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Brief Paper

Robust Performance of Systems with Structured Uncertainties in State Space*

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Key Words—Robust control; state feedback; convex programming.

Abstract—This paper considers robust performance analysis and state feedback design for systems with time-varying parameter uncertainties. The notion of a strongly robust $\mathcal{H}_\infty$ performance criterion is introduced, and its applications in robust performance analysis and synthesis for nominally linear systems with time-varying uncertainties are discussed and compared with the constant scaled small gain criterion. It is shown that most robust performance analysis and synthesis problems under this strongly robust $\mathcal{H}_\infty$ performance criterion can be transformed into linear matrix inequality problems, and can be solved through finite-dimensional convex programming. The results are in general less conservative than those using small gain type criteria.

1. *Introduction*

During the last decade, much progress has been made in the robust control analysis and synthesis of linear time-invariant systems with time-invariant uncertainties. In particular, the development of $\mathcal{H}_\infty$ theory and structured singular value computation algorithms has greatly simplified the robust stability, performance analysis and controller design (see Doyle, 1982; Packard and Doyle, 1988; Doyle *et al.*, 1989, 1991; Krause *et al.*, 1989; and references therein). For systems with time-varying uncertainties, some new results regarding the system robust stability have also been developed using the notion of quadratic stability (see Boyd and Yang, 1989; Khargonekar *et al.*, 1990; Packard and Doyle, 1990; Packard *et al.*, 1991; Becker and Packard, 1991). However, the robust performance problem for systems with time-varying uncertainties has not been sufficiently explored. The most commonly used criterion in this case is the so-called constant scaled small gain condition (Krause *et al.*, 1989). This paper is motivated by the need to improve the results that can be obtained by the constant scaled small gain criterion. Our approach is closely related to the notion of quadratic stability and is a further extension of the results presented in Boyd and Yang (1989), Khargonekar *et al.* (1990), Packard *et al.* (1991), Packard and Doyle (1990), Xie and Souza (1990a, b), Becker and Packard (1991), Geromel *et al.* (1991) and Peres *et al.* (1991).

In this paper we consider the notion of strongly robust $\mathcal{H}_\infty$ performance. This is a natural generalization of the concept of quadratic stability and is related to an analogous concept introduced in Xie and Souza (1990a,b). While some of the results presented in this paper are similar to those in Geromel *et al.* (1991) and Peres *et al.* (1991), we consider a more general class of uncertain systems and a more natural description of the uncertainty. More importantly, it should also be noted that our results are non-conservative, i.e. the conditions stated in the paper are necessary and sufficient under the defined stability and performance notions. We consider linear time-invariant systems with real time-varying parameter uncertainties which lie in compact intervals. The main results of this paper show that both analysis and state-feedback synthesis problems can be reduced to finite-dimensional convex programming problems.

The paper is organized as follows: the strongly robust $\mathcal{H}_\infty$ performance criterion is formally introduced in Section 2 and its implications in disturbance rejection and robust stability are discussed. It is also shown in this section that under this robust performance criterion the dynamic state feedback problem is equivalent to a static state feedback problem. Section 3 considers the analysis problem while Section 4 considers the state feedback controller synthesis problem. The results obtained in this paper are compared with the so-called scaled small gain condition in Section 5 and an example is shown in Section 6 to illustrate our results. The corresponding discrete-time results are also obtained in Section 7. Finally, Section 8 offers some conclusions.

2. *Preliminaries*

Consider a linear time-varying dynamical system with a state space representation

$$\dot{x} = A_\Delta x + B_\Delta w, \quad x(0) = 0 \tag{1}$$

$$z = C_\Delta x + D_\Delta w, \tag{2}$$

where A_Δ, B_Δ, C_Δ, and D_Δ are continuous matrix functions of $\Delta(t)$, and $\Delta(t) \in \mathbf{\Delta}$ is (possibly) a time-varying uncertain matrix. The symbol $\mathbf{\Delta}$ denotes a compact set of appropriately dimensioned matrices with a particular structure which will be specified later on. The function $\Delta(t)$ is assumed to be a measurable function of $t \in [0, \infty)$.

Definition 1. The system described by equations (1) and (2) with $w = 0$ is said to be quadratically stable if there exists a symmetric matrix $X > 0$ such that $V(x) = x'Xx$ is a Lyapunov function for the system, i.e. $\dot{V}(x(t)) < 0$ for all $x \neq 0$ and $\Delta \in \mathbf{\Delta}$.

The key point here is that the Lyapunov function is fixed and is independent of uncertainty. It should be noted that this stability notion is quite reasonable since the uncertainty

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$\Delta(t)$ could be time-varying. Moreover this stability notion is useful even in the case of time-invariant real uncertainty due to the lack of better analysis methods. This point will be further demonstrated in Section 5.

Definition 2. The time-varying uncertain dynamical system described by equations (1) and (2) is said to satisfy strongly robust $\mathcal{H}_\infty$ performance criterion if $\|D_\Delta\| < 1$ $\forall \Delta \in \mathbf{\Delta}$ and there exists a constant symmetric matrix $X > 0$ such that

$$A_\Delta' X + XA_\Delta + (XB_\Delta + C_\Delta' D_\Delta)R_\Delta^{-1}(B_\Delta' X + D_\Delta' C_\Delta) + C_\Delta' C_\Delta < 0 \quad (3)$$

for all $t \geq 0$ and $\Delta \in \mathbf{\Delta}$, where $R_\Delta = I - D_\Delta' D_\Delta > 0$.

