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From PA(X) to RPAM(X)

Steen Markvorsen

Dedicated to Professor Vagn Lundsgaard Hansen
on the occasion of his 70*th birthday*

Abstract. How can we use the well-established Public Awareness of some phenomenon X, i.e. PA(X), to Raise the Public Awareness of the Mathematics of – or within – this X, i.e. RPAM(X)? There are several examples illustrating particular assets for mathematics in this way within such phenomena X. Here we will discuss only one phenomenon, which, however, contains a particularly dramatic momentum for arousing awareness among all of us, namely X = wildfires. The mathematics of this and of similar phenomena range – among several other topics – from elementary K–12 studies of ellipses to deep research questions in Finsler geometry. Moreover, in this context RPAM(X) may even save lives!

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What is Public Awareness?

Public awareness in the large (or in the small) is usually (but not always!) concerned with and centered around *concrete* phenomena which for a variety of reasons attract the attention of many people (or just a few) for a shorter or a longer period of time – mostly shorter. This is definitely not a definition: it is only a rough attempt to apply the two descriptors *concrete* and *time limited* to what might be called public awareness. What we might call mathematics, on the other hand, is mostly *abstract* and mostly *timeless*. This apparent dichotomy is, however, just apparent. In fact there is ample room for mergers, as witnessed by the following quotations:

> In the teaching of mathematics, and when explaining the essence of mathematics to the public, it is important to get the abstract structures in mathematics linked to concrete manifestations of mathematical relations in the outside

world. Maybe the impression can then be avoided that abstraction in mathematics is falsely identified with pure mathematics, and concretization in mathematics just as falsely with applied mathematics. *Vagn Lundsgaard Hansen* [1]

I find it difficult to convince students – who are often attracted into mathematics for the same abstract beauty that brought me here – of the value of the messy, concrete, and specific point of view of possibility and example. In my opinion, more mathematicians stifle for lack of breadth than are mortally stabbed by the opposing sword of rigor. *Karen Uhlenbeck* [2]

Fig. 1 Forest firefighting

Guided by the spirit of these parallel quotations we will thence concentrate and focus upon only one single very specific, very concrete, and very messy example from real life, a phenomenon which, nevertheless, is known to have moved everybody into alert mode since – and probably even long before – the Mesolithic era (see Fig. 1).

The Phenomenon of Wildfires

A wildfire, also known as a forest fire, vegetation fire, grass fire, brush fire, or bush fire (in Australia), is an uncontrolled fire often occurring in wild land areas, but which can also consume houses or agricultural resources. Common causes include lightning, human carelessness and arson. One main component of Carboniferous north hemisphere coal is charcoal left over by forest fires. The earliest known evidence of a wildfire dates back to Late Devonian period (about 365 million years ago). [3]

Wildfires are frequent and extremely threatening phenomena that we must try to understand from first principles. How do we most effectively prevent them from happening? How do they evolve once they have started, how do we most effectively escape from them, or conversely – the firefighters' quest – how do we most effectively fight them?

> A working knowledge of the effects of wind and other weather elements on fire behaviour, supported by accurate fire intelligence, is vital for good suppression planning. Without good fire behaviour information firefighters are unable to:
> - determine the number of firefighters and level of equipment necessary;
> - identify the location of suitable areas for backburning; and
> - ensure that the general public is informed about the precise fire situation. [4]

The existence of a well-informed public as well as professional up-to-date awareness of the general behaviour of wildfires is a necessary prerequisite for tackling these tasks.

Mathematical Horizons

Clearly we have PA(X) en masse for this wildfire X. How do we then generate momentum into RPAM(X), which is our main concern here?

How do we engage the public, how do we engage students, teachers, and researchers through K–12, high school, and university, to look for, to appreciate, and to apply known mathematics and search for new mathematics and thereby contribute to the solution of such tasks?

