit*. This itself is open to differing interpretations by the subjects but less so than the other phrases. We shall regard a generic limit response as: $0.\dot{9}<1$, the curve in Q30 does not have limit 0, limit of 0.9, 0.99, ... is $0.\dot{9}$ and not 1, limit of 0.1, 0.01, ... does not exist and limit of the sequence of functions (Q25a) is a jagged line. We avoid analyzing sequences or functions that are not monotone here since, as we have seen, these bring in extra difficulties in interpretation.

**GAM** Subject gave generic limit responses in the first administration of Questionnaire 1 with the exception that 0.9, 0.99, ... had both $0.\dot{9}$ and 1 as a limit. In the first interview, however, he displayed a sophisticated understanding of the limit notion. He changed his mind in Q30 stating that it did have 0 as a limit. In discussion concerning Q28a he was asked what a limit was. Again he changed his mind since doing the first administration of Questionnaire 1, where he claimed 0.1, 0.01, ... does not have a limit:

> It means it seems to be approaching a certain number. It may not become that certain number but it is approaching it, the difference between them is getting less and less, between the two numbers. In this case it's going towards 0, it's getting smaller and smaller. It may not actually touch it though.

In the second administration of Questionnaire 1 the subject gave many responses indicating a post generic limit conception e.g. $0.\dot{9}=1$, limit of 0.1, 0.01, ... is 0, the limit of the sequence of functions (Q25a) is a straight line and the curve in Q30 has 0 as a limit. In the second interview these responses were supported by arguments such as *effectively* $0.\dot{9}$ *is 1* , etc. Although this use of *effectievly* may indicate finitism or willingness to accept approximation, it does appear that this subject was progressing towards a mathematicians' limit concept. Note that this first appears before a first calculus course could be said to have any effect.

PBM Subject gave generic limit responses in the first administration of Questionnaire 1 with the exception that the curve in Q30 has 0 as a limit. These views were supported in the first interview but he had since been told that $0.\dot{9}=1$. He backed up his $0.\dot{9}<1$ response saying

> Well, it's $0.\dot{9}$ and anything after that, the way we've been taught, if it's $0.\dot{9}$ anything after that won't change it. It's got to be less than 1, the way we've been taught, like in units, tens, hundreds. Anything less than the units column then it's not one. So I felt at the time it couldn't equal 1 but now I think it could equal 1.

Despite this the subject gave the same generic limit responses in the second administration of Questionnaire 1 (even changing his response to Q30). In the second interview he clearly saw recurring decimals as *improper numbers*, they were incomplete. In discussion of the limit of 0.9, 0.99, ... he says:

> Its limit is the final point it will get to. So I think the limit is $0.\dot{9}$ and then again the limit is 1, but it won't actually get to 1, so you can't have 1 as its limit.

Thus, although this is not a clear statement of generic limit ideas, it is very close. In this M group subject's case there appears no movement away from generic limit ideas.

JCN Subject gave generic limit responses in both administration of Questionnaire 1. Both interviews supported this:

> (First interview Q25a) There'd always be a slight wave. You can go on to infinity going 1/32, 1/64.
> (First interview Q28a Subject said 'No limit') Because you

can put as many noughts as you want with a 1 on the end and just carry on.

INT (Interviewer prompts the subject on jumping to the infinite case) What stops you believing the limit is 0 ?

SUB Well, it's not carrying on in the same sequence if you don't have a one.

**CEN** Subject gave generic limit responses, with the exception that the curve in Q30 did have 0 as a limit, in both administrations of Questionnaire 1. Both interviews supported this:

(First interview Q27) I've forgotten the mathematical definition of limit (*it is unlikely that he was ever given it*). When I think of limit now I think where it stops and it won't stop at 1, it will stop at $0.\dot{9}$ which is, if you had a little line 0.00..09, it would go on endlessly. So its limit is something that never ends.

This was stable over time:

(Second interview Q27) INT But how come its limit was $0.\dot{9}$ but 1 was confusing ?

SUB I'm not sure what you mean by limit but its limit is $0.\dot{9}$. That means it'll never get past $0.\dot{9}$. It'll never get to 1, obviously. Those were my lines of thought when I was thinking what to put for that one. You can't really put a 1 there because 0.9 could take a tremendous amount of time, amount of space. You said if you wanted to put a 1 here and $0.\dot{9}$ here. Well, that to me would seem impossible 'cos $0.\dot{9}$ it just goes on and your 1 would be at the end of it but you would never have an end, so you couldn't put a one in.

JHM Subject gave generic limit responses in the first administration of Questionnaire 1. It is difficult to evaluate his limit notions in the first interview because he attached many varying meanings to the four phrases e.g.:

(Q33) Tends to is just getting close to. Limit is what it actually is.

The curve in Q30 did not have 0 as a limit because:

It doesn't actually reach 0.

His reason why $0.\dot{9}<1$ was clearly prompted by generic limit ideas:

Well, I don't think $0.\dot{9}$ is 1 because however you go on you're always one little bit off.

In the second administration of Questionnaire 1 he retained most of the generic limit responses but changed his mind on the limit of 0.9, 0.99, ... The limit was now 1. Although the subject elsewhere seemed to have some mature mathematical limit notions (see above, pp.191-192) he was closer to generic limit notions than to mathematician limit notions

I didn't really see the limit as what it is. I saw the limit as what it's very close to but it isn't actually 1. So you have got $0.\dot{9}$ eventually but you haven't got 1. 1 is its limit it can't reach.

VMM Subject gave mainly generic responses in the first administration of Questionnaire 1. Exceptions were Q25a (the limit, however, she said, would merely look straight but would not really be) and Q27 which had both $0.\dot{9}$ and 1 as a limit:

SUB That's the proper limit (points to $0.\dot{9}$).

INT And that's the improper limit ?

SUB Well, if you've got that as a limit, you've got that as a limit too. Well if that's ($0.\dot{9}$) its limit, that's what it goes up to, I suppose if you rounded it up the limit would be 1. You've only to have one limit haven't you.

Changes in the second administration of Questionnaire 2 were that Q27 only had the limit 1 and the curve in Q30 did have 0 as a limit. The sequence of functions, she still claimed, would only look straight. The second interview did not examine other generic limit ideas. It is difficult to evaluate her beliefs as she appeared very willing to accept approximations.

PPM Subject gave generic limit responses in the first administration of Questionnaire 1 with the exception that the curve in Q30 did have 0 as a limit. He changed his mind on this question the second time (arguing in the second interview, as JH above, that it wouldn't reach 0) and also with the limit of 0.1, 0.01, ..., which then did exist (he wrote 1x10 on the question paper which is clearly a generic limit response). The first interview did not discuss limit notions except for 0.9 <1:

> Well, $0.\dot{9}$ is 0.9 carrying on forever, carrying on for a long time, so it must be less than 1.

In the second interview he showed signs of generic limit concepts, for example with Q33 he responded 'Yes' on the first administration of Questionnaire 1 and 'No' the second time:

> It just keeps going, keeps on fluctuating, until it becomes so small, but it'll never actually reach 0 though you come very close to it.

We are wary about ascribing generic limit notions to the subject, however, for it is possible to ascribe finitist interpretations to his ideas (as we saw above, p.189).

LSM Subject gave generic limit responses in the first administration of Questionnaire 1 and supported these in her first interview:

> INT How come $0.\dot{9}<1$ ?
>
> SUB You're not getting there. You've got to add something to $0.\dot{9}$ to get 1.

(Q25a) It'll never get down to a straight line. It'll never get down to 0. However far you divide a fraction by 2, keep on dividing by 2, it's never ever going to reach 0, which is my idea of a straight line I suppose.

(Q28a) (It does not have a limit) 'cos it never reaches a definite number. It carries on for infinity.

*Getting down to* appeared to be her criterion for a limit:

(Q30) It tends to 0. It's getting there nearer all the time but it's never actually going to get there.

INT But no for the rest ?

SUB It approaches to 0, it converges to 0 but no, it doesn't have a limit.

In the second administration of Questionnaire 2 the subject changed her response in Q29 (seeing the limit as 1 as well as $0.\dot{9}$) and with the curve in Q30 (i.e. it did have 0 as a limit). However, generic limit ideas do not seem to be affected:

(On why $0.\dot{9}<1$) 'cos it's not written the same. There must be a fraction added onto it that makes it equal to 1.

(Q27 Subject responded 'Yes' to all parts)

Well, when I last did this with you, you said that $0.\dot{9}=1$ 'cos it can't equal anything else (*I have checked. I did not*). Therefore that's what I'm going on here but when it comes down to this here I still can't appreciate that $0.\dot{9}$ does actually equal 1.

SWM Subject gave generic limit responses in the first administration of Questionnaire 1 with the exception of Q27 where she responded 'Yes' to all eight parts. This was repeated the second time with the exception of Q28a, where she responded the limit was 0, and Q30, where she responded that the curve has 0 as a limit. The only one of these questions discussed in either interview was Q25a in the first interview. Other than repeat her 'slightly jagged' response it did not, from this, appear that generic limit concepts were present. She did, however, reveal generic limit ideas in explaining why $1-0.\dot{9}$ is not 0:

> The 1 on the end is a value, so that must have a bigger value than 0, which I always think of doesn't have a value.

Recall that the remaining subjects were only interviewed once. The first two after the first administration of Questionnaire 1, the last two after the second administration.

GHN As we have seen above (p.190) the subject made responses that could be given finitist interpretations. She did, however, give generic limit responses in the first administration of Questionnaire 1. Generic limit views were only partially supported in the interview for she changed her mind there and saw the limit of 0.9, 0.99, ... as both $0.\dot{9}$ and 1. In Q28a generic limit ideas appeared to be present:

> Well, you can just keep adding 0 to that one can't you and it just gets a bit smaller each time but you can just keep adding 0 'cos there's no end to the amount of noughts you can add to it.

MWN Subject gave generic limit responses with the exception that the curve in Q30 did have 0 as a limit. He supported these generic limit views in the interview.

(Q27) The limit of that was $0.\dot{9}$ 'cos that was the farthest it could possibly reach, even though it can't actually reach it. That's the sort of hypothetical boundary it could get to.

INT And 1 isn't ?

SUB No, 1 isn't 'cos it'll never actually reach 1. It'll just about be 1. It'll never actually reach 1.

(Q28a,b) Yeh, because those two are going to that infinite value aren't they ? It's going to carry on to some infinite number. It won't actually reach that number but you suppose it does. There's a change all the way along and it'll carry on changing so it won't have a limit.

DGM Subject gave generic limit responses in the first administration of Questionnaire 1 with the exception that the curve in Q30 had 0 as a limit. In the second administration he gave non generic limit responses with the exception of Q25a where he claimed the limit of the functions was slightly jagged. He changed his reply during the interview.

SUB Well you would always get slight, it slightly jagged.

INT This term limit. What does it mean to you ?

SUB Well it means what a certain function tends to, or what a series tends to, so I don't know why I put that.

INT Would it tend to the minutest jagged line or a straight line?

SUB A straight line, so I don't know why.

Recall that this subject (and the subject below) were interviewed because they displayed high ability at A-level mathematics. Acknowledging that the limit of 0.9, 0.99, ... was either 1 or $0.\dot{9}$ (since $0.\dot{9}=1$) he attempted to explain why he put *limit not 1* in the first administration of Questionnaire 1:

> Well I probably thought that that one, that the limit is, if you like, the highest number that you can get and it never actually reaches 1, so its limit isn't 1.

**DLM** Subject gave generic limit responses in the first administration of Questionnaire 1, with the exception of Q25a, where he responded that the limit of the function was a straight line, and Q30, where he claimed the curve did have 0 as a limit. In the second administration he gave consistently non generic limit responses. It is difficult to evaluate, however, whether the subject was moving beyond generic limit ideas in the interview. The subject saw physical problems in evaluating infinite series:

> The sum's writing it out and therefore they couldn't ever write it all out 'cos it's so long.

He was thus clearly not conceptually working in a mathematicians' limit framework. The only relevant question asked in the interviews was Q28a. His response is not particularly illuminating:

Yeh, I was thinking of a limit because it's going towards 0.

This could be simply accepting approximation, however, as could the following very logical reason why $0.\dot{9}=1$:

> There's no number you can think of between $0.\dot{9}$ and 1. For that reason they must be the same.

Of course, as we mentioned, it is not possible to fully evaluate the theses concerning generic limits from the protocols alone. Let us, however, consider the evidence the protocols provide.

**Generic limit concepts are dominant.** There are several quotes that mirror exactly our characterization of the generic limit concept as being one where the limit cannot be qualitatively different from the terms:

> PBM1 Anything less than the units' column, then it's not 1.

> JCN1 (re 0.1, 0.01, ...) Well, it's not carrying on in the same sequence if you don't have a 1.

> PPM1 Well, $0.\dot{9}$ is 0.9 carrying on forever, carrying on for a long time, so it must be less than 1.

> LSM2 'cos it's not written the same. There must be a fraction added on to it that makes it equal to 1.

SWM1 The 1 on the end (of $1-0.\dot{9}$) is a value, so that must have a bigger value than 0, which I always think of doesn't have a value.

Many of the subjects, however, used language indicating that $0.\dot{9}$ or 0.0..1 or the minutest jagged line are reached, whereas 1 or 0 or a straight line is not reached (see above: PBM2, CEN1, JHM1, PPM2, MWN). Is this *reaching* idea part of the generic limit concept? We believe it is for it occurs with the more obvious generic limit verbalisations in the protocols. SWM's replies illustrate this. Recall that the question *Does 0.1, 0.01, ... get to 0 ?* was omitted from Questionnaire 2 because the language was misleading (it suggests actually reaching 0 or getting there). Now compare SWM's very clear generic limit response immediately above with her 'No' response, later in the same interview, to *Does 0.1, 0.01, ... get to 0 ?*:

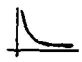
Because you always have a one on the end and that has a value that's not 0.

Similar, though less obvious, instances occur in the interviews with PBM, JCN, CEN, JHM, PPM and LSM. We are not clear if this reflects two related concepts or two aspects of the same concept.

We conclude that generic limit ideas are present and do dominate adolescent thought on limits.

Other concepts exist, however. We have seen responses that suggest finitist ideas and responses that suggest ideas close to those of mature mathematicians. There is no trichotomy, however. Subjects are not exclusively in only one of these three conceptual fields.

**Generic limit concepts are slightly stronger in arithmetic contexts than they are in geometric contexts.** Hypotheses claiming a slight difference are both vague and difficult to verify with any instrument other than a large scale sample. We cannot hope to come to conclusions in this chapter. Nevertheless this interesting claim can be examined here. Behind the wording of the hypothesis is the belief that subjects can immediately see the difference between $1 \times 10^{-10}$ and 0 but could not distinguish between them as points on a graph.

Unfortunately, for this aspect of our study, the interviews where the curves in questions 30 to 35 were discussed concentrated on the differences in the curves and in the four phrases rather than comparing the curves with similar numeric sequences. Only four subjects were asked about the limit of the curve in Q30  .

GAM1, who displayed post generic limit concepts in numeric contexts, changed his reply from 'No' in the first administration of Questionnaire 1 to 'Yes' in the interview but did not expand on this. JHM1 gave reasons similar to his replies to the numeric sequences:

INT Why does it tend to 0 but not limit 0 ?

SUB Because it doesn't actually reach 0.

PPM2 echoed this reply verbatim and LSM1 had similar thoughts as can be seen from her remarks quoted above (p.229). This evidence clearly points to rejecting the hypothesis.

The other question that could shed light on the hypothesis is Q25a (sequence of jagged functions). Again the protocols point to rejecting the hypothesis. As we have seen, the only subjects interviewed on this question who responded *straight line* were GAM, who appeared to have post generic limit concepts in numeric contexts; VMM, who was thinking in visual terms; and DGM, who again appeared to have post generic limit concepts in numeric contexts.

**There is a slight movement away from generic limit and to the mathematicians' limit concept in the M group.** The criticisms of the second hypothesis apply to this hypothesis. Nevertheless, let us sum up our findings above. None of the N group subjects interviewed appeared to progress beyond generic limit ideas. In the M group,PPM's concepts were difficult to categorize. Finitist, generic limit and nonstandard infinitist interpretations could be be given to his interview responses (see above, pp.189 and 238). To a lesser extent this applies to JHM though he appears to be mainly working with generic limit ideas. VMM used language that suggested she was content with finite approximation at times but again also appeared to be mainly working with generic limit ideas. PBM, LSM and SWM's ideas appeared to be wholly generic limit based. GAM appeared to be progressing towards standard mathematical limit ideas but these were evident in the first interview and so cannot be said to be the effect of a first calculus course. DLM's concepts were difficult to categorize though there did appear to be some movement away from generic limit ideas. Only DGM can be clearly said to be moving towards the mathematicians' limit concept as a result of the course but, unfortunately, he was only interviewed after the second administration of Questionnaire 1 (and if GAM was only interviewed after the second administration, then the same could have been said for him as his responses in the first administration also indicated he held generic limit ideas at the time).

Although these results are compatible with the hypothesis, all that can be said at this time is that the hypothesis is not refuted. We shall return to this (and to the other hypotheses) in the next chapter.

## SERIES

Q15 (picking the odd ones out from a given set of series) was not designed when the interviews were taking place. The following concern only Q12 (1+1+1+..), Q13 (0.1+0.01+..) and Q14 (Is 1/9=0.1+0.01+.. ?).

One of the most surprising results came from examining subjects' responses to the questions on series. As can be seen from Table Q12 and Table Q13, a slight distinction between convergence and divergence is observed by the M group (but notice with the MHS sample that this actually reduces over time). Moreover, when we compare the responses to questions 13 and 14 we observe that the overall opinion is that we cannot add 0.1+0.01+... and get an answer but we can define this infinite sum as 1/9 (again this is stronger in the M group and this time, with the MHS sample, it is stable over time). Why is 1/9 not an answer in Q13 ? Are the responses random ? The explanation appears to be that the same principle is involved in both series, both go on indefinitely and while they give an answer at any given point neither produces a final answer. 1/9 can be defined as 0.1+0.01+... but only because both are improper or incomplete in that they both never end.

Our hypotheses concerning infinite series are that convergence and divergence are not generally seen as the most important properties of series. Rather, the theoretical, physical and temporal problems of any infinite summation are seen as important. Thus subjects may notice that 0.1+0.01+... gets nearer to $0.\dot{1}$ while 1+1+... continually increases without seeing this as an important property. For this reason it is sometimes difficult to determine whether they notice this or not.

The following subjects were interviewed concerning this group of questions. The key to the annotations is: SP-same principle in both series; INF, $0.\dot{1}$-infinity in Q12, $0.\dot{1}$ in Q13; NNY, etc.- No, No, Yes to the questions; (NNY), etc.-responses in the first administration of Questionnaire 1 when interviewed after the second administration.

TABLE 8.1

| | | |
|---|---|---|
| GAM1 NNY SP | GAM2 YYY INF,$0.\dot{1}$ | PBM2 (NNN) NNY SP |
| JCN1 NNY SP | JCN2 YYY SP | CEN2 (YNY) NNY SP |
| VMM1 YYY SP | VMM2 NNY SP | PPM1 YYY INF,$0.\dot{1}$ |
| SWM2 (NNY) NNY SP | DGM (NYY) NYY (formally correct interpretation) | |
| DLM (NNY) NNY SP | JHM2 (YYY) NNY | |

GHN (who responded YN?) was asked about Q12 but was clearly confused and saw it as an answer at each stage (as did 3G in the first pilot interview, see p.59). Further questioning appeared counterproductive and was not pursued.

We begin by looking at subjects' explanation of the *same principle* schema.

By the *same principle* schema we mean the belief that any infinite summation (convergent or divergent) has the theoretical problem of never being able to reach a final answer. Convergence or divergence, if seen, thus reduce, respectively, to the partial sums not going beyond a certain number or extending without bounds. This is, however by the same principle schema, seen as secondary to the main issue - a final answer is never attained. The protocols show this principle to be clear and dominant in subjects' thought on infinite summation.

Please note that in the following the interviewer often says "The same there ?" for Q13. This is only said when the subject has given the same response to questions 12 and 13 on the questionnaire. It is thus not to be seen as prompting the subject:

GAM1 Well, because you're going to keep on adding you just add a one on all the time and so if you guess a number, you're just adding another one onto it, get a result from that and just add another one onto it. It's something without bounds isn't it ? Keeps on going.

INT (Q13) Same is it ?

SUB Same principle.

INT No difference at all ?

SUB No, not really because it's getting smaller isn't it, progressively smaller by a tenth. So you're still going to be adding something else onto it continually.

PBM2 (Q12) If you keep on adding 1 on every time, you can't get a final answer 'cos you're still adding the ones on.

INT Same for Q13 ?

SUB The same applies there because you'll be adding one onto the end of the series of numbers.

JCN1 You just carry on. Never arrive at a limit.

(Q13) No, for the same reason.

JCN2 I didn't like that question really. You get an answer all the time. But you can't go on forever and then stop. I mean it goes on forever. There's no stopping. A brick wall sort of thing. There's got to be something on the other side of it.

INT The other side of what ?

SUB The big wall. You can go on forever but you won't get an answer. Not at the end.

INT This one (Q13) ?

SUB It's the same as this really. There you can go on putting as many noughts as you want so you never get, get to the end.

CEN2 Well it's sort of quick mathematizing the word *forever*. I mean if you just keep adding one, if there's no definite end then there can be no answer surely. I can't explain my 'Yes' there (first questionnaire).

(Q13 Interviewer tries to point out the difference).

SUB Yeh, but the same principle applies because the fact that the addition will never come to an end therefore there can be no final answer.

VMM2 (Q12) If you go on forever and ever, you never stop to get an answer.

INT And the same there (Q13) ?

SUB The same there, yeh.

SWM2 (Q12) 'cos if you go on forever, you just don't stop. See what I mean ? If you go on adding them forever you're never going to reach the end of it.

INT And the same thing here (Q13) ?

SUB Yeh.

DLM (Q13) INT That one you didn't answer.

SUB Oh, I must have missed it. I would have put 'No'.

INT Is there a difference between 12 and 13 ?

SUB Well, the idea's the same.

INT What's the idea ?

SUB As you keep going higher and higher you're evening them out each time, then you can't get an answer. You just keep on going forever.

Protocols that did not clearly enunciate this principle in both Q12 and Q13 were VMM1, JHM2, PPM1 and DGM. VMM1 did, actually, state it but hesitated:

(Q12 Subject said 'Yes')

INT What would the answer be ?

SUB If you go on forever and you stop at a certain point you've got an answer. But if you are going on forever you don't really stop, do you, to get an answer.

(Q13 Subject said 'Yes') INT Is it the same ?

SUB It is the same ... but I'm just hesitating a bit because it's decimal, so they're smaller numbers. So there must be something different to, an answer ... It is the same. If you go on forever and ever, you don't get an answer.

DGM gave the conventionally correct response in both questionnaires and supported this in the interview:

Well that (Q13) tends to a limit, that tends to an answer, whereas that (Q12) doesn't tend to any number.

JHM2 and PPM1 were interesting in that both claimed infinity was the answer in Q12 and that $0.\dot{1}$ was the answer in Q13. This reveals that the *same principle* schema is not universal amongst subjects who do not give the mathematically proper answer (as DGM did). It is, however, from a naive position, very close to the mathematicians' answer:

JHM2 (Q13 Subject said 'No'. There is a pause).

INT What were you thinking ?

SUB I was thinking eventually you will get to the end of your infinity of noughts and they will add up.

INT And what will your answer be ?

SUB A row of noughts.

INT $0.\dot{1}$ ?

SUB Yes.

INT So now you're saying we can get to $0.\dot{1}$ ?

SUB I only think theoretically we can get to it.

(Subject asked to return to Q13 at the end of the interview).

SUB I think maybe it's because you will reach your endpoint, an infinity of noughts, and then you can add your ones up. You'll have an infinity of numbers of ones.

PPM1 (Subject said 'Yes' to both questions)

(Q12) Well if you go onto infinity you'll get the answer infinity.

(Q13) Answer would be $0.\dot{1}$.