It is easy to see that if a system satisfies strongly robust $\mathcal{H}_\infty$ performance criterion, then it is necessarily quadratically stable. This concept is also equivalent to a robust disturbance attenuation concept introduced in Xie and Souza (1990a,b). The strongly robust $\mathcal{H}_\infty$ performance criterion implies a standard $\mathcal{H}_\infty$ disturbance attenuation bound as shown in the following lemma:

Lemma 3. Suppose that $\mathbf{\Delta}$ is a compact set and the uncertain system in equations (1) and (2) satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. Then the system is quadratically stable and there exists an $\epsilon > 0$ such that

$$\|z\|_2 \leq (1-\epsilon)\|w\|_2.$$

Proof. Let $V(x) := x'Xx$ and define

$$-Q_\Delta := A_\Delta' X + XA_\Delta + (XB_\Delta + C_\Delta' D_\Delta)R_\Delta^{-1} \times (B_\Delta' X + D_\Delta' C_\Delta) + C_\Delta' C_\Delta.$$

Then there exists a (sufficiently small) $\epsilon_1 > 0$ such that for all $\Delta \in \mathbf{\Delta}$

$$\bar{R}_\Delta := R_\Delta - \epsilon_1 I > 0$$

and

$$\bar{Q}_\Delta := Q_\Delta (XB_\Delta + C_\Delta' D_\Delta)(\bar{R}_\Delta^{-1} - R_\Delta^{-1}) \times (B_\Delta' X + D_\Delta' C_\Delta) > 0.$$

It follows from the definition that the uncertain system is quadratically stable. Furthermore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(x'Xx) = -\|z\|^2 + (1-\epsilon_1)\|w\|^2 - \|\bar{R}_\Delta^{-1/2}[\bar{R}_\Delta w - (B_\Delta' X + D_\Delta' C_\Delta)x]\|^2 - x'\bar{Q}_\Delta x.$$

If $w \in \mathcal{L}_2$, then $x \in \mathcal{L}_2$, and integrating from $t = 0$ to $t = \infty$ gives

$$\|z\|_2^2 - (1-\epsilon_1)\|w\|_2^2 = -\|\bar{R}_\Delta^{-1/2}[\bar{R}_\Delta w - (B_\Delta' X + D_\Delta' C_\Delta)x]\|_2^2 - \int_0^\infty x(t)'\bar{Q}_\Delta x(t)\,\mathrm{d}t \leq 0.$$

Thus

$$\|z\|_2 \leq \sqrt{1-\epsilon_1}\,\|w\|_2 \leq (1-\epsilon)\|w\|_2$$

for some $\epsilon > 0$. □

Our objective in this paper is to derive some easily computable conditions for checking the satisfaction of the strongly robust $\mathcal{H}_\infty$ performance criterion for certain classes of uncertain systems. We shall also consider finding state feedback controllers to achieve the strongly robust $\mathcal{H}_\infty$ performance criterion. The following result is a generalization of Khargonekar *et al.* (1988):

Theorem 4. Consider the uncertain system

$$\dot{x} = A_\Delta x + B_\Delta w + B_{2\Delta} u \quad (4)$$

$$z = C_\Delta x + D_\Delta w + D_{2\Delta} u \quad (5)$$

$$y = x \quad (6)$$

and suppose that there exists a dynamic state feedback controller $u = K(s)y$ such that the closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. Then there exists a real matrix F such that with the static controller $u = Fy$, the closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion.

Proof. Suppose that there exists a dynamic state feedback controller

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}x$$

$$u = \hat{C}\hat{x} + \hat{D}x,$$

such that the closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. The closed-loop system has the following state space representation:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left(\begin{bmatrix} A_\Delta & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_{2\Delta} & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}\right)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_\Delta \\ 0 \end{bmatrix} w = \tilde{A}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \tilde{B}w$$

$$z = \left([C_\Delta \quad 0] + [D_{2\Delta} \quad 0]\begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}\right)\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_\Delta w = \tilde{C}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + D_\Delta w.$$

By definition, there exists a $\tilde{X} > 0$ such that

$$\tilde{A}'\tilde{X} + \tilde{X}\tilde{A} + (\tilde{X}\tilde{B} + \tilde{C}'D_\Delta)R_\Delta^{-1}(\tilde{B}'\tilde{X} + D_\Delta'\tilde{C}) + \tilde{C}'\tilde{C} < 0, \quad (7)$$

with $R_\Delta = I - D_\Delta' D_\Delta > 0$.