The following is but a brief account of the paper [5, pp. 50–61], which aims to show by example that this can be done. This example has recently appeared in a selection of similar examples in a book that is freely distributed on the web; it has also been distributed as an ordinary hardback book to the schools in Denmark, see Fig. 2 and [5]. In the book you can find exciting mathematical unfoldings (with exercises) of similar phenomena under headlines such as: "Mathematics through the millennia"; "Mathematics and evolution"; "Fire!"; "How a vending machine actually works"; "Wavelets"; "Secret codes made public"; "Math in medicine"; "Tour de France mathematics"; "Women and mathematics"; "Error correcting codes"; "Beer and flat screens"; "Mathematics in the computer and vice versa"; "The science of the better"; "Artificial intelligence"; "Mathematical modeling of climate and energy"; "The Mars mission"; "The mathematics of shape."

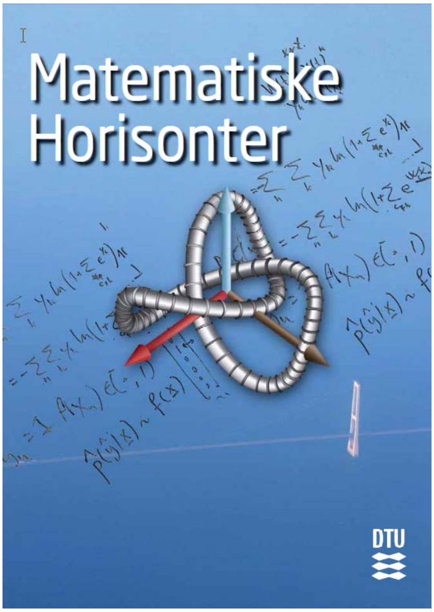

Fig. 2 Mathematical Horizons, 2009. Freely available via [5]

Some Details from the Wildfire Case

The simplest possible two-dimensional model of a fire front propagating in a perfectly homogeneous domain and in a wind that is directed along the y-axis is usually modeled as a parametrized time foliation through ellipses:

$$\begin{aligned} x(t, \phi) &= at\cos(\phi) \\ y(t, \phi) &= bt\sin(\phi) + ct\,, \end{aligned} \tag{1}$$

where a, $b \geq a$, and $c \leq b$ are constants depending on the fuel material and the speed of the wind. When $b = a$ and $c = 0$ this model gives elementary circular propagation with constant radial speed a from the initial center point.

In the general case of elliptic propagation we may define three values for the fire-front propagation speed, see Figs. 3 and 4:

$$\begin{aligned} v &= b + c & &\text{(Downwind front speed)} \\ u &= a & &\text{(Flank front speed)} \\ w &= b - c & &\text{(Upwind front speed)}\,. \end{aligned} \tag{2}$$

Note that the fire will also propagate upwind when $b - c > 0$.

This elliptic model is in actual use in Canada:

> The Canadian Forest Fire Behaviour Prediction System (CFF-BPS) assumes elliptical growth and has documented values

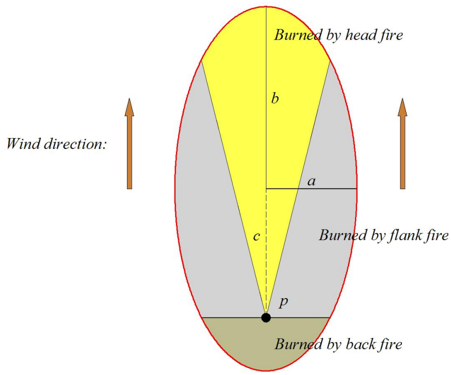

Fig. 3 Elliptic fire zone

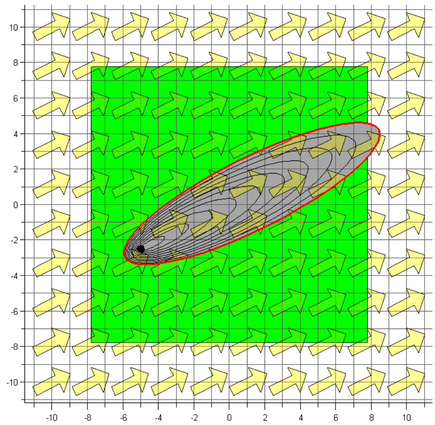

Fig. 4 Elliptic foliation of a homogeneous fire zone with a constant wind from the south-west