The word *forever* can be very important here. It was omitted in the larger scale survey as it was felt it might lead the subjects but the results for the two samples are very similar. *Forever*, however, seems to be implicit in the infinite sum and brings a temporal context with it. This is evident in the quotes from SWM2, JCN2, CEN2 and VMM1/2 above.

The fact that the converging series' terms got increasingly smaller was often noted, and when it wasn't this was, without exception, pointed out. This did not once, however, override the principle that both carried on, e.g.:

JCN1 (Prompted on difference. This is seen.)

INT Could we not say there was a limit of $0.\dot{1}$ here ?

SUB Well, $0.\dot{1}$ is just 0.1 with an infinite number of ones. It doesn't have a limit.

CEN2 (Prompted on difference) SUB Yeh, but the same principle applies because the fact that the addition will never come to an end, therefore there can be no final answer.

SWM2 (Prompted on difference).

SUB You still get slightly bigger. You still don't get a definite answer, somehow.

Not accepting $0.\dot{1}$ as a proper answer will be taken up in the next section of this chapter on subjects' conceptions of real numbers.

The last part of our thesis on series concerns the temporal and physical aspects evoked by infinite summation. The temporal aspect is part of the *forever* problem mentioned above. The physical aspect was not present in most protocols but, as the following shows, can arise. We believe other subjects had similar thoughts but couldn't state them as fluently as the following did. However, we have no evidence for this:

DLM INT In Q14 you said *think so*.

SUB Yeh, think I can. Well if this goes on for infinity , yes and then therefore there is no number between $0.\dot{1}$ and 1/9 so they must be the same.Same idea as that $0.\dot{9}$ .

INT So couldn't I get an answer in Q13 ?

SUB Ha, if you wrote that out I suppose ... well, when it says *get an answer* ... oh, I suppose, yeh.

INT Go on.

SUB I was thinking when it was adding point so on then I don't know if you could actually write it down. Somehow when it's added it just seems different. I can't explain why. Bit strange isn't it. When it's written out as $0.\dot{1}$ then I can think of it as 1/9, but when you just keep on adding it seems different in my head, the number. I don't know why.

INT ( $\sum_{i=1}^{\infty} 1/10^i$ is suggested) Is that O.K.?

SUB Yeh, I might put it in then ... What, you mean if this was the question ?

INT Yeh.

SUB I'd probably still put 'No'.

INT But we can still define it ?

SUB I don't know. It's just the way I think of it ... when it's a sum then I think of it as a different number as when you're just writing it as just straight away $0.\dot{1}$ .

INT What does the sum do then ?

SUB Well, I sort of imagine, I suppose, when they've got the sum, the sum's writing it out and therefore they couldn't ever write it all out 'cos it's so long. Whereas if you're writing it as $0.\dot{1}$ then you're saying it's written for ever and ever.

INT So it's kind of physical ?

SUB Physical. That's it I suppose.

Finally we come to the initially surprising acceptance of Q14. The explanation for its acceptance and the rejection of Q13 appears to be that both 0.1+0.01+... and 1/9 are improper and incomplete and as both are, we can define one in terms of the other. Some of the ideas raised in the following quotations will be taken up in the next section on subjects' conceptions of real numbers:

PBM2 (Subject said 'Yes' to Q14).

INT Why isn't $0.\dot{1}$ a final answer to Q13 ?

SUB Well you could call that an answer , but it's not a final answer really ... but it's not an answer in the way I meant ... that's not a final result really, 'cos it keeps on going. 10÷2=5 but this keeps on going.

INT What about 1÷3 ?

SUB Well, you get an answer but it's not a final answer really. You class it as an answer for simplicity to call it $0.\dot{3}$ but the answer never actually stops, it carries on going ... that's not as definite an answer but as you go on the threes become less and less significant and so it's not really as important, the ones as you're going on.

CEN2 (NNY) INT But 'Yes' on Q14 ?

SUB Yes because $0.\dot{1}$ to me isn't any particular number, if you see what I mean. It can be defined as that (1/9) providing you have your dots after 'cos that means it just keeps on going on.

SWM2 (Q13) You won't know where to stop putting your ones, would you. $0.\dot{1}$, still not like a definite answer is it. It's not like you could say 5. You know what 5 is.

As in other areas of adolescent thought on limits and infinity, subjects may accept finite approximations:

JCN2 INT Then 1/9 can be defined as that ?

SUB Well, it's as near as you can get using decimals. I suppose it's not absolutely the same as that.

INT How would it differ ?

SUB Well, it's always going to be just slightly smaller. Always getting nearer but never arriving there.

INT Why doesn't it arrive there ?

SUB Because you can keep putting on as many noughts after the decimal point before you add the one, as you like.

## REAL NUMBERS

Our interest in adolescent thought concerning real numbers is in their understanding (or lack of understanding) of the completeness of the real number system. There are many characterizations of the completeness of R : every bounded above subset of R has a supremum; every Cauchy sequence has a limit; etc. These characterizations are clearly in the domain of university, and not school, mathematics. Ideally, less rigorous formulations should be part of A-level mathematicians' cognitive framework. We shall regard an individual as having a basic understanding of the completeness of R if they view the limit of a convergent sequence of real numbers as a real number or if they regard any non terminating decimal as a real number. To be able to do this one must have a sense of the actual infinite or else non terminating decimals are always in a state of becoming and are never realised.

It is clear from considering subjects' conception of infinity as a process that adolescents' principle view of infinity is that of the potential infinity. It is possible, however, that some actual infinite ideas are present in subjects' thoughts. For example, in questions 17 and 18 both groups generally agreed that infinite sets could be considered as single sets. However, the move to considering a convergent infinite summation being carried out was not, as we have seen, as easily appreciated. Thus we must question whether there were actual infinite ideas present in subjects' minds when they agreed that N and R could be considered as single sets. Unfortunately only two subjects were directly asked what they meant by their replies to Q17 and Q18 (VMM2 and DGM). Neither gave responses that shed light on whether they appreciated the idea of the actual infinite or not.

Recall (pp.97-99) that Questionnaire 1 contained several questions designed to determine whether subjects were competent with decimal arithmetic. The responses indicated, with very few exceptions, that they were. Appreciation of the decimal system must, however, be combined with suitable limit ideas to form a proper conception of the real number system. In fact, decimal ideas can actually work against a mature understanding of R, as can be seen by one subject's use of decimal places in a generic limit style argument:

> PBM1 (Q11, Is $0.\dot{9}<1$ ?) Well, it's $0.\dot{9}$ and anything after that, the way we've been taught, if it's $0.\dot{9}$ anything after that won't change it. It's got to be less than 1, the way we've been taught. Like in units, tens, hundreds. Anything less than the units column, then it's not 1.

To mathematicians the natural division of the real numbers is into rational and irrational numbers. This was not the case with our subjects. The natural division for them was between terminating and non terminating decimals. Only one subject considered the rational /irrational distinction but this was an immature understanding and caused him to change his answer in Q18 from 'Yes' in the first administration of Questionnaire 1 to 'No' in the second administration:

DGM Because you've got rationals and irrationals. The rationals you've got numbers where, if you write them down as decimals, you can write them down on a piece of paper. Whereas other numbers, like pi, you can't write the decimals down on a piece of paper because an infinite number of.

INT And so it's several sets ?

SUB Yeh.

Our principle hypothesis concerning real numbers is that recurring decimals are generally seen as incomplete, dynamic entities which are qualitatively different to finite decimals. We have seen examples of this above in the protocols concerning generic limit concepts and series. Except for the three subjects who appeared, at the time, to be developing mathematicians' limit concepts (GAM, DGM and DLM) and also JHM there was evidence for this hypothesis in the interviews with all the subjects. We saw this very clearly with PBM on p.246. Others who gave clear satements of the incompleteness of recurring decimals were:

CEN1 I don't know what figures I'm talking about or what numbers I'm dealing with when I say $0.\dot{3}$. Well I certainly agree with this one, 1/3, because that is a specific number. $0.\dot{3}$ isn't a specific number. It could be any number really.

VMM1 (Discussion on contradiction with $0.\dot{9}<1$, $1/3=0.\dot{3}$, $0.\dot{3}\times 3=0.\dot{9}$, etc.)

INT Why wouldn't $0.\dot{3}\times 3=0.\dot{9}$ be right ?

SUB Because you don't know the exact answer. It goes on forever. That one's right because if you divide 3 into 1 you get 0.333...

LSM2 (Concerning $0.\dot{9}$ and 1)

SUB Well a number that recurs you can't really define as a number so I think you've got to bring it up to the nearest one.

SWM2 (Q13, 0.1+0.01+...) You won't know where to stop putting your ones, would you ? $0.\dot{1}$ is still not like a definite answer is it ? It's not like you could say 5. You know what 5 is.

The rest gave less clear statements but dynamic ideas were nevertheless present:

JCN2 (Subject was prompted on the difference between questions 12 and 13)

SUB I was thinking the same thing, you get an answer everytime.

INT But you won't get a final answer ?

SUB No.

INT Then 1/9 can be defined as that ?

SUB Yeh.

INT Wouldn't that be a final answer then ?

SUB Well it's as near as you can get using decimals. I suppose it's not absolutely the same as that.

INT How would it differ ?

SUB Well it's always going to be slightly smaller. Always getting nearer and nearer but never arriving there.

PPM1 (Discussion on contradiction with $0.\dot{9}<1$, $1/3=0.\dot{3}$, $0.\dot{3}\times 3=0.\dot{9}$, etc.)

INT What does that indicate ?

SUB $0.\dot{9}$ is exactly equal to 1.

INT What are you thinking as I'm doing it ?

SUB $0.\dot{9}$ can't equal 1.

INT Why not ?

SUB Well, $0.\dot{9}$ is 0.9 carrying on forever, carrying on for a long time. So it must be less than 1.

GHN Well, infinity with numbers, it's like $3.\dot{3}$ and it carries on.

MWN (Concerning 0.9, 0.99, ... Subject states the limit is $0.\dot{9}$ and not 1)

SUB No, 1 isn't 'cos it'll never actually reach 1 even though it'll keep on going. It'll just about be 1, it'll never actually reach 1.

JHM differentiated between $0.\dot{9}$ and 1 but did not use language to suggest it was incomplete or a qualitatively different dynamic entity.

The responses of the remaining three interviewed subjects support the thesis that there is a drift amongst A-level mathematicians towards a *mathematicians' understanding. We must not be too ready to jump to conclusions here, however, for as we have seen it is possible* to ascribe finitist interpretations to the protocols of GAM and DLM. The subjects below are all being asked about their responses 'No' to Q11 (Is $0.\dot{9}<1$ ?):

GAM2 Well again it's effectively the same, isn' it. They effectively equal each other.

INT What do you mean by effectively ?

SUB Well, because it's 0.999 going on into infinity if you like, it's going to be the same really. If you were using it in calculations it would be the same.

INT What about in pure, theoretical maths ?

INT Well .. well I think it is the same.

DGM Well all I thought was that you can't think of any number that's larger than that but smaller than that.

DLM Well I tried to think of a number between $0.\dot{9}$ and 1 and I thought there was no number in between them therefore it must equal 1 and so it's not less than 1.

## WORDS

Mathematics uses many everyday words and phrases with specialist meanings. As we have seen this can confuse many students. An amusing example is in Physics where students who know the word *conservation* in an everyday meaning often think *conservation of energy* is about saving trees. As we have seen the phrases *go on forever* and *get to* can mislead students by implying physical contexts. We shall, however, deal solely with the four phrases *tends to, approaches, converges* and *limit* here.

Our principle hypothesis is that the phrases often generate everyday connotations at odds with the mathematical meanings. Further to this we posit that *tends to* and *approaches* are seen as similar and are vague in that they describe general trends; *converges* causes confusion in that is often seen as inapplicable to numeric contexts; and *limit* is largely seen as an ultimate boundary. All these hypotheses are difficult to verify in a strong sense because the interpretations vary so much. These are, we hold, general trends in an area rich in multiple interpretations due to context and the mood of the subject. We thus merely support our hypotheses with examples. This section is intended to complement pp.146-174.

*Tends to* and *approaches* were often seen as the same. *Converges* was sometimes seen as synonomous as well. *Limit* was the odd one out:

> PBM1 (Q30-Q35, the four phrases applied to functions presented graphically)
>
> SUB I thought approaches is similar to tends to, but unlike a limit it just has to go nearer and nearer to it but it doesn't actually have to have that as a limit.
>
> PBM2 (Q27, the four phrases applied to 0.9,0.99,..)

SUB   Approaches and tends to are nearly the same.

(Q26c, 1,1/2,1/4,.. converges to ___)

SUB   To me converges means that it will approach it but it won't actually get there.

JHM1 (Q27, Subject replied YYYYYYYN)

INT   What does tends to mean to you ?

SUB   It approaches it .

INT   So i and ii are the same ?

SUB   Yes, I found all those meant the same thing.

INT   Tends to, approaches and converges all meant the same thing ?

SUB   Yeh.

INT   Limit meant something different ?

SUB   I thought if it tends to something it gets close but limit was the actual .. limit itself. The top.

VMM1 (Q27)   I think approaches and tends to mean the same thing.

INT   What about converges ?

SUB   That'll be the same as well .. converges is it goes towards it but it never reaches it.

INT   Limit ?

SUB   That's the proper limit ($0.\dot{9}$).

LSM1 (Q30,  )   It tends to 0. It's getting nearer all the time but it's never actually going to get there.

INT   But no for the rest ?

SUB It approaches to 0 . It converges to 0 but no it doesn't have a limit.

MWN (Q27) It approaches $0.\dot{9}$ 'cos approaches to me is the same as tends. It tends to go towards $0.\dot{9}$ and it approaches $0.\dot{9}$. It's going towards so they're both the same meaning. .. (later) ..converges again, I thought was the same and so I was unsure. I didn't know the difference you see.

INT Did limit seem the same ?

SUB No, its limit was its outer bounds really. That was a bit different to the others.

As we have seen above, *tends to* and *approaches* generally mean *going towards and never reaching*. Notice that subjects' interpretation of the words does not really affect their generic limit stance (where applicable) as the sequence 0.9, 0.99 ,... ,for example, can be seen as tending to either $0.\dot{9}$ or 1:

GAM1 (Q30-Q35) INT What do you mean by 'tends to'?

SUB Approaches, going to 0. That's (Q33, ) getting smaller and smaller, so eventually it's going to be 0.

INT And if Q30 suddenly stopped and continued along the x axis, would that tend to 0 ?

SUB No. It would be at 0 wouldn't it. Tend means it's going towards 0.

PBM1 (Q30) Well, tends to to me means it doesn't actually reach it but it gets very close to .. it would tends to 0 but it wouldn't actually reach 0.

PBM2 (Q27) I think the second part of the question (tends to 1), the tends part to it, the actual word *tends to* becomes more important. I mean it never actually gets there, which is what tends to means to me. It means it approaches it or comes close to it but it won't actually finally get there. I think the sequence is actually $0.\dot{9}$.

JCN2 (Q27b, Tends to 1 ?) Well it's always getting nearer to 1 but it never actually gets there. But it's always getting nearer. That's what tends means.

CEN1 (Q27) When I think of something approaching something I think of it getting nearer .. just like a car approaches a traffic light or something. Those numbers get nearer to one all the time. They will, of course, never get there.

JHM1 (Q30) INT Why does the curve tends to 0 but not limit 0 ?

SUB Because it doesn't actually reach 0.

INT Converges to 0 ?

SUB Well I wasn't too sure of that. I just put an answer down.

INT Approaches ?

SUB Because it gets closer as it goes along.

VMM1 (Q27 NYNYYNYY) 'cos it tends to $0.\dot{9}$ but as I think of it, it tends to 1 'cos it's getting nearer and nearer to 1.

PPM2 (Q27b, 'Yes' in the first administration of Questionnaire 1, 'No' the second time)

SUB Well it's going to tend to $0.\dot{9}$.

INT But why not 1 ?

SUB I think it's tending more to 0.9999 and going on rather than tending to 1.

*Converges* was the word generating the most uncertainty in the interviews. It seems very likely that this comes from everyday sense of two things actually coming together:

PBM1 (Q30-Q35) I wasn't sure what converges meant. I didn't know what the question meant.

(Q29, Say whether each of the following sequences converges.)

SUB Well I wasn't sure what converges meant.

CEN1 (Q27) When light converges, rays of light get closer together when they converge. So does it mean get closer to 1 ? Then I'd change my answer to 'Yes'.

JHM1 (Q29 Note that the subject was confused on *converges* in a geometric context as well. See his last quotation above).

SUB Well I'm not sure what converge means in this sense. I know what converge means but I don't know how it's used here.

INT Converges means ?

SUB *To come in at a point.*

INT I don't quite understand.

SUB Well if you have a converging lense it brings two rays of light in.

INT At a point ?

SUB Yeh, at a point. So converging would be saying it has a limit wouldn't it ?

JHM2 (Q29) I don't really see how numbers can converge.

INT Why ?

SUB Well really converge means light, from a thing, coming in, it's two separate parts.

INT (suggests two sequences).

SUB You'd have to have two sequences coming in on each other. I don't think you can have one sequence converging.

GAM1 (Q30) When I think of converge it seems to me that it's going to sort of touch 0. Two lines are going to touch each other.

JCN2 (Q27) I always think of two things converging on one. There's got to be two things converging, getting nearer to each other.

DLM (Q33, , YYYY in the first administration of Questionnaire 1, YNNY the second time)

SUB Converges to 0. Well I was thinking, I don't know why, I was thinking of the word converges as coming from two sides, whereas that's only coming from the top. But maybe it's just my misinterpretation of converges and approaches. Yeh, 'cos it does approach, even if only from one side.

*Limit* was, as we have seen, qualitatively different from *tends to*, *approaches* and *converges*. Despite being more specific it was dually seen as the final point and as an unreachable boundary point. Notice the generic limit concepts in the following quotations and that these concepts can be used to affirm or deny a specific limit's existence in accordance with the above dualism:

PBM2 (Q27) Its limit is its final point that it will get to. I think the limit is $0.\dot{9}$ and there again there the limit is 1 but it won't actually get to one, so you can't have 1 as its limit.

CEN1 (Q27) When I think of limit now I think where it stops and it won't stop at 1, it will stop at $0.\dot{9}$ .

JHM2 (Q27) INT Why is its limit 1 but not $0.\dot{9}$ ?
SUB I didn't really see the limit as what it is. I saw the limit as what it's very close to but it isn't actually 1. So you have got $0.\dot{9}$ eventually but you haven't got 1. 1 is its limit it can't reach.

PPM2 (Q30) INT Why not limit 0 ?
SUB 'cos it never actually reaches 0. It'll get very close to it but it'll never actually reach it.

INT But we can say it tends to, approaches and converges to 0 ?

SUB Yeh.

LSM1 (Q28a, Q29a Subject put no limit but yes it converges both times).

SUB A limit is a fixed point. Converges is to go towards that point. It doesn't mean to say it's ever going to reach that point.

MWN (Q27 This passage comes towards the end of the discussion).

INT Did limit seem the same ?

SUB No, its limit, that was its outer bounds really. That was a bit different to the others. So the limit of that was $0.\dot{9}$ 'cos it was the farthest it could possibly reach even though it can't actually reach it. That's the sort of hypothetical boundary that it could get to.

DGM (Q27h, Is the limit of the sequence 1 ? Subject put 'No' in the first administration of Questionnaire 1 and 'Yes' the second time. He is here asked why he put 'No' the first time.)

SUB Well I probably thought that that one, that the limit is, if you like, the highest number that you can get and it never actually reaches 1, so its limit isn't 1.

## REASONING AND CONFLICT

The previous observations are all forms of reasoning but here we are interested in the overall form of subjects' arguments rather than particular beliefs.

As Wason and Johnson-Laird (1972) have shown, subjects are not, as Piaget would largely have had us believe, logical in their mental acts. Comparison with logical canons is not our priority, however. Our purpose here is merely to note subjects' forms of reasoning, valid and invalid. This aim is partially frustrated by the design of the questionnaire which was intended to examine subjects' intuitions and was thus not problem solving orientated. Nevertheless several aspects of reasoning in this domain were present in the protocol data.

Most reasons for answers were simple instances of general principles held by the subjects. These principles have been documented in the previous sections of this chapter. As examples consider *the generic law* and *infinity as a process*, which we shall examine in more detail shortly:

PPM1 (Q19, comparing the cardinality of N with that of the even numbers)
SUB Well there is more numbers in the first row because all that is just alternate numbers so they'll be twice as many numbers in the first row as there are in the second.

CEN1 (Q19) I suppose I put the same in both because the definition of same there is an endless number. This sequence will never end, neither will this one, therefore you could say they are the same in that both stretch to infinity.

More sophisticated forms of reasoning involve formulating hypotheses. Curiously these were usually accompanied, in our data, with negatives. Standard logical arguments using negatives, reductio ad absurdum (RAA), and modus tollendo tollens (MTT), that is, ((A -> B) & B') -> A', were present though often not in a perfect form and not in great abundance:

GAM1 (Q1, Is there a largest number ? RAA)

SUB Well, if you think of a very large number that comes into your mind with so many noughts, you can always think of one number higher, higher than that. So there really isn't a largest number.

DGM (Q3, What is 1/0 ? RAA)

INT Why isn't infinity a numeric answer ?

SUB If you think of it as the highest number you can get then you can add one to it and get a higher number. So there's no numeric answer to it.

GHN (Q5, Is $\infty+1 > \infty$ ? RAA)

SUB Well if you add another number to it it couldn't have been infinity before could it, because it's then infinity, isn't it ?

JCN1 (Q1, MTT) If there is a limit, then there has to be something on the other side of it.

INT There wouldn't be anything on the other side of the largest number ?

SUB I thought that was an impossible situation.

JHM2 (Q2, Is there a smallest number ? MTT)

SUB Because you can have an infinite number of noughts before you have a one, so, since you can't reach infinity, you can't reach the smallest number.

An RAA type of answer and one that was implicit in many of the 'No' responses to questions 12 and 13 (Can you add 1+1+..., 0.1+0.01+...) used a hypothetical fixed point. This form of reasoning was first observed in the early pilot interviews (see p.55):

VMM1 (Q12, 1+1+...) If you go on forever and you stop at a certain point, you've got an answer but if you are going on forever then you don't really stop, do you, to get an answer.

There is a fine line between the fallacy of denying the antecedent and claiming indeterminancy of the consequence when the antecedent is false, as the following examples show:

MWN (Q22, comaparing the cardinalities of $R_{(0,1)}$ and $R_{(0,10)}$)

INT Is it not 10 times greater ?

SUB No, it would be ten times greater if you could find out what that one actually was. If you think of it in terms of infinity being an ultimate number, then you can think of that as ten times that, 'cos you can define what it is. But I would say you couldn't 'cos you don't know what it is.

PBM1 (Q19) I thought you couldn't really compare it. I thought there'd be the same number because it goes on indefinitely. I thought there'd probably be the same number but as that one's higher then I suppose that one will have more numbers 'cos you can't have a highest number. If you did have a highest number then that one, the first row, will have more numbers in it 'cos the second one is double the first. But I thought that as there isn't really a largest number you can't really compare.

VMM1 (Q22) There's 10 times more. If it wasn't infinity there'd be 10 times more numbers between 0 and 10 than 0 and 1. But since it's infinity you can't say how many there is.

Arguments were often missing in the interviews. This does not necessarily mean that arguments were not present in the subjects' conscious or unconscious thought but rather that they did not verbalize them (though, as we noted in Chapter Seven, every attempt was made to encourage subjects to verbalize their actual thought processes). Moreover arguments often noted many points but failed to gather them together, as the following demonstrates:

PBM2 (Series questions (12-14) are being discussed).

SUB Well 1/9 is $0.\dot{1}$, and that's what that part says. It can be defined as that but you have to continue going on forever and ever and ever. I take that to mean that that just carries on , which is the same thing as that (Q13), 'cos all that means is the same as that, I thought. You'll never get a final answer though. If you are actually going to say that 1/9 equals that, that's what I took the question to mean, if that actually is the case, if that is a definite fact, then $0.\dot{1}$ is the same as that. So I thought well, it never could be that.