Define a matrix W and a matrix $Y > 0$ as

$$\begin{bmatrix} W & W_2 \\ W_3 & W_4 \end{bmatrix} := \begin{bmatrix} \hat{D} & \hat{C} \\ \hat{B} & \hat{A} \end{bmatrix}\tilde{X}^{-1}, \quad \begin{bmatrix} Y & Y_2 \\ Y_2' & Y_3 \end{bmatrix} := \tilde{X}^{-1}$$

and, furthermore, define

$$X = Y^{-1} > 0, \quad F = WY^{-1}.$$

Then it can be shown using inequality (7) that X and F satisfy the following inequality

$$X(A_\Delta + B_{2\Delta}F) + (A_\Delta + B_{2\Delta}F)'X + [XB_\Delta + (C_\Delta + D_{2\Delta}F)'D_\Delta]R_\Delta^{-1} \times [B_\Delta' X + D_\Delta'(C_\Delta + D_{2\Delta}F)] + (C_\Delta + D_{2\Delta}F)'(C_\Delta + D_{2\Delta}F) < 0.$$

This implies that the following system:

$$\dot{x} = (A_\Delta + B_{2\Delta}F)x + B_\Delta w$$

$$z = (C_\Delta + D_{2\Delta}F)x + D_\Delta w$$

satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. In other words, $u = Fx$ is a strongly robust $\mathcal{H}_\infty$ performance state feedback controller. □

3. *Robust performance of uncertain systems*

In this section, we shall consider strongly robust $\mathcal{H}_\infty$ performance for a special class of uncertain systems. Suppose that the uncertain system admits a state space realization in the following form:

$$\dot{x} = A_\Delta x + B_\Delta w \quad (8)$$

$$z = C_\Delta x + D_\Delta w, \quad (9)$$

where

$$\begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_0 \\ D_{10} \end{bmatrix}\Delta[C_0 \quad D_{01}]$$

for some constant matrices A, B_0, B_1, C_0, C_1, D_{01}, D_{10}, and

D_{11}. For simplicity, we shall also assume that the uncertainty matrix $\Delta \in \mathbf{\Delta}$ is real-time varying and

$$\mathbf{\Delta} = \{block\ diag\ [\delta_1(t)I_{k_1}, \ldots, \delta_m(t)I_{k_m}] : \delta_i(t) \in [\underline{\delta}_i, \bar{\delta}_i]\}.$$

For future reference, we shall denote the vertex set of $\mathbf{\Delta}$ as

$$\mathbf{\Delta}_{\text{vex}} = \{block\ diag\ [\delta_1 I_{k_1}, \ldots, \delta_m I_{k_m}] : \delta_i = \underline{\delta}_i \text{ or } \delta_i = \bar{\delta}_i\}.$$

It is easy to see that there are 2^m vertices in $\mathbf{\Delta}_{\text{vex}}$.

Remark 5. It is interesting to note that the uncertainty operator Δ can in fact be a nonlinear time-varying bounded operator and not necessarily a linear time-varying gain matrix. As an example, consider a nonlinear uncertainty of the form

$$\xi := \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_q \end{bmatrix} = \Delta\eta := \begin{bmatrix} g_1(t, \eta_1) \\ g_2(t, \eta_2) \\ \vdots \\ g_q(t, \eta_q) \end{bmatrix},$$

where the nonlinear time-varying functions $g_i(t, \eta_i)$, $i = 1, \ldots, q$ satisfy

$$\underline{\delta}\eta_i^2 \le \eta_i g_i(t, \eta_i) \le \bar{\delta}_i \eta_i^2.$$

Then in this case, if we design a controller such that the closed-loop system with the time-varying uncertainty gain matrix $\Delta = \text{diag}(\delta_1, \ldots, \delta_q)$ with $\delta_i \in [\underline{\delta}_i, \bar{\delta}_i]$ satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion, then the same statement holds for the nonlinear uncertainty as well. It is also important to note that the operator Δ need not necessarily be uncertain for the analysis and synthesis approaches proposed here to be applicable. These approaches may also be useful for systems with known but complicated operator Δ in order to simplify the analysis and design. This is particularly true for the analysis and synthesis of nonlinear time-varying systems.

The following theorem is our main result of this section:

Theorem 6. Consider the uncertain system described by equations (8) and (9). Define

$$R_\Delta := I - D'_\Delta D_\Delta.$$

Then the following statements are equivalent:

(i) The system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion.

(ii) $R_\Delta > 0$, $\forall \Delta \in \mathbf{\Delta}_{\text{vex}}$ and there exists an $X = X' > 0$ such that

$$A'_\Delta X + XA_\Delta + (XB_\Delta + C'_\Delta D_\Delta)R_\Delta^{-1} \times (B'_\Delta X + D'_\Delta C_\Delta) + C'_\Delta C_\Delta < 0$$

for all $\Delta \in \mathbf{\Delta}_{\text{vex}}$.

(iii) $R_\Delta > 0$, $\forall \Delta \in \mathbf{\Delta}_{\text{vex}}$ and there exists an $X = X' > 0$ such that

$$\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_\Delta & B_\Delta \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A'_\Delta & 0 \\ B'_\Delta & 0 \end{bmatrix}\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} C'_\Delta C_\Delta & C'_\Delta D_\Delta \\ D'_\Delta C_\Delta & -R_\Delta \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{\text{vex}}$.