of u, v, and w for a very large set of constant parameters affecting a fire. It has also been observed that, within certain limits, the ratio a/b is a function of wind speed only; this is also an assumption of the CFFBPS. [6]

There are so-called pocket cards for firefighters, which recommend that the fire front should be attacked on the flanks. However, when the fuel density is not homogeneous, or when the topography is not perfectly flat, or when the basic model cannot be assumed elliptic but is some other oval-shaped generator as in Figs. 6 or 7, then the flank attack strategy may not be optimal.

The more advanced mathematics needed to see and understand this is concerned with geodesic sprays in Riemannian geometries (and Finslerian geometries when the wind is blowing). This will be discussed and exemplified in some detail in the next section.

In Fig. 5 we indicate how a fire front (without wind) may attack the fuel domain – and also the firefighters – from more than one side when propagating – like a pincer movement. This occurs precisely when the geodesic spray from the initial point of ignition creates so-called cut points in the domain. The set of these points are indicated by yellow dots in Fig. 5. The blue color in Fig. 5 indicates the moorish area through which the fire does not burn easily and where it therefore progresses only slowly.

The Cramer's Creek accident is but one such dramatic case featuring the formation of dangerous cut points. In [7] it is described in detail how fire unexpectedly approached from both sides of a ridge between two valleys and eventually killed two firefighters who were trapped by such a pincer movement.

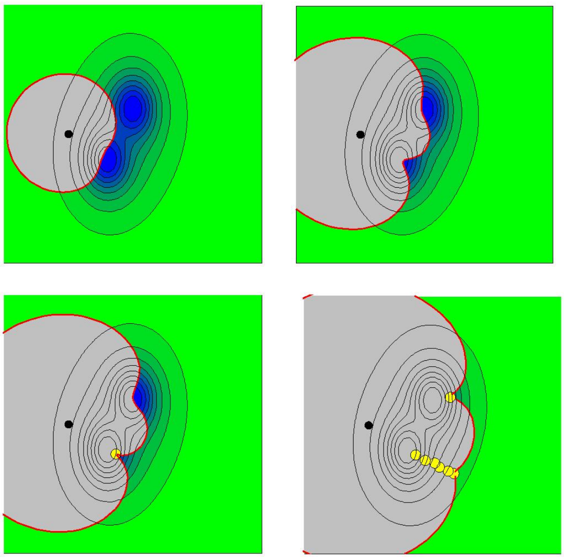

Fig. 5 A fire front developing pincer movements without wind but with varying fuel density

Finsler Geometric Analysis and Modeling

In this and the following sections we discuss some of the tools and concepts from Finsler geometry, which have only been alluded to above.

The possibilities of studying and applying asymmetric length functionals had been suggested by Riemann in his famous and foundational *Probevorlesung* in 1854 [8], but they were first developed in detail by Finsler in his *Inauguraldissertation* in 1918 [9]. Within the last 10 years the methods and results of global Finsler geometric analysis have experienced a renaissance – not least inspired by the seminal works of e.g. Chern and his collaborators and students. See for example the survey paper by Chern [10] and the works [11, 12, 13].

Like every Tour de France racing cyclist we all know that it is much harder to cycle uphill or against the wind than it is to freewheel downhill or with the wind pushing comfortably on your back – although the (classical Euclidean) *length* of the road of course is the same, whether we measure it in one direction or the other.