Our principle hypothesis in this section is: **reasoning schemes peculiar to problems dealing with limits and infinity are *infinity as a process* and the *generic law*. Both schemes have widespread application and subjects may switch from one scheme to the other in response to similar questions.**

We have already seen many instances of both schemes when we considered infinity as a process and generic limit concepts earlier in this chapter. Further support for this hypothesis is evident in subjects responses to the cardinality questions. Although cardinal arithmetic is not relevant to school calculus it does lend itself to clear expression of both schemes. Below we document occurrences of both schemes that occurred in responses to these questions in the protocols (recall that Q23 - comparing the cardinalities of a circle and enclosed square - had a Yes / ? / No format in Questionnaire 1).

It may, of course, be that there are schemes that we are not aware of. However, as can be seen, both schemes are widely used, neither appears dominant and subjects do change from one to the other. For ease of presentation we use the following abbreviations in the table:

GL - generic law; IP - infinity as a process; CC - can't compare; S - same in each; M - more in the superset; MC - measuring context evoked; ? - confused. We indicate the scheme employed in the initial response followed by change of response made during interview (changes to questionnaire responses that occurred during interviews are indicated by arrows).

TABLE 8.2

| | Q19 | Q20 | Q21 | Q22 | Q23 |
|---|---|---|---|---|---|
| GAM2 | IP S | IP S | IP S | | |
| PBM1 | IP S & CC | IP CC | GL MC M | GL MC M | GL MC M |
| PBM2 | | GL MC M | GL MC M | | |
| JCN1 | | IP CC | | | |
| CEN1 | IP S | GL CC | GL/IP CC/S | GL MC CC/S | GL MC ? |
| CEN2 | IP S | | | | |
| JHM2 | GL S->M | | | | |
| VMM1 | GL M->CC ? | ? | IP CC --GL M | ? | |
| VMM2 | IP CC | IP CC | IP S | ? | ? S/CC |
| PPM1 | GL M | IP S | | GL M | |
| PPM2 | IP S | | | IP S | IP S |
| LSM1 | IP S | ? M | GL M | GL M->IP S | |
| LSM2 | | | | | ? S |
| SWM1 | IP S->GL M | ? CC | GL M | GL M | GL M |
| SWM2 | | | GL M | GL M | GL M |
| MWN | GL M->IP S | ? CC | GL M | ? CC | ? M->? |
| DGM | IP CC | | | | |
| DLM | | | | | GL M->IP S |

Note that 'more' (in the superset) responses usually accompany generic law arguments and that this is more frequent in questions 21, 22 and 23. This is entirely natural. The questions evoke measuring contexts that appear to evoke the generic law. Moreover, the generic law naturally suggests more in the superset. Apart from these observations there appears to be no clear pattern to the responses. This does not mean the results are not open to analysis but calls, rather, for an analysis that accounts for diffuseness of responses. Such an analysis would require a theory similar to that of Path Dependent Logic developed in Appendix C. As we have mentioned, however, our data collection methods are not open to such an analysis. Nevertheless, it is useful to examine changes of mind that occurred during interviews:

PBM2 (Q20, Comparing the cardinalities of N and $R_{(0,1)}$ )

INT Why the same number of each ?

SUB Not sure. Both have an infinite number of numbers in them. I'll change my answer. You can't really compare these because both will go on to infinity.

CEN1 (Q20 Subject said 'can't compare').

SUB That seems to contradict what I said earlier,in the last question (Q19, subject said 'same'). I think I put that more on instinct...

INT Would you still agree ?

SUB Well, no. Maybe given time to think about it no I wouldn't 'cos again this carries on endlessly. But even though these are two specific numbers (meaning 0 and 1) the number of numbers you can have between them can also carry on endlessly. So there's an infinite number of numbers in that and that.

INT So what would you say now ?

SUB Well I suppose the same number of each but you can't sort of say a specific number, it's just a massive number. Well it's just infinity in each set.

SWM1 (Q19 Subject said 'same').

INT Any reason why ?

SUB Well it just goes on forever. Well if I looked at it again I would think there'd be more in the first one, 'cos those are even numbers and those are odd numbers.About half as many.

As well as interviews where subjects changed their minds several subjects expressed great uncertainty:

LSM2 (Q22, Com^paring the cardinalities of $R_{\langle 0,1\rangle}$ and $R_{\langle 0,10\rangle}$ . Subject said 'more').

INT Can one infinity be bigger than another ?

SUB Yeh, for example,you have 9 point something there, 9 point going on forever decimals. Whereas you're restricted to 0 point something decimal there between 0 and 1.

INT So this infinity is smaller than that infinity ?

SUB That's how I think of it, but in practice it can't be.

INT Why ?

SUB Well, infinity is infinity.

INT And there's only one infinity ?

SUB Yeh.

CEN1 (Q23 Comparing the cardinalities of the circle and square)

SUB I can't really explain why I put 'Yes'. I suppose it's really guesses because I don't know what I'm talking about when I say infinity.

PBM1 (Q19 Subject put 'can't compare' but then started saying the first row would have more.We come in in the middle of his response).

SUB I thought you couldn't really compare it. I thought there'd be the same number because it goes on indefinitely. I thought there'd be the same number but, as that one's higher, I suppose that one will have more numbers 'cos you can't have a highest number. If you did have a higher one then that one, the first row, will have more numbers in it 'cos the second one is double the first. But I thought as there isn't really a highest number you can't really compare.

VMM1 (Q19 Subject put 'more' in first row).

SUB Well I put 'Yes'. Well...I don't think it's right what I put here. I don't think there are more numbers now.

INT What do you think ?

SUB You go on to infinity but...like that one's gone up to 8 but you've used four numbers..(etc)

INT And why don't you think that now ?

SUB It beats me really, unless you can't compare. You don't go up to a limit so you can't count how many numbers there are. It's stupid.

VMM1 (Q21 Subject said 'same').

SUB Don't know...don't really know. At first I put more numbers than points 'cos for each point you've got two numbers. That's not right. I don't really know. The more you think, the more it confuses you.

Uncertainty in this area may be rational. Note the rational options below and the widespread use of *probably* and *I don't know*:

DLM (Q23 'Yes' in the first administration of Questionnaire 1, 'No' the second time)

SUB Well the first time I probably imagined there being a certain amount, maybe a defined value, the size of a pen or something. The second time I thought theoretically you could get any number of points there and any number of points in any of them 'cos it's infinity.

MWN (Q19) It seems on first looking at it that there's twice as many but when you try and complicate it because you don't know when the sequence ends, you can't think of it. You can't sort of define it. You can't think of it in terms of anything so you, (sic - subject is changing his mind) I suppose I've done it wrong really. I suppose at the time I thought, I just considered those numbers really and I considered that it would repeat itself all the time until you get to this great ending number whereupon you should have more there 'cos you've got only half as many numbers. So I suppose that's why. But thinking about it now I don't know what I'd put. I'd probably put the same in both I think. I'd probably put I don't know actually.

This was not just the case with responses to the cardinality questions:

CEN1 ($0.\dot{9}<1$, $0.\dot{3}\times3=0.\dot{9}$, etc. looked at. Contradiction noted).

SUB I imagine that probably this one may be wrong ($0.\dot{3} \times 3 = 0.\dot{9}$). I still agree with my $0.\dot{3}=1/3$.

INT Why should that one be wrong ?

SUB Maybe I used the wrong word there. I don't think...perhaps I shouldn't have said wrong. I would have said..oh dear..a difficult question..it's just that..I still agree with that what I put..Don't know. Maybe there is some very, very marginal difference between this $0.\dot{3}$ here, which equals 1/3, perhaps there is some very marginal difference between these.

Options and rational choices can, however, cause cognitive conflict:

VMM1 (Q22) I've sort of changed haven't I ? I must have thought about that one.

INT What do you think you thought ?

SUB Well if I hadn't of thought I most likely would have put more numbers between 1 and 10. Sounds like you have more numbers between 1 and 10, more decimal numbers, but you haven't really 'cos they go on to infinity so you can't really count them.

CEN1 (Q27 4 Yeses and 4 blanks) I don't really know. I can remember not putting anything. I think I was so completely baffled. Half of me said 'Yes' and half of me said 'No'. I suppose I should have put unsure really.

As has been said:

> the lability of the intuition of infinity can be explained by admitting its intrinsic contradictory nature as a psychological reality (Fischbein et al, 1979).

This can arise from a theoretical/concrete dichotomy or may arise from the many aspects of infinity:

LSM1 (*Achilles and the Tortoise* is explained)

SUB Well he would do wouldn't he but in practice he wouldn't because the tortoise would always be that tiny bit further than him.

INT Ah, but in practice he would, wouldn't he ?

SUB In practice he would but thinking about it mathematically he couldn't because he'd always be behind him.

MWN (After some discussion on Q21) When I try and do things like that I have a terrible job trying to understand, trying to put it in terms. 'cos whenever you deal with any other problems, it's always defined. But when you come on to something like infinity, where you can't actually imagine what it is, it sort of complicates you. Maybe that's why they don't seem to follow on from each other 'cos it depends which way you look at infinity. It's harder to try and play with it in the mind.

The clearest case of conflict came with *Is* $0.\dot{9} < 1$ *?* :

LSM2 (Q27 all responses 'Yes') INT Why ?
SUB Well, when *I last did this with you you did say that* $0.\dot{9}$ does equal 1 'cos it can't equal anything else (I did not. I have checked this). Therefore that's what I'm going on here but when it comes to this here I still can't appreciate that $0.\dot{9}$ does actually equal 1.

SWM1 ($0.\dot{9}<1$, $0.\dot{3}\times3=0.\dot{9}$, etc. examined, contradiction brought out)
INT What would that seem to indicate ?
SUB $0.\dot{9} = 1$ .
INT Where's the mistake ?
SUB Probably there ($0.\dot{9} < 1$) 'cos Mr X proved the other day that $0.\dot{9} = 1$ .
INT Did he ?
SUB Well he seemed to. But that gave me a bit of a shock and confusion. He was doing something like that, taking things away

and he came up with it. He did something. He came up $x=1/3$ or something. He took away the recurring, then something. He did it to 1/3 and to $0.\dot{3}$ and $0.\dot{9}$ and he came up with that but that just gets me in a flap.

INT (going back to the question) Where would the mistake be then ?

SUB I'd say with $0.\dot{9} = 1$ but now that I've seen it I'd say that $0.\dot{9} < 1$ is wrong.

The last subject gave an almost identical reply in the second interview. This indicates to us that teachers teaching mathematics related to limits and infinity must force subjects to confront their conflicts or, as here, their pupils will, in time, revert to their previous thought patterns. PPM was, perhaps, more typical. On seeing the contradiction in the first interview he was quite certain the mistake lay in $0.\dot{9}<1$, but on the second questionnaire put 'Yes' to $0.\dot{9}<1$.

The lability of subjects' thought on limits and infinity pervades all the aspects we have examined. We end this chapter with examples from many sections.

**Infinity as a number**

JHM2 (Q3, What is 1/0 ? Subject changed his mind from 'infinity' to 'impossible')

SUB That's from the A-level course

INT Has Mr X said that ?

SUB Yes.

INT Do you believe him ?

SUB Yeh, and the computer gives me an error as well.

INT It couldn't be infinity ?

SUB No. I don't think so really.

INT How come ?

SUB Well you can't really divide anything with 0. I don't know. I can't explain really why...(encouraged but not prompted)...Well I think there I must have thought that if you divide something by 0 you can just keep going and going and going.

INT And now you don't think you can ?

SUB Well, mainly because of what people told me. I don't know really.

VMM2 (Q3 Subject put 'infinity' this time)

SUB 'cos I've learnt that 1/0 is infinity and I didn't know that before.

INT Who told you that ?

SUB Mr X, to do with asympt^otes on a graph.

DLM (Q3 Subject put 'infinity' first time, 'undefined' second time)

I don't know. I might have seen that somewhere. Well, first I thought you can get any amount of noughts into one so it's infinity, but then I probably thought that since you can put any amount you can't really put a number to it so I put undefined. Basically I'm not too clear about that. I'd probably put a different answer to it every time.

## Generic limit concepts

DGM (Q25a Subject said 'slightly jagged' both times).

SUB Well you would always get slight, it slightly jagged.

INT This term limit. What does it mean to you ?

SUB Well it means what a certain function tends to or what a series tends to, so I don't know why I put that.

INT Would it tend to the minutest jagged line or a straight line ?

SUB A straight line, so I don't know why.

SWM2 (Q26c Subject put '$1/\infty$' the first time, '0' the second time)

INT Why 0 and why the change of mind?

SUB Converged to 0 because the number underneath gets bigger so that's more. It gets closer to 0. I don't think it ever gets to 0. So I'd agree with the first one more.

## Series

JHM2 (Q13 Subject said 'No'. There is a pause).

INT What were you thinking ?

SUB I was thinking eventually you will get to the end of your infinity of noughts and they will add up.

INT And what will your answer be ?

SUB A row of noughts.

INT $0.\dot{1}$ ? SUB Yes.

INT So now you're saying we can get to $0.\dot{1}$ ?

SUB I only think theoretically we can get to it.

# CHAPTER NINE

# DISCUSSION OF THE THESES

We review the 11 theses, outlined in the Introduction, in the light of all the findings. Our sights here are set at broader results supported by the data. Our findings fall into three categories in terms of *evidence for*: claims that we have high confidence that the data supports (either accepting or rejecting theses); claims that are compatible with the data but are not *proved* by the data; and claims that can only be evaluated via new data.

**1) SUBJECTS HAVE A CONCEPT OF INFINITY.**

This is manifested by:

i) Cognizance of non terminating processes (infinite subdivision of a line, infinite sequences and series, and, in general, infinite continuation of an operation).

ii) Cognizance of collections containing more than any given finite number of elements.

To answer the question *Do subjects have a concept of infinity ?* we must first agree what constitutes having a concept of infinity. Of the many aspects of infinity noted in this study the two that emerge as the most basic (in a subjective evaluation) are the notion of a non terminating process and the notion of a collection containing more than any given finite number of elements. We proceed on the premise that to apprehend these notions constitutes having a concept of infinity.

**1.i) Non terminating processes.**

**Infinite subdivision of a line.**

We do not focus here on the shape or nature of the ultimate elements (indeed, there may be no ultimate elements) nor on the reconstitution of the whole from the ultimate elements. Rather we are concerned only with subjects' recognition of unlimited subdivision. Evidence for perception of the notion was presented in the report of questions 1 to 4 of the first pilot study (p.55).

The questionnaires and interviews did not examine this notion. It was discarded, along with many more in the item design stage, as being of some interest but not essential (given the length of the questionnaire) because the question had largely been determined by other workers. In reflection we felt this to be an oversight and we administered the question to fourth year pupils at MHS. The data from this, reported at the end of Chapter Four (pp.65-67), adds weight to the argument that subjects can apprehend the infinite subdivision of a line.

**Non terminating sequences and series.**

Of all the interviews only the Third Year girl in the first pilot test displayed an inability to talk of infinite sequences and series and their infinite, non terminating, nature (recall that she appeared to see only the finite partial sums in 1+1+1+... and not the infinite sum, p.59). Moreover, if subjects did not appreciate the non terminating nature of infinite sequences, then it would seem to follow that there would be a largest number (the terminator of 1, 2, 3,...). However, subjects are strong in their rejection of a largest number. Moreover, if subjects did not appreciate the non terminating nature of infinite series then their responses to Q12 (1+1+1...) would be 'Yes', as this would be a finite sum. A minority, however, responded 'Yes' and of these, those interviewed indicated that although there was an answer at each stage there was no final answer.

**1.ii) Sets with more than any given finite number of elements.**

Our evidence that subjects can apprehend the notion of such sets comes from responses to Q17 *Is N a single set ?* and Q18 *Is $R_{(0,1)}$ a single set ?* Both questions resulted in strong 'Yes' responses. Q17 was particularly strong, over groups and questionnaires, and considered the natural numbers (which, we have seen, subjects view as non terminating). Moreover in all of the many protocols dealing with cardinality concepts there is no indication that subjects are having difficulty with the concept of an infinite collection.

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| **2) INFINITY AS A PROCESS AND AS AN OBJECT.** |
| i) Infinity exists as a process, and as an object. |
| ii) Infinity means *going on and on* and as such is used as an evaluatory scheme for judging whether a question determines an infinite answer. |
| iii) As an object there is a cognizance of a number at *the end* of the number line and the cardinality of infinite sets. |

**2.i) Process and Object**

A contradictory feature of infinity arises from it being seen both as a process, rather like the principle of induction or infinite loops in computing, and as an object, as a large number or the cardinality of a set. Standard phrases reflect this. Phrases that occurred repeatedly were *This goes on and on. It's infinite.*, seeing infinity phrases suggest that infinity is seen not as a thing but as the act of going on and on. Also *It's going towards infinity*, seeing infinity as the goal of the process. We must not be too keen to polarize the situation here for the borderline between the two interpretations is fuzzy. Thus, although subjects may say *towards infinity*, this does not rule out *infinity as a process* colouring subjects' thoughts. It may be that because something goes on and on it is infinite and thus goes towards infinity. *Infinite* and *infinity* had a very free interchangeable usage in the interviews. We must not assume, though usage is often correct, that *infinity* refers to an object, a noun, and *infinite* to a process, an adjective.

**2.i) Infinity means *going on and on*.**

As we noted in Chapter Eight (p.199), we cannot form questions to test this directly because this would involve asking subjects to theorize about concepts of infinity rather than simply asking them about their concept of infinity. Nevertheless, as we saw there, with two exceptions, subjects used this meaning of infinity in explaining their responses to a wide variety of questions. This alone supports the thesis that infinity is seen as a process. It does not, however, determine whether this view is dominant. We believe it is but further work, in the form of interviews, not questionnaires, must be carried out to test this hypothesis.

Not only is infinity as a process used as a definition of infinity, it is also used as an **evaluatory scheme** to decide whether a question determines an infinite answer. By this we mean the mode of reasoning:

This goes on and on

Infinity is going on and on

Therefore this is infinity (or infinite)

The protocol data showed us that this was the rationale behind the 'same principle scheme in answering problems on cardinality: *If this one goes on forever so must this one. So there's the same in both.* (p.194) Infinity as a process also led to 'can't compare' responses in these questions because as they go on forever we will never be able to stop to compare them. In cardinality problems this is a reasoning scheme at odds with the generic law, which leads to more in one set. The responses to the cardinality questions reveal that neither reasoning scheme is dominant and that subjects may use one for one question and another for another question (p.265). We examine these schemes further in the ninth thesis.

The rationale behind many real number conceptions is generated by infinity as a process: *You can't have an infinitesimally small number because infinity goes on forever.* This, most teachers would agree, is a satisfactory concept image, but it is virtually identical to the following which would not, in the mathematical community, be seen as satisfactory: *$0.\dot{9}$ is infinite and isn't a proper number because the nines go on forever.*

**2.ii) Infinity as an object.**

**Cognizance of a number at the end of the number line.**

Although Q7 (*Is this how you think of infinity ?* - following *Think* *of infinity as an enormous number)* shows that most subjects do not view infinity as an enormous number, though this does not reveal whether they think of it as an idea, an ideal element or a process. Moreover, of the third (roughly) who did see infinity as an enormous number, the responses do not reveal whether this is as a vague generalization of a large number or as a kind of one point compactification. We must rely on the protocol data for evidence. Referring to the responses there to the direct question, *What is infinity ?* (pp.204-205), we see, apart from *the largest number* and *the largest number, to simplify things*, several subjects claiming *Not really* a *number but ...* and *Not* a *specific thing but ...* This indicates to us that even when infinity is not seen as an object it is considered, and rejected, as a possibility. This indicates that infinity can be viewed as an object.

**Cognizance of the cardinality of a set.**

A set is an object. Q17 and Q18 show that subjects can consider infinite sets as objects. If the number of elements in a set can be referred to, then the cardinality of a set is being treated as an object. If, in the cardinality questions, we collapse responses i), ii) and iii) and compare these with 'can't compare' we find two thirds of the subjects are making comparisons, are comparing objects. Moreover, the protocol data reveals (p.195) that even those saying 'can't compare' use language in which the number of elements is treated as an object, albeit an object of unknown size.

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| 3) **INFINITY AS A NUMBER** |
| i) Infinity as a number is an indeterminate form, a generalization of a large number. |
| ii) Infinite numbers need not be numerically large. Recurring decimals and infinitesimals may also be granted the title 'infinite numbers' because they go on and on. |
| iii) Although there is general recognition of infinity as the largest number, cognitive belief in the existence of this number is low. |
| iv) Subjects' conceptions of infinity do not conform to infinite cardinal or ordinal paradigms. |

### 3.i) Infinity as an indeterminate form.

The protocols give a number of illustrations (p.204):

MWN you think of it as the largest number to simplify

LSM2 you're just generalizing a whole mass of numbers somewhere over there

The responses to Q5 (Is $\infty+1 > \infty$?) are interesting from this angle. The majority 'Yes' response, we argued, arose because infinity was taken as an enormous number and the principles of arithmetic apply to numbers (in particular $x+1>x$). However, though less than 50%, the 'No' response was not small in the MAIN sample. We believe the idea of infinity as an indeterminate form lay behind many of the 'No' responses. The subject LSM2 above is explaining her 'No' response to Q5.

Note that she explains that this is due to infinity being a generalization of a number rather than a number. To act against this very basic x+1>x principle requires the concept to be very strong. However, we cannot generalize from one instance. The hypothesis requires further research.

**3.ii) Recurring decimals and infinitesimals may be granted the title *infinite numbers.***

This was not expected and was not examined in the questionnaires. Remarks arose in interviews (pp.212-214) that exposed this. Behind this claim is *infinity as a process*: Infinity is going on forever, $0.\dot{3}$ goes on forever, therefore $0.\dot{3}$ is infinite. Subjects clearly see the difference between the three categories but all have a non terminating, infinite nature.

**3.iii) The largest number**

Responses to Q1 *Is there a largest number ?* establishes that subjects do not believe in a largest number. With one notable exception (PPM, p.204) interviews support this. We must be careful not to confuse this claim with the claim that subjects cannot apprehend the concept of a final number for, as we have seen above, subjects do conceive of a vague, large form, that corresponds to infinity. Moreover, subjects' denials, such as *There isn't actually a largest number ...*, reveal that they can apprehend the concept of a largest number. They simply reject it.

**3.iv) Subjects' conceptions do not conform to infinite cardinal or ordinal paradigms.**

It is useful to remind ourselves of the basic features of ordinal and cardinal numbers. An ordinal number, X, is a well ordered set such that

$$\forall a \in X,\ a = \{x \in X : x < a\} \ .$$

The basic picture of the ordinals is

$$0,\ 1,\ 2, \ldots,\ w,\ w+1, \ldots,\ w.2, \ldots,\ w\ , \ldots$$

Note that $w+1 > w$ but $1+w = w$

A basic concept image of the ordinals is of counting numbers. Tall and Stewart (1979) show how this aspect of number is often overlooked by post Piagetians. Nevertheless, Piaget and his followers have demonstrated that seriation (ordering by size) is acquired at about seven years of age. Our subjects can clearly count and in the sense that finite ordinals are counting numbers our subjects have a basic but true conception of finite ordinals. But what are their conceptions of limit ordinals ? A limit ordinal has no greatest member and is not the successor of any ordinal. w, for example, is a limit ordinal. It is not the successor of any ordinal but does, itself, have a successor, w+1. It is for this reason that $w+1 > w$ but $1+w = w$. We have seen above that subjects do not believe in the existence of a largest number. w is, in intuitive mathematics, the concept image of the largest natural number. Thus the limit ordinal most accessible to the imagination would probably not be granted cognitive existence by the subjects.

