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Kim, Oleksiy S.

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Oleksiy S. Kim

Abstract—Theoretically, the minimum radiation quality factor Q of an isolated resonance can be achieved in a spherical electrically small antenna by combining TM_{1m} and TE_{1m} spherical modes, provided that the stored energy in the antenna spherical volume is totally suppressed. Using closed-form expressions for the stored energies obtained through the vector spherical wave theory, it is shown that a magnetic-coated metal core reduces the internal stored energy of both TM_{1m} and TE_{1m} modes simultaneously, so that a self-resonant antenna with the Q approaching the fundamental minimum is created. Numerical results for a multiarm spherical helix antenna confirm the theoretical predictions. For example, a 4-arm spherical helix antenna with a magnetic-coated perfectly electrically conducting core ($ka = 0.254$) exhibits the Q of 0.66 times the Chu lower bound, or 1.25 times the minimum Q.

Index Terms—Chu lower bound, circular polarization, electric dipole, electrically small antennas, fundamental limitations, magnetic dipole, quality factor.

I. INTRODUCTION

THE radiation quality factor Q is one of the key characteristics that measures the performance of an electrically small antenna. It relates the total stored energy W—the sum of electric W_E and magnetic W_H stored energies—to the radiated power P and is defined at a resonance as

$$Q = \omega \frac{W}{P} = 2\omega \frac{\max(W_E, W_H)}{P}. \tag{1}$$

The second expression applies to the case when an antenna is not self-resonant, but tuned to the resonance by an ideal lumped component (capacitor or inductor).

The fractional bandwidth of an isolated resonance is inverse proportional to the Q, and since the latter is bounded from below, the former appears to be limited from above. This, however, does not preclude an antenna to have a wider fractional bandwidth than dictated by the limit, if several resonances occur close enough to each other [1]. On the other hand, the Q of a passive linear time-invariant antenna can never be less than a fundamental lower bound [2], and therefore there have always been intriguing questions about how low the Q can theoretically be for a given antenna size and how close this Q can be approached in practice.

A. Single-Mode Q

Wheeler and Chu were the first to systematically address the problem. In his paper from 1947 [3], Wheeler considered electrically small cylindrical antennas as circuit elements—an electric dipole antenna as a capacitor and a magnetic dipole antenna

as an inductor. For these antennas, he derived simple expressions for the "radiation power ratio," which in fact is the inverse of Q. Chu attacked the problem differently. He introduced an imaginary minimum sphere of radius a enclosing an antenna and disregarded the fields inside (and, thus, the internal stored energy) by assuming "the most favorable conditions to exist inside the sphere" [4]. The external fields were represented by the vector spherical wave functions—TM_{mn} and TE_{mn}. Then, an equivalent circuit was defined for each spherical wave, and the corresponding quality factors Q_n were obtained, though with some approximations. Since, the dipole modes ($n = 1$) have the lowest Q, Q_1 represents the lower bound on Q for a single-mode antenna. Later, in 1996, McLean showed how Q_1 can be exactly obtained from Chu's equivalent circuit [5].

As a matter of fact, Collin and Rothschild presented the exact expression for the lower bound already in 1964 [6]. Contrary to Chu, they did not consider the equivalent circuits, but operated directly with fields and energy densities. Thus, the Chu lower bound, as we know it today, was obtained as

$$Q_{\mathrm{Chu}} = \frac{1}{(ka)^3} + \frac{1}{ka} \tag{2}$$

where k is the free-space wave number. Collin and Rothschild [6] also gave exact expressions for the Q of higher order modes ($n > 1$).

A realistic Q accounting for the stored energy inside the minimum sphere is inevitably larger than the Q_{Chu}. In 1958, Wheeler derived the Q for vanishingly small spherical antennas filled with a magnetodielectric material with relative permittivity ε_r and permeability μ_r as [7]

$$Q = \left(1 + \frac{\varepsilon_r}{2}\right) \frac{1}{(ka)^3} \tag{3a}$$

$$Q = \left(1 + \frac{2}{\mu_r}\right) \frac{1}{(ka)^3} \tag{3b}$$

for electric and magnetic dipole antennas, respectively. Expressions (3) have several important implications:

1) for electric dipole antennas with air core, $Q \rightarrow 1.5 Q_{\mathrm{Chu}}$, as $ka \rightarrow 0$;
2) for magnetic dipole antennas with air core, $Q \rightarrow 3.0 Q_{\mathrm{Chu}}$, as $ka \rightarrow 0$;
3) dielectric material with $\varepsilon_r > 1$ increases the Q of electric dipole antennas;
4) magnetic material of infinitely high permeability allows a vanishingly small magnetic dipole antenna to reach the Chu lower bound.

In 2006, using the Chu equivalent circuits extended for the antenna internal region, Thal numerically computed the exact Q for air-core spherical antennas of finite size excited by a surface electric current [8]. However, exact explicit expressions for the Q of such antennas were provided later by Hansen and Collin in 2009 [9]. Numerical and experimental results reported for

various air-core electric and magnetic spherical dipole antennas are in full agreement with the theoretical bounds [10]–[15].

The resonances in a finite-size antenna core caused by finite internal fields may be of little importance in the air-core case (the first cavity resonance is at $k_s a \equiv ka\sqrt{\varepsilon_r \mu_r} = 2.744$ for TM_{1m} modes), but they start appearing in the electrically small antenna regime (usually, $ka < 0.5$), as the permittivity and/or permeability of the core increases. The cavity resonances are characterized by zero radiated power, and thus infinite Q. Therefore, Wheeler's formulas (3) are not valid for finite-size antennas. Instead, exact expressions, as derived in [16] for magnetic dipole (TE_{1m} mode) antennas and in [17] for general TM_{nm}- and TE_{nm}-mode antennas, must be applied. Moreover, as shown in [16] and [18], the infinite permeability is no longer optimal for a solid homogeneous core of a finite-size magnetic dipole antenna. In this case, the optimal value of permeability is limited by the first cavity resonance to

$$\mu_r = \frac{1}{\varepsilon_r} \frac{(2.816)^2}{(ka)^2} \tag{4}$$

which corresponds to the lower bound [18]

$$Q = \{1 + 0.366(ka)^2\} Q_{\mathrm{Chu}}. \tag{5}$$

In general, a solid homogeneous core cannot be regarded as an optimal distribution of material inside an antenna. For example, magnetic coating on the surface of a metal core can have much higher permeability than a solid core, before the first cavity resonance occurs. Consequently, the Q of a magnetic dipole antenna with such a core can drop below the bound (5) and even reach the Chu lower bound in the limit of infinitely thin coating and infinitely high permeability [19], [20], note, for arbitrary ka. Remarkably, the same limit applies to electric dipole antennas as well [21].