In a similar way (but note the important up-down reversal) a fire front will move much faster uphill(!) or with the wind than it moves downhill or against the wind.

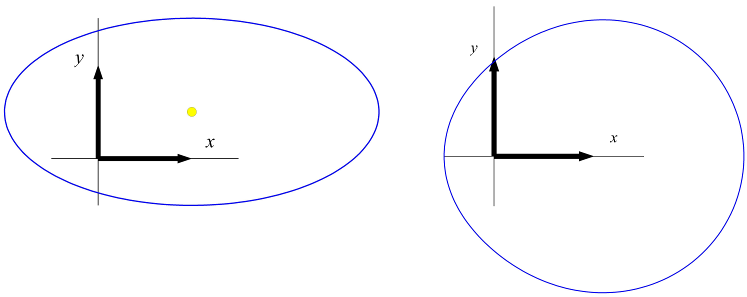

Fig. 6 Two indicatrices consisting of (the endpoints of) F-unit vectors. Every vector from the origin to the oval is in each case of F-length 1

A Finsler geometric model of a forest fire has this asymmetry built directly into the so-called indicatrix field $\mathcal{I}_q$ of unit vectors at each point q. Two such possible indicatrices are shown in Fig. 6, and indeed the oval shown in Fig. 7 is yet another possible indicatrix. Note thatin Fig. 7 the origin is very close to the indicatrix at the left. This indicates that in this model the upwind speed of the fire front is very small. See also [3] and Figs. 6 and 8.

In all cases the value of the Finsler length F_q is defined to be 1 for all the vectors connecting the origin to the points on the indicatrix oval:

$$\mathcal{I}_q = \{u \,|\, F_q(u) = 1\}\,. \tag{3}$$

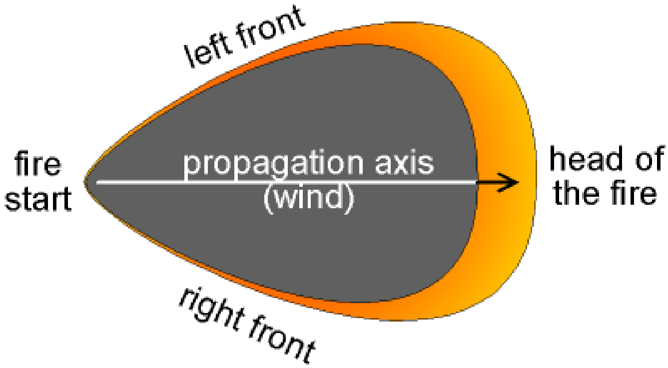

Fig. 7 An alternative oval propagation generator (indicatrix) model, see [3]

Note that in some directions the F_q-unit length is much larger (by ordinary Euclidean standards) than in the opposite direction. So if we measure the cost of transport – or the cost of propagation – by F_q-standards it may be much cheaper to go a long (Euclidean) distance in one direction than the same (Euclidean distance) in the opposite direction.

As illustrated in Figs. 6, 7, and 8 the origin (of the local vectors) need not be at the center of the respective indicatrix ovals (in fact the ovals need not even have well-defined centers). The shape and size of each oval as well as the position of the center inside it can be chosen to depend on the wind, on the topographical slope, and on the quality of the fuel at the point in question.

The Finsler Length Functional

Euclidean geometry is obtained in the special case where all indicatrices are identical circles with their centers at the origin. Riemannian geometry is obtained when all indicatrices are circles of possibly varying sizes but again with their centers at the origin. Riemannian geometry thus corresponds to the no-wind and no-slope situations because the winds and topographical slopes essentially shift the positions of the indicatrix centers and break the otherwise centered elliptic symmetry of the indicatrices. The possibility of varying the *radii* of the circles, however, corresponds to varying fuel conditions in the area.

In Fig. 8 small circles model the wet moorland shown in blue and the larger circles model the more homogeneous forest-like area, shown in green.

