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Pairs of dual periodic frames

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ABSTRACT

The time–frequency analysis of a signal is often performed via a series expansion arising from well-localized building blocks. Typically, the building blocks are based on frames having either Gabor or wavelet structure. In order to calculate the coefficients in the series expansion, a dual frame is needed. The purpose of the present paper is to provide constructions of dual pairs of frames in the setting of the Hilbert space of periodic functions $L^2(0,2\pi)$. The frames constructed are given explicitly as trigonometric polynomials, which allows for an efficient calculation of the coefficients in the series expansions. The generality of the setup covers periodic frames of various types, including nonstationary wavelet systems, Gabor systems and certain hybrids of them.

1. Introduction

Time–frequency representations play a fundamental role in many practical applications as they provide localized information of signals in time and frequency domains. Series representations in terms of frames capture such information at prescribed discrete points in the time–frequency plane. Gabor frames and wavelet frames are leading examples of frames from the perspective of time–frequency analysis, and both of them have their respective strengths, see for instance [3,7–9,15,20]. Technically, the series expansion provided by a frame requires knowledge of a dual frame, either for the synthesis or the analysis of the given signal. Therefore simultaneous constructions of a frame and a corresponding dual with desirable properties is a key issue. In addition, many signals of practical interest can be considered as periodic. Apart from signals that are inherently periodic, all signals resulting from experiments with a finite duration can in principle be modeled as periodic signals, see for example [18]. This motivates the current paper on periodic frames.

The purpose of this paper is to construct explicitly given frames and dual pairs of frames in $L^2(0,2\pi)$, the Hilbert space of 2π-periodic functions on $\mathbb{R}$ that are square-integrable over $(0,2\pi)$. The frames will be given as a collection of translates of a set of functions. Under suitable conditions we will also derive explicit expressions for associated dual frames. As concrete examples, we obtain frame constructions of Gabor type and wavelet type, as well as a certain hybrid of these. The practical relevance of the results is explained in the context of signal processing. More details on the premise of the paper will appear later in the introduction.

An outline of the paper is as follows. In the rest of this introduction we present a few basic definitions and facts about frames. We also give an example that motivates the theoretical results to follow. Then, in Section 2 we present sufficient conditions for a sequence of translates of a collection of functions in $L^2(0,2\pi)$ to be a Bessel sequence or a frame. In Section 3 we demonstrate how to explicitly construct dual pairs of frames. These dual pairs are frames comprised

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of trigonometric polynomials, which facilitate efficient analysis of periodic functions. The theoretical results are followed by various concrete examples, dealing with, e.g., Gabor analysis, stationary as well as nonstationary wavelet analysis, and various hybrids of these. The generating functions of these trigonometric polynomial frames have desirable properties such as being real-valued and symmetric, and possessing good time–frequency localization.

We now describe the setting of our study. Let I denote a subset of the integers $\mathbb{Z}$, let $\{L_k\}_{k\in I}$ be any countable sequence of positive integers, and define associated translation operators T_k acting on $L^2(0,2\pi)$ by

$$T_k f(x) := f\left(x - \frac{2\pi}{L_k}\right). \tag{1.1}$$

Note that composing T_k with itself leads to

$$T_k^{\ell} f(x) = f\left(x - \frac{2\pi\ell}{L_k}\right), \quad \ell \in \mathbb{Z}, \tag{1.2}$$

and so, $T_k^{L_k} f(x) = f(x-2\pi) = f(x)$, i.e., $T_k^{L_k}$ equals the identity operator. For each $k \in I$ we will apply the operators T_k^{ℓ} to a function ψ_k in $L^2(0,2\pi)$; thus, we consider the collection of functions $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$, where the index set is chosen to avoid repetitions. Note that this general setup allows us to apply different shifts to the involved functions ψ_k.

Our purpose is to consider frame properties for a collection of functions of the form $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ in $L^2(0,2\pi)$, so we will briefly recall a few standard results and facts about frames. We say that the collection $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ in $L^2(0,2\pi)$ forms a *frame* for $L^2(0,2\pi)$ if there exist positive constants A, B such that

$$A\|f\|^2 \leqslant \sum_{k\in I}\sum_{\ell=0}^{L_k-1} \left|\left\langle f, T_k^{\ell}\psi_k\right\rangle\right|^2 \leqslant B\|f\|^2, \quad f \in L^2(0,2\pi). \tag{1.3}$$

The constants A, B are called *bounds* of the frame. If at least the second inequality in (1.3) holds, then $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ is said to be a *Bessel sequence* and B its bound. Two collections $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^{\ell}\widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ in $L^2(0,2\pi)$ form a pair of *dual frames* if both collections are Bessel sequences and

$$f = \sum_{k\in I}\sum_{\ell=0}^{L_k-1} \left\langle f, T_k^{\ell}\psi_k\right\rangle T_k^{\ell}\widetilde{\psi_k}, \quad f \in L^2(0,2\pi). \tag{1.4}$$

It is well known that if $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^{\ell}\widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ are dual frames, the roles of $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^{\ell}\widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ can be interchanged in the analysis and synthesis of f.

The setup in the form of $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ encompasses periodic frames of various types. Different choices of $\{L_k\}_{k\in I}$, giving translation operators T_k of different shifts as defined in (1.1), determine the frame systems on hand. In particular, if $I=\mathbb{Z}$, $L_k = D$ for some positive integer D and $\psi_k = e^{ik\cdot}\psi_0$, then $T_k = T_0$ for all k, and by (1.2), $\{T_k^{\ell}\psi_k\}_{k\in\mathbb{Z},\ell=0,\dots,D-1}$ can be written as $\{e^{-2\pi ik\ell/D}e^{ik\cdot}T_k^{\ell}\psi_0\}_{k\in\mathbb{Z},\ell=0,\dots,D-1}$ which is a Gabor system generated by ψ_0, up to the constant factors $e^{-2\pi ik\ell/D}$. On the other hand, if $I = \mathbb{N}\cup\{0\}$ and $L_k = D^k$ for some integer $D \geqslant 2$, then T_k amounts to shifting by $\frac{2\pi}{D^k}$ and $\{T_k^{\ell}\psi_k\}_{k\geqslant 0,\ell=0,\dots,D^k-1}$ is a nonstationary wavelet system.

Let us motivate the constructions to follow from the perspective of signal processing. Here, and in the rest of the paper, the Fourier coefficients for a function $f \in L^2(0,2\pi)$ are denoted by

$$\widehat{f}(n) := \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx, \quad n \in \mathbb{Z}.$$

Let $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^{\ell}\widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ be a pair of trigonometric polynomial dual frames, and $f \in L^2(0,2\pi)$ a signal to be analyzed and synthesized. Using the frame $\{T_k^{\ell}\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ for the analysis of f, as in (1.4), we compute the frame coefficient

$$\left\langle f, T_k^{\ell}\psi_k\right\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{T_k^{\ell}\psi_k}(n)} = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)}e^{2\pi in\ell/L_k} \tag{1.5}$$

for $k \in I$, $\ell = 0,\dots,L_k-1$. This can be evaluated efficiently as each $T_k^{\ell}\psi_k$ is a trigonometric polynomial and so (1.5) is a finite sum. When explicit expressions for $\widetilde{\psi_k}$, $k \in I$, are available (which is the case in this paper), the signal f can be readily recovered from the reconstruction formula (1.4).

While the frame coefficients $\langle f, T_k^{\ell}\psi_k\rangle$, $k\in I$, $\ell = 0,\dots,L_k-1$, can be evaluated efficiently, one also needs to address whether they provide an effective time–frequency analysis of f. The following example highlights some of the issues involved in this.

Example 1.1. Let H be a function in $L^2(0,2\pi)$ which is well localized in time (for instance, a Dirichlet kernel or a Fejér kernel), and consider the signal $f \in L^2(0,2\pi)$ defined by

$$f(x) := \cos(n_1x)H(x-x_1) + \cos(n_1x)H(x-x_2) + \cos(n_2x)H(x-x_1) + \cos(n_2x)H(x-x_2) \tag{1.6}$$

for some distinct $n_1, n_2 \in \mathbb{N} \cup \{0\}$ and $x_1, x_2 \in [0, 2\pi)$. The signal f comprises four components of the form

$$f_{rs}(x) = \cos(n_rx)H(x-x_s),$$

a function that is localized in frequency around $n = \pm n_r$ and localized in time around $x = x_s$. A trigonometric polynomial frame $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ would resolve the components of f if for each component f_{rs} of f, we can find a frame element $T_k^\ell \psi_k$ that returns a large value of $|\langle f_{rs}, T_k^\ell \psi_k\rangle|$, while giving relatively small values for the other components.

In (1.6), if n_1 and n_2 as well as x_1 and x_2 are sufficiently far apart, then the analysis of f provided by (1.5) would successfully resolve the different components of f when $\widehat{\psi_k}$ and ψ_k are translated by appropriate constant amounts in frequency and time domains respectively. This is the typical setup of Gabor analysis, see Example 3.1.

If n_1 and n_2 are large values, then f is a high frequency signal which has rapid changes in time. In this case, for the analysis (1.5), it is natural to employ a finer shift $T_k^\ell \psi_k$ given by a larger value of L_k. On the other hand, if n_1 and n_2 are small values, then f is a low frequency signal, and it suffices to take a smaller value of L_k amounting to a coarser shift $T_k^\ell \psi_k$. This is precisely the flexibility provided by wavelet analysis where $L_k = D^k$ for some integer $D \geqslant 2$, see Example 3.2.