In fact, the metal core is not essential for an antenna to approach the Chu lower bound. A thin shell of high-permeable magnetic material performs equally well [22]–[24].

B. Dual-Mode Q

Although the Chu lower bound (2) has proven reachable by both electric and magnetic dipole antennas, it is not the lowest Q an antenna of a given size ka can achieve. Already Chu noted that a combination of TM_{1m} and TE_{1m} modes potentially yields the Q that is approximately one-half of the Q_{Chu} (again, neglecting the internal stored energy) [4]. In 1996, McLean presented the exact lower bound for a dual-mode TM-TE antenna as

$$Q_{\min} = \frac{1}{2}\left\{ \frac{1}{(ka)^3} + \frac{2}{ka} \right\}. \tag{6}$$

These results can also be obtained from the general expressions by Fante [25], who also noted that this is the lowest Q that can be achieved by a passive linear time-invariant antenna.[1]

When electric and magnetic dipole modes radiate equal power, electric and magnetic stored energy balance each other, and the antenna becomes self-resonant [28]. However, the balance does not hold if the internal energy is taken into account,

because for the same radiated power, the TE_{1m} mode stores four times more energy inside the minimum sphere than the TM_{1m} mode does (for an air-core antenna and $ka \to 0$). According to (3), the balance can be restored by a solid magnetodielectric core with $\varepsilon_r \mu_r = 4$ [7]. Of all possible combinations, $\varepsilon_r = 1$ $\mu_r = 4$ is the most advantageous, since it provides the lowest $Q = 0.75 Q_{\mathrm{Chu}}$, as $ka \to 0$.

Obviously, an even more advantageous core for a dual-mode antenna is ensuring no stored energy inside the minimum sphere for both TM_{1m} and TE_{1m} modes simultaneously. A magnetic-coated metal sphere is the core that can do the job [29].

This paper shows how the Q of a dual-mode TM–TE spherical antenna can be minimized using a material-coated perfectly electrically conducting (PEC) core. In Section II, the theory based on the vector spherical wave functions is presented. Optimal geometrical and material parameters of the antenna leading to the lowest possible Q are discussed in Section III. It is shown that the minimum $Q_{\min}$ can be reached in the limit of infinitely thin coating with infinitely high permeability. An expression for the excitation current for a dual-mode TM_{10}–TE_{10} antenna is derived in Section IV, and a numerical example for a multi-arm spherical helix antenna with an optimally designed magnetic-coated PEC core is given in Section V. Finally, the summary and conclusions are provided in Section VI.

In this work, the vector spherical wave functions in notation by Hansen [30] are used. The time factor $\exp(-i\omega t)$, where ω denotes the angular frequency and t is the time variable, is assumed and suppressed throughout the paper.

II. THEORY

The geometry of the problem is depicted in Fig. 1. It consists of a PEC spherical core of radius b covered by a uniform layer of a homogeneous isotropic and lossless material. The outer radius of the material shell is a, and the relative permittivity and permeability of the shell material are ε_r and μ_r, respectively. The Cartesian coordinate system (x, y, z) and the corresponding spherical coordinate system (r, θ, ϕ) defined in the usual manner so that the origin coincides with the core center. The antenna is excited by a superposition of two electric current densities $\mathbf{J}_{\mathrm{TM}_{nm}}$ and $\mathbf{J}_{\mathrm{TE}_{nm}}$ radiating the TM_{nm} and TE_{nm} modes, respectively. Since we are aimed at the minimum Q, our consideration is restricted to $n = 1$ modes, and thus m can take values 0 and ± 1. The energy and power relations are independent on m, and therefore it sufficient to consider $m = 0$ modes. Thus, the excitation currents are (see the Appendix for details)

$$\mathbf{J}_{\mathrm{TM}_{10}} = J_{\mathrm{TM}_{10}} \hat{\mathbf{a}}_\theta \sin\theta \tag{7a}$$

$$\mathbf{J}_{\mathrm{TE}_{10}} = J_{\mathrm{TE}_{10}} \hat{\mathbf{a}}_\phi \sin\theta \tag{7b}$$

where $\hat{\mathbf{a}}_\theta$ and $\hat{\mathbf{a}}_\phi$ are the unit vectors, $J_{\mathrm{TM}_{10}}$ and $J_{\mathrm{TE}_{10}}$ are the amplitudes of the current density for TM_{10} and TE_{10} modes, respectively.

The radiated fields and the fields in the shell are superposition of the corresponding TM_{10} and TE_{10} fields with modal expansion coefficients [20, eq. (4)], [21, eq. (4)]

$$C_{\mathrm{TM}_{10}}^- = J_{\mathrm{TM}_{10}} ia \frac{1}{\sqrt{\eta_s}} \frac{4\sqrt{\pi}}{\sqrt{6}} \left\{(ka) h_1^{(1)}(ka)\right\}' I_{\mathrm{TM}_{10}}^{-1} \tag{8a}$$

$$C_{\mathrm{TM}_{10}}^+ = J_{\mathrm{TM}_{10}} ia \frac{\sqrt{\eta}}{\eta_s} \frac{4\sqrt{\pi}}{\sqrt{6}} \left\{(k_s a) z_1^{(6)}(k_s a)\right\}' I_{\mathrm{TM}_{10}}^{-1} \tag{8b}$$
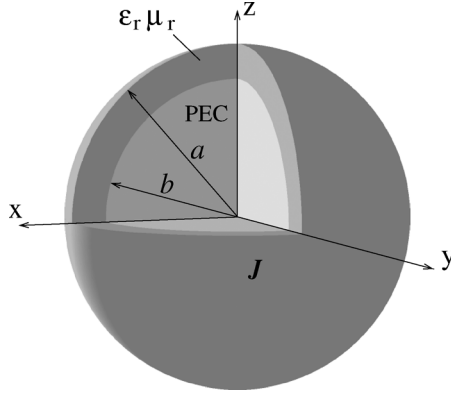
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[1] This is also shown by Collin [26] and Geyi [27].