Proof. First note that $R_\Delta > 0$, $\forall \Delta \in \mathbf{\Delta}$ if and only if

$$\begin{bmatrix} I & -D'_\Delta \\ -D_\Delta & I \end{bmatrix} > 0 \quad \forall \Delta \in \mathbf{\Delta}$$

if and only if

$$\begin{bmatrix} I & -D'_\Delta \\ -D_\Delta & I \end{bmatrix} > 0 \quad \forall \Delta \in \mathbf{\Delta}_{\text{vex}}.$$

The proof for (i)$\Rightarrow$(ii) is trivial since $\mathbf{\Delta}_{\text{vex}} \subset \mathbf{\Delta}$ and the implication of (ii)$\Leftrightarrow$(iii) follows from the Schur complement formula. To show (ii)$\Rightarrow$(i), let us first define

$$\tilde{Q}_\Delta := \begin{bmatrix} A'_\Delta X + XA_\Delta & XB_\Delta & C'_\Delta \\ B'_\Delta X & -I & D'_\Delta \\ C_\Delta & D_\Delta & -I \end{bmatrix}.$$

Since Δ appears affinely in $\tilde{Q}_\Delta$, it is easy to see by convexity that

$$\max_{\Delta \in \mathbf{\Delta}} \lambda_{\max}(\tilde{Q}_\Delta) = \max_{\Delta \in \mathbf{\Delta}_{\text{vex}}} \lambda_{\max}(\tilde{Q}_\Delta).$$

This implies that $\tilde{Q}_\Delta < 0$ $\forall \Delta \in \mathbf{\Delta}$ if and only if $\tilde{Q}_\Delta < 0$ $\forall \Delta \in \mathbf{\Delta}_{\text{vex}}$. On the other hand, it is easy to see from the Schur complement formula that $\tilde{Q}_\Delta < 0$ $\forall \Delta \in \mathbf{\Delta}$ is equivalent to

$$A'_\Delta X + XA_\Delta + (XB_\Delta + C'_\Delta D_\Delta)R_\Delta^{-1} \times (B'_\Delta X + D'_\Delta C_\Delta) + C'_\Delta C_\Delta < 0$$

for all $\Delta \in \mathbf{\Delta}$, i.e. strongly robust $\mathcal{H}_\infty$ performance criterion is satisfied. □

Now the key point is that finding a positive definite symmetric matrix $X > 0$ such that condition (iii) holds can be done through *convex programming*. In particular, the numerical algorithm described in Boyd and Yang (1989) can be modified easily for this problem.

Remark 7. In fact, the above results (and the results presented in the rest of the paper) apply to a much more general class of uncertain systems. For example, suppose the uncertain system matrices satisfy the following conditions: for each δ_i, there exist appropriately dimensional matrix functions E_i, H_i, and scalar functions α_i, β_i which are all independent of δ_i such that

$$\begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} = \frac{E_i + \delta_i H_i}{\alpha_i + \delta_i \beta_i}.$$

Then it is easy to see that the following relation used in the proof is still true:

$$\max_{\Delta \in \mathbf{\Delta}} \lambda_{\max}(\tilde{Q}_\Delta) = \max_{\Delta \in \mathbf{\Delta}_{\text{vex}}} \lambda_{\max}(\tilde{Q}_\Delta).$$

Hence, the theorem holds for uncertain systems satisfying the above conditions.

Remark 8. It is not hard to show that for the class of uncertain systems considered above, the system matrices can be written in a matrix linear fractional form:

$$\begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} = \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_0 \\ D_{10} \end{bmatrix} \times \Delta(I - D_{00}\Delta)^{-1}[C_0 \quad D_{01}]$$

for some matrices A, B_1, B_0, C_1, C_0, D_{10}, D_{10}, D_{00} and $\Delta \in \mathbf{\Delta}$. Hence this problem can also be treated in the general linear fractional framework and the constant scaled small gain condition can be used as suggested in Krause *et al.* (1989). The advantages of these approaches will be further discussed in Section 5.

4. *Robust state feedback control*

In this section, we shall consider state feedback controller design such that the closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. For technical reason, we shall only consider the following class of uncertain systems:

$$\dot{x} = A_\Delta x + B_\Delta w + B_{2\Delta} u, \quad \Delta \in \mathbf{\Delta} \qquad (10)$$

$$z = C_\Delta x + D_\Delta w + D_{2\Delta} u \qquad (11)$$

$$y = x, \qquad (12)$$

where A_Δ, B_Δ, $B_{2\Delta}$, C_Δ, D_Δ, and $D_{2\Delta}$ are any affine matrix functions of Δ as assumed in the last section and $\mathbf{\Delta}$ is the same compact set defined in the last section. In fact, they can be more complicated matrix functions as pointed out in the last section.