In the typical wavelet setup, the length of the support of $\widehat{\psi_k}$ often grows rapidly in multiples of D^k as k increases. So when n_1 and n_2 are large values but near to each other, they may both end up in the same support of $\widehat{\psi_k}$ for some large value of k. This can be avoided if the support of $\widehat{\psi_k}$ expands at a less rapid rate as k increases. On the other hand, when x_1 and x_2 in (1.6) are close to each other, we should utilize a fine shift $T_k^\ell \psi_k$, given by a large value of L_k, to resolve the components localized at these points in time. The balancing of these desirable features is incorporated into the construction of Example 3.3, which attempts to combine the strengths of Gabor analysis and wavelet analysis.

Previous work on periodic frames in the literature includes [13,14] in which periodic wavelet frames are obtained from multiresolution analyses, and [5] where pairs of oblique duals are constructed for finite-dimensional spaces of periodic functions. On the other hand, for the space $L^2(\mathbb{R})$, explicit pairs of dual Gabor frames are constructed in [4,6] (with corresponding results for dual wavelet frames reported in [19]) and [16,17]. Here we focus on the space $L^2(0,2\pi)$, and we adapt, unify and further develop these ideas to construct pairs of trigonometric polynomial frames. Some of our extensions are made possible only by the periodic setting on hand, i.e., corresponding results for $L^2(\mathbb{R})$ are not available.

In contrast to [4,19], our approach is based on a general nonstationary setup where different values of k may correspond to rather different functions ψ_k and parameters L_k. It also does not assume the multiresolution analysis framework in [13, 14], which typically takes $L_k = D^k$ for some integer $D \geqslant 2$.

2. Bessel sequences and frames of the form $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$

We first present a general sufficient condition for a system of functions of the form $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L^k-1}$ to be a Bessel sequence or form a frame for $L^2(0,2\pi)$. Similar results are known for Gabor systems and wavelet systems in $L^2(\mathbb{R})$ (see [2,3]). While our proof adapts appropriately the main ideas in establishing [3, Theorem 11.2.3] on wavelet frames for $L^2(\mathbb{R})$ to the space $L^2(0,2\pi)$, the nonstationary setting on hand gives a general result that is applicable to periodic Gabor systems, periodic wavelet systems, as well as other periodic systems of interest. This proof is provided in Appendix A.

Theorem 2.1. *Consider functions $\{\psi_k\}_{k\in I} \subset L^2(0,2\pi)$, let $\{L_k\}_{k\in I}$ be a sequence of positive integers, and assume that*

$$B := \sup_{n\in\mathbb{Z}} \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}} \left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_kq)\right| < \infty. \tag{2.1}$$

Then $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ is a Bessel sequence with bound B. If in addition,

$$A := \inf_{n\in\mathbb{Z}} \left(\sum_{k\in I} L_k \left|\widehat{\psi_k}(n)\right|^2 - \sum_{q\in\mathbb{Z}\setminus\{0\}} \sum_{k\in I} L_k \left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_kq)\right| \right) > 0, \tag{2.2}$$

then $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ is a frame for $L^2(0,2\pi)$ with bounds A, B.

For $L_k = D^k$ for some integer $D \geqslant 2$, another sufficient condition for $\{T_k^\ell \psi_k\}_{k\geqslant 0,\ell=0,\dots,D^k-1}$ to be a Bessel sequence can be found in [12, Theorem 4.1].

While Theorem 2.1 provides a condition for $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ to form a frame for $L^2(0,2\pi)$, it does not contain information about how an appropriate dual frame can be obtained. In the next section we will construct pairs of dual frames explicitly, and in that context the Bessel condition will play an important role. For this reason we now state some easily accessible conditions, based on Theorem 2.1, for $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots L_k-1}$ to be a Bessel sequence. Here, and in the rest of the paper, we put

$$S_k := \operatorname{supp} \widehat{\psi_k}, \tag{2.3}$$

i.e., S_k is the set of all $n \in \mathbb{Z}$ for which $\widehat{\psi_k}(n) \neq 0$. We will subsequently assume that ψ_k is a trigonometric polynomial, meaning that S_k is a finite set.

Corollary 2.1. *Consider functions $\{\psi_k\}_{k\in I} \subset L^2(0, 2\pi)$ and a sequence $\{L_k\}_{k\in I}$ of positive integers, and assume the following:*

(i) *For each $k \in I$, the set S_k is contained in an interval of length strictly less than JL_k for some $J \in \mathbb{N}$;*
(ii) *There exists a constant $C > 0$ such that*

$$\left|\widehat{\psi_k}(n)\right| \leqslant \frac{C}{\sqrt{L_k}}, \quad k \in I,\ n \in \mathbb{Z};$$

(iii) *There exists a number $K \in \mathbb{N}$ such that each $n \in \mathbb{Z}$ belongs to at most K of the sets S_k.*

Then $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ is a Bessel sequence with bound $KC^2(2J-1)$.

Proof. Fix $n \in \mathbb{Z}$. Assume that $n \in S_k$ for $k \in \{k_1, k_2, \dots, k_\mu\}$, where $k_1, \dots, k_\mu \in I$; by assumption $\mu \leqslant K$. Then

$$\begin{aligned}\sum_{k\in I} L_k \sum_{q\in\mathbb{Z}} \left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right| &= \sum_{k\in\{k_1,\dots,k_\mu\}} \sum_{q=-J+1}^{J-1} \left(\sqrt{L_k}\left|\widehat{\psi_k}(n)\right|\right)\left(\sqrt{L_k}\left|\widehat{\psi_k}(n+L_k q)\right|\right) \\ &\leqslant K(2J-1)\Bigl(\sup_{k\in I,n\in\mathbb{Z}} \sqrt{L_k}\left|\widehat{\psi_k}(n)\right|\Bigr)^2 \\ &\leqslant KC^2(2J-1).\end{aligned}$$

The result now follows from Theorem 2.1. $\square$

We will now use Corollary 2.1 to check the Bessel condition for some special classes of functions. In Section 3 we return to these examples and construct dual pairs of frames. Our first example leads to a system of functions with Gabor structure.

Example 2.1. Let g be a nontrivial, bounded, and real-valued function on $\mathbb{R}$ with support in an interval $[M, N]$ for some $M, N \in \mathbb{Z}$, $M < N$. For the sequence $\{L_k\}_{k\in I}$ in Corollary 2.1, take $I = \mathbb{Z}$ and $L_k = D$ for some positive integer D. We define a family of trigonometric polynomials

$$\psi_k(x) = \sum_{n\in\mathbb{Z}} \widehat{\psi_k}(n) e^{inx}, \quad k \in \mathbb{Z}, \tag{2.4}$$

by

$$\widehat{\psi_k}(n) := \frac{g(n-k)}{\sqrt{D}}, \quad n \in \mathbb{Z}. \tag{2.5}$$

Then for $k \in \mathbb{Z}$, $\ell = 0, \dots, D-1$, using (1.2), (2.4) and (2.5), a calculation gives

$$T_k^\ell \psi_k(x) = e^{-2\pi ik\ell/D} e^{ikx} \psi_0\left(x - \frac{2\pi\ell}{D}\right) = e^{-2\pi ik\ell/D} e^{ikx} T_k^\ell \psi_0(x). \tag{2.6}$$

Thus, up to the constant factors $e^{-2\pi ik\ell/D}$, we are dealing with a Gabor system generated by the function ψ_0.

Checking the conditions of Corollary 2.1, we see that (i) clearly holds because by (2.5), the set S_k is contained in an interval of length $N - M$ which is independent of k. Since g is bounded and $L_k = D$, (ii) is also satisfied. Finally, it follows from (2.5) that (iii) holds with $K = N - M + 1$. Hence, by Corollary 2.1, $\{T_k^\ell \psi_k\}_{k\in\mathbb{Z},\ell=0,\dots,D-1}$ is a Bessel sequence.

In Example 2.1 which leads to Gabor frames, the size of the support of $\widehat{\psi_k}$ is independent of k. The next example shows how to construct Bessel sequences consisting of trigonometric polynomials such that the support of $\widehat{\psi_k}$ grows exponentially with k, which sets the scene for wavelet frames.

Example 2.2. Let g be a nontrivial, bounded, and real-valued function on $\mathbb{R}$ with support in an interval $[M,N]$ for some $M,N\in\mathbb{R}$, $M<N$. Let $I=\mathbb{N}\cup\{0\}$, and consider the sequence $\{L_k\}_{k\geqslant 0}$ given by $L_k=D^k$ for some integer $D\geqslant 2$. Choose any integer $k_0>0$ and define a family of trigonometric polynomials

$$\psi_k(x)=\sum_{n\in\mathbb{Z}}\widehat{\psi_k}(n)e^{inx},\quad k=0,1,\ldots,\tag{2.7}$$

by

$$\widehat{\psi_k}(0):=\frac{1}{\sqrt{D^k}}\delta_{k,k_0},$$

and, for $n\neq 0$,

$$\widehat{\psi_k}(n):=\begin{cases}0, & \text{if } k=0,\ldots,k_0-1,\\ \frac{g(\log_D(|n|))}{\sqrt{D^k}}, & \text{if } k=k_0,\\ \frac{g(\log_D(\frac{|n|}{D^{-k+k_0}}))+g(\log_D(\frac{|n|}{D^{k-k_0}}))}{\sqrt{D^k}}, & \text{if } k>k_0.\end{cases}\tag{2.8}$$

Then the functions $\widehat{\psi_k}$ are real-valued and symmetric on $\mathbb{Z}$; and this gives generators ψ_k that are real-valued and symmetric.