Fig. 1. A spherical antenna with a material-coated PEC core. The excitation is a time-harmonic electric surface current density **J** distributed over the surface of the antenna.

$$C_{\text{TE}_{10}}^- = J_{\text{TE}_{10}} ia \frac{\sqrt{\eta_s}}{\eta} \frac{4\sqrt{\pi}}{\sqrt{6}} kah_1^{(1)}(ka) I_{\text{TE}_{10}}^{-1} \qquad (8c)$$

$$C_{\text{TE}_{10}}^+ = J_{\text{TE}_{10}} ia \frac{1}{\sqrt{\eta}} \frac{4\sqrt{\pi}}{\sqrt{6}} k_s a z_1^{(5)}(k_s a) I_{\text{TE}_{10}}^{-1} \qquad (8d)$$

where

$$I_{\text{TM}_{10}} = \frac{\eta}{\eta_s} kah_1^{(1)}(ka) \left\{ (k_s a) z_1^{(6)}(k_s a) \right\}'$$
$$\qquad - k_s a z_1^{(6)}(k_s a) \left\{ (ka) h_1^{(1)}(ka) \right\}' \qquad (9a)$$
$$I_{\text{TE}_{10}} = \frac{\eta_s}{\eta} kah_1^{(1)}(ka) \left\{ (k_s a) z_1^{(5)}(k_s a) \right\}'$$
$$\qquad - k_s a z_1^{(5)}(k_s a) \left\{ (ka) h_1^{(1)}(ka) \right\}'. \qquad (9b)$$

In (8) and elsewhere, the superscripts "+" and "−" denote quantities corresponding to radiated fields in free space and fields in the shell, respectively; η and η_s are the intrinsic admittances of free space and the shell, respectively; and $k_s = \sqrt{\varepsilon_r \mu_r} \, k$. The radial functions $z_1^{(6)}(k_s a)$ and $z_1^{(5)}(k_s a)$, being a linear combination of the spherical Bessel j_1 and Neumann y_1 functions, ensure that the boundary condition on the surface of the PEC sphere is satisfied for TM_{10} and TE_{10} modes, respectively:

$$z_1^{(6)}(k_s r) = y_1(k_s r) \{ (k_s b) j_1(k_s b) \}'$$
$$\qquad - j_1(k_s r) \{ (k_s b) y_1(k_s b) \}' \qquad (10a)$$
$$z_1^{(5)}(k_s r) = y_1(k_s r) j_1(k_s b) - j_1(k_s r) y_1(k_s b). \qquad (10b)$$

The first derivative of the function $f(x)$ with respect to its argument x is denoted by the prime symbol, as $\{f(x)\}'$.

The radiated power for each mode is expressed in terms of modal expansion coefficients of the radiated field C^+ as [30]

$$P_{\text{TM}_{10}} = \frac{1}{2} \left| C_{\text{TM}_{10}}^+ \right|^2, \quad P_{\text{TE}_{10}} = \frac{1}{2} \left| C_{\text{TE}_{10}}^+ \right|^2. \qquad (11)$$

By imposing the condition

$$P_{\text{TM}_{10}} = P_{\text{TE}_{10}} \qquad (12)$$

which is necessary to achieve the minimum Q (6), and using (8b), (8d), and (11), we can derive the relation between the amplitudes of the TM_{10} and TE_{10} excitation currents as

$$J_{\text{TM}_{10}} = \pm J_{\text{TE}_{10}} \frac{\eta_s}{\eta} \frac{k_s a z_1^{(5)}(k_s a)}{\left\{ (k_s a) z_1^{(6)}(k_s a) \right\}'} \frac{I_{\text{TM}_{10}}}{I_{\text{TE}_{10}}} \equiv \gamma J_{\text{TE}_{10}}. \qquad (13)$$

Formally, the condition (12) allows the right part of (13) to be multiplied by a factor $e^{j\xi}$, where ξ is an arbitrary phase. In other words, the phasing between the TM_{10} and TE_{10} excitation currents is not essential to achieve the balance between the respective radiated powers. However, as far as the minimum Q is the goal, $J_{\text{TM}_{10}}$ and $J_{\text{TE}_{10}}$ have to be in phase or opposite in phase ($\xi = 0$ or π), because any other phase shift would require additional reactive energy, and thus, would increase the Q [26], [31].

The balance of the radiated power implies also the balance for the external electric and magnetic stored energies, $W_E^+ = W_H^+$. Indeed, denoting $C^+ \equiv C_{\text{TM}_{10}}^+ = C_{\text{TE}_{10}}^+$, we obtain [20], [21]

$$W_E^+ = W_{E,\text{TM}_{10}}^+ + W_{E,\text{TE}_{10}}^+ = \frac{1}{4\omega} |C^+|^2 \left(\frac{1}{(ka)^3} + \frac{2}{ka} \right)$$
$$W_H^+ = W_{H,\text{TM}_{10}}^+ + W_{H,\text{TE}_{10}}^+ = \frac{1}{4\omega} |C^+|^2 \left(\frac{1}{(ka)^3} + \frac{2}{ka} \right) \quad (14)$$

where $W_{E/H,\text{TM}_{10}}^+$ and $W_{E/H,\text{TE}_{10}}^+$ are external electric/magnetic stored energy of the TM_{10} and TE_{10} modes, respectively.

The internal stored energies do not possess the same symmetry [20], [21]:

$$W_{E/H}^- = \frac{1}{4\omega} \left\{ \left| C_{\text{TM}_{10}}^- \right|^2 \hat{W}_{E/H,\text{TM}_{10}}^- + \left| C_{\text{TE}_{10}}^- \right|^2 \hat{W}_{E/H,\text{TE}_{10}}^- \right\} \qquad (15)$$

with

$$\hat{W}_{E,\text{TM}_{10}}^- = \frac{1}{2} \left\{ L(k_s a) - L(k_s b) + (k_s a) \{ R_1(k_s a) \}'^2 \right\}$$
$$\hat{W}_{H,\text{TM}_{10}}^- = \frac{1}{2} \left\{ L(k_s a) - L(k_s b) - (k_s a) R_2(k_s a) \{ R_1(k_s a) \}' \right\}$$
$$\hat{W}_{E,\text{TE}_{10}}^- = \frac{1}{2} \left\{ K_1(k_s a) - K_1(k_s b) \right\}$$
$$\hat{W}_{H,\text{TE}_{10}}^- = \frac{1}{2} \left\{ K_0(k_s a) - K_0(k_s b) - 2(k_s a) U_1^2(k_s a) \right\}$$

where

$$L(k_s r) = R_1(k_s r) \left[(k_s r) R_1(k_s r) - R_2(k_s r) \right]$$
$$R_l(k_s r) = (k_s r) y_l(k_s r) \{ (k_s b) j_1(k_s b) \}'$$
$$\qquad - (k_s r) j_l(k_s r) \{ (k_s b) y_1(k_s b) \}'$$
$$K_l(k_s r) = (k_s r)^3 \left[U_l^2(k_s r) - U_{l-1}(k_s r) U_{l+1}(k_s r) \right]$$
$$U_l(k_s r) = y_l(k_s r) j_1(k_s b) - j_l(k_s r) y_1(k_s b).$$