To the left in Fig. 8: The field is Riemannian in the sense that all indicatrices are circles. Note, however, that the circles are smaller in the blue area, so that it takes effort (long time for the fire) to go straight through the moorland. To the right in Fig. 8 is shown a genuine Finslerian field of indicatrices, which consists of wind-shifted ellipses. Note that the shifted ellipses are again smaller in the blue moorland area.

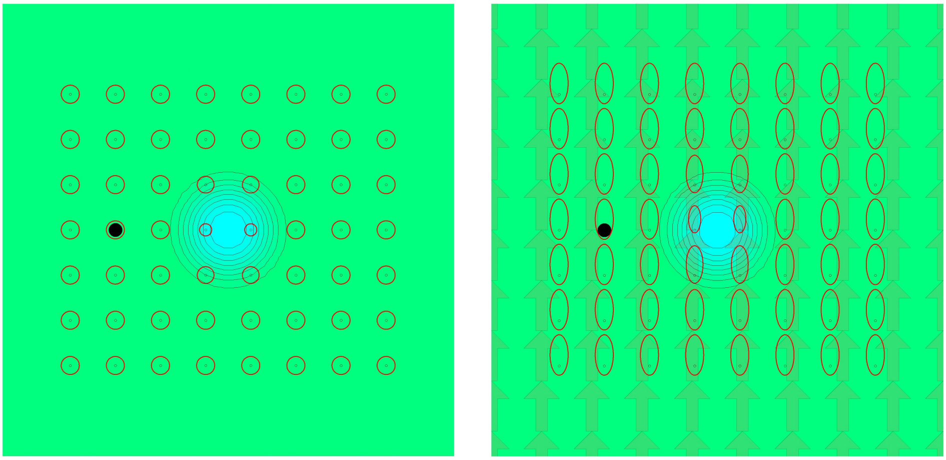

Fig. 8 Two simple fields of indicatrices, one without and one with wind

In general, when we choose an indicatrix $\mathcal{I}_q$ of F_q-unit vectors at every point q, then the F_q-length of any other vector at q is simply defined by homogeneous scaling:

Definition 1. Suppose we know the F_q-unit vectors at every point q, i.e. we assume that we have chosen the indicatrix field $\mathcal{I}_q$ already – as exemplified in Fig. 8. Then, since every (other) vector y is a factor λ times some unit vector u, $y = \lambda \cdot u$, we simply define the F_q-length of y to be that factor:

$$F_q(y) = F_q(\lambda \cdot u) = \lambda \cdot F_q(u) = \lambda\,, \quad u \in \mathcal{I}_q\,. \tag{4}$$

The F-length of a *curve* is then (as usual) the integral of the F-length of its tangent vectors:

Definition 2. Suppose $c(t) = (c^1(t), c^2(t))$, $t \in [0, T]$, denotes a regular smooth curve in the plane. Then the F-*length* of c is given by:

$$\mathcal{L}(c) = \int_0^T F_{c(t)}(\dot{c}(t))\,dt\,, \tag{5}$$

where $\dot{c}(t)$ denotes the tangent vector of the curve c at the point $c(t)$.

Note that the length of $c(t)$, $t \in [0, T]$, is not necessarily the same as the length of the reversed curve $\hat{c}(t) = c(T - t)$, $t \in [0, T]$. This is because $F_{c(t)}(\dot{c}(t))$ is not necessarily the same as $F_{c(t)}(-\dot{c}(t))$. And this is exactly what we want! The length functional $\mathcal{L}$ measures and takes into account that it is easy to go one way along the curve ($\mathcal{L}(c)$ is small) but possibly difficult to go the other way ($\mathcal{L}(\hat{c})$ is large).