Theorem 9. There exists a state feedback controller such that the above closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if $R_\Delta := I - D_\Delta' D_\Delta > 0$, $\forall \Delta \in \mathbf{\Delta}_{vex}$ and there exists a matrix W and a matrix $Y = Y' > 0$ such that

$$\begin{bmatrix} YA_\Delta' + A_\Delta Y + W'B_{2\Delta}' + B_{2\Delta}W & B_\Delta + YC_\Delta' D_\Delta + W'D_{2\Delta}' D_\Delta & YC_\Delta' + W'D_\Delta' \\ B_\Delta' + D_\Delta' C_\Delta Y + D_\Delta' D_{2\Delta} W & -R_\Delta & 0 \\ C_\Delta Y + D_{2\Delta} & 0 & -I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$. Moreover, the scalar feedback controller can be taken as a constant gain as

$$F = WY^{-1}.$$

Proof. ($\Rightarrow$) By Theorem 4, it can be assumed without loss of generality that there exists F such that the closed-loop system with $u = Fx$ satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. The closed-loop system can be written as

$$\dot{x} = (A_\Delta + B_{2\Delta}F)x + B_\Delta w$$
$$z = (C_\Delta + D_{2\Delta}F)x + D_\Delta w.$$

By the definition of strongly robust $\mathcal{H}_\infty$ performance criterion and Theorem 5, there exists an $X = X' > 0$ such that

$$X(A_\Delta + B_{2\Delta}F) + (A_\Delta + B_{2\Delta}F)'X + (C_\Delta + D_{2\Delta}F)'(C_\Delta + D_{2\Delta}F) + [XB_\Delta + (C_\Delta + D_{2\Delta}F)'D_\Delta]R_\Delta^{-1} \times [B_\Delta' X + D_\Delta'(C_\Delta + D_{2\Delta}F)] < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$. Now define

$$Y := X^{-1}, \quad W = FX^{-1}.$$

Then the above inequality can be written as

$$YA_\Delta' + A_\Delta Y + W'B_{2\Delta}' + B_{2\Delta}W + (YC_\Delta' + W'D_{2\Delta}')(C_\Delta Y + D_{2\Delta}W) + [B_\Delta + YC_\Delta' D_\Delta + W'D_{2\Delta}']R_\Delta^{-1} \times [B_\Delta' + D_\Delta' C_\Delta Y + D_\Delta' D_{2\Delta} W] < 0$$

or equivalently

$$\begin{bmatrix} YA_\Delta' + A_\Delta Y + W'B_{2\Delta}' + B_{2\Delta}W & B_\Delta + YC_\Delta' D_\Delta + W'D_{2\Delta}' D_\Delta & YC_\Delta' + W'D_{2\Delta}' \\ B_\Delta' + D_\Delta' C_\Delta Y + D_\Delta' D_{2\Delta} W & -R_\Delta & 0 \\ C_\Delta Y + D_{2\Delta}W & 0 & -I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$.

($\Leftarrow$) This follows easily by reversing the above steps and using the state-feedback gain $F = WY^{-1}$. □

This theorem shows that the problem of state feedback synthesis can be reduced to searching for the matrices W, Y satisfying the linear matrix inequality above. The main point is that this matrix inequality is convex in W, Y and thus convex programming techniques can be used to solve for W, Y. This result is similar to Becker and Packard (1991), and Packard *et al.* (1991). The above theorem can be simplified considerably if $B_{2\Delta}$, C_Δ, and $D_{2\Delta}$ are all independent of uncertainty Δ and, furthermore, $D_{11} = 0$.

Corollary 10. Suppose $D_{11} = 0$, $B_{2\Delta} = B_2$, $C_\Delta = C_1$, and $D_{2\Delta} = D_{12}$. Define $R_2 := D_{12}' D_{12} > 0$ and let $D_\perp$ be any matrix such that

$$D_\perp D_\perp' := I - D_{12}R_2^{-1}D_{12}'.$$

Then there exists a state feedback controller such that the above closed-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if there exists a $Y = Y' > 0$ such that

$$\begin{bmatrix} A_\Delta - B_2R_2^{-1}D_{12}'C_1 & 0 \\ D_\perp' C_1 & 0 \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (A_\Delta - B_2R_2^{-1}D_{12}'C_1)' & C_1'D_\perp \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_\Delta B_\Delta' - B_2R_2^{-1}D_{12}' & 0 \\ 0 & -I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$. Moreover, the state feedback controller can be taken as a constant gain as

$$F = -R_2^{-1}(D_{12}'C_1 + B_2'Y^{-1}).$$

Proof. ($\Rightarrow$) Similar to the proof of Theorem 9, there exists an $X = X' > 0$ such that

$$(A_\Delta + B_2F)'X + X(A_\Delta + B_2F) + XB_\Delta B_\Delta' X + (C_1 + D_{12}F)'(C_1 + D_{12}F) < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$. Now complete the square with respect to F to get

$$(A_\Delta - B_2R_2^{-1}D_{12}'C_1)'X + X(A_\Delta - B_2R_2^{-1}D_{12}'C_1) + C_1'(I - D_{12}R_2^{-1}D_{12}')C_1 + XB_\Delta B_\Delta' X - XB_2R_2^{-1}B_2'X + (F + R_2^{-1}(D_{12}'C_1 + B_2'X)'R_2 \times (F + R_2^{-1}(D_{12}'C_1 + B_2'X)) < 0.$$

Then we have

$$(A_\Delta - B_2R_2^{-1}D_{12}'C_1)'X + X(A_\Delta - B_2R_2^{-1}D_{12}'C_1) + C_1'(I - D_{12}R_2^{-1}D_{12}')C_1 + XB_\Delta B_\Delta' X - XB_2R_2^{-1}X < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$. Now define $Y := X^{-1}$. We have

$$Y(A_\Delta - B_2R_2^{-1}D_{12}'C_1)' + (A_\Delta - B_2R_2^{-1}D_{12}'C_1)Y + YC_1'D_\perp D_\perp' C_1 Y + B_\Delta B_\Delta' - B_2R_2^{-1}B_2' < 0$$

or equivalently

$$\begin{bmatrix} A_\Delta - B_2R_2^{-1}D_{12}'C_1 & 0 \\ D_\perp' C_1 & 0 \end{bmatrix}\begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} (A_\Delta - B_2R_2^{-1}D_{12}'C_1)' & C_1'D_\perp \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_\Delta B_\Delta' - B_2R_2^{-1}B_2' & 0 \\ 0 & -I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{vex}$.