Consequently, to make the antenna self-resonant, another condition,

$$W_E^- = W_H^-, \qquad (16)$$

in addition to (12), must be satisfied. Upon substituting (15) and (8), (16) reduces to

$$\frac{(ka)^2 \left| h_1^{(1)}(ka) \right|^2}{(k_s a)^2 \left| z_1^{(5)}(k_s a) \right|^2} \left(W_{\mathrm{E,TE_{10}}}^{-} - W_{\mathrm{H,TE_{10}}}^{-} \right)$$

$$= \frac{\left| \left\{ (ka) h_1^{(1)}(ka) \right\}' \right|^2}{\left| \left\{ (k_s a) z_1^{(6)}(k_s a) \right\}' \right|^2} \left(W_{\mathrm{H,TM_{10}}}^{-} - W_{\mathrm{E,TM_{10}}}^{-} \right). \quad (17)$$

Finally, the Q is found by using (14), (15), and (11) together with (8) in (1), so that

$$Q = 2\omega \frac{\max \left(W_{\mathrm{E}}^{+} + W_{\mathrm{E}}^{-}, W_{\mathrm{H}}^{+} + W_{\mathrm{H}}^{-} \right)}{P_{\mathrm{TM_{10}}} + P_{\mathrm{TE_{10}}}} \equiv \max(Q_{\mathrm{E}}, Q_{\mathrm{H}}). \quad (18)$$

III. MINIMUM Q PARAMETERS

A standard way to minimize the ratio $Q/Q_{\min}$ would be to take the derivative of it with respect to an antenna parameter, such as μ_{r} or b/a, set the result equal to zero, and solve the equation. This approach has been successfully applied to electric [29] and magnetic [21] dipole antennas, that is, to antennas with clearly dominating electric or magnetic energy. In the present case, however, it is expected that the minimum of Q is located in the vicinity of the energy balance, where the (18) becomes nondifferentiable due to the max function. The problem is illustrated by an example in Fig. 2(a), where the $Q/Q_{\min}$, $Q_{\mathrm{E}}/Q_{\min}$, and $Q_{\mathrm{H}}/Q_{\min}$ are plotted as functions of μ_{r} for $ka = 0.5$, $b/a = 0.8$, and $\varepsilon_{\mathrm{r}} = 1$. It is seen that the minimum of $Q/Q_{\min}$ is found at a salient point, that is, at the resonance [Fig. 2(b)]. In Fig. 2(b), it is also observed that these are the odd (radiating) resonances that yield low Q, whereas the even resonances, corresponding to the cavity resonances in the core, make the Q go to infinity.

Obviously, we cannot be certain that $Q/Q_{\min}$ is always minimized at the radiating resonance. Therefore, we compare parameters found by a direct minimum search in (18) and by solving (17) in Fig. 3(a), where $k_s a$ found by both methods in the vicinity of the first four odd (radiating) resonances are plotted versus b/a for $ka = 0.25$. We can see that the optimal and resonant parameters coincide for the first resonance ($p = 1$) in the entire range of b/a, whereas for higher odd resonances ($p = 2, 3, 4$) a slight deviation occurs for small radii of the PEC sphere $b/a < 0.2$. Nevertheless, the corresponding ratios $Q/Q_{\min}$ shown in Fig. 3(b) clearly illustrate that the lowest $Q/Q_{\min}$ can always be achieved by the resonant parameters, and thus with a self-resonant antenna, for any b/a: for $b/a \leq 0.13$, with $p = 1$ parameters, and then, for $b/a > 0.13$, with parameters for higher radiating resonances ($p > 1$). As the thickness of the material shell decreases and b/a goes to 1, the Q approaches $Q_{\min}$ for all radiating resonances with $p > 1$; among them, the lowest one ($p = 2$) is preferable, since it is based on the lowest $k_s a$, and thus, the lowest μ_{r}.

Note that the optimal parameters do not depend separately on ε_{r} and μ_{r}, but only on the product $\varepsilon_{\mathrm{r}} \mu_{\mathrm{r}}$, and thus sufficiently defined by a value of $k_s a$ for given ka and b/a. Consequently, if the permittivity of the shell ε_{r} is higher than 1, the optimal
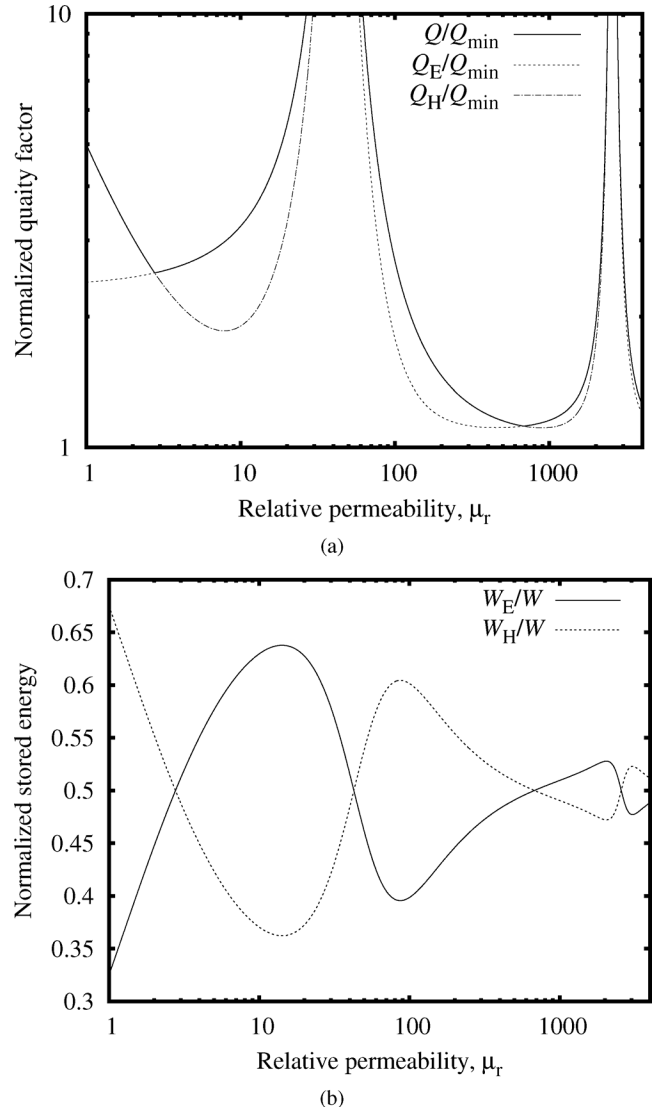