In order to check condition (i) in Corollary 2.1, it is clearly enough to consider $k>k_0$. Then it is only possible that $\widehat{\psi_k}(n)\neq 0$ if

$$M\leqslant\log_D\left(\frac{|n|}{D^{-k+k_0}}\right)\leqslant N\quad\text{or}\quad M\leqslant\log_D\left(\frac{|n|}{D^{k-k_0}}\right)\leqslant N,\tag{2.9}$$

i.e., if

$$D^{M-k+k_0}\leqslant|n|\leqslant D^{N-k+k_0}\quad\text{or}\quad D^{M+k-k_0}\leqslant|n|\leqslant D^{N+k-k_0}.\tag{2.10}$$

Thus the support of $\widehat{\psi_k}$ is contained in an interval of length $D^{N+k-k_0}-(-D^{N+k-k_0})=2D^{N-k_0}D^k$ which is strictly less than JD^k with $J=2D^{N-k_0}+1$. The condition (ii) in Corollary 2.1 is trivially satisfied.

On the other hand, given any $n\in\mathbb{Z}\backslash\{0\}$ it follows from (2.9) that it is only possible that $\widehat{\psi_k}(n)\neq 0$ if

$$M\leqslant\log_D\big(|n|\big)-(-k+k_0)\leqslant N\quad\text{or}\quad M\leqslant\log_D\big(|n|\big)-(k-k_0)\leqslant N.$$

This holds for at most $N-M+1$ values of k, so condition (iii) in Corollary 2.1 is satisfied. Thus $\{T_k^\ell\psi_k\}_{k\geqslant 0,\ell=0,\ldots,D^k-1}$ is a Bessel sequence.

In our third example, $\operatorname{supp}\widehat{\psi_k}$ is the union of two disjoint sets of constant cardinality that move farther apart as k increases. This provides the setup for constructing pairs of dual trigonometric frames which are hybrids of Gabor and wavelet systems.

Example 2.3. Let g be a nontrivial, bounded, and real-valued function on $\mathbb{R}$ with support in an interval $[M,N]$ for some $M,N\in\mathbb{Z}$, $M<N$. With the index set $I=\mathbb{N}\cup\{0\}$, let $\{L_k\}_{k\geqslant 0}$ be a sequence of positive integers. We now fix some integer $k_0>0$ and define a family of trigonometric polynomials ψ_k, $k=0,1,\ldots,$ of the form (2.7) by

$$\widehat{\psi_k}(n):=\begin{cases}0, & \text{if } k=0,\ldots,k_0-1,\\ \frac{g(n)}{\sqrt{L_k}}, & \text{if } k=k_0,\\ \frac{g(n-k+k_0)+g(n+k-k_0)}{\sqrt{L_k}}, & \text{if } k>k_0,\end{cases}\tag{2.11}$$

for $n\in\mathbb{Z}$.

The assumption (iii) in Corollary 2.1 is satisfied with $K=N-M+1$. Due to the assumption that g is bounded, we can choose a constant $C>0$ such that $|g(x)|\leqslant C$ for all $x\in\mathbb{R}$; by the choice of $\widehat{\psi_k}(n)$ in (2.11) this implies that

$$\left|\sqrt{L_k}\widehat{\psi_k}(n)\right|\leqslant 2C,\quad k\geqslant 0,\ n\in\mathbb{Z},$$

i.e., condition (ii) in Corollary 2.1 is satisfied. Finally, note that if only one of the terms $g(n-k+k_0)$ and $g(n+k-k_0)$ appeared in (2.11) for $k>k_0$, then the set S_k would be contained in a translate of the interval $[M,N]$ of length $N-M$ which is strictly less than JL_k with $J=N-M+1$. Thus, by Corollary 2.1 we can conclude that if ψ_k is modified to just contain one of these terms for $k>k_0$, then we have a Bessel sequence. This means that the sequence $\{T_k^\ell\psi_k\}_{k\geqslant 0,\ell=0,\ldots,L_k-1}$ generated by our ψ_k as defined in (2.11) can be considered as a sum of two Bessel sequences, and therefore it is a Bessel sequence itself.

3. Dual pairs of trigonometric polynomial frames

The purpose of this section is to provide a construction of pairs of dual trigonometric polynomial frames $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$, $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ for $L^2(0,2\pi)$. The approach is very general and it can be tailored to give, among others, periodic Gabor frames and periodic wavelet frames. In the entire section, we will assume that the index set I is either $I=\mathbb{N}\cup\{0\}$ or $I=\mathbb{Z}$. As in (2.3), we let S_k denote the support of $\widehat{\psi_k}$.

Theorem 3.1. *Let $\{\psi_k\}_{k\in I}$ be a collection of trigonometric polynomials with real-valued Fourier coefficients, and consider any sequence $\{L_k\}_{k\in I}$ of positive integers. Assume that the following conditions are satisfied*:

(i) *There exists a constant $P\in\mathbb{N}$ such that $S_k\cap S_{k+\nu}=\emptyset$ for $\nu\geqslant P$ and all $k\in I$*;
(ii) *The collection $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ forms a Bessel sequence*;
(iii) *For any $n\in\mathbb{Z}$,*
$$1=\sum_{k\in I}\sqrt{L_k}\,\widehat{\psi_k}(n);$$
(iv) *For any $k\in I$,*
$$\rho_k:=\max\left\{|n-m|\;\middle|\;n\in S_k,\ m\in\bigcup_{\nu=0}^{P-1}S_{k+\nu}\right\}<L_k.$$

For $k\in I$, let $\widetilde{\psi_k}$ be defined by
$$\widehat{\widetilde{\psi_k}}(n):=\widehat{\psi_k}(n)+\frac{2}{\sqrt{L_k}}\sum_{\nu=1}^{P-1}\sqrt{L_{k+\nu}}\,\widehat{\psi_{k+\nu}}(n),\quad n\in\mathbb{Z}. \tag{3.1}$$

If $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ is also a Bessel sequence, then the functions $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ form a pair of dual frames for $L^2(0,2\pi)$.

Proof. Fix $n\in\mathbb{Z}$. It follows from assumption (i) that $n\in S_k$ can only hold for finitely many $k\in I$. Choose k_n as the smallest integer such that $n\in S_{k_n}$; then, if $n\in S_k$ for some k, we have
$$k\in\{k_n,k_n+1,\dots,k_n+P-1\}.$$
Using (iii), a standard but rather involved argument via induction on P shows that
$$\begin{aligned}1&=\left(\sum_{k=k_n}^{k_n+P-1}\sqrt{L_k}\,\widehat{\psi_k}(n)\right)^2\\&=L_{k_n}\widehat{\psi_{k_n}}(n)\left[\widehat{\psi_{k_n}}(n)+\frac{2\sqrt{L_{k_n+1}}}{\sqrt{L_{k_n}}}\widehat{\psi_{k_n+1}}(n)+\cdots+\frac{2\sqrt{L_{k_n+P-1}}}{\sqrt{L_{k_n}}}\widehat{\psi_{k_n+P-1}}(n)\right]\\&\quad+L_{k_n+1}\widehat{\psi_{k_n+1}}(n)\left[\widehat{\psi_{k_n+1}}(n)+\frac{2\sqrt{L_{k_n+2}}}{\sqrt{L_{k_n+1}}}\widehat{\psi_{k_n+2}}(n)+\cdots+\frac{2\sqrt{L_{k_n+P-1}}}{\sqrt{L_{k_n+1}}}\widehat{\psi_{k_n+P-1}}(n)\right]\\&\quad+\cdots+L_{k_n+P-1}\widehat{\psi_{k_n+P-1}}(n)\left[\widehat{\psi_{k_n+P-1}}(n)\right].\end{aligned}$$
Collecting the terms via finite sums and adding zeros leads to
$$\begin{aligned}1&=L_{k_n}\widehat{\psi_{k_n}}(n)\left[\widehat{\psi_{k_n}}(n)+\frac{2}{\sqrt{L_{k_n}}}\sum_{\nu=1}^{P-1}\sqrt{L_{k_n+\nu}}\,\widehat{\psi_{k_n+\nu}}(n)\right]\\&\quad+L_{k_n+1}\widehat{\psi_{k_n+1}}(n)\left[\widehat{\psi_{k_n+1}}(n)+\frac{2}{\sqrt{L_{k_n+1}}}\sum_{\nu=1}^{P-1}\sqrt{L_{k_n+1+\nu}}\,\widehat{\psi_{k_n+1+\nu}}(n)\right]+\cdots\\&\quad+L_{k_n+P-1}\widehat{\psi_{k_n+P-1}}(n)\left[\widehat{\psi_{k_n+P-1}}(n)+\frac{2}{\sqrt{L_{k_n+P-1}}}\sum_{\nu=1}^{P-1}\sqrt{L_{k_n+P-1+\nu}}\,\widehat{\psi_{k_n+P-1+\nu}}(n)\right].\end{aligned}$$
For any $k\in I$, $\widehat{\widetilde{\psi_k}}(n)$ is defined by (3.1); using that $\widehat{\psi_k}(n)=0$ for $k\notin\{k_n,\dots,k_n+P-1\}$, the above calculation shows that

$$1 = \sum_{k=k_n}^{k_n+P-1} L_k \widehat{\psi_k}(n)\widehat{\widetilde{\psi_k}}(n) = \sum_{k\in I} L_k \widehat{\psi_k}(n)\widehat{\widetilde{\psi_k}}(n). \tag{3.2}$$