The Geodesic Paradigm for the Fire Front Propagation

One key ingredient in the forest-fire model presented here is that the fire front is formed by the union of all F-shortest curves – of the same F-length – issuing from a given ignition point. Such F-shortest curves are called F-geodesics. Due to the asymmetric measure of the F-length they extend further in the direction of the far ends of the local indicatrices than in the opposite directions. Therefore the endpoints of these geodesics (the geodesic "circle," the fire front) initially (for small radii) look like the indicatrix at the ignition point, but as the geodesic circles and fire front extend to further radii they may take on very different shapes as illustrated by the simple examples in Figs. 9 and 10. In both figures the fire front will progress through and around the symmetric blue moorland area and create pincer cut points along a curve east of the moorland. When the wind is blowing from the south then the pincer curve is clearly shifted towards the north of the moorland.

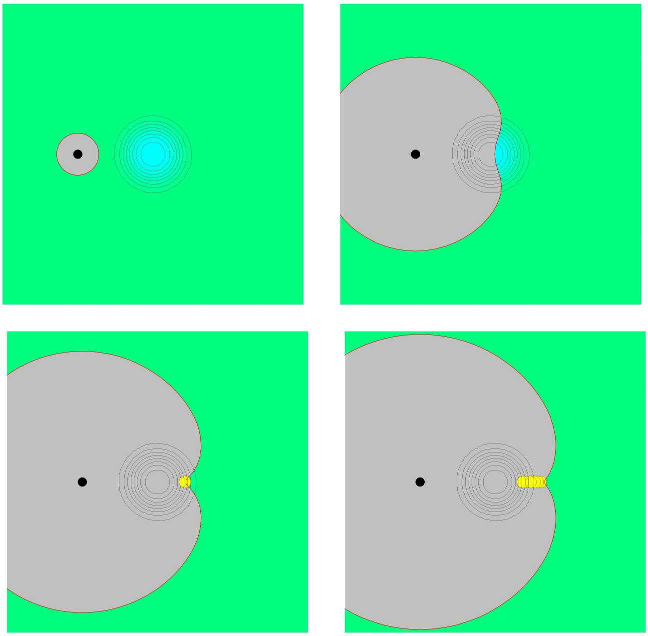

Fig. 9 Without any wind the fire front will penetrate through and around the symmetric blue moorish area

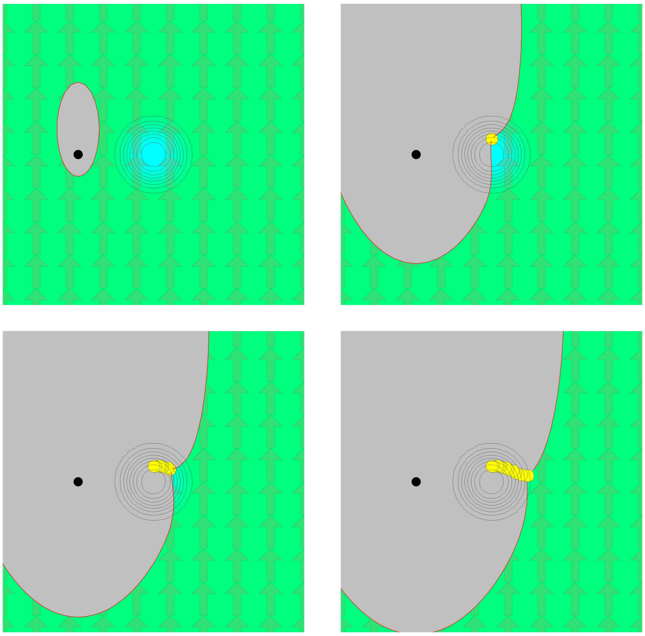

Fig. 10 With a wind from the south (shown by the arrows) the fire front will be elliptical and the first part of the cut locus will be shifted

The F-geodesics satisfy a system of nonlinear ordinary differential equations:

Proposition 3. A given curve $c(t) = (c^1(t), c^2(t))$ is an F-geodesic, i.e. a locally F-shortest curve between any pair of its points, if it satisfies the differential equations:

$$\ddot{c}^i(t) + \gamma^i_{jk}(c(t), \dot{c}(t)) \cdot \dot{c}^j(t) \cdot \dot{c}^k(t) = 0\,, \quad i = 1, 2\,, \tag{6}$$

where the so-called connection (or Christoffel) functions $\gamma^i_{jk}(c(t), \dot{c}(t))$ are given by a (somewhat complicated) mixture of suitable derivatives of the Finsler function F at the point $c(t)$, see e.g. [14].