To examine these ideas we performed a short test to see if subjects possessed limit ordinal conceptions of infinity. 34 Lower

Sixth A-level mathematicians were asked to give 'True' or 'False' responses to seven questions. The test was administered in the first five minutes of a mathematics lecture period at Morecambe High School in October 1985. Subjects were asked to *Imagine infinity as the ultimate natural number, the thing at the end of forever.* The administrator (the author) answered several questions on what this meant. It was stressed that subjects should answer according to what this meant to them but that a finite number was not the object in mind. This is, of course, vague and we must not place too much value on the test. The responses, however, are of interest. The seven questions were: 1) $\infty+1 > 1+\infty$ 2) $\infty+1 = \infty$ 3) $1+\infty = \infty+1$ 4) $1+\infty = \infty$ 5) $\infty+1 > \infty$ 6) $1+\infty > \infty$ 7) $\infty+1 < 1+\infty$

The responses were:

**TABLE 9.1**

| | Response | Frequency |
|---|---|---|
| a) | FFTFTTF | 14 |
| b) | FTTTFFF | 8 |
| c) | FFTTFFF | 2 |
| d) | FFTFFFF | 2 |
| e) | others | 8 |

**TABLE 9.2** '*' denotes the formally correct response

| Question | True | False |
|---|---|---|
| 1) $\infty+1 > 1+\infty$ | 3 * | 31 |
| 2) $\infty+1 = \infty$ | 13 | 21 * |
| 3) $1+\infty = \infty+1$ | 29 | 5 * |
| 4) $1+\infty = \infty$ | 14 * | 20 |
| 5) $\infty+1 > \infty$ | 19 * | 15 |
| 6) $1+\infty > \infty$ | 21 | 13 * |
| 7) $\infty+1 < 1+\infty$ | 0 | 34 * |

None of the subjects gave the formally correct response. The fact that 22 out of 34 subjects gave the response FFTFTTF or FTTTFFF (there are 128 permutations) indicates that most subjects were not responding randomly. The strong responses to 1), 3) and 7) are due to the belief that $1+\infty = \infty+1$. The responses to 2), 4), 5) and 6) are consistent with the MAIN responses to Q5 (Is $\infty+1 > \infty$?), that is, a roughly 60% agreement that $\infty+1$ is indeed greater than $\infty$. We conclude that subjects' concepts do not conform to infinite ordinal paradigms.

The cardinal number, X, of an ordinal, X, is the least ordinal for which there is a bijection onto X. Thus w+1 = 1+w = w. The essential characteristic to the mathematician is one-to-one correspondence. As for finite ordinals there is no doubt that subjects of the age and ability of ours do possess a basic but true conception of finite cardinals (indeed, for finite cardinals the conception is virtually identical to that of finite ordinals). This is not to say that subjects can explain their conceptions in terms of one-to-one correspondence but merely to say that they can discard the form and order of any finite set and abstract the number of elements in a set as a number. There are problems involved in such basic characterizations of cognitive number theory, as Stewart and Tall note (1979, Part 2, p.5), but we shall not go into these as we are interested here in subjects' conceptions of infinite cardinals.

As we have seen above (thesis 2.ii) subjects can apprehend the notion of the cardinality of a infinite set in that they can refer to the number of elements. This is a start but do they use one to one correspondence to compare cardinals ? We did note, in the protocol data (p.195), that this was used by one subject but there is no indication that this isolated instance had widespread use. Instead *infinity as a process* and the generic law were commonly used for comparing infinite cardinals. This is not the method of formal cardinal mathematics. Moreover, as the responses to the five cardinality questions show, although subjects often respond correctly, in terms of transfinite arithmetic, they more often respond incorrectly. We conclude that subjects' conceptions do not conform to formal cardinal paradigms.

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| 4) INFINITESIMALS |
| i) Infinitesimals are not generally accepted but may be seen as useful fictions. When they are accepted they are seen as dynamic entities that exist in the process of a sequence of numbers, or a function, decreasing. Static infinitesimals do not conform with subjects' conceptions. A cognitive framework ripe for the introduction of the concepts of non standard analysis does not exist amongst subjects. |
| ii) Willingness to accept approximations is strong with small numbers. |

**4.i) Infinitesimals are not generally accepted.**

The strong 'No' response to Q2 (Is there a smallest number ?) shows clearly that subjects do not believe in an ultimate infinitesimal. Thus the nonstandard approaches of Tall or Keisler, that start by considering the real number system with an additional infinitesimal of this kind, would meet with initial cognitive opposition. Protocols suggest (p.209) that the main reason for this rejection was *infinity as a process*, that is, it is possible to go on forever getting smaller and smaller.

Subjects' thoughts on infinitesimals may stop here but protocols reveal (pp.209-211) that some, at least (we are not in a position to quantify), did consider them as useful fictions or considered $1/\infty$ as a generalization of a small number. This may appear as a possible starting point for nonstandard ideas but the strong 'No' response to

Q8 *(Does 2+s=2 ?)* indicates that great care would have be taken in presenting the crucial concept of taking standard parts (that amounts to treating 2+s as 2). It is interesting, however, to note that the equally strong rejection of Q9 *(Does 2xs=s ?)* is compatible with the ideas of nonstandard analysis. Responses to both questions arise because affirmation contradicts some of the most basic laws of arithmetic.

Despite this, 45% of the N group and 35% of the M group claimed they could believe in such a number (Q10). We did not interview these subjects and did not, unfortunately, include this question in the earlier questionnaires. We thus cannot give definite reasons for this apparent anomaly. As we suggested in Chapter Six (p.121), however, we believe the reason for this is that subjects can conceive of, but actually do not believe in, infinitesimals.

Other than this, $1/\infty$ would appear a possible candidate for an infinitesimal. This construct was referred to, unprompted, by subjects in response to Q26c *(1,1/2,1/4,... converges to ____ )* and was strongest in the M group. This arose again in the interviews and was seen by some (again stronger in the M group) as a proper number in Q16. As we saw on the last page, however, some subjects saw $1/\infty$ as a generalization of a small number. An inspection of the protocols on pp.210-211 reveals a dynamic concept image is dominant here. In fact, apart from viewing infinitesimals as useful fictions, only one subject (3B in the pilot tests) had a static concept image:

Down to the smallest line you can have a line two atoms long.

The dynamic context suggested by decreasing sequences has two interesting consequences. On the one hand it leads to a rejection of static infinitesimals because the halving, tenthing, etc. always leads to smaller numbers but it may also lead to viewing infinitesimals as dynamic entities that continuously decrease. This is close to Cauchy's view that an infinitesimal is a variable that converges to 0. This view is derived from seeing infinity as a process and is part of the same phenomenon we observed in the last thesis where some subjects saw infinitesimals as infinite numbers because they go on and on.

**4.ii) Willingness to accept approximations.**

Despite the general rejection of static infinitesimals some subjects display a tendency to accept approximations when small numbers are involved. Our study was not designed to examine this but isolated protocols (p.212, LSM1 and GHN) show this to be so. One may have guessed this from the fact that so much of O-level mathematics uses approximations as exact answers (4/x=7 —> x=0.571, pi=3.14, sin $33^\circ$=0.545, etc.).

With respect to the teaching of calculus this may mean that even if a nonstandard approach will meet with initial cognitive rejection or have to play on *useful fiction* concepts there is a possible approach that simply ignores small, but real, numbers. We are not sympathetic to such an approach but this possibility is worthy of further research.

**5) INFINITE SEQUENCES AND SERIES.**

i) Basic convergence/divergence properties of infinite sequences are generally noted though subjects often focus on mathematically unimportant features such as oscillations in evaluating convergence.

ii) The generic limit concept is dominant in subjects' conceptions of the limit of an infinite sequence. There is a small shift to the mathematicians' limit concept amongst A-level mathematicians.

iii) The convergence or divergence of an infinite series is not generally seen as its most important feature. Theoretical, physical and temporal problems of any infinite summation often override them as important features.

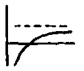
Infinite sequences and series are considered in a wider context in theses 1 and 4 above. The peculiarities of language used to describe them are considered in thesis 7 below. Here results concerning their nature qua sequences and series are collected.

**5.i) Basic convergence/divergence properties of infinite sequences are generally noted.**

We have seen in thesis 1.i above that infinite sequences are seen as infinite in the sense of being non terminating. We thus have an assurance that the subjects appreciate the nature of the entities they were presented with.

Responses to Q27 (the four phrases applied to $0.\dot{9}$ and 1 with the sequence 0.9,0.99,...) show that subjects generally note convergence in monotone sequences but that the phrases used to describe convergence do affect the responses (*tends* to and *approaches* being more acceptable than *limit* and *converges* - this is examined in detail in thesis 8). In a geometric context this applies to functions too, as we saw in Q30 .

It may be objected that Q27 presented two alternatives (limit $0.\dot{9}$ and limit 1) both of which would lead us to claim that subjects recognize the convergence of monotone sequences (though, in defence of such a claim, subjects were free to put 'No'). Q26 (1+h tends to ___ as h tends to 0, etc.) is useful here because it is an open question. Although the first two sequences (or functions) generated are implicit (they are not presented in the form a,b,c,...) the questions do show that subjects recognize the convergent nature of the sequences. Note that this was especially strong in the M group and that again *converges* caused more problems (p.145).

Divergence concepts are harder to analyse. For functions, subjects were presented with examples that did not converge to 0 (Q31  and Q32 ). As we have seen, with provisos noted on p.168 concerning the wording, subjects clearly rejected any claim that these converged to 0. Though compatible with the thesis that basic convergence / divergence properties are noted this cannot be used as evidence for this thesis for although mathematicians may deduce the sequence result from the function result here, it does not follow that students see the implication. Moreover, although the functions in these questions do not converge to 0, they do converge to 1. They are not, then, general examples of divergent functions.

The fact that *tends to* and *approaches* best described monotone convergence complicates an examination of subjects' perceptions of divergence since we examined the four sequences (1,0.1,0.01,.. 1,0, 0.1,0,.. 1,0.1,1,0.01,.. 1,1,1,..) in Q26 and Q27 using *limit* and *converges*. In retrospect it would have been better to use *approaches* or *tends to* in place of *converges* or, better still, to have used all four phrases. Concern for the length of the questionnaire, at the time, caused this to be omitted. Although the divergent sequence in these questions generated the overall strongest 'No' response this is, by itself, insufficient evidence to claim that subjects recognize that divergent sequences have no limit (though, again, it is compatible with the claim).

Evidence against the claim that divergence is seen comes from Q12 (1+1+1..). Subjects claimed (pp.239-241), on the whole, that the series 1+1+1+... and 0.1+0.01+... were the same in principle. Are the sequences of partial sums (1,2,3,.. and 0.1,0.11,0.111,..) not clear ? We suspect they are and, moreover, that the unbounded nature of *1,2,3,..* and the bounded nature of *0.1,0.11,0.111,..* are also clear. Several subjects expressed this explicitly on being asked *Is there not a difference between 1+1+1+.. and 0.1+0.01+.. ?* PBM2 (p.239), for example, clearly expresses a *same principle* stance. Nevertheless, on further questioning he states:

> SUB On that one you'll never get beyond 0.12, or whatever. With that one it'll carry on getting bigger and bigger.

INT But there's not really.. (interrupted)

SUB There's a limit to where that one can get to but there isn't to that one.

The subject, however, still saw the *same principle* applying to both series (and thus to both sequences).

It is a failing of our research that we did not probe further with all the subjects. Further research is needed here.

The data provided by the questionnaires (pp.159-163) is unable to shed any light on the effect of oscillations on convergent or divergent sequences. Protocols (pp.215-217), however, reveal that oscillations can shift subjects' thoughts away from convergence and divergence and onto the nature of the oscillations themselves. As has been seen (p.217), subjects focus on mathematically unimportant features

You're alternating between 0 and another number. It sort of confuses the issue somehow.

Similar focussing on mathematically unimportant features was observed with regard to the constant sequence 1,1,1,... (p.217) :

When it approaches it goes towards. This is already at 1.

You have to move towards a limit.

All we can claim here is that some subjects are confused by mathematically unimportant features. We suspect this is general.

Further research may not be particularly useful here for a huge variation in the kind of variant features that interfere with subjects' thought is possible and likely.

**5.ii) The generic limit concept is dominant.**

This has been noted in discussions on subjects' strong belief that $0.\dot{9} < 1$ and in Q27 where there was a significant response from both groups that the limit of 0.9, 0.99, ... was $0.\dot{9}$ and not 1. Protocols fully support the questionnaires (pp.221-237). In fact we did not find one subject in the interviews who could be said to be fully removed from the influence of generic limit ideas (though there appears to be the beginnings of a shift away in some subjects).

Responses to Q24 (nested triangles) and Q25 (sequence of jagged functions) suggest that generic limit concepts are stronger in arithmetic contexts than they are in a geometric contexts. As we have seen (p.235), however, the protocols do not support this thesis. It may be that approximation is simply more widely used in geometric contexts. We postpone further discussion on these points until thesis 9 (where we consider the effect of context).

As we have mentioned several times we believe a movement away from the generic limit concept to the mathematicians' limit concept occurs in the conceptions of some A-level mathematicians. Recall that in Chapter Six (p.144) we saw a significant shift in the M group, relative to the N group, in non generic limit responses over questions 24, 25, 27 and 28. There is insufficient data in the protocols to back this view up (though there are instances - see p.236). A problem with claims such as *there is a small shift...* is that a very large sample is needed to verify it. Moreover, to detail the shift to a mathematicians' view would require following subjects to their University courses. Such a study would be of very great interest.

**5.iii) The convergence or divergence of an infinite series is not generally seen as its most important feature.**

As we saw in the discussion of Q15 (p.128) there appears to be no recognition of convergence / divergence properties in the N group and less than a third of the M group appear to recognize the distinction. Moreover, as we have seen with Q12 and Q13, the series *1+1+..* and *0.1+0.01+..* are seen as the same in principle (they go on forever). We stop short of concluding that the convergence and divergence of infinite series is not generally noted because we believe many (we cannot specify how many) note this but give it only secondary importance. We did design the Q15 so that fraction and decimal groups obviously stood out and these parts of mathematics have been the all important features of the subjects prior mathematical experience. Subjects, then, may have simply focussed on the obvious (as, indeed, they did) while being quite capable of discerning the convergence of some of the series.

A dynamic view of series may lead to observations of convergence and divergence taking second place to the unifying fact that both types of series go on and on. Also dynamic interpretations of series may lead to physical and temporal factors coming into mathematical arguments. Many interpretations coexist. Some subjects clearly fail to note convergence / divergence properties, some recognize them but fail to see them as important and some recognize them and their importance.

Subjects understanding of infinite series deserves further study. Knowledge of the extent to which each of the three interpretations mentioned above exist in subjects' conceptions would be useful. Questionnaire data alone is insufficient for such a task for subjects can then opt for obvious patterns (e.g. grouping series with fraction terms together).

| 6) REAL NUMBERS |
|---|
| i) Subjects' ontological framework includes infinite recurring decimals but they are interpreted in a dynamic context and seen as qualitatively different from finite decimals. This leads to an inconsistent model and, ultimately, to cognitive conflict. |
| ii) Subjects' concepts of the continuum do not correspond to mature mathematicians' models of the continuum. |

We must begin by ensuring that our subjects understand the the basic theory of decimals. We performed checks in the first pilot study (p.59) and in Questionnaire 1 (p.97) to ensure that subjects could insert a decimal between two close decimals; they were. Moreover, our subjects all obtained O-level mathematics passes. They are thus, roughly speaking, in the top 25% of the mathematics ability range. The CSMS team (1981), in their chapter on decimals, conclude that:

> The top 50% of pupils are likely by the time they leave school to have a reasonable if not complete understanding of decimals.

They did not, unfortunately, consider recurring decimals. The only study in this field that we are aware of is that of Vinner and Kidron (1985).

Vinner and Kidron examined able Israeli High School pupils' conceptions of the construction of infinite repeating and non-repeating decimals. Few subjects (4% in the Tenth Grade and 33% in the Eleventh Grade) displayed an awareness of the existence of non-repeating infinite decimals. The result is interesting but not all that relevant to us since they do not examine whether any infinite decimal has *proper* status.

We confess that we carried out our investigations assuming recurring decimals were understood, for years of work with pupils aged 11+ has convinced us of this. Moreover, in all the protocols there is not one indication that subjects were unable to grasp the concept of infinite decimals. This is consonant with thesis 1.i that subjects have a cognizance of non terminating processes.

**6.i) Recurring decimals.**

We have seen in the protocols (p.250) that a dynamic interpretation of recurring decimals is common. This view was also dominant in subjects' conceptions of the series 0.1 + 0.01 + ... (p.246) and was the reason why $0.\dot{9} < 1$ (p.251). Nevertheless, as we saw in Q16 (p.130), $0.\dot{9}$ is given *proper* numeric status by 72% of the M group though only 54% of the N group. As we noted there, the idea of a *proper number* is vague and we must not read too much into these results. They do show, however, a hesitancy but a general acceptance of recurring decimals as *proper numbers*.

An interesting observation in Q16 is the 13% drop in 'Yes' responses in the M group for $1-0.\dot{9}$. Although this result is not significant it does suggest that basic closure properties of numbers do not apply to recurring decimals for some subjects. The explanation would appear to be that although $0.\dot{9}$, etc. may be acceptable, *nought point nought recurring one* is not. The concept of *nought point nought recurring one* is interesting and worthy of discussion in school mathematics.

A question omitted, unfortunately, from Questionnaire 2 was *Is* $1/3=0.\dot{3}$ *?* The responses from Questionnaire 1 are, however, significant. The 'Yes' responses for the M group (out of a possible 27) were 25 in the first administration and 24 (out of 27) in the second administration. The 'Yes' responses for the N group (out of a possible 27) were 25 in the first administration and 18 in the second administration. In the interviews, four subjects were asked *Does* $1/3=0.\dot{3}$ *?* All responded 'Yes'. This question was asked each time within the context of revealing the contradiction: $1/3=0.\dot{3}$, therefore $0.\dot{9} = 3\times0.\dot{3} = 3\times1/3 = 1$, but $0.\dot{9} < 1$. None claimed the error lay in $1/3=0.\dot{3}$. The reason for the strong acceptance of this is, we posit, partially familiarity and partially the shape of the decimals. Our subjects had been told, for about five years, that $1/3=0.\dot{3}$. Its cognitive strength is thus very strong. Moreover the shape of $0.\dot{9}$ suggests it is less than 1. It starts *nought point* and anything starting like this is less than 1 (p.224). This is not so with $0.\dot{3}$ and 1/3. We conclude that subjects are often inconsistent in their interpretations of recurring decimals. This may give rise to conflict as in the case of $0.\dot{3}=1/3$ but $0.\dot{9}<1$.

The inconsistencies present in the protocols led us to the following hypothesis which we are unable to verify here and which we thus leave for further research:

> Subjects operate in a mixture of the following mathematical universes:
>
> i) A finite universe where 1/3=0.333333. $0.\dot{3}$ does exist here but is a finite number.
>
> ii) A finite decimal representation world where $0.\dot{3}$ does not exist but all finite approximations do ( and do not equal $0.\dot{3}$).
>
> iii) A generic limit universe where $0.\dot{3}$ exists but $0.\dot{9} < 1$.

**6.ii) Conceptions of the continuum.**

Essential to the mathematicians' view of R is the completeness of the real numbers. Subjects' generic limit interpretations are not consonant with this (a classic example being that an infinite sequence of rationals determines the irrational $\sqrt{2}$ - this is quite alien to generic limit concepts). Moreover, as we have seen above,a dynamic view of limits can, because the limit is never attained, lead to viewing the limit of a sequence as existing on a different ontological plane to the finite terms of a sequence. Again this is quite alien to mature mathematical thought.

These remarks apply to standard and nonstandard mathematical thought. We have seen, moreover, that subjects' views of infinitesimals do not conform to non standard models. We conclude that subjects' conceptions of the continuum do not conform to mathematicians' views.

| |
|---|
| 7) **LANGUAGE** |
| i) Phrases such as *gets to* and *goes on forever* suggest impossible situations. |
| ii) Mathematical phrases often used in calculus courses have everyday connotations that affect subjects' mathematical interpretations. |

### 7.i) Impossible situations.

We believe most school mathematics teachers are guilty of occasionally refering to sequences such as 0.1, 0.01, ... *getting to 0*. Even such an experienced researcher as Orton (1980a) uses questions such as:

> *Can you use this formula to obtain the 'final term' or limit of the sequence.*

Recall (p.98) that the question *Will 0.1, 0.01, ... ever get to 0 ?* was omitted from Questionnaire 2 because MHS subjects were 99% (107 out of a possible 108 responses) certain that it did not get to 0.

This is an intelligent response. The sequence clearly does not get to 0, in any everyday sense. Even a mathematician would have to qualify a claim that it did by saying that s/he really meant *the limit is 0*.

Similarly *going on forever* is not possible in the physical world. The series protocols (pp.243-245) reveal that this caused difficulties, that it is not possible to go on forever. Again this is an intelligent response. Infinite series have a special meaning to mathematicians. We think of converging series as having a limit. This is quite different to *going on forever*.

**7.ii) Interference of everyday meanings of phrases.**

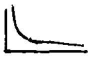
To the mathematician the phrases *tends to*, *approaches*, *converges* and *limit* are more or less interchangeable. To a large extent this is seen by the subjects but there are many disturbances to the pattern. We have documented these in detail in Chapter Six (pp.146-174) and Chapter Eight (pp.252-260). We shall not repeat all the many individual variations here but summarize the main findings.

*Tends to* and *approaches* present the least difficulty to subjects because they are vague. *Tends to* usually connotes a general trend. *Approaches* is similar but may cause problems because in everyday language the approached object is often arrived at, unlike, say, $y=1/x$. Also the approaching objects may remain a fixed distance away. Thus $y=1+1/x$ may approach 0, as in *the man approaches the dog* example (p.149).

*Converges* was the most confusing phrase to subjects. It was, almost without exception, seen in terms of light rays or lines converging.

Many subjects could not see how a sequence of numbers could converge. This was reflected in the questions concerning *converges* having the least number of correct responses.

*Limit* has a very strong connotation to the subjects in that a sequence or function may tend to 0 without having a limit 0. The limit of a sequence or function was seen both as a generic final point (and thus unreachable) and as an unattainable non generic boundary point (such as *1* in the sequence 0.9, 0.99, ...).

Mathematically irrelevant differences sometimes affected responses. An example we have witnessed (p.165) are equivalent sequences and functions which have the same limit but which cause subjects to respond 'Yes' in one context but 'No' in the other. Another example is oscillations. Whereas Q30  may approach 0, the oscillating functions of Q33  and Q34  may not because they approach 0 and go away again. Similar comments apply to the constant sequence 1,1,1,... (pp.217-218).

**8) REASONING**

Reasoning schemes peculiar to problems dealing with limits and infinity are *infinity as a process* and the generic law. Both schemes have widespread application. Subjects may change from one scheme to the other in response to similar questions.

As we noted in Chapter Eight (p.261), we do not believe that subjects are logical, in any formal sense, in their mental acts and that comparisons with logical canons is not a priority in this study.

Moreover, the design of the study was geared towards examining intuitions and was not multiple-step-problem-solving-orientated and is thus limited in the extent to which it can examine reasoning. Nevertheless, several aspects of subjects' reasoning in this area can be extracted from the data obtained from the questionnaires and, much more so, from the protocols.

**Infinity as a process and the generic law.**

As we noted in thesis 2.i, not only is *infinity as a process* used to define the concept of infinity it is also used as an evaluatory scheme to decide whether a question determines an infinite answer. We noted there that the protocol data showed us that this was the rationale behind the *same principle* scheme in answering problems on cardinality (leading to 'same number in each set') and to 'can't compare' responses. Tirosh (1985) notes the same phenomenon:

> The main argument given by the students for the equivalence claim was that *only one kind of infinity exists, therefore, all the infinite sets have the same number of elements*. This idea of equivalence corresponds to the primary intuitive understanding of the infinite as an endless process.

In cardinality problems this reasoning scheme is at odds with the generic law, which leads to more in one set. The responses to the cardinality questions reveal that neither reasoning scheme is dominant and that subjects may use one for one question and another for another question (pp.265-266).

In the cardinality questions the distinction between the two schemes is clear. Elsewhere this is not always the case. For example, in thesis 7.i we noted that the rationale behind many real number conceptions is generated by *infinity as a process*. Thus $0.\dot{9}$ is an infinite number, in that it goes on indefinitely, and is thus qualitatively different to 1 and so cannot equal 1. Protocols (p.224), however, reveal that $0.\dot{9} < 1$ may be obtained via the generic law *0.9 < 1, 0.99 < 1, ... and thus $0.\dot{9}$ must be less than 1.*

The documentation of occurrences of both schemes in Chapter Eight (p.266) indicates that the first year of an A-level course does not lead to a increased use of either scheme. We must treat this claim with some care, however, for although we were aware of each scheme at the time of the interviews, our knowledge was less mature then and subjects were not probed as strongly as they would be now.