Employing the notation in (A.2) and (A.3) in the proof of Theorem 2.1 and a similar one with $\langle h, T_k^\ell \widetilde{\psi_k}\rangle = \sum_{r=0}^{L_k-1} \widehat{\beta_k}(r) e^{2\pi i r\ell/L_k}$, we have that for all trigonometric polynomials f, h,

$$\begin{aligned}\sum_{\ell=0}^{L_k-1} \langle f, T_k^\ell \psi_k\rangle \overline{\langle h, T_k^\ell \widetilde{\psi_k}\rangle} &= \sum_{\ell=0}^{L_k-1}\sum_{j=0}^{L_k-1} \widehat{\alpha_k}(j) e^{2\pi i j\ell/L_k} \sum_{r=0}^{L_k-1} \overline{\widehat{\beta_k}(r)} e^{-2\pi i r\ell/L_k} \\ &= \sum_{j=0}^{L_k-1}\sum_{r=0}^{L_k-1} \widehat{\alpha_k}(j)\overline{\widehat{\beta_k}(r)} \sum_{\ell=0}^{L_k-1} e^{2\pi i(j-r)\ell/L_k} = L_k \sum_{j=0}^{L_k-1} \widehat{\alpha_k}(j)\overline{\widehat{\beta_k}(j)}.\end{aligned}$$

Then applying (A.3) gives

$$\begin{aligned}\sum_{\ell=0}^{L_k-1} \langle f, T_k^\ell \psi_k\rangle \overline{\langle h, T_k^\ell \widetilde{\psi_k}\rangle} &= L_k \sum_{j=0}^{L_k-1} \Big(\sum_{p\in\mathbb{Z}} \widehat{f}(j+L_k p)\widehat{\psi_k}(j+L_k p)\Big)\Big(\sum_{q\in\mathbb{Z}} \overline{\widehat{h}(j+L_k q)\widehat{\widetilde{\psi_k}}(j+L_k q)}\Big) \\ &= L_k \sum_{j=0}^{L_k-1}\sum_{p\in\mathbb{Z}} \widehat{f}(j+L_k p)\widehat{\psi_k}(j+L_k p) \sum_{q\in\mathbb{Z}} \overline{\widehat{h}(j+L_k p+L_k q)\widehat{\widetilde{\psi_k}}(j+L_k p+L_k q)} \\ &= L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{\psi_k}(n) \sum_{q\in\mathbb{Z}} \overline{\widehat{h}(n+L_k q)\widehat{\widetilde{\psi_k}}(n+L_k q)} \\ &= L_k \sum_{n\in\mathbb{Z}} \Big[\widehat{f}(n)\widehat{\psi_k}(n)\overline{\widehat{h}(n)\widehat{\widetilde{\psi_k}}(n)} + \sum_{q\in\mathbb{Z}\setminus\{0\}} \widehat{f}(n)\widehat{\psi_k}(n)\overline{\widehat{h}(n+L_k q)\widehat{\widetilde{\psi_k}}(n+L_k q)}\Big].\end{aligned} \tag{3.3}$$

In this expression the second term $\sum_{q\in\mathbb{Z}\setminus\{0\}} \widehat{f}(n)\widehat{\psi_k}(n)\overline{\widehat{h}(n+L_k q)\widehat{\widetilde{\psi_k}}(n+L_k q)}$ actually vanishes for all $k \in I$. If $n \notin S_k$ this is trivial; and if $n \in S_k$, then for any $q \in \mathbb{Z}\setminus\{0\}$ we have $|n-(n+L_k q)| = L_k|q| \geqslant L_k$, which by assumption (iv) implies that $n + L_k q \notin \bigcup_{\nu=0}^{P-1} S_{k+\nu}$ and thus $n + L_k q \notin \operatorname{supp}\widehat{\widetilde{\psi_k}}$.

Hence, considering the sum over $k \in I$ of the terms in (3.3) and using (3.2), we arrive at

$$\sum_{k\in I}\sum_{\ell=0}^{L_k-1} \langle f, T_k^\ell \psi_k\rangle \overline{\langle h, T_k^\ell \widetilde{\psi_k}\rangle} = \sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\widehat{\psi_k}(n)\overline{\widehat{h}(n)\widehat{\widetilde{\psi_k}}(n)} = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{h}(n)} = \langle f, h\rangle.$$

From here, since both the collections $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ and $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ are Bessel sequences, a standard duality argument implies that they form a pair of dual frames for $L^2(0,2\pi)$. □

Note that in order to apply Theorem 3.1, we need to check the assumptions (i) to (iv) as well as that the functions $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,L_k-1}$ form a Bessel sequence. In the following corollaries, we provide several ways of satisfying this, either in terms of conditions on the sequence $\{L_k\}_{k\in I}$ or by appropriate conditions that imply (i) to (iv) and the Bessel condition simultaneously.

Corollary 3.1. *In the special case where $L_k = D$ for some positive integer D, the assumptions* (i) *to* (iv) *in Theorem 3.1 produce a dual frame $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,D-1}$ from the functions $\widetilde{\psi_k}$ defined by*

$$\widehat{\widetilde{\psi_k}}(n) := \widehat{\psi_k}(n) + 2\sum_{\nu=1}^{P-1} \widehat{\psi_{k+\nu}}(n), \quad n \in \mathbb{Z}. \tag{3.4}$$

Proof. Note that, as a finite linear combination of Bessel sequences, $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I,\ell=0,\dots,D-1}$ is a Bessel sequence. Thus the result follows from Theorem 3.1. □

Corollary 3.2. *In the special case where $I = \mathbb{N}\cup\{0\}$ and $L_k = D^k$ for some integer $D \geqslant 2$, the assumptions* (i) *to* (iv) *in Theorem 3.1 yield a dual frame $\{T_k^\ell \widetilde{\psi_k}\}_{k\geqslant 0,\ell=0,\dots,D^k-1}$ generated by the functions $\widetilde{\psi_k}$ given by*

$$\widehat{\widetilde{\psi_k}}(n) := \widehat{\psi_k}(n) + 2\sum_{\nu=1}^{P-1} \sqrt{D^\nu}\,\widehat{\psi_{k+\nu}}(n), \quad n \in \mathbb{Z}. \tag{3.5}$$

Proof. For $\nu = 1, \ldots, P-1$, observe that $\{T_k^\ell \psi_{k+\nu}\}_{k\geqslant 0, \ell=0,\ldots,D^k-1}$ is a Bessel sequence. This is because

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{\ell=0}^{D^k-1}\left|\left\langle f, T_k^\ell \psi_{k+\nu}\right\rangle\right|^2 &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{D^k-1}\left|\left\langle f, T_{k+\nu}^{\ell D^\nu} \psi_{k+\nu}\right\rangle\right|^2 \\ &\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{D^{k+\nu}-1}\left|\left\langle f, T_{k+\nu}^{l} \psi_{k+\nu}\right\rangle\right|^2 = \sum_{\kappa=\nu}^{\infty}\sum_{l=0}^{D^\kappa-1}\left|\left\langle f, T_\kappa^{l} \psi_{\kappa}\right\rangle\right|^2 \\ &\leqslant \sum_{\kappa=0}^{\infty}\sum_{l=0}^{D^\kappa-1}\left|\left\langle f, T_\kappa^{l} \psi_{\kappa}\right\rangle\right|^2,\end{aligned}$$

and $\{T_k^\ell \psi_k\}_{k\geqslant 0, \ell=0,\ldots,D^k-1}$ is a Bessel sequence. The rest of the proof is the same as in the proof of Corollary 3.1. □

Corollaries 3.1 and 3.2 deal with special values of L_k. For the general case, our strategy is first to construct ψ_k that satisfy the assumptions in Corollary 2.1 and then consider the extra assumptions in Theorem 3.1. As observed in the following corollary, for general values of L_k, it is possible to impose a stronger condition in one of these assumptions and hereby ensure that $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I, \ell=0,\ldots,L_k-1}$ automatically is a Bessel sequence.

Corollary 3.3. *With $\{\psi_k\}_{k\in I}$ and $\{L_k\}_{k\in I}$ as in Theorem 3.1, suppose that the assumptions* (ii) *and* (iii) *in Corollary 2.1 and* (i) *and* (iii) *in Theorem 3.1 hold. In addition, assume that for any $k \in I$,*

$$\sigma_k := \max\left\{|n-m| \;\middle|\; n, m \in \bigcup_{\nu=0}^{P-1} S_{k+\nu}\right\} < L_k. \tag{3.6}$$

For $\widetilde{\psi_k}$ defined by (3.1), the collections $\{T_k^\ell \psi_k\}_{k\in I, \ell=0,\ldots,L_k-1}$ and $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I, \ell=0,\ldots,L_k-1}$ form a pair of dual frames for $L^2(0, 2\pi)$.

Proof. Note that the assumption (3.6) implies condition (i) in Corollary 2.1 as well as condition (iv) in Theorem 3.1. Then by Corollary 2.1, $\{T_k^\ell \psi_k\}_{k\in I, \ell=0,\ldots,L_k-1}$ is a Bessel sequence. The result will follow from Theorem 3.1 once we establish that $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I, \ell=0,\ldots,L_k-1}$ is also a Bessel sequence.