(a)

(b)

Fig. 2. Quality factors and stored energies as functions of μ_{r} ($\varepsilon_{\mathrm{r}} = 1$) for fixed $ka = 0.25$: (a) Quality factors normalized to $Q_{\min}$ and (b) electric and magnetic stored energies normalized to the total stored energy.

μ_{r} should be decreased to keep $k_s a$ unchanged, and the corresponding Q increases.

As shown in Fig. 4(a), the variation of resonant $k_s a$ with the antenna electrical size ka is rather marginal for $p > 1$, whereas for $p = 1$, the $k_s a$ scales almost linearly with ka. The latter is due to the fact that the first resonance is located in the vicinity of $\varepsilon_{\mathrm{r}} \mu_{\mathrm{r}} \approx 4$ for $ka < 0.5$, which is also in agreement with (3) when $b/a \to 0$. The corresponding variation of the ratio $Q/Q_{\min}$ is shown in Fig. 4(b), where we can see that for $p = 1$ and 2 (and for other p, as well), the lowest achievable $Q/Q_{\min}$ depends weakly on the antenna electrical size ka.

Thus, from Fig. 4(a), given the antenna electrical size ka and radius of the PEC core b, we can always find optimal resonant material parameters of the shell $\varepsilon_{\mathrm{r}} \mu_{\mathrm{r}} = (k_s a/ka)^2$, ensuring the lowest $Q/Q_{\min}$. If the material parameters ε_{r} and μ_{r} are fixed, which is a more realistic situation in practice, the optimal radius of the PEC core b can also be determined from Fig. 4(a) for a given $k_s a = ka\sqrt{\varepsilon_{\mathrm{r}} \mu_{\mathrm{r}}}$, but only if $k_s a > 3.65$ so that the radiating resonance $p = 2$ can be used. This resonance is always
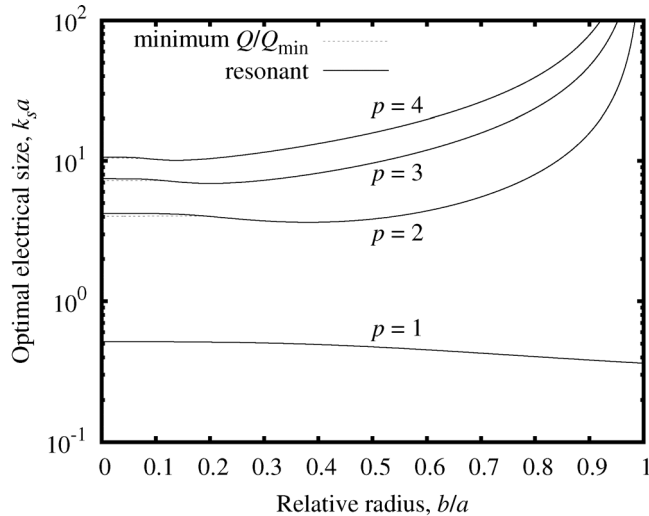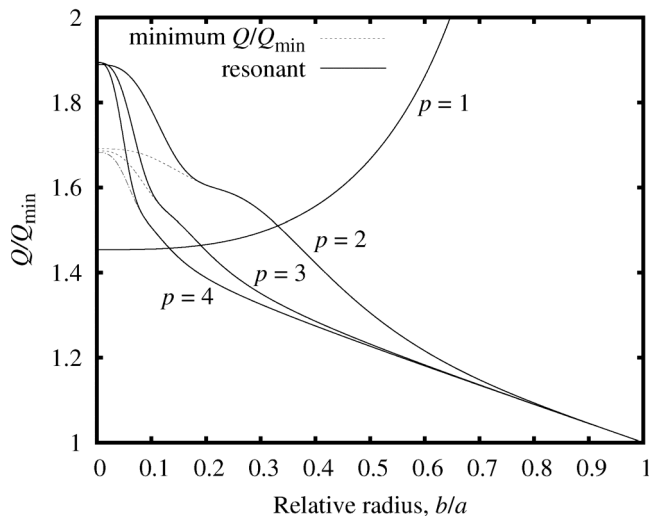
Fig. 3. (a) Optimal and resonant electrical size $k_s a$ and (b) the corresponding lowest achievable ratio $Q/Q_{\min}$ in the vicinity of first four odd (radiating) resonances $p = 1, 2, 3, 4$ for $ka = 0.25$ and $\varepsilon_r = 1$.

Fig. 4. (a) Resonant electrical size $k_s a$ and (b) the corresponding lowest achievable ratio $Q/Q_{\min}$ for $ka = 0.1$, 0.25, and 0.5; $\varepsilon_r = 1$.