($\Leftarrow$) This again follows easily by reversing the above steps. □

5. *Comparison with small gain type criterion*

In this section, we will analyze the conservativeness of the proposed analysis and synthesis framework. In particular, we will compare the proposed method with constant scaled small gain type analysis and synthesis methodology, i.e. time varying μ framework. We will focus on a simple class of uncertain systems where the system can be shown as in Fig. 1 with

$$M(s) := \left[\begin{array}{c|cc} A & B_0 & B_1 \\ \hline C_0 & 0 & 0 \\ C_1 & 0 & 0 \end{array}\right]$$

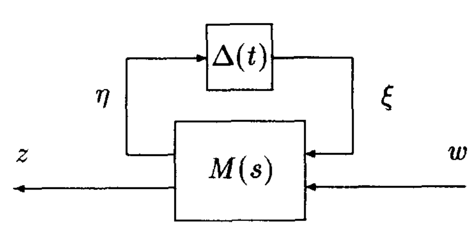

Fig. 1. Uncertainty description.

and $\Delta = \mathrm{diag}[\delta_1(t), \delta_2(t), \ldots, \delta_m(t)]$. We shall also assume that the uncertainty is normalized so that $\bar{\delta}_i = -\underline{\delta}_i = 1$, i.e. $\|\Delta\| \le 1$.

Now define the constant scaling matrix set as

$$\mathcal{T} = \{\text{block diag}(T_1, T_2, \ldots, T_m): 0 < T_i = T_i' \in \mathcal{R}^{k_i \times k_i}\}.$$

It is clear that for any $T \in \mathcal{T}$ and $\Delta \in \mathbf{\Delta}$, we have $T\Delta T^{-1} = \Delta$. By small gain type of criterion, the system is robustly stable and $\|z\|_2 < \|w\|_2$ for all $\Delta \in \mathbf{\Delta}$ if there exists a $T \in \mathcal{T}$ such that

$$\left\| \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} M(s) \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1.$$

By 'bounded real lemma', the above is true if and only if there is an $X = X' > 0$ such that

$$XA + A'X + X[B_0T^{-1} \quad B_1][B_0T^{-1} \quad B_1]'X + \begin{bmatrix} TC_0 \\ C_1 \end{bmatrix}' \begin{bmatrix} TC_0 \\ C_1 \end{bmatrix} < 0. \quad (13)$$

We now show that the inequality (13) implies the strongly robust $\mathcal{H}_\infty$ performance condition. To do that, we note that for any $T \in \mathcal{T}$, we have

$$XB_0\Delta C_0 + C_0'\Delta' B_0'X \le XB_0T^{-1}(T')^{-1}B_0'X + C_0'T'TC_0 \quad (14)$$

for all $\Delta \in \mathbf{\Delta}$. Using inequalities (13) and (14), we have immediately

$$X(A + B_0\Delta C_0) + (A + B_0\Delta C_0)'X + XB_1B_1'X + C_1'C_1 < 0, \quad \forall \Delta \in \mathbf{\Delta},$$

i.e. the strongly robust $\mathcal{H}_\infty$ performance criterion is satisfied. However, it should be pointed out that the strongly robust $\mathcal{H}_\infty$ performance criterion condition does not in general imply the constant scaled small gain condition. This should be clear from the fact that quadratic stability for systems with structured real-time varying uncertainty does not in general imply the scaled small gain condition, see Packard and Doyle (1990). Hence, the proposed method is in general less conservative than the constant scaled μ method. They are equivalent if $\mathbf{\Delta}$ is an unstructured full real block. This fact is a generalization of an analogous result on the equivalence between quadratic stability and the small gain theorem for unstructured real uncertainty (Khargonekar *et al.*, 1990), and follows essentially from Fu *et al.* (1991). For completeness, we shall give a very short proof. We need a matrix fact which is referred to as Finsler's Lemma, see, e.g. Petersen (1987).