To this end, we apply Corollary 2.1 to the functions $\{\widetilde{\psi_k}\}_{k\in I}$. Indeed, for $k \in I$, we define $\widetilde{S_k} := \operatorname{supp} \widehat{\widetilde{\psi_k}}$. Then (3.1) implies that $\widetilde{S_k} \subseteq \bigcup_{\nu=0}^{P-1} S_{k+\nu}$, and by (3.6), $\widetilde{S_k}$ is contained in an interval of length strictly less than L_k. Using condition (ii) in Corollary 2.1, we obtain from (3.1) that

$$\left|\widehat{\widetilde{\psi_k}}(n)\right| \leqslant \left|\widehat{\psi_k}(n)\right| + \frac{2}{\sqrt{L_k}}\sum_{\nu=1}^{P-1}\sqrt{L_{k+\nu}}\left|\widehat{\psi_{k+\nu}}(n)\right| \leqslant \frac{(2P-1)C}{\sqrt{L_k}}, \quad n \in \mathbb{Z}.$$

Now, for a fixed $n \in \mathbb{Z}$, condition (iii) in Corollary 2.1 implies that $n \in S_k$ only for $k \in \{k_1, k_2, \ldots, k_\mu\}$, where $k_1, \ldots, k_\mu \in I$ and $\mu \leqslant K$. Take any $k \in \{k_1, \ldots, k_\mu\}$. By (i) in Theorem 3.1, $S_k \cap \bigcup_{\nu=0}^{P-1} S_{\kappa+\nu} = \emptyset$ for $\kappa \geqslant k+P$ or $\kappa \leqslant k-2P+1$. Thus S_k intersects at most $(k+P-1)-(k-2P+2)+1 = 3P-2$ sets of the form $\bigcup_{\nu=0}^{P-1} S_{\kappa+\nu}$. Since $\widetilde{S_\kappa} \subseteq \bigcup_{\nu=0}^{P-1} S_{\kappa+\nu}$, it follows that S_k intersects at most $3P-2$ of the sets $\widetilde{S_\kappa}$. This in turn shows that n lies in at most $K(3P-2)$ of the sets $\widetilde{S_\kappa}$ as $\mu \leqslant K$. Hence we conclude from Corollary 2.1 that $\{T_k^\ell \widetilde{\psi_k}\}_{k\in I, \ell=0,\ldots,L_k-1}$ is a Bessel sequence. □

With these results in place, we are ready to construct various general classes of dual pairs of trigonometric polynomial frames. As the following examples will demonstrate, a key issue turns out to be various partition of unity conditions.

Example 3.1. We continue the analysis of the setup in Example 2.1 with some minor adjustments in order to adapt to the assumptions of Theorem 3.1. First, we further assume that $\sum_{k\in\mathbb{Z}} g(k) = 1$. This can always be achieved by multiplying g with a nonzero constant, provided that $\sum_{k\in\mathbb{Z}} g(k) \neq 0$. Note that this assumption implies that

$$\sum_{k\in\mathbb{Z}} g(x-k) = 1, \quad x \in \mathbb{Z}. \tag{3.7}$$

Second, we also assume that the number D and the length of the interval $[M, N]$ are related by

$$D > 2(N-M). \tag{3.8}$$

Checking conditions (i) to (iv) in Theorem 3.1, we note that for $k \in \mathbb{Z}$ and $\nu \geqslant 1$, $S_k \subseteq \{M+k, \ldots, N+k\}$ and $S_{k+\nu} \subseteq \{M+k+\nu, \ldots, N+k+\nu\}$. Then $S_k \cap S_{k+\nu} = \emptyset$ if $N+k < M+k+\nu$, i.e., if $\nu > N-M$. So (i) holds with $P = N-M+1$. Condition (ii) has already been established in Example 2.1.

Using (2.5) and (3.7), for every $n \in \mathbb{Z}$,

$$\sum_{k\in\mathbb{Z}} \sqrt{D}\widehat{\psi_k}(n) = \sum_{k\in\mathbb{Z}} g(n-k) = 1,$$

which is condition (iii). In view of $\bigcup_{\nu=0}^{P-1} S_{k+\nu} \subseteq \{M+k, \dots, N+k+P-1\}$, we see that ρ_k in (iv) satisfies

$$\rho_k \leqslant (N+k+P-1)-(M+k) = 2(N-M) < D,$$

where (3.8) gives the final inequality.

Hence, we may apply Corollary 3.1 to construct a pair of dual periodic Gabor frames. In particular, for $\widetilde{\psi_k}$ defined from its Fourier coefficients $\widehat{\widetilde{\psi_k}}$ as in (2.4), it follows from a calculation via (3.4) and (2.5) that

$$\widetilde{\psi_k}(x) = e^{ikx} \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{D}} \left(g(n) + 2\sum_{\nu=1}^{N-M} g(n-\nu) \right) e^{inx} = e^{ikx}\widetilde{\psi_0}(x).$$

So for $k \in \mathbb{Z}$, $\ell = 0, \dots, D-1$, using (1.2),

$$T_k^\ell \widetilde{\psi_k}(x) = e^{-2\pi ik\ell/D} e^{ikx} \widetilde{\psi_0}\left(x - \frac{2\pi\ell}{D} \right) = e^{-2\pi ik\ell/D} e^{ikx} T_k^\ell \widetilde{\psi_0}(x), \tag{3.9}$$

which also has the Gabor structure. As (2.6) and (3.9) have the same constant factors $e^{-2\pi ik\ell/D}$, the frame expansion (1.4) reduces to the Gabor expansion

$$f = \sum_{k\in\mathbb{Z}} \sum_{\ell=0}^{D-1} \langle f, e^{ik\cdot} T_k^\ell \psi_0 \rangle e^{ik\cdot} T_k^\ell \widetilde{\psi_0}, \quad f \in L^2(0,2\pi).$$

Note that our construction of dual Gabor frames for $L^2(0,2\pi)$ in Example 3.1 originates from the frequency domain, while the approach for Gabor systems in $L^2(\mathbb{R})$ in [4] takes place in the time domain. As the frequency domain for $L^2(0,2\pi)$ is the integers $\mathbb{Z}$, we only require condition (3.7), which, as we have discussed, can be satisfied after a simple modification of the function g. On the other hand, the construction for $L^2(\mathbb{R})$ requires a partition of unity over the real line $\mathbb{R}$ that is more complicated to satisfy.

Next, we use Corollary 3.2 to construct pairs of dual periodic wavelet frames.

Example 3.2. We continue the analysis of the functions ψ_k in Example 2.2, with the extra assumption that g satisfies the partition of unity condition

$$\sum_{k\in\mathbb{Z}} g(x-k) = 1, \quad x \in \mathbb{R}.$$

We will also assume that k_0 is a positive integer satisfying

$$k_0 > N + \log_D\left(D^{N-M}+1\right). \tag{3.10}$$

(The reason for this choice of k_0 will be revealed later in the example.) Note that in contrast to the situation in Example 3.1, see (3.7), we now need the partition of unity to hold for all $x \in \mathbb{R}$. This is more restrictive, but it is satisfied, e.g., for any B-spline or any scaling function. We will verify conditions (i) to (iv) in Theorem 3.1 and then apply Corollary 3.2. First note that (2.10) shows that $\widehat{\psi_k}$ might consist of two "bumps" on the positive axis and two bumps on the negative axis. If $N-k+k_0 < 0$, i.e., if $k > N+k_0$, there will be only one bump on each of the positive axis and the negative axis.

We now check condition (i) in Theorem 3.1. To this end, let $\lfloor\cdot\rfloor$ denote the floor function, and let $P = \lfloor N-M+1 \rfloor$. If $k > k_0$, then for $\nu \geqslant P > N-M$, we have $D^{N-(k+\nu)+k_0} < D^{M-k+k_0}$ and $D^{N+k-k_0} < D^{M+(k+\nu)-k_0}$. Applying both (2.10) directly as well as (2.10) with $k+\nu$ in place of k, we see that $S_k \cap S_{k+\nu} = \emptyset$. In addition, by (2.8), $\widehat{\psi_{k_0}}(n) \neq 0$ only if $D^M \leqslant |n| \leqslant D^N$. For $\nu \geqslant P > N-M$, we have $D^{N-\nu} < D^M$ and $D^N < D^{M+\nu}$, which show that $S_{k_0} \cap S_{k_0+\nu} = \emptyset$. As $S_k = \emptyset$ for $k = 0, \dots, k_0-1$, condition (i) in Theorem 3.1 holds for all $k \geqslant 0$.

Condition (ii) in Theorem 3.1 has already been verified in Example 2.2. Using that $\log_D(\frac{|n|}{D^\kappa}) = \log_D(|n|) - \kappa$, the definition of the functions $\widehat{\psi_k}$ shows that for $n \neq 0$,

$$\begin{aligned}\sum_{k=0}^{\infty} \sqrt{D^k}\widehat{\psi_k}(n) &= g\left(\log_D\left(|n|\right)\right) + \sum_{k=k_0+1}^{\infty} \left(g\left(\log_D\left(|n|\right) + (k-k_0)\right) + g\left(\log_D\left(|n|\right) - (k-k_0)\right)\right) \\ &= \sum_{\kappa\in\mathbb{Z}} g\left(\log_D\left(|n|\right) - \kappa\right) = 1.\end{aligned}$$

Thus condition (iii) in Theorem 3.1 is satisfied for $n \neq 0$. Clearly, it is satisfied for $n = 0$ as well.