Although these observations show the widespread application of both schemes we are unable to clearly delineate the scope of each scheme. As we shall consider in the next set of theses, on contexts, however, it may be that dynamic contexts lead to use of *infinity as a process* and measuring contexts lead to use of the generic law.

Other than these two schemes we suspected that a recognition of the nature of infinity as a pure construct, which no direct experience could support, would lead to more abstract reasoning. In particular that it would lead to an increased casual (as oppossed to formal) use of hypothesis testing and inferential reasoning (that may be valid or invalid in form). While we have seen some evidence to suggest that this is so (pp.261-264), we do not have protocols from other areas of mathematics to compare them to. Should future research take up this questions we would suggest it be part of a wider hypothesis that able subjects' modes of reasoning generally rise to the level required by the mathematics. Thus, more abstract reasoning will be employed by subjects in areas of mathematics beyond simple empirical verification or finite computation.

**9) CONTEXTS**

Subjects' responses may be affected by the context of a question. There are three notable divisions:

i) **Numeric and Geometric.** Subjects' sense of the existence of a limit of a convergent function, presented graphically, is stronger than their sense of the existence of a limit of a convergent numeric sequence. Generic limit ideas appear less pronounced in geometric contexts.

ii) **Counting and Measuring.** A measuring context encourages subjects to ascribe a greater cardinality to the superset in cardinality questions.

iii) **Static and Dynamic.** A dynamic interpretation of recurring decimals leads subjects to a view of the continuum which is often at odds with the static real complete continuum of higher mathematics. A dynamic interpretation of series often leads subjects to overlook the convergence and divergence of series and see them as similar because they both go on and on. Such interpretations also lead to physical and temporal factors affecting subjects considerations of series.

By *context* we mean the sum of the linguistic, social and mathematical conventions that give a concept (or *cognitive proposition* - see Appendix C) meaning. We shall think of questions suggesting contexts in that certain connotations are suggested in the subject's mind. Of course, a single question may suggest different contexts to different subjects. The results obtained on contexts were partially sought after, as described in Chapter Five, and partially obtained by accident, in the course of examining other factors.

## 9.i) Numeric and geometric

By a *numeric context* we mean a situation that evokes the general principles of number and the basic operations of arithmetic applied to numbers and numeric variables. By a *geometric context* we mean a situation that evokes knowledge of curves and spatial figures.

Questionnaire responses suggested that subjects' senses of the existence of a limit of a convergent function, presented graphically, is stronger than their sense of the existence of a limit of a convergent numeric sequence. Questions 28a, 28b, 30b and 33b examine whether strictly monotone convergent and oscillating convergent numeric sequences and functions could be said to *have a limit*. We repeat the response tables below and insert the raw figures, in brackets, after the percentages. We do not examine the similar questions with *converges* for, as we have seen, many of the subjects did not understand the application of this phrase to mathematical questions.

Q28 For each of the sequences below say whether it has a limit.

**TABLE Q28a** 1, 0.1, 0.01, 0.001, ...

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 0 | 0 | 4 | 44 |
| ? | 4 | 0 | 0 | 0 |
| N | 96 | 100 | 96 | 56 |

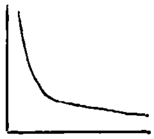
| MAIN | N | M |
|---|---|---|
| Y | 4 (3) | 43 (49) |
| ? | 5 (4) | 2 (2) |
| N | 91 (69) | 55 (63) |

**TABLE Q28b** 1, 0, 0.1, 0, 0.01, ...

| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 7 | 15 | 7 | 33 |
| ? | 0 | 7 | 0 | 0 |
| N | 96 | 78 | 96 | 67 |

| MAIN | N | M |
|---|---|---|
| Y | 12 (9) | 33 (38) |
| ? | 13 (10) | 4 (4) |
| N | 75 (57) | 63 (72) |

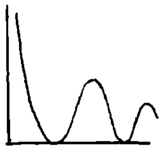
**TABLE Q30b** Does this curve have 0 as a limit ?



| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 56 | 52 | 48 | 85 |
| ? | 0 | 4 | 0 | 0 |
| N | 44 | 44 | 52 | 15 |

| MAIN | N | M |
|---|---|---|
| Y | 34 (26) | 46 (53) |
| ? | 13 (10) | 9 (10) |
| N | 53 (40) | 45 (51) |

**TABLE Q33b** Does this curve have 0 as a limit ?



| MHS | N 1 | N 2 | M 1 | M 2 |
|---|---|---|---|---|
| Y | 56 | 63 | 85 | 67 |
| ? | 0 | 0 | 7 | 0 |
| N | 44 | 37 | 7 | 33 |

| MAIN | N | M |
|---|---|---|
| Y | 63 (48) | 75 (85) |
| ? | 8 (6) | 4 (5) |
| N | 29 (22) | 21 (24) |

Our hypothesis was inspired by the MHS responses. Notice the tendency in their responses to ascribe a limit in the geometric cases but not in the numeric cases. However, the MAIN responses are not completely consistent with the MHS responses. Although, with the MAIN sample, neither group was biased to either pole in the monotone function question and the M group was not biased to either pole in the monotone sequence question, the N group was very strong in its 'No' response to the sequence.

The hypothesis that subjects' sense of the existence of a limit is stronger in a geometric context is supported by the responses in the oscillating cases. In the arithmetic question the sequence did not generally have a limit whereas the function did in the geometric question. There was agreement between groups on these questions. Although the responses were only strongly marked for the N group in the arithmetic question and for the M group in the geometric question the shift in both groups over context is strong enough to rule out random replies ('Yes' responses changing from 12% to 63% in the N group and from 33% to 75% in the M group). This consistency is further supported by responses, in both the monotone and oscillating cases, over contexts. We display below responses for each case. Geo denotes the geometric context, Num denotes the arithmetic context.

TABLE 9.3

Monotone case

| Num | Geo | N | M |
|---|---|---|---|
| Y | Y | 2 | 31 |
| Y | N | 1 | 14 |
| N | Y | 22 | 22 |
| N | N | 37 | 36 |

Oscillating case

| Num | Geo | N | M |
|---|---|---|---|
| Y | Y | 3 | 30 |
| Y | N | 6 | 6 |
| N | Y | 37 | 52 |
| N | N | 15 | 17 |

Note the relatively high incidence of 'NY' compared to 'YN'. What appears to emerge is a stronger sense of limits existing in a geometric context. As we have seen (pp.218-221), however, the protocols do not support this hypothesis. We do not dismiss the hypothesis altogether but leave it for further research.

Our second hypothesis concerning numeric and geometric contexts is that generic limit ideas are less pronounced in geometric contexts.

We may compare generic limit ideas in these contexts by comparing responses to the geometric questions, Q24 (nest of triangles) and Q25 (sequence of jagged functions), with the numeric *limit is* $0.\dot{9}$ and *limit is 1* parts of Q27 ( 0.9,0.99,..). Both pairs of questions present convergent sequences and offer subjects a choice between generic and non generic limits. The responses show a 2:1 generic bias in both groups in the arithmetic questions diminishing to a 1:1 ratio in both groups in the geometric questions. The responses are such that we must acknowledge a difference in context. The results for the MAIN group, however, are not what was suggested by the MHS results (where generic ideas appeared to remain dominant in geometric contexts). We must, then, hold a certain scepticism here and probe deeper. Unfortunately subjects were not sufficiently interrogated concerning this hypothesis and the protocols do not really provide supporting or contrary evidence (p.235). The hypothesis that generic limit concepts are stronger in arithmetic contexts thus requires further research.

A possibility that further research could investigate is that it is not actually the case that generic limit ideas are less strong in geometric contexts but that approximation is more widely used.

Finally it should be noted that the boundary between the two contexts are not always finely drawn for subjects may arithmetize geometric questions:

PBM1 (Q25, Sequence of jagged functions)
SUB I thought each time you're dividing by 2, that 1,1/2,1/4, so you must have something left.

JCN1 (Q25) There'd always be a slight wave. You can go on to infinity going 1/32, 1/64.

VMM1 (Q25) Well it looks straight but really it won't get down to a straight line ... 'cos you've always got 1 over a number. You'd never get to 0.

## 9.ii) Counting and measuring

By a counting context we mean a situation in which problems are solved via counting or one to one correspondence. This is usually generated by the problem dealing with discrete sets though a subject may impose such a scheme onto the problem (such as Cantor with the continuum). By a measuring context we mean a situation in which problems are solved by comparing continuous quantities or sets in one, two or three dimensions.

We anticipated that these two contexts may come into opposition in subjects' concepts of infinity: that a counting, end of the integers, concept of infinity may be different from a measuring, end of the continuum, concept. We found no evidence of this, however. The main area where we found this a genuine division was in the cardinality problems. Tall (1980b) discusses this and shows that the reasonable idea that a line twice another line's length, has twice the number of points can be extended to a coherent nonstandard system.

Re-examining the responses to the cardinality questions, ignoring '?' and silly responses (e.g. *more in the square* in Q21) and grouping 'same' and 'can't compare' together we can compare 'more' with 'other' responses. Table 9.4, below, displays the percentages for the MAIN sample.

TABLE 9.4

| | Q19 | | Q20 | | Q21 | | Q22 | | Q23 | |
|---|---|---|---|---|---|---|---|---|---|---|
| | N vs Evens | | N vs R | | __ vs □ | | _ vs _____ | | □ vs ○ | |
| | N | M | N | M | N | M | N | M | N | M |
| More | 18 | 15 | 46 | 27 | 29 | 32 | 51 | 35 | 50 | 40 |
| Other | 76 | 82 | 54 | 71 | 69 | 67 | 48 | 64 | 43 | 58 |

Q19 clearly suggests a counting context whereas questions 22 and 23 clearly suggest a measuring context. Questions 20 and 21 are not obviously one or the other. We expected that a 'more' response would be stronger in a measuring context. Looking at the responses this appears to be the case but the figures alone merely indicate a trend. Is this significant ? $X^2$ tests refute the hypothesis that there is no difference between the questions. If we compare each group, N and M, with itself over questions 19 and 22 and over 19 and 23, we obtain:

N group: 19/22, $X^2_1$=15.5 ; 19/23, $X^2_1$=23.4

M group: 19/22, $X^2_1$=11.1 ; 19/23, $X^2_1$=17.0

$P < 0.001$ in each case. This strongly suggests that a measuring context encourages subjects to ascribe a greater cardinality to the superset in cardinality questions.

An alternative hypothesis is that a measuring context may encourage greater use of the generic law and the generic law encourages subjects to ascribe a greater cardinality to the superset. We believe there is some truth in this but there is something more than just this at work here for certainly the generic law is applicable to Q19, where we are comparing a set with a set derived from it by deleting half its members, but this has significantly fewer 'more' responses than any of the other questions.

Another objection to the thesis is that questions 22 and 23 encourage subjects to think in terms of physical points and that in these terms the rules of finite mathematics dictate that the longer line and the larger area have more points. Again we believe there is some truth in this but that it is not quite so simple. We have seen throughout the study that subjects are well beyond crude finitism and thus, in general, the points they would consider are abstractions of finite points. An abstract, infinite extension of the concrete, finite situation of a longer line having more points than a shorter line is precisely, however, the idea we believe is behind subjects' conceptions surrounding measuring contexts.

## 9.iii) Static and dynamic

This is a very subjective distinction and depends on how the subject interprets a problem. If an indefinite process is evoked, especially one suggesting motion in some sense, in determining a response then we shall call the context *dynamic*. If an indefinite process suggesting motion is not evoked then we shall call the context *static*. To see how it is the interpretation rather than simply the problem which determines which of these contexts applies, consider the question *What is* $1/(1-0.\dot{9})$ *?*. A response from a static context (not necessarily the only one) would be $1-0.\dot{9}$ *is infinitesimally small. The reciprocal is thus infinitely large*. A response from a dynamic context (again, not necessarily the only one) would be *1/0.1=10, 1/0.01=100, ... The answer becomes infinitely large.*

Tall (1981a) calls the old style of school calculus (where expressions such as 3xh are said to get closer and closer to 0 as h does) the *dynamic limit method*. He argues, with examples of students' responses to calculations using the algebra of limits, that this method has strong cognitive appeal. Cornu (1983) details obstacles to formal limit notions arising from the shift from the static to the dynamic (passing from finite terms of a sequence to the limit at infinity) and later from the dynamic introductory notion to the static quantified definition of a limit ( $\forall \varepsilon > 0, \exists N \ldots$ ). Both of these reseachers offer important insights into students' static and dynamic limit concepts. We cannot build on their work, however, for the mathematics they deal with is too advanced for us to have presented to all of our subjects. We can, however, examine the effect of these interpretations at a lower mathematical level.

Behind dynamic interpretations of infinite phenomena is the idea of *infinity as a process*. Dynamic contexts are, however, less general than infinity as a process. Thus, as we have argued in thesis 2.i, infinity as a process is behind the *same principle* scheme in responses to cardinality questions but this is not a dynamic context as we mean it here. (Subjects may, however, see the real interval (0,1) as a whole mass of decimal numbers splitting off from each other and multiplying rather like cells multiplying in a biological colony. This analogy was suggested by a Sixth Form pupil in an informal discussion. The interval is, nevertheless, fixed and has no external movement). Of the items we presented to subjects, the ones that appeared to suggest movement to them were those to do with recurring decimals and the limits of sequences and series.

As protocols reveal the reason why $0.\dot{9} < 1$ (p.251) and the reason behind the 'No' responses to *Can we add 0.1 + 0.01 +...?* (pp.239-241) is that recurring decimals and series are seen as *becomings* rather than *beings*. They are always on the move and do not deliver a final proper answer. As we saw in thesis 4.i, infinitesimals are seen in this same dynamic light. This dynamic interpretation of a subset of the real number line is at odds with the mathematicians' view. To the university trained mathematician recurring decimals and convergent infinite series are seen as completed instantaneously. Formal limit ideas have clearly been considered at one time but this static concept image is certainly the residual intuition left by such methods. Teachers must be very careful in their explanations or their internal static representations will not make sense to students' dynamic models. A dynamic view of recurring decimals may also alter the domain of definition of an expression. Thus $1/(1-0.\dot{9})$ may, in a static interpretation be seen as *1/0* and thus as *undefined*, but the terms 1/0.1, 1/0.01, ... ,generated in a dynamic model, are defined and may lead to $1/(1-0.\dot{9})$ being seen as defined.

A dynamic view of series may lead to observations of convergence and divergence taking second place to the unifying fact that both types of series go on and on. This was behind the rather surprising results of questions 13 and 14, noted in Chapter Six (pp.245-246), where the series 0.1 + 0.01 + ... did not necessarily produce an answer but 1/9 could be defined as this series. The fact that many subjects noted the convergence of the series but gave this second place to its dynamic nature (it was the same in principle as 1+1+...) was amply recorded in the protocols (pp.239-244). Dynamic interpretations of series may, moreover, lead to physical and temporal factors coming into mathematical arguments for subjects are use to calculations taking time and 0.1 + 0.01 + ... would then, clearly, need an infinite amount of time. These aspects were considered in thesis 5.iii.

Finally we note that dynamic considerations may lead to sequences such as 1,1,1,... etc. being seen as improper because they don't move (see protocols, pp.217-218).

**10) SUBJECTS' CONCEPTIONS OF LIMITS AND INFINITY ARE CONTRADICTORY AND LABILE.**

i) Subjects' conceptions of limits and infinity are contradictory in that subjects are drawn to two opposing views, e.g. : infinity is the largest number but you can't have a largest number; the limit of a sequence is the final number in the list but there is no final number; there are more natural than even numbers but there are the same (infinite) number of each.

ii) Subjects' responses are often not stated with great confidence and may be easily changed by context, reasoning and suggestion.

Of course subjects' conceptions in many areas of mathematics may be contradictory and labile. We have not made a comparative study and thus are not in a position to compare the quality of conceptions here with that in other areas of mathematics.

## 10.i) Contradictory.

As we have seen in thesis 6.i (concerning real numbers), open contradictions are held by subjects with respect to $0.\dot{3}$, 1/3, $0.\dot{9}$ and 1. Holding logical contradictions is not, however, peculiar to infinite phenomena. Wason and Johnson-Laird (1972) consider contradiction in abstract and practical tests. They show (p.189) that subjects will latch onto any convenient proposition to get themselves out of holding a contradiction. They also show that subjects do not necessarily see a contradiction as a contradiction:

> Wait a minute .. You have proved one thing and then you have proved the other .. There is only one card which needs to be turned over to prove the statement exactly. (ibid., p.195)

As our subjects are similar in age and ability to theirs it seems likely that these remarks apply to our study; although we have not made a thorough examination of this phenomenon.

The occurrence and perception of formal contradictions was not, however, our main interest here. Rather we were interested in the inherent contradictions to be located in our subjects' conceptions, the pull of both thesis and antithesis in subjects' thoughts. The following illustrate subjects' recognition of two poles:

Q1 Is there a largest number ?

GAM2 If you count infinity as the largest number you could say that's the largest number, but if it's an actual number there's always one more and 100 times more.

Q2 Is there a smallest number ?

JCN2 You could say the smallest number is $1/\infty$ . You can't get any smaller than $1/\infty$ . But infinity is going on forever so it just carries on getting smaller.

Q5 Is $\infty+1 > \infty$ ?

MWN Well say, for argument, it exists as an enormous number, right. When you think of it as a number then, if you add one to that enormous number then it complicates things because you're beginning to think of it as something greater than that.

Q8 Does 2+s=2 ? 2xs=s ?

GAM1 ('No', 'No') I just thought of it as a number.
GAM2 ('Yes', 'Yes') I just thought s is something infinitely small and so there's nothing smaller.

Q13 Can you add 0.1+0.01+... and get an answer ?

DLM (subject said 'No' to this but thought this equals 1/9)
There is no number between $0.\dot{1}$ and 1/9, so they must be the same.

INT So couldn't I get an answer to 0.1+0.01+... ?

SUB Ha, if you wrote that out I suppose... well when it says *get an answer*... Oh, I suppose - yeh. I was thinking...when it's written out as $0.\dot{1}$ , then I can think of it as 1/9. But when you just keep adding it seems different in my head.

Q17-Q21 Cardinality questions.

PBM1 (Q17) I thought you couldn't really compare it. I thought there'd be the same number because it goes on indefinitely, but as that one's higher I suppose that one will have more numbers.

Questions on the four phrases.

PBM2 Its limit is its final point that it will get to. So I think its limit is $0.\dot{9}$. And then again there the limit is 1, but it won't actually get to 1, so you can't have 1 as its limit.

Quine (1966) documents three types of contradictions: truth telling paradoxes, that resolve themselves on futher explanation or tell of an impossibility; fallacies, such as misproofs of 2=1; and antimonies, that produce a contradiction by accepted modes of reasoning such as *This sentence is false*.

Our subjects enter into all three types of contradiction: truth telling such as GAM2 above convincing himself by his contradiction that there is no largest number; fallacies such as LSM1 who, on seeing the contradiction with $0.\dot{3}$, 1/3, $0.\dot{9}$ and 1, maintains that *you've got* *to add something to* $0.\dot{9}$ *to get 1* ; antimonies such as subjects noting (p.196) that although in finite terms there are more numbers between 0 and 10 than there are between 0 and 1, that this type of reasoning fails when we consider the infinite case.

The kinds of contradiction experienced by subjects is thus many sided. Interestingly they mirror the types of contradiction that caused the three crises in mathematics outlined in Chapter One. Let it be clear, however, that these are psychological conflicts. Whatever *line* one takes on the foundation of mathematics, the mathematical concept of infinity is consistent. Our thesis is that the psychological intuition of infinity is inherently contradictory in that it *pulls* thought to two opposing views.

An alternative thesis is that teaching creates concepts of infinity that are opposed to our primary intuitions. We have seen instances of this (p.224). If this is true then this must be pre A-level schooling for there is very little difference between the N and M groups in the questionnaire responses. Moreover, the contradictions noted above occur in both groups and in both the September and May interviews.

Pre A-level schooling rarely, if ever, makes reference to infinity, although approximate limits of sequences and fractional representation of recurring decimals occurs in SMP O-level courses. It may well be, however, that topics not immediately concerned with limits or infinity contribute in forming concepts that affect subjects' conceptions of limits and infinity. This is an area worthy of further study. Not having investigated this we cannot draw conclusions. Whatever the actuality it is reasonable to claim that these influences form part of the normal mathematical framework of the mathematically above average adolescent that is our typical subject and that the inherent contradictory nature of infinity is a reality for him/her.

Another possible explanation of the contradictory nature of infinite phenomena is that subjects may interpret statements dealing with limits and infinity as theoretical or ideal statements and as practical or approximate statements. Again there is some truth in this in that some subjects did appear to refer to both interpretations (pp.190-193). However, the overall approach of subjects in the interviews was to support their responses with theoretical, ideal arguments. This applied to both groups.

Closely associated with this dichotomy is the argument of Fisch^bein et al. (1979), that subjects are drawn, more or less equally after the age of 12, to finitist and infinitist positions. This we have not witnessed. Our protocols show clearly that subjects talk about limits and infinity in an *infinitist* manner. We discussed this disparity with Fischbein et al. on p.65 and concluded that mathematically able subjects are drawn to infinitist positions. It is interesting, in this light, that the only interviewee we felt we could label finitist (GHN, see p.190) was the only subject, of all those taking the questionnaires, who failed her O-level the first time she took it (we included her in the interview precisely for this reason).

## 10.ii) Labile

Although the lack of confidence shown by subjects in their responses to questions is affected by their ignorance of logical limit notions, their uncertainty is also a result of the contradictory nature of limits and infinity. If we were carrying out an investigation in an area where subjects' basic conceptions or intuitions were not formed then we would expect responses to be easily changed simply because they did not know (or have a belief about) any correct response. We have seen, however, that our subjects do have a concept of infinity. Thus, although they are ignorant of formal notions, they are not ignorant of their own conceptions. We posit, then, that subjects' conceptions are labile because of the contradictory nature of infinity, not because of subjects' uncertainty. However, there may be other factors at work complementing this.

Context can affect subjects' responses, as we have seen in our ninth thesis. Although numeric / geometric and counting / measuring contexts are largely determined by the question itself (rather than the subjects' interpretation) we have seen examples where a numeric evaluation is forced on a question having a geometric context (p.311). We have not, however, seen such an effect in counting / measuring contexts.

Whether a question has a static or a dynamic context is much more subjective. Subjects may turn from viewing an infinitesimal or recurring decimal as having existence in a dynamic ontological framework, to viewing these as having no real existence in a static ontological framework (p.190).

Another context, one interconnected with subjects' reasoning, is whether the question or problem is presented in an open or closed situation. This is discussed in Appendix C with regard to Path Dependent Logic. A simple *Yes/No* question is an example of a closed context. A *Why ?* question is an example of an open context. The former, by virtue of having given *possible ends, may require less* reasoning. The latter usually requires more discriminating reasons. This recourse to deeper rationales may make the problem appear *afresh* to the subject. This, however, is a conjecture in the field of information processing that our study is not designed to evaluate. As we have seen, however, it is consistent with the protocols (p.276).

More central to the lability of subjects' conceptions of limits and infinity, but difficult to determine directly, is the inherent contradictory nature of these conceptions. This global feature of the psychological concept of infinity is evident in the protocols, as we have seen above (thesis 10.i). The following is completely typical:

GAM1 (Subject responded 'Yes' to *Is* $\infty+1 > \infty$ *?*).

SUB I just thought of it as a number.

INT If that infinity meant your idea of infinity would you still put 'Yes' ?

SUB Well, if infinity is so large, then if you add one to it it's still large, isn't it ? So I wouldn't. I'd put 'No'.