Constant Indicatrix Fields

In particular, if F has a constant indicatrix field, i.e. if the background fuel and topography is completely homogeneous, then the Christoffel functions all vanish, and the differential equations for the geodesics reduce to

$$\ddot{c}^i(t) = 0\,, \quad i = 1, 2\,, \tag{7}$$

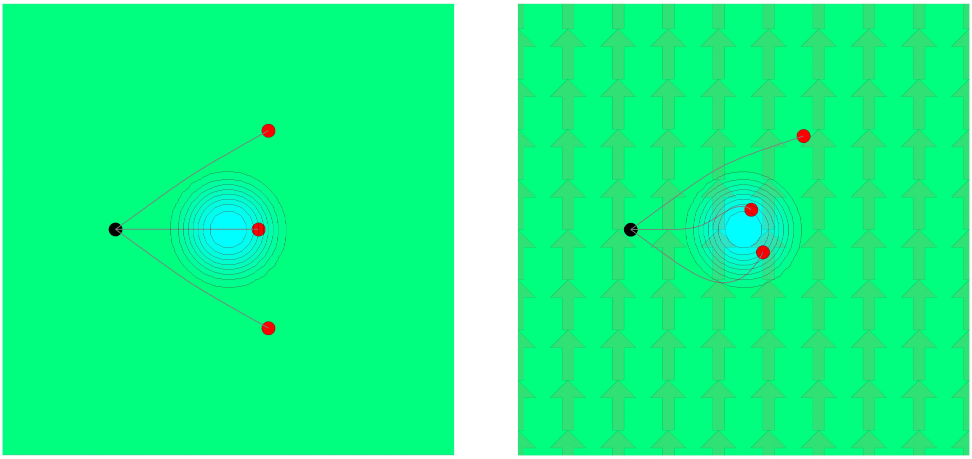

Fig. 11 Three geodesics of the *same* F-length without wind and with wind (shown by the arrows), respectively

so that every geodesic issuing from the ignition point $p = c(0)$ is a straight line; the solutions are linear functions in t:

$$\begin{aligned} c^1(t) &= \alpha_1 t + \beta_1 \\ c^2(t) &= \alpha_2 t + \beta_2 \,. \end{aligned} \tag{8}$$

This is in precise accordance with the model equation (1) if we let

$$\begin{aligned} \alpha_1 &= a\cos(\phi) \\ \alpha_2 &= b\sin(\phi) + c \\ \beta_1 &= 0 \\ \beta_2 &= 0\,. \end{aligned} \tag{9}$$

This linearity of geodesics is displayed in Figs. 11, 12, and 13. Initially the geodesics, the fire particle tracks, from the point p of ignition are directed straight away from p because the region in front of the "moor" is essentially homogeneous and flat. Note (in Fig. 11) that when you approach the moorland area you get further to the right side of the moor when you go around it. When the wind is blowing it clearly also matters which way you choose to go around it!

In Figs. 12 and 13 we show *all* the geodesics issuing from a common ignition point and up to a common propagated F-length. The different cases with or without wind are also indicated. The ensuing endpoints of the geodesic fire tracks (which together form the geodesic "circle" fire front) are marked together with the burnt-out area in the accompanying Figs. 9 and 10.