In order to verify condition (iv) in Theorem 3.1, we note that the largest number in $\bigcup_{\nu=0}^{P-1} S_{k+\nu}$ is not more than

$$D^{N+k+P-1-k_0} = D^{N+k+\lfloor N-M+1\rfloor-1-k_0} \leqslant D^{N+k+N-M-k_0} = D^{2N-M+k-k_0}.$$

The minimal number in S_k is not less than $-D^{N+k-k_0}$. Thus

$$\rho_k \leqslant D^{2N-M+k-k_0} - \left(-D^{N+k-k_0}\right) = D^k\left(D^{2N-M-k_0} + D^{N-k_0}\right).$$

For condition (iv) in Theorem 3.1 to hold, it suffices to have $D^{2N-M-k_0} + D^{N-k_0} < 1$ which is equivalent to (3.10), as assumed in this example. Hence, it follows from Corollary 3.2 that $\{T_k^\ell \psi_k\}_{k\geqslant 0,\ell=0,\ldots,D^k-1}$ and $\{T_k^\ell \widetilde{\psi_k}\}_{k\geqslant 0,\ell=0,\ldots,D^k-1}$ form a pair of dual wavelet frames for $L^2(0, 2\pi)$.

Concrete constructions based on the above can easily be realized by taking the function g to be a centered B-spline supported on $[-N, N]$. As an illustration, we set

$$g(x) := B_3(x) = \begin{cases} \frac{1}{2}x^2 + \frac{3}{2}x + \frac{9}{8}, & \text{if } -\frac{3}{2} \leqslant x \leqslant -\frac{1}{2}, \\ -x^2 + \frac{3}{4}, & \text{if } -\frac{1}{2} \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{2}x^2 - \frac{3}{2}x + \frac{9}{8}, & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{3}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

Then g is nonnegative, bounded, real-valued, and supported on $[-\frac{3}{2}, \frac{3}{2}]$, i.e., $M = -\frac{3}{2}$ and $N = \frac{3}{2}$. We take the dilation factor D to be 2. Thus we can let $P = \lfloor N - M + 1\rfloor = 4$, and take $k_0 > N + \log_2(2^{N-M} + 1) = \frac{3}{2} + \log_2 9$, e.g., $k_0 = 5$. With this choice, $\widehat{\psi_k}$ only has one bump on each of the positive and negative axes if $k > N + k_0 = \frac{13}{2}$.

Fig. 1 shows the plots of ψ_k and $\widehat{\psi_k}$ defined by (3.11) and (2.8) for $D = 2$ and $k = 8$, and the corresponding $\widetilde{\psi_k}$ and $\widehat{\widetilde{\psi_k}}$ from (3.5).

Note that Example 3.2 shows that Corollary 3.2 can be used to construct wavelet frames $\{T_k^\ell \psi_k\}_{k\geqslant 0,\ell=0,\ldots,D^k-1}$ for $L^2(0, 2\pi)$ with only one generator ψ_k for each level k. This is in contrast to the approaches based on multiresolution analyses in [13,14]: typically, for a dilation factor $D \geqslant 2$, a periodic wavelet frame constructed from a multiresolution analysis would have at least $D - 1$ generators at each level.

Returning to more general values of L_k, we now employ Corollary 3.3 to obtain explicit pairs of periodic frames that are hybrids of Gabor and wavelet types.

Example 3.3. We continue further the setup in Example 2.3 with a few minor modifications. As in Example 3.1, if $\sum_{k\in\mathbb{Z}} g(k) \neq 0$, by multiplying g with a nonzero constant, we can ensure that (3.7) holds. In addition, we assume that for an appropriate integer $k_0 > 0$,

$$3(N - M) + 2k < L_k, \quad k \geqslant k_0. \tag{3.12}$$

This is possible, e.g., if we assume that $L_k - 2k \to \infty$ as $k \to \infty$. We now verify the conditions in Corollary 3.3.

As conditions (ii) and (iii) in Corollary 2.1 have already been verified in Example 2.3, to apply Corollary 3.3, it remains to check conditions (i) and (iii) in Theorem 3.1 and the inequality (3.6). Note that $\operatorname{supp}\widehat{\psi_{k_0}} \subseteq \{M, \ldots, N\}$, and for $k > k_0$,

$$\begin{aligned}\operatorname{supp}\widehat{\psi_k} &\subseteq \left\{M - (k - k_0), \ldots, N - (k - k_0)\right\} \cup \{M + k - k_0, \ldots, N + k - k_0\} \\ &\subseteq \left\{M - (k - k_0), \ldots, N + k - k_0\right\}.\end{aligned} \tag{3.13}$$

It follows that for $\nu \geqslant 1$,

$$S_{k+\nu} \subseteq \left\{M - (k + \nu - k_0), \ldots, N - (k + \nu - k_0)\right\} \cup \{M + k + \nu - k_0, \ldots, N + k + \nu - k_0\}.$$

Consequently, $S_k \cap S_{k+\nu} = \emptyset$ if $N + k - k_0 < M + k + \nu - k_0$ and $N - (k + \nu - k_0) < M - (k - k_0)$, i.e., if $\nu \geqslant N - M + 1$. So condition (i) in Theorem 3.1 is satisfied with $P = N - M + 1$. Also, for each $n \in \mathbb{Z}$,

$$\sum_{k=0}^{\infty} \sqrt{L_k}\,\widehat{\psi_k}(n) = g(n) + \sum_{k=k_0+1}^{\infty} \left(g(n - k + k_0) + g(n + k - k_0)\right) = \sum_{\kappa\in\mathbb{Z}} g(n - \kappa) = 1,$$

where (3.7) is applied in the last equality. Thus condition (iii) in Theorem 3.1 is satisfied.

In order to check (3.6), we note that the largest element in $\bigcup_{\nu=0}^{P-1} S_{k+\nu}$ is at most $N + k + N - M - k_0$ and that the minimal element in $\bigcup_{\nu=0}^{P-1} S_{k+\nu}$ is at least $M - (k + N - M - k_0)$. Thus, for σ_k in (3.6), we have

$$\sigma_k \leqslant N + k + N - M - k_0 - \left(M - (k + N - M - k_0)\right) = 3(N - M) + 2(k - k_0) \leqslant 3(N - M) + 2k.$$

As a result, choosing k_0 such that (3.12) holds, the inequality (3.6) is satisfied. Hence, by Corollary 3.3, this setup leads to pairs of dual periodic frames that are hybrids of Gabor frames and wavelet frames.

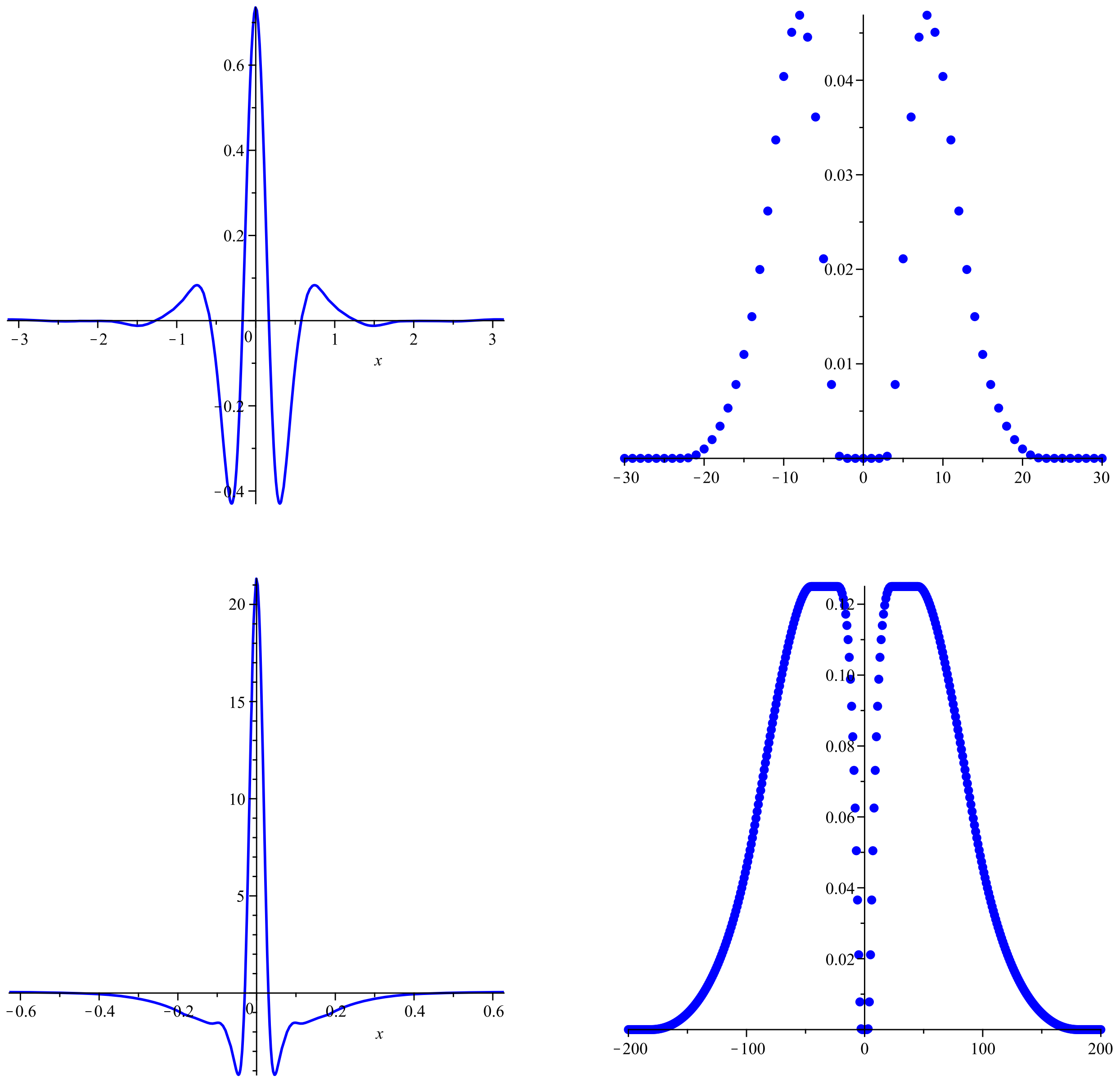

Fig. 1. Plots of ψ_k (top left) and $\widehat{\psi_k}$ (top right) defined by (3.11) and (2.8) for $D=2$, $k_0=5$ and $k=8$, and the corresponding $\widetilde{\psi_k}$ (bottom left) and $\widehat{\widetilde{\psi_k}}$ (bottom right) from (3.5) as in Example 3.2.