The inherent lability of the psychological concept of infinity causes each question on the questionnaire to be more independent from the other questions than it may otherwise be (in other areas of mathematics). Each question indexes particular contexts, aspects of infinity and rationales. Consider, for example Q22 (which compared the real intervals (0,1) and (0,10) ) and Q23 (which compared the coordinate points in a square and enclosing circle). Both questions may evoke a measuring context leading to a response 'More' (in the superset). Now the majority responding 'More' in one of these questions responded consistently in both: 26, of the 38 for each question, in the N group and 30, of the 37 in Q22 and 46 in Q23, in the M group. Although some of the non consistent responses may be put down to misunderstanding it would not be consistent with the general intelligent responses of the subjects to put them all down to this. Rather, we hold, the non consistent responses are due to the lability of subjects' conceptions. We are in complete agreement with Fisch~bein et al. (1979) here:

> The natural intuition of infinity is highly labile, depending on conjectural and contextual influences. The lability of the intuition of infinity can be explained if admitting its intrinsic contradictory nature as a psychological reality.

**11) THE EFFECT OF TEACHING.**

**The first year of an A-level mathematics course which includes an introduction to all the basic ideas of calculus does not, generally, affect subjects' conceptions of limits, infinity or real numbers.**

As a general summary the above is true. Indeed, the absence of a difference between groups is striking. Although the whole body of the data obtained in this study (the questionnaire results and the protocol data) bears witness to this, it is, nevertheless, remarkable that nine months of intensive exposure to mathematics involving infinite sequences and series, asymptotes, derivatives (real and vector valued), integrals, Newton-Raphson and Taylor approximations seems to have such a negligible effect on subjects' basic conceptions of limits and infinity. Why is this ?

Test design may have been ineffective in omitting areas of concern or in its handling of areas covered. This is always a possibility in research. We detailed our attempts at objectivity and comprehensiveness in Chapter Five.

Another possibility is that subjects may have reached their final level of comprehension of infinite phenomena by 16 years of age. To strict Piagetians, subjects such as ours are certainly well established in the period of formal operational by this age. However, as we argued at the beginning of Chapter Three (p.30), few mature students consistently function at this level and they generally function at a concrete level when the subject matter is not familiar. In terms of cognitive functioning, then, there would appear to be room for improved reasoning in this area. Moreover, if our subjects do represent a final stage then mature mathematical thought with regards to infinite processes would be impossible, as it patently is not.

Another possibility to explain the apparent negligible effect of teaching is that concepts from pre A-level mathematics may remain dominant in the first year of an A-level course. This is very plausible, mathematical concepts mature slowly and the finite schemes of elementary mathematics have had many years of successful application. We have seen, moreover, that subjects' conceptions of recurring decimals are naive. This may reflect teacher explanations presented earlier in the subjects' schooling.

The areas of pre A-level mathematics and subjects' general experiences that affect their concepts of infinity have not been dealt with in this study but are worthy of research.

In retrospect it would have been useful to test Upper Sixth students. The timimg of the questionnaires was the main obstacle here as it would have been improper to disrupt their exam preparation. However, it was felt that the main impact of infinite methods would be made in the first year of a calculus course when the calculus and related concepts would be most thoroughly discussed (later work being largely *methods*).

Another possible explanation of the negligible effect of an A-level mathematics course is that the teaching of calculus and related topics at this level avoids forcing subjects to question their basic conceptions of limits and infinity. We believe there is a great deal of truth in this. In all the protocols only four subjects expressed sentiments that they had considered the question in the course of their mathematical studies (JHM2, VMM2 and DLM, all with regard to 1/0, and SWM2, with regard to $0.\dot{9}$). True the questions were designed to avoid A-level type questions so that the control group could answer them too, but a great number of the questions were very closely tied to A-level topics. It was particularly surprising that in all the questions regarding the four phrases no subject mentioned class discussion of the concepts. While not conclusive this evidence cetainly points to an avoidance of *difficult* ideas in A-level classes.

At a subjective level the author has found that despite his own interest and knowledge in this area, discussion on these difficult *limit* ideas can be very confusing for the students and that a vague understanding coupled with faith in the teachers' results causes them less anxiety. This has been attested in conversation with many graduate mathematics teachers. Regardless of the truth of this belief it would be of great value to the mathematical community to develop an instructional *Introduction to Limits and Infinity* that presented paradigms that could be successfully comprehended by A-level students.

**Although the generalization, that teaching does not affect these conceptions, is true as a generalization, there are, however, some areas where a difference between groups was apparent.**

In the MAIN sample the N group were less certain in that 12.0% of their responses were blank or '?', whereas only 5.4% of the M groups' responses were blank or '?'. This is highly significant, $X_1^2=141$, $P<0.001$. This posed some problems in analysing responses. We discussed this on p.108 and p.153. The main explanation we believe is behind this is that students who take A-level mathematics are more confident about their abilities and thus of their responses.

The alternatives are that ability or gender plays a role. These are possibilities but ones we are unable to ascertain. Referring to Table 5.1 on p.104 it is clear that there were proportionally more males than females and more A grades than B or C grades in the M group and the exact opposite applied in the N group. There were indications that females were more likely to give a blank or '?' response (10.8% of female responses were such as opposed to 6.2% of male responses). Similarly there were indications that subjects with grades B or C were more likely to give a blank or '?' response (10.6% of their responses were such as opposed to 5.0% of responses from those with grade A). There is, however, insufficient data to draw definite conclusions.

Questions 3 and 4, which asked what 1/0 and $1/(1-0.\dot{9})$ were, displayed a very significant difference between groups. We reproduce the results below.

TABLE 9.5

| | Q3 | | Q4 | |
|---|---|---|---|---|
| | N | M | N | M |
| Infinite | 38 | 76 | 25 | 75 |
| Indeterminate | 4 | 16 | 1 | 5 |
| Formally wrong | 55 | 6 | 53 | 16 |
| ? | 3 | 2 | 21 | 4 |

We ascribe this difference to the more recently improved and practised calculating abilities amongst the mathematics group. Moreover the M group will have me^t this very question in the context of asymptotes of graphs. Indeed, as we have seen above, this question accounted for three of the four *I've met this before* responses witnessed in the protocols. The questions are also very difficult from the subjects' viewpoint. There is a certain *naturalness* in translating it as 0/1 or as *0 into 1 won't go, therefore it is nothing.*

The other area where a marked difference existed was in Q26 where subjects were asked to complete sentences using the phrases *tends to, limit* and *converges.* Table Q26 shows that the M group mainly got the answer right and the N group gave an unprecedented number of blank or '?' responses, as well as many responses that are not strictly correct (though not always *wrong,* such as *1+h tends to decrease* or *1,1/2,1/4,... converges to infinity*). As we pointed out in Chapter Five, this set of questions was included because it represented a kind of question that would be covered in an A-level course and could be given to the control group. The great number of blank or '?'

responses, however, indicates that it is on the borderline of what can intelligibly be given to the control group (in the technical language of test/item design it has a high discriminatory power). The results show that A-level mathematics courses do help subjects answer standard questions on limits and infinity. The alternative is to accept that these results indicate that A-level mathematics affects subjects' basic conceptions of limits and infinity. If we accept this then we must question the validity of the rest of the results for these show that the groups differ very little. We have argued in Chapter Five, however, that the questionnaire does test subjects' conceptions of limits and infinity. We conclude, then, that this set of questions merely shows that subjects can improve their answers to certain questions on limits by following an A-level mathematics course.

There were other areas where small differences were noted. We have commented in detail on all these cases in Chapter Six and do not repeat that detail here. As can be seen, however, in almost every case the difference is between a very strong response from one group as opposed to only a moderately strong, but similar, response from the other group.

What is remarkable about all the above questions is that subjects can improve their number of correct answers in certain areas without substantially altering their basic concepts of limits and infinity and, indeed, of R as well (as we have seen with subjects' conceptions of $0.\dot{9}$ and $\sum_{i=1}^{\infty} 10^{-i}$ , $\sqrt{2}$, etc.)

# CHAPTER TEN

# CONCLUSIONS

Chapter One presented the aims of this investigation. In this last chapter we summarize our findings, examine the extent to which the initial aims have been fulfilled, consider the implications for teaching that arise from the study and consider questions for further research.

Our aim was to investigate *mathematically* **able adolescents'** conceptions of the basic notions behind the **Calculus: infinity** (including the infinitely large, the infinitely small **and infinite** aggregates); limits (of sequences, series and functions); **and real** numbers. The effect of a one year calculus course on these conceptions was also to be examined.

Infinity

We agree with Fischbein et al. (1979, p.30) that generally mature adolescents do have a concept of infinity that is inherently contradictory and labile (our theses 1 and 10).

We agree with Tall (1980d) that the dominant concept of infinity is of a dynamic *process* (theses 2, 4, 6.i and 9.iii). This is compatible with the traditional view of *potential infinity* as opposed to *actual infinity* in that infinity is *going on and on.*

The infinitely large exists as a indeterminate form, a vague generalization of a large number. Infinity as this large number can be conceived but is not granted real cognitive existence (thesis 3).

The infinitely small, infinitesimals, exist only in the process of a sequence of numbers, or a function decreasing and as such are dynamic entities. Static infinitesimals may be seen as useful fictions and approximations with very small numbers may be seen as generally acceptable (thesis 4). Further research is needed to verify the extent to which subjects hold these views.

Infinite aggregates can be considered as single sets by the subjects but comparison of cardinalities is pre-Cantorian (theses

1.ii, 3.iv and 9.ii). Subjects derive their responses here from the reasoning schemes *infinity as a process* and the generic law (thesis 8). Although cardinal arithmetic is not, at A-level, relevant to subjects' understanding of concepts behind the calculus, the results do help us form a global picture of subjects conceptions of infinity.

**Limits**

Subjects generally note the convergence or divergence of infinite sequences but generic limit concepts dominate their conceptions (thesis 5). There appears to be a slight shift away from the generic limit concept and to the mathematician's limit concept amongst some A-level mathematicians, but evidence here is far from conclusive.

Several factors, irrelevant to the mature mathematician, can affect adolescents' responses to questions on limits.

i) Sequences that do not conform to a monotone convergent paradigm (thesis 5.i):

Oscillations, e.g. 1, 0, 0.1, 0, 0.01, ...

Constant values, e.g. 1, 1, 1, ...

ii) The everyday connotations of the phrases *tends to, limit, converges* and *approaches* (thesis 8). *Tends to* and *approaches* are largely seen as general tendencies, *converges* is largely seen as inapplicable to sequences and *limit* is often seen as a very strong condition (an ultimate boundary which may or may not be the generic limit).

iii) The context of the question. In thesis 9 we considered three dichotomies: numeric/geometric, counting/measuring and static/dynamic.

Only the first and last are relevant to subjects' conceptions of limits. Our data was inconclusive with regard to the numeric/geometric division (though we suspect that the existence of limits is more readily accepted in geometric contexts). With regard to the static/dynamic division we argued that subjects' limit conceptions were primarily dynamic in nature and that teachers must be very careful in this area or their static conceptions will not make sense to students with dynamic limit concepts.

Our data and analysis is consistent with the work of Cornu in that his *obstacles* in the limit notion (p.43) were noted here. In particular subjects find limit ideas *surrounded in mystery*, angst over whether the limit is obtained or not and experience difficulty in the passage from the finite to the infinite.

Series present many cognitive problems for subjects. Their convergence or divergence is often not seen or not regarded as important compared to the physical and/or temporal problems of infinite summation.

**Real numbers**

Subjects do not appreciate the concept of the completeness of the real number system. The completeness of the reals gives equal ontological status to finite decimals, recurring decimals and infinite non recurring decimals. We have assumed that the concept of infinite recurring decimals is a cognitively easier concept than the concept of infinite non recurring decimals. Examining subjects' conceptions of recurring decimals we have seen that they are perceived as *incomplete* or *improper* in the sense that they do not represent uniquely defined numbers. They exist as dynamic entities, perpetually in a state of *becoming*.

A much observed phenomenon is that students see $0.\dot{9}$ as strictly less than 1. We agree with Schwarzenberger and Tall (1978, p.46) that this is due to their lack of understanding of the limit concept. More precisely it is a result of their generic limit concepts and their dynamic view of infinity. $0.\dot{9}$ is the generic limit of 0.9, 0.99, ... Each term is less than 1, thus, by the generic law, $0.\dot{9}$ is less than 1. Moreover, not only is $0.\dot{9}$ qualitatively different from 1 because $0.\dot{9}$ is the generic limit and 1 is not, but $0.\dot{9}$ is qualitatively different, and thus not equal to, 1 because $0.\dot{9}$ is a dynamic entity whereas 1 is a fixed static entity.

## IMPLICATIONS FOR TEACHING

This study was primarily cognitive in nature and was not intended to evaluate the affect of instructional variables in the learning of elementary calculus. Nevertheless several factors emerge that are relevant to the teaching of calculus at A-level.

Infinity is a vague and contradictory notion to Sixth Formers. As teachers we must take great care to explain, without using the word *infinity*, what we mean when we say something *approaches* infinity or use $\int_{-\infty}^{\infty} f(x)dx$, etc.

Similarly Sixth Formers do not grant infinitesimals real cognitive existence, although they may exist as dynamic entities. Nevertheless, students may view them as useful fictions and may accept mathematical arguments that ignore very small numbers.

This is not to say that infinity and infinitesimals should be banned from A-level mathematics. Mathematics teachers must enlarge and refine students' ideas and correct false intuitive conceptions. The fact that students have concepts that do not fit neatly into standard approaches to the calculus should not put us off confronting and amending these conceptions.

Paraphrasing Bruner's celebrated hypothesis that *any subject can be taught effectively in some intellectually honest form to any child at any stage of development* we believe that any legitimate method of teaching calculus to able adolescents can be made effective and intellectually honest. What must be avoided is mixing ideas from different approaches to the calculus, where the terms have different meanings. This can cause both mathematical and cognitive conflict.

Our work has shown that the dominant limit concept amongst A-level mathematicians is the generic limit concept. If A-level mathematics is seen as an end in itself, as opposed to as a prelude to higher mathematics, then it is not clear, with respect to differentiation, that this concept needs to be overcome. In teaching differentiation at an elementary level generic limit ideas can be accommodated into the common dynamic limit method (where 2x+h gets closer to 2x as h gets closer to 0).

Two areas where post generic limit methods are required for a resonable understanding are real numbers and series. As we have seen these concepts are largely misunderstood by students. Not until generic limit ideas are overcome and replaced by mathematicians' concepts can recurring decimals (and then irrational numbers) be seen as *proper* points on the real number line. This requires A-level syllabi allocating space for the study of R . If such a study is to take place in the classroom, then questions on the structure of R must be included in examinations, otherwise teachers may choose to ignore this.

Work on infinite series *per se* is largely lacking in the SMP texts (see appendix B). Finite series are covered in detail and infinite series arise in integration but work on infinite series alone occupies very little space. This may account for subjects' lack of understanding. We suggest more emphasis be placed on infinite series and that teachers take pains to bring out the limit of partial sums. The irrelevance, from the pure mathematical point of view, of physical and temporal problems of infinite summation should be made very clear.

Given that teachers should devote more time to developing an intelligent understanding of limits in A-level pupils, how is this to be done ? Schwarzenberger and Tall (1978) advocate a conflict free approach where instruction commences from the students' concepts and builds up ideas gradually so as to avoid conscious or subconscious conflict. Cornu (1983) suggests that teachers lead classes to explore and discuss their own ideas on being introduced to limit ideas. Orton (1983a, 1985) suggests that early numeric (calculator based) and graphical work may help build greater understanding of limits. Robert thinks it best if students are made aware of their erroneous ideas sometime *after* a course.

We feel there are insights, truths and problems in all these suggestions. The approach of Schwarzenberger and Tall would, correctly presented, result in a spiral calculus curriculum, continually building on established concepts. But is a *conflict free* approach totally desirable ? Certainly there is a sense in which we want to introduce conflict so that students can meet obstacles under the guidance of a teacher. This is why Cornu suggests that teachers lead classes to explore their ideas at the time of learning limit concepts. Robert's proposal of introducing discussion after a course, however, is highly dubious. Certainly such an approach courts the danger of allowing immature paradigms to become fixed in subjects' conceptions. Orton's suggestion is implemented by many teachers. By *playing* with numbers and gradients subjects are able to form concepts with concrete objects before going onto abstract arguments. The SMP approach, at both O and A-level, adopts this approach. There are dangers, however, for the work may reinforce generic limit concepts. For example consider using a calculator to calculate the sequence converging to $f'(1)$, from below, where $f(x)=x^2$. We would obtain the sequence 1.9, 1.99, 1.999, ... The limit could be taken as $1.\dot{9}$ rather than 2.

Our results on subjects' interpretations of the phrases *tends to, approaches, converges, limit* and *gets to*, moreover, show that teachers must take great care that their use of language is understood by the students.

Finally, we feel that nonstandard analysis is unsuitable for school calculus because: i) to understand the structure of $R^*$ one must understand the structure of R, and, as we have seen, students *do not* understand the completeness of R; ii) in general students' interpretations of infinitesimals are either as improper or as dynamic entities. Neither of these interpretations is particularly suitable as a starting point for nonstandard analysis.

## SUGGESTIONS FOR FURTHER RESEARCH

A number of questions arose during the study that we were unable to fully evaluate with our data. These questions are summarized below.

i) **Are generic limit concepts less strong in geometric contexts ?** As we saw in thesis 9.i, this claim is suggested by the data but requires further investigation. Interviews would be necessary to study this for students may believe that approximations are more valid in geometric contexts and questionnaires would have difficulty monitoring this.

ii) **Can students distinguish between convergent and divergent sequences ?** All the evidence points to the answer 'Yes'. There was a fault in our questionnaire design, however: we did not consider monotone divergent sequences. Further questionnaire items designed to examine this question would be useful.

iii) **Is there actually a small shift away from generic limit concepts in some A-level mathematicians ?** A long term, large scale investigation into ideas related to those presented here could profitably examine this. It would involve following subjects through the Sixth Form and into university mathematics.

iv) **Do students fail to note convergence/divergence with respect to infinite series or are these properties simply seen as less important than other properties ?** We have seen that subjects fail to isolate convergent series as essentially similar from sets of series with a variety of mathematical forms (pp.126-128). Moreover we have seen that convergent and divergent series are seen as essentially similar in that they go on adding forever (pp.239-241). There remains a possibility, however, that subjects do recognize convergence and divergence but do not see it as the most important property of series. Interviews would be necessary to examine this as all possible properties seen by subjects would have to be taken into account.

A possible factor influencing our subjects' responses to the series questions is the fact (mentioned earlier) that work on infinite series *per se* is largely lacking in SMP texts. It would thus be useful to do a comparative study, across several boards and syllabi, of A-level students' understanding of series.

v) **Do subjects operate in a mixture of *finite, finite decimal representation,* and *generic limit* universes ?** We hypothesized on p.301 that they do. An evaluation of this would be a useful complement to the present study.

vi) The present study examined the immediate responses of subjects. A further study that examines **students' *information processing powers*** in this area is needed. Although this study examined students' understanding of concepts behind the calculus it nevertheless leads onto questions concerning the calculus itself.

vii) **What are the criteria for evaluating the success of a period of calculus instruction ?** Do they include a real understanding of limits, infinity and real numbers or are tools for further application all that is necessary ?

viii) Related to this is the question **Does school calculus place too much emphasis on applications and not enough on understanding ?** By *application* we mean use within pure mathematics (e.g. the quotient rule for differentiation, integration by parts, etc.). We may, however, ask the same question with *application* meaning *real world problem solving*.

ix) Is there a need to approach the study of calculus earlier in the mathematics curriculum ? With the arrival of GCSE there is a distinct move away from pre A-level calculus instruction. There may, however, be a place for a greater study of the limit ideas of calculus without the development of the methods of differentiation and integration.

Apart from questions directly relevant to the object of this thesis there arise more general cognitive questions.

x) There is the thesis, considered on p.307, that able subjects' reasoning rises to the level required by the mathematics (that more abstract reasoning will be employed in areas of mathematics beyond simple empirical verification or finite computation).

xi) The concept of path dependent logic arose during this study. The study was an unsuitable vehicle to examine it. A full independent study to examine it as a model of mathematical thought would be very interesting.

xii) We have referred several times to subjects' ontological commitments. Ontology is a field largely avoided by cognitive scientists. This is a pity for it holds many interesting questions: What mathematical entities are granted existence by students ? What is their working meaning of *existence* ? If a mathematical entity is not granted cognitive existence (i.e. it is a fiction to the student), then can students still do meaningful mathematics with them ?

xiii) Does gender affect subjects' conceptions of infinity, limits and real numbers ? We initially examined our data in an ad hoc manner in the early stages and did not note a difference. It is, nevertheless, possible that a difference in geometric interpretations exist for there is evidence that girls' geometric sense differs from boys' (see Shuard's Appendix 2, B18, in Cockcroft, 1981).

xiv) **Are subjects' conceptions of infinity, limits and real numbers affected by their ability ?** We did not investigate this largely because we were not satisfied that subjects' O-level grades (which is all we had to work on) are fine enough measures to base an assessment on. The evaluation of this question would undoubtably be of worth but would have to develop tools to measure the relative abilities of A-level students.

Apart from proposals that can be examined under more or less clinical conditions there are more pratical projects that could be usefully evaluated.

xv) We believe it would benefit students if calculus books sharply differentiated between what goes on in the physical world and in the mathematical world. Thus a text with one page dealing (say) with differentiation as an approximation and a facing page dealing with it as a pure mathematical construct, may help resolve conflicts. It would be interesting to produce and evaluate such an approach.

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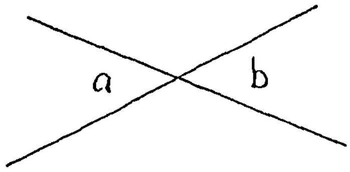
# APPENDIX A

Questionnaire 1

Questionnaire 2

NAME........................

1) The two lines in the figure on the right intersect. Does $a = b$ ?




YES / think so / ?unsure? / think not / NO

---

2) Is there an answer to every mathematical question ?

YES / think so / ?unsure? / think not / NO

---

3) Is there a largest number ? YES / think so / ?unsure? / think not / NO

---

4) Is there a smallest number ? YES / think so / ?unsure? / think not / NO

---

5) Write down a number between 2.105931 and 2.10604 ____________

---

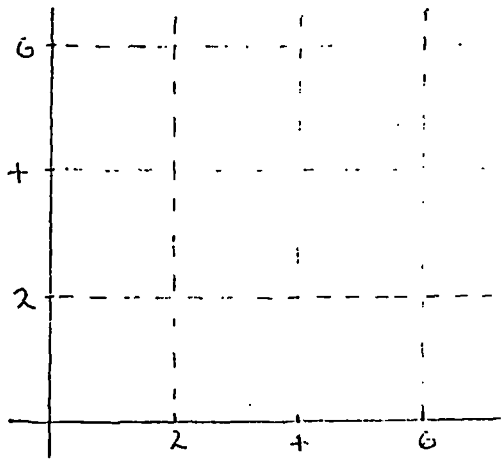
6) What is $\frac{1}{0.001}$ ? ____________

---

7) What is $\frac{1}{0}$ ? ____________

---

8) Sketch the curve $y = \frac{1}{x}$ , for $x > 0$, on the grid on the right.




---

9) 1,2,3,4,....  are there: i) more numbers in the first row
2,4,6,8,....  ii) more numbers in the second row
iii) same in both
iv) can't compare

---

10) Is $0.\dot{9} < 1$ ? YES / think so / ?unsure / think not / NO

11) Is there a number smaller than $1 - 0.\dot{9}$ ?

YES / think so / ?unsure? / think not / NO

---

12) Consider the whole numbers 1,2,3,4,5,.....

Can we think of these as a single set ? YES / think so / ?unsure? / think not / NO

Consider also all the decimal numbers between 0 and 1 .

Can we think of these as a single set? YES / think so / ?unsure? / think not / NO

Are there: i) more whole numbers ii) more decimal numbers
iii) same number of each iv) can't compare

---

13) What is $\frac{1}{1-0.99}$ ?

---

14) What is $\frac{1}{1-0.\dot{9}}$ ?

---

15) Consider all the decimal numbers between 0 and 1 and all the decimal numbers between 0 and 10 .

Are there: i) more between 0 and 1 ?
ii) more between 0 and 10 ?
iii) same number of each ?
iv) can't compare ?

---

16) Does $0.\dot{3} = \frac{1}{3}$ YES / think so / ?unsure? / think not / NO

---

17) What is $0.\dot{3} \times 2$ ?