preferable to higher resonances ($p > 2$), because it corresponds to the largest b/a, and thus to the lowest $Q/Q_{\min}$, as illustrated in Fig. 5 for $ka = 0.25$.
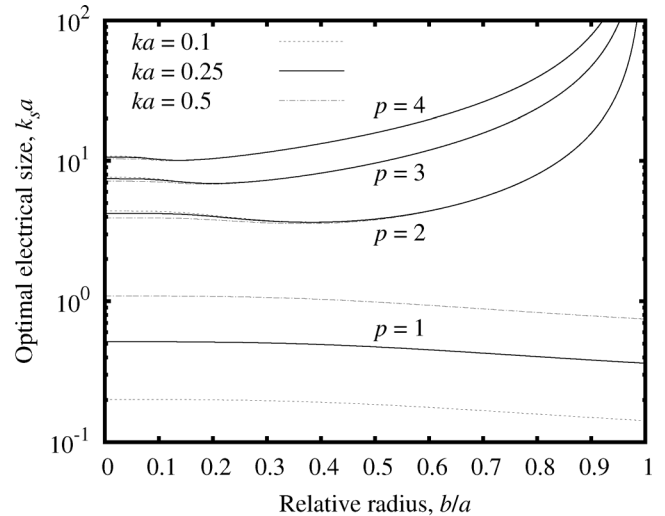
If the parameters are such that $k_s a < 3.65$, then either a solid material core or no core at all can be more advantageous. However, in this case, either the condition of equal radiated power for the TE_{10} and TM_{10} modes (12) or the self-resonant condition (16) have to be discarded in order to achieve a low $Q/Q_{\min}$ (Fig. 5).
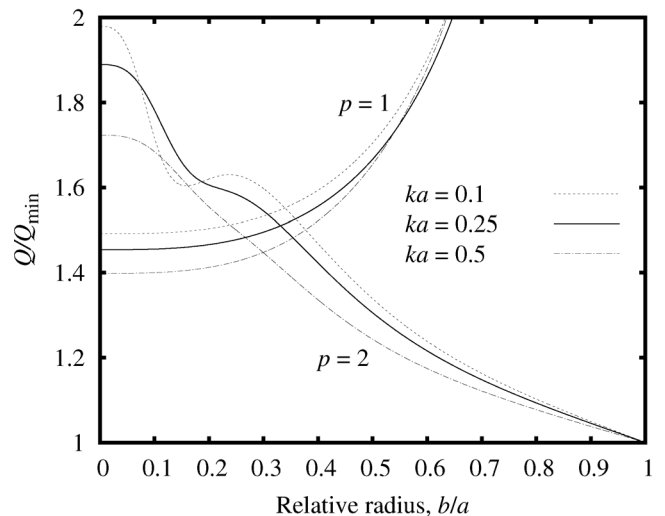
IV. DUAL-MODE TM_{10}–TE_{10} SPHERICAL ANTENNA

The total current density for a combination of TM_{10} and TE_{10} modes on the surface of a spherical antenna can be written using (7) and (13) as

$$\mathbf{J} = J_0 \sin\theta (\gamma \hat{\mathbf{a}}_\theta + \hat{\mathbf{a}}_\phi) \qquad (19)$$

where J_0 is an amplitude and γ is as defined in (13). Such an antenna yields perfect circular polarization—left-hand or right-

hand, depending on the sign of γ—in all directions, except the poles, where the radiation is zero.[2]

In (19), the terms in the parentheses in fact describe a set of spherical helices (Fig. 6) as

$$\theta = 2\tan^{-1}\left(e^{\gamma(\phi+c)}\right) \qquad (20)$$

where c is an arbitrary real constant. The argument ϕ runs from $-\infty$ to ∞, where θ asymptotically approaches 0 and π, respectively. This implies that each helix contains an infinite number of turns. In practice, the helices can be truncated by setting the range of θ from θ_0 to $\pi - \theta_0$, which corresponds to $\phi \in [\ln(\tan(\theta_0/2))/\gamma + c : -\ln(\tan(\theta_0/2))/\gamma + c]$. Then, the number of turns N_t in each helix is

$$N_t = -\frac{\ln(\tan(\theta_0/2))}{\pi\gamma}. \qquad (21)$$

[2]The maximum directivity $D = 1.5$ occurs at $\theta = 90°$ Consequently, for the antenna parameters ensuring $Q \to Q_{\min}$, the ratio D/Q approaches the limit $1.5/Q_{\min}$ found in [4], [27] for omnidirectional antennas radiating dipole modes.
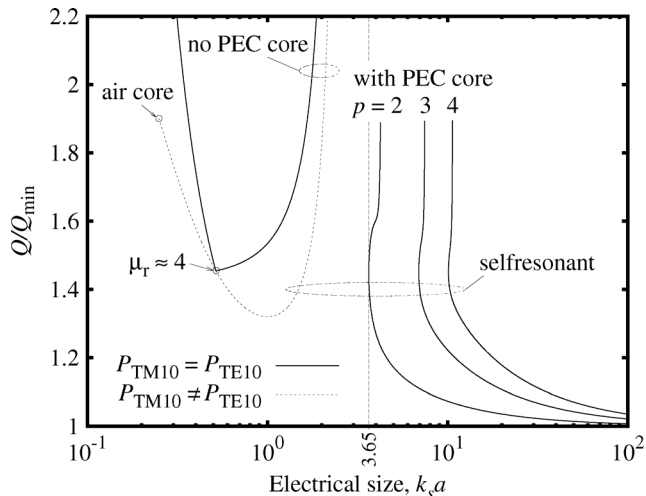
Fig. 5. Lowest achievable ratio $Q/Q_{\min}$ corresponding to the optimal resonant b/a selected in Fig. 4(a) for a given $k_s a$. The ratio $Q/Q_{\min}$ is also shown for the case of a solid material core (no PEC core). In all cases, $ka = 0.25$ and $\varepsilon_r = 1$.
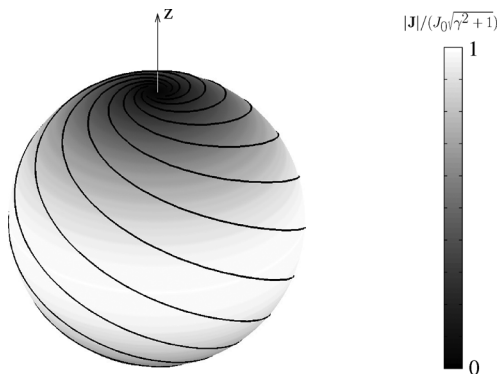

Fig. 6. Distribution of the electric current density (19) on the surface of a dual-mode TM_{10}–TE_{10} spherical antenna. The amplitude is encoded in gray scale and the current paths are shown by lines (number of lines is arbitrary).

If we put conducting strips or wires along these helices, we obtain a multi-arm spherical helix antenna. This antenna resembles similar folded spherical helix antennas reported in [10], [11], [14], [29], [32], but with a different winding governed by (20), which has neither z-uniform pitch, as in [10] and [11], nor θ-uniform pitch, as in [14], [29], and [32].