In a special case the setup in Example 3.3 can be tailored to a construction of real-valued and symmetric frame generators ψ_k.

Example 3.4. Employing the setup in Example 3.3, we impose the additional assumptions that g is continuous, symmetric, and supported on $[-N,N]$ for some $N \geqslant 1$. This is in line with our aim of constructing frame generators ψ_k that are real-valued and symmetric, which amounts to the requirement that the functions $\widehat{\psi_k}$ are real-valued and symmetric on $\mathbb{Z}$.

For a fixed positive integer k_0, defining $\widehat{\psi_k}(n)$ as in (2.11), we now check that for every k it holds that $\widehat{\psi_k}(n)$ is real-valued and that $\widehat{\psi_k}(-n)=\widehat{\psi_k}(n)$ for all $n \in \mathbb{Z}$. We note that for $k=k_0$, these statements are obvious because $g(-x)=g(x)$. Next, consider any $k>k_0$. Then, for any $n \in \mathbb{Z}$,

$$\widehat{\psi_k}(-n)=\frac{g(-n-k+k_0)+g(-n+k-k_0)}{\sqrt{L_k}}=\frac{g(n+k-k_0)+g(n-k+k_0)}{\sqrt{L_k}}=\widehat{\psi_k}(n).$$

Since g is continuous and supported on $[-N,N]$, we have $g(-N)=g(N)=0$. Thus $\operatorname{supp}\widehat{\psi_{k_0}} \subseteq \{-N+1,\dots,N-1\}$, and it follows from (3.13) that for $k>k_0$,

$$\operatorname{supp}\widehat{\psi_k} \subseteq \{-N+1-(k-k_0),\dots,N-1+k-k_0\}.$$

Consequently, we can take P in Theorem 3.1 as $P = (N-1)-(-N+1)+1 = 2N-1$.

As an illustration of this construction, for any positive integer N, we take

$$g(x) := \begin{cases} \tan\frac{\pi}{4N}\cos\frac{\pi x}{2N}, & \text{if } -N \leqslant x \leqslant N, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

Then g is nonnegative, continuous, real-valued, symmetric, and supported on $[-N, N]$. By a straightforward calculation,

$$\sum_{k=-N}^{N}\cos\frac{k\pi}{2N} = \sum_{k=-N+1}^{N-1}\cos\frac{k\pi}{2N} = \cot\frac{\pi}{4N},$$

which shows that $\sum_{k\in\mathbb{Z}} g(k) = 1$. Thus, (3.7) holds. For the sequence $\{L_k\}_{k\geqslant 0}$, we put $L_k = k^2$. Then we can find an appropriate k_0 for which

$$6(N-1)+2k < k^2, \quad k \geqslant k_0. \tag{3.15}$$

This gives (3.12).

Based on the function g in (3.14), the trigonometric polynomial ψ_k obtained from (2.11) achieves optimal time-localization in the following sense. In [1], time-localization of a function f in $L^2(0,2\pi)$ with $\sum_{n\in\mathbb{Z}}\widehat{f}(n)\overline{\widehat{f}(n+1)} \neq 0$ is measured by its angular variance defined by

$$\Delta_\theta(f)^2 = \frac{(\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|^2)^2 - |\sum_{n\in\mathbb{Z}}\widehat{f}(n)\overline{\widehat{f}(n+1)}|^2}{|\sum_{n\in\mathbb{Z}}\widehat{f}(n)\overline{\widehat{f}(n+1)}|^2}. \tag{3.16}$$

It is shown in [10,11] that among all $h \in L^2(0,2\pi)$ with $\operatorname{supp}\widehat{h} = \{-N+1,\dots,N-1\}$,

$$\Delta_\theta(h)^2 \geqslant \Delta_\theta(\psi_{k_0})^2,$$

i.e., the minimum angular variance is attained by ψ_{k_0}. Using this fact and the definition (3.16), a calculation then shows that for each $k \geqslant k_0+N$, ψ_k gives the minimum angular variance among all f whose Fourier coefficients are of the form

$$\widehat{f}(n) = \widehat{h}(n-k+k_0) + \widehat{h}(n+k-k_0), \quad n \in \mathbb{Z},$$

where $h \in L^2(0,2\pi)$ with $\operatorname{supp}\widehat{h} = \{-N+1,\dots,N-1\}$.

Taking $N = 5$, (3.15) holds for all $k \geqslant 7$, so we choose $k_0 = 7$. Fig. 2 shows the plots of ψ_k and $\widehat{\psi_k}$ defined by (3.14) and (2.11) for $N = 5$ and $k = 12$, and the corresponding $\widetilde{\psi_k}$ and $\widehat{\widetilde{\psi_k}}$ from (3.1).

We end the paper with a comment on application of the constructed dual periodic frames to practical problems. Certain applications involve thresholding, such as during denoising and deconvolution, or require visualization of the time–frequency plane. In these instances, the frame coefficients $\langle f, T_k^\ell\psi_k\rangle$ given by (1.5) have to be compared over all $k \in I$ and $\ell = 0,\dots,L_k-1$. However, our periodic frames $\{T_k^\ell\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ are nonstationary; in particular, the norm $\|T_k^\ell\psi_k\|$, which equals $\|\psi_k\|$, changes as k varies. Thus, in order to obtain a meaningful analysis of f, the frame coefficients $\langle f, T_k^\ell\psi_k\rangle$ need to be normalized during the processing. The calibration can be achieved via dividing $\langle f, T_k^\ell\psi_k\rangle$ by $\|\psi_k\|$. Note that this is just a practical measure for processing in applications. After the required analysis, we would still apply (1.4) to the original coefficients $\langle f, T_k^\ell\psi_k\rangle$ for the synthesis of f.

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Appendix A. Proof of Theorem 2.1

To show that $\{T_k^\ell\psi_k\}_{k\in I,\ell=0,\dots,L_k-1}$ is a Bessel sequence with bound B, it suffices to establish that the inequality

$$\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\left|\langle f, T_k^\ell\psi_k\rangle\right|^2 \leqslant B\|f\|^2$$

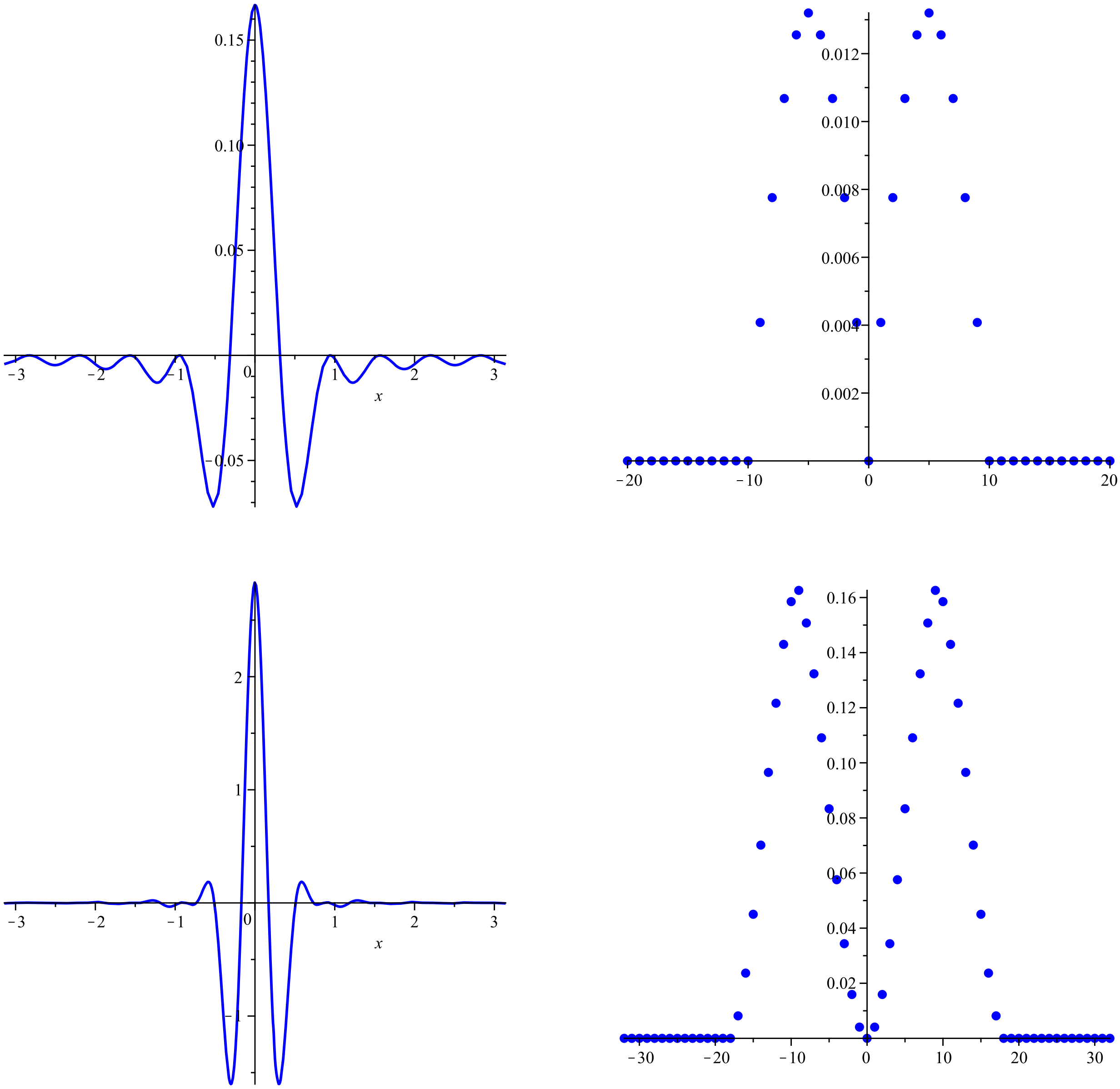

Fig. 2. Plots of ψ_k (top left) and $\widehat{\psi_k}$ (top right) defined by (3.14) and (2.11) for $L_k = k^2$, $N = 5$, $k_0 = 7$ and $k = 12$, and the corresponding $\widetilde{\psi_k}$ (bottom left) and $\widehat{\widetilde{\psi_k}}$ (bottom right) from (3.1) as in Example 3.4.