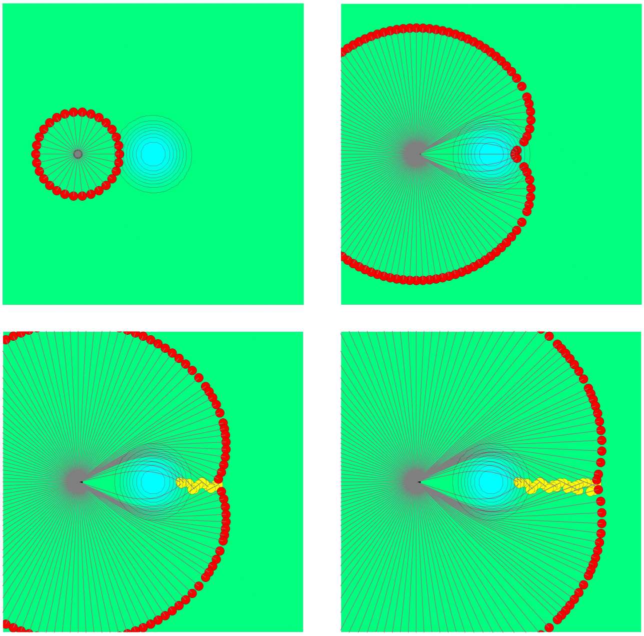

Fig. 12 The individual geodesics issuing from the ignition point bend around the moorland to form the fire front movement in Fig. 9

Bear Hugs

A fire front initially forms an oval, which is very similar to the F_p indicatrix $\mathcal{I}_p$ at the ignition point, but as it develops further away from the point of ignition the front will bend around lakes or any other moorish domains that cannot be easily penetrated by the fire, and the front may thus create self-intersections, pincer movements (also known as “bear hugs”) behind these obstacles – as already mentioned and observed in Fig. 5. As before we mark such “pincer points” by yellow (warning) dots on the figures. They are the positions where firefighters may be in danger of attack from multiple (at least two) sides by the fire front. Technically these points of self-intersection form a continuous set of points $\mathrm{Cut}(p)$, which is called the *cut locus* of p.

Conclusion

Cut loci are of current research interest for several other reasons and for many other applications than the particular one addressed here, but in the present setting, where we have been concerned with understanding wildfires, they are obviously of particular direct impact and importance. With a suitable geometric analysis and an aroused and raised public awareness, i.e. $\mathrm{RPAM}(X)$

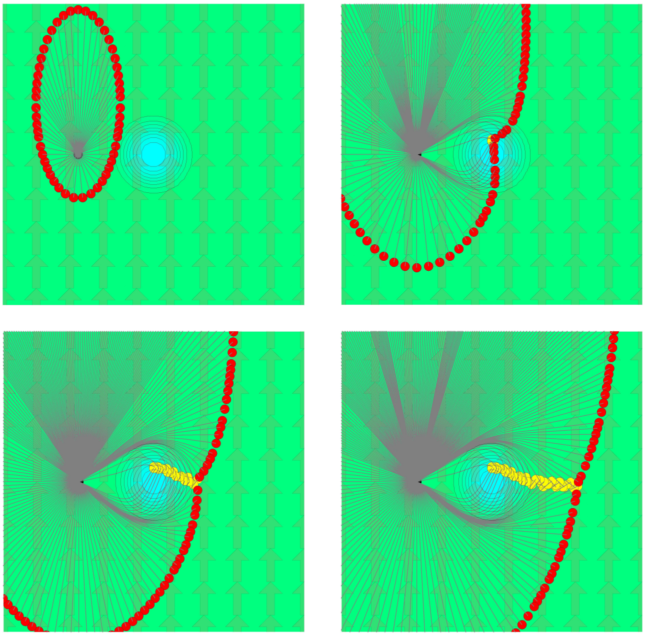

Fig. 13 The individual fire particles that create the fire front movement in Fig. 10. The influence of the wind is evident

to hand, their formation may even be predicted – and catastrophes thus prevented.

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