Lemma 11. Let P, Q, and R be $n \times n$ symmetric matrices and $P \ge 0$, $Q < 0$, and $R \ge 0$. Assume

$$(z'Qz)^2 - 4(z'Pz)(z'Rz) > 0$$

for all $0 \ne z \in \mathcal{R}^n$. Then there exists a constant $\lambda > 0$ such that

$$P + \lambda Q + \lambda^2 R < 0.$$

Theorem 12. Suppose $\mathbf{\Delta} = \mathcal{BR}^{m \times m}$, i.e. $\mathbf{\Delta} = \{\Delta \in \mathcal{R}^{m \times m}, \|\Delta\| \le 1\}$. Then the system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if there exists a constant $d > 0$ such that

$$\left\| \begin{bmatrix} dI_m & 0 \\ 0 & I \end{bmatrix} M(s) \begin{bmatrix} \frac{1}{d} I_m & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1.$$

Proof. The 'if' part is obvious from the previous discussion. We only need to show the 'only if' part. Suppose that the system satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion, i.e. there exists an $X = X' > 0$ such that

$$X(A + B_0\Delta C_0) + (A + B_0\Delta C_0)'X + XB_1B_1'X + C_1'C_1 < 0, \quad \forall \Delta \in \mathbf{\Delta}$$

or equivalently for all $z \in \mathcal{R}^n$, we have

$$z'(XA + A'X + XB_1B_1'X + C_1'C_1)z < -2 \max_{\Delta \in \mathbf{\Delta}} z'XB_0\Delta C_0 z.$$

The maximum on the right-hand side can be computed easily and we have

$$z'(XA + A'X + XB_1B_1'X + C_1'C_1)z < -2\sqrt{z'XB_0B_0'Xzz'C_0'C_0z}.$$

By Finsler's Lemma, there exists a constant $d > 0$ such that

$$(XB_0B_0'X) + d^2(XA + A'X + XB_1B_1'X + C_1'C_1) + d^4C_0'C_0 < 0$$

or equivalently

$$d^{-2}(XB_0B_0'X) + (XA + A'X + XB_1B_1'X + C_1'C_1) + d^2C_0'C_0 < 0,$$

i.e.

$$XA + A'X + X[B_0d^{-1} \quad B_1][B_0d^{-1} \quad B_1]'X + \begin{bmatrix} dC_0 \\ C_1 \end{bmatrix}' \begin{bmatrix} dC_0 \\ C_1 \end{bmatrix} < 0.$$

The last inequality implies by 'bounded real lemma' that

$$\left\| \begin{bmatrix} dI_m & 0 \\ 0 & I \end{bmatrix} M(s) \begin{bmatrix} \frac{1}{d} I_m & 0 \\ 0 & I \end{bmatrix} \right\|_\infty < 1. \qquad \square$$

6. *A numerical example*

In this section, we shall use a simple example to illustrate the results obtained in the previous sections. We shall adopt the notation in Section 4 and assume

$$A_\Delta = \begin{bmatrix} 0 & 1 \\ -10 & -10 \end{bmatrix} + \delta_1 \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + \delta_2 \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$B_\Delta = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad B_{2\Delta} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$D_{11} = 0, \quad D_{2\Delta} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where $|\delta_i| \le \bar{\delta}$ and $\delta_i \in \mathcal{R}$. The ellipsoid algorithm has been implemented here to solve this problem.

It is found that the open-loop system (without applying state feedback) is quadratically stable if and only if $\bar{\delta} < 0.548$ and $V(x) = x'Xx$ with

$$X = \begin{bmatrix} 0.1214 & 0.0865 \\ 0.0865 & 0.1035 \end{bmatrix}$$

is a Lyapunov function for $\bar{\delta} = 0.54$.

On the other hand, the open-loop system satisfies the strongly robust $\mathcal{H}_\infty$ performance if and only if $\bar{\delta} < 0.225$ and

$$X = \begin{bmatrix} 6.10309 & 5.04167 \\ 5.04167 & 5.95436 \end{bmatrix}$$

in a solution to the inequality (3) for $\bar{\delta} = 0.224$.

Finally there exists a strongly robust $\mathcal{H}_\infty$ performance state feedback if and only if $\bar{\delta} < 2.618$. In fact, we find a positive definite matrix

$$Y = \begin{bmatrix} 0.0826 & 0.0156 \\ 0.0156 & 0.0707 \end{bmatrix}$$

which satisfies the inequality in Corollary 10 for $\bar{\delta} = 2.61$ and this gives a state feedback law

$$u = [-12.635 \quad 2.7946]x$$

which makes the closed-loop systems satisfy the strongly robust $\mathcal{H}_\infty$ performance.

7. *Discrete-time systems*

Having discussed the robust performance problem for continuous-time systems, a natural question to pose is whether similar results hold in the discrete-time case. In studying discrete-time systems, one can use the bilinear transformation to convert the problem into a continuous-time problem. In the present setting, this transformation complicates the description of the uncertain matrices A_Δ, B_Δ, etc. Consequently, we shall address the discrete-time problem directly. It is shown below that the robust performance problem for an uncertain discrete-time system can also be solved using finite-dimensional convex optimization.

Consider the discrete-time uncertain system

$$x_{k+1} = A_\Delta x_k + B_\Delta w_k + B_{2\Delta} u_k \tag{15}$$

$$z_k = C_\Delta x_k + D_\Delta w_k + D_{2\Delta} u_k \tag{16}$$

$$y_k = x_k, \tag{17}$$

where again A_Δ, B_Δ, etc are assumed to be affine matrix functions of $\Delta \in \mathbf{\Delta}$ and $\mathbf{\Delta}$ is the compact set defined in Section 3.