V. SPHERICAL HELIX ANTENNA WITH A MATERIAL-COATED PEC CORE

Here, we consider a numerical example, for which we choose the outer radius of the core to be $a = 39$ mm and the material of the coating to be perfectly magnetic with relative permeability $\mu_r = 900$ and permittivity $\varepsilon_r = 1$. At 300 MHz, the electrical radius of the core becomes $ka = 0.245$, or $k_s a = \sqrt{\varepsilon_r \mu_r}\, ka = 7.351$. Given this, from the data in Fig. 4(a), we can determine the resonant $b/a = 0.782$ for $p = 2$, which corresponds to the radius of the PEC core $b = 30.5$ mm. Finally, from (13), we find $\gamma = 0.26$.

To excite the antenna, we wind wire helices around the core according to (20) at the radius $a_0 = 40$ mm, as sketched in Fig. 7; the diameter of the wires is 1 mm. An offset of 1 mm between the wire center and the outer surface of the coating was
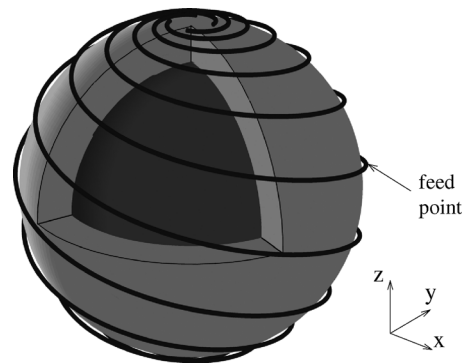

Fig. 7. A dual-mode TM_{10}–TE_{10} spherical helix antenna composed of four wire helices and a magnetic-coated PEC core. (Part of the coating is removed to show the PEC core.)

TABLE I
RESULTS FOR THE MULTI-ARM SPHERICAL HELIX ANTENNA WITH A MAGNETIC-COATED PEC CORE

N_a	N_t	R_0	Q	Q/Q_{Chu}	$Q/Q_{\min}$
2	3.26	41	87.56	1.35	2.54
3	3.30	89	45.84	0.71	1.33
4	3.32	196	42.97	0.66	1.25

introduced to ensure numerical stability of the surface integral equation technique [33] employed to solve the problem. Thus, an overall electrical radius of the antenna becomes $ka_0 \equiv k(a + 1.5\text{ mm}) = 0.254$.

By terminating the helices at $\theta_0 = 7°$, we limit the number of turns to $N_t = 3.43$, as determined from (21). However, the final N_t has to be fine-tuned to correct for the finite diameter of the wires and the air-gap between the wires and the core.

Table I summarizes the results for two, three and four helices, or helical arms ($N_a = 2, 3, 4$). We can see that the number of turns N_t obtained in each case is very close to the theoretically predicted value. The input resistance R_0 varies between 41 and 196 Ω, depending on the number of arms. As the number of arms increases, the wire current approximation of the ideal excitation current distribution (19) improves and the Q approaches the $Q/Q_{\min}$ limit. It should be noted that the bounds Q_{Chu} and $Q_{\min}$ in Table I are calculated for the overall antenna electrical size $ka_0 = 0.254$.

The frequency variation of the input impedance and the corresponding reflection coefficient (when matched at the resonance) for the four-arm configuration are shown in Fig. 8(a) and (b), respectively. A clear resonance is observed at 300 MHz with the fractional bandwidth of 1.5% at the -10-dB level and 2.7% at the -6-dB level of the reflection coefficient.

The radiation pattern is omnidirectional [Fig. 9(a)]—typical for an electrically small dipole antenna. The maximum directivity of 1.76 dBi occurs in the $\theta = 90°$ plane with the azimuthal variation within ± 0.06 dB due to the coarse approximation of the ideal excitation (19) by the wire currents. The polarization is circular over the entire far-field sphere except the polar regions, where the radiation is low anyway [Fig. 9(b)]. The axial ratio in the $\theta = 90°$ plane averaged over ϕ is -0.33 dB with the variation ± 0.02 dB.

VI. DISCUSSION

As the Chu lower bound (2) defines the lowest Q for a single-mode antenna, the minimum Q (6) sets the lower bound on ra-
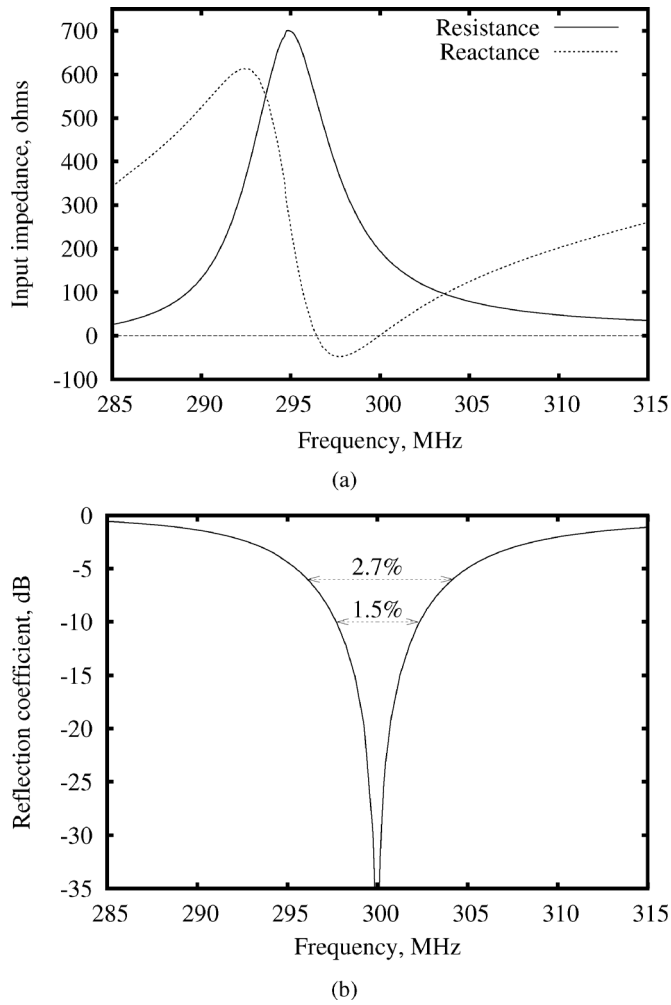
Fig. 8. (a) Input impedance and (b) reflection coefficient for the four-arm spherical helix antenna with a magnetic-coated PEC core.

Fig. 9. (a) Radiation pattern and (b) axial ratio for the four-arm spherical helix antenna with a magnetic-coated PEC core.

diation Q for any linear passive time-invariant lossless antenna. To achieve the minimum Q, an antenna must satisfy certain conditions, as follows.