holds for all f in a dense subspace of $L^2(0, 2\pi)$; we will consider the subspace formed by all trigonometric polynomials.

By the same arguments as in the first part of the proof of [12, Theorem 4.1],

$$\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\left|\left\langle f, T_k^\ell \psi_k\right\rangle\right|^2 = \sum_{k\in I} L_k \sum_{j=0}^{L_k-1}\left|\sum_{p\in\mathbb{Z}} \widehat{f}(j+L_k p)\overline{\widehat{\psi_k}(j+L_k p)}\right|^2. \tag{A.1}$$

Indeed, as we have seen already in (1.5), we can write

$$\left\langle f, T_k^\ell \psi_k\right\rangle = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)} e^{2\pi i n\ell/L_k} = \sum_{j=0}^{L_k-1} \widehat{\alpha_k}(j) e^{2\pi i j\ell/L_k}, \tag{A.2}$$

where $\widehat{\alpha_k}$ is the L_k-periodic sequence defined by

$$\widehat{\alpha_k}(j) := \sum_{p\in\mathbb{Z}} \widehat{f}(j+L_k p)\overline{\widehat{\psi_k}(j+L_k p)}, \quad j=0,\dots,L_k-1; \tag{A.3}$$

now the expression in (A.1) is obtained by applying the inverse finite Fourier transform to $\widehat{\alpha_k}$, followed by an application of Parseval's identity.

Now it follows from (A.1) that

$$\begin{aligned}\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\left|\left\langle f, T_k^\ell \psi_k\right\rangle\right|^2 &= \sum_{k\in I} L_k \sum_{j=0}^{L_k-1}\sum_{p\in\mathbb{Z}} \widehat{f}(j+L_k p)\overline{\widehat{\psi_k}(j+L_k p)} \sum_{q\in\mathbb{Z}} \overline{\widehat{f}(j+L_k q)}\widehat{\psi_k}(j+L_k q) \\ &= \sum_{k\in I} L_k \sum_{p\in\mathbb{Z}}\sum_{j=0}^{L_k-1} \widehat{f}(j+L_k p)\overline{\widehat{\psi_k}(j+L_k p)} \sum_{q\in\mathbb{Z}} \overline{\widehat{f}(j+L_k q)}\widehat{\psi_k}(j+L_k q).\end{aligned} \tag{A.4}$$

Note that with the exception of the sum over k, all the above summations are finite sums as f is a trigonometric polynomial. The L_k-periodicity of $\widehat{\alpha_k}$ implies that for every $j=0,\dots,L_k-1$ and $p\in\mathbb{Z}$,

$$\overline{\widehat{\alpha_k}(j+L_k p)} = \overline{\widehat{\alpha_k}(j)} = \sum_{q\in\mathbb{Z}} \overline{\widehat{f}(j+L_k q)}\widehat{\psi_k}(j+L_k q);$$

thus (A.4) can be rewritten as

$$\begin{aligned}\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\left|\left\langle f, T_k^\ell \psi_k\right\rangle\right|^2 &= \sum_{k\in I} L_k \sum_{p\in\mathbb{Z}}\sum_{j=0}^{L_k-1} \widehat{f}(j+L_k p)\overline{\widehat{\psi_k}(j+L_k p)} \sum_{q\in\mathbb{Z}} \overline{\widehat{f}\big((j+L_k p)+L_k q\big)}\widehat{\psi_k}\big((j+L_k p)+L_k q\big) \\ &= \sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)} \sum_{q\in\mathbb{Z}} \overline{\widehat{f}(n+L_k q)}\widehat{\psi_k}(n+L_k q).\end{aligned} \tag{A.5}$$

We shall now interchange the infinite sum over k with some of the other sums in (A.5). This can be justified by replacing all the terms in (A.5) by their absolute values and following the arguments below. Interchanging the sums in (A.5), we have

$$\begin{aligned}\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\left|\left\langle f, T_k^\ell \psi_k\right\rangle\right|^2 &= \sum_{q\in\mathbb{Z}}\sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)}\,\overline{\widehat{f}(n+L_k q)}\widehat{\psi_k}(n+L_k q) \\ &= \sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)}\overline{\widehat{f}(n)}\widehat{\psi_k}(n) + \sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)}\overline{\widehat{f}(n+L_k q)}\widehat{\psi_k}(n+L_k q) \\ &= \sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right|^2 \sum_{k\in I} L_k\left|\widehat{\psi_k}(n)\right|^2 + R,\end{aligned} \tag{A.6}$$

where

$$R := \sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{k\in I} L_k \sum_{n\in\mathbb{Z}} \widehat{f}(n)\overline{\widehat{\psi_k}(n)}\overline{\widehat{f}(n+L_k q)}\widehat{\psi_k}(n+L_k q).$$

Applying the Cauchy–Schwarz inequality twice, we have

$$\begin{aligned}|R| &\leqslant \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\overline{\widehat{f}(n+L_k q)}\overline{\widehat{\psi_k}(n)}\widehat{\psi_k}(n+L_k q)\right| \\ &\leqslant \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}\setminus\{0\}} \left(\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right|^2\left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right|\right)^{1/2} \left(\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n+L_k q)\right|^2\left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right|\right)^{1/2} \\ &\leqslant \sum_{k\in I} L_k \left(\sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n)\right|^2\left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right|\right)^{1/2} \left(\sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n+L_k q)\right|^2\left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right|\right)^{1/2}.\end{aligned} \tag{A.7}$$

Observe that by the substitution $m=n+L_k q$ for each fixed $q\in\mathbb{Z}\setminus\{0\}$,

$$\begin{aligned}\sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{n\in\mathbb{Z}}\left|\widehat{f}(n+L_k q)\right|^2\left|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\right| &= \sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{m\in\mathbb{Z}}\left|\widehat{f}(m)\right|^2\left|\widehat{\psi_k}(m-L_k q)\widehat{\psi_k}(m)\right| \\ &= \sum_{q\in\mathbb{Z}\setminus\{0\}}\sum_{m\in\mathbb{Z}}\left|\widehat{f}(m)\right|^2\left|\widehat{\psi_k}(m)\widehat{\psi_k}(m+L_k q)\right|;\end{aligned}$$

therefore (A.7) implies that

$$|R| \leqslant \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|^2 \big|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\big|. \tag{A.8}$$

Applying this estimate to (A.6), we obtain

$$\begin{aligned}\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\big|\langle f, T_k^\ell \psi_k\rangle\big|^2 &\leqslant \sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 \sum_{k\in I} L_k \big|\widehat{\psi_k}(n)\big|^2 + \sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}\setminus\{0\}} \big|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\big| \\ &= \sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}} \big|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\big|.\end{aligned}$$

Hence, it follows from (2.1) that

$$\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\big|\langle f, T_k^\ell \psi_k\rangle\big|^2 \leqslant B\sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 = B\|f\|^2,$$

proving that $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\ldots,L_k-1}$ is a Bessel sequence with bound B.

If (2.2) also holds, let again f be a trigonometric polynomial. Then by (A.6) and (A.8),

$$\begin{aligned}\sum_{k\in I}\sum_{\ell=0}^{L_k-1}\big|\langle f, T_k^\ell \psi_k\rangle\big|^2 &\geqslant \sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 \sum_{k\in I} L_k \big|\widehat{\psi_k}(n)\big|^2 - \sum_{k\in I} L_k \sum_{q\in\mathbb{Z}\setminus\{0\}} \sum_{n\in\mathbb{Z}} \big|\widehat{f}(n)\big|^2 \big|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\big| \\ &= \sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 \left(\sum_{k\in I} L_k \big|\widehat{\psi_k}(n)\big|^2 - \sum_{q\in\mathbb{Z}\setminus\{0\}} \sum_{k\in I} L_k \big|\widehat{\psi_k}(n)\widehat{\psi_k}(n+L_k q)\big|\right) \\ &\geqslant A\sum_{n\in\mathbb{Z}}\big|\widehat{f}(n)\big|^2 = A\|f\|^2.\end{aligned}$$

Since this holds for all trigonometric polynomials f, we conclude that $\{T_k^\ell \psi_k\}_{k\in I,\ell=0,\ldots,L_k-1}$ is a frame for $L^2(0,2\pi)$ with bounds A, B.

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