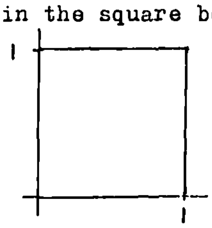
---

18) Does $0.\dot{3} \times 3 = 0.\dot{9}$ ? YES / think so / ?unsure? / think not / NO

19) Does $0.\dot{9}=1$ ?  YES / think so / ?unsure? / think not / NO

---

20) Consider all the decimal numbers between 0 and 1 and all the coordinate points in the square below. Are there:



i) more points than numbers ?
ii) more numbers than points ?
iii) same number of each ?
iv) can't compare ?

---

21) Consider the sequence 0.1, 0.01, 0.001, 0.0001, .....

Will it ever get to 0 ?  YES / think so / ?unsure? / think not / NO

---

22) Infinity, $\infty$, means different things to different people. Suppose, for the sake of argument, it exists as an enormous number. Then:

i) Is $\infty+1>\infty$ ?  YES / think so / ?unsure? / think not / NO

ii) Does $\frac{1}{\infty}=0$ ?  YES / think so / ?unsure? / think not / NO

---

23) Suppose, for the sake of argument, that there is a number smaller than any other number but not zero. Call it s. Then:

i) Does $2+s=2$ ?  YES / think so / ?unsure? / think not / NO

ii) Does $2\times s=s$ ?  YES / think so / ?unsure? / think not / NO

---

24) Can you add 1 + 1 + 1 + 1 + ..... and go on forever and get an answer ?

YES / think so / ?unsure? / think not / NO

---

25) Can you add 0.1+0.01+0.001+..... and go on forever and get an answer ?

YES / think so / ?unsure? / think not / NO

---

26) $\frac{1}{9}=0.\dot{1}$ . Can $\frac{1}{9}$ be defined as 0.1+0.01+0.001+...... ?

YES / think so / ?unsure? / think not / NO

27) Consider the sequence of functions in Fig.1.Is the limit of this sequence a perfectly straight line or is it everso slightly jagged ?

i) perfectly straight ......... YES / think so

ii) slightly jagged........ YES / think so

iii) don't know

---

28) Sentence question

---

29) Consider the sequence 0.9 , 0.99 , 0.999 , 0.9999 , ........

Which of the following sentences are true of this sequence ?

| | | |
|---|---|---|
| i) | It tends to $0.\dot{9}$ | YES / think so / ?unsure? / think not / NO |
| ii) | It tends to 1 | YES / think so / ?unsure? / think not / NO |
| iii) | It approaches $0.\dot{9}$ | YES / think so / ?unsure? / think not / NO |
| iv) | It approaches 1 | YES / think so / ?unsure? / think not / NO |
| v) | It converges to $0.\dot{9}$ | YES / think so / ?unsure? / think not / NO |
| vi) | It converges to 1 | YES / think so / ?unsure? / think not / NO |
| vii) | Its limit is $0.\dot{9}$ | YES / think so / ?unsure? / think not / NO |
| viii) | Its limit is 1 | YES / think so / ?unsure? / think not / NO |

---

The next four questions refer to the curves in Fig.2.Please put your answers (Y,y,?,n or N)in the table below.

30) Can we say "the curve TENDS TO 0" as x gets larger and larger ?

---

31) Can we say "the curve has 0 as a LIMIT" as x gets larger and larger ?

---

32) Can we say "the curve CONVERGES to 0" as x gets larger and larger ?

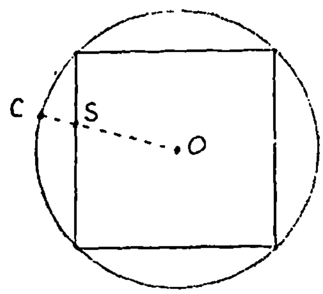
---

33) Can we say "the curve APPROACHES 0" as x gets larger and larger ?

---

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 30 | | | | | | |
| 31 | | | | | | |
| 32 | | | | | | |
| 33 | | | | | | |

34) To get to any point, say C, on the circumference of the circle I must first find a point on the perimeter of the square, call this point S, so that a line from the centre of the circle, O, through S passes through C.




i) Can I get to every point on the circumference this way ?

YES / think so / ?unsure? / think not / NO

ii) Suppose two points are very close on the square. Will the corresponding points on the circle be very close ?

YES / think so / ?unsure? / think not / NO / it depends

iii) Are there more coordinate points in the circle ?

YES / think so / ?unsure? / think not / NO

---

35) Complete the following sentences:

i) 1 + h tends to ________ as h tends to 0.

ii) The limit of $(2 + h)^2$ as h tends to 0 is ________

iii) 1 , $\frac{1}{2}$ , $\frac{1}{4}$ , $\frac{1}{8}$ , ....... converges to ________

---

36) For each of the sequences below say whether or not it has a limit. If it does and you know what the limit is, then write this underneath the sequence.

i) 1 , 0.1 , 0.01 , 0.001 ,..... YES / think so / ?unsure? / think not / NO

ii) 1 , 0 , 0.1 , 0 , 0.01 , 0... YES / think so / ?unsure? / think not / NO

iii) 1 , 0.1 , 1 , 0.01 , 1 , .... YES / think so / ?unsure? / think not / NO

iv) 1 , 1 , 1 , 1 , 1 , ......... YES / think so / ?unsure? / think not / NO

---

37) For each of the sequences below say whether of not it converges. If it does and you know what it converges to, then write this underneath the sequence.

i) 1 , 0.1 , 0.01 , 0.001 ,..... YES / think so / ?unsure? / think not / NO

ii) 1 , 0 , 0.1 , 0 , 0.01 , 0... YES / think so / ?unsure? / think not / NO

iii) 1 , 0.1 , 1 , 0.01 , 1 , .... YES / think so / ?unsure? / think not / NO

iv) 1 , 1 , 1 , 1 , 1 , .......... YES / think so / ?unsure? / think not / NO

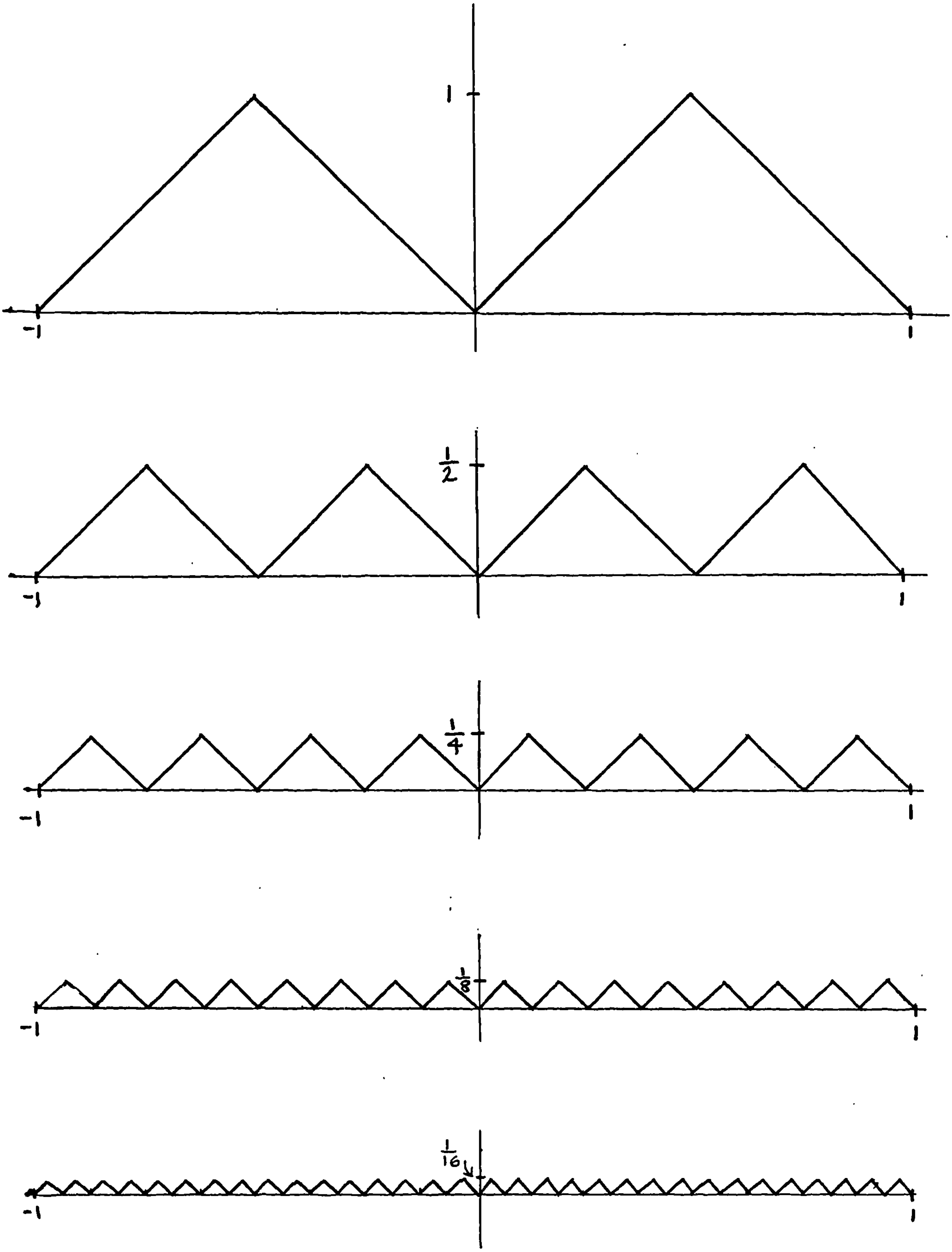



Fig. 1

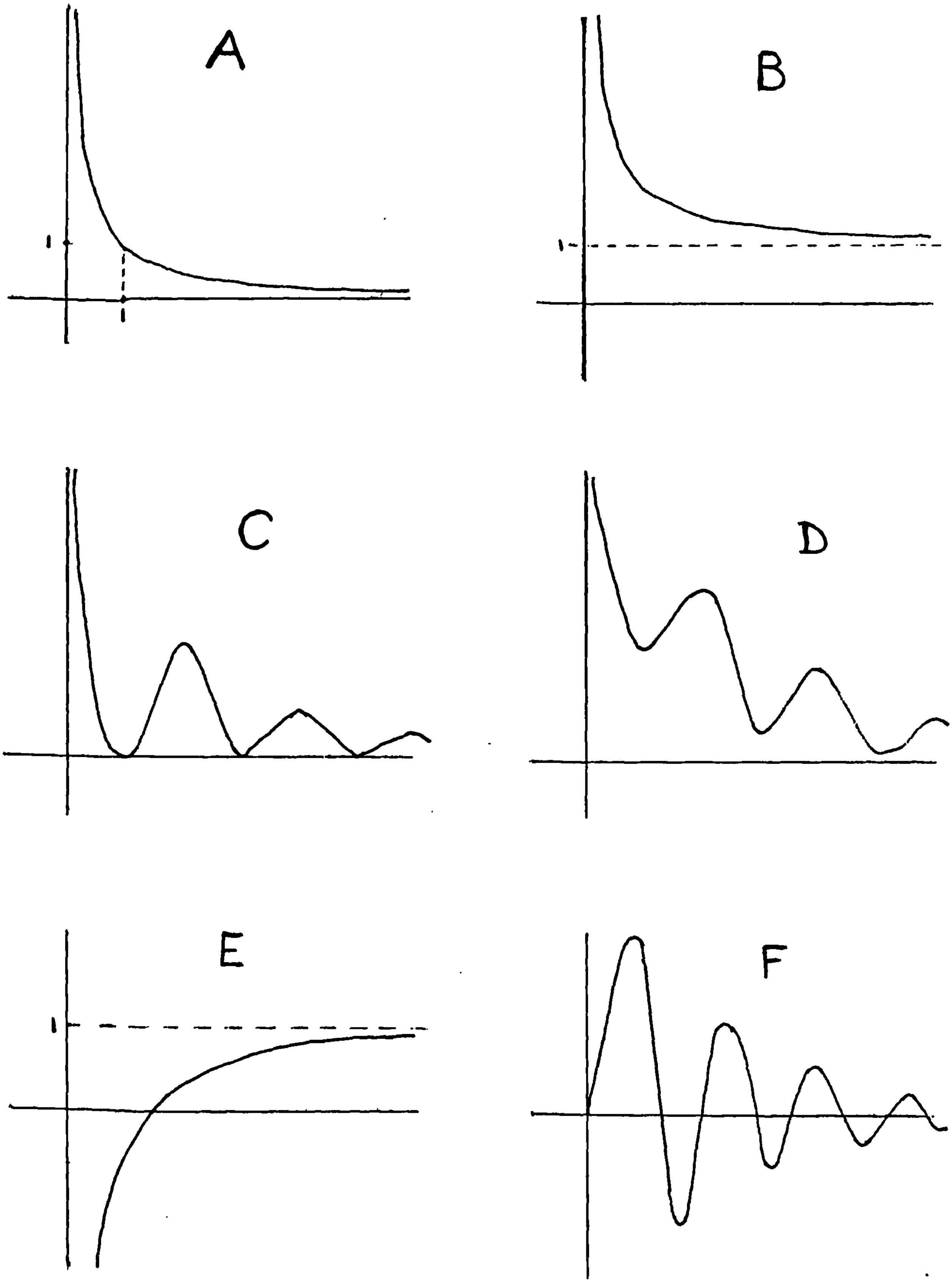



Fig.2

UNIVERSITY OF WARWICK

DEPARTMENT OF SCIENCE EDUCATION

The following is a questionnaire,not a test.The replies will be used to build up a picture of how sixth formers think of infinity.They will not be used to assess or grade you.

NAME ..........................................

YEAR L6 / U6 SEX male / female (please circle)

O level Mathematics grade A / B / C

Did you do Additional O level Mathematics in your fifth year ? Yes / No

If your last answer was 'Yes' ,then please give your grade ______________

Are you doing SMP A level Mathematics? Yes / No

If your last answer was 'Yes' ,then please answer the following two questions.

Are you doing Further Mathematics ? Yes / No

Are you doing S level Mathematics ? Yes / No

Most of the following questions seem a little strange to people.Don't let this put you off and don't worry about getting things wrong - we are interested in what you think,not in what you know.

It is tempting to put unsure,(?),to every question.Try and avoid this.Only use ? when you are really unsure.

We are interested in your immediate response.Don't spend too long on any question or look at you friend's answer.

Most of the questions have Y / ? / N after them (meaning yes,unsure,no). Answer by circling one of these.If you change your mind simply cross out your first answer and circle another one.

Is there a largest number ? Y / ? / N

---

Is there a smallest number ? Y / ? / N
greater than 0

---

What is $\frac{1}{0}$ ? __________

---

Consider the two sequences of numbers 1,2,3,4,.... and 2,4,6,8,....
(the dots indicate that the sequences carry on)
Are there
i) more numbers in the first row
ii) more numbers in the second row
iii) same in both
iv) can't compare

---

Is $0.\dot{9} < 1$ ? Y / ? / N

---

Consider the whole numbers 1,2,3,4,.....
Can we think of these as a single set Y / ? / N

Consider also all the decimal numbers between 0 and 1 .
Can we think of these as a single set Y / ? / N

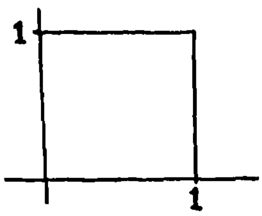
Are there
i) more whole numbers
ii) more decimal numbers
iii) same number of each
iv) can't compare

---

What is $\frac{1}{1-0.\dot{9}}$ ? __________

---

Consider all the decimal numbers between 0 and 1 and all the coordinate points in the square below. Are there
i) more points than numbers
ii) more numbers than points
iii) same number of each
iv) can't compare



Infinity , $\infty$, means different things to different people. Suppose, for the sake of argument, it exists as an enormous number. Then :

Is $\infty + 1 > \infty$ ? Y / ? / N

Does $\frac{1}{\infty} = 0$ ? Y / ? / N

Is this how you think of infinity ? Y / ? / N

---

Suppose, for the sake of argument, that there is a number smaller than any other number but bigger than zero. Call it s . Then :

Does $2 + s = 2$ ? Y / ? / N

Does $2 \times s = s$ ? Y / ? / N

Can you believe in such a number ? Y / ? / N

---

Can you add $1 + 1 + 1 + 1 + .....$ (the dots indicate continuation) and get an answer ? Y / ? / N

---

Can you add $0.1 + 0.01 + 0.001 + .....$ and get an answer ? Y / ? / N

---

Just as we often write $\frac{1}{3} = 0.\dot{3}$ , we can write $\frac{1}{9} = 0.\dot{1}$

Can $\frac{1}{9}$ be defined as $0.1 + 0.01 + 0.001 + .....$ ? Y / ? / N

---

Consider all the decimal numbers between 0 and 1 and all the decimal numbers between 0 and 10. Are there :
i) more between 0 and 1
ii) more between 0 and 10
iii) same number of each
iv) can't compare

---

Complete the following sentences :

$1 + h$ tends to ________ as h tends to 0 .

The limit of $(2 + h)^2$ as h tends to 0 is ________

$1 , \frac{1}{2} , \frac{1}{4} , \frac{1}{8} , .....$ converges to ________

The next set of questions refer to the four curves below and the two on the opposite page. For each curve you are asked four questions.

Can we say the curve { TENDS TO 0 / HAS 0 AS A LIMIT / CONVERGES TO 0 / APPROACHES 0 } as x gets larger and larger ?

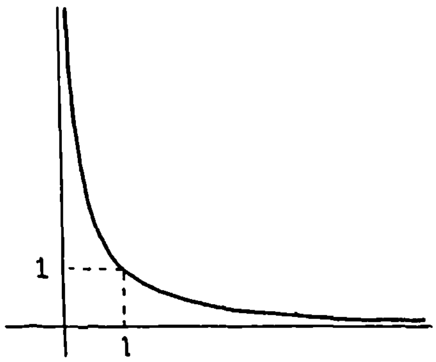
Don't spend too long on any question, just put your first reaction down.

tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

converges to 0 Y / ? / N

approaches 0 Y / ? / N

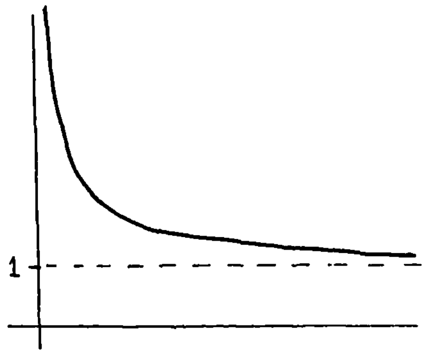



tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

converges to 0 Y / ? / N

approaches 0 Y / ? / N

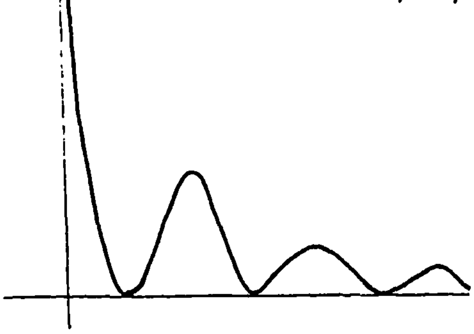



tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

converges to 0 Y / ? / N

approaches 0 Y / ? / N

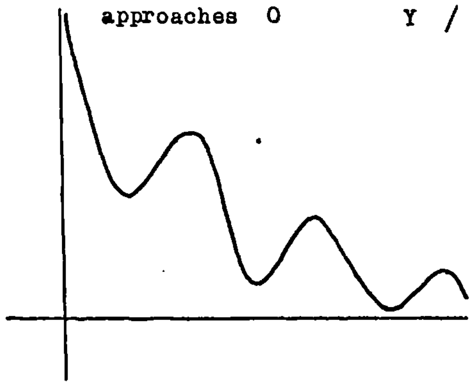


tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

converges to 0 Y / ? / N

approaches 0 Y / ? / N

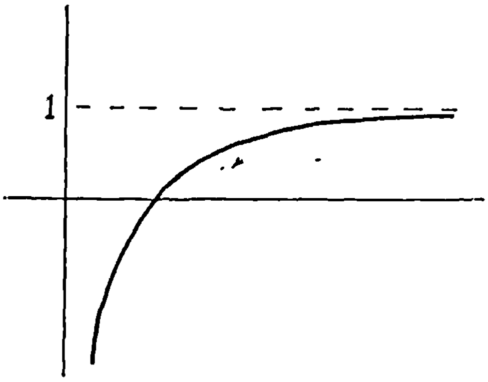


tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

converges to 0 Y / ? / N

approaches 0 Y / ? / N

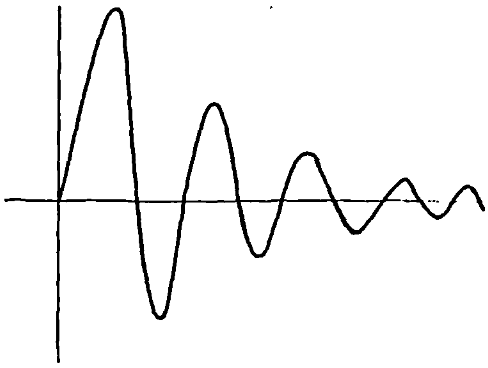



tends to 0 Y / ? / N

has 0 as a limit Y / ? / N

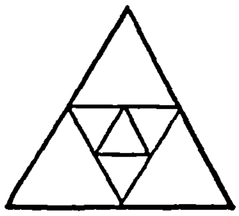
converges to 0 Y / ? / N

approaches 0 Y / ? / N



---

Consider the triangles below(one inside the other,inside the other,etc.)
We have only shown three but imagine the process continuing.



Is the limit i) a triangle

ii) don't know

iii) a point

Did you answer the last question theoretically or in terms of actually drawing the triangles ? theoretically / ? / in terms of drawing

---

Two of these 'sums' don't belong to the rest.Put the letters of the odd ones out in the boxes (the dots indicate that the process continues) .

A) $0.1 + 0.1 + 0.1 + 0.1 + \ldots\ldots$

B) $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$

C) $1 + 2 + 3 + 4 + \ldots\ldots$

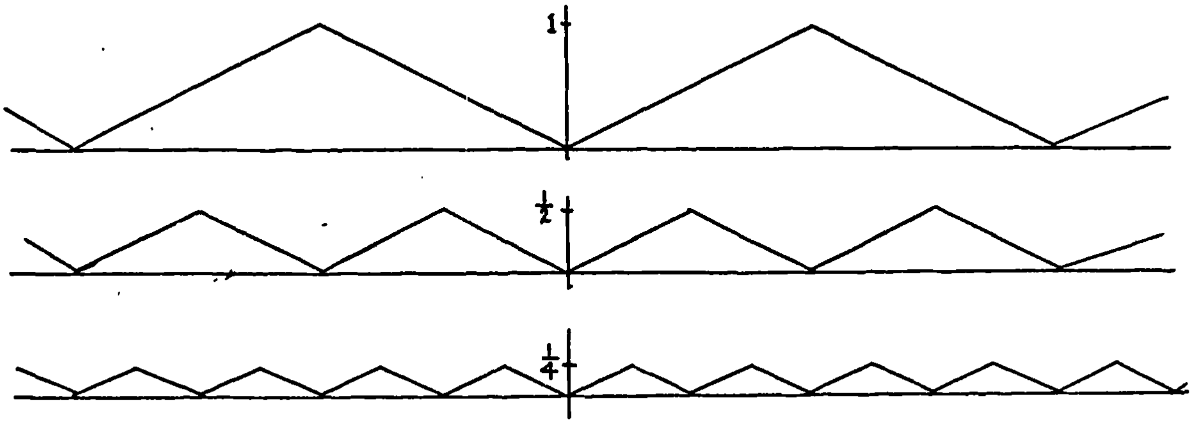
D) $0.1 + 0.01 + 0.001 + 0.001 + \ldots\ldots$

E) $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots\ldots$

| |
|---|
| |
| |

Consider the sequence of graphs below. We have only shown the first three but imagine the sequence continuing.

Is the limit of this sequence i) a perfectly straight line
or ii) everso slightly jagged ?