1) It must excite at least one electric dipole mode (TM$_{10}$ or TM$_{11}$) and one magnetic dipole mode (TE$_{10}$ or TE$_{11}$).

2) The amplitudes of the excitation currents must be related by the factor γ, as defined in (13). The latter must be real, because any phase shift other than 0° or 180° introduces extra stored energy into the system, and thus increases the Q [26], [31]. If an antenna, besides being excited as prescribed above, also has the moments of the electric and magnetic dipoles aligned, its radiation is perfectly circular over the entire far-field sphere with the sense of polarization (left-hand or right-hand) depending on the sign of γ.

3) There must be no energy stored inside the antenna minimum sphere, which implies that 1) electric and magnetic dipole modes, being considered individually, both reach the Chu lower bound and 2) the antenna becomes self-resonant.

The exact theory presented in this paper shows that condition 3) can be satisfied in a spherical antenna having a PEC spherical core coated with infinitely thin layer of magnetic material of infinitely high permeability and excited by an electric current density impressed on the outer surface of the core. With finite
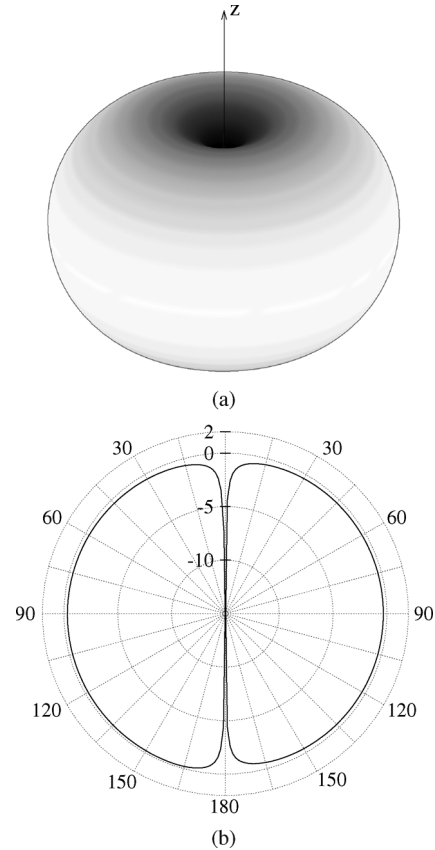
values of the coating thickness and permeability, the Q does not reach the $Q_{\min}$ but still can closely approach it. Using closed-form expressions for the stored energies, radiated power, and the Q, the optimal parameters of the coating minimizing the Q and ensuring a self-resonant configuration are found and presented in Fig. 4(a). Given an antenna electrical size ka and the relative radius of the PEC core b/a, the optimal material parameters of the coating can be found from the plot, as $\varepsilon_r \mu_r = (k_s a)^2 / (ka)^2$. Out of a variety of materials satisfying the obtained value $\varepsilon_r \mu_r$, the one with the lowest permittivity ε_r is preferable, since it ensures the lowest stored energy for both electric and magnetic dipole modes [20], [21], and thus the lowest Q. Alternatively, an optimum radius of the PEC core b can be determined from the same Fig. 4(a) for given ka and $k_s a = ka\sqrt{\varepsilon_r \mu_r}$.

The relative importance of the PEC core in reducing the internal stored energy for electric and magnetic dipole modes was discussed in [21] and [24], respectively. In both cases, an infinitely thin shell of infinitely permeable magnetic material reduces the internal stored energy to zero with and without PEC core. However, a central metal core has a number of other advantages. First, as noted in [24], it ensures mechanical robustness of the antenna. Second, if made hollow, it provides a screened volume inside the antenna that can room various devices, such as batteries, transmitters–receivers, integrated circuits, and sensors.

The multi-arm spherical helix antenna with a magnetic-coated PEC core, presented as a numerical example, illustrates how a minimum Q antenna can be implemented. The antenna is of the overall electrical size $ka_0 = 0.254$, and its radiation quality factor is $Q = 0.66Q_{\text{Chu}} = 1.25Q_{\min}$. In

practice, the Q can be further improved by placing thin conducting strips right on the surface of the coating, for example, by direct writing with conductive ink [34] or by direct transfer patterning [35].

APPENDIX
EXCITATION CURRENTS FOR TM_{10} AND TE_{10}
SPHERICAL MODES

An excitation current for the TM_{10} spherical mode must satisfy the following boundary condition on the surface of the spherical antenna (Fig. 1):

$$\mathbf{J}_{TM_{10}}(\theta, \phi) = \hat{\mathbf{a}}_r \times \left(\mathbf{H}^+_{TM_{10}} - \mathbf{H}^-_{TM_{10}} \right) \qquad (22)$$

where $\hat{\mathbf{a}}_r$ is the radial unit vector and $\mathbf{H}^{\pm}_{TM_{10}}$ is the magnetic field of the TM_{10} spherical mode. Using [21, eq. (4)] for $\mathbf{H}^{\pm}_{TM_{10}}$, expression (22) can be reduced to

$$\mathbf{J}_{TM_{10}}(\theta, \phi) = i \frac{\sqrt{6}}{4\sqrt{\pi}} \left(k\sqrt{\eta} C^+ h_1^{(1)}(ka) \right.$$
$$\left. - k_s \sqrt{\eta_s} C^- z_1^{(6)}(k_s a) \right) \hat{\mathbf{a}}_\theta \sin\theta \quad (23a)$$
$$= J_{TM_{10}} \hat{\mathbf{a}}_\theta \sin\theta. \qquad (23b)$$

Due to the spherical geometry of the problem (Fig. 1) and the orthogonality property of the vector spherical wave functions on any spherical surface concentric to the origin, no other modes except TM_{10} are excited by the current (23b).

Similarly, using [20, eq. (4)], it can be shown that the current (7b) excites the TE_{10} spherical mode.

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Oleksiy S. Kim received the M.Sc. and Ph.D. degrees from the National Technical University of Ukraine, Kiev, in 1996 and 2000, respectively, both in electrical engineering.

In 2000, he joined the Antenna and Electromagnetics Group at the Technical University of Denmark (DTU), Lyngby, where he is currently an Associate Professor with the Department of Electrical Engineering, Electromagnetic Systems, DTU. His research interests include electrically small antennas, computational electromagnetics, metamaterials, photonic bandgap, and plasmonic structures.