To derive the discrete-time results, we need a discrete-time $\mathcal{H}_\infty$ norm characterization (Doyle *et al.*, 1991).

Lemma 13. Let $G(z) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stable discrete-time system. Then $\|G(z)\|_\infty < 1$ if and only if there exists a nonsingular matrix T such that

$$\left\| \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I \end{bmatrix} \right\| < 1$$

or equivalently there exists an $X > 0$ such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}' \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0.$$

Note that the matrix inequality characterization of a bounded real function is equivalent to the following Riccati inequality characterization: there exists an $X > 0$ such that $I - D'D - B'XB > 0$ and

$$A'XA - X + (B'XA + D'C)'(I - D'D - B'XB)^{-1} \times (B'XA + D'C) + C'C < 0.$$

Now we can introduce the definition of strongly robust $\mathcal{H}_\infty$ performance criterion for a discrete-time system.

Definition 14. The time-varying uncertain dynamical system described by equations (15) and (16) with $u = 0$ is said to satisfy strongly robust $\mathcal{H}_\infty$ performance criterion if there exists a constant symmetric matrix $X > 0$ such that

$$\begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix}' \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}$.

We also need a simple matrix fact, which follows from the standard Schur complement result, to prove our results.

Lemma 15. Let A be any square matrix. Then $P > 0$ and $A'PA - P < 0$ if and only if

$$\begin{bmatrix} -P^{-1} & -A \\ -A' & -P \end{bmatrix} < 0.$$

Now the following result is obvious.

Theorem 16. Suppose A_Δ, B_Δ, C_Δ, and D_Δ are affine matrix functions of $\Delta \in \mathbf{\Delta}$. Then the uncertain system described by equations (15) and (16) with $u = 0$ satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if there exists an $X > 0$ such that

$$\begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix}' \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{\text{vex}}$.

The state feedback results can also be obtained analogously.

Theorem 17. There exists a state feedback controller such that the system described by equations (15) and (16) satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion if and only if there exist a matrix W and a matrix $Y = Y' > 0$ such that

$$\begin{bmatrix} -Y & 0 & -(A_\Delta Y + B_{2\Delta} W) & -B_\Delta \\ 0 & -I & -(C_\Delta Y + D_{2\Delta} W) & -D_\Delta \\ -(YA'_\Delta + W'B'_{2\Delta}) & (YC'_\Delta + W'D'_{2\Delta}) & -Y & 0 \\ -B'_\Delta & -D'_\Delta & 0 & -I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{\text{vex}}$. Moreover, the state feedback controller can be taken at a constant gain as

$$F = WY^{-1}.$$

Proof. ($\Rightarrow$) Note that the discrete version of Theorem 4 holds and can be proved along the same lines. Hence it can be assumed without loss of generality that there exists a F such that the closed-loop system with $u_k = Fx_k$ satisfies the strongly robust $\mathcal{H}_\infty$ performance criterion. The closed-loop system can be written as

$$x_k = (A_\Delta + B_{2\Delta} F)x_k + B_\Delta w_k$$
$$z_k = (C_\Delta + D_{2\Delta} F)x_k + D_\Delta w_k.$$

By definitions and Theorem 16, there exists an $X = X' > 0$ such that

$$\begin{bmatrix} A_\Delta + B_{2\Delta}F & B_\Delta \\ C_\Delta + D_{2\Delta}F & D_\Delta \end{bmatrix}' \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\Delta + B_{2\Delta}F & B_\Delta \\ C_\Delta + D_{2\Delta}F & D_\Delta \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$$

for all $\Delta \in \mathbf{\Delta}_{\text{vex}}$. Now using Lemma 15, we have

$$\begin{bmatrix} -\begin{bmatrix} X^{-1} & 0 \\ 0 & I \end{bmatrix} & -\begin{bmatrix} A_\Delta + B_{2\Delta}F & B_\Delta \\ C_\Delta + D_{2\Delta}F & D_\Delta \end{bmatrix} \\ -\begin{bmatrix} A_\Delta + B_{2\Delta}F & B_\Delta \\ C_\Delta + D_{2\Delta}F & D_\Delta \end{bmatrix}' & -\begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} < 0.$$

Now the result follows by pre- and post-multiplying the above inequality by

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & X^{-1} & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and defining $Y := X^{-1}$, $W = FX^{-1}$.

($\Leftarrow$) This follows easily by reversing the above steps and using the state feedback gain $F = WY^{-1}$. □

8. *Conclusions*

In this paper we considered the robust performance analysis and state feedback synthesis for a certain class of uncertain systems with time-varying parameter uncertainties. A notion of robust performance for systems with time-varying uncertainties—strongly robust $\mathcal{H}_\infty$ performance criterion—was introduced.

It was shown that for the class of uncertain systems considered in this paper the strongly robust $\mathcal{H}_\infty$ performance problem can be formulated as a convex programming problem and gives, in general, less conservative results than those obtained using the scaled small gain condition. Parallel results were also obtained for discrete-time systems. However, the strongly robust $\mathcal{H}_\infty$ performance problem is still unsolved for systems with general linear fractional uncertainty although we could immediately generalize our results to some classes of systems with linear fractional uncertainty as we did at the end of Section 3.

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