Did you answer the last question theoretically or in terms of actually drawing the graphs ? theoretically / ? / in terms of drawing

---

Consider the sequence 0.9 ,0.99 , 0.999 , 0.9999 , ......

Which of the following sentences are true of this sequence ?

| | | | |
|---|---|---|---|
| It tends to $0.\dot{9}$ | Y / ? / N | It tends to 1 | Y / ? / N |
| It approaches $0.\dot{9}$ | Y / ? / N | It approaches 1 | Y / ? / N |
| It converges to $0.\dot{9}$ | Y / ? / N | It converges to 1 | Y / ? / N |
| Its limit is $0.\dot{9}$ | Y / ? / N | Its limit is 1 | Y / ? / N |

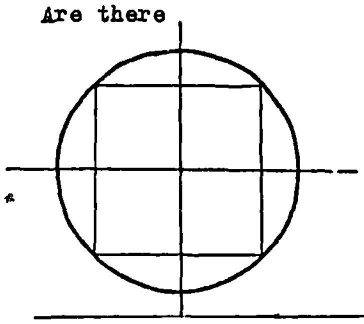
---

Use the five possible answers to each of the following to indicate whether you think the following are proper numbers, for example you may think $0.\dot{9}$ is a number but not be completely sure, then put 'think so'.

| | |
|---|---|
| $0.\dot{9}$ | yes/think so/ ? /think not/ no |
| $\infty$ | yes/think so/ ? /think not/ no |
| $\sqrt{2}$ | yes/think so/ ? /think not/ no |
| $1-0.\dot{9}$ | yes/think so/ ? /think not/ no |
| $\frac{1}{\infty}$ | yes/think so/ ? /think not/ no |
| $\sqrt{-1}$ | yes/think so/ ? /think not/ no |

Consider the circle and square below.

Are there



i) more coordinate points in the circle

ii) more coordinate points in the square

iii) same amount in each

iv) can't compare

---

For each of the sequences below say whether or not it has a limit.

| | |
|---|---|
| 1 , 0.1 , 0.01 , 0.001 , ...... | Y / ? / N |
| 1, 0 , 0.1 , 0 , 0.01 , 0 , ...... | Y / ? / N |
| 1 , 0.1 , 1 , 0.01 , 1 , ...... | Y / ? / N |
| 1 , 1 , 1 , 1 , ...... | Y / ? / N |

---

For each of the sequences below say whether or not it converges.

| | |
|---|---|
| 1 , 0.1 , 0.01 , 0.001 , ...... | Y / ? / N |
| 1, 0 , 0.1 , 0 , 0.01 , 0 , ...... | Y / ? / N |
| 1 , 0.1 , 1 , 0.01 , 1 , ...... | Y / ? / N |
| 1 , 1 , 1 , 1 , ...... | Y / ? / N |

---

We have used the phrases 'tends to ' , 'has a limit' , 'approaches' and 'converges' several times.Do you find any of these confusing in the way we have used them?If you do say which one(or which ones), and why you do.

---

---

---

(continue over the page if necessary)

END OF QUESTIONNAIRE

# APPENDIX B

## THE SCHOOL MATHEMATICS PROJECT (SMP) A-LEVEL SCHEME

SMP is a national project founded in 1961 with the aim of providing courses that reflect modern usages of mathematics. It provides books and support materials for primary and secondary mathematics. The SMP A-level scheme offers academic 16 to 18 year olds a mathematics course that integrates pure mathematics, mechanics and statistics.

The SMP A-level scheme generally places more emphasis on understanding than other A-level courses do. To do this it offers a slightly reduced calculus component, compared to other pure mathematics A-level courses (hyperbolic functions and curvature are omitted; formal limit methods, series, integration and differential equations are not taken as far as they are in some courses).

All subjects in the experimental group taking Questionnaire 1 were doing SMP A-level mathematics. Due to SMP's differences from other courses it was considered necessary to ensure that subjects taking Questionnaire 2 were also following the SMP scheme. We provide this appendix to give readers unfamiliar with SMP an indication of its nature.

The SMP A-level scheme usually runs for two years and uses three books (SMP, 1978). Subjects in the present study will have covered just under half of the course at the time they took the questionnaires. We thus deal only with the relevant (calculus or limit orientated) chapters that can reasonably be expected to have been covered in this time. We outline the various topics with brief notes as to their usual exposition. Teachers will, obviously, bring their own ideas into class expositions but we cannot account for these. For an analysis of problems inherent in the SMP approach to limits see Tall & Vinner, 1981.

**Chapter 1 Flow charts and sequences.**

An infinite loop is defined as one that will *cause the sequence to go on being printed out indefinitely* (p.4). Note that this is introducing infinity as a process. Infinite sequences are then studied but are simply called sequences. N is introduced (p.13) as an entity but there is no discussion as to whether infinite sets are legitimate. The idea of a sequence as a function with domain N is introduced. The behaviour of sequences for large values of n are studied. The word *limit* is introduced:

> In some sequences $u_n$ approaches some fixed number. In *1/2, 1/5, 1/8, 1/11,...* the limit is 0 and in $u_n = 6+1/n$ the limit is 6 ... For some sequences, $u_n$ tends to a limit as n increases without limit, but there are several alternatives. Look back at the sequences of Exercise D ... Which oscillate infinitely ? Which increase without limit ? (p.15)

It is left to the teacher to clarify the meanings of the words. Notice that three of the four phrases we have studied arise on this page.

**Chapter 3 Graphs**

Asymptotes and discontinuities are introduced. Infinity is not, initially, referred to; rather, the text says *the graph approaches the line y=x as x becomes larger and larger* (p.39) and *x=0 presents a problem since we cannot divide by 0, and we say 3/0 is undefined* (p.40). Standard notation using infinity comes later *We say that y tends to 1/2 as x gets larger and larger, written* $y \to 1/2$ *as* $x \to \infty$ (p.46). The approach is standard to many texts at this level.

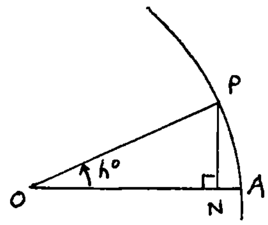
**Chapter 5 Derivatives**

The initial approach, after a set of exercises on rates of change, is via scale factors with parallel axes graphs illustrating the basic ideas. The chapter puts early emphasis on working out average scale factors but suddenly jumps to the local scale factor as the limit. Standard notations, gradients and maximum points all too quickly follow. The text concentrates on functional notation in its exposition but also uses Leibnizian notation.

The SMP books are supposedly self contained but it seems improbable that any student could gain an intelligent grounding of the basic ideas of differential calculus from this chapter.

**Chapter 6 Circular functions**

$\lim\limits_{b \to a} \dfrac{\sin b - \sin a}{b - a}$ is examined in some depth. Practical numeric and graphical investigations are carried out before the theoretical result is explained. The argument is suasive and may encourage students' dynamic conceptions.




From the diagram, it seems that as P approaches A and $h \rightarrow 0$, the lengths of PN and the arc PA become more and more nearly equal to each other so that $\lim_{h \to 0}$ (PN/PA) = 1 (p.122).

The importance of radian measure is then seen. The remainder of the chapter is given to deriving trigonometric formulae.

**Chapter 7 Kinematics**

The approach is based on vectors. The introduction emphasises practical examples and slowly builds up to the idea of instantaneous velocity and acceleration. Here, however, it curtails the discussion and simply utilizes the idea of a derivative to obtain the instantaneous results (in vector form). The chapter ends by considering angular velocity. Again, however, the limits are presented but not discussed in detail.

**Chapter 9 Sigma notation and series**

Series are introduced via flowcharts. **The text only considers finite series.** Infinite series are only introduced via exercises at the end of the chapter.

**Chapter 10 Area and Integration**

Approximate methods of obtaining the area enclosed by a curve are examined and used to give upper and lower bounds for the true area. The text emphasises the difference between the upper and lower bounds:

> What can you say about the limits of the sequences of upper and lower bounds ? (p.204)

This leads to the integral as the limiting sum (p.208).

The chapter, in the opinion of the author, is far superior to the introduction of integration in similar books. Nevertheless, nowhere is the nature of this limit explored in more than a cursory manner. Definite integrals, indefinite integrals and the Fundamental Theorem of Calculus follow.

**Chapter 13 The chain rule and integration by substitution**

The introduction is, as for the first chapter on differentiation, via parallel axes graphs. This has an advantage here over Cartesian graphs as the local scale factor (the derivative) of the composite function can be seen to be the product of the local scale factors of the component functions. Only after this has been established is the more usual Leibniz form introduced. Some attempt is made here to consider what is really happening:

> To find the value of the derived function, dy/dx, we have to consider the limit as $\delta x \to 0$; $\delta y/\delta x \to dy/dx$, $\delta y/\delta u \to dy/du$ and $\delta u/\delta x \to du/dx$ so it would be reasonable to conclude that $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
> To prove the result formally, which we shall not attempt to do here, we have to cope with the limit of a product and also the possibility that $\delta u$ might be zero even if $\delta x \neq 0$. (What difficulty would this create ?) (pp.305-306)

Integration by substitution is introduced via the transformation of areas. Although this is a distinct improvement on many other texts that regard it, more or less, as an algebraic trick, there is no discussion on limits. The result is then formalised:

> Suppose we are trying to find $y = \int f(x)\,dx$ where we can express $f(x)$ in the form $g(u) \times \frac{du}{dx}$ ($u$ being a function of $x$), then $y = \int g(u) \times \frac{du}{dx}\,dx$; so $\frac{dy}{dx} = g(u) \times \frac{du}{dx}$.
> But $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.
> Comparing these, we find $\frac{dy}{du} = g(u)$, so $y = \int g(u)\,du$.
> That is $\int f(x)\,dx = \int g(u)\frac{du}{dx}\,dx = \int g(u)\,du$. (p.317)

Note that limits are not explicitly considered. The example that follows is $\int \sin^4 x \cos^3 x\,dx$. The text states:

> Substitute $u=\sin x$, giving $\frac{du}{dx}=\cos x$. Replace $\cos x\,dx$ by $du$.

The text is presenting *functional* theory but *differential* practice. This is highly likely to cause cognitive difficulties for the students.

**Chapter 16 Product rules for differentiation and integration.**

**Chapter 20 Calculus techniques and applications**

Both of these chapter are *calculus techniques* chapters. They present nothing new by way of theory and are comparable, in presentation, to other texts at this level.

## Chapter 22 Local approximation

The Newton-Raphson and other methods of finding an approximate roots of an equation are introduced graphically. The emphasis is on getting an approximation to any desired accuracy.

Finite polynomial approximations are gradually developed. These lead into Taylor polynomials. Only nth order approximations are considered, i.e. a theory of infinite series is not developed. This, given the emphasis on finite series in Chapter 9, means that students following the SMP scheme may rarely meet an infinite series.

A general analysis of errors is not developed; rather, errors arising in particular cases are examined in the exercises.

# APPENDIX C

## PATH DEPENDENT LOGIC

The idea of a Path Dependent Logic (PDL) has been referred to several times in this thesis. The concept arose in the process of analysing the data. As we have mentioned (p.267), however, our data collection methods did not permit an evaluation of the notion. The theory presented here is thus a conjecture of which the basis in reality remains to be tested. We include it because we feel it to be an interesting theory but, because it is a conjecture, we include it as an appendix and not as part of the main text.

We present the main ideas in two sections. The first deals with the general idea of a PDL. The second considers how such a system might function in the domain of adolescents' conceptions of limits, infinity and real numbers.

### The general theory of a Path Dependent Logic

Consider the following arguments:

i) People should have equal rights, thus people should have equal property, thus people should not be allowed to own any property they like.

ii) People should have equal rights, thus people should have equal liberties, thus people should be allowed to own any property they like.

The first argument may be schematized as A, A —> B, B —> D therefore D; the second as A, A —> C, C —> D' therefore D' (D' meaning *not D*). We conclude that D and D'. This is not, of course, logic in the formal sense. We use the term *logic* in the sense of human rationalization - which is often inconsistent. Moreover, and this is the important feature of PDL, the same assumptions can lead to different conclusions depending on the progression from the assumptions. We give these ideas a syntactic form.

By the term *cognitive proposition* we mean not only the classical linguistic vehicles for expressing truth or falsity (propositions) but also the thoughts behind speech acts, be they instances of concepts or schemas, *2 < 5*, or isolated facts, *Jean hates curry*. For brevity we shall call these *propositions* hereafter. By a *rule of inference* we mean a transformation rule that maps from the set of propositions into the set of propositions. By a *context* we mean the sum of linguistic, social and mathematical conventions that give a proposition meaning. We may consider a context as a label on a proposition. Several contexts can be attached to the same proposition. For example *(0,10) has more numbers than (0,1)* may be viewed in a measuring context, in which case it appears to be true, or in an arithmetic context,in which case it appear that we can continue counting forever in both intervals.

A PDL consists of a set of propositions (not necessarily consistent) with contexts and a set of rules of inference (not necessarily logically valid). A *belief* is either a proposition or follows from propositions or previously established beliefs by application of a rule of inference. We shall refer to these as propositions hereafter except when we wish to call attention to beliefs proper. A *path* is an ordered chain of propositions and rules of inference.

As a simple example let proposition one (P1) be *there is no number less than 1 and greater than* $0.\dot{9}$, P2 be *$0.\dot{9}$ is a non terminating decimal*, P3 be *1 is a terminating decimal* and P4 be $0.\dot{9}=1$. Let rule of inference one (R1) be *if there is no number less than a and greater than b, then a=b* and R2 be *if a is terminating and b is non terminating then* $a \neq b$. Two paths in this PDL are:

P1, R1 therefore P4 and P2 , P3 , R2 therefore not P4.

Neither mortals nor logicians like accepting A and A'. Logicians rule it inconsistent and dismiss it. Humans, however, on realizing they have claimed or implied A and A' either decide to accept one of them or claim indeterminancy in the form of *I don't know*. The word *realize* is important here for it is not unusual for an individual to hold two contrary beliefs but not realize it because they are not evoked at the same time.

The definition of a PDL can be amended to take account of these points. First we differentiate between three kinds of propositions - *latent, evoked* and *compared*. Latent propositions are propositions that lie dormant in an individual. They may be used but do not enter a path unless they are evoked. Evoked propositions may be used in paths (they may be seen as irrelevant and not used, however). Compared propositions must be evoked but are also such that they are compared to other propositions with this status. Conflict occurs only when two compared propositions imply A and A'.

The second amendment is the *degree of certainty* to which a proposition is held by an individual. As a convention let this be a number between 0 and 1, where 0 is complete disbelief and 1 is complete certainty. Context will generally be a factor in determining the certainty of an evoked proposition (e.g. the proposition considered at the bottom of p.372 will have a high degree of certainty in a measuring context but a low degree of certainty in an arithmetic context). Let us denote the certainty of a proposition P in a context C by c(P,C) or just c(P) when the context is clear or unimportant. The degree of certainty with which propositions P and P' are held certainly affects whether one accepts P, P' or claims indeterminancy. A possible candidate for a *decision rule* for resolving conflict is :

accept P' ............... if c(P) is much less than c(P')

accept P ............... if c(P') is much less than c(P)

accept indeterminancy.... otherwise

Apart from a decision rule it seems reasonable that there should also be a rule to the effect that *the degree of certainty of a proposition should not be greater than the maximum degree of certainty of any of the propositions determining the proposition*. Whether one wants to refine the quantitative relationship here depends on the application to which the PDL is put. For our introductory purposes the above is adequate. A computer model of some behaviour would require further refinements.

The above definition of a PDL is our starting point. It may be necessary to amend it by constructing a theory of subpaths where at any stage in a path a new set of conditions can come in and replace, partially or wholly, existing propositions and beliefs. It would appear that there are links here with Fuzzy Sets and Many Valued Logics but our interests are mainly syntactic while these variants of standard logic are semantic deviations. It is the paths that characterize a PDL. The idea of a Path Dependent Logic has been suggested in relation to Catastrophe Theory models of mental activity (Tall, 1977). This exposition, however, makes no use of this. We proceed to consider these notions in relation to the present thesis.

**PDL applied to adolescents' conceptions of limits and infinity.**

The model below is intended to explain both trends and diffusion in responses and from this locate a structure for adolescents' conceptions of limits and infinity. As has been mentioned

> The lability of the intuition of infinity can be explained by admitting its intrinsic contradictory nature as a psychological reality. (Fischbein et al., 1979)

The model is intended to go beyond this and detail how the intrinsic factors come into play and interact. Unfortunately the evaluation of this intention is beyond the scope of the present work. Nevertheless, it is of interest to see how we could proceed.

The limitations of explanatory models in cognitive science and the extent to which they model reality must always be kept in mind. Moreover, care must be taken in making inferences about cognitive structure from subjects responses (more so when using both questionnaire and interview data collection methods since the questionnaire tends to ask closed questions, e.g. *Is there a largest number ?*, while interviews constantly ask the open and ubiquitous *Why ?*). Nevertheless, such models are necessary if we are to analyse, and not merely report on, the data.

We differentiate between general and specific forms of knowledge. General forms we call *schemas*, specific forms we call *principles*.

By a *schema* we mean a cognitive structure which permits concepts to be understood, new and related concepts to be assimilated and which allows for adaption (accommodation) to new situations. A basic cognitive schema in mathematics is that of the number system (with the operations +, -, x, ÷). As a child develops repeated addition leads to the multiplication tables which in turn lead to long multiplication. This is assimilation. When fractions are introduced the schema must adapt to cope with the new situation. This is accommodation. We use the term *schema* in a somewhat looser sense to cover concepts such as one-to-one association, the principles of logic (valid or not) and in general, mental constructs which are not specific facts or beliefs.

By a *principle* we mean any proposition accepted (on authority or by rationalization) by a subject in evaluating a proposition. At one extreme a principle may be *Mr X says 1/0 is undefined*. At the other extreme a principle can be part of a schema. *Atomic weights have integer values* may be a critical factor used by a child in evaluating a proposition be it although it is part of the child's schema of atomic structure. Principles may be implicit in the cognitive structure without holding in the mathematical structure. For example, $n < n^2$ is held, erroneously, by many subjects, as universally true.

We can now state the assumptions of our model. We pose them in terms of questions and answers because this is all we have to work on.

1) Questions are question pairs consisting of question plus context.

2) Answers are path-dependent deductions based on cognitive propositions and rules of inference belonging to a subjects' set of schemas and principles.

3) Answers are answer pairs consisting of answer plus certainty with both parts dependent on question and context. High certainty answers are stable, others are liable to change.

We now move on to apply these ideas to an analysis of adolescents' conceptions of infinity.

## CONTEXTS , SCHEMAS AND PRINCIPLES

relevant in analysing subjects' concepts of infinity.

### Contexts

1) Geometric, Arithmetic and Measuring These are some of the contexts of elementery mathematics and are concerned, respectively, with spatial figures, counting (and the fundamental operations of arithmetic) and with the comparison of continuous quantities.

2) Universe of Discourse The context here is supplied by the subject and is genuine or Real depending on whether the question is

interpreted in the genuine world where 1/3=0.333333, to all intents and purposes, or interpreted in the Real world of pure mathematics where 1/3 = 0.333333. The universe of discourse may change from question to question and both worlds may be evoked simultaneously causing conflict, uncertainty and confusion of meaning.

3) Language Not a context itself but a giver of cues and a pointer to contexts. We single out three aspects:

i) Connotations of the words *tends to, approaches, converges* and *limit*.

ii) Cues or pointers given by words or phrases, e.g. *Will 0.1, 0.01, ... ever get to 0 ?*. This is very different to *Does it approach 0 ?*. Of course it will never get there (*get* implying finite attainment) unless one uses an approximation in the genuine world.

iii) Open and closed questions. The basic distinction here is between the questionnaire, where most of the questions were multiple choice or simple answers, and the interviews, where most of the questions were of the form *Why ?*. In terms of paths the former, by virtue of having given possible ends to the paths, require shorter paths. We call this a context for path reason in that a more discriminating choice of propositions and rules of inference must be made with open questions and this can make the question appear afresh in the subjects' mind.

4) Static and Dynamic This is a very subjective distinction and depends on how the subject interprets the question. If a process is evoked, the end result of which is the answer, then the context giving rise to this interpretation is dynamic. For example *What is* $1/(1-0.\dot{9})$ *?*

A response from a static context would be *$1-0.\dot{9}$ is infinitesimally small, the reciprocal is thus infinitely large.* A response from a dynamic context would be *$1/0.1 = 10$, $1/0.01 = 100$, ..., the answer becomes infinitely large.*

**Schemas and principles**

We attach a certainty (high, medium and low - H, M and L) to each stated principle. This is hypothetical and comes from close familiarity with the protocol data. This is also *averaged out* over the subjects and will actually vary from subject to subject.

1) The Piagetian schemas of conservation, seriation, classification, number and space. At a higher cognitive level are the schemas of proportionality and function. The latter do not have a strong cognitive effect in determining principles used in evaluating propositions encountered in this study. The fact that the more elementary schemas have long been established in subjects' cognitive framework gives a very high certainty to cognitive propositions derived from them. Important principles deriving from these schemas are: i) 1-1 association (M). This needs to be qualified. Certainly acceptance of 1-1 association is high but use of it as a principle in evaluating propositions is not high; ii) infinite repetition of arithmetic operations, this includes infinite subdivision. (H in arithmetic contexts, M in geometric contexts where physical and finitist interpretations can come in). Physical and temporal interpretations may arise, however, so that infinite repetition of arithmetic operations is not seen as giving a final answer;

iii) number properties, in particular ordering ($a+1>a$, $n^2>n$), place value (giving rise to $0.\dot{9}<1$), existence of different types of numbers, especially that of fractions and terminating decimals versus non terminating decimals. These all have high degrees of certainty attached to them.

2) Concepts of infinity. The fact that we are attempting to determine these should not obscure the fact that, whatever they are, their existence determines answers that we use in determining them. Principles isolated from the protocol data are: i) infinity as a generalization of a large number and infinitesimals as generalizations of small numbers. This is accepted with medium certainty and presents itself in arithmetic contexts; ii) *infinity as a process* is present in all contexts and has a high degree of certainty attached to it. It can lead to subjects viewing two infinite processes as incomparable, because of their indefinite nature. This is held with medium certainty. We hypothethize that incomparability is strengthened in the A-level mathematics group but we are unable to state a cause; iii) *infinity as a process* can also lead subjects to the idea that there is only one infinity. This is held with medium certainty; iv) The generic law is a high certainty schema when it is applied. It appears particularly strong in arithmetic and measuring contexts; v) generic limit concepts have a high certainty and have widespread use.

3) Reasoning schemas. Short deductive paths (syllogistic or propositional) are, as noted, used by subjects. These include invalid and indeterminate inferences (all with medium certainty). Other schemes

are reductio ad adsurdum (L), inductive generalizations (L) and, most commonly, simple instances of principles. There appear again *infinity as a process* and the generic law (both of which are held with high certainty when applied).

4) Taught concepts. Reception learnt concepts, concept definitions and concept images. Worthy of particular note in our investigations are the following: i) 1/0 is undefined (M); ii) 1/0 is infinity (M); iii) $1/3=0.\dot{3}$ (H); iv) taught limit results (L, applicable to the mathematics group alone); v) $0.\dot{9}=1$ (L applicable to the mathematics group alone).

**How we might proceed**

Let us consider, for example, the cardinality questions (questions 20 to 23). The important propositions here would appear to be:

P1 There is only one infinity (M).

P2 Each set contains an infinite number of elements (H).

P3 We can't compare infinities (M).

P4 This set is a subset of that set (H).

P5 There are the same number in each.

P6 There are more in this one.

P7 We can't compare them.

All questions are open to the inferences:

1) P2,P1 —> P5 With medium certainty.

2) P2,P3 —> P7 With medium certainty.

The generic law is applicable in the arithmetic context of Q20 and the measuring contexts of Q22 and Q23, thus the following inference is open in these questions:

3) P2,P4 —> P6 With high certainty for those who hold the generic law.

Subjects would be interviewed to determine the degree of certainty with which they held the propositions (and to determine other propositions they might hold) and the rationales by which they determined their replies. Interviews would probe propositions held, degrees of certainty held, rules of inferences used and paths followed.